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Needs["Theorema`"]
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Vickrey Auctions

Preliminaries

Deviation from a vector

I'm not yet sure whether we need *both* of the following definitions, or just one of them (and if so, which one).

TODO: Theorema will provide a built-in operation for "deviation".

Definition(Deviation)

$$\forall_{b,\,alternativeValue,\,alternativeVector,\,index}$$

$$\text{deviation}[b,\ alternativeValue,\ index] :=\ \left\langle \begin{cases} alternativeValue & j == index \\ b_j & \text{otherwise} \end{cases} \ \Big|_{j=1,\ldots,|b|} \right\rangle \qquad (deviation)$$

Part::partw : Part 14 of {1, 2, 1, 1, 9, 1, 1, 4, 1, 1, 1} does not exist. ≫

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General::stop : Further output of Part::partw will be suppressed during this calculation. ≫

$$\text{deviationVector}[b,\ alternativeVector,\ index] := \text{deviation}[b,\ alternativeVector_{index},\ index] \qquad (deviationVector)$$

Strategy (bids), Allocation, Payment

TODO: Once we have "where" as a global quantifier, say "where n=|b|".

Definition(Bids, Allocation and Payment)

$$\forall_{b,x,p,v}$$

$$\text{valuation}[v] :\Leftrightarrow \forall_{j=1,\ldots,|v|} v_j > 0 \qquad (valuation)$$

$$\text{bids}[b] :\Leftrightarrow \forall_{j=1,\ldots,|b|} b_j \geq 0 \qquad (bids)$$

$$\text{allocation}[b,\ x] :\Leftrightarrow \underset{n=|b|}{\text{where}}\ \text{bids}[b] \wedge \left(\exists_{k=1,\ldots,n} (x_k == 1) \wedge \forall_{\substack{j=1,\ldots,n \\ j\neq k}} x_j == 0 \right) \wedge (|x| == n) \qquad (allocation)$$

???

$$\text{vickreyPayment}[b, p] :\Longleftrightarrow \underset{n=|b|}{\text{where}}\ \text{bids}[b] \wedge \left(\underset{i=1,\ldots,n}{\forall} p_i \geq 0\right) \wedge (|p| == n)$$

Sample Definition (a vector of “1” bids)

$$\text{allBid1} := \langle 1, 1, 1, 1, 1\rangle \qquad (allBid1)$$

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bids[allBid1]
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...?

$$\underset{j=1,\ldots,|\langle 1,1,1,1,1\rangle|}{\forall} \langle 1, 1, 1, 1, 1\rangle_j \geq 0$$

Interesting part

From now on, …

$$\underset{\text{valuation}[v]}{\forall}\ \underset{\substack{\text{bids}[b] \\ |b|==|v|}}{\forall}\ \underset{\substack{x \\ \text{allocation}[b,x]}}{\forall}\ \underset{\substack{p \\ \text{vickreyPayment}[b,p]}}{\forall}\ \underset{n=|v|}{\text{where}}$$

Lemma (A deviation from a bid is still a well-formed bid)

$$\underset{\substack{\text{bids}[\text{alternativeVector}] \\ |\text{alternativeVector}|==n}}{\forall}\ \underset{i=1,\ldots,n}{\forall}\ \text{bids}[\text{deviationVector}[b, \text{alternativeVector}, i]] \qquad (4)$$

Lemma (An allocation uniquely determines the winner)

$$\underset{\substack{\text{winner} \\ x_{\text{winner}}==1}}{\forall}\ x_j == 1 \Longrightarrow j == \text{winner} \qquad (5)$$

Outcome

We don’t define the outcome (allocation and payment) explicitly.

Valuation

Lemma (A valuation is a well-formed bid)

$$\text{bids}[v] \qquad (6)$$

Payoff

TODO: document that defining payoff[...]$_i$ does not at the same time define a partially evaluated payoff[...] (as it would in HOL)

Definition(Payoff)

$$\text{payoff}[v, x, p] := \left\langle v_i * x_i - p_i \;\Big|_{i=1,\ldots,n} \right\rangle \tag{7}$$

Second-price auctions

TODO: How will max for the empty set be defined? (Recall the discussion with Colin, on second-price auctions with a single bidder, and our different approaches to formalising them.)

