UNIT 3

ERROR ANALYSIS

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1.0 INTRODUCTION

In the last unit we studied about errors in measurements due to imprecision of measuring devices. The results of measurements were expressed as approximate numbers. We also learn about performing basic operations of addition, subtraction, multiplication and division of approximate numbers and expressing results using correct number of significant digits. We assumed that the measuring instruments as well as the observers were perfect. However, as you are aware, there can be defects in measuring instruments and also humans are not perfect. If the environment is not perfectly controlled its changes will affect the object to be measured thereby introducing errors in measurements. In this unit we will familiarise ourselves with these and other sources of errors. We will also learn how to estimate and possibly eliminate or account for such errors. In most of the physics experiments our objective is to determine relationship between physical quantities. Therefore, we will estimate the errors in the measurement of various physical quantities and make efforts to determine valid relationships as mentioned above. In the next couple of experimental write-ups we will apply our knowledge of errors and its propagation to actual measurements and deduce relationships. We will first concentrate on the measurements of fundamental quantities such as mass, length and time, and then do experiments involving two or more of these quantities.

2.0 OBJECTIVES

After studying this unit you should be able to

- distinguish between random errors and systematic errors
- eliminate to some extent the systematic errors
- compute errors in the measurement of various physical quantities
- analyse data by calculation and by plotting graphs to determine functional relationship
- interpret the slope of a graph and to determine the value of certain physical quantities from the slope of a straight-line graph.

3.1 TYPES OF ERRORS

Every measuring instrument has a limitation in that it cannot measure physical quantities smaller than a certain value known as the least count of instrument. For
example, a meter scale can measure only up to 1 mm (smallest division of the scale). A vernier calliper can generally measure up to 0.1 mm whereas a spherometer and screw gauge can measure lengths up to 0.01 mm. Similarly a thermometer usually has the least count of half a degree. In addition to these limitations, which are inherent in a measuring device, there are other sources of error. These arise due to changes in environment, faults in observational techniques, malfunctioning of measuring devices etc. The

errors in any measurement can be classified into two broad headings namely - Systematic errors and Random errors.
Let us now study the causes of such errors, and see how they are eliminated or minimised.

3.1.1 SYSTEMATIC ERRORS

The systematic errors, also called determinant errors, are due to causes which can be identified. Therefore, these errors, in principle, can be eliminated. Errors of this type result in measured values which are consistently too high or consistently too low. Let us discuss these errors one by one.

(i) Zero Error

In the case of vernier callipers, for example, when the jaws are in contact, the zero of the vernier may not coincide with the zero of the main scale. The magnitude and sign of the `zero error' can be determined for the scale readings. We can easily eliminate this error from the measurement by subtracting or adding the zero error.

(ii) Back lash Error

While measuring a physical quantity there may be an error due to wear and tear in the instruments like screw gauge or spherometer due to defective fittings. Such an error is called back lash error and can be minimised in a particular set of measurements by rotating the screw head in only one direction.

(iii) End Correction

Sometimes the zero marking of the metre scale may be worn out. Unless we are careful, this will lead to incorrect measurements. We must therefore compensate for this by shifting our reference point.

(iv) Errors due to changes in the Instrument parameters

Usually, in experiments involving electrical quantities, the value of the electrical quantities change during the course of the experiment due to heating or other causes. For example, the value of the resistance of a wire will increase because of current passing through it. This will lead to errors, which are generally difficult to calculate and compensate for. To some extent this can be avoided by not allowing current to flow through the circuit while observations are not being taken.

(v) Defective Calibration

Occasionally instruments may not be properly calibrated leading to errors in the results of measurement. This type of error is not easily detected and compensated for. This is a manufacturer's defect and if possible the instrument should be calibrated against a standard equipment.

(vi) Faulty Observation

This could be due to causes like parallax in reading a metre scale. These errors are eliminable by using proper techniques.

3.1.3 RANDOM ERRORS

You must have noticed that many times repeated measurements of the same quantity do not yield the same value. The readings obtained show a scatter of values. Some of those values are high while others are low. This function is due to random errors whose possible sources are:

(i) Observational

These arise due to errors in judgment of an observer when reading a scale to the smallest division.

