Stable sets in three agent pillage games*

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Abstract

We consider pillage games as in Jordan [2006, “Pillage and property”, JET] and show that, for anonymous, three agent pillage games: when the core is non-empty, it must take one of five forms; all such pillage games with an empty core represent the same dominance relation; when a stable set exists, and the game also satisfies a continuity and a responsiveness assumption, it is unique and contains no more than 15 elements, a tight bound. This result uses a three step procedure: first, if a single agent can defend all of the resources against the other two, these allocations belong to the stable set; dominance is then transitive on the loci of allocations on which the most powerful agent can, with any ally, dominate the third, adding the maximal elements of this set to the stable set; finally, if any allocations remain undominated or not included, the game over the remaining allocations is equivalent to the ‘majority pillage game’, which has a unique stable set [Jordan and Obadia, 2004, “Stable sets in majority pillage games”, mimeo]. Non-existence is easily determined by applying Zorn’s lemma to the locus of allocations for which the most powerful agent is as powerful as the other two combined.

Key words: pillage, cooperative game theory, core, stable sets, algorithm

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> [T]here is a feeling that procedures are not really all that relevant; that it is the possibilities for coalition forming, promising and threatening that are decisive, rather than whose turn it is to speak. [Aumann, 2008]

1 Introduction

Opponents of a new government's taxes nonetheless pay, knowing its supporters are more powerful than they are; international alliances are formed and weaker nations propped up to maintain a balance of power; nations pay tribute or Danegeld to avoid war – or reparations following war; merchant ships surrender rather than fight pirates or privateers Kontorovich [2004]; junior employees often tolerate abusive behaviour from their seniors without protest, knowing that a challenge would become a power contest which they would lose.

All the above are examples of power contests in which power is derived from both agents' inalienable attributes and their transferrable resources. Further, while resources may be transferred, none are actually consumed by the contest. This latter feature seems most consistent with common knowledge of relative strengths.[1]

Their study therefore presents two problems for typical analyses. First, Aumann's critique that non-cooperative game forms may be of limited relevance is especially strong when, in the absence of strong institutions, the game forms themselves are contested. Second, while cooperative game theory works directly with a dominance relation defined on outcomes, abstracting from game forms, its two most common formulations are limited by their dichotomous treatment of power and utility: the contested goods yield utility, but cannot contribute to the contest of power.

Games in characteristic function form[2] allow power to depend only on coalitions' absolute inalienable attributes; consequently, analyses of such games tend to predict that the grand coalition of all agents forms, ruling out the very real possibility of conflict between coalitions "from the beginning" [Maskin, 2003].

Games in partition function form generalise those in characteristic function form, allowing power to depend on coalitions' relative inalienable attributes, and therefore for externalities across coalitions [Thrall and Lucas, 1963, Maskin, 2003, de Clippel and Serrano, 2008a]. However, they remain unable to model the pos-

[1]If so, the literature's usage of terms like 'pillage' [Jordan, 2006a] and 'jungle' [Piccione and Rubinstein, 2007] may be misleading, calling to mind exceptional interactions between relatively unknown parties.

[2]While a small subset of von Neumann and Morgenstern's abstract games, the dominance of these games within cooperative game theory is so complete that they are often taken to define cooperative game theory.

sibility that power may depend on holdings of transferrable resources, even when explicitly considering transferrable utility.

Jordan [2006a] introduced *pillage games*, a class of cooperative games whose dominance relations are represented by power functions, increasing in both coalitional membership and members' resource holdings. Thus, such games allow power to depend both on inalienable attributes and holdings of transferrable resources without the imposition of restrictive game forms. As a result, they offer an alternative and possibly fruitful tool for the study of power contests.[3] To emphasise their relation to the better known classes of cooperative games, we also refer to pillage games as *games in power function form*.

To compare more concretely, consider a game in characteristic function form that allowed any sufficiently large coalition to split a pie. In an analogous game in partition function form, the *majority game*, absolute size is insufficient: instead, the coalition must be larger than any other to split the pie. Extending the game further to power function form allows ties between coalitions of equal size to be broken by comparison of their existing holdings (the *majority pillage game* of Jordan and Obadia [2004]).

Although games in power function form are closely related to and richer than those in partition function form, they are nevertheless disjoint, replacing a constancy condition in partition function forms with a strict monotonicity condition. The original theory of partition function form games [Thrall and Lucas, 1963] also made the "bloodthirsty" assumption [Ray and Vohra, 1997] that the residual agents opposed any coalition while, in power function form, a coalition is only opposed by those agents with opposed interests.[4] While the former condition is restrictive, the latter seems more realistic than its counterpart.

Section 2 of this paper formally introduces pillage games. While their independence of game forms and rich ability to model power are attractive, their behaviour is correspondingly sensitive to the choice of power function. Jordan [2006a] presented results for stable sets (no allocation in the set dominates any other; any allocation outside the set is dominated by one in it) for three particular power functions: in the first, a unique stable set was derived; in the second, a (possibly non-unique) stable set was derived under some circumstances; in the third, a non-existence result was proven.

As there are an infinite number of power functions, a number of questions therefore arose naturally. How are these three results related? How many other

[3] Models of contest success functions [q.v. Skaperdas, 1992, 1996], a popular non-cooperative approach, differ in two ways from those based on cooperative games: game forms must be specified, and the power contest is costly, eliminating the possibility of efficiency from the outset.

[4] Kóczy [2007] reviewed more recent approaches to the residual agents in partition function form games; its own model imposes commitment assumptions on agents, allowing cores to be defined recursively from those of singleton agents.

classes of stable sets are possible? What determines uniqueness or existence? Are there equivalence classes among power functions?

This paper seeks to address these questions. Its principal technical challenge arises from stable sets' intractability. While von Neumann and Morgenstern [1944] assumed that stable sets existed for games in characteristic function form - the focus of their analysis - it took a quarter century for a counterexample to be found [Lucas, 1968].[5] However, Jordan [2006a] proved that a set is stable if there is a consistent expectation for which it is the (farsighted) core in expectation.[6] Thus, to the extent that the core and forward looking agents are compelling, stable sets bear study.[7]

Consequently, exhaustive analyses of stable sets have been limited to three agent examples.[8] For the classical theory of games in characteristic function form, this was performed by von Neumann and Morgenstern [1944] and, for games in partition function form, by Thrall and Lucas [1963]. We provide a similar analysis for pillage games satisfying three further axioms.

First, though, Section 3 presents results on the core (the set of undominated allocations). For any number of agents, the core is either empty, or takes one of five forms. Strikingly, all power functions yielding empty cores represent the same dominance relation, and therefore yield the same stable set.

For three agent games, Section 4 derives a three step procedure for determining stable sets. First, non-empty cores contain the 'tyrannical' allocations which give the whole unit pie to a single agent [Jordan, 2006a]. As core allocations necessarily belong to any stable set, these three allocations therefore seed our candidate stable sets.

Second, decreasing a tyrant's holdings may eventually yield a balance of power with the other two agents. When power is continuous in resources, allocations along this locus – if it exists – are only dominated by allocations themselves dominated by tyrannical allocations. As dominance is transitive along this locus, external stability not only requires inclusion of allocations from the locus, but uniquely determines the allocations. For each agent, there are up to three of these.

Third, further decreasing the tyrant's holdings may then yield allocations for which any two agents can defeat a single agent. Over such allocations, dominance

[5] For abstract cooperative games, we are grateful to Oleg Itskhoki for noting that Condorcet cycles produce counterexamples.

[6] See Anesi [2006] and the references therein for more motivations of stable sets based on farsightedness.

[7] See de Clippel and Serrano [2008b], Kalai and Kalai [2009] and Brandt et al. [2009] for recent work in cooperative game theory that makes use of other solution concepts.

[8] Lucas [1971] notes that stable sets have only been "described for a few very particular games with arbitrary n". Lucas [1992, §5] lists six conjectures based on $n = 3$ analysis that were overturned for $n > 3$. See Maskin [2003] and de Clippel and Serrano [2008a] for an example of the difficulties of generalising beyond three agents.

is equivalent to that in the majority pillage game. This has a unique stable set of three allocations [Jordan and Obadia, 2004], producing a unique stable set on this last domain.

When power satisfies responsiveness, anonymity and continuity axioms, this procedure both yields a unique stable set, when one exists, and sets a tight upper bound of fifteen allocations on it. It also identifies the source of non-existence: stable sets do not exist when the balance of power locus lacks a maximal element, a new application of an old consequence of Zorn's lemma [von Neumann and Morgenstern, 1944, §65.4.2]. In Jordan [2006a], this occurred when the element that would be maximal was dominated by a tyrannical allocation. We show that it may also occur when the power function is discontinuous in resource holdings. While non-existence occurs under some plausible conditions, it is not possible to comment on the measure of these conditions without further progress in identifying equivalent power functions, which might reduce their dimension of their space.

Similar arguments have been made elsewhere to demonstrate the existence and uniqueness of stable sets [Gillies, 1959, Chang and Chang, 1991]. The power function's monotonicity, however, aids their application to the actual computation of stable sets. In the case of anonymous, continuous, responsive $n = 3$ power functions, the above procedure also improves on the algorithm in Jordan and Obadia [2004] (derived from Roth [1976]), which provided limited guidance on how to begin with an empty core, whether a stable set existed and, if so, how to compute it.

Section 5 first applies the preceding theory to examples satisfying the anonymity, continuity and responsiveness axioms; it then presents results for stable sets violating those axioms. In doing so, it shows that the stable set of the 'wealth is power' function introduced by Jordan [2006a] generalises to a class of power functions that weaken its conservatism axiom to a point symmetry axiom; all such power functions represent the same dominance ordering. It also shows that the continuity axiom is restrictive: not all pillage games can be represented by a continuous power function.

Section 6 concludes. An appendix presents some auxiliary results not used in the main arguments.

Taken together, the paper provides further reason to believe that pillage games are well-behaved examples of cooperative games. The monotonicity axioms that underpin dominance impose considerable structure on pillage games relative to abstract cooperative games, which may have multiple stable sets, each infinite [Shapley, 1959].[9] As yet, no examples of multiple stable sets have been found for

[9] A positive fraction of non-zero-sum games in characteristic function form have unique stable sets [Gillies, 1959], as do "most" $n = 3$ and $n = 4$ games in partition function form [Lucas, 1971].

pillage games; further, various finite bounds on them are known [Jordan, 2006a, Kerber and Rowat, forthcoming, Saxton, 2010].[10]

2 Pillage games

Let $I = \{1, \ldots, n\}$ be a finite set of *agents*. An *allocation* divides a unit resource among them, so that the feasible set is a compact, continuous $n-1$ dimensional simplex:

$$\mathcal{X} \equiv \left\{ \{x_i\}_{i \in I} \middle| x_i \geq 0, \sum_{i \in I} x_i = 1 \right\}.$$

2.1 Power functions

Let $\subset$ denote a proper subset, and use $\subseteq$ to allow the possibility of equality. Jordan [2006a] defined a *power function* over subsets of agents and allocations, so that $\pi : 2^I \times \mathcal{X} \to \mathbb{R}$ satisfies:

(WC) if $C \subset C' \subseteq I$ then $\pi(C', \boldsymbol{x}) \geq \pi(C, \boldsymbol{x}) \, \forall \boldsymbol{x} \in \mathcal{X}$;

(WR) if $y_i \geq x_i \forall i \in C \subseteq I$ then $\pi(C, \boldsymbol{y}) \geq \pi(C, \boldsymbol{x})$; and

(SR) if $\emptyset \neq C \subseteq I$ and $y_i > x_i \forall i \in C$ then $\pi(C, \boldsymbol{y}) > \pi(C, \boldsymbol{x})$.

Axiom WC requires weak monotonicity in coalitional inclusion; WR requires weak monotonicity in resources; SR requires strong monotonicity in resources. These axioms imply the following representation:

Lemma 1. *Any power function, $\pi(C, \boldsymbol{x})$, can be represented by another, $\pi'(C, \{x_i\}_{i \in C})$, which depends only on the resource holdings of its coalition members.*

Proof. Consider arbitrary $\boldsymbol{x}, \boldsymbol{y}$ such that $x_i = y_i \forall i \in C \subseteq I$. Then $y_i \geq x_i$ and $x_i \geq y_i$ so that axiom WR requires $\pi(C, \boldsymbol{y}) \geq \pi(C, \boldsymbol{x}) \geq \pi(C, \boldsymbol{y})$. For this to hold, $\pi(C, \boldsymbol{x})$ cannot depend on x_j for any $j \notin C$. □

The axioms also imply that $\pi(\emptyset, \boldsymbol{x})$ is the smallest value that π can take, and is independent of $\boldsymbol{x}$.[11] Without loss of generality, we normalise $\pi(\emptyset, \boldsymbol{x}) = 0$.

