INVERSE PROBLEMS FOR STURM–LIOUVILLE OPERATORS WITH BESSEL-TYPE SINGULARITY INSIDE AND INTERVAL

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What is Inverse Problem?

Inverse problems are concerned with determining causes for a desired or an observed effect.

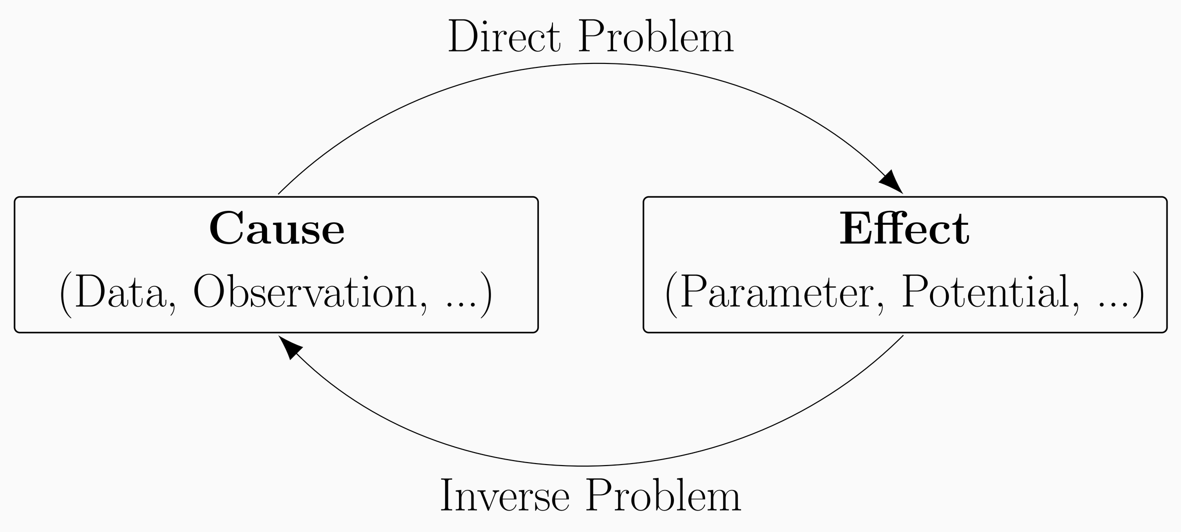

Fig. 1: A relation between direct and inverse problems

The situation in which the parameters or sources are directly accessible by measurement is called direct measurement. However, in many cases, the information searched for is not directly measurable. Instead, it is physically connected to other measurable values. We call this situation indirect measurement. The laws of physics that link observable information to the values searched for are generally mathematically complex, using, for example, integral equations or partial derivatives. The solution of a problem that calculates observable effects from unknown values, or a direct problem, is often simpler than that of an inverse problem, which calculates unknown values from observable effects.

The Planet

The discovery of the planet Neptune is a famous example of an inverse problem. At the beginning of the 19th century, the most distant discovered planet was Uranus. When astronomers applied Newton's theory of universal gravitation to the movement of Uranus, the calculations did not match the observed orbit and movement.

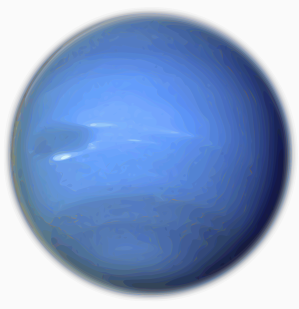

Fig. 2: The planet Neptune

In 1844 a French mathematician Urbain Le Verrier began to work at this calculation, using inverse perturbations of features (mass, orbit, current position) of the hypothetical planet assumed to produce Uranus' observed irregularities. In his report to the Academy of Sciences on August 31, 1846, Le Verrier presented the orbital elements of this new planet. Later, using the predicted position, the Berlin astronomer Galle successfully observed the planet Neptune.

The First Result

The inverse spectral problems theory was discovered and introduced by Soviet-Armenian astrophysicist Victor Ambartsumian.

Just after his graduation from the University of Leningrad he was interested in what degree do the eigenvalues of an ordinary differential operator determine the functions and parameters entering into that operator? Ambartsumian published in 1929 in Zeitschrift für Physik a paper which contained the statement of the general problem and the proof of the theorem that among all strings the homogeneous string is uniquely defined by the set of its oscillation-frequencies.

Theorem

Consider the boundary value problem

$$-y'' + q(x)y = \lambda y, \quad y'(0) = y'(\pi) = 0. \qquad (1)$$

If the eigenvalues of (1) are $\lambda_n = n^2$, $n \geq 0$, then $q(x) = 0$ a.e. on $(0, \pi)$.

During the fifteen years nobody has taken notice of that paper. However in 1944 that work was found by Swedish mathematicians who have obtained many interesting results related to the "inverse Sturm-Liouville problem" and thus that paper became the foundation of an entire discipline.

Acknowledgements

This work was supported by the Ministry of Education and Science of the Russian Federation (Grant 1.1436.2014K) and NANUM 2014 Travel Grant.

