Using Theorema in the Formalization of Theoretical Economics

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Overview

Motivation:

- Proofs in economics use typically undergraduate level proofs
- Proofs in economics are error prone (just as in other theoretical fields)
- Formalization should be achievable
- Automation (or minimization of user interactions) as goal

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Outline

- Basic Theory
- Examples
- Two Lemmas and Theorema
- Demo
- Pseudo Algorithm
- Summary

Power Function

$\mathcal{X} \equiv \{\{x_i\}_{i\in I} | x_i \geq 0, \sum_{i\in I} x_i = 1\}$., the following axioms can be defined. A power function satisfies

WC if $C \subset C' \subseteq I$ then $\pi(C, \boldsymbol{x}) \leq \pi(C', \boldsymbol{x})\ \forall \boldsymbol{x} \in \mathcal{X}$;

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WR if $y_i \geq x_i \forall i \in C \subseteq I$ then $\pi(C, \boldsymbol{y}) \geq \pi(C, \boldsymbol{x})$;

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WR if $y_i \geq x_i \forall i \in C \subseteq I$ then $\pi(C, \boldsymbol{y}) \geq \pi(C, \boldsymbol{x})$; and

SR if $\emptyset \neq C \subseteq I$ and $y_i > x_i \forall i \in C$ then $\pi(C, \boldsymbol{y}) > \pi(C, \boldsymbol{x})$.

The Same in Theorema (WC)

WC if $C \subset C' \subseteq I$ then $\pi\left(C, \boldsymbol{x}\right) \leq \pi\left(C', \boldsymbol{x}\right) \forall \boldsymbol{x} \in \mathcal{X}$

Definition[“WC”, any[π, n], bound[allocation$_n$[x]],

$$\mathrm{WC}[\pi, n] :\Leftrightarrow n \in \mathbb{N} \wedge \underset{\substack{C1,C2 \\ C1 \subset C2 \wedge C2 \subseteq I[n]}}{\forall} \ \underset{x}{\forall} \ \pi[C2, x] \geq \pi[C1, x]]$$

]

The Same in Theorema (WR)

WR $\quad$ if $y_i \geq x_i \forall i \in C \subseteq I$ then $\pi\left(C, \boldsymbol{y}\right) \geq \pi\left(C, \boldsymbol{x}\right)$

Definition[“WR”, any[π, n], bound[allocation$_n$[x], allocation$_n$[y]],

$$\mathrm{WR}[\pi, n] :\Leftrightarrow n \in \mathbb{N} \wedge \left(\underset{\substack{C \\ C \subseteq I[n]}}{\forall} \; \underset{x,y}{\forall} \left(\left(\underset{i \in C}{\forall} y_i \geq x_i \right) \implies \pi[C, y] \geq \pi[C, x] \right) \right)$$
]

The Same in Theorema (SR)

SR if $\emptyset \neq C \subseteq I$ and $y_i > x_i \forall i \in C$ then $\pi\left(C, \boldsymbol{y}\right) > \pi\left(C, \boldsymbol{x}\right)$.

Definition[“SR”, any[π, n], bound[allocation$_n$[x], allocation$_n$[y]],

$$\mathrm{SR}[\pi, n] :\Leftrightarrow n \in \mathbb{N} \wedge \Big(\underset{C \subseteq I[n] \wedge C \neq \emptyset}{\underset{C}{\forall}} \ \underset{x,y}{\forall} \Big(\Big(\underset{i \in C}{\forall}\ y_i > x_i\Big) \Longrightarrow \pi[C, y] > \pi[C, x]\Big)\Big) \Big]$$

Properties

Other important properties that power functions may have:

- AN — if $\sigma : I \rightarrow I$ is a 1:1 onto function permuting the agent set, $i \in C \Leftrightarrow \sigma(i) \in C'$, and $x_i = x'_{\sigma(i)}$ then $\pi(C, \boldsymbol{x}) = \pi(C', \boldsymbol{x}')$.
- CX — $\pi(C, \boldsymbol{x})$ is continuous in $\boldsymbol{x}$.
- RE — if $i \notin C$ and $\pi(\{i\}, \boldsymbol{x}) > 0$ then $\pi(C \cup \{i\}, \boldsymbol{x}) > \pi(C, \boldsymbol{x})$.

Domination

Def$_{\succ}$ An allocation $\boldsymbol{y}$ dominates an allocation $\boldsymbol{x}$, written $\boldsymbol{y} \succ \boldsymbol{x}$, iff $\pi(W, \boldsymbol{x}) > \pi(L, \boldsymbol{x})$; where $W \equiv \{i \mid y_i > x_i\}$ and $L \equiv \{i \mid x_i > y_i\}$. W = win set & L lose set.

Def$_D$ For $\mathcal{Y} \subset \mathcal{X}$, let $D(\mathcal{Y}) \equiv \{\boldsymbol{x} \in \mathcal{X} \mid \exists \boldsymbol{y} \in \mathcal{Y} \text{ s.t. } \boldsymbol{y} \succ \boldsymbol{x}\}$ be the dominion of $\mathcal{Y}$. $U(\mathcal{Y}) = \mathcal{X} \backslash D(\mathcal{Y})$, the set of allocations undominated by any allocation in $\mathcal{Y}$.

