Stable sets

TS_In[157]:=
```
Definition["internal stability", any[π, n, S],
 IS[S, π, n] :⇔ (S ⋂ D[S, π, n] = ∅)]
```

TS_In[158]:=
```
Definition["internal stability On", any[π, n, S, Z],
 ISOn[S, π, n, Z] :⇔ (S ⋂ DOn[S, π, n, Z] = ∅)]
```

TS_In[159]:=
```
Definition["external stability", any[π, n, S],
 ES[S, π, n] :⇔ (S ⋃ D[S, π, n] = X[n])]
```

TS_In[160]:=
```
Definition["external stability on", any[π, n, S, Z],
 ESOn[S, π, n, Z] :⇔ (S ⋃ DOn[S, π, n, Z] = Z)]
```

TS_In[161]:=
```
Definition["stable", any[π, n, S],
 stable[S, π, n] :⇔ (IS[S, π, n] ⋀ ES[S, π, n])]
```

TS_In[162]:=
```
Definition["stable On", any[π, n, S, Z],
 stableOn[S, π, n, Z] :⇔ (ISOn[S, π, n, Z] ⋀ ESOn[S, π, n, Z])]
```

TS_In[163]:=
```
Lemma["stable lemma", any[π, n, S], with[stable[S, π, n]],
 S = X \ D[S, π, n]]
```

TS_In[164]:=
```
Definition["self protection", any[π, n, S],
 SP[S, π, n] :⇔ (S ⊆ U[U[S, π, n], π, n])]
```

TS_In[165]:=
```
Algorithm["Roth-Jordan", any[S, K, π, n],
 RothJordan[S, π, n] := where[aux = U[U[S, π, n], π, n],
   { { aux              ⇐ ES[aux, π, n]
       "no stable set"  ⇐ otherwise        ⇐ aux = S
     RothJordan[aux, π, n]                 ⇐ otherwise  ]
]
```

Call the algorithm as **RothJordan[K, π, n]** where **K** is the core w.r.t. π and n.

Empty core

TS_In[166]:=
```
Theorem["SINequivalence", any[π1, π2, 3],
 with[AN[π1, 3] ⋀ AN[π2, 3] ⋀ (K[π1, 3] = ∅) ⋀ (K[π2, 3] = ∅)],
          ∀
         x,y
 allocation_n[x]⋀allocation_n[y]
        (dominates[y, x, π1, 3] ⇔ dominates[y, x, π2, 3])]
```

TS_In[167]:=
```
Corollary["emptyCoreStableA", any[π], with[AN[π, 3] ⋀ (K[π, 3] = ∅)],
 stable[dyadicSet[1, 3] \ dyadicSet[0, 3], π, 3]]
```

TS_In[168]:=
```
Corollary["emptyCoreStableB", any[π, S],
 with[AN[π, 3] ⋀ (K[π, 3] = ∅) ⋀ stable[S, π, n]],
 S = (dyadicSet[1, 3] \ dyadicSet[0, 3])]
```

Non-empty core (case n=3)

TS_In[169]:=
```
Definition["powerBalance", any[π, i],
 B[i, π] := {x ∈ X[3] | (π[{i}, x] = π[I[3] \ {i}, x])}]
```

TS_In[170]:= Definition["midpoint of B", any[π, i, j, k], with[{i, j, k} = {1, 2, 3}],

$$q[i, \pi] = \underset{x \in B[i,\pi]}{\underset{x}{\ni !}} \left(q[i]_j = q[i]_k\right)]$$

(*Definition["midpoint of B"]//InputForm*)

Definition["maxBalanced", any[π, i],

$$M[i, \pi] = \left\{x \in B[i, \pi] \setminus D[t[i, 3]] \;\middle|\; \left(x_i = \max_{y \in B[i,\pi] \setminus D[t[i,3]]} y_i\right)\right\}]$$

TS_In[171]:= Definition["r", any[π, i, j],

$$r[i, j, \pi] = \underset{z \in M[i,\pi]}{\ni} \left(\underset{x \in M[i,\pi]}{\forall} z_j \geq x_j\right)]$$

