The core

$$\text{Definition}\Big[\text{"allAllocations"},\ \text{any}[n],\ X[n] := \Big\{x \ \Big|_{\text{allocation}_n[x]}\Big\}\Big]$$

$$\text{Definition}\Big[\text{"dominion"},\ \text{any}[\pi, n, Y],\ D[Y, \pi, n] := \Big\{x \ \Big|_{\text{allocation}_n[x]} \ \underset{y}{\exists}\,(y \in Y \bigwedge \text{dominates}[y, x, \pi, n])\Big\}\Big]$$

$$\text{Definition}\Big[\text{"dominionOn"},\ \text{any}[\pi, n, Y],\ DOn[Y, \pi, n, Z] := \Big\{x \ \Big|_{\text{allocation}_n[x]} \ x \in Z \bigwedge \underset{y}{\exists}\,(y \in Y \bigwedge \text{dominates}[y, x, \pi, n])\Big\}\Big]$$

$$\text{Definition}[\text{"Un-dominion"},\ \text{any}[\pi, n, Y],\ U[Y, \pi, n] := X[n] \setminus D[Y, \pi, n]]$$

$$\text{Definition}\Big[\text{"core-paper"},\ \text{any}[\pi, n],\ K[\pi, n] := \Big\{x \ \Big|_{\text{allocation}_n[x]} \ \underset{i \in I[n]}{\exists} x_i > 0 \Leftrightarrow \pi[\{i\}, x] \geq \pi[I[n] \setminus \{i\}, x]\Big\}\Big]$$

$$\text{Definition}[\text{"core"},\ \text{any}[\pi, n],\ K[\pi, n] := U[X[n], \pi, n]]$$

$$\text{Definition}\Big[\text{"tyrannical"},\ \text{any}[i, n],\ t[i, n]_j := \begin{cases} 1 \Leftarrow j = i \\ 0 \Leftarrow \text{otherwise} \end{cases}\Big]$$

$$\text{Definition}\Big[\text{"tyrannical"},\ \text{any}[i, n],\ t[i, n] := \Big\langle \begin{cases} 1 \Leftarrow j = i \\ 0 \Leftarrow \text{otherwise} \end{cases} \Big|_{j \in I[n]} \Big\rangle\Big]$$

$$\text{Lemma}\Big[\text{"tyrannicalSum"},\ \text{any}[j, n],\ \sum_{i \in I[n]} t[j, n]_i = 1\Big]$$

$$\text{Lemma}[\text{"tyrannical allocation"},\ \text{any}[i, n],\ \text{with}[n \in \mathbb{N} \bigwedge i \in I[n]],\ \text{allocation}_n[t[i, n]]]$$

$$\text{Definition}\Big[\text{"split allocation"},\ \text{any}[i, j, n],\ s[j, k, n]_i := \begin{cases} \frac{1}{2} \Leftarrow ((i = j) \bigvee (i = k)) \\ 0 \Leftarrow \text{otherwise} \end{cases}\Big]$$

$$\text{Definition}\Big[\text{"split allocation"},\ \text{any}[k, j, n],\ s[j, k, n] := \Big\langle \begin{cases} \frac{1}{2} \Leftarrow ((i = j) \bigvee (i = k)) \\ 0 \Leftarrow \text{otherwise} \end{cases} \Big|_{i \in I[n]} \Big\rangle\Big]$$

