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Shuffla: fast and scalable full-text search engine

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Oświadczam, że pracę magisterską wykonałem samodzielnie i zgłaszam ją do oceny.

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Streszczenie

Celem tej pracy jest stworzenie efektywnego i skalowalnego narzędzia do dokładnego przeszukiwania zbioru użytkowników dużego portalu społecznościowego. Przechowywane dane to imię, nazwisko, wiek, oraz miejsce zamieszkania.

Użytkownik portalu może wyszukiwać innych użytkowników poprzez:

1. podanie imienia (jego prefiksu lub pod słowa),
2. podanie nazwiska (jego prefiksu lub pod słowa),
3. podanie wieku (dokładnie bądź przez określenie dolnej i górnej granicy),
4. podanie miejsca zamieszkania (jego prefiksu lub pod słowa).

Kolejnym założeniem jest konieczność personalizacji wyników wyszukiwania. Na przykładu użytkownik z San Francisco szukający Jane powinien dostać w pierwszej kolejności osoby o imieniu Jane z San Francisco, następnie osoby z Kaliforni, Stanów Zjednoczonych, a dopiero na samym końcu z innych części świata.

Jednym z możliwych rozwiązań tak postawionego problemu jest zastosowanie jednej z popularnych relacyjnych baz danych. Inną możliwością jest użycie wyszukiwarki tekstowej takiej jak Apache Solr czy Sphinx Search. Jeszcze innym pomysłem jest zaimplementowanie własnego narzędzia. Celem tej pracy jest przedstawienie Shuffli, wyspecjalizowanej wyszukiwarki stworzonej przeze mnie z myślą o tym konkretnym problemie. Oprócz przedstawienia architektury i szczegółów implementacji, porównam jej efektywność z PostgreSQL, Apache Solr, oraz Sphinx Search.

Pracuję nad portalem nk.pl (który jest największym polskim portalem społecznościowym) od początku roku 2011. W sierpniu 2011 roku zostało mi powierzono rozwiązanie tego problemu.

Słowa kluczowe: bazy danych, nosql

Shuffla: fast and scalable full-text search engine

Jarosław Gomułka

October 12, 2012

Paweł: to w zasadzie jest nie abstract, a introduction

Jarek: Trochę tak wyszło, spróbuje część tego przenieść do rozdziału 1 a tu napisze kilka zdań o tym, że temat to wyszu

Abstract

Given a list of people (their first name, last name, age, home town) we want to provide high-available, scalable and very fast solution that can search through such data. User can narrow search results by:

1. defining first name (its prefix or substring)
2. defining last name (its prefix or substring)
3. narrowing age (user can provide lower or/and upper bound for age)
4. defining city (its prefix or substring)

Furthermore, it should be possible to personalize the order of the results. For example, if user from San Francisco is looking for Jane, top results should show Janes from San Francisco, then Janes from California, then rest of U.S. One of possible solution is to use relational database. Other solution is to use full-text search engine like Apache Solr or Sphinx Search. Another solution is to develop own specialized search engine dedicated to solving such task. In this paper I will describe Shuffla, full-text search engine created by me. I will compare it to Sphinx, Solr, and PostgreSQL.

I am working on nk.pl (which is biggest polish social networking website) since the beginning of 2011. In August 2011 I was appointed to solve the task described above.

Keywords: database, nosql

1 Problem definition

Jarek: Fix: Zaczynasz od definicja, ale właściwie nie wiadomo po co ta definicja. Może najpierw napisz, że chcesz mieć

Like in many databases, Shuffla should provide possibility for storing different types of objects. SQL solves it by giving possibility to create multiple tables and so does Shuffla. Lets define table definition by combination of table name and list of pairs <column name, type of column>. Type of column can be either a string or a number.

1.1 Selecting data

Shuffla like every other database provides possibility for searching data. There are different possible conditions for different types. Possible conditions for numbers are equality check, inequalities and strict inequalities. Strings are compared using lexicographic order so equality check, inequalities and strict inequalities should be supported. There are two additional conditions

1. Checking if given string is prefix of selected column value.
2. Checking if given string is substring of selected column value.

Jarek: fix: a może napisz to pełnym zdaniem, użytkownik ma możliwość wyboru kolejności blablabla User can define order in which results will be returned. Size of the results may be narrowed by defining offset and limit. These search parameters must work exactly the same as their equivalents in SQL.

Narrowing result described in abstract is possible with defined model. GPS coordinates may be fetched by the modern web browsers [?]. GPS coordinates can be added to scheme. For selecting users in some range we can add conditions

1. $\text{someLatitude} - \text{radius} \leq \text{latitude} \wedge \text{latitude} \geq \text{someLatitude} + \text{radius}$
2. $\text{someLongitude} - \text{radius} \leq \text{longitude} \wedge \text{longitude} \geq \text{someLongitude} + \text{radius}$

Jarek: Fix: to jest troszeczkę naciągane moim zdaniem.

Jarek: Napisałem to, bo to chyba najrozsądniejsza metoda na sortowanie userów. Próbowaliśmy coś takiego w nk robić.

2 Shuffla's competition

Problem described in this paper is solvable by various existing software. There are two most common kinds of software which can be used:

2.1 Relational databases

Relational Database Management System data is structured in database tables, fields and records. Each RDBMS table consists of database table rows. Each database table row consists of one or more database table fields. RDBMS store the data into collection of tables, which might be related by common fields (database table columns). RDBMS also provide relational operators to manipulate the data stored into the database tables. Most RDBMS use SQL as database query language. The most popular RDBMS are MySQL, MSSQL Server, PostgreSQL Jarek: Ściągnięte z google . In this paper I am only testing PostgreSQL which is in my opinion best RDBMS nowadays.

2.1.1 PostgreSQL

PostgreSQL has really cool features like: point in time recovery, tablespaces, asynchronous replication, online/hot backups, a sophisticated query planner/optimizer, and write ahead logging for fault tolerance. It supports international character sets, multibyte character encodings, Unicode, and it is locale-aware for sorting, case-sensitivity, and formatting. It is highly scalable both in the sheer quantity of data it can manage and in the number of concurrent users it can accommodate. There are active PostgreSQL systems in production environments that manage in excess of 4 terabytes of data

Jarek: Kopiowane ze strony Postgresa, mam nadzieję, że to nie plagiat . PostgreSQL keeps its data on disk. PostgreSQL implements their own master-slave replication, there are also few middlewares which could provide master-master replication [1]. There is a slow log, error log, PostgreSQL performance can be easily monitored by software like Munin and New Relic. SQL implies very strong typing system. PostgreSQL processes comparisons and inequalities very fast as long as proper indexes are created. Big disadvantage of using PostgreSQL for this problem is performance on substring queries. B-tree based indexes are not supporting fast processing of substring queries. There are attempts to create full-text indexes (gist, gin) which could outperform classical index based on B-tree. Example of processing query by B-tree: `first name = John and substring(last name) = qwertyabc and limit = 1`. In this case engine is finding first person with first name John. It takes $O(\log n)$ time Jarek: fix: co to jest n ? where n is number of rows in the table. Finding another person with first name John takes $O(1)$. Unfortunately all these rows must be processed one by one for checking if they contain substring `qwertyabc`. Since there is no one with such name, engine will process all guys named John, and it will find nothing.

2.2 Full-text search engine

In text retrieval, full text search refers to techniques for searching a single computer-stored document or a collection in a full text database. Full text search is distinguished from searches based on metadata or on parts of the original texts represented in databases (such as titles, abstracts, selected sections

or bibliographical references) **Jarek: Ściagniete z wiki**. The most popular full-text search engines are Sphinx Search and Apache Solr. Almost all big players are choosing Sphinx and Solr over other engines like Elastic Search and Xapian so I am going to consider only the first two. Both engines are based on similar data structure called inverted index. Inverted index keeps list of every occurring word in database. For each word inverted index remembers where this word occurs. This is similar to glossaries which we can often see at the end of books. Such inverted index is stored on disk. When processing query, engine selects best strategy to follow. Strategy can be seen as procedure

1. Narrow as much as possible words from inverted index which needs to be processed.
2. Process remaining words.

For example if query is `first name = John and limit 1` then we are only interested in word John from inverted index. Engine finds first occurrence and returns it. Inequalities, prefix queries and substring queries are treated similarly to equality checks. They are narrowing number of words in dictionary that algorithm needs to process. If query is `first name = John and substring(last name) = qwertyabc and limit = 1` then it is likely that there are more Johns than all people with last name containing `qwertyabc`. By using sophisticated data structure, engine may find out that there aren't many people with name containing `qwertyabc`. Such output is rendered very fast¹ In worst case scenario algorithm works in linear time. Example for such case could be `substring(last name)=a and substring(last name) = b and ... and substring(last name) = z and limit = 1`. Almost whole inverted index must be processed and result set will be empty.