The following axioms are not necessary for a function to be a power function, but will be analytically useful:

[10] While games in partition function form generally have fewer stable sets than those in characteristic function form, their stable sets are often larger [Lucas, 1971].

[11] See Beardon and Rowat [2010] for a slightly longer discussion.

(AN) if $\sigma : I \to I$ is a 1:1 onto function permuting the agent set, $i \in C \Leftrightarrow \sigma(i) \in C'$, and $x_i = x'_{\sigma(i)}$ then $\pi(C, \boldsymbol{x}) = \pi(C', \boldsymbol{x}')$.

(CX) $\pi(C, \boldsymbol{x})$ is continuous in $\boldsymbol{x}$.

(RE) if $i \notin C$ and $\pi(\{i\}, \boldsymbol{x}) > 0$ then $\pi(C \cup \{i\}, \boldsymbol{x}) > \pi(C, \boldsymbol{x})$.

Axiom AN is an anonymity axiom which means that power does not depend on the identity of agents, merely their cardinality and resources.[12] Axiom RE is the responsiveness axiom of Jordan [2007]: the addition of an agent which has power even as a singleton strictly increases the power of its new coalition.

A *pillage game* is then a triple, $(n, \mathcal{X}, \pi)$.

The three power functions defined in Jordan [2006a] are 'wealth is power' (WIP), 'strength in numbers' (SIN) and 'Cobb-Douglas' (CD):

$$\pi_w(C, \boldsymbol{x}) = \sum_{i \in C} x_i; \tag{1}$$

$$\pi_v(C, \boldsymbol{x}) = \pi_w(C, \boldsymbol{x}) + v\|C\| = \sum_{i \in C} (x_i + v); \tag{2}$$

$$\pi_c(C, \boldsymbol{x}) = \|C\|^{\alpha} \cdot \pi_w(C, \boldsymbol{x})^{1-\alpha}; \tag{3}$$

where $v \geq 0, \|C\|$ denotes the cardinality of C and $\alpha \in (0, 1)$. When $v > 1$, SIN is the majority pillage game of Jordan and Obadia [2004]. These are clearly related:

$$\begin{aligned}
&\lim_{v \to 0} \pi_v(C, \boldsymbol{x}) = \pi_w(C, \boldsymbol{x}); \\
&\lim_{\alpha \to 0} \pi_c(C, \boldsymbol{x}) = \pi_w(C, \boldsymbol{x}); \text{ and} \\
&\lim_{\alpha \to 1} \pi_c(C, \boldsymbol{x}) = \lim_{v \to \infty} \frac{1}{v} \pi_v(C, \boldsymbol{x}) = \|C\|.
\end{aligned}$$

All three power functions satisfy additional axioms AN, CX and RE.

2.2 Dominance

An allocation $\boldsymbol{y}$ dominates an allocation $\boldsymbol{x}$, written $\boldsymbol{y} \succ \boldsymbol{x}$, iff

$$\pi(W, \boldsymbol{x}) > \pi(L, \boldsymbol{x});$$

where $W \equiv \{i \,| y_i > x_i\}$ and $L \equiv \{i \,| x_i > y_i\}$. By the strict inequality, domination is irreflexive; by axiom SR, it is asymmetric.

We shall use the following immediate result later:

[12] Jordan and Obadia [2004] called this axiom 'symmetry'.

Lemma 2. *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$ such that $W = \{i \,| y_i > x_i\} = \{1\}$ and $L = \{i \,| y_i < x_i\} = \{2\}$. Then, for any power function satisfying axiom AN, $\boldsymbol{y} \succ \boldsymbol{x} \Leftrightarrow x_1 > x_2$.*

Multiple power functions may represent the same dominance relation, in the same way that multiple utility functions may represent the same preference relation.[13] Monotonic transformations of power functions clearly represent the same dominance relations [Jordan and Obadia, 2004];[14] we would like to know how much broader these equivalence classes are.

For $\mathcal{Y} \subset \mathcal{X}$, let

$$D(\mathcal{Y}) \equiv \{\boldsymbol{x} \in \mathcal{X} \,| \exists \boldsymbol{y} \in \mathcal{Y} \text{ s.t. } \boldsymbol{y} \succ \boldsymbol{x}\};$$

be the *dominion* of $\mathcal{Y}$, the set of allocations dominated by an allocation in $\mathcal{Y}$. Similarly, $U(\mathcal{Y}) = \mathcal{X} \backslash D(\mathcal{Y})$, the set of allocations undominated by any allocation in $\mathcal{Y}$.

3 The core

The *core*, $\mathcal{K}$, is the set of undominated allocations, $U(\mathcal{X}) = \mathcal{X} \backslash D(\mathcal{X})$. Following Jordan [2006a], let $\boldsymbol{t}^i \in \mathcal{X}$ be a *tyrannical* allocation such that $t^i_i = 1$ and $t^i_j = 0 \forall j \neq i \in I$. Then:

Theorem 1 (Jordan [2006a], 2.6)**.** *The core is the set*

$$\mathcal{K} = \{\boldsymbol{x} \in \mathcal{X} \,| x_i > 0 \Leftrightarrow \pi(\{i\}, \boldsymbol{x}) \geq \pi(I \backslash \{i\}, \boldsymbol{x})\}.$$

In particular,

(a) for each $i \in I, \boldsymbol{t}^i \in \mathcal{K}$ iff $\pi\left(\{i\}, \boldsymbol{t}^i\right) \geq \pi\left(I \backslash \{i\}, \boldsymbol{t}^i\right)$; and

(b) if $\pi\left(\{i\}, \boldsymbol{t}^i\right) < \pi\left(I \backslash \{i\}, \boldsymbol{t}^i\right) \forall i \in I$ then $\mathcal{K} = \emptyset$.

The inequality in item (b), above, was called the *no-tyranny* condition in Jordan [2006b]. The main result presented in this subsection, Theorem 2, will be an explicit enumeration of all possible cores for anonymous $n = 3$ pillage games. To present this, we first proceed with some definitions.

Let $\boldsymbol{s}^{jk} \in \mathcal{X}$ be a *split* allocation with $s^{jk}_i = 0$ and $s^{jk}_j = s^{jk}_k = \frac{1}{2}$ for all distinct i, j and k. Some implications follow immediately:

Lemma 3. *When $n = 3$:*

[13] The crucial distinction is that a rational preference ordering is transitive, a property not required of dominance relations.

[14] Thus, the restriction $\pi(C, \boldsymbol{x}) \geq 0$ is without loss of generality.

1. *$\mathcal{K} = \emptyset$ implies $\boldsymbol{t}^i \in D\left(\boldsymbol{s}^{jk}\right)$; and*

2. *axiom AN implies that $\boldsymbol{s}^{jk} \notin D\left(\boldsymbol{t}^i\right)$;*

for distinct $i, j, k \in I$.

Proof. 1. As $\mathcal{K} = \emptyset$, no agent can defend its holdings against both others, so that $\pi\left(\{i\}, \boldsymbol{t}^i\right) < \pi\left(\{j,k\}, \boldsymbol{t}^i\right)$ for distinct i, j and k. As $\{j,k\}$ prefers $\boldsymbol{s}^{jk}$ to $\boldsymbol{t}^i$, this ensures that $\boldsymbol{s}^{jk} \Subset\!\!- \boldsymbol{t}^i$.

2. Suppose the contrary. Then, for disjoint i, j and k,

$$\pi\left(\{j,k\}, \boldsymbol{s}^{jk}\right) \geq \pi\left(\{j\}, \boldsymbol{s}^{jk}\right) > \pi\left(\{j\}, \boldsymbol{s}^{ik}\right) = \pi\left(\{i\}, \boldsymbol{s}^{jk}\right) > \pi\left(\{j,k\}, \boldsymbol{s}^{jk}\right);$$

by axioms WC, SR and AN, and the assumption that $\boldsymbol{s}^{jk} \in D\left(\boldsymbol{t}^i\right)$, respectively. This is a contradiction.

□

Following Jordan [2006a], a scalar $x_i \in [0,1]$ is *dyadic* if $x_i = 0$ or $x = 2^{-k}$ for some nonnegative integer k. An allocation $\boldsymbol{x} = (x_1, \ldots, x_n)$ is *dyadic* if each x_i component is dyadic. Let $\mathcal{D}$ denote the set of dyadic allocations. For each nonnegative integer k, let

$$\mathcal{D}_k \equiv \left\{\boldsymbol{x} \in \mathcal{X} \,\middle|\, \boldsymbol{x} \in \mathcal{D}, \text{ for each } i, x_i > 0 \Rightarrow x_i \geq 2^{-k}\right\}.$$

Then $\mathcal{D}_k \subset \mathcal{D}_{k+1}$ for each k, and $\mathcal{D} \equiv \cup_k \mathcal{D}_k$.

Thus, $\mathcal{D}_0 = \left\{\boldsymbol{t}^1, \boldsymbol{t}^2, \boldsymbol{t}^3\right\}$, $\mathcal{D}_1 = \left\{\boldsymbol{t}^1, \boldsymbol{t}^2, \boldsymbol{t}^3, \boldsymbol{s}^{12}, \boldsymbol{s}^{13}, \boldsymbol{s}^{23}\right\}$ and $\mathcal{D}_1 \backslash \mathcal{D}_0 = \left\{\boldsymbol{s}^{12}, \boldsymbol{s}^{13}, \boldsymbol{s}^{23}\right\}$.

Finally, define the *champion power function* as

$$\bar{\pi}(C, \boldsymbol{x}) \equiv \max_{i \in C} x_i; \tag{4}$$

and let the *leader power function* be

$$\hat{\pi}(C, \boldsymbol{x}) \equiv \left\{ \begin{array}{ll} \max_{i \in C} x_i & \text{if } \max_{i \in C} x_i \leq \frac{1}{3} \\ \sum_{i \in C} \left[(1-v)\, x_i + v\right] & \text{otherwise} \end{array} \right\}; \tag{5}$$

where $v \in (0,1)$. The leader power function therefore functions as the champion when no agent holds more than a third of the resources; once an agent's holdings increase beyond this level, the leader power function behaves according to strength in numbers. Satisfying axiom SR requires $v < 1$; were $v = 0$, the second expression would reduce to wealth is power. Both $\bar{\pi}$ and $\hat{\pi}$ satisfy the anonymity axiom AN; neither satisfy the responsiveness axiom RE; while $\bar{\pi}$ satisfies the continuity axiom, CX, $\hat{\pi}$ generally does not.

To derive the cores for these new power functions, we denote by $\bar{\boldsymbol{p}} \equiv \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ the simplex' centroid:

Lemma 4. *Under* $\bar{\pi}(C, \boldsymbol{x}), \mathcal{K} = \mathcal{D}_1 \cup \{\bar{\boldsymbol{p}}\}$.

Proof. First use Theorem 1 to establish that $\mathcal{D}_1 \cup \{\bar{\boldsymbol{p}}\} \subseteq \mathcal{K}$: $\bar{\pi}\left(\{1\}, \boldsymbol{t}^1\right) = 1 > 0 = \bar{\pi}\left(\{2,3\}, \boldsymbol{t}^1\right)$; $\bar{\pi}\left(\{1\}, \boldsymbol{s}^{12}\right) = \frac{1}{2} = \bar{\pi}\left(\{2,3\}, \boldsymbol{s}^{12}\right)$; $\bar{\pi}(\{1\}, \bar{\boldsymbol{p}}) = \frac{1}{3} = \bar{\pi}(\{2,3\}, \bar{\boldsymbol{p}})$.

Now establish that no other allocation is undominated. Any allocation (x_1, x_2, x_3) whose maximal element, x_1 without loss of generality, is greater than its next largest lies in $D\left(\boldsymbol{t}^1\right)$. This leaves only the following allocations, $\boldsymbol{x} = (x, x, 1-2x)$ where $x \in \left(\frac{1}{3}, \frac{1}{2}\right)$; in these cases, $x > 1 - 2x$ so that $\boldsymbol{x} \in D\left(\boldsymbol{s}^{12}\right)$. □

Lemma 5. *Under* $\hat{\pi}(C, \boldsymbol{x}), \mathcal{K} = \mathcal{D}_0 \cup \{\bar{\boldsymbol{p}}\}$.

Proof. First add allocations to the core: $\hat{\pi}\left(\{1\}, \boldsymbol{t}^1\right) = 1 > 0 = \hat{\pi}\left(\{2,3\}, \boldsymbol{t}^1\right)$; $\hat{\pi}(\{1\}, \bar{\boldsymbol{p}}) = \frac{1}{3} = \hat{\pi}(\{2,3\}, \bar{\boldsymbol{p}})$.

