Prove:

(Lemma (powerfunction-independent))

$$\underset{C,n,\pi,x,y}{\forall}\left(\text{allocation}_n[x]\wedge\text{allocation}_n[y]\wedge C\subseteq \text{I}[n]\wedge\text{powerfunction}[\pi,\ n]\Rightarrow\left(\underset{i}{\forall}\left(i\in C\Rightarrow(x_i=y_i)\right)\Rightarrow(\pi[C,\ x]=\pi[C,\ y])\right)\right)$$

under the assumptions:

(Definition (powerfunction)) $\underset{n,\pi}{\forall}(\text{powerfunction}[\pi,\ n]:\Leftrightarrow\ \text{WC}[\pi,\ n]\wedge\text{WR}[\pi,\ n]\wedge\text{SR}[\pi,\ n])$,

(Definition (WR))

$$\underset{n,\pi}{\forall}\left(\text{WR}[\pi,\ n]:\Leftrightarrow\ n\in\mathbb{N}\bigwedge\underset{C,x,y}{\forall}\left(\text{allocation}_n[x]\wedge\text{allocation}_n[y]\wedge C\subseteq\text{I}[n]\Rightarrow\left(\underset{i}{\forall}(i\in C\Rightarrow y_i\geq x_i)\Rightarrow\pi[C,\ y]\geq\pi[C,\ x]\right)\right)\right)$$

(Axiom (trichotomy)) $\underset{a,b}{\forall}((a=b)\Leftrightarrow a\geq b\wedge b\geq a)$.

For proving (Lemma (powerfunction-independent)) we take all variables arbitrary but fixed and prove:

(1) $\text{allocation}_{n_0}[x_0]\wedge\text{allocation}_{n_0}[y_0]\wedge C_0\subseteq\text{I}[n_0]\wedge\text{powerfunction}[\pi_0,\ n_0]\Rightarrow\left(\underset{i}{\forall}(i\in C_0\Rightarrow(x_{0_i}=y_{0_i}))\Rightarrow(\pi_0[C_0,\ x_0]=\pi_0[C_0,\ y_0])\right)$

We prove (1) by the deduction rule.

We assume

(2) $\text{allocation}_{n_0}[x_0]\wedge\text{allocation}_{n_0}[y_0]\wedge C_0\subseteq\text{I}[n_0]\wedge\text{powerfunction}[\pi_0,\ n_0]$

and show

(3) $\underset{i}{\forall}(i\in C_0\Rightarrow(x_{0_i}=y_{0_i}))\Rightarrow(\pi_0[C_0,\ x_0]=\pi_0[C_0,\ y_0])$.

The formula (2.4) is expanded by the definition (Definition (powerfunction)) into:

(4) $\text{WC}[\pi_0,\ n_0]\wedge\text{WR}[\pi_0,\ n_0]\wedge\text{SR}[\pi_0,\ n_0]$.

The formula (4.2) is expanded by the definition (Definition (WR)) into:

(5) $n_0 \in \mathbb{N} \bigwedge \underset{C,x,y}{\forall} \Big(\texttt{allocation}_{n_0}[x] \wedge \texttt{allocation}_{n_0}[y] \wedge C \subseteq \mathrm{I}[n_0] \Rightarrow$.
$\Big(\underset{i}{\forall} (i \in C \Rightarrow y_i \geq x_i) \Rightarrow \pi_0[C, y] \geq \pi_0[C, x]\Big)\Big)$

We prove (3) by the deduction rule.

We assume

(6) $\underset{i}{\forall} (i \in C_0 \Rightarrow (x_{0_i} = y_{0_i}))$

and show

(7) $\pi_0[C_0, x_0] = \pi_0[C_0, y_0]$.

From (2.3), by (5.2), we obtain:

(10)
$\underset{x,y}{\forall} \Big(\texttt{allocation}_{n_0}[x] \wedge \texttt{allocation}_{n_0}[y] \Rightarrow \Big(\underset{i}{\forall} (i \in C_0 \Rightarrow y_i \geq x_i) \Rightarrow \pi_0[C_0, y] \geq \pi_0[C_0, x]\Big)\Big)$.

From (2.2), by (10), we obtain:

(18) $\underset{y}{\forall} \Big(\texttt{allocation}_{n_0}[y] \Rightarrow \Big(\underset{i}{\forall} (i \in C_0 \Rightarrow y_i \geq y_{0_i}) \Rightarrow \pi_0[C_0, y] \geq \pi_0[C_0, y_0]\Big)\Big)$.