TS_In[172]:= Definition["RMaxBalanced", any[π, i],

$$R[i, \pi] := \left\{r \in M[i, \pi] \;\middle|\; \underset{j \in I[3] \setminus \{i\}}{\exists} \; \underset{s \in M[i,\pi]}{\forall} r_j \geq s_j\right\}]$$

- **Lemma 6**

TS_In[173]:= Lemma["powerBalanceContained", any[π, i, j, k],

$$B[i, \pi] \subset \left\{x \underset{\text{allocation}_n[x]}{|} x_i \geq \text{Max}[\{x_j, x_k\}]\right\}]$$

- **Lemma 7**

TS_In[174]:= Definition["BProp", any[π, i, j, k, x, y],
with[$x \in B[i, \pi] \wedge y \in B[i, \pi] \wedge y_i > x_i \wedge \{i, j, k\} = \{1, 2, 3\}$],
$y_j > x_j \vee y_k > x_k$]

Next we define a ray from a tyrannical element through the simplex.

TS_In[175]:= Definition["RayTyrannical", any[i, α], with[$0 \leq \alpha \wedge \alpha \leq 1$],

$$\text{Ray}[i, \alpha] = \left\{ \begin{array}{ll} \left\{\langle \beta, (1-\beta) * \alpha, (1-\beta) * (1-\alpha) \rangle \underset{\beta \in \mathbb{R}}{|} 0 \leq \beta \leq 1\right\} & \Leftarrow i = 1 \\ \left\{\langle (1-\beta) * \alpha, \beta, (1-\beta) * (1-\alpha) \rangle \underset{\beta \in \mathbb{R}}{|} 0 \leq \beta \leq 1\right\} & \Leftarrow i = 2 \\ \left\{\langle (1-\beta) * \alpha, (1-\beta) * (1-\alpha), \beta \rangle \underset{\beta \in \mathbb{R}}{|} 0 \leq \beta \leq 1\right\} & \Leftarrow i = 3 \end{array} \right.]$$

- **Corollary 2**

TS_In[176]:= Corollary["Ray hits B", any[π, i],

$$\underset{0 \leq \alpha \leq 1}{\underset{\alpha}{\forall}} (\text{Ray}[i, \alpha] \cap B[i, \pi]) \neq \emptyset]$$

(*Definition["BPlus",any[π,i,j,k], with[{i,j,k}={1,2,3}],

$$\text{BPlus}[i,\pi] := \left\{x \underset{x \in B[i,\pi,3]}{|} x_i > x_j \wedge x_i > x_k\right\}]*)$$

TS_In[177]:=

$$\mathbf{Definition}\Big[\text{"BPlus"}, \mathbf{any}[\pi],$$

$$\mathbf{BPlus}[1, \pi] := \Big\{x \underset{x\in B[1,\pi,3]}{|} x_1 > x_2 \wedge x_1 > x_3\Big\}$$

$$\mathbf{BPlus}[2, \pi] := \Big\{x \underset{x\in B[2,\pi,3]}{|} x_2 > x_1 \wedge x_2 > x_3\Big\}\Big]$$

$$\mathbf{BPlus}[3, \pi] := \Big\{x \underset{x\in B[3,\pi,3]}{|} x_3 > x_1 \wedge x_3 > x_2\Big\}$$

TS_In[178]:=

$$\mathbf{Definition}\Big[\text{"BPlusVar"}, \mathbf{any}[\pi, i],$$

$$\mathbf{BPlus}[i, \pi] := \begin{cases} \Big\{x \underset{x\in B[1,\pi,3]}{|} x_1 > x_2 \wedge x_1 > x_3\Big\} & \Leftarrow\ i = 1 \\ \Big\{x \underset{x\in B[2,\pi,3]}{|} x_2 > x_1 \wedge x_2 > x_3\Big\} & \Leftarrow\ i = 2 \\ \Big\{x \underset{x\in B[3,\pi,3]}{|} x_3 > x_1 \wedge x_3 > x_2\Big\} & \Leftarrow\ i = 3 \end{cases}\Big]$$

