Stable sets

TS_In[80]:=
```
Definition["internal stability", any[π, n, S],
 IS[S, π, n] :⇔ (S ⋂ D[S, π, n] = ∅)]
```

TS_In[81]:=
```
Definition["internal stability On", any[π, n, S, Z],
 ISOn[S, π, n, Z] :⇔ (S ⋂ DOn[S, π, n, Z] = ∅)]
```

TS_In[82]:=
```
Definition["external stability", any[π, n, S],
 ES[S, π, n] :⇔ (S ⋃ D[S, π, n] = X[n])]
```

TS_In[83]:=
```
Definition["external stability on", any[π, n, S, Z],
 ESOn[S, π, n, Z] :⇔ (S ⋃ DOn[S, π, n, Z] = Z)]
```

```
General::spell1 :
 New symbol name "ESOn" is similar to existing symbol "ISOn" and may be misspelled. >>
```

TS_In[84]:=
```
Definition["stable", any[π, n, S],
 stable[S, π, n] :⇔ (IS[S, π, n] ⋀ ES[S, π, n])]
```

TS_In[85]:=
```
Definition["stable On", any[π, n, S, Z],
 stableOn[S, π, n, Z] :⇔ (ISOn[S, π, n, Z] ⋀ ESOn[S, π, n, Z])]
```

TS_In[86]:=
```
Lemma["stable lemma", any[π, n, S], with[stable[S, π, n]],
 S = X \ D[S, π, n]]
```

TS_In[87]:=
```
Definition["self protection", any[π, n, S],
 SP[S, π, n] :⇔ (S ⊆ U[U[S, π, n], π, n])]
```

TS_In[88]:=
```
Algorithm["Roth-Jordan", any[S, K, π, n],
 RothJordan[S, π, n] := where[aux = U[U[S, π, n], π, n],
   { { aux                ⇐ ES[aux, π, n]
     { "no stable set"    ⇐ otherwise        ⇐ aux = S
   { RothJordan[aux, π, n]                   ⇐ otherwise    ]
]
```

Call the algorithm as **RothJordan[K, π, n]** where **K** is the core w.r.t. π and n.

Empty core

TS_In[89]:=
```
Theorem["SINequivalence", any[π1, π2, 3],
 with[AN[π1, 3] ⋀ AN[π2, 3] ⋀ (K[π1, 3] = ∅) ⋀ (K[π2, 3] = ∅)],
            ∀                 (dominates[y, x, π1, 3] ⇔ dominates[y, x, π2, 3])]
           x,y
 allocation_n[x]⋀allocation_n[y]
```

TS_In[90]:=
```
Corollary["emptyCoreStableA", any[π], with[AN[π, 3] ⋀ (K[π, 3] = ∅)],
 stable[dyadicSet[1, 3] \ dyadicSet[0, 3], π, 3]]
```

TS_In[91]:=
```
Corollary["emptyCoreStableB", any[π, S],
 with[AN[π, 3] ⋀ (K[π, 3] = ∅) ⋀ stable[S, π, n]],
 S = (dyadicSet[1, 3] \ dyadicSet[0, 3])]
```

Non-empty core (case n=3)

TS_In[92]:= $\mathbf{Definition}[\text{"powerBalance"}, \mathbf{any}[\pi, i], B[i, \pi] := \{x \in X[3] \mid (\pi[\{i\}, x] = \pi[I[3] \setminus \{i\}, x])\}]$

TS_In[93]:= $$\mathbf{Definition}\Big[\text{"midpoint of B"}, \mathbf{any}[\pi, i, j, k], \mathbf{with}[\{i, j, k\} = \{1, 2, 3\}], q[i, \pi] = \underset{\substack{x \\ x \in B[i,\pi]}}{\ni !} \left(q[i]_j = q[i]_k\right)\Big]$$