Now show that no other allocations may belong to the core: let $\boldsymbol{x} = (x_1, x_2, x_3)$ such that, without loss of generality, $x_1 \geq x_2 \geq x_3$. Then, as we need not consider $\bar{\boldsymbol{p}}$ again, $x_1 > \frac{1}{3}$; similarly, as $\boldsymbol{t}^1$ has already been considered, $x_1 < 1$, so that $x_2 > 0$. There are then one of three possibilities:

1. $\hat{\pi}(\{1\}, \boldsymbol{x}) > \hat{\pi}(\{2,3\}, \boldsymbol{x})$, implying that $\boldsymbol{x}$ is dominated by $\boldsymbol{t}^1$, excluding it from the core.

2. $\hat{\pi}(\{1\}, \boldsymbol{x}) = \hat{\pi}(\{2,3\}, \boldsymbol{x})$ which, as $\hat{\pi}(\{1,3\}, \boldsymbol{x}) > \hat{\pi}(\{1\}, \boldsymbol{x})$ when $x_1 > \frac{1}{3}$, and $\hat{\pi}(\{2\}, \boldsymbol{x}) \leq \hat{\pi}(\{2,3\}, \boldsymbol{x})$ by axiom WC, implies that $\hat{\pi}(\{1,3\}, \boldsymbol{x}) > \hat{\pi}(\{2\}, \boldsymbol{x})$, so that agent 2 cannot defend its non-zero holdings, violating the core condition.

3. $\hat{\pi}(\{1\}, \boldsymbol{x}) < \hat{\pi}(\{2,3\}, \boldsymbol{x})$, so that it is now agent 1 that cannot defend its positive holdings.

□

We may now extend Theorem 1 to fully describe the core for all anonymous power functions when $n = 3$. The examples of power functions considered in Jordan [2006a] favour concentrated core allocations, dividing resources between one or two agents. The next result shows that, when $n = 3$ and the anonymity axiom holds, there is one other possible allocation, and two other possible cores.

Theorem 2. *When* $n = 3$ *and axiom AN holds, the core is one of: (1)* $\emptyset$*; (2)* $\mathcal{D}_0$*; (3)* $\mathcal{D}_0 \cup \{\bar{\boldsymbol{p}}\}$*; (4)* $\mathcal{D}_1$*; and (5)* $\mathcal{D}_1 \cup \{\bar{\boldsymbol{p}}\}$.

Thus, all allocations in the core, $\boldsymbol{x} \in \mathcal{K}$, lie on symmetry axes, so that $x_j = x_k$ for some $j \neq k$. We shall later see that this need not be true of allocations in stable sets.

Proof. First show that the cases listed are, in fact, feasible:

1. the 'strength in numbers' power function, defined in equation 2, yields an empty core [Jordan, 2006a, Proposition 4.2] when $\nu > 1$.

2. for the same power function, $\nu \in (0, 1)$ yields $\mathcal{K} = \mathcal{D}_0$ [Jordan, 2006a, Proposition 4.2].

3. the 'leader' power function, defined in equation 5, yields $\mathcal{K} = \mathcal{D}_0 \cup \{\bar{\boldsymbol{p}}\}$ by Lemma 5.

4. the 'wealth is power' power function, defined in equation 1, yields $\mathcal{K} = \mathcal{D}_1$ [Jordan, 2006a, Proposition 3.2].

5. the 'champion' power function, defined in equation 4, yields $\mathcal{K} = \mathcal{D}_1 \cup \{\bar{\boldsymbol{p}}\}$ by Lemma 4.

Now show that no other allocations than the $\boldsymbol{t}^i$, the $\boldsymbol{s}^{jk}$ and $\bar{\boldsymbol{p}}$ may be in the core. Trivially, the only $\boldsymbol{x} \in \mathcal{X}$ such that $x_j = x_k = 0$ are the $\boldsymbol{t}^i$, which have already been considered. Of those $\boldsymbol{x} \in \mathcal{X}$ such that $x_i, x_j > 0, x_k = 0$, only those setting $\pi(\{i\}, \boldsymbol{x}) = \pi(\{j\}, \boldsymbol{x})$ may be in $\mathcal{K}$; by axiom AN, these are the $\boldsymbol{s}^{jk}$. Finally, consider the strictly positive $\boldsymbol{x} \in \mathcal{X}$ such that $\boldsymbol{x} > \boldsymbol{0}$. In order to guard against the richer domination possibilities, these must satisfy $\pi(\{i\}, \boldsymbol{x}) = \pi(\{j\}, \boldsymbol{x}) = \pi(\{k\}, \boldsymbol{x}) = \pi(\{j, k\}, \boldsymbol{x})$; by axiom AN, this uniquely implies $\bar{\boldsymbol{p}}$.

Finally, show no other combination of these allocations is permissible. By the first clause in Theorem 1, a non-empty core must contain the $\boldsymbol{t}^i$. This ends the proof. □

When $n > 3$, the 'champion' power function's core contains the generalisation of $\bar{\boldsymbol{p}}$, namely, $\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$. Usually, core allocations do not grant all agents positive resources as this requires that the allocation must be undominated by perturbations in any direction; under the champion power function, agents with positive resources need not contribute to their coalition's power.

4 Stable sets

A set of allocations, $\mathcal{S} \subseteq \mathcal{X}$, is a *stable set* iff it satisfies *internal stability*,

$$\mathcal{S} \cap D(\mathcal{S}) = \emptyset; \tag{IS}$$

and *external stability*,

$$\mathcal{S} \cup D(\mathcal{S}) = \mathcal{X}. \tag{ES}$$

The conditions combine to yield $\mathcal{S} = \mathcal{X} \backslash D(\mathcal{S})$. While stable sets may not exist, or may be non-unique in general cooperative games, the core necessarily belongs to any stable set; when the core also satisfies external stability, it is the unique stable set [Jordan, 2006a, Proposition 2.4].[15]

As stable sets are more complex objects than the core, Jordan [2006a] derived results only for the WIP, SIN and CD power functions defined in equations 1 to 3, above. WIP has a unique stable set; a stable set for some instances of SIN is derived; no stable set exists for Cobb-Douglas.

Deriving more general results may benefit from an explicitly algorithmic approach. However, the complexity of stable sets is such that results only exist for special cases. Brandt et al. [2009] considered irreflexive, asymmetric dominance relations on abstract games containing finite sets of allocations. By reducing the problem of whether an allocation belongs to a stable set (assumed to exist) to the Boolean satisfiability problem, which is known to be hard ($\mathcal{NP}$-complete), they proved that the stable set problem is also $\mathcal{NP}$-complete, requiring exponentially increasing resources (in n and $\|\mathcal{X}\|$) to solve. Thus, when the cardinality of $\mathcal{X}$ is infinite, as here, the approach of Brandt et al. [2009] does not provide a general algorithm. On the question of whether a stable set even exists, Deng and Papadimitriou [1994] warned that the question "is not even known to be decidable" as it "is not obvious at all that there is *any* algorithm (however slow) for deciding this!"

The existing algorithm for constructing stable sets in pillage games owes to Roth [1976] via Jordan and Obadia [2004].[16] Before introducing it, define a weaker version of condition ES, namely *self-protection*:

$$\mathcal{S} \subseteq U^2(\mathcal{S}); \tag{SP}$$

where $U^2(\mathcal{S}) \equiv U(U(\mathcal{S}))$. Whereas external stability requires that all allocations outside the set in question be dominated by one inside it, self-protection only requires this of those external allocations that dominate allocations within the set.

Algorithm 1 takes as input a non-empty set satisfying conditions IS and SP; if the core is non-empty, this is the natural candidate to use. Each iteration of the algorithm generates weakly larger sets $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \ldots$ satisfying both IS and SP. As, by Jordan [2006a], stable sets are finite, the algorithm will terminate after finitely many iterations. If its final iterate also satisfies condition ES, then it is the unique stable set. The algorithm may, however, terminate before this point without finding a stable set.[17]

The algorithm is therefore incomplete in four respects. First, it provides no means for calculating the core. Second, when the core is empty, it provides no

[15] Asilis and Kahn [1992] expressed this result in terms of *transparency*.

[16] See also Asilis and Kahn [1992] and Jordan [2006b, Proposition 3.9].

[17] If $\mathcal{S}_0 = \emptyset$, which trivially satisfies IS and SP, then the algorithm will terminate immediately.

Algorithm 1 The Roth-Jordan algorithm

$\mathcal{S}_0 \equiv \mathcal{K}$
$i = 1$
repeat
  $\mathcal{S}_i \equiv U^2(\mathcal{S}_{i-1})$
  $i = i + 1$
until $\mathcal{S}_i = \mathcal{S}_{i-1}$
if $\mathcal{S}_i$ satisfies ES **then**
  $\mathcal{S} = \mathcal{S}^i$
end if

clue for finding an internally stable and self-protecting initial iterate, $\mathcal{S}_0$. Indeed, for some $n > 2$ pillage games the only such sets are themselves stable [Jordan and Obadia, 2004], in which case the algorithm cannot help find them. Third, it does not specify an efficient way to compute $U^2(\mathcal{S}_{i-1})$. Finally, it is not clear whether terminating at an $\mathcal{S}_i$ which is not externally stable means that no stable set exists, or merely that further steps must be taken independently of the algorithm.

Thus, the Roth-Jordan algorithm leaves open the Deng and Papadimitriou [1994] question about the decidability of stable set problems, even in the context of pillage games.[18] In Examples 2 and 9, below, the algorithm makes no progress beyond $\mathcal{S}_0$; in the first case, no stable set exists; in the second case, one does. In Example 3, it terminates at $\mathcal{S}_2$, without finding the unique stable set.

The next subsection proves that all anonymous, $n = 3$ pillage games with an empty core represent the same dominance relation. The subsequent subsection then proves that all such games with a non-empty core, also satisfying the CX and RE axioms, have either no stable set or a unique one containing no more than 15 elements. It also presents a new algorithm that both decides the stable set question for the class of pillage games considered, and allows computation of their stable sets.

4.1 Empty core

When the core is empty, the Roth-Jordan algorithm can obviously not use the core as its first iterate. This subsection demonstrates that these cases do not actually pose a problem: any anonymous three agent pillage game with an empty core yields a unique stable set, $\mathcal{S} = \mathcal{D}_1 \backslash \mathcal{D}_0$. The argument proceeds in two steps: an example of the preceding is found; Theorem 3 then shows that all other anony-

[18] A *decision problem* may be answered by a 'yes' or a 'no'; an algorithm *decides* a decision problem if, for every instance of the problem, it returns 'yes' if the input represents a 'yes'-instance, and a 'no' otherwise [Kellerer et al., 2004].

mous three agent pillage games with an empty core represent the same dominance ordering as that in the example. Intuitively, the empty core leaves $D\left(\boldsymbol{s}^{ij}\right)$ defined only by the coalitional membership inequalities $x_i, x_j < \frac{1}{2}$, which do not distinguish between power functions.

Our example shall be the SIN power function defined in equation 2. Jordan [2006a, Proposition 4.2] proved a general result which, in the $n = 3$ case, implies that: when $v > 1$, so that the core is empty, $\mathcal{D}_1 \backslash \mathcal{D}_0 = \left\{\boldsymbol{s}^{12}, \boldsymbol{s}^{13}, \boldsymbol{s}^{23}\right\}$ is stable. This may be confirmed by deriving the dominion of a split allocation,

$$D\left(\boldsymbol{s}^{ij}\right) = \left\{\boldsymbol{x} \in \mathcal{X} \left| \left(\frac{1}{2} > x_i, x_j; x_i + x_j > \frac{1-v}{2}\right) \cup \left(\frac{1}{2} = x_i > x_j > x_k\right) \cup \left(\frac{1}{2} = x_j > x_i > x_k\right)\right.\right\},$$

as illustrated in Figure 1.

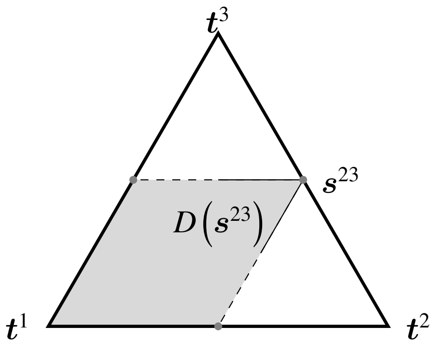

Figure 1: SIN when $v > 1$: $\mathcal{S} = \mathcal{D}_1 \backslash \mathcal{D}_0$

Theorem 3 (SIN equivalence)**.** *When $n = 3$, all power functions satisfying axiom AN and yielding an empty core represent the same dominance ordering.*

Proof. Fix an $\boldsymbol{x}$ and a $\boldsymbol{y}$, in turn fixing W and L. By axiom AN and Lemma 2, these power functions do not discriminate between dominance orderings when the induced W and L are singletons.