(ii) Environmental

These arise due to causes like unpredictable fluctuations inline voltage, variation in temperature etc. They could also be due to mechanical vibrations and wear and tear of the systems. There could be a random spread of readings due to friction say, wear and tear of mechanical parts of a system.

Self Assessment Question 1

Which of the figures 1(a) or (b) show random errors only?

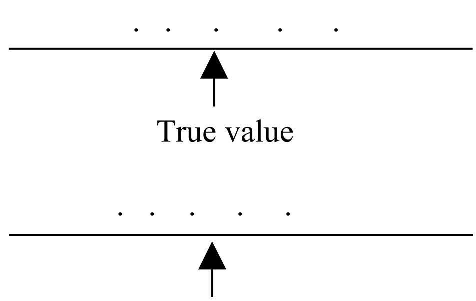

Fig. 1 Set of measurements. Each point indicates the result of a measurement

SOLUTION
1(a)

Unlike systematic errors, random errors can be quantified by statistical analysis. Let us now learn to determine the size of such error.

3.2 DETERMINING SIZE OF ERROR

When we measure a quantity it is important to take several readings. It may be preferable that readings are taken by independent observer. This has the advantage that bias of a single observer is eliminated. The value obtained will indicate whether the data is scale limited or random. An error analysis can be made to determine the size of error from these readings. A typical set of values of a measurement is given below in table l. The quantity to be measured as a `true' value is independent of our measuring process. But the imperfection of our measuring process prevents us from obtaining that value every time. Which on of the values listed in table I would be `true' value? It is impossible to tell that from the measurements because of this spread. Under the circumstances the average A value can be quoted. To get the average value we simply add up all the measurements and divide the sum by the total measurements. As you can see from the table l, the average is 3.68. Also notice that most of the data in table I deviates

from the average $\bar{d}$ we first take the difference of each data from the average to

get individual deviations $\bar{d}$. These deviations are then added and their sum is divided by the number of observations to obtain d. As you can see from table 1 the average deviation in this case is 0.009.

Table 1

S/No	Data	Deviation (d)
1.	3.69	0.01
2.	3.67	0.01
3.	3.68	0.0
4.	3.69	0.01
5.	3.68	0.0
6.	3.69	0.01
7.	3.66	0.02
8.	3.67	0.01
	A = 3.68	$\bar{d}$ = 0.009

As you are aware, repeated measurements of the same quantity yield results with better precision. A measure of this precision index S whose definition (without proof) is

$$S = \frac{\bar{d}}{\sqrt{n}}$$

where $\bar{d}$ is the average derivation and n is the number of observations. The precision index S is a measure of uncertainty of average. Using the data of table l, the precision index is

$$S = \frac{d}{\sqrt{n}} = \frac{0.009}{\sqrt{8}} = 0.003$$

Thus the final result can be expressed as A ± S. In this case the result of random data analysis gives 3.68 ± 0.003. We can see that this error is much less than the possible error, which is ± 0.005. Thus in such cases we will consider the possible error only.

Self Assessment Question 2

The measurement of the length of a table yields the following data. 11=135.0cm,/2=136.5cm,13=134.0 cm, /4=134.5 cm Calculate (a) the average value and (b) precision index. How does the precision index compare with possible error? How will you express the final result?

SOLUTION

(a) 135.0 cm
(b) 0.375 cm

3.3 PROPAGATION OF ERROR

We have so far learnt how to determine the error in the measurement of a quantity, which can be measured directly. In actual practice, however, we determine values of a quantity from the measurements of two or more independent quantities. In such cases the error in the value of the quantity to be determined will depend on the errors in other independent quantities. In other words, the error will `propagate'. The actual analysis of propagation of error is beyond the scope of this course. We shall therefore, quote some rules which can be used in our laboratory.

3.3.1 ERROR PROPAGATION IN ADDITION AND SUBTRACTION

What will be the error, in quantity E, defined by $E = x + y + z$?
Let us take the differential of this quantity, we get $\delta E = \delta x + \delta y + \delta z$
if the error is small compared to the measurement we can replace the differential by `delta' to get $\delta E = \delta x + \delta y + \delta z$
which is simply the sum of errors in x, y and z. It, therefore, is the maximum error in E. Statical analysis shows that a better approximation is

$$\delta E = \sqrt{(\delta x)^2 + (\delta y)^2 + (\delta z)^2}$$

We only consider the magnitude of errors in the above calculation. Therefore, the error in the quantity $(x + y - z)$ will also be the same.