(*Definition["midpoint of B"]//InputForm*)

$$\mathbf{Definition}\Big[\text{"maxBalanced"}, \mathbf{any}[\pi, i], M[i, \pi] = \Big\{x \in B[i, \pi] \setminus D[t[i, 3]] \;\Big|\; \Big(x_i = \max_{y \in B[i,\pi] \setminus D[t[i,3]]} y_i\Big)\Big\}\Big]$$

TS_In[94]:= $$\mathbf{Definition}\Big[\text{"r"}, \mathbf{any}[\pi, i, j], r[i, j, \pi] = \underset{z \in M[i,\pi]}{\ni} \Big(\underset{x \in M[i,\pi]}{\forall} z_j \geq x_j\Big)\Big]$$

TS_In[95]:= $$\mathbf{Definition}\Big[\text{"RMaxBalanced"}, \mathbf{any}[\pi, i], R[i, \pi] := \Big\{r \in M[i, \pi] \;\Big|\; \underset{j \in I[3] \setminus \{i\}}{\exists} \; \underset{s \in M[i,\pi]}{\forall} r_j \geq s_j\Big\}\Big]$$

- **Lemma 6**

TS_In[96]:= $$\mathbf{Lemma}\Big[\text{"powerBalanceContained"}, \mathbf{any}[\pi, i, j, k], B[i, \pi] \subset \Big\{x \underset{\text{allocation}_n[x]}{\Big|} x_i \geq \mathrm{Max}[\{x_j, x_k\}]\Big\}\Big]$$

- **Lemma 7**

TS_In[97]:= $\mathbf{Definition}[\text{"BProp"}, \mathbf{any}[\pi, i, j, k, x, y], \mathbf{with}[x \in B[i, \pi] \bigwedge y \in B[i, \pi] \bigwedge y_i > x_i \bigwedge \{i, j, k\} = \{1, 2, 3\}], y_j > x_j \bigvee y_k > x_k]$

Next we define a ray from a tyrannical element through the simplex.

TS_In[98]:= $$\mathbf{Definition}\Big[\text{"RayTyrannical"}, \mathbf{any}[i, \alpha], \mathbf{with}[0 \leq \alpha \bigwedge \alpha \leq 1], \mathrm{Ray}[i, \alpha] = \begin{cases} \Big\{\langle \beta, (1-\beta)*\alpha, (1-\beta)*(1-\alpha)\rangle \underset{\beta \in \mathbb{R}}{\Big|} 0 \leq \beta \leq 1\Big\} & \Leftarrow i = 1 \\ \Big\{\langle (1-\beta)*\alpha, \beta, (1-\beta)*(1-\alpha)\rangle \underset{\beta \in \mathbb{R}}{\Big|} 0 \leq \beta \leq 1\Big\} & \Leftarrow i = 2 \\ \Big\{\langle (1-\beta)*\alpha, (1-\beta)*(1-\alpha), \beta\rangle \underset{\beta \in \mathbb{R}}{\Big|} 0 \leq \beta \leq 1\Big\} & \Leftarrow i = 3 \end{cases}\Big]$$

- **Corollary 2**

TS_In[99]:= $$\mathbf{Corollary}\Big[\text{"Ray hits B"}, \mathbf{any}[\pi, i], \underset{\substack{\alpha \\ 0 \leq \alpha \leq 1}}{\forall} (\mathrm{Ray}[i, \alpha] \bigcap B[i, \pi]) \neq \emptyset\Big]$$

```
(*Definition["BPlus",any[π,i,j,k], with[{i,j,k}={1,2,3}],
 BPlus[i,π]:={x | x∈B[i,π,3] : xi>xj ∧ xi>xk}]*)
```