2.2.1 Solr

Solr is a popular open source search platform from the Apache Lucene project. It has lots of very cool features like faceted searches, geospatial searches, string tokenizers (makes John and Johnny to be interpreted as the same name). Solr has great admin interface with comprehensive statistics on cache utilization, updates, and queries. **Jarek: Jedno zdanie zgapione od Solra**. It is easily extendable for new features. Solr implements master-slave replication which solves scalability problems. Everything is perfect except one thing.

Solr does not modify database in real-time. Data is not added immediately to database after insert/update commands. Data is actually inserted after calling commit command. Commit is very costly operation because it requires all caches to be invalidated. Even for table with few entries, commit takes more than 0.1 sec. Time for commit grows with growing size of data. Instance of Solr in nk contains 40 million entries and commit takes about 20 seconds. Commit does not block database which is very important. After commit, search engine is in warming stage. It means that all caches are flushed. First queries after commit can run slowly. After number of processed queries, engine is past warming stage and queries are handled much faster.

2.2.2 Sphinx Search

Jarek: fix: Może zacznij ten kawałek od stwierdzenia, że Sphinx ma dużo minusów, i teraz je wymienisz? Sphinx Search is very good and fast search engine. Sphinx provides good relevance ranking through combination of phrase proximity ranking and statistical ranking **Jarek: To zdanie skopiowane ze strony Sphinx**. It is possible to install Sphinx as MySQL or PostgreSQL plugin. Unfortunately our specification emphasizes Sphinx disadvantages. Sphinx does not have replication. It is unacceptable for big social website. There is a way to simulate replication. Inverted index is stored on disk. Directory containing index could be replicated by file system (for example network file system). But if index is big than syncing such directories would take too much time and would be inefficient and problematic.

There are several high-traffic websites with Sphinx search. They horizontally partition searched data across search nodes and then process it in parallel [2].

Replication is very important to us because in case of server failure we want to avoid downtime. This eliminated Sphinx Search from our list of potential search engines.

Except of replication sphinx has another problem. There are two different kinds of indexes in sphinx.

1. Real-time index - enables inserts

¹with table containing 8 million rows Solr is rendering output for such query in less than 0.01 second

2. Normal index - modifications possible only by rebuilding whole index (which takes too much time for our requirements)

Real-time index has many caveats [?]. Prefix and substring queries are not supported yet. Periodical rebuild of index with more than 40 million elements is problematic. Apache Solr and Sphinx search are using similar data structures. Since number of features implemented by Solr is much greater than in Sphinx search I decided to look into only Solr.

2.3 Why am I creating new search engine?!

It seems that none of existing solutions are processing search queries very fast. In worst case scenario it is always $\mathcal{O}(n)$. All solutions are working on data stored on disk. In age when RAM is cheap and servers with +32GB RAM are considered average, solutions working in RAM memory are very appreciated. Although data stored on disk could be stored in RAM by ramdisks, it is not primary target to software creators. Solr has designated class `solr.RAMDirectoryFactory` which keeps all data in RAM memory but it does not support replication. For me it seems that this area has room for improvement. That is why I decided to create my own search engine.

3 Algorithm description

We can clearly see analogy to multidimensional range searches. All conditions (except substring condition) can be represented as orthogonal range queries. There are three data structures for performing efficient searches in a multidimensional space:

1. Range Tree [11]
2. Inverted Index
3. K-D tree [11]

Complexities of these algorithms are depending on number of elements (n) and number of dimensions (d). In our case number of dimensions is corresponding to number of columns in the table. Here is a table of complexities for listed algorithms.

Algorithm	Insert (without balancing)	Delete (without balancing)	Search	Space
K-D tree	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(n^{1-1/d} + k)$	$\mathcal{O}(n)$
Inverted index	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$
Range tree	$\mathcal{O}(\log^{d-1} n)$	$\mathcal{O}(\log^{d-1} n)$	$\mathcal{O}(\log^d n)$	$\mathcal{O}(n \log^{d-1} n)$

Jarek: fix: W źródle do którego się odwołuje ([11]) złożoności są takie jakie podałem (strona 110, końcówka kartki)

Since both Apache Solr and Sphinx Search are using inverted index I decided not to follow this direction and try something new. It seems that K-D tree could outperform inverted index on search queries.

I chose to implement K-D trees over Range trees because:

1. They are only worse in processing search requests. Operations which modify data i.e. insert/delete are more important. If they are very fast, scaling can be easily achieved (for example by implementing master-slave replication). Moreover, while the search bounds are worse in theory, they seem to be significantly better in practice.
2. Space complexity of range trees is unacceptable for this kind of problem. With 5 columns and 10 million rows, algorithm would require more than 100GB of RAM.

At the beginning let's start with basic description of K-D tree. Later, in chapter 5, I will describe how this basic version can be improved. K-D tree is a binary tree. Each node contains a pivot, for our purpose we can view this as a condition $row_e \leq C$ where e is pointing to some dimension. Most important functions of our data structure are insert and search. Both these functions are recursive. Insert call of `row` to `node` is handled by following procedure:

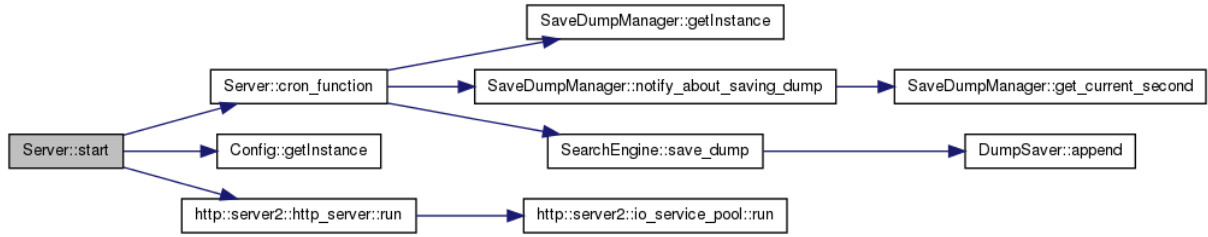


Figure 1: Processing request begins

1. If **node** is a leaf then add **row** to the node. Choose a pivot and create **node** sons if necessary
2. Otherwise if **row** is matching condition defined by **node**->**pivot** then recursively insert **row** to left son of **node**
3. Otherwise recursively insert **row** to right son of **node**

At this point we do not know nothing about how pivots are choosed and how/when rebuilding of a tree is processed. I will get back to this topics later.

In classical K-D tree we are searching points in some orthogonal search region. Parameters for search functions are **node** and **searchRegion**. If **node** is a leaf then we are processing all rows inside **node**. Otherwise we recursively searching through **node** sons with two exceptions. If intersection of region represented by **node** and **searchRegion** is empty then we can stop processing this subtree. If region represented by **node** is fully contained in **searchRegion** then we don't have to process all it's children, we already know that all rows will be inside **searchRegion**.

As I mentioned before almost all conditions can be represented as orthogonal range queries. But we also have to handle queries with substring conditions. In each node of K-D tree we could store structure which efficiently process substring queries. We could use suffix trees, for example Ukkonen's algorithm [16]. Unfortunately adding linear structure to every node of K-D tree increases the space complexity to $\mathcal{O}(n \log n)$. There are other problems with suffix trees. If we have to return data ordered by some column then all we can do is fetch all rows matching substring query and sort them. Not mentioning that we can process this way only one substring condition in a query.