$$\text{Theorem}\Big[\text{"Jordan[2006a],2.6a"},\ \text{any}[\pi, n],\ \underset{i \in I[n]}{\forall}\,(t[i, n] \in K[\pi, n]) \Leftrightarrow \pi[\{i\}, t[i, n]] \geq \pi[I[n] \setminus \{i\}, t[i, n]]\Big]$$

```
Theorem["Jordan[2006a],2.6a,contra", any[π, n],
  ∀_{i∈I[n]} ((t[i, n] ∉ K[π, n]) ⇒ π[I[n] \ {i}, t[i, n]] > π[{i}, t[i, n]])]

Theorem["Jordan[2006a],2.6b", any[π, n], with[∀_{i∈I[n]} π[{i}, t[i, n]] < π[I[n] \ {i}, t[i, n]]],
  K[π, n] = ∅]

Lemma["Core n=3,a", any[π],
  (K[π, 3] = ∅) ⇒ ∀_{i,j,k∈I[3]} (distinct[i, j, k] ⇒ t[i, 3] ∈ D[{s[j, k, 3]}, π, 3])]

Compute[W[3, t[1, 3], s[2, 3, 3]], using → {Definition["agents"],
    Definition["WinLose"], Definition["tyrannical"], Definition["split allocation"]},
  builtin → {Builtin ["Connectives"], Builtin ["Quantifiers"],
    Builtin ["Tuples"], Builtin ["Sets"], Builtin ["Numbers"]}]

{2, 3}

Compute[L[3, t[1, 3], s[2, 3, 3]], using → {Definition["agents"],
    Definition["WinLose"], Definition["tyrannical"], Definition["split allocation"]},
  builtin → {Builtin ["Connectives"], Builtin ["Quantifiers"],
    Builtin ["Tuples"], Builtin ["Sets"], Builtin ["Numbers"]}]

{1}

Definition["distinct", any[i, j, k],
  distinct[i, j, k] :⇔ (i ≠ j ⋀ i ≠ k ⋀ j ≠ k)]

Compute[W[3, t[1, 3], s[2, 3, 3]] = I[3] \ {1},
  using → {Definition["distinct"], Definition["agents"], Definition["WinLose"],
    Definition["tyrannical"], Definition["split allocation"]},
  builtin → {Builtin ["Connectives"], Builtin ["Quantifiers"],
    Builtin ["Tuples"], Builtin ["Sets"], Builtin ["Numbers"]}]

True

Definition["agents"]

def [agents, range [simpleRange [n]], True, flist [lf [, I[n] := {i | _{i=1,…,n}}]]]

Compute[∀_{i,j,k∈I[3]} (distinct[i, j, k] ⇒ (W[3, t[i, 3], s[j, k, 3]] = I[3] \ {i})),
  using → {Definition["distinct"], Definition["agents"], Definition["WinLose"],
    Definition["tyrannical"], Definition["split allocation"]},
  builtin → {Builtin ["Connectives"], Builtin ["Quantifiers"],
    Builtin ["Tuples"], Builtin ["Sets"], Builtin ["Numbers"]}]

True

Compute[∀_{i,j,k∈I[3]} (distinct[i, j, k] ⇒ (L[3, t[i, 3], s[j, k, 3]] = {i})),
  using → {Definition["distinct"], Definition["agents"], Definition["WinLose"],
    Definition["tyrannical"], Definition["split allocation"]},
  builtin → {Builtin ["Connectives"], Builtin ["Quantifiers"],
    Builtin ["Tuples"], Builtin ["Sets"], Builtin ["Numbers"]}]

True

ResetComputation[]
```

```
$Context

Tma`

Lemma["s dominates t", any[i, j, k ∈ I[3]], with[distinct[i, j, k]],
 W[3, t[i, 3], s[j, k, 3]] = I[3] \ {i}
 L[3, t[i, 3], s[j, k, 3]] = {i}              ]

Prove[Lemma["s dominates t"],
 builtin  → {Definition["distinct"], Definition["agents"], Definition["WinLose"],
   Definition["tyrannical"], Definition["split allocation"],
   Builtin ["Connectives"], Builtin ["Quantifiers"], Builtin ["Tuples"],
   Builtin ["Sets"], Builtin ["Numbers"]}, by → SetTheoryPCSProver,
 ProverOptions → {DisableProver → {STKBR, STP, QR, PND, CDP}, TransformRanges → False}]

{- ProofObject -, - ProofObject -}

Prove[Lemma["Core n=3,a"],
 using → {Theorem["Jordan[2006a],2.6a,contra"], Lemma["s dominates t"],
   Lemma["tyrannical allocation"], Definition["dominion"], Definition["dominates"]},
 by → SetTheoryPCSProver, ProverOptions → {DisableProver → {STKBR}},
 builtin  → {Builtin ["Number Domains"], Builtin ["Connectives"]}, SearchDepth → 110]

- ProofObject -

Block[{$RecursionLimit = ∞},
 Transform[%, TransformerOptions → {branches → Proved, steps → Useful}]]

- ProofObject -

Lemma["Core n=3,b", any[π], with[AN[π, 3] ⋀ powerfunction[π, 3]],
    ∀
 i,j,k∈I[3]  (distinct[i, j, k] ⇒ s[j, k, 3] ∉ D[{t[i, 3]}, π, 3])]

Lemma["AN perm,n=3", any[π], with[AN[π, 3]],
  ∀
 i,j,k  (distinct[i, j, k] ⇒ (π[{i}, s[j, k, 3]] = π[{j}, s[i, k, 3]]))]

Lemma["t dominates s", any[i, j, k], with[distinct[i, j, k]],
 W[3, s[j, k, 3], t[i, 3]] = {i}
 L[3, s[j, k, 3], t[i, 3]] = I[3] \ {i}]

Lemma["distinct agents", any[i, j, k], with[distinct[i, j, k]],
 I[3] \ {i} = {j, k}]

Lemma["SR,n=3", any[π], with[powerfunction[π, 3]],
   ∀
 i,j,k,α  (distinct[i, j, k] ⋀ π[{j}, s[i, k, 3]] > α ⇒ π[{j}, s[j, k, 3]] > α)]

Lemma["WC,n=3", any[π], with[powerfunction[π, 3]],
   ∀
 i,j,k,α  (distinct[i, j, k] ⋀ π[{j}, s[j, k, 3]] > α ⇒ π[{j, k}, s[j, k, 3]] > α)]

Lemma["irreflexive>", any[a],
 ¬ (a > a)]
```

```
Prove[Lemma["Core n=3,b"], using →
  {Lemma["AN perm,n=3"], Lemma["t dominates s"], Lemma["distinct agents"], Lemma["SR,n=3"],
   Lemma["WC,n=3"], Definition["dominion"], Definition["dominates"], Lemma["irreflexive>"]},
 by → SetTheoryPCSProver, ProverOptions → {DisableProver → {STP, STC, STS, CDP},
   RWSetOperators → True, AllowIntroduceQuantifiers → True,
   DisableInferenceRule → {¢KBInferNonEmpty, ¢KBSetEquality, ¢KBInclusion}}, SearchDepth → 90]
```