For the non-singleton W and L, make use of $\mathcal{K} = \emptyset$. First consider $W = \{1\}$ and $L = \{2, 3\}$. There are no $\boldsymbol{y}$ and $\boldsymbol{x}$ for which $\boldsymbol{y} \succ \boldsymbol{x}$ as

$$\pi(\{1\}, \boldsymbol{x}) \leq \pi\left(\{1\}, \boldsymbol{t}^1\right) < \pi\left(\{2, 3\}, \boldsymbol{t}^1\right) \leq \pi(\{2, 3\}, \boldsymbol{x})$$

for all $\boldsymbol{x} \in \mathcal{X}$, with the strict inequality following from $\mathcal{K} = \emptyset$. Reading the inequalities above from right to left yields the case of $W = \{2, 3\}$ and $L = \{1\}$. This always sets $\boldsymbol{y} \succ \boldsymbol{x}$ without further conditions, just as for SIN. $\square$

Corollary 1. *When $n = 3$, all power functions satisfying axiom AN and yielding an empty core have $\mathcal{D}_1 \backslash \mathcal{D}_0$ as their unique stable set.*

Proof. Theorem 3.4 of Jordan and Obadia [2004] proved that $\mathcal{D}_1 \backslash \mathcal{D}_0$ is the unique stable set of SIN when $v > 1$ and $n = 3$. As this pillage game has an empty core, the result follows from the theorem. $\square$

4.2 Non-empty core

When the core is non-empty, $D\left(\boldsymbol{s}^{jk}\right)$ will be defined not just by the coalitional membership inequalities, but by particular features of the power function. Thus, unlike the previous case, power functions admitting non-empty cores cannot generally be expected to represent the same dominance orderings, or even to yield the same stable sets. In this subsection, we show that all anonymous $n = 3$ pillage games with non-empty cores whose power functions also satisfy the continuity and responsiveness axioms either have no stable set, or have a unique stable set. In the former case, we isolate the source of non-existence; analysis of both cases yields an algorithm for computing as well as deciding the stable set question in these games.

First define the locus of allocations lying between $D\left(\boldsymbol{t}^i\right)$ and $D\left(\boldsymbol{s}^{jk}\right)$:

$$B^i \equiv \{\boldsymbol{x} \in X \,|\pi(\{i\}, \boldsymbol{x}) = \pi(\{j,k\}, \boldsymbol{x})\}.$$

Note that B^i may be empty. When it is non-empty, a number of objects are often of interest. The first are the $\boldsymbol{q}^i \in B^i$, the midpoint of B^i, so that $q^i_j = q^i_k$. The second is the set of maximal elements from agent i's point of view:

$$M^i \equiv \left\{\boldsymbol{x} \in B^i \,\middle|\, x_i = \max_{\boldsymbol{y} \in B^i} y_i\right\}$$

with elements $\boldsymbol{m}^i \in M^i$. Finally, there are the elements of the maximal set most favourable to each of the other two agents,

$$R^i \equiv \left\{\boldsymbol{r}^i_j, \boldsymbol{r}^i_k \in M^i \,\middle|\, r^i_{jj} \geq x_j, r^i_{kk} \geq y_k \forall \boldsymbol{x}, \boldsymbol{y} \in M^i\right\};$$

where r^i_{jj} is the j^{th} coordinate of $\boldsymbol{r}^i_j$. Let $\mathcal{M} \equiv \left\{M^i\right\}_{i=1}^3$, $\mathcal{R} \equiv \left\{R^i\right\}_{i=1}^3$ and $\mathcal{Q} = \left\{\boldsymbol{q}^i\right\}_{i=1}^3$.

Clearly, when $B^i = \emptyset$, M^i and R^i are also empty, and $\boldsymbol{q}^i$ does not exist. These objects may also be empty or fail to exist if B^i is open in the right way, as will be demonstrated in Examples 8 and 9. Finally, it may also be the case that $\boldsymbol{q}^i = \boldsymbol{r}^i_j = \boldsymbol{r}^i_k$, or that $\boldsymbol{r}^i_j = \boldsymbol{s}^{ij}$.

Before proceeding, we establish a number of conditions satisfied along the B^i, when they exist.

Lemma 6. *When $n = 3$, all power functions satisfying axiom AN yield*

$$B^i \subset \left\{\boldsymbol{x} \in X \,\middle|\, x_i \geq \max\left\{x_j, x_k\right\}\right\}.$$

Proof. Suppose otherwise, so that $x_i < x_j$ for some $\boldsymbol{x} \in B^i$. Then, by axioms SR and AN, $\pi(\{i\}, \boldsymbol{x}) < \pi(\{j\}, \boldsymbol{x})$ which, by axiom WC, is weakly less than $\pi(\{j,k\}, \boldsymbol{x})$. Thus, $\boldsymbol{x} \notin B^i$. $\square$

The lemma's bound on x_i is tight, as exemplified by the champion power function of equation 4, illustrated in Figure 9.

Lemma 7. *$\forall \boldsymbol{x}, \boldsymbol{y} \in B^i$, if $y_i > x_i$ then either $y_j > x_j$ or $y_k > x_k$ for distinct i, j and k.*

Proof. Suppose otherwise, so that $y_j \leq x_j$ and $y_k \leq x_k$. Then, by definition of B^i and axioms WR and SR:

$$\pi(\{i\}, \boldsymbol{y}) > \pi(\{i\}, \boldsymbol{x}) = \pi(\{j,k\}, \boldsymbol{x}) \geq \pi(\{j,k\}, \boldsymbol{y});$$

contradicting $\boldsymbol{y} \in B^i$. $\square$

Corollary 2. *When $n = 3$, a ray from $\mathbf{t}^i$ through the simplex cuts B^i at most once.*

Proof. If $\boldsymbol{x}$ and $\boldsymbol{y}$ lie on such a ray then $y_i > x_i$ implies $y_j < x_j$ and $y_k < x_k$, contradicting Lemma 7. $\square$

For the next lemma, define $B^i_+ \equiv \left\{\boldsymbol{x} \in B^i \middle| x_i > \max\left\{x_j, x_k\right\}\right\}$, the subset of B^i such that agent i has strictly more power than either of j or k, alone. (In the champion power function, $B^i_+ = \emptyset$; in SIN with $v \in (0,1)$, $B^i_+ = B^i$.) We may now state conditions under which dominance follows from inequality:

Lemma 8. *When $n = 3$ and π satisfies axiom AN, if either*

1. *$\boldsymbol{x}, \boldsymbol{y} \in B^i$ are such that $y_i > x_i \geq x_j > x_k$ and $y_j > x_j$; or*
2. *$\boldsymbol{x} \in B^i_+$ and $\boldsymbol{y} \in B^i$ are such that $y_i > x_i$;*

then $\boldsymbol{y} \in\!\!- \boldsymbol{x}$.

Proof. In the first case, $W \equiv \{l | y_l > x_l\} = \{i, j\}$ and $L \equiv \{l | x_l > y_l\} = \{k\}$ by assumption. In the second case, the same W and L can be assumed without loss of generality by Lemma 7.

In the first case, $x_i > x_k$ is assumed; in the second, it follows from $\boldsymbol{x} \in B^i_+$. In both cases, axioms AN and SR then ensure that $\pi(\{i\}, \boldsymbol{x}) > \pi(\{k\}, \boldsymbol{x})$; by axiom WC, it follows that $\pi(W, \boldsymbol{x}) > \pi(L, \boldsymbol{x})$, hence $\boldsymbol{y} \in\!\!- \boldsymbol{x}$. $\square$

When axiom RE holds, $B^i \backslash B^i_+$ may disappear:

Lemma 9. *When $n = 3$ and π satisfies axioms AN and RE, $B^i \backslash B^i_+ \subseteq \mathcal{D}_1 \backslash \mathcal{D}_0$.*

Proof. First consider the $\boldsymbol{x} > \mathbf{0}$ in B^i. By axiom SR, $\pi(\{i\}, \boldsymbol{x}), \pi(\{j\}, \boldsymbol{x}), \pi(\{k\}, \boldsymbol{x}) > 0$ so that, by axiom RE, $\pi(\{j,k\}, \boldsymbol{x}) > \max\{\pi(\{j\}, \boldsymbol{x}), \pi(\{k\}, \boldsymbol{x})\}$. If $\boldsymbol{x} \in B^i \backslash B^i_+$ then $x_i = \max\left\{x_j, x_k\right\}$ so that, by axiom AN, $\pi(\{i\}, \boldsymbol{x}) = \max\{\pi(\{j\}, \boldsymbol{x}), \pi(\{k\}, \boldsymbol{x})\} < \pi(\{j,k\}, \boldsymbol{x})$, so that $\boldsymbol{x} \notin B^i$.

Now consider the $\boldsymbol{x} \in B^i$ that set at least one component to 0. This can't be x_i as axioms AN and WC would imply $\pi(\{i\}, \boldsymbol{x}) < \pi(\{j,k\}, \boldsymbol{x})$. Without loss of generality, then, let $x_k = 0$ so that $\pi(\{i\}, \boldsymbol{x}) = \pi(\{j,k\}, \boldsymbol{x})$. Axiom AN then requires that $x_i = x_j = \frac{1}{2}$. ☐

Lemma 10. *When $n = 3$ and π satisfies axioms AN and RE, if $R^i \neq \emptyset$ and $\boldsymbol{x} > \mathbf{0}$ lies in $B^i \backslash \left(R^i \cup \left\{\boldsymbol{q}^i\right\}\right)$, then $\boldsymbol{x} \in D\left(R^i\right)$.*

Proof. First consider the case in which $\boldsymbol{x}$ lies in $M^i \backslash \left(R^i \cup \left\{\boldsymbol{q}^i\right\}\right)$, so that agent i is indifferent between allocation $\boldsymbol{x}$ and those in R^i. As $\boldsymbol{x} \neq \boldsymbol{q}^i$ let, without loss of generality, $x_j > x_k$, so that, by axiom AN, $\pi(\{j\}, \boldsymbol{x}) > \pi(\{k\}, \boldsymbol{x})$ so that $\boldsymbol{r}^i_j \succ \boldsymbol{x}$, establishing the result.

Now let $\boldsymbol{x}$ lie in $B^i \backslash \left(M^i \cup \left\{\boldsymbol{q}^i\right\}\right)$. First consider the subcase in which $\boldsymbol{x} \in B^i_+$. The set of agents preferring $\boldsymbol{r}^i_j$ to $\boldsymbol{x}$ therefore includes i (by definition of M^i) and at least one other (by Lemma 7). As $\boldsymbol{x} \in B^i_+$, $x_i > \max\left\{x_j, x_k\right\}$ so that $\pi(\{i\}, \boldsymbol{x}) > \pi\left(\left\{\arg\max\left\{x_j, x_k\right\}\right\}, \boldsymbol{x}\right)$. By axiom WC, $\boldsymbol{r}^i_j \succ \boldsymbol{x}$.

Finally, let $\boldsymbol{x} \in B^i \backslash \left(B^i_+ \cup M^i \cup \left\{\boldsymbol{q}^i\right\}\right)$ so that, without loss of generality, $r^i_{ji} > x_i = x_j > x_k > 0$. Then the set of agents preferring $\boldsymbol{r}^i_j$ to $\boldsymbol{x}$ includes i and at least one other; in the worst case, this is k. As $x_k > 0$ and axiom RE holds, $\pi(\{i,k\}, \boldsymbol{x}) > \pi(\{i\}, \boldsymbol{x}) = \pi(\{j\}, \boldsymbol{x})$ so that $\boldsymbol{r}^i_j \succ \boldsymbol{x}$. ☐

The assumption that $R^i \neq \emptyset$ is made in addition to the axiomatic assumptions as no conditions have been presented as yet for ruling out B^i behaving like the topologist's sine curve, $y = \sin\frac{1}{x}$.

A set of allocations $\mathcal{K}^i \subseteq \mathcal{Y} \subseteq \mathcal{X}$ that satisfies $\mathcal{K}^i = \mathcal{Y} \backslash D(\mathcal{Y})$ is the *core on* $\mathcal{Y}$. Similarly, a set of allocations $\mathcal{S}^i \subseteq \mathcal{Y} \subseteq \mathcal{X}$ satisfying $\mathcal{S}^i \cap D\left(\mathcal{S}^i\right) = \emptyset$ and $\mathcal{S}^i \cup D\left(\mathcal{S}^i\right) = \mathcal{Y}$, is *stable on* $\mathcal{Y}$.[19]

Theorem 4. *Let $n = 3$, and π satisfy axioms AN and RE. When $R^i \neq \emptyset$, the unique stable set to exist on $\left(B^i_+ \cup R^i\right)$ is $\mathcal{S}^i \equiv R^i \cup \left(\left\{\boldsymbol{q}^i\right\} \cap M^i\right)$; otherwise, if $R^i = \emptyset$, there is no stable set on B^i_+.*

Proof. Consider the restriction of $\left(B^i_+ \cup R^i\right)$ to, without loss of generality, those allocations, $\boldsymbol{x}$, for which $x_j > x_k$. When $R^i \neq \emptyset$, Lemmas 8 and 10 together establish that dominance is a complete ordering on this restricted domain. Thus,

[19] Gillies [1959] called this $\mathcal{Y}$ − *stable* while Thrall and Lucas [1963] called this stable for $\mathcal{Y}$.

if $\boldsymbol{r}_j^i \neq \boldsymbol{q}^i$, it is the maximal element of the domain, and therefore the unique stable set on it; otherwise, there is no stable set on it [von Neumann and Morgenstern, 1944, §65.4.2].