Core and stable set

Def$_{\mathcal{K}}$ The core, $\mathcal{K}$, is the set of undominated allocations, $U(\mathcal{X}) = \mathcal{X} \backslash D(\mathcal{X})$.

Def$_{\mathcal{S}}$ A set of allocations, $\mathcal{S} \subseteq \mathcal{X}$, is a stable set iff it satisfies

internal stability,	$\mathcal{S} \cap D(\mathcal{S}) = \emptyset$	(IS)
external stability,	$\mathcal{S} \cup D(\mathcal{S}) = \mathcal{X}$	(ES)

The conditions combine to yield $\mathcal{S} = \mathcal{X} \backslash D(\mathcal{S})$. The core necessarily belongs to any existing stable set.

Wealth Is Power

$$\mathrm{WIP}\pi[C, x] := \sum_{i \in C} x_i$$

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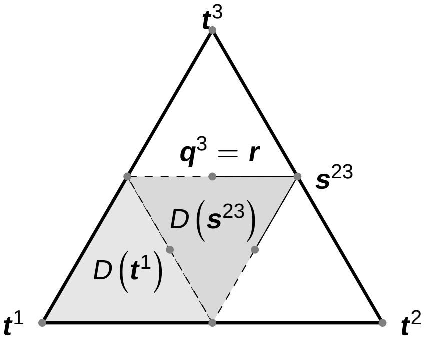
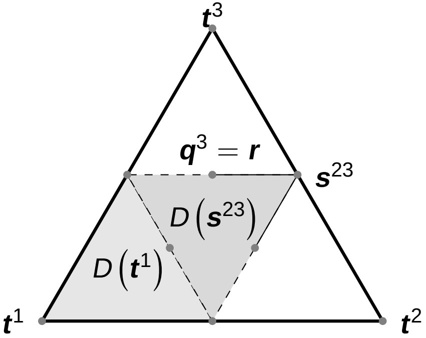


Wealth Is Power

$$\text{WIP}\pi[C, x] := \sum_{i \in C} x_i$$

Stable Set: $S =$

$$\left\{ \begin{array}{l} (0,0,1),(0,1,0),(1,0,0), \\ (0,\frac{1}{2},\frac{1}{2}),(\frac{1}{2},0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2},0), \\ (\frac{1}{4},\frac{1}{4},\frac{1}{2}),(\frac{1}{4},\frac{1}{2},\frac{1}{4}),(\frac{1}{2},\frac{1}{4},\frac{1}{4}), \end{array} \right\}$$



Strength In Numbers with $\nu > 1$

$$\mathrm{SIN}\pi_\nu[C, x] := \sum_{i \in C} (x_i + \nu)$$

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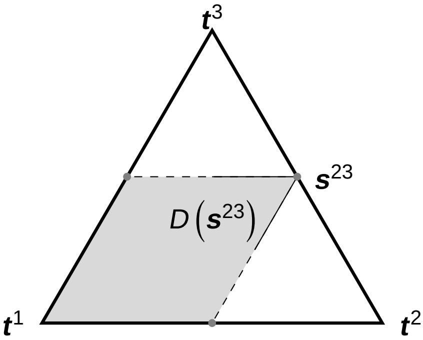

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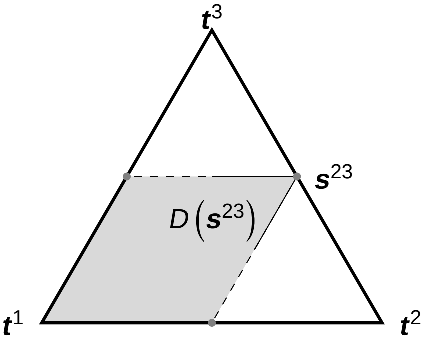
Stable Set: $S = \left\{(0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, 0)\right\}$

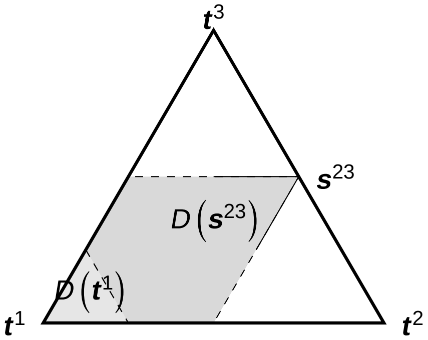


Strength In Numbers with $0 < \nu < 1$

$$\mathrm{SIN}\pi_\nu[C, x] := \sum_{i \in C} (x_i + \nu)$$

with $0 < \nu < 1$

no stable set exists



Some Explicit Dependencies of Statements

	WR	SR	WC	AN	CX	RE	Def1	Thm	Lem	Ext
Lem1	×									
Lem2				×						
Thm1										×
Lem3		×	×	×						
Lem4							×	1		
Lem5			×				×			
Thm2				×				1	4,5	×
Thm3				×					2	
Cor1								3		×
Lem6		×	×	×						
Lem7	×	×								
Cor2									7	