- ProofObject -

```
Block[{$RecursionLimit = Infinity}, ProofShow[]]
```

Definition["dyadic number", any[x], with[$x \in \mathbb{R}$],

$$\text{dyadicNumber}[x] :\Leftrightarrow \left((x = 0) \bigvee \underset{k\in\mathbb{N}}{\exists} \left(x = 2^{-k}\right)\right)]$$

Definition["dyadic allocation", any[x, n],

$$\text{dyadicAllocation}[x, n] :\Leftrightarrow \underset{i\in I[n]}{\forall} \text{dyadicNumber}[x_i]]$$

Definition["dyadic set", any[k, n],

$$\text{dyadicSet}[k, n] = \left\{\underset{x}{x \mid} \text{dyadicAllocation}[x, n] \bigwedge \underset{i\in I[n]}{\forall} x_i > 0 \Rightarrow x_i \geq 2^{-k}\right\}]$$

Lemma["D03",
 dyadicSet[0, 3] = {t[1, 3], t[2, 3], t[3, 3]}]

Lemma["D13",
 dyadicSet[1, 3] = {t[1, 3], t[2, 3], t[3, 3], s[1, 2, 3], s[1, 3, 3], s[2, 3, 3]}]

Definition["championPowerfunction", any[C, x],

$$\text{champion}\pi[C, x] = \max_{i\in C} x_i]$$

$$\text{Lemma}\left[\text{"max"}, \underset{k\in C}{\exists} \left(\max_{i\in C} x_i = x_k\right) \bigwedge \underset{k\in C}{\forall} \max_{i\in C} x_i \geq x_k\right]$$

Definition["leaderPowerfunction", any[C, x, ν], with[$0 < \nu \wedge \nu < 1$],

$$\text{leader}\pi_\nu[C, x] = \begin{cases} \max_{i\in C} x_i & \Leftarrow \max_{i\in C} x_i \leq \frac{1}{3} \\ \sum_{i\in C} ((1-\nu) * x_i + \nu) & \Leftarrow \text{otherwise} \end{cases}]$$

Definition["centre allocation",

$$\overline{p} = \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle]$$

Lemma["championCore",
 K[championπ, 3] = dyadicSet[1, 3] ⋃ {$\overline{p}$}]

Lemma["leaderCore", any[ν], with[$0 < \nu \wedge \nu < 1$],
 K[leaderπ_ν, 3] = dyadicSet[0, 3] ⋃ {$\overline{p}$}]

Theorem["coreCharacterization", any[π], with[powerfunction[π, 3] ⋀ AN[π, 3]],
 (K[π, 3] = ∅) ⋁ (K[π, 3] = dyadicSet[0, 3]) ⋁
  (K[π, 3] = dyadicSet[0, 3] ⋃ {$\overline{p}$}) ⋁ (K[π, 3] = dyadicSet[1, 3]) ⋁
  (K[π, 3] = dyadicSet[1, 3] ⋃ {$\overline{p}$})]

```
Theorem["coreCharacterization:⊆",
 {K[π, 3] | powerfunction[π, 3] ⋀ AN[π, 3]} ⊆
        π
  {∅, dyadicSet[0, 3], dyadicSet[0, 3] ⋃ {p̅}, dyadicSet[1, 3], dyadicSet[1, 3] ⋃ {p̅}}]

Theorem["coreCharacterization:⊇",
 {K[π, 3] | powerfunction[π, 3] ⋀ AN[π, 3]} ⊇
        π
  {∅, dyadicSet[0, 3], dyadicSet[0, 3] ⋃ {p̅}, dyadicSet[1, 3], dyadicSet[1, 3] ⋃ {p̅}}]

Prove[Theorem["coreCharacterization:⊆"], using → {}, by → SetTheoryPCSProver,
 ProverOptions → {AllowIntroduceQuantifiers → False}, SearchDepth → 40]

ES

- ProofObject -
```