- **Lemma 8**

TS_In[179]:=

$$\mathbf{Lemma}[\text{"dominance on B.1"}, \mathbf{any}[\pi, x, y, i],$$
$$\mathbf{with}[\mathbf{AN}[\pi, 3] \wedge \mathbf{allocation}_3[x] \wedge \mathbf{allocation}_3[y]],$$
$$x \in \mathbf{B}[i, \pi] \wedge y \in \mathbf{B}[i, \pi] \wedge y_i > x_i \wedge x_i \geq x_j \wedge x_j > x_k \wedge y_j > x_j \Rightarrow \mathbf{dominates}[y, x, \pi, 3]]$$

TS_In[180]:=

$$\mathbf{Lemma}[\text{"dominance on B.2"}, \mathbf{any}[\pi, x, y, i],$$
$$\mathbf{with}[\mathbf{AN}[\pi, 3] \wedge \mathbf{allocation}_3[x] \wedge \mathbf{allocation}_3[y]],$$
$$x \in \mathbf{BPlus}[i, \pi] \wedge y \in \mathbf{BPlus}[i, \pi] \wedge y_i > x_i \Rightarrow \mathbf{dominates}[y, x, \pi, 3]]$$

- **Lemma 9**

TS_In[181]:=

$$\mathbf{Lemma}[\text{"Characterisation under AN and RE"}, \mathbf{any}[\pi], \mathbf{with}[\mathbf{AN}[\pi, 3] \wedge \mathbf{RE}[\pi, 3]],$$
$$\mathbf{B}[i, \pi] \setminus \mathbf{BPlus}[i, \pi] \subseteq \mathbf{dyadicSet}[1, 3] \setminus \mathbf{dyadicSet}[0, 3]]$$

- **Lemma 10**

TS_In[182]:=

$$\mathbf{Lemma}\Big[\text{"Lemma10"}, \mathbf{any}[\pi, x],$$

$$\mathbf{with}\Big[\mathbf{AN}[\pi, 3] \bigwedge \mathbf{RE}[\pi, 3] \bigwedge \mathbf{R}[i, \pi, 3] \neq \emptyset \bigwedge \mathbf{allocation}_3[x] \bigwedge \Bigg(\underset{i\in\{1,2,3\}}{\underset{i}{\forall}} x_i > 0\Bigg)\Big],$$

$$x \in \mathbf{B}[i, \pi] \setminus (\mathbf{R}[i, \pi] \bigcup \{\mathbf{q}[i, \pi]\}) \Rightarrow x \in \mathbf{dominion}[\mathbf{R}[i, \pi], \pi, 3]\Big]$$

- **Theorem 4**

TS_In[183]:=

$$\mathbf{Theorem}\Big[\text{"Theorem 4.a"}, \mathbf{any}[\pi], \mathbf{with}[\mathbf{AN}[\pi, 3] \wedge \mathbf{RE}[\pi, 3] \wedge \mathbf{R}[i, \pi] \neq \emptyset],$$

$$\underset{Si}{\exists!}\ (\mathbf{stableOn}[Si, \pi, 3, \mathbf{BPlus}[i, \pi] \bigcup \mathbf{R}[i, \pi]] \wedge (Si = (\mathbf{R}[i, \pi] \bigcup (\{\mathbf{q}[i, \pi]\} \bigcap \mathbf{M}[i, \pi]))))\Big]$$

TS_In[184]:=

$$\mathbf{Theorem}\Big[\text{"Theorem 4.a"}, \mathbf{any}[i \in \mathbf{I}[3], \pi], \mathbf{with}[\mathbf{AN}[\pi, 3] \wedge \mathbf{RE}[\pi, 3] \wedge \mathbf{R}[i, \pi] \neq \emptyset],$$

$$\underset{S}{\exists!}\ (\mathbf{stableOn}[S, \pi, 3, \mathbf{BPlus}[i, \pi] \bigcup \mathbf{R}[i, \pi]] \wedge (S = \mathbf{S}[i]))\Big]$$

TS_In[185]:=

$$\mathbf{Theorem}[\text{"Theorem 4.a,exists"}, \mathbf{any}[i \in \mathbf{I}[3], \pi], \mathbf{with}[\mathbf{AN}[\pi, 3] \wedge \mathbf{RE}[\pi, 3] \wedge \mathbf{R}[i, \pi] \neq \emptyset],$$
$$\mathbf{stableOn}[\mathbf{S}[i], \pi, 3, \mathbf{BPlus}[i, \pi] \bigcup \mathbf{R}[i, \pi]]]$$