When $R^i = \left\{\boldsymbol{q}^i\right\}$, the preceding argument applies to $\left(B_+^i \cup R^i\right)$ such that $x_j \geq x_k$. Otherwise, consider the restriction of $\left(B_+^i \cup R^i\right)$ to $x_j = x_k$, the singleton, $\boldsymbol{q}^i$. When $\boldsymbol{q}^i$ exists, it is trivially the unique stable set on itself.

Finally, as neither $\boldsymbol{r}_j^i \succ \boldsymbol{r}_k^i$ nor the reverse, both must belong to any stable set on $\left(B_+^i \cup R^i\right)$; $\boldsymbol{q}^i$ will also belong to any stable set on $\left(B_+^i \cup R^i\right)$ iff $\boldsymbol{q}^i \in M^i$ as it will, otherwise, belong to $D\left(R^i\right)$. □

The proof may be compared to that in Jordan [2006a, Proposition 4.6], which established non-existence of a stable set for the Cobb-Douglas power function, π_c, and the SIN power function when $v \in (0, 1)$.[20] In that case, the proof demonstrated that any stable set that might exist was necessarily infinite, a contradiction; the proof here establishes a complete ordering over $B_+^i \cup R^i$.

The next lemma will be used to prove Theorem 5, which presents conditions under which the $\mathcal{S}^i$ must belong to any stable set:

Lemma 11. *Let $\mathcal{S}^i$ be the unique stable set on $\left(B_+^i \cup R^i\right)$. If $\mathcal{K} \neq \emptyset, \boldsymbol{x} \in \mathcal{S}^i, \pi$ satisfies axiom CX and $\boldsymbol{y} \succ \boldsymbol{x}$ then $\boldsymbol{y} \in D\left(\boldsymbol{t}^i\right)$.*

Proof. As $\boldsymbol{x} \in \mathcal{S}^i$ it is also in B^i. If $\boldsymbol{y} \succ \boldsymbol{x}$ then one of the following three disjoint cases applies:

1. $\boldsymbol{y}$ sets $y_i > x_i$ so that, by axiom SR, $\pi(\{i\}, \boldsymbol{y}) > \pi(\{i\}, \boldsymbol{x})$. As $\boldsymbol{x} \in \mathcal{S}^i$ it is also in M^i. Therefore, it cannot be that $\pi(\{i\}, \boldsymbol{y}) = \pi(\{j, k\}, \boldsymbol{y})$ as then $\boldsymbol{y} \in B^i$ with $y_i > x_i$, so that $\boldsymbol{x} \notin M^i$, a contradiction. If $\pi(\{i\}, \boldsymbol{y}) > \pi(\{j, k\}, \boldsymbol{y})$ then $\boldsymbol{y} \in D\left(\boldsymbol{t}^i\right)$, the desired result. Finally, suppose that $\pi(\{i\}, \boldsymbol{y}) < \pi(\{j, k\}, \boldsymbol{y})$; then, as $\mathcal{K} \neq \emptyset$, $\pi\left(\{i\}, \boldsymbol{t}^i\right) \geq \pi\left(\{j, k\}, \boldsymbol{t}^i\right)$ so that axioms SR and CX guarantee that there exists a $\boldsymbol{z}$ such that $1 \leq z_i > y_i$ and $\pi(\{i\}, \boldsymbol{z}) = \pi(\{j, k\}, \boldsymbol{z})$, again contradicting $\boldsymbol{x} \in M^i$.

2. $\boldsymbol{y}$ sets $y_i = x_i$ and, permuting indices j and k if necessary, $y_j > x_j > x_k > y_k$. (Without $x_j > x_k$, $\boldsymbol{y}$ would not dominate $\boldsymbol{x}$ under axiom AN.) As this precludes $\boldsymbol{x} = \boldsymbol{q}^i$, it follows from Theorem 4 that $\boldsymbol{x} = \boldsymbol{r}_j^i$. It cannot be that $\pi(\{i\}, \boldsymbol{y}) = \pi(\{j, k\}, \boldsymbol{y})$ as then $\boldsymbol{y}$ is an element of M^i with $y_j > x_j$, contradicting the assumption that $\boldsymbol{x} = \boldsymbol{r}_j^i$. Finally, $\pi(\{i\}, \boldsymbol{y}) < \pi(\{j, k\}, \boldsymbol{y})$ is ruled out by implying the existence of the same $\boldsymbol{z}$ that yielded the contradiction in the previous case.

[20]There, the allocations indexed by λ define a generalised version of B^i. The present lemma is obviously more restrictive insofar as it only applies to the $n = 3$ case, but more general in applying to a broader class of power functions.

3. $\boldsymbol{y}$ sets $y_i < x_i$. As $\boldsymbol{x} \in B^i, \pi(\{i\}, \boldsymbol{x}) = \pi(\{j,k\}, \boldsymbol{x})$, so that $\boldsymbol{y} \not\succ \boldsymbol{x}$.

□

The following example shows that, when axiom CX is violated, a $\boldsymbol{y} \notin D\left(\boldsymbol{t}^i\right)$ may be found to dominate $\boldsymbol{x} \in B^i$:

Example 1. *Consider a discontinuous version of SIN, so that*

$$\pi(C, x) \equiv \sum_{i \in C} \left\{ \begin{array}{ll} \frac{1}{2}x_i + v & \text{if } x_i \leq \frac{1-v}{2} \\ x_i + v & \text{otherwise} \end{array} \right\}; \tag{6}$$

where $v \in \left(\frac{1}{7}, 1\right)$. Feasibility of $\boldsymbol{x}$ requires that $x_i > \frac{1-v}{2} \geq x_j, x_k$:

$$B^i = \left\{ \boldsymbol{x} \in \mathcal{X} \middle| x_i = \frac{1+2v}{3}, \max\left\{x_j, x_k\right\} \leq \frac{1-v}{2} \right\}.$$

Thus, depicted in the simplex, B^i is a line that stops short of the edges. As x_i remains constant over B^i, $M^i = B^i$; $\boldsymbol{q}^i$ sets $q^i_i = \frac{1+2v}{3}$ and $q^i_j = q^i_k = \frac{1-v}{3}$; finally, $R^i = \left\{\boldsymbol{r}^i_j, \boldsymbol{r}^i_k\right\}$, where $\boldsymbol{r}^i_j$ sets $r^i_{ji} = \frac{1+2v}{3}$, $r^i_{jj} = \frac{1-v}{2}$ and $r^i_{jk} = \frac{1-v}{6}$. There is therefore a $\boldsymbol{y} \in D\left(\boldsymbol{s}^{jk}\right)$ with $y_i = r^i_{ji}$ and $y_j > r^i_{jj}$ such that $\boldsymbol{y} \succ \boldsymbol{x}$ for any $\boldsymbol{x} \in B^i$ with $x_i = r^i_{ji}$ and $x_j > x_k$.

While the lemma therefore does not apply to this example, $\mathcal{D}_1 \cup \mathcal{R} \cup \mathcal{Q}$ is, nonetheless, stable.

Finally, then:

Theorem 5. *If $\mathcal{K} \neq \emptyset$, π satisfies axioms AN, CX and RE, and $R^i \neq \emptyset$, then any stable set, $\mathcal{S}$, must contain $\mathcal{S}^i$, as defined in Theorem 4.*

Proof. As $R^i \subseteq B^i$, the assumed conditions ensure that B^i exists. By Theorem 4, $\mathcal{S}^i$ is the unique stable set on it. Finally, as $\boldsymbol{t}^i$ must belong to $\mathcal{S}$ (as $\mathcal{D}_0 \subseteq \mathcal{K} \subseteq \mathcal{S}$) so that no allocation in $D\left(\boldsymbol{t}^i\right)$ can belong to $\mathcal{S}$. Lemma 11 then ensures that including $\mathcal{S}^i$ is necessary for external stability. This establishes the result. □

The preceding theorem and the existing result that core allocations must belong to a stable set now force two types of allocations into any stable set that exists. With the next theorem, a third and final type is forced in. This shall establish our uniqueness result. We begin with a special case that introduces the required equivalence.

Theorem 6. *For the anonymous, $n = 3$ SIN pillage game with $v = 1$, the unique stable set is $\mathcal{S} = \mathcal{D}_1$.*

Proof. It is immediate that $\mathcal{K} = \mathcal{D}_0$ so that $U(\mathcal{K}) = \{\boldsymbol{x} \in \mathcal{X} \,|\, \boldsymbol{x} > \mathbf{0} \cup \boldsymbol{x} \in \mathcal{D}_1 \backslash \mathcal{D}_0\}$. Adding the $\mathcal{D}_1 \backslash \mathcal{D}_0$ to the core therefore produces a stable set, $\mathcal{S} = \mathcal{D}_1$.

To establish uniqueness, recall that, for $v > 1$, Jordan and Obadia [2004, Theorem 3.4] prove that $\mathcal{D}_1 \backslash \mathcal{D}_0$ is the unique stable set on $\mathcal{X}$. Two steps allow direct application of this result. First, when $v > 1$, dominance on $\mathcal{X}$ is equivalent to dominance on $U(\mathcal{D}_0)$ when $v = 1$: for any $\boldsymbol{y}, \boldsymbol{x}, W \equiv \{i \,|y_i > x_i\}$ and $L\{i \,|y_i < x_i\}, \boldsymbol{y} \succ \boldsymbol{x}$ iff either ($\|W\| = 2 > \|L\| = 1$) or, if $\|W\| = \|L\| = 1$ such that the W member is granted a larger endowment in $\boldsymbol{x}$ than is the L member.

Second, it must be ensured that the geometry of $U(\mathcal{D}_0)$ allows the arguments used by Jordan and Obadia [2004] on $\mathcal{X}$. This is ensured by replacing $\left(\frac{3}{4}, 0, \frac{1}{4}\right)$ with $\left(\frac{3}{4}, \varepsilon, \frac{1}{4} - \varepsilon\right)$ for small $\varepsilon > 0$. Thus, $\mathcal{D}_1 \backslash \mathcal{D}_0$ is the unique candidate for addition to $\mathcal{K} = \mathcal{D}_0$ when $v = 1$; as $\mathcal{D}_1$ is stable, the result follows. □

In this degenerate case, $\left\{\boldsymbol{q}^i\right\} = \left\{\boldsymbol{t}^i\right\} = R^i = M^i = B^i$, a singleton. The Roth-Jordan algorithm fails to make progress beyond $\mathcal{S}_0 = \mathcal{D}_0$, setting $U^2(\mathcal{S}_0) = \mathcal{S}_0$.

To establish our final result, define

$$\mathcal{X}_- \equiv \left\{\boldsymbol{x} \in \mathcal{X} \,\middle|\, x_1 > r^2_{11}, x_2 > r^3_{22}, x_3 > r^1_{33}\right\};$$

and $\boldsymbol{p}^{jk} \equiv \frac{1}{2}\left(\boldsymbol{r}^j_i + \boldsymbol{r}^k_i\right)$, so that $\mathcal{P} \equiv \left\{\boldsymbol{p}^{12}, \boldsymbol{p}^{13}, \boldsymbol{p}^{23}\right\}$. Allocations in the restricted domain, $\mathcal{X}_-$, are undominated by those in R^i; they form a contraction of the original domain, $\mathcal{X}$. The $\boldsymbol{p}^{jk}$ are the midpoints of the edges of the contracted simplices; they are also undominated by allocations in R^i. When the $\boldsymbol{p}^{jk} = \bar{\boldsymbol{p}}$, the centroid, $\mathcal{P}$ is a degenerate singleton. As the WIP power function shows, $\mathcal{P}$ may be empty (q.v. the left panel of Figure 4).