References

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The Problem

Consider the boundary value problem $\mathcal{L} = \mathcal{L}(q)$

$$-y'' + \left(\frac{\nu_0}{(x-a)^2} + q(x)\right)y = \lambda y, \quad x > 0, \qquad (2)$$

$$y(0) = 0$$

on the half-line with a Bessel-type singularity at an interior point $a > 0$ and additional *matching conditions* near the singular point $x = a$ (see [1]). Here $q(x)$ is a complex-valued function and ν_0 is a complex number. Let $\nu_0 = \nu^2 - 1/4$ and to be definite, we assume that $Re\,\nu > 0$, $\nu \neq 1, 2, \ldots$ We also assume that $q(x)|x - a|^{\min(0,1-2Re\,\nu)} \in L(0,T)$ for some $T > a$ and $q(x) \in L(T, \infty)$.

Fig. 3: $a \in (0, \infty)$

Inverse problems for differential equations with singularities inside an interval are important in mathematics and its applications. A wide class of differential equations with turning points can be reduced to equations with singularities. For example, inverse problems for such equations occur in electronic engineering in designing heterogeneous transmission channels with given characteristics [2]. Boundary value problems with a discontinuity at an interior point also arise in geophysical models of the Earth's crust [3]. Inverse problems for equations with singularities and turning points are used in investigations of the discontinuous solutions of some integrable nonlinear equations of mathematical physics [4].

Matching Conditions

We consider in some sense arbitrary matching conditions with a transition matrix $A = [a_{jk}]_{j,k=1,2}$, that connects solutions of (2) in a neighborhood of the singular point.

Let $\lambda = \rho^2$ and $\operatorname{Im} \rho \geq 0$. Consider the functions

$$C_j(x,\lambda) = (x-a)^{\mu_j} \sum_{k=0}^{\infty} c_{jk}(\rho(x-a))^{2k}, \; j = 1, 2,$$

where $c_{10}c_{20} = (2\nu)^{-1}$,

$$\mu_j = (-1)^j \nu + 1/2, \; c_{jk} = (-1)^k c_{j0} \left(\prod_{s=1}^{k} ((2s + \mu_j)(2s + \mu_j - 1) - \nu_0) \right)^{-1}.$$

Here $z^{\mu} = \exp(\mu(\ln|z| + i \arg z))$, $\arg z \in (-\pi, \pi]$. If $x > a$ or $x < a$ then the functions $C_j(x, \lambda)$ are solutions of (2) with $q(x) \equiv 0$. Let the functions $s_j(x, \lambda)$, $j = 1, 2$ be solutions of the following integral equations for $x > a$ and $x < a$:

$$s_j(x,\lambda) = C_j(x,\lambda) + \int_a^x g(x,t,\lambda) q(t) s_j(t,\lambda)\, dt,$$

where $g(x,t,\lambda) = C_1(t,\lambda)C_2(x,\lambda) - C_1(x,\lambda)C_2(t,\lambda)$. For each fixed x the functions $s_j(x,\lambda)$ are entire in λ of order 1/2 and form a fundamental system of solutions of (2).

Let $A = [a_{jk}]_{j,k=1,2}$, $\det A \neq 0$ be a given matrix with complex numbers a_{jk}. We introduce the functions $\{\sigma_j(x,\lambda)\}_{j=1,2}$, $x \neq a$ by the formula

$$\sigma_j(x,\lambda) = \begin{cases} s_j(x,\lambda), & x < a, \\ \sum_{k=1}^{2} a_{kj} s_k(x,\lambda), & x > a. \end{cases}$$

The fundamental system of solutions $\{\sigma_j(x,\lambda)\}$ is used to match solutions in a neighborhood of the singular point $x = a$.

We consider the most difficult special case when the discrete spectrum is unbounded and essential qualitative modifications in the investigation of the direct and inverse problems are arised.

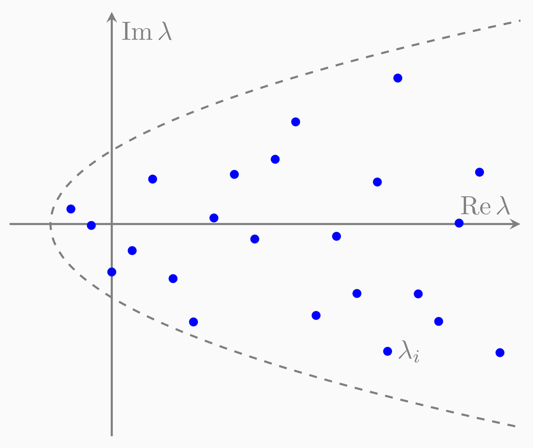

Fig. 4: The spectrum of the boundary value problem $\mathcal{L}$.

The Inverse Problem

Let the function $\Phi(x, \lambda)$ be a solution of (2) under conditions

$$\Phi(0,\lambda) = 1, \quad \Phi(x,\lambda) = O(e^{i\rho x}), \quad x \to \infty.$$

where $\operatorname{Im} \rho \geq 0$, $\rho \neq 0$.

The function

$$M(\lambda) := \Phi'(0, \lambda)$$

is called the *Weyl function* for the problem (2).

Inverse Problem

Recover $q(x)$ by the given Weyl function $M(\lambda)$.

Uniqueness Theorem

The Weyl function uniquely determines the boundary value problem $\mathcal{L}$.