TS_In[186]:=

$$\mathbf{Theorem}[\text{"Theorem 4.a,unique"}, \mathbf{any}[i \in \mathbf{I}[3], \pi], \mathbf{with}[\mathbf{AN}[\pi, 3] \wedge \mathbf{RE}[\pi, 3] \wedge \mathbf{R}[i, \pi] \neq \emptyset],$$
$$\ldots]$$

TS_In[186]:=
```
Definition["Si", any[i, π],
  S[i, π] := R[i, π] ⋃ (Sq[i, π] ⋂ M[i, π])]
```

A less cryptic formulation could be:

$$\text{Definition}\Big[\text{"Si case"},\ \text{any}[i, \pi],\quad S[i, \pi] := \begin{cases} R[i, \pi] \bigcup \{q[i, \pi]\} & \Leftarrow\ q[i, \pi] \in M[i, \pi] \\ R[i, \pi] & \Leftarrow\ \text{otherwise} \end{cases}\Big]$$

TS_In[187]:=

$$\text{Theorem}\Big[\text{"Theorem 4.b"},\ \text{any}[\pi],\ \text{with}[AN[\pi, 3] \bigwedge RE[\pi, 3] \bigwedge (R[i, \pi] = \emptyset)],\quad \neg \underset{S}{\exists}\ \text{stableOn}[S, \pi, 3, \text{BPlus}[i, \pi]]\Big]$$

- **Lemma 11**

TS_In[188]:=
```
Lemma["Lemma 11", any[π, Si, x, y],
  with[stableOn[Si, π, 3, BPlus[i, π] ⋃ R[i, π]] ⋀
      K[π, 3] ≠ ∅ ⋀ x ∈ Si ⋀ CX[π, 3] ⋀ dominates[y, x, π, 3]]
    y ∈ dominion[{t[i, 3]}, π, 3]]
```

- **Theorem 5**

TS_In[189]:=
```
Theorem["Theorem 5", any[π, S, Si],
  with[AN[π, 3] ⋀ CX[π, 3] ⋀ RE[π, 3] ⋀ R[i, π] ≠ ∅ ⋀
      (Si = (R[i, π] ⋃ ({q[i, π]} ⋂ M[i, π]))) ⋀ stable[S, π, 3]]
    Si ⊆
   S]
```

- **Definition SIN = Strength In Numbers**

TS_In[190]:=

$$\text{Definition}\Big[\text{"SIN"},\ \text{any}[C, x, v],\quad \pi\text{SIN}_v[C, x] = \sum_{i \in C} (x_i + v)\Big]$$

- **Theorem 6**

TS_In[191]:=
```
Theorem["Theorem 6a",
  stable[dyadicSet[1, 3], πSIN1, 3]]
```

TS_In[192]:=
```
Theorem["Theorem 6b", any[S], with[stable[S, πSIN1, 3]],
  S = dyadicSet[1, 3]]
```

TS_In[193]:=

$$\text{Theorem}\Big[\text{"Theorem 7a"},\ \text{any}[\pi],\quad \text{with}[AN[\pi, 3] \bigwedge CX[\pi, 3] \bigwedge RE[\pi, 3] \bigwedge (K[\pi, 3] \neq \emptyset) \bigwedge R[1, \pi] \neq \emptyset],\quad \underset{S}{\exists!}\ \text{stable}[S, \pi, 3]\Big]$$

TS_In[194]:=

$$\text{Definition}\Big[\text{"pij"},\ \text{any}[\pi, j, k],\quad p[j, k, \pi] := \begin{cases} \frac{1}{2} * (r[1, 3, \pi] + r[2, 3, \pi]) & \Leftarrow\ (j = 1) \bigwedge (k = 2) \\ \frac{1}{2} * (r[1, 2, \pi] + r[3, 2, \pi]) & \Leftarrow\ (j = 1) \bigwedge (k = 3) \\ \frac{1}{2} * (r[2, 1, \pi] + r[3, 1, \pi]) & \Leftarrow\ (j = 2) \bigwedge (k = 3) \end{cases}\Big]$$

Variant with permuted indices. Which one is the correct one?