TS_In[100]:=

$$\text{Definition}\Big[\text{"BPlus"},\ \text{any}[\pi],$$
$$\text{BPlus}[1,\pi] := \{x_{x\in B[1,\pi,3]} \mid x_1 > x_2 \wedge x_1 > x_3\}$$
$$\text{BPlus}[2,\pi] := \{x_{x\in B[2,\pi,3]} \mid x_2 > x_1 \wedge x_2 > x_3\}\Big]$$
$$\text{BPlus}[3,\pi] := \{x_{x\in B[3,\pi,3]} \mid x_3 > x_1 \wedge x_3 > x_2\}$$

```
General::spell1 :
 New symbol name "BPlus" is similar to existing symbol "+" and may be misspelled. >>
```

TS_In[101]:=

$$\text{Definition}\Big[\text{"BPlusVar"},\ \text{any}[\pi, i],$$
$$\text{BPlus}[i,\pi] := \begin{cases} \{x_{x\in B[1,\pi,3]} \mid x_1 > x_2 \wedge x_1 > x_3\} & \Leftarrow i = 1 \\ \{x_{x\in B[2,\pi,3]} \mid x_2 > x_1 \wedge x_2 > x_3\} & \Leftarrow i = 2 \\ \{x_{x\in B[3,\pi,3]} \mid x_3 > x_1 \wedge x_3 > x_2\} & \Leftarrow i = 3 \end{cases}\Big]$$

- **Lemma 8**

TS_In[102]:=

$$\text{Lemma}[\text{"dominance on B.1"},\ \text{any}[\pi, x, y, i],\ \text{with}[\text{AN}[\pi, 3] \wedge \text{allocation}_3[x] \wedge \text{allocation}_3[y]],$$
$$x \in B[i,\pi] \wedge y \in B[i,\pi] \wedge y_i > x_i \wedge x_i \geq x_j \wedge x_j > x_k \wedge y_j > x_j \Rightarrow \text{dominates}[y, x, \pi, 3]]$$

TS_In[103]:=

$$\text{Lemma}[\text{"dominance on B.2"},\ \text{any}[\pi, x, y, i],\ \text{with}[\text{AN}[\pi, 3] \wedge \text{allocation}_3[x] \wedge \text{allocation}_3[y]],$$
$$x \in \text{BPlus}[i,\pi] \wedge y \in \text{BPlus}[i,\pi] \wedge y_i > x_i \Rightarrow \text{dominates}[y, x, \pi, 3]]$$

- **Lemma 9**

TS_In[104]:=

$$\text{Lemma}[\text{"Characterisation under AN and RE"},\ \text{any}[\pi],\ \text{with}[\text{AN}[\pi, 3] \wedge \text{RE}[\pi, 3]],$$
$$B[i,\pi] \setminus \text{BPlus}[i,\pi] \subseteq \text{dyadicSet}[1, 3] \setminus \text{dyadicSet}[0, 3]]$$

- **Lemma 10**

TS_In[105]:=

$$\text{Lemma}\Big[\text{"Lemma10"},\ \text{any}[\pi, x],$$
$$\text{with}\Big[\text{AN}[\pi, 3] \bigwedge \text{RE}[\pi, 3] \bigwedge R[i,\pi,3] \neq \emptyset \bigwedge \text{allocation}_3[x] \bigwedge \Big(\forall_{i \in \{1,2,3\}} x_i > 0\Big)\Big],$$
$$x \in B[i,\pi] \setminus (R[i,\pi] \bigcup \{q[i,\pi]\}) \Rightarrow x \in \text{dominion}[R[i,\pi], \pi, 3]\Big]$$

- **Theorem 4**

TS_In[106]:=

$$\text{Theorem}\Big[\text{"Theorem 4.a"},\ \text{any}[\pi],\ \text{with}[\text{AN}[\pi, 3] \wedge \text{RE}[\pi, 3] \wedge R[i,\pi] \neq \emptyset],$$
$$\exists!_{Si}\,(\text{stableOn}[Si, \pi, 3, \text{BPlus}[i,\pi] \bigcup R[i,\pi]] \wedge (Si = (R[i,\pi] \bigcup (\{q[i,\pi]\} \bigcap M[i,\pi]))))\Big]$$