SOLVED EXAMPLE: Let the measured value of two lengths be
$L_1 + \delta L_1 = 1.746 \pm 0.010$ m
$L_2 + \delta L_2 = 1.507 \pm 0.010$ m
The error in the quantity $L = L_1 + L_2$ will be

$$\delta L = \sqrt{(0.010 \text{ m})^2 + (0.0l0 \text{ m})^2} = 0.014 \text{ m}$$

3.3.2 ERROR PROPAGATION IN MULTIPLICATION AND DIVISION

If a quantity $E = A \times B$ and the result of measurement of A & B is $A \pm \delta A$ and $B \pm \delta B$ then what will be the error δE in E? Here if we take differentials we get

$$dE = B\, dA + A\, dB$$

Dividing by $E = AB$ and changing differentials by `deltas' we get

$$\frac{\delta E}{E} = \frac{\delta A}{A} + \frac{\delta B}{B}$$

Self Assessment Question 3

Take logarithm of E = AB and then differentiate to show that $\frac{\delta E}{E} = \frac{\delta A}{A} + \frac{\delta B}{B}$ which is generally known as the logarithmic error.

SOLUTION

E = AB
Taking logarithm on both sides
log E = log A + log B

Differentiating partially

$$\frac{\delta E}{E} = \frac{\delta A}{A} + \frac{\delta B}{B}$$

The statistical analysis, however, gives the following better result of the fractional error in E.

$$\frac{\delta E}{E} = \sqrt{\left(\frac{\delta A}{A}\right)^2 + \left(\frac{\delta B}{B}\right)^2}$$

RULE 1: When independent measurements are multiplied or divided the fractional error in error in the result is the square root of the sum of squares of fractional errors in individual quantities.

SOLVED EXAMPLE: In an experiment we calculate velocity from measurement of distance and time. If the distance is $S \pm \delta S = 0.63 \pm 0.02$ m

$\frac{\delta S}{S} = 0.03$

and time is $T \pm \delta T = 1.71 \pm 0.10$

$\frac{\delta T}{T} = 0.06$

Then the velocity (V)

$V = \frac{S}{T} = 0.368$ MS^{-1}

The fractional error in V is given by

$$\frac{\delta V}{V} = \sqrt{\left(\frac{\delta S}{S}\right)^2 + \left(\frac{\delta T}{T}\right)^2} = 0.07$$

$\delta V = 0.368\ ms^{-1} \times 0.07 = 0.02\ ms^{-1}$

Thus the final result becomes
$V \pm \delta V = 0.37 \pm 0.02\ ms^{-1}$

3.3.3 ERROR PROPAGATION IN OTHER MATHEMATICAL OPERATIONS

Errors in exponential quantity: Let us first consider a special case where a quantity
appears with an exponent. For example, $S = A^2 = A \times A$. Here the two numbers multiplied together are identical and hence not independent. The rule mentioned above does not apply. Detailed analysis shows that logarithmic error gives a good estimate. Taking the logarithm of the above equation we get,
log S = 2 log A
on differentiation and changing differentials to `deltas' we get

$$\frac{\delta S}{S} = 2\frac{\delta A}{A}$$

Therefore, the fractional error in A^2 would be twice the error in A, the fractional error in A^3 will be 3 times the fractional error in A, and the fractional error in $\sqrt{A}$ will be 1/2 the fractional error in A.
RULE: The fractional error in the quantity An is given by n times the fractional error in A.

EXAMPLE: Suppose two measurements of mass are $M_1 \pm \delta M_l = 0.743 \pm 0.005$ kg and $M_2 \pm \delta M_2 = 0.384 \pm 0.005$ kg.

Determine the value of $M = 2M_l + 5M_2$ along with δM.
What will be the error in $(M_l + M_2)^2$ and $(M_l - M_2)^{-3}$.

HINT: The error in $2M_1$ is $2\ \delta M_1$ and in $5M_2$ is $5\ \delta M_2$.