There is another possibility. Engine may process search query without substring parts. In the end, if row is matching query without it substring parts then pattern matching is performed. In this solution rows should be provided to pattern matching in order defined within query.

Solution with suffix tree is faster when there aren't many rows with given substring. If situation is opposite then pattern matching will work faster. I decided to implemented solution with pattern matching because of its simplicity and superiority in space complexity. As you may see in tests results at the end of this paper, even without suffix trees memory usage is high.

4 Implementation

4.1 HTTP Service

Jarek: fix:Odnosińiki do rysunków powinny być choć minimalnie zintegrowane z tekstem

I decided that it would be best if search engine could work as simple HTTP server. Every query should be send as HTTP GET request. As a base of HTTP server I used **example2** from **boost::asio** documentation [3]. In Shuffla configuration file user should define number of available threads. One of the server start procedure is to create all this threads. Every HTTP request is assign to one of these threads by **boost::asio::ip::tcp::acceptor**. There is another thread for periodical functions. Currently its invoking only saving snapshots. All dependencies of server start procedure are presented on Figure 1.

4.2 Validation and passing queries to SearchEngine class

http::server2::request_handler::handle_request passes requesting urls to **QueryDispatcher**. **QueryDispatcher** converts every request to corresponding class by using **QueryCreate**. For example, search request is con-

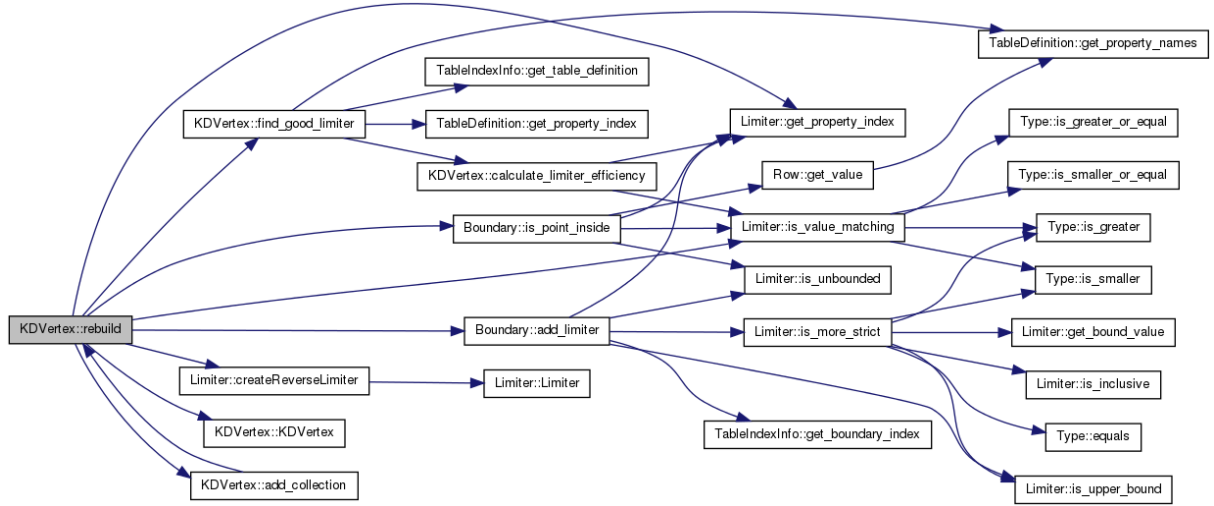


Figure 2: Rebuilding process

verted to **SearchQuery**, insert is converted **InsertQuery**, and so on. At this level, syntax validation is performed. In case of an error, HTTP error code 400 is returned, and error message is appended to the log file with errors.

Instances of query classes are sent to **SearchEngine** class. **SearchEngine** class is responsible for:

1. data validation (types, correspondence to table definition),
2. measuring running time (engine should report queries which executing is taking too much time),
3. locking tables in case of modifying query,
4. passing valid requests to K-D Trees.

4.3 Processing insert/delete requests

Inserting/deleting without tree balancing is pretty straight-forward. Just simple recursion. Balancing is tricky as you may see in dependencies visualisation on (Figure 2).

Balancing can be done using any linear algorithm for finding median in set. I have decided to take different approach. In **KDVertex::find_good_pivot** I am iterating over all points. Based on coordinate value in currently processed row I am creating a **Pivot**. **Pivot** is simple class which takes **Row** and **property_name**. For any **Row**, **pivot** can check if given **Row** satisfy condition defined inside **Pivot**. Algorithm iterates over all rows in node and in linear time calculates pivot efficiency. If pivot efficiency is good enough (See definition of well-balanced tree) than I am stopping calculations. I have got a pivot and I am recursively rebuilding two subtrees. I have decided to use this solution because of space efficiency which is $\mathcal{O}(1)$. Space complexity is crucial because if it reaches linear size than rebuilding root may cause memory usage to double. Such behaviour is very unwelcome in server environment. Implemented solution is $\mathcal{O}(n^2)$ but in average case it works well because $\frac{1}{5}$ of rows are creating good pivots.

Pawel: Nie do końca kupuję to wyjaśnienie. Jarek: Wyjaśnienia nie mam innego. Tutaj tylko pamięć chciałem optyma

4.4 Processing search requests

When request data is validated, control is passed to **TableIndex** class. Actual computations begins here. Lets look at processing search request. Method **TableIndex::search** contains priority queue which processes all search events. There are different objects for different tasks. There is **SearchTaskSearchNode** which contains node to process. There is also **SearchTaskFoundRow** which always contains one row. This row may be inserted to task queue if and only if it satisfy all conditions defined in search request. Both these classes are extending abstract class **SearchTask**. When processing **SearchTaskFoundRow**, queue just adds row to result set. When processing **SearchTaskSearchNode** queue passes control to

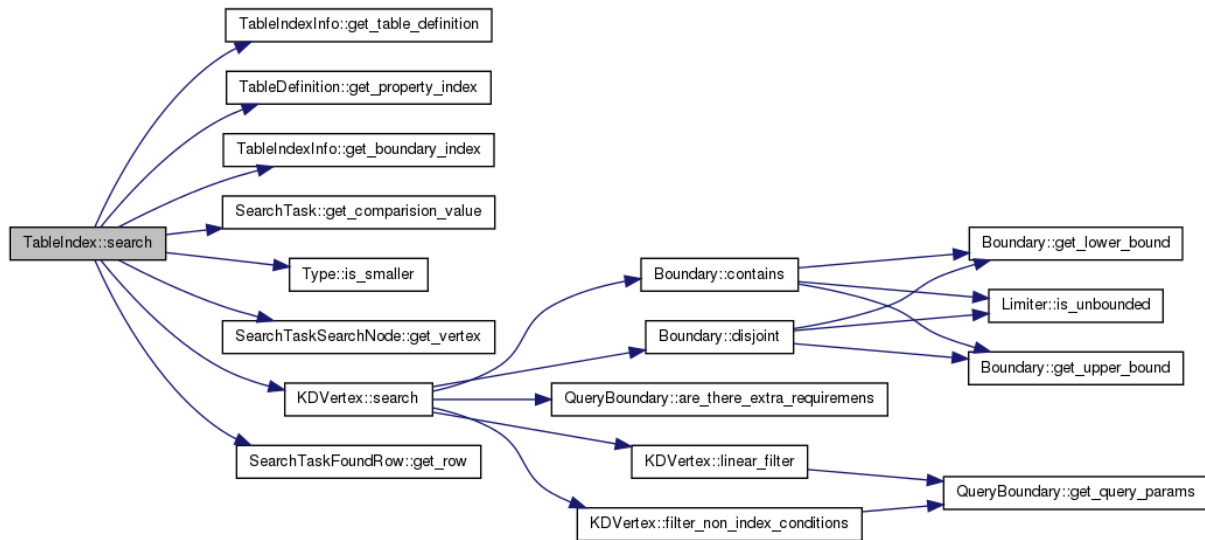


Figure 3: Processing search request

`KDVertex::search` which returns all events that should be added to queue. `KDVertex::search` checks if boundary defined by query contains/is disjoint with current node's boundary. If it's disjoint there is nothing more to do. If not then algorithm simply adds two sons of current node to queue. There was idea to add possibility for multiple index creation. Such index could be created for some subset of table's properties. Queries with conditions containing only properties from such subset would run much faster. For this feature I introduced two functions `KDVertex::filter_non_index_conditions` and `QueryBoundary::are_there_extra_requirements`. Possibility of creating additional indexes is not implemented yet.