Theorem 7. *When $n = 3$, π satisfies axioms AN, CX and RE, $\mathcal{K} \neq \emptyset$ and $R^i \neq \emptyset$, then there exists a unique stable set, $\mathcal{S} = \mathcal{K} \cup \left\{\mathcal{S}^i\right\}_{i=1}^3 \cup \mathcal{P}$, where $\mathcal{S}^i$ is defined in Theorem 4.*

Proof. The set of allocations undominated by $\mathcal{K} \cup \left\{\mathcal{S}^i\right\}_{i=1}^3$ is $\mathcal{X}_- \cup \mathcal{P}$. By the same argument as in the proof of Theorem 6, Jordan and Obadia [2004, Theorem 3.4] establishes that $\mathcal{P}$ is the unique stable set on $\mathcal{X}_- \cup \mathcal{P}$. The result then follows from now familiar arguments: any stable set must contain $\mathcal{D}_0 \subseteq \mathcal{K}$, which excludes $D\left(\boldsymbol{t}^i\right)$, forcing, in turn, the inclusion of the $\mathcal{S}^i$ (by Theorem 5); finally, this leaves $\mathcal{X}_- \cup \mathcal{P}$, which has just been addressed. □

When $\mathcal{P}$ is empty, the unique $\mathcal{S}$ may therefore be computed in two, rather than three steps.

Corollary 3. *When $n = 3$, and π satisfies axiom AN, CX and RE, if a stable set, $\mathcal{S}$, exists, then $\|\mathcal{S}\| \leq 15$.*

Proof. When the core is empty, $\|\mathcal{S}\| = 3$. Otherwise, the three tyrannical allocations belong to the core, and therefore to $\mathcal{S}$. Each $\mathcal{S}^i$ contributes up to three more. Finally, when $\mathcal{P}$ is non-empty, it adds either one (if degenerate) or three (otherwise) more allocations. □

This bound improves enormously on the finite bound in Jordan [2006a], the Ramsey bound in Kerber and Rowat [forthcoming], and the doubly exponential bound of Saxton [2010]. The first two of those bounds derived implications of axiom SR alone; the last also took axiom WC into account. Further, the bound is tight, as demonstrated by the following power function:

$$\pi_k(C, \boldsymbol{x}) \equiv \sum_{i \in C} \left[x_i + \frac{\sin(k\varpi x_i)}{k\varpi} \right]; \tag{7}$$

for $k \in \mathbb{Z}$, where ϖ is the constant pi.[21] When $k = 9$ all allocations enumerated by the Corollary are present.

To conclude this section, Algorithm 2 presents an alternative approach to generating stable sets in $n = 3$ pillage games.[22] Like Roth-Jordan, the algorithm adds elements to a small, undominated set in a number of steps. Further, all sets generated are self-protected, and generated by the $U^2(\cdot)$ operation: if $\mathcal{S}$ exists, then $\mathcal{S}_1 = U^2(\mathcal{S}_0)$ and $\mathcal{S}$, the final iterate, is equal to $U^2(\mathcal{S}_1)$. Our algorithm differs from Roth-Jordan in being able to add core elements at all three steps by distinguishing between core allocations at which one agent is strictly more powerful than the other two, and those at which the first agent is just as powerful as the others.[23] While their $\mathcal{S}_0 = \mathcal{K}$, ours is $\mathcal{S}_0 = \mathcal{D}_0$.[24]

The algorithm therefore both decides the stable set question for an $n = 3$ pillage game satisfying additional axioms AN, CX and RE, and computes its unique stable set. As such, it overcomes three of the four ways in which the Roth-Jordan algorithm is incomplete, albeit over a narrow class of pillage games: the first two weaknesses are addressed as the new algorithm starts with $\mathcal{D}_0$, not the core and, thus, never needs to compute the core; and the last shortcoming is dealt with since the algorithm always either returns that no stable set exists or returns the unique stable set. The Roth-Jordan algorithm's third problem, that of efficiently computing the interim sets, remains but reduced in complexity as the U^2 operator is replaced by easier calculations.

The above procedure provides guidance on handling the $n > 3$ case as well: there may be a $D\left(\boldsymbol{t}^i\right)$ region around each $\boldsymbol{t}^i$, separated from the rest of X by a

[21] We use ϖ to denote the constant pi $\approx 3.14\ldots$ to avoid confusion with the power function.

[22] The pseudo-code is presented for clarity rather than coding efficiency.

[23] q.v. the champion power function, displayed in Figure 9.

[24] Thus, Roth-Jordan appears "greedy", adding more elements in its first iteration; this is irrelevant, though, as that presented here eventually adds all core allocations.

Algorithm 2 The stable set in $n = 3$ pillage games satisfying axioms AN, CX and RE

1: **if** $\pi\left(\{i\}, \boldsymbol{t}^i\right) \geq \pi\left(\{j,k\}, \boldsymbol{t}^i\right)$ **then**
2: $\quad \mathcal{S}_0 = \mathcal{D}_0$
3: $\quad$ **if** $M^i = \emptyset$ **then**
4: $\quad\quad$ **return** "no stable set exists"
5: $\quad$ **else**
6: $\quad\quad \mathcal{S}_1 = U^2(\mathcal{S}_0) = \mathcal{S}_0 \cup \left\{\mathcal{S}^i\right\}_{i=1}^3$
7: $\quad\quad$ **if** $\mathcal{S}_1 \cup D(\mathcal{S}_1) \neq \mathcal{X}$ **then**
8: $\quad\quad\quad \mathcal{S}_2 = U^2(\mathcal{S}_1) = \mathcal{S}_1 \cup \mathcal{P}$
9: $\quad\quad$ **else**
10: $\quad\quad\quad \mathcal{S} = \mathcal{S}_1$
11: $\quad\quad$ **end if**
12: $\quad$ **end if**
13: **else**
14: $\quad \mathcal{S} = \mathcal{D}_1$
15: **end if**
16: **return** $\mathcal{S}$

B^i. Analysis of $D\left(\boldsymbol{t}^i\right)$ and B^i likely proceeds as in the $n = 3$ case. On the other side of the B^i, the game will again be equivalent to the SIN game with $v > 1$, for which few results are known [Jordan and Obadia, 2004]. Thus, progress in analysing $n > 3$ pillage games depends, in general, on progress in analysing these SIN games.

5 Examples

In this section we first illustrate the theory presented above in cases satisfying axioms AN, CX and RE. We then derive results for cases violating axioms CX and RE.

We first present some additional axioms that may or may not be satisfied by particular power functions:

(C0) if $x_i = 0 \forall i \in C$ then $\pi(C, \boldsymbol{x}) = 0$.

(CN) for all partitions on I, $\left\{C^1, \ldots, C^M\right\}$, $\sum_{m=1}^M \pi(C^m, \boldsymbol{x})$ is constant.

(AD) $\pi(C, \boldsymbol{x}) = \sum_{i \in C} \pi(\{i\}, \boldsymbol{x})$.

Axiom CN is a conservatism axiom; the additivity axiom, AD, is stronger than axiom RE.

5.1 Jordan examples and variants

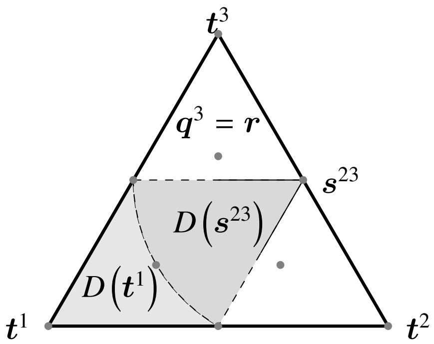

Figure 2: Strictly concave power function, $\pi(C, \boldsymbol{x}) = \sum_{i \in C} \sqrt{x_i}$

When π satisfies axioms AN, CX and RE and $\left\{\boldsymbol{q}^i\right\} = R^i = M^i \subset B^i$ – a non-degenerate locus – then, by Theorem 7, $\mathcal{S} \equiv \mathcal{D}_1 \cup \mathcal{Q}$. The next theorem shows that any strictly concave π satisfying all additional axioms except CN has this as its unique stable set:

Theorem 8. *When $n = 3$, π satisfies axioms AN, CX, AD and C0, and is strictly concave in $x_i \forall i \in C$, $\mathcal{S} \equiv \mathcal{D}_1 \cup \mathcal{Q}$ is the unique stable set.*

Proof. By Jordan [2006a, Proposition 2.6], the core is the set $\mathcal{D}_1$, which must therefore belong to any stable set that exists. Furthermore, B^i is non-empty. Thus, the result follows if we can prove that $M^i = \left\{\boldsymbol{q}^i\right\}$.

By the additional axioms and the notational abuse allowed by Lemma 1, allocations along B^i may be represented by

$$\pi(x_i) = \pi\left(x_j\right) + \pi\left(1 - x_i - x_j\right);$$

for distinct i, j and k. As π is strictly increasing and concave, it has an inverse, say ρ, that is also strictly increasing. Thus, the allocations in B^i may be represented by $x_i = \rho\left(\pi\left(x_j\right) + \pi\left(1 - x_i - x_j\right)\right)$ so that allocations in M^i solve $\max_{\boldsymbol{x} \in \mathcal{X}} \pi\left(x_j\right) + \pi\left(1 - x_i - x_j\right)$. As π is increasing and strictly concave, the expression's unique maximiser equates the two arguments, yielding $x_j = \frac{1-x_i}{2}$; feasibility then requires $x_k = \frac{1-x_i}{2}$, the proof is complete. □

Figure 2 illustrates an example of this.[25]

Jordan [2006a, Proposition 4.6] proves that no stable set exists for SIN with $\nu \in (0, 1)$. The next example interprets that result in terms of the present theory:

[25] In Jordan [2007], power depends on production, a concave function of wealth.

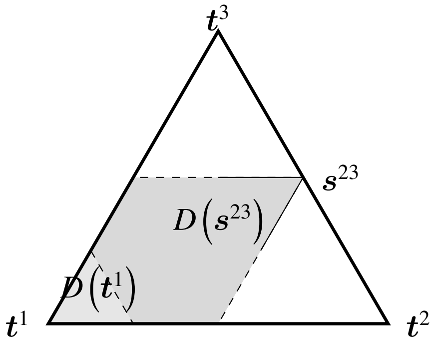

Figure 3: SIN when $v \in (0, 1)$

Example 2. *Under the SIN power function with $v \in (0, 1)$, illustrated in Figure 3, $\mathcal{K} = \mathcal{D}_0$. Further, $M^i = B^i = \left\{\boldsymbol{x} \in \mathcal{X} \,\middle|\, \boldsymbol{x} > \mathbf{0}, x_i = \frac{1+v}{2}\right\}$. Thus, B^i is an open set as $\left(\frac{1+v}{2}, \frac{1-v}{2}, 0\right) \in D\left(\boldsymbol{t}^1\right)$. As R^i is empty, Theorem 4 demonstrates that no stable set exists on B^i_+; by Theorem 5, then, no stable set can exist.*

Applying the Roth-Jordan algorithm, $\mathcal{S}_0 = \mathcal{K} = \mathcal{D}_0, U(\mathcal{S}_0) = \mathcal{D}_1 \cup \mathcal{X} \backslash D(\mathcal{D}_0)$ and $U^2(\mathcal{S}_0) = \mathcal{S}_0$. Thus, the algorithm fails to make progress, indicating – in this case – non-existence of $\mathcal{S}$.

In the next example, the unique stable set is not found by the Roth-Jordan algorithm; furthermore, and unusually, it does not contain the split allocations, $\mathcal{D}_1 \backslash \mathcal{D}_0 = \left\{\boldsymbol{s}^{12}, \boldsymbol{s}^{13}, \boldsymbol{s}^{23}\right\}$.

Example 3. *Let*

$$\pi(C, \boldsymbol{x}) = \sum_{i \in C} \left[\sqrt{x_i} + v\right]$$

where $v \in (0, 1)$. It is easily confirmed that $\mathcal{K} = \mathcal{D}_0$. By Theorem 4, $\mathcal{Q} = \mathcal{M} = \mathcal{R}$ must also be elements of any stable set, where $\boldsymbol{q}^i$ implicitly solves $\sqrt{q^i_i} = \sqrt{2\left(1 - q^i_i\right)} + v$ and $q^i_j = q^i_k$.

Now search for $\mathcal{P} \subseteq U(\mathcal{D}_0 \cup \mathcal{Q})$ such that $\mathcal{P}$ is stable on $(\mathcal{D}_0 \cup \mathcal{Q}) \cup U(\mathcal{D}_0 \cup \mathcal{Q})$ and that $\mathcal{S} \equiv \mathcal{D}_0 \cup \mathcal{Q} \cup \mathcal{P}$ is internally stable. The problem is identical to that addressed by Theorem 6 when $U(\mathcal{D}_0)$ is scaled down to $U(\mathcal{D}_0 \cup \mathcal{Q})$. Letting $\mathcal{P}$ be $\left(\frac{1+q^i_i}{4}, \frac{1+q^i_i}{4}, \frac{1-q^i_i}{2}\right)$ and its permutations, it follows that the unique stable set is $\mathcal{S} = \mathcal{D}_0 \cup \mathcal{Q} \cup \mathcal{P}$. Unusually, the split allocations, $\mathcal{D}_1 \backslash \mathcal{D}_0$ do not belong to $\mathcal{S}$ as they lie within $D(\mathcal{P})$.