Thus error in $2M_1 + 5M_2$ is $\delta M = \sqrt{(2\delta M_1)^2 + (5\delta M_2)^2}$

Error in $(M_1 + M_2)^2 = 2\sqrt{(\delta M_1)^2 + (\delta M_2)^2}$

Error in $(M_1 - M_2)^{-3} = 3\sqrt{(\delta M_1)^2 + (\delta M_2)^2}$

Similarly in other mathematical operations and deducing results from graphs (about this you will learn in the next subsection) the flowing rule is used.

RULE: The error in the result is found by determining how much change occurs in the result when the maximum error occurs in the data.

EXAMPLE: Let us compute the error in the sine of 30° ± 0.5°. Using the logarithmic tables we get,
sin 30° = 0.5, sin 30.5° = 0.508, sin 29.5° = 0.492
The difference between sin 30° and sin 30.5° is 0.008, and the difference between sin 30° and sin 29.5° is also 0.008. Thus the error in sin 30° would be ± 0.008.

Self Assessment Question 4

Determine the error of sine of 90°, when the error in the angle is 0.5°. Compare your result with that of the example above.

SOLUTION

(i) sin 90° = 1.000, sin 90.5° = cos 0.5° = 1.000, sin 89.5° = 1.000

In this case error in sin 90° is zero.

3.3.4 ERROR PROPAGATION IN GRAPHING

Very often we can better visualise the functional relationship between two physical quantities by plotting a graph between them. This is another useful way of handling experimental data because the values of some quantities can be obtained from the slope. While plotting a graph we will use the following guidelines:

1. A brief title may be given at the top

2. Label the axes with the names of the physical quantities being presented along with units. It is customary to plot the independent variable (the quantity which is varied during the experiment at one's will) on the x-axis and the dependent variable on the y-axis (the dependent variable is the one that varies as a result of change in the independent variable). We would write to the name of the variable represented on each axis along with units in which they are measured.

3. We should choose the range of the scales on the axis so that the points are suitably spread out on the graph paper and not cramped into one

corner. Check for the minimum and maximum values of the data that has to be plotted. We may then round off these two numbers to slightly less than the minimum and slightly more than the maximum. Their difference may be divided by the number of divisions on the graph paper. For example, if we are to plot 5.2 and 17.7 it would be convenient to allow the scale to run from 5 to 20 rather than from 0 to 18.

Each set of data points is indicated by a point within a circle on the graph paper and the error is shown by using bars above and below this point as shown in fig. 2. The graphed data show that velocity V is the linear function of time T.

We recall that the general equation of a straight line is y = mx + c where m is the slope of line and c the vertical intercept in the value of y when x = 0. From the graph we can thus write $V = aT + V_0$. By comparing the above equation we can conclude that the slope of the graph gives the acceleration and the intercept gives the velocity V_o at T =0. From the graph V_o = 0.32 ms^{-1}. To determine the slope we consider two points on the straight line, which are well separated.

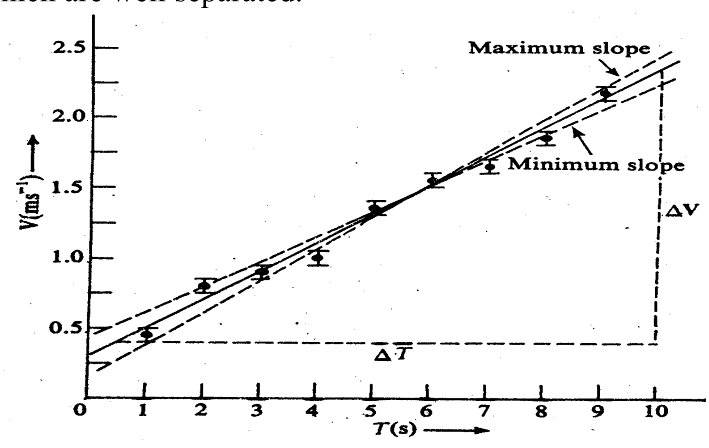

Fig. 2: Graph between velocity and time

Then,

$$a = \text{Slope} = \frac{V}{T} = \frac{2.35 - 0.40\ (ms^{-1})}{10.0 - 0.5\ (s)} = 0.20\ ms^{-2}$$

In the above example, we have plotted the variable V, which is a linear function of T in a linear graph paper. In some experiments we may get data where the relationship between the measured variable is not linear. Suppose a man gets a salary of ₦ 20, 000.00 on the 1st of every month and he decides that each day he will spend half the money he has with him on that day. Then the amount of money, which the man will have over a period of first seven days of any month, will be given as in table 2.