Dependencies of search process are visualized on (Figure 3).

4.5 Locking tables

Jarek: fix: mógłbyś napisać to pełnym zdaniem? I zdefiniować co to znaczy locking?

To correct work in multi-threaded environment we must ensure thread safety. It requires us to eliminate all data races, especially simultaneous writes which can make our data inconsistent. That is why I introduced locking. Basically by acquiring lock we are restricting access to some part of code.

Our goal is to create following schema: Any number of read-only queries can be run concurrently as long as no write query is being performed at the moment, write queries can be performed if no other query is being performed at the moment.

This locking schema has the advantage that it can be implemented using only `mutexes` and `locks` from `boost`. A `mutex` object facilitates protection against data races and allows thread-safe synchronization of data between threads. A thread obtains ownership of a mutex object by calling one of the lock functions and relinquishes ownership by calling the corresponding unlock function. Mutexes may be either recursive or non-recursive, and may grant simultaneous ownership to one or many threads. `Boost.Thread` supplies recursive and non-recursive mutexes with exclusive ownership semantics, along with a shared ownership (multiple-reader or single-writer) mutex.

Jarek: 3 ostatnie zdania są przeklezione z dokumentacji boosta, nie wiem czy ta. There are three most common kinds of locks. All locks are working in similar fashion. Mutex is passed as constructor parameter to lock. Ownership of mutex is acquired in constructor. Ownership of mutex is released when lock object is destroyed.

1. `unique_lock` - When mutex is acquired by `unique_lock`, then no one can acquire this lock until unique lock will release ownership of mutex.
2. `shared_lock` - Multiple shared locks can acquire the same mutex.
3. `upgrade_lock` - Acquires upgradable ownership. Upgradable ownership may be at any time upgraded from shared lock to unique lock and vice versa.

My solution requires every table to have its own mutex. Read-only request use `shared_lock` and write request use `unique_lock`. With such solution write queries cannot be executed along with other queries. Using `shared_lock` allows simultaneous execution of other read-only queries. So everything works exactly as it suppose to.

4.6 How data is stored

Architecture of data storage in databases is very important. It influences application performance and architecture of all other components. I defined requirement for data storage as:

1. all data must be serializable,
2. chosen architecture should force type safety,
3. very good space efficiency,
4. it must be fast!

Values can be strings or numbers, therefore I created 3 classes:

1. abstract class `Type`,
2. class `TypeString` which extends `Type`,
3. class `TypeNumber` which extends `Type`.

If we take a closer look at specification we may notice that we are never changing values of existing numbers and strings. Therefore these classes may be immutable. Immutable object is an object whose state cannot be modified after it is created. Immutability trivially guarantees thread-safety and allows for some storage optimization. Lets take a look at `TypeString`. Storing text values in `std::string` would be inefficient because of overhead caused by `std::string` being not immutable. Text is actually stored in `char*`. This array is created in `TypeString` constructor because at this point we know text length.

Both `TypeString` and `TypeNumber` are implementing all supported functions. `TypeNumber` implements equality check, inequalities and strict inequalities. `TypeString` implements equality check, inequalities, strict inequalities, substring and prefix function. Pattern matching is done by function `strstr` from `cstring` [5].

Single row stores pointer to the table definition. Besides that there is a `std::vector<Type>` which stores values of all fields.

4.7 Database persistence

Jarek: fix: zdefiniuj persistence In case of a crash there must be a way to restore database. In this chapter I will describe how Shuffla preserves data. Guaranteeing persistence of database which holds data in RAM is difficult (as you can see by example of Redis [6]). After every modifying query, engine should not only write such data to disk but ensure that OS will actually write such data to disk. It is a very costly operation which drastically decreases application speed. Instead of such solution, preferred way is to have “almost durable” system. Nowadays there are two most popular ways of having almost durable database in RAM:

1. AOF - append only file, every modifying query is saved to disk (without forcing OS to actually do it),
2. Snapshotting - in constant time intervals, snapshots of database are saved to disk.

I implemented both possibilities.

4.8 Presenting results

Jarek: fix: tutaj nie wiadomo czego dotyczą te formaty. Rozumiem, że wyników, ale.. jest to niejasne :(w szczególności

Presenting results may sound like a trivial issue. Actually it's not and not many libraries would fit for requirements presented below. Modern databases and search engines have to support most popular formats. Nowadays these are JSON, XML and (maybe) YAML. Chosen library should provide the same interface for feeding data to render, without any assumptions of the selected output format. Support for unicode characters is necessary. For some simple queries rendering response could be a bottleneck. That is why I chose `boost::property_tree`. This library does not support unicode by default. It is possible to fix it, by following [7] .

5 Implemented optimizations

Jarek: fix: tutaj widziałbym jakiś akapicik, w którym mówisz, że o ostatecznej efektywności decydują szczegóły, a niżej

What makes an algorithm really fast are optimizations. Here is a description of core optimizations implemented in Shuffla. Most important are first two optimizations which are making Shuffla few times faster.

5.1 Queue event

Web search-engines usually shows maximum of 10-20 results per page. High percentage of users would not look at more results anyway. Lets input random english word into google search engine. Google shows, that they have found millions of results. Rather than showing all results they are showing us only 10 first (most relevant) results. Such cases are very typical and they must be processed very fast, i.e. $\mathcal{O}((\text{offset} + \text{limit}) * ?)$ instead of $\mathcal{O}(\text{total number of results} * ?)$.

In classical K-D tree implementations, search is performed by using a simple recurrence. Lets define a single recurrence call as a search event. Each search event is performed on a single node of K-D tree. For each node we could store boundary of its nodes. Such boundary should store the values of lower and upper bound for each column.

Instead of running search events right away we could insert such events into a priority queue. Consider case when user requested data ordered by value of column X . In such case, priority queue should always return events with minimal lower bound of value X . As soon as sufficiently many results have been found, engine can stop processing search events from the queue. Hence we can hope to substantially restrict the number of actually visited nodes of the K-D tree. This optimization alone may seem like big improvement. It actually isn't. Number of nodes added to the queue may be large.

Imagine that we want to find point with smallest value of the x coordinate. We need to process all nodes that do not have a lower bound on x . Lets count them:

$$T(1) = 1$$
$$T(n) \geq 2^{d-1} * T(n/2^d)$$

Pawel: ta rekurencja zakłada, że drzewo jest w pełni zbalansowane, nie? Ale u Ciebie raczej nie jest?

Jarek: Ja tu chciałem zobrazować ile jest takich nodów. To że pisze $T(n) \geq$, a później Ω nie wystarczy?

There are $\Omega(n^{1-1/d})$ such nodes. This number can be decreased by optimization described in next chapter.

5.2 Defined boundary and real boundary

Instead of storing whole area covered by the node (see Figure 4), we could calculate area covered by points inside each node (Figure 5).