In the rest of the document is the variant of the core characterisation, in which the lemma is not proved for arbitrary disjoint i,j,k, but 1,2,3

```
Lemma["Core n=3,a,123", any[π], with[distinct[1, 2, 3]],
 (K[π, 3] = ∅) ⇒ t[1, 3] ∈ D[{s[2, 3, 3]}, π, 3]]

Lemma["s dominates t,123",
 W[3, t[1, 3], s[2, 3, 3]] = I[3] \ {1}
 L[3, t[1, 3], s[2, 3, 3]] = {1}          ]

Prove[Lemma["s dominates t,123"],
 builtin → {Definition["distinct"], Definition["agents"], Definition["WinLose"],
   Definition["tyrannical"], Definition["split allocation"],
   Builtin ["Connectives"], Builtin ["Quantifiers"], Builtin ["Tuples"],
   Builtin ["Sets"], Builtin ["Numbers"]}, by → SetTheoryPCSProver,
 ProverOptions → {DisableProver → {STKBR, STP, QR, PND, CDP}, TransformRanges → False}]

Theorema::noEnvironment : No environment of type def  named "distinct" available.

{- ProofObject -, - ProofObject -}

Prove[Lemma["Core n=3,a,123"],
 using → {Theorem["Jordan[2006a],2.6a,contra"], Lemma["s dominates t,123"],
   Lemma["tyrannical allocation"], Definition["dominion"], Definition["dominates"]},
 by → SetTheoryPCSProver, ProverOptions → {DisableProver → {STKBR}},
 builtin → {Definition["agents"], Builtin ["Numbers"], Builtin ["Number Domains"],
   Builtin ["Quantifiers"], Builtin ["Connectives"]}, SearchDepth → 110]

Block[{$RecursionLimit = Infinity}, ProofShow[]]

Compute[L[3, t[1, 3], s[2, 3, 3]], using → {Definition["agents"],
   Definition["WinLose"], Definition["tyrannical"], Definition["split allocation"]},
 builtin → {Builtin ["Connectives"], Builtin ["Quantifiers"],
   Builtin ["Tuples"], Builtin ["Sets"], Builtin ["Numbers"]}]

{1}
```

```
Compute[W[3, t[1, 3], s[2, 3, 3]] = I[3] \ {1},
 using → {Definition["distinct"], Definition["agents"], Definition["WinLose"],
   Definition["tyrannical"], Definition["split allocation"]},
 built-in → {Built-in ["Connectives"], Built-in ["Quantifiers"],
   Built-in ["Tuples"], Built-in ["Sets"], Built-in ["Numbers"]}]
```

```
True
```

```
Lemma["t dominates s,123",
 W[3, s[2, 3, 3], t[1, 3]] = {1}
 L[3, s[2, 3, 3], t[1, 3]] = I[3] \ {1}]
```

```
Lemma["SR, n=3,123", any[π, α], with[powerfunction[π, 3]],
  π[{2}, s[1, 3, 3]] > α ⇒ π[{2}, s[2, 3, 3]] > α]
```

```
Lemma["WC n=3,b,123", any[π, α], with[powerfunction[π, 3]],
  π[{2}, s[2, 3, 3]] > α ⇒ π[{2, 3}, s[2, 3, 3]] > α]
```

```
Lemma["Core n=3,b,123", any[π], with[powerfunction[π, 3]],
 AN[π, 3] ⇒ s[2, 3, 3] ∉ D[{t[1, 3]}, π, 3]]
```

```
Lemma["I[3]\{1}",
 I[3] \ {1} = {2, 3}]
```

```
Prove[Lemma["Core n=3,b,123"],
 using → {Definition["dominion"], Definition["dominates"], Lemma["t dominates s,123"],
   Lemma["AN perm,n=3,123"], Lemma["SR, n=3,123"], Lemma["WC n=3,b,123"],
   Lemma["I[3]\{1}"], Lemma["irreflexive>"]}, by → SetTheoryPCSProver,
 ProverOptions → {DisableProver → {STP, STC, STS, CDP}, RWSetOperators → True,
   AllowIntroduceQuantifiers → True, DisableInferenceRule →
    {¢KBInferNonEmpty, ¢KBSetEquality, ¢KBInclusion}}, SearchDepth → 60]
```