TS_In[195]:= Definition["pij", any[π, j, k],

$$p[j, k, \pi] := \begin{cases} \frac{1}{2} \odot (r[3, 1, \pi] \oplus r[3, 2, \pi]) & \Leftarrow (j = 1) \wedge (k = 2) \\ \frac{1}{2} \odot (r[2, 1, \pi] \oplus r[2, 3, \pi]) & \Leftarrow (j = 1) \wedge (k = 3) \\ \frac{1}{2} \odot (r[1, 2, \pi] \oplus r[1, 3, \pi]) & \Leftarrow (j = 2) \wedge (k = 3) \end{cases}]$$

TS_In[196]:= Definition["tup+*", any[x, y],

$$x \oplus y := \left\langle x_i + y_i \;\Big|_{i=1,\ldots,|y|} \right\rangle$$
$$x \odot y := \left\langle x * y_i \;\Big|_{i=1,\ldots,|y|} \right\rangle]$$

TS_In[197]:= Definition["P", any[π],
P[π] := {p[1, 2, π], p[1, 3, π], p[2, 3, π]}]

TS_In[198]:= Theorem["Theorem 7b", any[π],
with[AN[π, 3] ⋀ CX[π, 3] ⋀ RE[π, 3] ⋀ (K[π, 3] ≠ ∅) ⋀ R[1, π] ≠ ∅],
stable[K[π, 3] ⋃ S[1, π] ⋃ ({q[1, π]} ⋂ M[1, π]) ⋃ S[2, π] ⋃
({q[2, π]} ⋂ M[2, π]) ⋃ S[3, π] ⋃ ({q[3, π]} ⋂ M[3, π]) ⋃ P[π], 3]]

Corollary["Corollary 3", any[π, S], with[AN[π, 3] ⋀ CX[π, 3] ⋀ RE[π, 3] ⋀ stable[S, π, 3]],
|S| ≤ 15]

TS_In[199]:= Algorithm["StableSet", any[π],
stableSet[π] :=

$$\begin{cases} \text{dyadicSet}[1, 3] \setminus \text{dyadicSet}[0, 3] & \Leftarrow \pi[\{1\}, \\ \begin{cases} \text{"no stable set exists"} & \Leftarrow \text{empty}[R[1, \pi]] \\ \text{where}\Big[\mathcal{S} = \text{dyadicSet}[0, 3] \bigcup \bigcup_{i=1,\ldots,3} S[i, \pi], \begin{cases} \mathcal{S} & \Leftarrow \pi[\{2\}, s[2, 3, 3]] \geq \pi[\{1, 3\}, s[2, 3, 3]] \\ \mathcal{S} \bigcup P[\pi] & \Leftarrow \text{otherwise} \end{cases} \Big] & \Leftarrow \neg\, \text{empty}[R[1, \pi]] \\ \text{"unknown R"} & \Leftarrow \text{otherwise} \end{cases} & \Leftarrow \text{otherw} \end{cases}$$

]

TS_In[200]:= Definition["empty", any[$\mathcal{M}$],
empty[$\mathcal{M}$] := $\mathcal{M}$ = ∅]

TS_In[201]:= Definition["midpoint of B", any[π, i],

$$q[i, \pi] := \text{where}\Big[j = æ[I[3] \setminus \{i\}], k = æ[I[3] \setminus \{i, j\}], \underset{x \in B[i,\pi]}{\ni !} (x_j = x_k)\Big]\Big]$$

TS_In[202]:= Use[{Builtin ["Connectives"], Builtin ["Quantifiers"],
Builtin ["Tuples"], Builtin ["Sets"], Builtin ["Numbers"]}]

TS_In[203]:= UseAlso[{Algorithm["StableSet"], Lemma["D13"], Lemma["D03"],
Definition["tyrannical"], Definition["agents"], Definition["Si"],
Definition["centre allocation"], Definition["r"], Definition["P"],
Definition["pij"], Definition["tup+*"], Definition["split allocation"]}]