TS_In[107]:=
```
Theorem["Theorem 4.a", any[i ∈ I[3], π], with[AN[π, 3] ⋀ RE[π, 3] ⋀ R[i, π] ≠ ∅],
  ∃!_S (stableOn[S, π, 3, BPlus[i, π] ⋃ R[i, π]] ⋀ (S = S[i]))]
```

TS_In[108]:=
```
Theorem["Theorem 4.a,exists", any[i ∈ I[3], π], with[AN[π, 3] ⋀ RE[π, 3] ⋀ R[i, π] ≠ ∅],
  stableOn[S[i], π, 3, BPlus[i, π] ⋃ R[i, π]] ]
```

TS_In[109]:=
```
Theorem["Theorem 4.a,unique", any[i ∈ I[3], π], with[AN[π, 3] ⋀ RE[π, 3] ⋀ R[i, π] ≠ ∅],
  ... ]
```

TS_In[109]:=
```
Definition["Si", any[i, π],
  S[i, π] := R[i, π] ⋃ (Sq[i, π] ⋂ M[i, π])]
```

A less cryptic formulation could be:

```
Definition["Si case", any[i, π],
  S[i, π] := { R[i, π] ⋃ {q[i, π]}  ⇐  q[i, π] ∈ M[i, π]
             { R[i, π]              ⇐  otherwise          ]
```

TS_In[110]:=
```
Theorem["Theorem 4.b", any[π], with[AN[π, 3] ⋀ RE[π, 3] ⋀ (R[i, π] = ∅)],
  ¬ ∃_S stableOn[S, π, 3, BPlus[i, π]]]
```

- **Lemma 11**

TS_In[111]:=
```
Lemma["Lemma 11", any[π, Si, x, y],
  with[stableOn[Si, π, 3, BPlus[i, π] ⋃ R[i, π]] ⋀
      K[π, 3] ≠ ∅ ⋀ x ∈ Si ⋀ CX[π, 3] ⋀ dominates[y, x, π, 3]]
    y ∈ dominion[{t[i, 3]}, π, 3]]
```

- **Theorem 5**

TS_In[112]:=
```
Theorem["Theorem 5", any[π, S, Si],
  with[AN[π, 3] ⋀ CX[π, 3] ⋀ RE[π, 3] ⋀ R[i, π] ≠ ∅ ⋀
      (Si = (R[i, π] ⋃ ({q[i, π]} ⋂ M[i, π]))) ⋀ stable[S, π, 3]]
    Si ⊆
   S]
```

- **Definition SIN = Strength In Numbers**

TS_In[113]:=
```
Definition["SIN", any[C, x, ν],
  πSIN_ν[C, x] = ∑_{i∈C} (x_i + ν)]
```

- **Theorem 6**

TS_In[114]:=
```
Theorem["Theorem 6a",
  stable[dyadicSet[1, 3], πSIN_1, 3]]
```

TS_In[115]:=
```
Theorem["Theorem 6b", any[S], with[stable[S, πSIN_1, 3]],
  S = dyadicSet[1, 3]]
```

TS_In[116]:=
```
Theorem["Theorem 7a", any[π],
  with[AN[π, 3] ⋀ CX[π, 3] ⋀ RE[π, 3] ⋀ (K[π, 3] ≠ ∅) ⋀ R[1, π] ≠ ∅],
  ∃!_S stable[S, π, 3]]
```

TS_In[117]:= Definition["pij", any[π, j, k],

$$p[j, k, \pi] := \begin{cases} \frac{1}{2} * (r[1, 3, \pi] + r[2, 3, \pi]) & \Leftarrow (j = 1) \wedge (k = 2) \\ \frac{1}{2} * (r[1, 2, \pi] + r[3, 2, \pi]) & \Leftarrow (j = 1) \wedge (k = 3) \\ \frac{1}{2} * (r[2, 1, \pi] + r[3, 1, \pi]) & \Leftarrow (j = 2) \wedge (k = 3) \end{cases}]$$

Variant with permuted indices. Which one is the correct one?