Lets revised this idea on example where point with minimal value of coordinate x is requested. First event contains root of a tree. It is processed. It causes to push two of its sons to the queue. Then we chose node with smaller minimum value of coordinate x . This event is processed and another two nodes are added. Point with minimal value of coordinate x belongs to exactly one of these nodes, and this

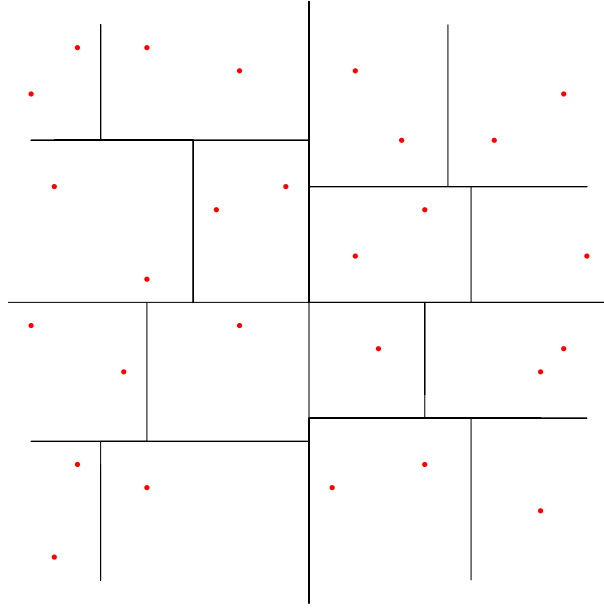


Figure 4: Storing whole area covered by nodes

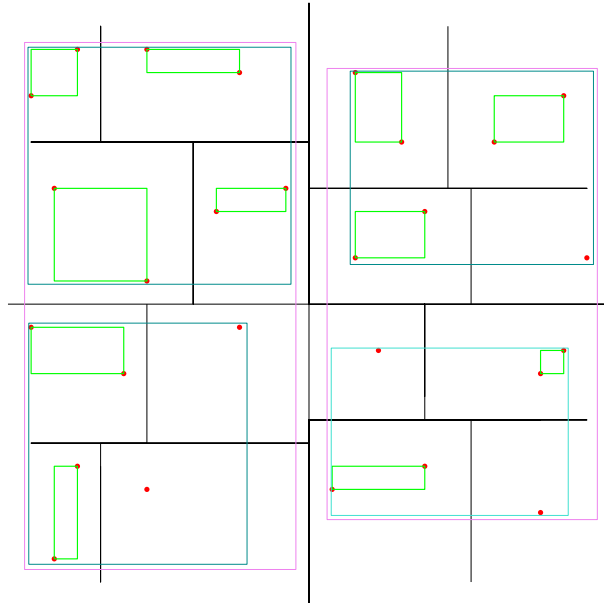


Figure 5: Area covered by points inside nodes

node is processed. This situation goes on until leaf containing point with minimum value of coordinate x is processed. At this point we found point with minimal value of coordinate x . Search process can be stopped and results may be rendered. Summarizing $\mathcal{O}(\log n)$ nodes are added to queue, and $\mathcal{O}(\log n)$ nodes are processed. Finding minimum value of coordinate x of node takes $\mathcal{O}(1)$ since we are storing this information in boundary. Processing one node takes $\mathcal{O}(1)$ time. Bottleneck of this solution is priority queue. Priority queue may be implemented as a binary heap. We processed $\mathcal{O}(\log n)$ nodes so complexity of this solution is $\mathcal{O}(\log n \log \log n)$. It can be improved by using binomial heaps [13]. Binomial heap has an insert in amortized time of $\mathcal{O}(1)$ and finding minimum in $\mathcal{O}(1)$ which reduces complexity to $\mathcal{O}(\log n)$.

Pawel: sorry, ale ten kawałek jest dla mnie średnio czytelny

Jarek: Dużo tu nie poprawilem, mógłbyś mi powiedzieć którą część poprawić?

5.3 Raw pointers instead of shared pointers

In modern C++ pointers should not be used, and shared pointers are recommended instead. Idea behind shared pointer is to automate releasing resources. Basically shared pointer keeps counter representing the number of copies of current pointer. When shared pointer is created, counter is incremented. When shared pointer is deleted, then counter is decremented. When counter is decreased to zero then resource is released. With such a tool developer is free from most common cause of memory leaks. Unfortunately shared pointers are coming with time and space overhead. I decided to use raw pointers which gives better running time and space usage. Right now I think it was biggest mistake in the project. Source code became really messy, even small change in legacy code could create bugs.

5.4 Balancing K-D tree

K-D tree has similar problem to binary trees. Binary tree can be unbalanced and so can K-D tree. My idea for balancing K-D tree is very simple. When number of nodes in one subtree is 2 times larger than number of nodes in opposite subtree then I am rebuilding their parent subtree from the beginning. This is known as partial rebuilding, a simple yet very powerful paradigm often used for making data structures dynamic, for example in weight-balanced trees [14]. While the worst case complexity of a single operations can be large because of this rebalancing, the amortized complexity [15] is actually rather good.

Pawel: zdefiniuj amortized complexity

Jarek: Ten reference nie wystarczy?

6 Complexities

Terminology:

1. $left(v)$ - left subtree of vertex v
2. $right(v)$ - right subtree of vertex v

Jarek: fix: ja bym ten warunek napisał zdaniem, zamiast \geq i \leq powinny być specjalne znaczniki

Jarek: Zmienna X , porównywanie wielkości poddrzew, jak próbowałem to robić zdaniem to było nieczytelne; /

Let n be number of points in the tree. Tree is X -balanced if and only if

$$\forall v : X * |left(v)| \leq |right(v)| \wedge X * |left(v)| \geq |right(v)|$$

Tree is well-balanced if it is 2-balanced. Tree is super-balanced if it is $\frac{3}{2}$ -balanced. Height of both well and super balanced trees are $\mathcal{O}(\log n)$. Every point belongs to $\mathcal{O}(\log n)$ nodes.

Assumptions:

1. all values of all columns are unique
2. rebuilding subtree with root θ causes subtree with root θ to be super-balanced.

6.1 Inserting/deleting point X

Lemma 1. *Rebuilding tree of size n takes $\mathcal{O}(n \log n)$.*

Proof. We can find the median in linear time [12]. Hence in order to rebuild a tree, we first find the median, and then recursively rebuild two subtrees. Complexity of such procedure can be described by $T(n) = \mathcal{O}(n) + T(\alpha n) + T((1 - \alpha)n)$, where $\frac{3}{8} \leq \alpha \leq \frac{5}{8}$. The recurrence solves to $T(n) = \Theta(n \log n)$. This can be proved by drawing recurrence call tree. At each level work done sums to $\mathcal{O}(n)$. There are $\mathcal{O}(\log n)$ levels therefore the complexity is $\mathcal{O}(n \log n)$. \square

Lemma 2. *Amortized time of insert/delete is $\mathcal{O}(\log^2 n)$.*

Proof. We will give $6 \log n$ credits to each node visited during insert/delete. As the depth of the whole tree is $\mathcal{O}(\log n)$, the total number of credits allocated during a single insert/delete operation is just $\mathcal{O}(\log^2 n)$. The goal is to make sure that the following invariant holds: a node such that the left subtree is of size n_l and the right subtree is of size n_r has at least $6|n_l - n_r| \log n$ credits available (observe that because the tree is well-balanced, this is at most $n \log n$). We must prove that the invariant still holds after each insert/delete and that we always have enough credits to amortize the rebuilding. We consider those two parts separately.

1. Each node visited during an insert/delete operation gets $6 \log n$ credits. Notice that after the visit we either increase or decrease n_l or n_r . Hence we increase $6|n_l - n_r| \log n$ by at most $6 \log n$, and so we can afford to pay for this increase using the new credits allocated to the node (it can also happen that we decrease $|n_l - n_r| \log n$, which is even better).
2. We rebuild a node as soon as one subtree is two times larger than the other, say $n_l = 2n_r$. Then the number of credits accumulated at the node is at least $6(2n_r - \frac{3}{2}n_r) \log n = 6\frac{1}{2}n_r \log n = n \log n$, where $n = n_l + n_r$. After we reconstruct the tree, the required number of credits will be at most $\log n$. Hence we can use $(n - 1) \log n$ credits to pay for the reconstruction, and by Lemma 1 this is enough.

There is one additional details. The value of $\log n$ can (and will) actually change during the execution of the algorithm. We actually care about its integer part, though. In order for it to increase, we need to perform n insert/delete operations. Hence if we charge each such operation with additional $c \log^2 n$ credits, whenever the integer part of $\log n$ increases, we will have $cn \log^2 n$ credits available.

Now, for each node of the tree we need to add $n \log n$ credits, hence the total required number of credits is described by the recurrence $T(n) = n \log n + T(\alpha n) + T((1 - \alpha)n)$, where $\frac{1}{3} \leq \alpha \leq \frac{2}{3}$. The recurrence solves to $T(n) = \Theta(n \log^2 n)$. This can be proved by drawing recurrence call tree. At each level work done sums to $\mathcal{O}(n \log n)$. There are $\mathcal{O}(\log n)$ levels therefore the complexity is $\mathcal{O}(n \log^2 n)$.