Definition(second-price auction)

$$\text{secondPriceAuction}[b, x, p] :\Longleftrightarrow \underset{i=1,\ldots,n}{\exists} \text{secondPriceAuctionWinner}[b, x, p, i] \bigwedge \underset{\substack{j=1,\ldots,n \\ j\neq i}}{\forall} \text{secondPriceAuctionLoser}[x, p, j] \tag{secondPriceAuction}$$

$$\underset{i=1,\ldots,n}{\forall}$$

$$\text{secondPriceAuctionWinner}[b, x, p, i] :\Longleftrightarrow b_i == \max_{j=1,\ldots,n} b_j \bigwedge x_i == 1 \bigwedge p_i == \max_{\substack{j=1,\ldots,n \\ j\neq i}} b_j \tag{secondPriceAuctionWinner}$$

$$\text{secondPriceAuctionLoser}[x, p, i] :\Longleftrightarrow x_i == 0 \wedge p_i == 0 \tag{secondPriceAuctionLoser}$$

Properties of second-price auctions

$$\text{secondPriceAuction}[b, x, p] \Rightarrow$$

Lemma(A second-price auction has only one winner)

???

$$\underset{\substack{i=1,\ldots,n \\ \text{secondPriceAuctionWinner}[b,x,p,i]}}{\forall} \; \underset{\substack{j=1,\ldots,n \\ \text{secondPriceAuctionWinner}[b,x,p,j]}}{\forall} i == j$$

Lemma (The participant to whom the good gets allocated also satisfies the further properties of a second-price auction winner)

???

$$\forall_{\substack{winner = 1, \ldots, n \\ x_{winner} == 1}} \texttt{secondPriceAuctionWinner}[b, x, p, winner]$$

Lemma (Participants to whom the good doesn't get allocated also satisfy the further properties of a second-price auction loser)

???

$$\forall_{\substack{loser = 1, \ldots, n \\ x_{loser} == 0}} \texttt{secondPriceAuctionLoser}[x, p, loser]$$

Lemma (If there is only one highest bidder, that bidder wins)

???

$$\forall_{\substack{maxBidder = 1, \ldots, n \\ b_{maxBidder} > \max_{\substack{j = 1, \ldots, n \\ j \neq maxBidder}} b_j}} \texttt{secondPriceAuctionWinner}[b, x, p, maxBidder]$$

Lemma (A formula for computing the payoff of the winner of a second-price auction)

???

$$\forall_{\substack{winner = 1, \ldots, n \\ x_{winner} == 1}} \texttt{payoff}[v, b, x, p]_{winner} == v_{winner} - \max_{\substack{j = 1, \ldots, n \\ j \neq winner}} b_j$$

Lemma (A formula for computing the payoff of a loser of a second-price auction)

???

$$\forall_{\substack{loser = 1, \ldots, n \\ x_{loser} == 0}} \texttt{payoff}[v, b, x, p]_{loser} == 0$$

Lemma (Payoff of the winner when deviating from his/her valuation)

???

$$\texttt{payoff}[v, b, x, p]_{winner} == v_{winner} - \max_{\substack{j = 1, \ldots, n \\ j \neq winner}} \texttt{deviationVector}[b, v, winner]_j$$

Properties of single-good auctions

Definition(Efficiency)

efficiency

$$\texttt{efficient}[v, x] :\Longleftrightarrow \forall_{i=1,\ldots,n} \; x_i == 1 \Longrightarrow v_i == \max_{j=1,\ldots,n} v_j$$

TODO: find out in what way $\text{whateverBid}_i \neq b_i$ makes the proof harder/ easier

Definition(WeaklyDominantStrategy)

equilibriumWeaklyDominantStrategy

$$\texttt{equilibriumWeaklyDominantStrategy}[v, b] :\Longleftrightarrow$$

$$\forall_{i=1,\ldots,n} \; \forall_{\substack{\texttt{bids}[\texttt{whateverBid}] \\ \texttt{whateverBid}_i \neq b_i}} \; \underset{\texttt{iSticksWithBid} = \texttt{deviationVector}[\texttt{whateverBid}, b, i]}{\texttt{where}}$$

$$\forall_{\substack{x \\ \texttt{allocation}[\texttt{iSticksWithBid}, x]}} \; \forall_{\substack{p \\ \texttt{vickreyPayment}[\texttt{iSticksWithBid}, p]}} \; \forall_{\substack{y \\ \texttt{allocation}[\texttt{whateverBid}, y]}} \; \forall_{\substack{q \\ \texttt{vickreyPayment}[\texttt{whateverBid}, q]}}$$

$$\texttt{payoff}[v, b, x, p]_i \geq \texttt{payoff}[v, b, y, q]_i$$

Vickrey’s Theorem

Theorem(Vickrey)

$$\forall_{\texttt{valuation}[v]} \; \forall_{\substack{x \\ \texttt{allocation}[v, x]}} \; \forall_{\substack{p \\ \texttt{vickreyPayment}[v, p]}} \; \texttt{secondPriceAuction}[v, x, p] \Rightarrow$$

$$\texttt{equilibriumWeaklyDominantStrategy}[v, v] \qquad (\textit{vickreyA})$$

$$\texttt{efficient}[v, x] \qquad (\textit{vickreyB})$$