Champion Powerfunction

Wealth is Power Powerfunction

TS_In[209]:=
```
Definition["WIP", any[C, x],
  WIPπ[C, x] = ∑_{i∈C} x_i]
```

TS_In[210]:=
```
Lemma["WIPR",
  R[1, WIPπ] = {⟨1 / 2, 0, 1 / 2⟩, ⟨1 / 2, 1 / 2, 0⟩}
  R[2, WIPπ] = {⟨0, 1 / 2, 1 / 2⟩, ⟨1 / 2, 1 / 2, 0⟩}
  R[3, WIPπ] = {⟨1 / 2, 0, 1 / 2⟩, ⟨0, 1 / 2, 1 / 2⟩}]
```

TS_In[211]:=
```
Lemma["midpoint of B,WIP",
  Sq[1, WIPπ] ⋂ M[1, WIPπ] = {⟨1 / 2, 1 / 4, 1 / 4⟩}
  Sq[2, WIPπ] ⋂ M[2, WIPπ] = {⟨1 / 4, 1 / 2, 1 / 4⟩}
  Sq[3, WIPπ] ⋂ M[3, WIPπ] = {⟨1 / 4, 1 / 4, 1 / 2⟩}]
```

TS_In[212]:=
```
Lemma["emptyR,WIP",
  ¬ empty[R[1, WIPπ]]]
```

TS_In[213]:=
```
UseAlso[{Definition["WIP"], Lemma["midpoint of B,WIP"], Lemma["emptyR,WIP"]}]
```

```
Compute[dyadicSet[0, 3] ⋃ ⋃_{i=1,…,3} S[i, WIPπ]]
```

$$\left\{\langle 0, 0, 1\rangle, \left\langle 0, \frac{1}{2}, \frac{1}{2}\right\rangle, \langle 0, 1, 0\rangle, \left\langle \frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right\rangle, \left\langle \frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\rangle, \left\langle \frac{1}{2}, 0, \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{2}, 0\right\rangle, \langle 1, 0, 0\rangle\right\}$$

```
Compute[S[1, WIPπ]]
```

$$\left\{\left\langle \frac{1}{2}, 0, \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{2}, 0\right\rangle\right\}$$

TS_In[214]:=
```
Lemma["WIPD", any[S],
  fullSet[{⟨0, 0, 1⟩, ⟨0, 1/2, 1/2⟩, ⟨0, 1, 0⟩, ⟨1/4, 1/4, 1/2⟩, ⟨1/4, 1/2, 1/4⟩, ⟨1/2, 0, 1/2⟩,
    ⟨1/2, 1/4, 1/4⟩, ⟨1/2, 1/2, 0⟩, ⟨1, 0, 0⟩} ⋃ D[{⟨0, 0, 1⟩, ⟨0, 1/2, 1/2⟩, ⟨0, 1, 0⟩, ⟨1/4, 1/4, 1/2⟩,
    ⟨1/4, 1/2, 1/4⟩, ⟨1/2, 0, 1/2⟩, ⟨1/2, 1/4, 1/4⟩, ⟨1/2, 1/2, 0⟩, ⟨1, 0, 0⟩}, WIPπ, 3]]]
```

TS_In[215]:=
```
UseAlso[{Lemma["WIPD"], Lemma["WIPR"]}]
```

```
Compute[R[2, WIPπ]]
```

$$\left\{\left\langle 0, \frac{1}{2}, \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{2}, 0\right\rangle\right\}$$

```
Compute[S[3, WIPπ]]
```

$$\left\{\left\langle 0, \frac{1}{2}, \frac{1}{2}\right\rangle, \left\langle \frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}, 0, \frac{1}{2}\right\rangle\right\}$$

```
Compute[WIPπ[{1, 2, 3}, ⟨1, 0, 0⟩]]
```

$$\sum_{i\in\{1,2,3\}} \langle 1, 0, 0\rangle_i$$

```
Compute[WIPπ[{1, 2, 3}, ⟨1 / 2, 0, 1 / 2⟩]]
```