We have enough credits to pay for the increase. \square

6.2 Search

Recall that the complexity of searching in a static K-D tree is $\mathcal{O}(n^{1-1/d} + k)$. The usual proof of such bound is by analyzing a query of the form $x \geq c$. We want to bound the number of nodes whose cells intersect the boundary of such query. Because every d -th level of the tree partitions the points according to the x coordinate, we can write the recurrence $T(n) = 1 + 2^{d-1}T(\frac{n}{2^d})$, which by the master theorem solves to $T(n) = \mathcal{O}(n^{\frac{\log 2^{d-1}}{\log 2^d}}) = \mathcal{O}(n^{1-1/d})$.

Unfortunately, such worst case bound seems difficult to achieve in our dynamic settings, where the tree is just almost balanced. Nevertheless, a slightly weaker bound holds under the assumption that a child of a node corresponding to n points corresponds to at most $\frac{2}{3}n$ points itself.

Lemma 3. *Upper bound for number of processed nodes on level d is 2^{d-1} .*

Proof. K-d tree is a binary tree. Root has height 0, its children have height 1, and so on. Lets count maximal possible number of visited nodes on level d by a recursive search algorithm where region is defined by $X_e \geq C$. Node will be recursively processed iff it intersects search region but is not fully contained in it. At one of these levels there will be a pivot on dimension e . Such line intersects either the region represented by left son or right son, not both. Therefore only one of its sons will be recursively

called. Then the upper bound for processed nodes is 2^{d-1} . It can be easily generalized to every region. Adding another condition may only reduce number of processed nodes. Hence upper bound for number of processed nodes on level d is 2^{d-1} . \square

Lemma 4. *Upper bound for average number of points inside a processed node on level d is $\frac{2}{3}n$.*

Proof. The total number of points on all nodes on level d is n . The total number of points in processed nodes on level d is at most $\frac{2}{3}n$. Worst case scenario occurs when number of processed nodes is maximal. From lemma 3 **TODO: zrób z tego reference** we know that it is the scenario with search region defined as $X_e \geq C$. For all nodes with pivots on dimension e it is true that nodes from one son will be processed and nodes from other son will not be processed. Tree is well-balanced so ratio between processed points and not processed points is not greater than 2:1. Hence the upper bound for average number of points inside a processed node on level d is $\frac{2}{3}n$. \square

I will prove that the worst case scenario occurs when all subproblems are the same size. Then the complexity will be $T(n) = 2^{d-1}T(\frac{2}{3}n)$.

Lemma 5. *The recurrence describing the worst-case scenario is $T(n) = 1 + 2^{d-1}T(\frac{2}{3}n)$ which solves to $\mathcal{O}(n^{1-\log_{3/2} 2^{d-2}} \frac{3}{2})$.*

Proof. Our goal will be to prove that $T(n) \leq cn^a$ for some constant $a < 1$ using induction on n . For small values of n it is true because we can choose constant c . Now let's prove that $T(n) \leq cn^a$ assuming that $T(n') \leq cn'^a$ for all $n' < n$. We have:

$$T(n) = \sum_{i=1}^{2^{d-1}} T(x_i) \leq \sum_{i=1}^{2^{d-1}} c(x_i)^a \quad \text{where} \quad \sum_{i=1}^{2^{d-1}} x_i = \frac{2}{3}n$$

As we are interested in upperbounding $T(n)$, we are interested in the maximum possible value of the sum of all $(x_i)^a$ under the condition that all x_i sum up to $\frac{2}{3}n$. Because the function $f(x) = x^a$ is convex for $a < 1$, from Jensen's inequality we get that the sum is maximized when all x_i are equal. Hence:

$$\begin{aligned} T(n) &= \sum_{i=1}^{2^{d-1}} T(x_i) = 2^{d-1}T\left(\frac{2}{3}n\right) \leq 2^{d-1}c\left(\frac{2}{3}n\right)^a \\ &= c2^{d-1}\left(\frac{2}{3}\right)^a 2^{-a(d-1)}n^a = 2^{(1-a)(d-1)}\left(\frac{2}{3}\right)^a cn^a \end{aligned}$$

Hence $T(n) \leq cn^a$ as long as $2^{(1-a)(d-1)}\left(\frac{2}{3}\right)^a \leq 1$.

Pawel: ten kawałek trzeba posprzątać

Jarek: Troche nie wiem, w jaki sposób posprzątać;/ Zmniejszyć ilość przejść?

$$\begin{aligned} 2^{(1-a)(d-1)}\left(\frac{2}{3}\right)^a &\leq 1 \\ 2^{(1-a)(d-1)}2^a &\leq 3^a \\ 2^{d-1-ad+2a} &\leq 2^{a\log_2 3} \\ d-1-ad+2a &\leq a\log_2 3 \\ d-1 &\leq a(\log_2 3 + d-2) \\ a &\geq \frac{d-1}{\log_2 3 + d-2} \\ \frac{d-1}{\log_2 3 + d-2} &< 1 \text{ so for all } d \text{ exists } a < 1. \end{aligned}$$

\square

$T(n) = 1 + 2^{d-1}T(\frac{2}{3}n)$ solves to (by the master theorem) $T(n) = \mathcal{O}(n^{\frac{\log 2^{d-1}}{\log 3/2}})$.
 $\frac{\log 2^{d-1}}{\log 3/2} = \log_{3/2} 2^{d-1} = \log_{3/2} 2^{d-1} + \log_{3/2} 2^{d-2} - \log_{3/2} 2^{d-2} = \log_{3/2} 2^{d-2} 3 + 2^{d-2} - \log_{3/2} 2^{d-2} \frac{3}{2} =$
 $1 - \log_{3/2} 2^{d-2} \frac{3}{2}$. Therefore $T(n) = \mathcal{O}(n^{1-\log_{3/2} 2^{d-2} \frac{3}{2}})$.

Here is table with a few values of d and complexity associated with this d .

d	Complexity
2	$\mathcal{O}(n^{0.631})$
3	$\mathcal{O}(n^{0.774})$
4	$\mathcal{O}(n^{0.837})$
6	$\mathcal{O}(n^{0.896})$
10	$\mathcal{O}(n^{0.939})$

Jarek: Nie wiem czy ta tabelka ma sens. A może warto to rozszerzyć o jakiś wykresik?

While the worst case bound might seem somehow disappointing when d is large, one should remember that in practice we will observe a much better performance. For average queries algorithm behaves like $\mathcal{O}(S \log n)$ where $S = \text{offset} + \text{limit}$.

Let me explain. Lets take for example data with random distribution. When algorithm recursively enters node of K-D tree then with some probability there are some rows fitting query. If we define this probability as a constant then finding single row takes $\mathcal{O}(\log n)$. **Paweł: rozwiń to proszę**

7 Tested engines

All engines must be tested on the same schema and by the same queries. Yet presented engines have different syntax. In chapters below I'm describing table configurations and how queries were translated during their testing.

7.1 Solr

During tests I was using Solr 3.6.1. Tables in Solr are defined via xml file called schema.xml. Below is part of this file which is responsible for creating table.

```
<fields>
  <field name="id" type="uuid" indexed="true" stored="true" default="NEW"/>
  <field name="first_name" type="text_general" indexed="true"
    stored="true" required="true" />
  <field name="last_name" type="text_general" indexed="true"
    stored="true" required="true"/>
  <field name="age" type="int" indexed="true" stored="true" />
  <field name="city" type="text_general" indexed="true" stored="true"/>
</fields>

<uniqueKey>id</uniqueKey>
```

Solr requires unique key. Combining all properties is not enough. For example there are lots of guys named Jan Kowalski in Warsaw. Therefore I created another property named id.

Below I'm presenting syntax for all types of queries:

1. Equality check: `property:value`
2. Inequalities are realized by range queries: `property:[from T0 to]`. From and to can be replaced by *. For example searching number greater then or equal to 2 is performed by `property:[2 T0 *]`. `property:[* T0 *]` matches everything.
3. Strict inequalities are realized by excluding some range `-property:[from T0 to]`. For example searching numbers smaller than 2 is performed by excluding range `[2,∞]`: `-property:[2 T0 *]`.
4. Prefix queries: `property:value*`.
5. Substring queries: `property:*value*`.