Interpreting the above in light of Roth-Jordan, $\mathcal{S}_0 = \mathcal{D}_0$, so that

$$U(\mathcal{S}_0) = \mathcal{D}_0 \cup \{\boldsymbol{x} \in \mathcal{X} \,|\, \pi(\{i\}, \boldsymbol{x}) < \pi(\{j, k\}, \boldsymbol{x})\};$$

and $\mathcal{S}_1 = \mathcal{D}_0 \cup \mathcal{Q}$. But $U(\mathcal{S}_1) = U(\mathcal{D}_0 \cup \mathcal{Q})$ and $\mathcal{S}_2 = \mathcal{S}_1$, so that the algorithm terminates before finding $\mathcal{S}$. By contrast to Example 2, however, a unique stable set exists in this case.

We now turn to cases in which power functions are convex in the x_i. This allows a coalition to increase its power by transferring resources from the poorer to the richer member, even if its aggregate resources are diminished:

Example 4. *Let $\pi(C, \boldsymbol{x}) = \sum_{i \in C} x_i^2$. Then, for all $\varepsilon \in \left(0, \frac{1}{4}\right)$,*

$$\pi\left(\{2,3\}, \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)\right) = \frac{1}{8} < \pi\left(\{2,3\}, \left(\frac{1}{2}, \frac{1}{4} + \varepsilon, \frac{1}{4} - \varepsilon\right)\right) = \frac{1}{8} + 2\varepsilon^2.$$

We may also state a parallel result to Theorem 8 for strictly convex functions:

Theorem 9. *When $n = 3$, π satisfies axioms AN, CX, AD and C0, and is strictly convex in x_i for all i in C, $\mathcal{D}_1$, the core, is the unique stable set.*

Proof. The core is $\mathcal{D}_1$. By Jordan [2006a, Proposition 2.4], any stable set contains $\mathcal{K}$, the core. As

$$B^i \subseteq D\left(\boldsymbol{s}^{ij}\right) \cup D\left(\boldsymbol{s}^{ik}\right) \forall i \in I;$$

there exists no $\boldsymbol{x} \in \mathcal{X} \backslash \mathcal{K}$ such that $\boldsymbol{x} \notin D(\mathcal{K})$. □

Thus, as $\pi(C, \boldsymbol{x})$ varies from strictly concave to strictly convex, the interior (non-core) allocations in the unique stable set distort towards the WIP stable set, and then disappear, leaving just the core allocations.

The right panel of Figure 4 depicts the case when $\pi(C, \boldsymbol{x})$ is strictly convex in the x_i; for sake of reference, the WIP case is depicted in its left panel.

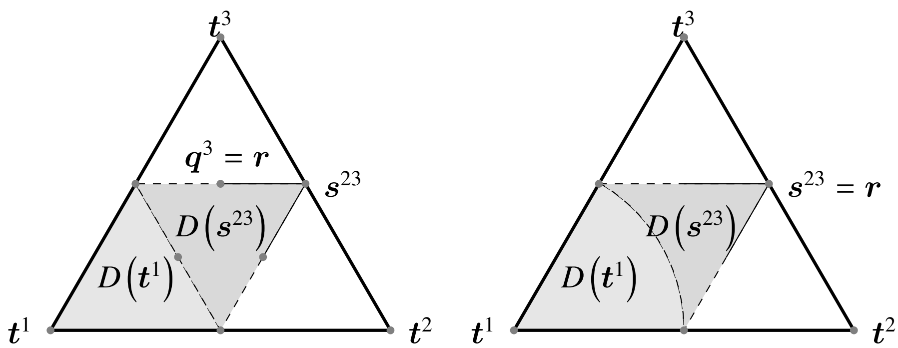

Figure 4: WIP, and strictly convex $\pi(C, \boldsymbol{x})$

Power functions need not be convex or concave over their whole domain to produce stable sets identical to those that are. The next two examples illustrate this. Both use the following lemma:

Lemma 12. *When $n = 3$ and π satisfies axioms AN, C0 and AD then, for distinct i, j and k, $\pi\left(\{i\}, \boldsymbol{s}^{ij}\right) = \pi\left(\{j,k\}, \boldsymbol{s}^{ij}\right)$.*

Proof. By axioms AD and C0 respectively:

$$\pi\left(\{j,k\}, \boldsymbol{s}^{ij}\right) = \pi\left(\{j\}, \boldsymbol{s}^{ij}\right) + \pi\left(\{k\}, \boldsymbol{s}^{ij}\right) = \pi\left(\{j\}, \boldsymbol{s}^{ij}\right).$$

Axiom AN ensures the result. □

Example 5. *Consider*

$$\pi_h(C, \boldsymbol{x}) \equiv \sum_{i \in C} \text{haversin}(\varpi x_i) \equiv \sum_{i \in C} \frac{1 - \cos(\varpi x_i)}{2}.$$

This satisfies all of the additional axioms except CN, conservatism. It is locally convex for all $x_i \in \left(0, \frac{1}{2}\right)$ and locally concave for all $x_i \in \left(\frac{1}{2}, 1\right)$. By Lemma 12 and axiom SR, $x_1 = \frac{1}{2}$ is the maximal allocation for agent 1 satisfying $\pi(\{1\}, \boldsymbol{x}) = \pi(\{2,3\}, \boldsymbol{x})$. Thus, π is convex for all $\boldsymbol{x} \in D\left(\boldsymbol{t}^i\right)$, which is then – as in the right panel of Figure 4 – a convex set. Therefore, as $\mathcal{D}_1$ is the unique stable set in that case, it is in this case as well.

Example 6. *Parametrically define the power function $\pi_p(C, \boldsymbol{x}) \equiv \sum_{i \in C} \pi_p(\{i\}, \boldsymbol{x})$ as follows:*

$$\begin{aligned} x_i &= \frac{2t_i - \sin t_i}{8\varpi}; \\ \pi_p(\{i\}, \boldsymbol{x}(\boldsymbol{t})) &= \frac{2t_i + \sin t_i}{8\varpi}; \end{aligned} \tag{8}$$

for $t_i \in [0, 4\varpi]\ \forall i \in I$. Thus, $\pi_p(\{i\}, \boldsymbol{x})$ is the sine curve, attenuated and then rotated counterclockwise $\frac{\varpi}{4}$ radians. It satisfies all the additional axioms except conservatism, CN. Again, Lemma 12 holds, but now the dominance regions are as in the left panel of Figure 4. Thus, this power function belongs to an equivalence class with WIP.

The key feature in both of the above examples is a symmetry axiom:

(QS) $\pi(\{j\}, \boldsymbol{x})|_{x_j = \frac{1}{4} - \varepsilon} + \pi(\{k\}, \boldsymbol{x})|_{x_k = \frac{1}{4} + \varepsilon} = \frac{1}{2}$ for all $\varepsilon \in \left[0, \frac{1}{4}\right]$.

Then the general result is that:

Theorem 10. *When $n = 3$, and π satisfies axioms AN, C0, AD and QS, $\mathcal{D}_2$ is the unique stable set.*

Proof. Property QS and axioms AD and C0 ensure that $\pi(\{1\}, \boldsymbol{x})|_{x_1=\frac{1}{2}} = \frac{1}{2}$. Axioms C0 and AD then ensure that $\pi(\{2,3\}, \boldsymbol{x})|_{x_1=\frac{1}{2}} = \frac{1}{2}$. Therefore $\left(\frac{1}{2}, \frac{1}{4}+\varepsilon, \frac{1}{4}-\varepsilon\right)$, for $\varepsilon \in \left[0, \frac{1}{4}\right]$, defines B^1. For all $\varepsilon \neq 0$ these allocations are dominated by $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$. As axiom AN implies that the problem is symmetric in the agents, the result follows. □

Corollary 4 (WIP equivalence)**.** *When $n = 3$, all power functions satisfying axioms AN, C0, AD and QS represent the same dominance ordering.*

Proof. As in Theorem 3, fix an $\boldsymbol{x}$ and a $\boldsymbol{y}$. As before, when these sets are both singletons, Lemma 2 notes that all power functions satisfying axiom AN represent the same dominance orderings.

There are two non-singleton W and L. First consider $W = \{1\}$ and $L = \{2,3\}$. Thus $\boldsymbol{y} \succ \boldsymbol{x}$ is equivalent to $\pi(\{1\}, \boldsymbol{x}) > \pi(\{2,3\}, \boldsymbol{x})$. As, under the assumed axioms, the inequality holds with equality when $x_1 = \frac{1}{2}$, dominance is equivalent to $x_1 > \frac{1}{2}$ for any power function satisfying these axioms.

The case when $W = \{2,3\}$ and $L = \{1\}$ reverses the inequality above. □

As $\sin x$ is point symmetric about $x = \varpi$, the power function defined in Example 6 is point symmetric about $\left(\frac{1}{4}, \frac{1}{4}\right)$. Thus, Theorem 10 applies. It also applies to other transformations of $\sin x$, including the family defined by equation 7 when $\frac{1}{4}k \in \mathbb{Z}$. Modifications that destroy the symmetry about $x_i = \frac{1}{4}$, such as that displayed in Figure 5, destroy the result.

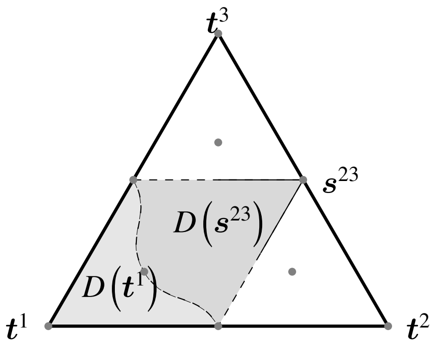

Figure 5: $\pi_3(C, \boldsymbol{x}) : \mathcal{S} = \mathcal{D}_1 \cup \mathcal{Q} = \mathcal{D}_1 \cup \mathcal{R}$

For completeness, we present an example in which $\mathcal{S}^i \subset B^i$ lacks the central $\boldsymbol{q}^i$ allocations. The asymmetric allocations in $\mathcal{R}$ were present when the π_k class of power functions were introduced in equation 7, but have not yet been explicitly illustrated. Relative to the cases studied in Jordan [2006a], they are new.

Example 7. *For $n = 3$ and $\pi_7(C, \boldsymbol{x})$, the unique stable set is $\mathcal{D}_1 \cup \mathcal{R}$, as illustrated in Figure 6.*

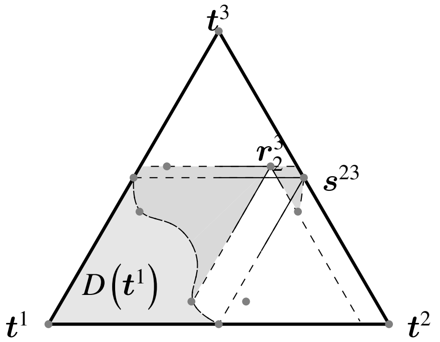

Figure 6: $\pi_7\left(C, \boldsymbol{x}\right) : \mathcal{S} = \mathcal{D}_1 \cup \mathcal{R}$

These sinusoidal power functions have 'wealth is power' as a limit case:

$$\lim_{k \to \infty} \pi_k\left(C, \boldsymbol{x}\right) = \sum_{i \in C} x_i.$$

As $k \to \infty$, the range of the sinusoidal B^i thus collapses.

5.2 Discontinuous examples

When axiom CX is violated, the results presented above do not apply, as demonstrated by Example 1. In this subsection, we therefore present two examples of discontinuities: in the first case, no stable set exists; in the second, a unique stable set is computed.

The first example introduces a discontinuity to the power functions considered in Theorem 8 so that the most powerful agent just reaches it at the apex of B^i, removing the $\boldsymbol{q}^i$. Reducing the jump point increases the segment missing from B^i.

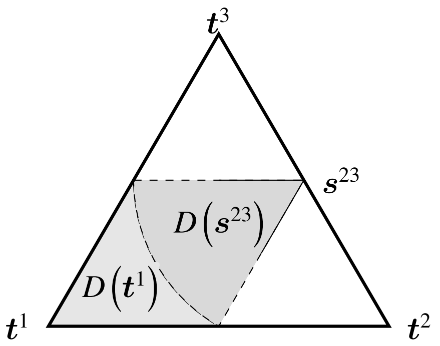

Figure 7: Example 8: discontinuity prevents existence of $\mathcal{S}$

Example 8. *Consider*

$$\pi(C,x) \equiv \sum_{i\in C} \left\{ \begin{array}{ll} \frac{1}{2}\sqrt{x_i} & \textit{if } x_i < \frac{2}{3} \\ \sqrt{x_i} & \textit{otherwise} \end{array} \right\}. \tag{9}$$

This satisfies all of the additional axioms except CX, continuity, C0, continuity at zero, and CN, conservatism. This example, for which $\mathcal{K} = \mathcal{D}_1$, is depicted in Figure 7. In this case, B^i is identical to its continuous analog (q.v. the bottom panel of Figure 4) other than excluding the point $x_i = \frac{2}{3}, x_j = x_k = \frac{1}{6}$ on the symmetry axis. Consequently, $M^i = \emptyset$ so that the $\boldsymbol{r}^i_j$ do not exist. The $\boldsymbol{q}^i$ also do not exist. As the allocation removed from each B^i now belongs to $D\left(\boldsymbol{t}^i\right)$, it can no longer be used to dominate allocations in B^i. Thus, as the core contains both the $\boldsymbol{t}^i$ and the $\boldsymbol{s}^{jk}$, no stable set exists.