1

```
Compute[¬ empty[R[1, WIPπ]]]
```

True

$$\text{Compute}\left[\text{dyadicSet}[0, 3] \cup \bigcup_{i=1,\ldots,3} S[i, \text{WIP}\pi]\right]$$

$$\left\{\langle 0, 0, 1\rangle, \left\langle 0, \frac{1}{2}, \frac{1}{2}\right\rangle, \langle 0, 1, 0\rangle, \left\langle \frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right\rangle, \left\langle \frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\rangle, \left\langle \frac{1}{2}, 0, \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{2}, 0\right\rangle, \langle 1, 0, 0\rangle\right\}$$

```
Compute[WIPπ[{2}, s[1, 3, 3]] ≥ WIPπ[{1, 3}, s[1, 3, 3]]]
```

False

```
Compute[WIPπ[{1}, t[1, 3]] < WIPπ[{2, 3}, t[1, 3]]]
```

False

```
Compute[stableSet[WIPπ]]
```

$$\left\{\langle 0, 0, 1\rangle, \left\langle 0, \frac{1}{2}, \frac{1}{2}\right\rangle, \langle 0, 1, 0\rangle, \left\langle \frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right\rangle, \left\langle \frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\rangle, \left\langle \frac{1}{2}, 0, \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{2}, 0\right\rangle, \langle 1, 0, 0\rangle\right\}$$

Strength in Numbers Powerfunction

TS_In[216]:= $$\text{Definition}\left[\text{"SIN"}, \text{any}[C, x, \nu], \text{SIN}\pi_\nu[C, x] = \sum_{i\in C} (x_i + \nu)\right]$$

TS_In[217]:=
```
UseAlso[{Definition["SIN"]}]
```

```
Compute[SINπ2[{1}, ⟨1, 0, 0⟩]]
```

3

```
Compute[SINπ2[{2, 3}, ⟨1, 0, 0⟩]]
```

4

```
Compute[stableSet[SINπ2]]
```

$$\left\{\left\langle 0, \frac{1}{2}, \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}, 0, \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{2}, 0\right\rangle\right\}$$

Cobb Douglas Powerfunction

TS_In[218]:= **Definition**["Cobb Douglas", any[C, x, ν],

$$CD\pi_{\nu}[C, x] = |C|^{\nu} * \left(\sum_{i\in C} x_i\right)^{1-\nu}$$]

TS_In[219]:= **UseAlso**[{Definition["Cobb Douglas"]}]

Compute[$CD\pi_{1/3}$[{1}, ⟨1, 0, 0⟩]]

1

Compute[$CD\pi_{1/3}$[{2, 3}, ⟨1, 0, 0⟩]]

0

TS_In[220]:= **Lemma**["emptyRCobbDouglas", any[ν],
empty[R[1, $CD\pi_{\nu}$]]]

TS_In[221]:= **UseAlso**[{Lemma["emptyRCobbDouglas"]}]

Compute[stableSet[$CD\pi_{1/3}$]]

no stable set exists

Sqrt Powerfunction

TS_In[222]:= **Definition**["SqrtP", any[C, x, ν],

$$Sqrt\pi_{\nu}[C, x] = \sum_{i\in C}\left(\nu + \sqrt{x_i}\right)$$]

TS_In[223]:= **UseAlso**[{Definition["SqrtP"]}]

TS_In[226]:= **Compute**[$Sqrt\pi_{1/3}$[{1}, ⟨1, 0, 0⟩]]

TS_Out[226]=

$$\frac{4}{3}$$

TS_In[227]:= **Compute**[$Sqrt\pi_{1/3}$[{2, 3}, ⟨1, 0, 0⟩]]

TS_Out[227]=

$$\frac{2}{3}$$

TS_In[224]:= **Lemma**["emptySqrtP", any[ν],
¬ empty[R[1, $Sqrt\pi_{\nu}$]]]

TS_In[228]:= **UseAlso**[{Lemma["emptySqrtP"]}]

TS_In[229]:= **Compute**[stableSet[$Sqrt\pi_{1/3}$]]