TS_In[118]:= Definition["pij", any[π, j, k],

$$p[j, k, \pi] := \begin{cases} \frac{1}{2} \odot (r[3, 1, \pi] \oplus r[3, 2, \pi]) & \Leftarrow (j = 1) \wedge (k = 2) \\ \frac{1}{2} \odot (r[2, 1, \pi] \oplus r[2, 3, \pi]) & \Leftarrow (j = 1) \wedge (k = 3) \\ \frac{1}{2} \odot (r[1, 2, \pi] \oplus r[1, 3, \pi]) & \Leftarrow (j = 2) \wedge (k = 3) \end{cases}]$$

TS_In[119]:= Definition["tup+*", any[x, y],

$$x \oplus y := \left\langle x_i + y_i \;\Big|_{i=1,\dots,|y|} \right\rangle$$
$$x \odot y := \left\langle x * y_i \;\Big|_{i=1,\dots,|y|} \right\rangle]$$

TS_In[120]:= Definition["P", any[π],
P[π] := {p[1, 2, π], p[1, 3, π], p[2, 3, π]}]

TS_In[121]:= Theorem["Theorem 7b", any[π],
with[AN[π, 3] ⋀ CX[π, 3] ⋀ RE[π, 3] ⋀ (K[π, 3] ≠ ∅) ⋀ R[1, π] ≠ ∅],
stable[K[π, 3] ⋃ S[1, π] ⋃ ({q[1, π]} ⋂ M[1, π]) ⋃ S[2, π] ⋃
({q[2, π]} ⋂ M[2, π]) ⋃ S[3, π] ⋃ ({q[3, π]} ⋂ M[3, π]) ⋃ P[π], 3]]

Corollary["Corollary 3", any[π, S], with[AN[π, 3] ⋀ CX[π, 3] ⋀ RE[π, 3] ⋀ stable[S, π, 3]],
|S| ≤ 15]

Algorithm["StableSet", any[π],
stableSet[π] :=

$$\begin{cases} \text{dyadicSet}[1, 3] \setminus \text{dyadicSet}[0, 3] & \Leftarrow \pi[\{1\}, t[\\ \begin{cases} \text{"no stable set exists"} & \Leftarrow \text{empty}[R[1, \pi]] \\ \text{where}\Big[\mathcal{S} = \text{dyadicSet}[0, 3] \cup \bigcup_{i=1,\dots,3} S[i, \pi], & \Leftarrow \neg\,\text{empty}[R[1, \pi]] \\ \quad \begin{cases} \mathcal{S} & \Leftarrow \pi[\{2\}, s[1, 3, 3]] \geq \pi[\{1, 3\}, s[1, 3, 3]] \\ \mathcal{S} \cup P[\pi] & \Leftarrow \text{otherwise} \end{cases} \Big] & \\ \text{"unknown M"} & \Leftarrow \text{otherwise} \end{cases} & \Leftarrow \text{otherwise} \end{cases}$$

]

General::spell1 :
New symbol name "empty" is similar to existing symbol "Empty" and may be misspelled. >>

TS_In[122]:= Definition["empty", any[$\mathcal{M}$],
empty[$\mathcal{M}$] := $\mathcal{M}$ = ∅]

General::spell1 :
New symbol name "empty" is similar to existing symbol "Empty" and may be misspelled. >>