From (2.1), by (10), we obtain:

(17) $\underset{y}{\forall} \Big(\texttt{allocation}_{n_0}[y] \Rightarrow \Big(\underset{i}{\forall} (i \in C_0 \Rightarrow y_i \geq x_{0_i}) \Rightarrow \pi_0[C_0, y] \geq \pi_0[C_0, x_0]\Big)\Big)$.

From (2.1), by (18), we obtain:

(25) $\underset{i}{\forall} (i \in C_0 \Rightarrow x_{0_i} \geq y_{0_i}) \Rightarrow \pi_0[C_0, x_0] \geq \pi_0[C_0, y_0]$.

From (2.2), by (17), we obtain:

(24) $\underset{i}{\forall} (i \in C_0 \Rightarrow y_{0_i} \geq x_{0_i}) \Rightarrow \pi_0[C_0, y_0] \geq \pi_0[C_0, x_0]$.

From (2.1), by (17), we obtain:

(23) $\underset{i}{\forall} (i \in C_0 \Rightarrow x_{0_i} \geq x_{0_i}) \Rightarrow \pi_0[C_0, x_0] \geq \pi_0[C_0, x_0]$.

Formula (23) is transformed into:

(27) $\neg \underset{i}{\forall} (i \in C_0 \Rightarrow x_{0_i} \geq x_{0_i}) \bigvee \pi_0[C_0, x_0] \geq \pi_0[C_0, x_0]$.

We prove (7) by case distinction using (27).

Case (27.1) $\neg \underset{i}{\forall} (i \in C_0 \Rightarrow x_{0_i} \geq x_{0_i})$:

Formula (24) is transformed into:

(41) $\neg \underset{i}{\forall} (i \in C_0 \Rightarrow y_{0_i} \geq x_{0_i}) \bigvee \pi_0[C_0, y_0] \geq \pi_0[C_0, x_0]$.

We prove (7) by case distinction using (41).

Case (41.1) $\neg \underset{i}{\forall} (i \in C_0 \Rightarrow y_{0_i} \geq x_{0_i})$:

Formula (41.1) is simplified to:

(42) $\underset{i}{\exists} (\neg (i \in C_0 \Rightarrow y_{0_i} \geq x_{0_i}))$.

By (42) we can take appropriate values such that:

(43) $\neg \left(i_1 \in C_0 \Rightarrow y_{0_{i_1}} \geq x_{0_{i_1}}\right)$.

Formula (43) is expanded into

(44) $i_1 \in C_0 \bigwedge y_{0_{i_1}} \ngeq x_{0_{i_1}}$.

From (44.1), by (6), we obtain:

(45) $x_{0_{i_1}} = y_{0_{i_1}}$.

From (45), by (Axiom (trichotomy)), we obtain:

(47) $x_{0_{i_1}} \geq y_{0_{i_1}} \bigwedge y_{0_{i_1}} \geq x_{0_{i_1}}$.

Formula (7) is proved because (47.2) and (44.2) are contradictory.

Case (41.2) $\pi_0[C_0, y_0] \geq \pi_0[C_0, x_0]$:

Formula (25) is transformed into:

(51) $\neg \underset{i}{\forall} (i \in C_0 \Rightarrow x_{0_i} \geq y_{0_i}) \bigvee \pi_0[C_0, x_0] \geq \pi_0[C_0, y_0]$.

We prove (7) by case distinction using (51).

Case (51.1) $\neg \underset{i}{\forall} (i \in C_0 \Rightarrow x_{0_i} \geq y_{0_i})$:

Formula (51.1) is simplified to:

(52) $\underset{i}{\exists} (\neg (i \in C_0 \Rightarrow x_{0_i} \geq y_{0_i}))$.

By (52) we can take appropriate values such that:

(53) $\neg \left(i_2 \in C_0 \Rightarrow x_{0_{i_2}} \geq y_{0_{i_2}}\right)$.

Formula (53) is expanded into

(54) $i_2 \in C_0 \bigwedge x_{0_{i_2}} \ngeq y_{0_{i_2}}$.

From (54.2), by (Axiom (trichotomy)), we obtain:

(56) $x_{0_{i_2}} \neq y_{0_{i_2}}$.

From (54.1), by (6), we obtain:

(55) $x_{0_{i_2}} = y_{0_{i_2}}$.