TS_Out[229]=

$$\{\langle 0,0,1\rangle,\langle 0,1,0\rangle,\langle 1,0,0\rangle\}\bigcup \mathrm{R}\left[1,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]\bigcup\left(\mathrm{Sq}\left[1,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]\right)\bigcap\left(\mathrm{M}\left[1,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]\right)\bigcup$$

$$\mathrm{R}\left[2,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]\bigcup\left(\mathrm{Sq}\left[2,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]\right)\bigcap\left(\mathrm{M}\left[2,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]\right)\bigcup \mathrm{R}\left[3,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]\bigcup$$

$$\left(\mathrm{Sq}\left[3,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]\right)\bigcap\left(\mathrm{M}\left[3,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]\right)\bigcup\left\{\left\langle\frac{1}{2}*\left\langle\left(\underset{z\in \mathrm{M}\left[1,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\exists}\ \underset{x\in \mathrm{M}\left[1,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_2\geq x_2)\right)_i+\right.\right.\right.$$

$$\left.\left(\underset{z\in \mathrm{M}\left[1,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\exists}\ \underset{x\in \mathrm{M}\left[1,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_3\geq x_3)\right)_i\ \Big|\ i=1,\ldots,\left|\underset{z\in \mathrm{M}\left[1,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\exists}\ \underset{x\in \mathrm{M}\left[1,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_3\geq x_3)\right|_i\right\rangle$$

$$\Big|\ i=1,\ldots,\left|\left\langle\left(\underset{z\in \mathrm{M}\left[1,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\exists}\ \underset{x\in \mathrm{M}\left[1,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_2\geq x_2)\right)_i+\left(\underset{z\in \mathrm{M}\left[1,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\exists}\ \underset{x\in \mathrm{M}\left[1,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_3\geq x_3)\right)_i\ \Big|\ i=1,\ldots,\left|\underset{z\in \mathrm{M}\left[1,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\exists}\ \underset{x\in \mathrm{M}\left[1,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_3\geq x_3)\right|\right\rangle\right|\Bigg\rangle,$$

$$\left\langle\frac{1}{2}*\left\langle\left(\underset{z\in \mathrm{M}\left[2,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\exists}\ \underset{x\in \mathrm{M}\left[2,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_1\geq x_1)\right)_i+\right.\right.$$

$$\left.\left(\underset{z\in \mathrm{M}\left[2,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\exists}\ \underset{x\in \mathrm{M}\left[2,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_3\geq x_3)\right)_i\ \Big|\ i=1,\ldots,\left|\underset{z\in \mathrm{M}\left[2,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\exists}\ \underset{x\in \mathrm{M}\left[2,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_3\geq x_3)\right|_i\right\rangle$$

$$\Bigg| \qquad \Bigg\rangle,$$

$$i=1,\ldots,\left|\left\langle\left(\underset{z\in M\left[2,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\ni}\ \underset{x\in M\left[2,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_1\geq x_1)\right)_i+\left(\underset{z\in M\left[2,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\ni}\ \underset{x\in M\left[2,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_3\geq x_3)\right)_{i\ i=1,\ldots,}\left|\underset{z\in M\left[2,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\ni}\ \underset{x\in M\left[2,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_3\geq x_3)\right|\right\rangle\right|$$

$$\left\langle\frac{1}{2}*\left\langle\left(\underset{z\in M\left[3,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\ni}\ \underset{x\in M\left[3,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_1\geq x_1)\right)_i+\right.\right.$$

$$\left(\underset{z\in M\left[3,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\ni}\ \underset{x\in M\left[3,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_2\geq x_2)\right)_{i\ i=1,\ldots,}\left|\underset{z\in M\left[3,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\ni}\ \underset{x\in M\left[3,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_2\geq x_2)\right|\Bigg\rangle_i$$

$$\Bigg| \qquad \Bigg\rangle\Bigg\}$$

$$i=1,\ldots,\left|\left\langle\left(\underset{z\in M\left[3,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\ni}\ \underset{x\in M\left[3,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_1\geq x_1)\right)_i+\left(\underset{z\in M\left[3,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\ni}\ \underset{x\in M\left[3,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_2\geq x_2)\right)_{i\ i=1,\ldots,}\left|\underset{z\in M\left[3,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\ni}\ \underset{x\in M\left[3,\mathrm{Sqrt}\pi_{\frac{1}{3}}\right]}{\forall}(z_2\geq x_2)\right|\right\rangle\right|$$