An Example

(One Lemma of 14 lemmas, 12 theorems, and 4 corollaries)

Lemma[“powerfunction-independent”, any[π, n, C, x, y],
with[allocation$_n[x]\wedge$allocation$_n[y]\wedge C \subseteq I[n]\wedge$powerfunction$[\pi, n]$],

$$\underset{i \in C}{\forall} (x_i = y_i) \implies (\pi[C, x] = \pi[C, y])\ ]$$

An Example

(One Lemma of 14 lemmas, 12 theorems, and 4 corollaries)

Lemma[“powerfunction-independent”, any[π, n, C, x, y],
with[allocation$_n$[x]$\wedge$allocation$_n$[y]$\wedge C \subseteq I[n] \wedge$powerfunction[$\pi, n$]],
$\underset{i \in C}{\forall} (x_i = y_i) \implies (\pi[C, x] = \pi[C, y])$]

Demo

Another Example

Lemma

When $n = 3$: 1. $\mathcal{K} = \emptyset$ *implies* $\mathbf{t}^i \in D\left(\mathbf{s}^{jk}\right)$ *for distinct* $i, j, k \in I$.

Proof.

1. As $\mathcal{K} = \emptyset$, no agent can defend its holdings against both others, so that $\pi\left(\{i\}, \mathbf{t}^i\right) < \pi\left(\{j, k\}, \mathbf{t}^i\right)$ for distinct i, j and k. As $\{j, k\}$ prefers $\mathbf{s}^{jk}$ to $\mathbf{t}^i$, this ensures that $\mathbf{s}^{jk} \succ \mathbf{t}^i$.

$\square$

An Example (Cont'd)

Make this proof more formal

- AN: use 1, 2, and 3 instead of i,j, and k.
- In $\boldsymbol{t}^1$, 2 and 3 together are more powerful than 1 on its own: $\mathcal{K} = \emptyset$ means that $\boldsymbol{t}^1 \notin \mathcal{K}$, that is, there exists an $\boldsymbol{x}$ such that $\boldsymbol{x} \succ \boldsymbol{t}^1$. For $\boldsymbol{x} = (x_1, x_2, x_3)$ distinguish 3 cases:

  Case 1: $x_1, x_2 \neq 0$. Since $\boldsymbol{t}^1 \notin \mathcal{K}$ we have $\pi(\{2,3\}, \boldsymbol{t}^1) > \pi(\{1\}, \boldsymbol{t}^1)$, hence we get $\boldsymbol{s}^{23} \succ \boldsymbol{t}^1$.

  Case 2: Without loss of generality $x_2 > x_3 = 0$. With axiom WC we have $\pi(\{2,3\}, \boldsymbol{t}^1) > \pi(\{2\}, \boldsymbol{t}^1)$.

  Case 3: $x_2 = x_3 = 0$. This would mean $\boldsymbol{x} = \boldsymbol{t}^1$, which cannot be.

Pseudo Algorithm

Algorithm["StableSet2", any[π],

$$
\mathrm{stableSet}[\pi] :=
\begin{cases}
\begin{cases}
\text{"no stable"} & \Leftarrow \mathrm{empty}[M[1,\pi]] \\
\mathrm{where}\Big[\mathcal{S} = \mathrm{dyadicSet}[0,3] \cup \bigcup\limits_{i=1,\dots,3} S[i,\pi], & \\
\quad\begin{cases}
\mathcal{S} \cup P[\pi] & \Leftarrow \neg\mathrm{fullSet}[\mathcal{S} \cup D[\mathcal{S},\pi,3]] \\
\mathcal{S} & \Leftarrow \mathrm{fullSet}[\mathcal{S} \cup D[\mathcal{S},\pi,3]] \\
\text{"unknown X"} & \Leftarrow \mathrm{otherwise}
\end{cases} \Big] & \Leftarrow \neg\mathrm{empty}[M[1,\pi]] \\
\text{"unknown M"} & \Leftarrow \mathrm{otherwise}
\end{cases} & \Leftarrow (*) \\
\mathrm{dyadicSet}[1,3] \backslash \mathrm{dyadicSet}[0,3] & \Leftarrow \mathrm{otherwise}
\end{cases}
$$

with $(*)$ to be replaced by $\pi[\{1\}, t[1,3]] \geq \pi[\{2,3\}, t[1,3]]$.

Pseudo Algorithm (Cont'd)

Demo

Pseudo Algorithm (Cont'd)

Demo

- Non-computational in several aspects
- Evaluation by a mixture of reasoning and computing. Can compute the stable set of WIP, SIN, assumed the corresponding lemmas are available.
- Plan: Extend the computational part, e.g., represent infinite set in a finite way. Use underlying Mathematica to compute solutions of equations.

Summary

- Formalisation and proof in Theorema possible.
- Axiomatic approach in theoretical economics valuable.
- Good field with non-trivial but not very deep mathematics.
- Automation at least partially possible.
- Theorema offers mixture of reasoning and computation. Very useful for determining stable sets.