7.2 PostgreSQL

I am using PostgreSQL 9.1. Schema used in testing is:

```
CREATE TABLE users (  
    first_name TEXT not null,  
    last_name TEXT not null,  
    age INT not null,  
    city TEXT not null  
);  
  
CREATE INDEX index1 ON users (first_name, last_name, age, city);  
CREATE INDEX index2 ON users (last_name, age, city);  
CREATE INDEX index3 ON users (city, age, first_name);  
CREATE INDEX index4 ON users (age, first_name, last_name);
```

Syntax:

1. Equality check, inequalities and strict inequalities are realized by =, >=, >, <=, <
2. Prefix queries are handled by `property LIKE "text%"`
3. Substring queries are handled by `property LIKE "%text%"`

For text-search PostgreSQL implemented special type of indexes: GIN [8] and GIST [9]. Unfortunately I was unsuccessful in making them work with polish words. Therefore I am testing PostgreSQL with default indexes.

7.3 Shuffla

Table is created by sending HTTP request:

```
/create_table/test_table/?first_name=string&last_name=string&age=int&city=string
```

Shuffla's query format is:

```
/search/<table name>/?subquery(&subquery)*
```

Where subquery can be:

1. `field=value` for equality check
2. `GREATER_THAN(field)=value` means that given value is greater than value in `field`
3. `SMALLER_THAN`, `SMALLER_OR_EQUAL`, `GREATER_OR_EQUAL` are working in similar fashion to `GREATER_THAN`
4. `PREFIX(field)=value` requires `field` to have prefix `value`
5. `CONTAINS(field)=value` requires `field` to contain `value` as a substring

8 Tested results

8.1 How results are tested

Testing environment is machine with Intel i5-2500 processor (4 cores, 3.5GHz), 8GB RAM and SSD drive. Comparing engines with default settings is pointless. I have tried to configure them all to production use. Most important change is increased memory limit for Solr and PostgreSQL to 1GB.

In nk we have got several millions registered users. This number does not grow in spectacular fashion. I want to simulate engines behaviour in similar conditions.

At the beginning I am inserting $8 * 10^6$ randomly generated users. Then I am running a script which creates random insert/search queries. It sends created queries for few hours and calculates statistics.

Inserted data is generated in a very simple way. I have gather lists of most popular first names and last names in Poland (with counts). Probability of selecting name is proportional to its count. Age is selected with linear distribution from set [5, 100].

Search queries are generated by following procedure:

1. Pick N - number of expressions in search query
2. Column in expression is selected from set [first name, last name, age, city]. Each of them has 25 percent chance to be chosen.
3. If selected column has int type (only age is an integer) then possible queries are equality check, inequalities and strict inequalities. Each of them has 20 percent chance to be chosen.
4. If selected column has string type then possible queries are comparison, inequalities, prefix queries and substring queries. Prefix and substring queries are having 25 percent chance to be chosen. Each of the remaining conditions has 10 percent chance to be chosen. Selected column is always compared to another string. This string is selected by selecting random word with 1 to 5 characters.

Then I am selecting random column and I am setting it as order by column. Then I am choosing offset and limit from range [0, 50].

8.2 Inserting 8 million rows

Average value of fields `first_name`, `last_name`, `city` contains 7 characters. It means that size of inserted data is about 250 MB. There is always some overhead of keeping data in data structures. It causes shuffla to use 6 GB RAM.

Inserting 8 million rows is not most important part of testing. It is performed before production deployment and we are doing this part only once. I am happy as long as indexing time does not last longer than a few hours.

Shuffla handles insert requests with speed of hunderds requests per second. Using single inserts with other engines is not so efficient. All considered engines are able to read csv file and import data from such file. For them it is the only way to import data within few hours.

8.3 Statistics

For purpose of testing in true multi-threaded environment I used tsung. Tsung is a distributed load testing tool written in erlang. It is protocol-independent and can be used to stress HTTP, WebDAV, SOAP, PostgreSQL, MySQL, LDAP, and Jabber/XMPP servers.

I am using 4 core processor for testing so I decided to test all engines using 4 threads. Each thread is sending 10.000 randomly created queries.

Solr and PostgreSQL are using less than 1 GB of RAM. It is possible because both these engines are keeping data on a hard drive. Shuffla stores everything in RAM and is consuming 6GB of RAM.

Engine	Req / sec	Memory consumption
Solr	6.89	1GB
Shuffla	3.86	6GB
PostgreSQL	2.55	1GB

Solr is superior over Shuffla in every way. Solr is faster. Solr uses less memory. It is more configurable, for example we could increase Solr memory limit. Solr is just better choice, especially for big social website purposes.

9 Few words about Solr in nk

9.1 Search architecture in nk

In nk we have 3 servers with Solr. There is master-slave replication between them. One server is a master, another two others are slaves. Only master processes modifying queries. When user does anything related to searching, PHP code handles it.