TS_In[123]:= Definition["midpoint of B", any[π, i],

$$q[i, \pi] := \text{where}\Big[j = æ[I[3] \setminus \{i\}],\ k = æ[I[3] \setminus \{i, j\}],\ \underset{x \in B[i,\pi]}{\ni !} (x_j = x_k)\Big]\Big]$$

```
Use[{Builtin ["Connectives"], Builtin ["Quantifiers"],
  Builtin ["Tuples"], Builtin ["Sets"], Builtin ["Numbers"]}]

UseAlso[{Algorithm["StableSet2"], Lemma["D13"], Lemma["D03"], Definition["tyrannical"],
  Definition["agents"], Definition["Si"], Definition["RMaxBalanced"],
  Definition["centre allocation"], Definition["r"], Definition["P"], Definition["pij"],
  Definition["tup+*"], Definition["empty"], Definition["split allocation"]}]
```

Champion Powerfunction

```
TS_In[124]:= Lemma["midpoint of B,champion", any[i],
  Sq[i, championπ] := {p̅}]

TS_In[125]:= Lemma["championM",
  M[1, championπ] = {⟨1 / 2, 0, 1 / 2⟩, ⟨1 / 2, 1 / 2, 0⟩}
  M[2, championπ] = {⟨0, 1 / 2, 1 / 2⟩, ⟨1 / 2, 1 / 2, 0⟩}
  M[3, championπ] = {⟨1 / 2, 0, 1 / 2⟩, ⟨0, 1 / 2, 1 / 2⟩}]
```

TS_In[126]:= Lemma["championDn", any[S],

$$\neg\ \text{fullSet}\left[\left\{\langle 0,0,1\rangle, \left\langle 0,\frac{1}{2},\frac{1}{2}\right\rangle, \langle 0,1,0\rangle, \left\langle \frac{1}{2},0,\frac{1}{2}\right\rangle, \left\langle \frac{1}{2},\frac{1}{2},0\right\rangle, \langle 1,0,0\rangle\right\} \cup \right.$$
$$\left. D\left[\left\{\langle 0,0,1\rangle, \left\langle 0,\frac{1}{2},\frac{1}{2}\right\rangle, \langle 0,1,0\rangle, \left\langle \frac{1}{2},0,\frac{1}{2}\right\rangle, \left\langle \frac{1}{2},\frac{1}{2},0\right\rangle, \langle 1,0,0\rangle\right\}, \text{champion}\pi, 3\right]\right]\Big]$$

TS_In[127]:= Lemma["championD", any[S],

$$\text{fullSet}\left[\left\{\langle 0,0,1\rangle, \left\langle 0,\frac{1}{2},\frac{1}{2}\right\rangle, \langle 0,1,0\rangle, \left\langle \frac{1}{3},\frac{1}{3},\frac{1}{3}\right\rangle, \left\langle \frac{1}{2},0,\frac{1}{2}\right\rangle, \left\langle \frac{1}{2},\frac{1}{2},0\right\rangle, \langle 1,0,0\rangle\right\} \cup \right.$$
$$D\left[\left\{\langle 0,0,1\rangle, \left\langle 0,\frac{1}{2},\frac{1}{2}\right\rangle, \langle 0,1,0\rangle, \left\langle \frac{1}{3},\frac{1}{3},\frac{1}{3}\right\rangle,\right.\right.$$
$$\left.\left.\left\langle \frac{1}{2},0,\frac{1}{2}\right\rangle, \left\langle \frac{1}{2},\frac{1}{2},0\right\rangle, \langle 1,0,0\rangle\right\}, \text{champion}\pi, 3\right]\right]\Big]$$

```
UseAlso[{Definition["championPowerfunction"],
  Lemma["championM"], Lemma["midpoint of B,champion"], Lemma["championDn"]}]

Compute[R[2, championπ]]
```

$$\left\{\left\langle 0,\frac{1}{2},\frac{1}{2}\right\rangle, \left\langle \frac{1}{2},\frac{1}{2},0\right\rangle\right\}$$

```
Compute[S[3, championπ]]
```

