1 Introduction

Our purpose is the formalization of [Mas04, Proposition 2], whose proof's wordiness hide some substantial difficulties when pouring it into formal symbols. A major one is that, after showing the independence of a function t_i^L on the i-th component of a vector b, it is inferred that a function t_i depending only on the remaining components can be introduced and used instead. Besides the minor problem of defining how the remaining components can be extracted, the formal way to do so is quotientation by a suitable equivalence relation. A related problem is how the thesis is spelled out: p_i is equated to a branched function whose side conditions read 'if buyer i wins' and 'if buyer i loses'. How to render that? Luckily, thanks to the quotientation approach, a natural way will show up.

Secondarily, the mathematical definition of payoff is never spelled out.

2 General results

This section recalls the abstract notions of compatibility (a generalization of morphism, or class invariance, see `http://en.wikipedia.org/wiki/Equivalence_class#Invariants`) and of quotient. These, although in varying forms and generality, are standard/mainstream mathematical matters, and are expected to be present in any decent library; any not including those should. The treatment here is somehow low-level and is leaning towards set theoretical foundations, but I hope that anyhow it will give the reader, besides the writer, a guide towards real formalization. It is an excerpt of my Ph.D. thesis, which indeed described a Mizar formalization.

2.1 Notations

1. $X \times Y$ is the cartesian product of the sets X and Y:
$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

2. $\mathbb{N}$, $\mathbb{Z}$ are the sets of natural numbers (including $0 = \emptyset$) and of the integers, respectively. We also write $\mathbb{Z}^+$ for $\mathbb{N} \setminus \{0\}$.

3. $\operatorname{dom} P$ and $\operatorname{rng} P$ denote the domain and range of a given relation P.

4. We will use the terms function, map and mapping interchangeably.

5. Y^X is the set of the maps from X into Y.

6. 2^X is the power set of X.

7. $\}\{_X$ is the map:
$$\{x\} \mapsto x \in X.$$

8. $\mathcal{I}_X$ is the identity map on the set X: $\mathcal{I}_X := \bigcup_{x \in X} \{x\} \times \{x\}$.

Notation 2.1.1. Consider a relation P and a set X. We write $P|_X$ for the restriction of P to X:

$$P|_X := (X \times \operatorname{rng} P) \cap P,$$

and $P[X]$ for the set of those elements of $\operatorname{rng} P$ corresponding through P to some element of X:

$$P[X] := \operatorname{rng}\left(P|_X\right).$$

Notation 2.1.2. $\bullet$ is the infix symbol for the composition of relations: $(Q \bullet P)[X] = P[Q[X]]$.
$\circ$ is the infix symbol for the composition of functions: $g \circ f : x \mapsto g(f(x))$.

Remark 2.1.3. Note that $\bullet$ and $\circ$ operate in reverse order: $f \bullet g = g \circ f$.

Notation 2.1.4. If P is a relation such that $\operatorname{rng} P \subseteq \operatorname{dom} P$, we can refer to the n-th iteration of P for any given $n \in \mathbb{N}$. We write it as

$$P^{(n)}.$$

Notation 2.1.5. Given any relation P, it is natural to denote with $P^{(-1)}$ its inverse relation, and with $P^{(-n)}$ the relation $P^{(-1)}{}^{(n)}$.

Notation 2.1.6 ('Functional pasting with right-hand precedence')**.** Given relations Q, P, set

$$Q \lhd P := Q \backslash (\operatorname{dom} P \times (\operatorname{rng} Q)) \cup P.$$

Remark 2.1.7. Given two functions f, g:

- $f \lhd g$ is a function;

- if f and g agree on $\operatorname{dom} f \cap (\operatorname{dom} g)$, then $f \lhd g = f \cup g$.

Notation 2.1.8 (Pointwise function alteration)**.** Given x, y and a relation P, denote: $P_x^y := P \lhd (\{x\} \times \{y\})$.

2.2 Quotients

Definition 2.2.1. Let P, Q be relations, f be a function. We say that f is (P, Q)-compatible if, given $(x, y) \in \operatorname{dom} f \times (\operatorname{dom} f) \cap P$, it is $(f(x), f(y)) \in Q$.

Definition 2.2.2. Given a non empty relation P, we consider the map

$$\pi_P : \operatorname{dom} P \ni x \mapsto P[\{x\}] \in 2^{\operatorname{rng} P}.$$

Given a set X and a relation P such that $X = \operatorname{dom} P$, we set

$$X/P := \operatorname{rng}(\pi_P).$$

Remark 2.2.3. If P is an equivalence relation over X, X/P is the set of the equivalence classes of P (hence a partition of X), and π_P maps each element of the domain of P to the unique equivalence class including it.

Definition 2.2.4 (Quotient of a relation). Let O, P, Q be relations, with P and Q non empty. The quotient of O by (P, Q) is defined as:

$$\frac{O}{P \;\; Q} := \{(p, q) \in \text{rng}\,(\pi_P) \times (\text{rng}\,(\pi_Q)) : p \times q \cap O \neq \emptyset\}.$$

Proposition 2.2.5. *Let E, F be non empty equivalence relations.*
If $f \in (\text{dom}\,F)^{\text{dom}\,E}$ is (E, F)-compatible, then

$$\frac{f}{E \;\; F} \in (\text{rng}\,\pi_F)^{\text{rng}\,\pi_E}.$$

Proof. Set $g := \frac{f}{E\;F}$. Since $g \subseteq \text{rng}\,\pi_E \times \text{rng}\,\pi_F$ by 2.2.4, it is $\text{rng}\,g \subseteq \text{rng}\,\pi_F$, hence we are left with two points to prove:

1. g is a function.