TABLE 2

Day of any month	Money left with the man (M) ₦ K x 10^2
1st	200.00
2nd	100.00
3rd	50.00
4th	25.00
5th	12.50
6th	6.25
7th	3.12
8th	1.56

Let us plot these data on a linear graph paper. The paper will be of the type shown in fig. 3. Look at the graph carefully. You will find that seven of the ten experimental points are clustered together near the bottom right-hand corner of the graph. The shape of the curve we have drawn also involves a bit of guesswork. Therefore, we have to find some method so that these data can be plotted in a better way.

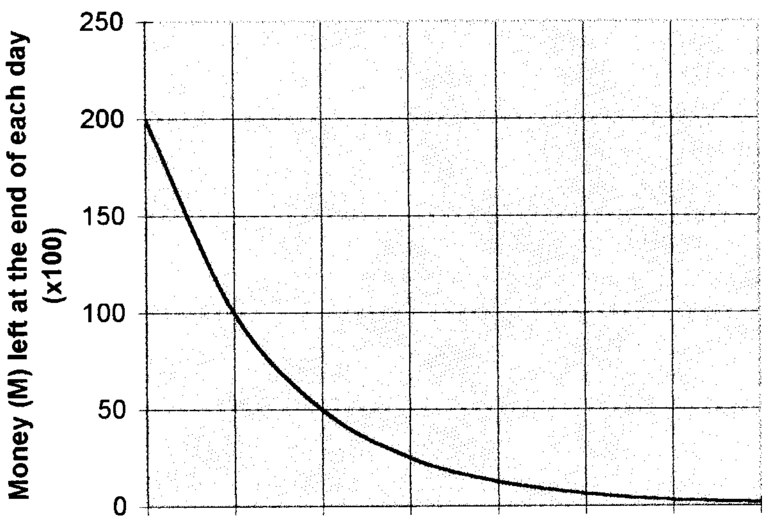

Fig. 3 Graphical representation of Table 2

Try to recollect what you used to do in school when you used to come across data like this, which range over a few orders of magnitude or having big gaps between the points. We will tell you, in such cases you used to take the logarithm of the data and then plot those data in a linear graph paper. When you did this, you must have found that the result was a straight line. So let us take the logarithm of the data of table 2 and tabulate them as shown in table 3.

Day of any month	Money left in the man (m) N K x 10^2	Log N
1st	200. 00	4.301ø
2nd	100. 00	4.000 ø
3rd	50. 00	3. 699 ø
4th	25. 00	3. 398
5th	12. 00	3.097
6th	6. 00	2.706
7th	3. 12	2.494
8th	1. 00	2.193

Day of any month	Log M
1st	3.301
2nd	2.000
3rd	1.699
4th	1.397
5th	1.097
6th	0.796
7th	0.494
8th	0.193

Now plot log M against days as shown in fig. 4. You obtain a graph in which points are more clearly spaced evenly and hence you can more easily draw a straight line through the points.

