Formalising Mathematics in Lean

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Demo!

Please head to:

`https://github.com/formal-methods-nl/uu-geometry-2025`

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We will use the interactive theorem prover *Lean* and its mathematical library *mathlib*.

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and the most important one:

- It is a lot of fun!

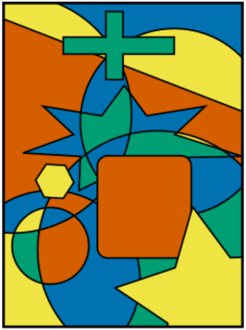
Some examples

Disclaimer: There exist many proof assistants (Automath, Mizar, Rocq, Isabelle, HOL, Agda, Lean, etc.) and they all come with important and successful formalisation projects.

What follows is not a comprehensive list!

Four colour theorem

- 1852: Guthrie conjectures



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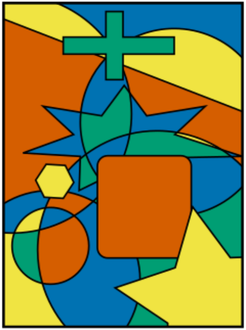

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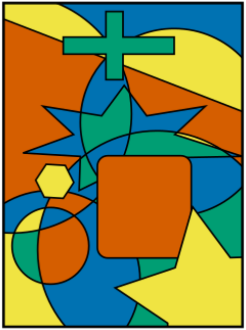
- 1976: Appel-Haken proof using extensive computer calculation
- 2005: Gonthier-Werner formalised proof in Roq



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- 2022: Full challenge completed in Lean
- Joint work of a dozen people, led by Johan Commelin and Adam Topaz

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- *Type theory* is an alternative foundation.
- It has the same logical strengh, but some practical advantages.

Naive type theory

Type theory	Set theory
Type `X`	Set X
Term `x : X`	Element $x \in X$
Function `X → Y`	Function $X \to Y$
Product `X × Y`	Product $X \times Y$
Sum `X ⊕ Y`	Disjoint union $X \sqcup Y$

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- The type checker can verify `x : X`.

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Enough theory! Let's go back to Lean.