2. $\text{rng}\,\pi_E \subseteq \text{dom}\,g$.

The two corresponding proofs are given.

1. Consider sets X, Y_1, Y_2 such that $\{(X, Y_1), (X, Y_2)\} \subseteq g$. The goal is to show $Y_1 = Y_2$. By 2.2.4, consider x_1, x_2, y_1, y_2 such that $(x_1, y_1) \in X \times Y_1 \cap f$ and $(x_2, y_2) \in X \times Y_2 \cap f$. Since X is an equivalence class of E, this implies $(x_1, x_2) \in E$ which in turn, by 2.2.1, gives $(y_1, y_2) \in F$. Hence y_1 and y_2 must belong to the same equivalence class of F, which gives $Y_1 = Y_2$.

2. Let $X \in \text{rng}\,\pi_E$. X being an equivalence class of the non empty equivalence relation E, there is $x \in X \subseteq \text{dom}\,E$. Set

$$\begin{aligned} y &:= f(x) \in \text{dom}\,F \\ Y &:= \pi_F(y) \in \text{rng}\,F. \end{aligned} \tag{1}$$

Since $(x, y) \in f$ by (1), and $y \in Y$, we draw $(X, Y) \in g$ by 2.2.4.

$\square$

3 Application to auctions

Let I be a set (of indices), R be an ordered ring (e.g., $\mathbb{R}$), and $x, p \in R^{\left(R^I\right)}$ satisfying

$$r x(b) - p(b) \leq r x(b_i^r) - p(b_i^r) \tag{2}$$

for any $b \in R^I, r \in R$.

Definition 3.0.6. Consider the map $e_i := R^I \ni b \mapsto (b\backslash(\{i\} \times \operatorname{rng} b), x(b)) \in R^{I\backslash i} \times \operatorname{rng} x$, and denote with E_i its kernel[1], i.e., the equivalence relation given by $b \sim b'$ iff $b\backslash(\{i\} \times \operatorname{rng} b) = b'\backslash(\{i\} \times \operatorname{rng} b')$ and $x(b) = x(b')$.

Lemma 3.0.7. (2) *implies that p is $(E_i, \mathcal{I}_R)$-compatible.*

Proof. It is to be shown that, given b, b' satisfying $x(b) = x(b')$ and $b\backslash(\{i\} \times \operatorname{rng} b) = b'\backslash(\{i\} \times \operatorname{rng} b')$, we have $p(b) = p(b')$. This is easily done by applying (2) twice to get

$$b'(i)x(b) - p(b) \leq b'(i)x(b') - p(b')$$
$$b(i)x(b') - p(b') \leq b(i)x(b) - p(b)$$

which, thanks to $x(b) = x(b')$, reduce to $-p(b) \leq -p(b') \leq -p(b)$. $\qquad\square$

By 3.0.7 and 2.2.5, $\frac{p}{E_i \ \mathcal{I}_R} \in \left(\operatorname{rng} \pi_{(\mathcal{I}_R)}\right)^{\left(\operatorname{rng} \pi_{E_i}\right)}$, so that $\mathcal{P}_{p,i} := \pi_{\left(e_i^{(-1)}\right)} \bullet \frac{p}{E_i \ \mathcal{I}_R} \bullet \}\{_R$ maps $R^{I\backslash\{i\}} \times \operatorname{rng} x$ to R and $p(b) = \mathcal{P}_{p,i}(e_i(b))$.

Above, we have shown that, if i changes his bid, the price he pays does not change as long as the allocation of the good to i does not change. This is a good part of [Mas04, Proposition 2]. It remains to show that the winning price also has the property of being the sum of the losing price and of the second price. This, of course, imposes requirements on x. A first requirement is that a bid vector whose i-th component is the second price is a sort of accumulation point:

Lemma 3.0.8. *Consider $b \in R^I, r \in R$, and assume that for any $\delta > 0_R$ there are $\delta > \epsilon_1, \epsilon_2 > 0_R$ such that*

1. *$b_i^{b(i)-\epsilon_1} \in x^{(-1)}[\{0\}]$, and*

2. *$b_i^{b(i)+\epsilon_2} \in x^{(-1)}[\{r\}]$.*

Then (2) implies that

$$\mathcal{P}_{p,i}(b\backslash(\{i\} \times \operatorname{rng} b), r) = rb(i) + \mathcal{P}_{p,i}(b\backslash(\{i\} \times \operatorname{rng} b), 0_R).$$

Proof. ... $\qquad\square$

Now, the wanted completion of [Mas04, Proposition 2] is easily obtained by realizing that, upon requiring efficiency and truthfulness, the above accumulation properties are satisfied by any bid vector b whose i-th component is the second price.

References

[Mas04] E. Maskin. "The unity of auction theory: Milgrom's master class". In: *Journal of Economic Literature* 42.4 (2004), pp. 1102–1115. URL: `http://scholar.harvard.edu/files/maskin/files/unity_of_auction_theory.pdf`.

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[1]See `http://en.wikipedia.org/wiki/Equivalence_class#Invariants`.