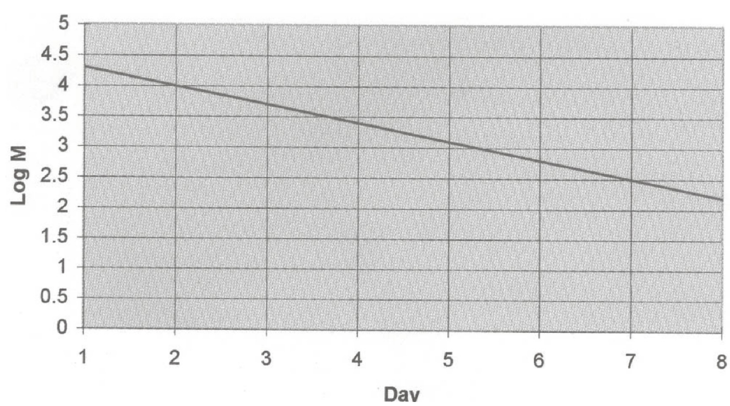

Fig. 4 Graphical representation of Table 3

You might have realised that working out the log values for each data is tedious and it also introduces another step, which may introduce error between the data and the graph. Therefore, to plot such a data we use a graph paper called semi-logarithmic or log-linear graph paper where the lines on one axis have been drawn in a logarithmic fashion. On a semi-log paper (see the graph paper of fig. 5) the horizontal scale is an ordinary one, in which the large division are divided into tenths and each division has the same size. The vertical scale is a logarithmic scale (is automatically takes logarithms of data plotted), in which each power of ten or decade (also called frequency) corresponds to the same length of scale. In each decade, the divisions become progressively compressed towards the upper end. Now to the semi-log graph paper we plot the data of table 1. We obtain a straight line as shown in fig. 5. If you compare fig. 4 & 5 you will see that the points plotted on semi-log paper are distributed on a linear graph paper. A question may strike your mind that how to calculate the slope of the straight line of fog. 5? Also what is the equation of the straight line? Let log M be represented by y and by t then we have a straight line graph of y against t. Let the question be represented as

$y = b + kt$

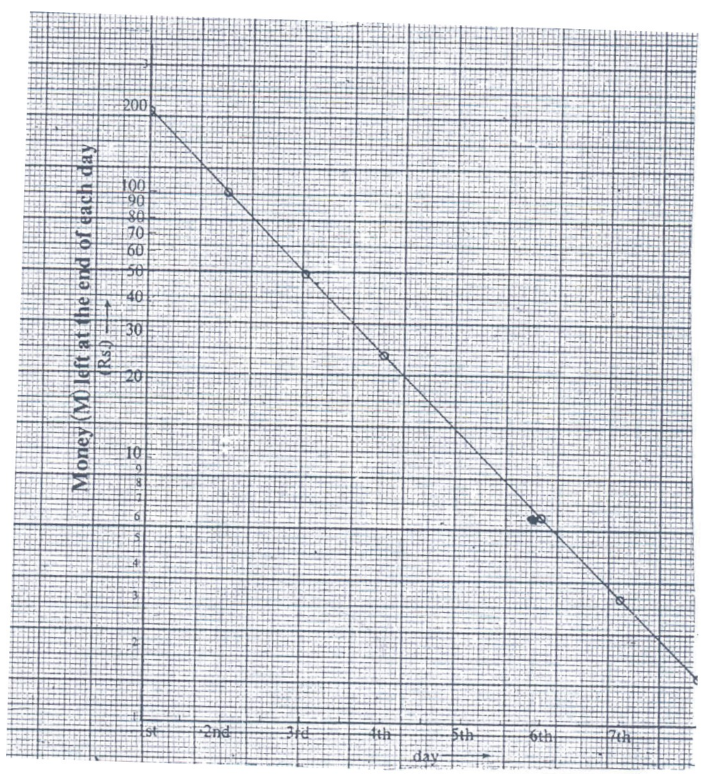

Fig. 5 Representation of Table 3 graphically on semi-log paper

Where b is the intercept of the line on the y-axis and k the slope of the line. We can find the values of b and k from the graph as follows. When t = 0, M = 200 then log M

=log 200=2.30=y

$$2.30=b+0 \text{ or } b=2.30$$

$$\therefore y=2.30+kt$$

When t = 7th day, M = 1.56 and log M = log 1.56 = 0.193 = y

Putting these values in equation (1) we get

0.193=2.30+7k

$\therefore$ The slope $k = -\frac{2.1}{7} = -0.3$

and the equation of the straight line is

$$y = 2.3-0.3t \qquad (2)$$

From the graph of fig. 5 or in other words from equation (2) can you find the equation of the curve plotted in fig. 3?

Let the value of M at t = 0 be denoted as M_o then equation (2) becomes

$$\log M = \log M_O + kt$$

or $$\log M - \log M_o = kt$$

or $$\text{Log} \frac{M}{M_O} = kt$$

or $$\frac{M}{M_O} = 10^{kt}$$

or $$M = M_O 10^{kt}$$

or $$M = 200 \times 10^{-0.3t} \qquad (3)$$

This is the equation of curve plotted in fig. 3. It tells us that the money is decreasing logarithmically (also called exponentially) with each day.

In science, you will come across many logarithmic or exponential relations of the form of equation (3). In such cases it would be convenient to plot the data on semilogarithmic graph paper because the graph will be convenient to plot the data on semilogarithmic graph because the graph will be a straight line if the relationship is logarithmic. Also the slope of the line (which may give you the value of any physical constant) can be read simply and directly from the graph.