Formula (7) is proved because (55) and (56) are contradictory.

Case (51.2) $\pi_0[C_0, x_0] \geq \pi_0[C_0, y_0]$:

From (51.2), by (Axiom (trichotomy)), we obtain:

(59) $(\pi_0[C_0, x_0] = \pi_0[C_0, y_0]) \Leftrightarrow \pi_0[C_0, y_0] \geq \pi_0[C_0, x_0]$.

From (41.2) and (59) we obtain by modus ponens

(60) $\pi_0[C_0, x_0] = \pi_0[C_0, y_0]$.

Formula (7) is true because it is identical to (60).

Case (27.2) $\pi_0[C_0, x_0] \geq \pi_0[C_0, x_0]$:

Formula (24) is transformed into:

(64) $\neg \underset{i}{\forall} (i \in C_0 \Rightarrow y_{0_i} \geq x_{0_i}) \bigvee \pi_0[C_0, y_0] \geq \pi_0[C_0, x_0]$.

We prove (7) by case distinction using (64).

Case (64.1) $\neg \underset{i}{\forall} (i \in C_0 \Rightarrow y_{0_i} \geq x_{0_i})$:

Formula (64.1) is simplified to:

(65) $\underset{i}{\exists} (\neg (i \in C_0 \Rightarrow y_{0_i} \geq x_{0_i}))$.

By (65) we can take appropriate values such that:

(66) $\neg \left(i_3 \in C_0 \Rightarrow y_{0_{i_3}} \geq x_{0_{i_3}}\right)$.

Formula (66) is expanded into

(67) $i_3 \in C_0 \bigwedge y_{0_{i_3}} \ngeq x_{0_{i_3}}$.

From (67.1), by (6), we obtain:

(68) $x_{\theta_{i_3}} = y_{\theta_{i_3}}$.

From (68), by (Axiom (trichotomy)), we obtain:

(70) $x_{\theta_{i_3}} \geq y_{\theta_{i_3}} \bigwedge y_{\theta_{i_3}} \geq x_{\theta_{i_3}}$.

Formula (7) is proved because (70.2) and (67.2) are contradictory.

Case (64.2) $\pi_\theta[C_\theta, y_\theta] \geq \pi_\theta[C_\theta, x_\theta]$:

Formula (25) is transformed into:

(74) $\neg \underset{i}{\forall} (i \in C_\theta \Rightarrow x_{\theta_i} \geq y_{\theta_i}) \bigvee \pi_\theta[C_\theta, x_\theta] \geq \pi_\theta[C_\theta, y_\theta]$.

We prove (7) by case distinction using (74).

Case (74.1) $\neg \underset{i}{\forall} (i \in C_\theta \Rightarrow x_{\theta_i} \geq y_{\theta_i})$:

Formula (74.1) is simplified to:

(75) $\underset{i}{\exists} (\neg (i \in C_\theta \Rightarrow x_{\theta_i} \geq y_{\theta_i}))$.

By (75) we can take appropriate values such that:

(76) $\neg \left(i_4 \in C_\theta \Rightarrow x_{\theta_{i_4}} \geq y_{\theta_{i_4}}\right)$.

Formula (76) is expanded into

(77) $i_4 \in C_\theta \bigwedge x_{\theta_{i_4}} \not\geq y_{\theta_{i_4}}$.

From (77.2), by (Axiom (trichotomy)), we obtain:

(79) $x_{\theta_{i_4}} \neq y_{\theta_{i_4}}$.

From (77.1), by (6), we obtain:

(78) $x_{\theta_{i_4}} = y_{\theta_{i_4}}$.

Formula (7) is proved because (78) and (79) are contradictory.

Case (74.2) $\pi_\theta[C_\theta, x_\theta] \geq \pi_\theta[C_\theta, y_\theta]$:

From (74.2), by (Axiom (trichotomy)), we obtain:

(82) $(\pi_\theta[C_\theta, x_\theta] = \pi_\theta[C_\theta, y_\theta]) \Leftrightarrow \pi_\theta[C_\theta, y_\theta] \geq \pi_\theta[C_\theta, x_\theta]$.

From (64.2) and (82) we obtain by modus ponens

(83) $\pi_0[C_0, x_0] = \pi_0[C_0, y_0]$.

Formula (7) is true because it is identical to (83).

□

■ Additional Proof Generation Information