$$\left\{\left\langle 0,\frac{1}{2},\frac{1}{2}\right\rangle, \left\langle \frac{1}{2},0,\frac{1}{2}\right\rangle\right\}$$

```
Compute[r[1, 2, championπ]]
```

$$\left\langle \frac{1}{2},\frac{1}{2},0\right\rangle$$

```
Compute[P[championπ]]
```

$$\left\{\left\langle \frac{1}{4},\frac{1}{4},\frac{1}{2}\right\rangle, \left\langle \frac{1}{4},\frac{1}{2},\frac{1}{4}\right\rangle, \left\langle \frac{1}{2},\frac{1}{4},\frac{1}{4}\right\rangle\right\}$$

```
Compute[dyadicSet[0, 3] ⋃ ⋃_{i=1,…,3} S[i, championπ]]
```

$$\left\{\langle 0, 0, 1\rangle, \left\langle 0, \frac{1}{2}, \frac{1}{2}\right\rangle, \langle 0, 1, 0\rangle, \left\langle \frac{1}{2}, 0, \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{2}, 0\right\rangle, \langle 1, 0, 0\rangle\right\}$$

```
Compute[championπ[{1, 2, 3}, ⟨1, 0, 0⟩]]
```

1

```
Compute[championπ[{1, 2, 3}, ⟨1 / 2, 0, 1 / 2⟩]]
```

$$\frac{1}{2}$$

```
Compute[stableSet[championπ]]
```

$$\left\{\langle 0, 0, 1\rangle, \left\langle 0, \frac{1}{2}, \frac{1}{2}\right\rangle, \langle 0, 1, 0\rangle, \left\langle \frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right\rangle, \left\langle \frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\rangle, \left\langle \frac{1}{2}, 0, \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{2}, 0\right\rangle, \langle 1, 0, 0\rangle\right\}$$

With Lemma ChampionDn:

```
π[{1}, t[1, 3]] < π[{2, 3}, t[1, 3]]
```

```
stableSet[championπ]
```

Wealth is Power Powerfunction

```
TS_In[128]:= Definition["WIP", any[C, x],
               WIPπ[C, x] = ∑_{i∈C} x_i]
```

```
TS_In[129]:= Lemma["WIPR",
               R[1, WIPπ] = {⟨1 / 2, 0, 1 / 2⟩, ⟨1 / 2, 1 / 2, 0⟩}
               R[2, WIPπ] = {⟨0, 1 / 2, 1 / 2⟩, ⟨1 / 2, 1 / 2, 0⟩}
               R[3, WIPπ] = {⟨1 / 2, 0, 1 / 2⟩, ⟨0, 1 / 2, 1 / 2⟩}]
```

```
TS_In[130]:= Lemma["midpoint of B,WIP",
               Sq[1, WIPπ] ⋂ M[1, WIPπ] = {⟨1 / 2, 1 / 4, 1 / 4⟩}
               Sq[2, WIPπ] ⋂ M[2, WIPπ] = {⟨1 / 4, 1 / 2, 1 / 4⟩}
               Sq[3, WIPπ] ⋂ M[3, WIPπ] = {⟨1 / 4, 1 / 4, 1 / 2⟩}]
```

```
TS_In[131]:= Lemma["emptyM,WIP",
               ¬ empty[M[1, WIPπ]]]
```

```
UseAlso[{Definition["WIP"], Lemma["midpoint of B,WIP"], Lemma["WIPR"], Lemma["emptyM,WIP"]}]
```

```
Compute[dyadicSet[0, 3] ⋃ ⋃_{i=1,…,3} S[i, WIPπ]]
```

$$\left\{\langle 0, 0, 1\rangle, \left\langle 0, \frac{1}{2}, \frac{1}{2}\right\rangle, \langle 0, 1, 0\rangle, \left\langle \frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right\rangle, \left\langle \frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\rangle, \left\langle \frac{1}{2}, 0, \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{2}, 0\right\rangle, \langle 1, 0, 0\rangle\right\}$$