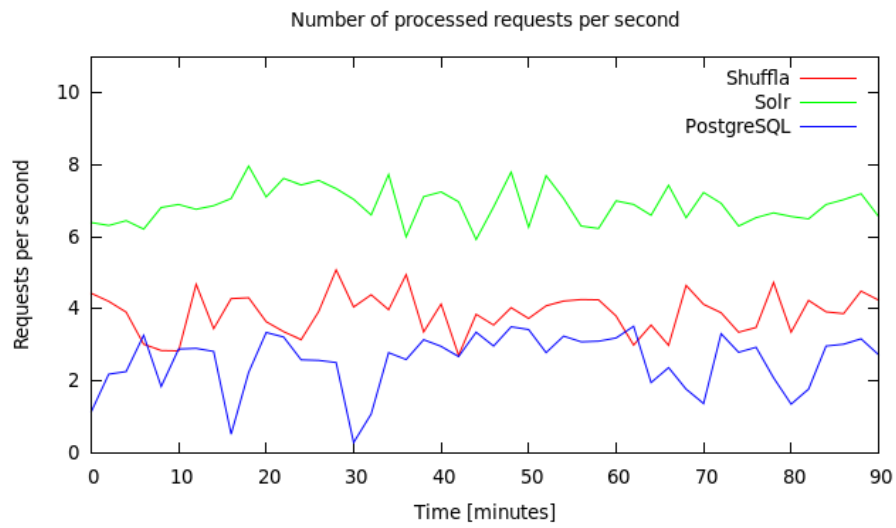


Figure 6: Number of search requests per second

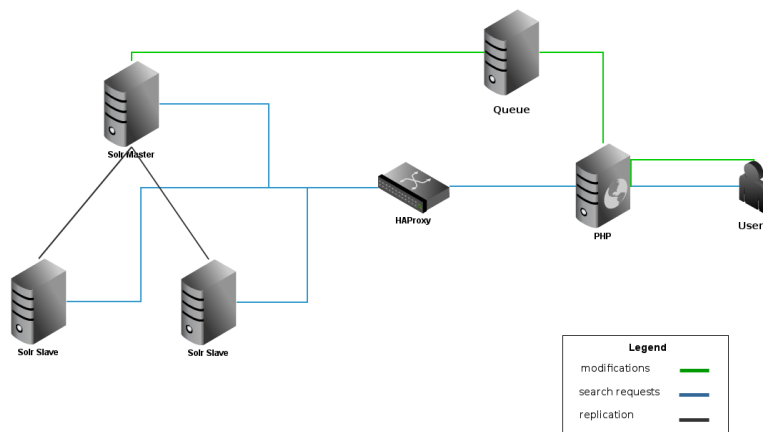
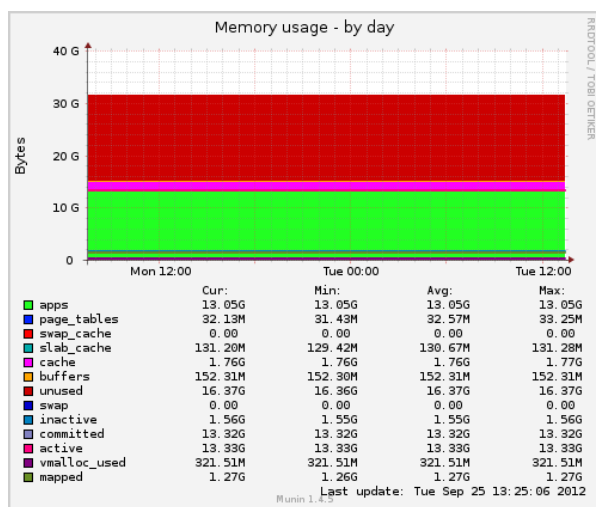
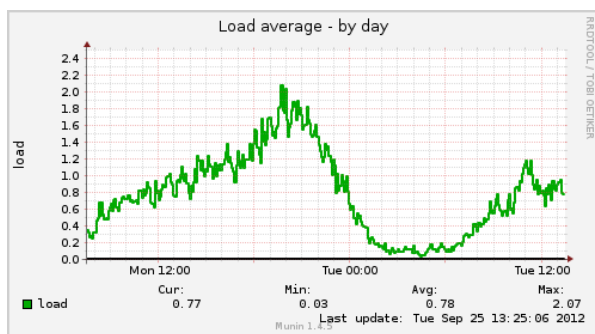
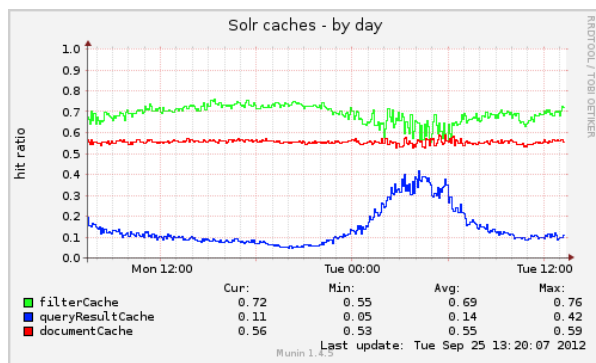
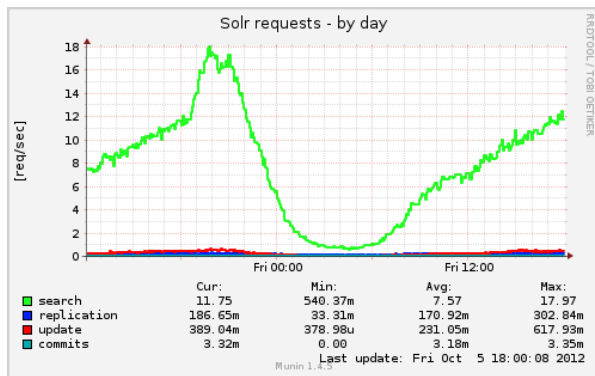


Figure 7: Architecture of searching



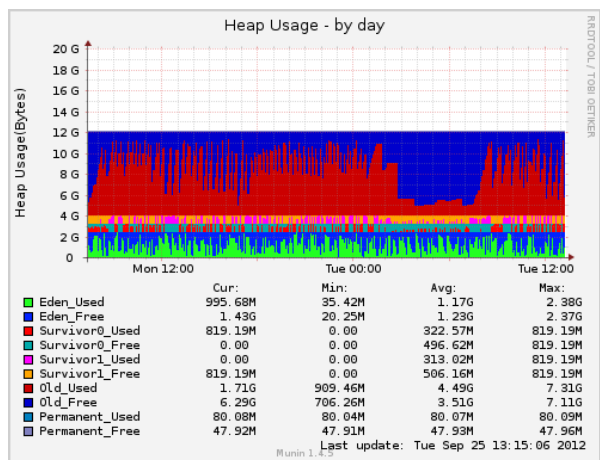
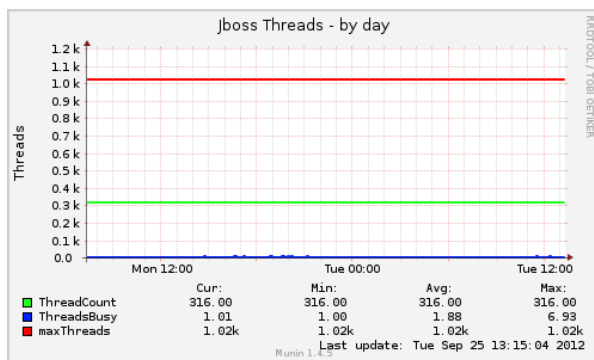
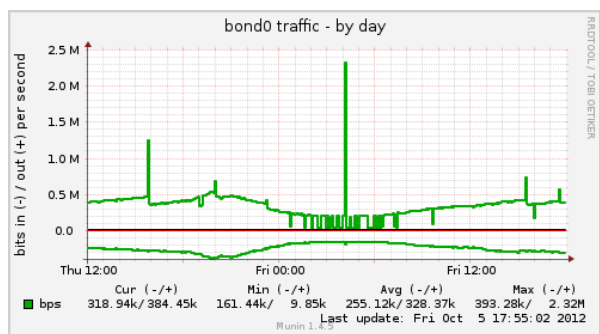
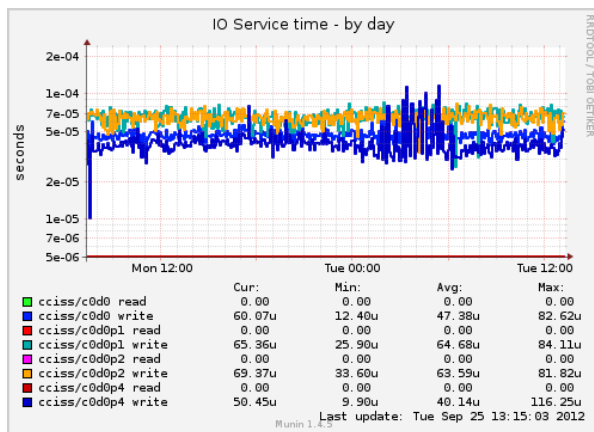
When some action requires modifying data in Solr, PHP code is creating task for it. This task is inserted into a queue. There is a background job which processes all tasks from queue one by one. We need queue to avoid unexpected spikes. For example uncaught spammer attack may cause 1000 modifying requests per second ². If this happens master could easily crash or at least experience some kind of slowness due to a big load. We want to avoid such situations.

In case of search request we need to distribute requests between servers. For this purpose search requests are going through HAProxy [10]. This load balancer is distributing requests between our 3 servers.

9.2 Statistics from master server

At night Solr does not do much. Load grows as the day progress. At the peek our master server handles 18 search requests per second (about 8 PM). Number of other requests like updates/replication/commit is negligible. Do not forget that we have three Solr servers on production and this chart is only from one of them. Solr has advanced cache system. Usually about 70 percent of requests are rendered based on values from cache. Load is a measure of the amount of computational work that a computer system performs. **Jarek: Zdanie z wiki** Load X means that we need at least $\text{ceil}(X)$ CPUs to run all processes without any of them waiting for a turn. We are using half of available memory because the other half is reserved for unexpected events, for example crash of one server. It may require increasing memory limits on two other machines. When we are closer to the peek we have more connections to Solr. It is close to a 100 at the peek. Thread pool contains more than 300 threads. Usually only small amount of this number is used. Java on our production environment is executed with `-Xms24G -Xmx24G` which means that memory used by heap plus memory used by stack cannot exceed 24GB. You may find our heap usage very strange. Most of heap memory is used by caches. We are committing changes every 15 minutes

²Actually this is highly unlikely, nk crashes spammers activity extremely well



which causes all caches to be flushed. For another 15 minutes cache is filling by search queries and then flushed again. At night there is small number of request and Solr is unable to get passed warming stage before next commit. You can see it at "Solr caches - by day" chart. Hit ratio is slightly smaller, about 60 percent at night.

10 Summary

After this discussion you may think that Shuffla is useless. I do not think it is useless. Shuffla fits great for smaller projects like searching through internet forum or blog. Both Solr and PostgreSQL are requiring not easy installation and configuration. Installation of shuffla takes 2 minutes, creating table takes 10 seconds and searching is very easy. In short/simple projects having possibility of integrating search engine in less than 5 minutes is more than welcome.

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