Sometimes we find that we wish to plot a graph where both variables range over several powers of ten. For example, you know that according to Keplar's law, the semi-major axis of the orbit of a planet (R) is related to its period (time for one revolution around the sun) T by the following power-law relation:

$$R^3 = kT^2 \tag{4}$$

Where k is another constant.

If you consider the experimental data that shows how the quantity T depends on quantity R you will observe that R varies by two orders of magnitude and T varies by three orders of magnitude. In other words the experiment data follows equation (4). For a moment, suppose you do not know the exact relationship between the variables T and R. The you can suppose that

$$R = kT^n \tag{5}$$

where n is another constant.
Using the conventional method to find the value of n, you will take logarithm of equation (5) as follows:

$$\log R = \log k + n \log T$$

Now you will plot log R vs. log T on a linear graph paper. The slope of straight line obtained will give the value of exponent n. But again, as mentioned above, taking logarithm of each experimental date is rather tedious so it would be convenient to plot both the variables T and R on a logarithmic scale where the lines on both axes are drawn in a logarithmic fashion. A log-log graph is shown in fig. 6. The points lie upon a straight line. The slope of the straight line will give the exponent (n) of the powerlaw relation and hence the exact relationship between R and T will be found out.
To determine the error in the value of the slope of the straight line drawn in any graph paper (linear or semi-log or log - log) we draw two dashed lines representing the greatest and the least possible slopes which reasonably fit the data as shown in fig. 2. Thus the error in the slope is defined as

$$\text{Error in slope} = \frac{\text{maximum slope - minimum slope}}{2}$$

Thus from the graph we get the error in the slope as

$$\delta a = \frac{0.23 - 0.19 \text{ ms}^{-2}}{2} = 0.02 \text{ ms}^{-2}$$

Thus the experimental value of acceleration from the graph is

$a \pm \delta a = 0.20 \pm 0.02\ ms^{-2}$.

Similarly the error in intercept = (intercept of minimum slope line - intercept of maximum slope line)/2

$$\delta V_O = \frac{(0.45 - 0.17)}{2}\ ms^{-1} = 0.14\ ms^{-1}$$

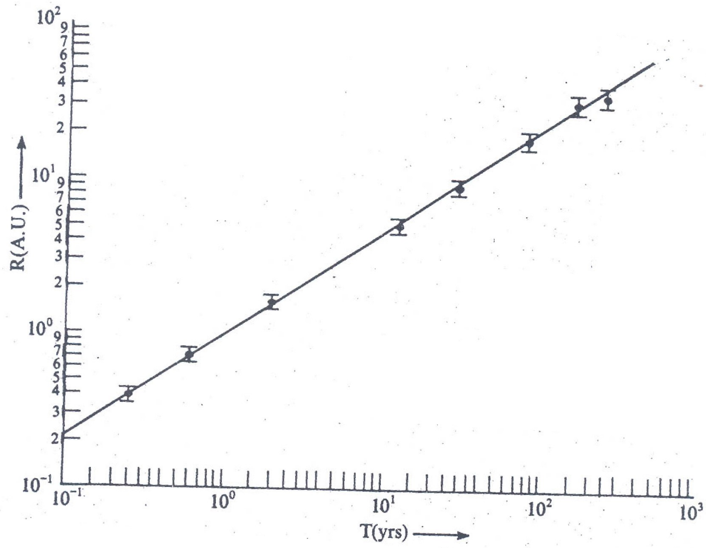

Fig 6

3.4 USES OF π

It appears that most of our students are under the impression that the value of π is equal to $\frac{22}{7}$ exactly. Unfortunately many books writers also have contributed to perpetuate and establish this false idea by setting numerical problems with data cooked up that using $\pi = \frac{22}{7}$, the factor always happily cancels out and the simplification becomes very easy. However, in the real world the values of actual physical quantities are not such as to facilitate cancellation with 7. Also, we may as well acknowledge that the value of π cannot be expressed exactly in terms of any whole number. The value of $\pi = \frac{22}{7}$ is one of the many approximations that can be used. In fact, a better approximation is $\frac{355}{113} = 3.1415928$. Compare this