CS 352 (Advanced Algorithms)	**Spring 2011**
Lecture #9: Traveling Salesman Problem and Randomization Algorithms	*Scribe: Betsy Cannon*

1 A Recap of the Traveling Salesman Algorithm

Last lecture, we introduced the Traveling Salesman Problem.

METRIC TRAVELING SALESMAN PROBLEM:

Given: a set of n cities and non-zero distances $d_{i,j}$ between cities, such that the triangle inequality and symmetry holds.
Output: an order $\pi_1, ..., \pi_n$ of cities such that the distance $\sum_{i=1}^{n} d_{\pi_i,\pi_{i+1}}$ is minimized.

In order to approximate the optimal solution, we proposed the following algorithm using minimum spanning trees.

1. let T be a minimum spanning tree of the complete graph G with nodes representing the n cities and using $d_{i,j}$ as the weight of the edge $\langle i, j \rangle$.
2. Let D be a depth-first-ordering of T.
3. Let C be D with duplicate cities deleted.
4. Return C.

Claim: This algorithm is a 2-approximation of the metric-TSP.

Proof. Let OPT denote the optimal solution. Then,

$$\begin{aligned} OPT &\geq \text{cost of } MST \\ &= \frac{1}{2} \cdot \text{cost of } D \end{aligned}$$

Since removing a vertex from a path can only shorten the path due to the triangle inequality,

$$\begin{aligned} OPT &\geq \frac{1}{2} \cdot \text{cost of } C \\ 2 \cdot OPT &\geq \text{cost of } C. \end{aligned}$$

□

2 Minimum Cost Perfect Matching for TSP

Let O be the set of all nodes u with odd degree in T.

Claim: $|O|$ is even.

Proof. Each edge in T adds 1 to the degree of two nodes. Therefore, the total degree of all nodes in T must be even. All nodes v in T but not in O have even degree, so the sum of their degrees is even. Thus, the sum of the degree of all nodes u in O must be even. Since the degree of each u is odd, the total degree can only be even if $|O|$ is even. □

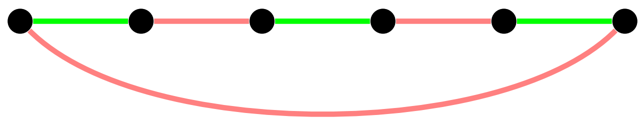
Given that $|O|$ is even, there must be a perfect matching of nodes in O. Let P be a minimum cost perfect matching for O, and take OPT as the cost of the optimal tour through all nodes.

Claim: cost of $P \leq \frac{1}{2} \cdot OPT$.

Proof. By definition, OPT is the cost of the cheapest tour of all nodes. Let N^* be the cost of the cheapest tour of only the nodes in O. Then, by the triangle inequality,

$$OPT \geq N^*.$$

In the example shown below, N^* is comprised of the pink and the green edges connecting the nodes of O.



Therefore,

$$\begin{aligned} OPT &\geq \text{cost of pink and green steps} \\ &\geq 2 \cdot \min(\text{cost of pink edges, cost of green edges}) \end{aligned}$$

The minimum cost perfect matching P is the cheaper of the pink or green edges.

$$OPT \geq 2 \cdot \text{cost of } P,$$
$$\Rightarrow \frac{1}{2}OPT \geq \text{cost of } P.$$

$\square$

An Eulerian circuit (or tour) is a path from node s to s such that every edge in the graph is traversed exactly once.

Theorem 1. *An Eulerian circuit exists if and only if all nodes have even degree, and the graph is connected. Furthermore, it is easy to find.*

By using minimum cost perfect matching and Eulerian circuits, we propose the following algorithm to solve metric-TSP.

1. Let T be an MST of the complete graph G.
2. Let O be the set of all odd-degree nodes in T. Let $P =$ min-cost perfect matching in O.
3. Let $E' = T \cup P$. (If P adds an edge already in O, add as a parallel edge.) Compute an Eulerian circuit of the graph (V, E). Let X be the tour.
4. Return X, with duplicated nodes removed.

Adding P to to T will increase the degree of all odd-degree nodes by 1. Thus, we are ensured that all nodes have an even degree, and by Theorem 1, there must be an easily found Eulerian circuit.

Claim: This algorithm is a $\frac{3}{2}$−approximation.

Proof. The algorithm is no worse than X, as it only removes unnecessary nodes, which can only shorten the distance by the triangle inequality.

$$\begin{aligned}\text{cost of algorithm} &\leq \text{cost of } X. \\ &\leq \text{cost of } T + \text{cost of } P\end{aligned}$$

The cheapest connected subgraph, T, cannot cost more than OPT, and the cost of $P \leq \frac{1}{2} \cdot OPT$ is proven above. Thus,

$$\begin{aligned}\text{cost of algorithm} &\leq OPT + \frac{1}{2} \cdot OPT \\ &\leq \frac{3}{2} \cdot OPT.\end{aligned}$$

$\square$