Compute[**S**[**1**, **WIP**π]]

$\left\{\left\langle \frac{1}{2}, 0, \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{2}, 0\right\rangle\right\}$

TS_In[132]:= **Lemma**["**WIPD**", **any**[**S**],
fullSet[$\left\{\langle 0, 0, 1\rangle, \left\langle 0, \frac{1}{2}, \frac{1}{2}\right\rangle, \langle 0, 1, 0\rangle, \left\langle \frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right\rangle, \left\langle \frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\rangle, \left\langle \frac{1}{2}, 0, \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{2}, 0\right\rangle, \langle 1, 0, 0\rangle\right\} \cup D\left[\left\{\langle 0, 0, 1\rangle, \left\langle 0, \frac{1}{2}, \frac{1}{2}\right\rangle, \langle 0, 1, 0\rangle, \left\langle \frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right\rangle, \left\langle \frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\rangle, \left\langle \frac{1}{2}, 0, \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{2}, 0\right\rangle, \langle 1, 0, 0\rangle\right\}, \text{WIP}\pi, 3\right]$]]

UseAlso[{**Lemma**["**WIPD**"]}]

Compute[**R**[**2**, **WIP**π]]

$\left\{\left\langle 0, \frac{1}{2}, \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}, \frac{1}{2}, 0\right\rangle\right\}$

Compute[**S**[**3**, **WIP**π]]

$\left\{\left\langle 0, \frac{1}{2}, \frac{1}{2}\right\rangle, \left\langle \frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}, 0, \frac{1}{2}\right\rangle\right\}$

Compute[**WIP**π[{**1**, **2**, **3**}, ⟨**1**, **0**, **0**⟩]]

$\sum_{i\in\{1,2,3\}} \langle 1, 0, 0\rangle_i$

Compute[**WIP**π[{**1**, **2**, **3**}, ⟨**1 / 2**, **0**, **1 / 2**⟩]]

1

Compute[**WIP**π[{**1**}, **t**[**1**, **3**]] < **WIP**π[{**2**, **3**}, **t**[**1**, **3**]]]

$\sum_{i\in\{1\}} t[1, 3]_i < \sum_{i\in\{2,3\}} t[1, 3]_i$

Compute[**stableSet**[**WIP**π]]

stableSet[WIPπ]

Strength in Numbers Powerfunction

TS_In[133]:= **Definition**["**SIN**", **any**[**C**, **x**, ν],
$\text{SIN}\pi_\nu[C, x] = \sum_{i\in C} (x_i + \nu)$]

UseAlso[{**Definition**["**SIN**"]}]

Compute[**SIN**π_2[{**1**}, ⟨**1**, **0**, **0**⟩]]

3

```
Compute[SINπ_2[{2, 3}, 〈1, 0, 0〉]]
```

4

TS_In[136]:= `Compute[stableSet[SINπ_2]]`

TS_Out[136]=
stableSet[SINπ_2]

Cobb Douglas Powerfunction

TS_In[134]:= Definition["Cobb Douglas", any[C, x, ν],

$$CD\pi_\nu[C, x] = |C|^\nu * \left(\sum_{i\in C} x_i\right)^{1-\nu}$$]

```
UseAlso[{Definition["Cobb Douglas"]}]
```

```
Compute[CDπ_1/3[{1}, 〈1, 0, 0〉]]
```

1

```
Compute[CDπ_1/3[{2, 3}, 〈1, 0, 0〉]]
```

0

TS_In[135]:=
```
Lemma["emptyM,Cobb Douglas", any[ν],
  empty[M[1, CDπ_ν]]]
```

```
UseAlso[Lemma["emptyM,Cobb Douglas"]]
```

TS_In[137]:= `Compute[stableSet[CDπ_1/3]]`

TS_Out[137]=
stableSet[CD$\pi_{\frac{1}{3}}$]