The next example introduces a discontinuity so that the most powerful agent aligned against i reaches it just when its partner runs out of resources; in turn, this causes B^i not to exist. By reducing the jump point below $\frac{1-v}{2}$, B^i can be extended to the whole interval; as $v \to 0$, the example reduces to the continuous Example 2.

Example 9. *Add a discontinuity to the power function in Example 2, so that*

$$\pi(C,x) \equiv \sum_{i\in C} \left\{ \begin{array}{ll} \frac{1}{2}x_i + v & \textit{if } x_i < \frac{1-v}{2} \\ x_i + v & \textit{otherwise} \end{array} \right\}; \tag{10}$$

for $v \in (0,1)$. This fails to satisfy the same additional axioms that Example 8 does. As above, $\mathcal{K} = \mathcal{D}_0$. The dominance regions resemble those depicted in Figure 3, with an important difference: when $\boldsymbol{x} > \boldsymbol{0}$ such that $x_i = \frac{1+v}{2}$,

$$\pi(\{i\}, \boldsymbol{x}) = \frac{1}{2} + \frac{3}{2}v > \frac{1}{4} + \frac{7}{4}v = \pi(\{j,k\}, \boldsymbol{x}).$$

When either x_j or x_k is zero, however, equality is restored. As $D\left(\boldsymbol{t}^i\right)$ contains $x_i = \frac{1+v}{2}$ when $\boldsymbol{x} > \boldsymbol{0}$, B^i does not exist, so that M^i and R^i do not either. Our algorithm cannot proceed in this case: as R^i does not exist, $\mathcal{P}$ does not either. However, the Jordan and Obadia [2004] arguments for SIN with $v > 1$ can be applied on $\mathcal{X}\backslash D(\mathcal{D}_0)$, so that the unique stable set is $\mathcal{S} = \mathcal{D}_1$.

In terms of Roth-Jordan, $\mathcal{S}_0 = \mathcal{K} = \mathcal{D}_0$ and

$$U(\mathcal{S}_0) = \mathcal{D}_0 \cup \left\{ \boldsymbol{x} \in \mathcal{X} \left| \left\{ x_i < \frac{1+v}{2} \forall i \right\} \cup \left\{ x_k = 0 \Rightarrow x_i = x_j \textit{ for distinct } i,j,k \right\} \right. \right\}.$$

The non-core allocations in $U(\mathcal{S}_0)$ all have the property that a two-member W can dominate a one-member L; as, for each of these allocations, it is possible to find another that dominates it. Thus, $\mathcal{S}_1 = \mathcal{D}_0 = \mathcal{S}_0$ and the algorithm terminates without finding the stable set.

The above examples provide the intuition for this subsection's conclusion, namely that the continuity axiom is restrictive. We state one further lemma before proving this:

Lemma 13. *When $n = 3$, π satisfies axiom CX and $B^i \neq \emptyset$, then B^i is a connected set containing sequences $\left\{\boldsymbol{x}^h\right\}$ and $\{\boldsymbol{y}^m\}$ such that $\lim_{h\to\infty} x_j^h = 0$ and $\lim_{m\to\infty} y_k^m = 0$ for distinct i, j and k.*

Proof. Proceed by contradiction, assuming an $\boldsymbol{x} > \mathbf{0}$ in B^i such that there is no $\boldsymbol{y} \in B^i$ with $y_k = x_k - \varepsilon$ for $\varepsilon > 0$. By axiom WR, we may use Lemma 1 to abuse notation to write

$$\begin{aligned} \pi(x_i) &= \pi\left(x_j, x_k\right); \\ \pi(x_i) &\geq \pi\left(x_j, x_k - \varepsilon\right). \end{aligned} \tag{11}$$

When inequality 11 holds with strict inequality, it generates three subcases:

1. $\pi(x_i) < \pi\left(x_j + \varepsilon, x_k - \varepsilon\right)$ so that, by the intermediate value theorem, there is a $\delta \in (0, \varepsilon)$ such that $\pi(x_i + \delta) = \pi\left(x_j + \varepsilon - \delta, x_k - \varepsilon\right)$, contradicting the assumption that $x_k > 0$ is the least value of the k^{th} component.

2. $\pi(x_i) = \pi\left(x_j + \varepsilon, x_k - \varepsilon\right)$, a contradiction immediately.

3. $\pi(x_i) > \pi\left(x_j + \varepsilon, x_k - \varepsilon\right)$. As $\boldsymbol{x} \in B^i$ it follows from axiom SR that $\pi(x_i - \varepsilon) < \pi\left(x_j + \varepsilon, x_k\right)$ so that the IVT ensures the existence of a $\delta \in \left(0, \frac{1}{2}\varepsilon\right)$ for which $\pi(x_i - \varepsilon + 2\delta) = \pi\left(x_j + \varepsilon - \delta, x_k - \delta\right)$, again yielding a contradiction.

When inequality 11 holds with equality axiom SR implies $\pi(x_i + \varepsilon) > \pi\left(x_j, x_k - \varepsilon\right)$. As $\pi(x_i) < \pi\left(x_j + \frac{1}{2}\varepsilon, x_k - \frac{1}{2}\varepsilon\right)$, the IVT again ensures the contradiction.

The above argument establishes that no component of B^i is the union of two disjoint non-empty closed sets; thus, B^i is locally connected. By Corollary 2, there can only be a single component, so that B^i is also connected. □

Theorem 11. *Not all pillage games can be represented by a power function satisfying continuity axiom CX.*

Proof. Example 8 provided an example of a pillage game for which $B^i \neq \emptyset$, but which was not a connected set: there was no $\boldsymbol{x} \in B^i$ such that $x_2 = x_3$. By Lemma 13, this cannot be the case for a pillage game satisfying axiom CX. □

As dominance, a binary operator, is necessarily discontinuous in both the relative power and composition of W and L, this may be surprising.[26] There is no evidence as yet for the stronger result that discontinuous power functions can generate stable sets that continuous power functions cannot.

5.3 Irresponsive examples

We now consider examples violating axiom RE. The subsection's main result is that any anonymous power function with the same core as the champion power function, has the same unique stable set as it.

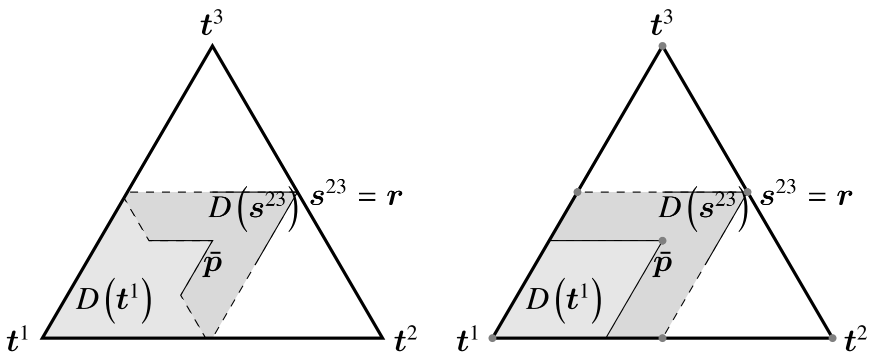

Figure 8: $\hat{\pi}(C, \boldsymbol{x})$ when $v \in \left(0, \frac{1}{4}\right)$, and when $v \in \left[\frac{1}{4}, 1\right)$

Theorem 12. *When $n = 3$ and power is defined by the leader power function, $\hat{\pi}(C, \boldsymbol{x})$:*

1. *for $v \in \left(0, \frac{1}{4}\right)$, no stable set exists.*
2. *for $v \in \left[\frac{1}{4}, 1\right)$, $\mathcal{S} = \mathcal{D}_1 \cup \{\bar{\boldsymbol{p}}\}$ is the unique stable set.*

Proof. From Lemma 5, $\mathcal{K} = \mathcal{D}_0 \cup \{\bar{\boldsymbol{p}}\}$ for the leader power function. It is immediate that $D(\bar{\boldsymbol{p}}) = \emptyset$.

For any value of $v \in (0, 1)$,

$$\left\{\boldsymbol{x} \in \mathcal{X} \,\middle|\, \{x_1 > x_2; x_3 = 0\} \cup \{x_1 > x_2; x_3 = 0\} \cup \left\{x_1 > \frac{1}{3} \geq x_2, x_3\right\}\right\} \subseteq D\left(\boldsymbol{t}^1\right):$$

for the first two sets, $\boldsymbol{t}^1 \mathrel{\in\!\!-} \boldsymbol{x}$ by the anonymity axiom; the third restricts $\hat{\pi}(\{2, 3\}, \boldsymbol{x}) \leq \frac{1}{3} < x_1 + (1 - v)\, x_1 = \hat{\pi}(\{1\}, \boldsymbol{x})$. When $v \in \left(0, \frac{1}{4}\right)$, there is an additional component

[26] See Bridges and Mehta [1995, Chapter 8] for a treatment of continuous utility function representations of preference relations.

to $D\left(t^1\right)$ as the SIN effect is no longer masked by the champion effect, namely $\left\{x \in \mathcal{X} \middle| x_1 > \frac{1}{2(1-\nu)}\right\}$. Thus

1. for $\nu \in \left(0, \frac{1}{4}\right)$, consider $\left\{x \in B^i \middle| x_j > x_k\right\}$. Within this set, π satisfies axioms CX and RE. The arguments in Lemmas 10 and 11 and Theorem 4 may therefore be easily modified to prove that, as B^i lacks a maximal element, no stable set exists.
2. for $\nu \in \left[\frac{1}{4}, 1\right)$, $D\left(s^{23}\right)$ is

$$\left\{x \in \mathcal{X} \middle| \left\{x_3 = \frac{1}{2}, x_2 > x_1\right\} \cup \left\{x_2 = \frac{1}{2}, x_3 > x_1\right\} \cup \left\{x_2 \in \left(\frac{1}{3}, \frac{1}{2}\right), x_3 < \frac{1}{2}\right\} \cup \left\{x_3 \in \left(\frac{1}{3}, \frac{1}{2}\right), x_2 < \frac{1}{2}\right.\right.$$

It is then straightforward to show that $\mathcal{K} \cup D(\mathcal{K}) = \mathcal{D}_1 \cup \{\bar{p}\} \cup D(\mathcal{D}_1) = \mathcal{X}$. Thus, the core is externally stable, making it the unique stable set [Jordan, 2006a, Proposition 2.4].

□

Finally, we demonstrate that any anonymous three agent pillage game whose core is $\mathcal{D}_1 \cup \{\bar{p}\}$ has the core as its unique stable set. An example is first presented; it is then shown that, for any such power function, the core is externally stable, and therefore the unique stable set.

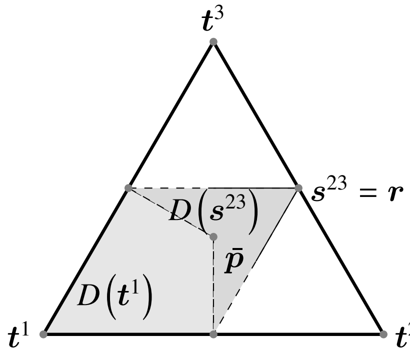

Figure 9: $\bar{\pi}(C, x) : \mathcal{S} = \mathcal{D}_1 \cup \{\bar{p}\}$

The simplex corresponding to the 'champion' power function is depicted in Figure 9, with $q = \bar{p}$ as it lies on the boundary between $D\left(t^i\right)$ and $D\left(s^{jk}\right)$.

Then:

Lemma 14. *When $n = 3$, the unique stable set of the champion power function is*

$$\mathcal{S} = \mathcal{K} = \mathcal{D}_1 \cup \{\bar{p}\}.$$

The lemma might be more properly stated as a corollary to Lemma 4, which established both that $\mathcal{K} = \mathcal{D}_1 \cup \{\bar{\boldsymbol{p}}\}$, and that $\mathcal{K}$ is externally stable; thus, the core is uniquely stable.

Theorem 13 (champion equivalence)**.** *When $n = 3$ all anonymous power functions such that $\mathcal{K} = \mathcal{D}_1 \cup \{\bar{\boldsymbol{p}}\}$ yield $\mathcal{S} = \mathcal{K}$ as their unique stable set.*

This is a weaker result than Theorem 3 as it implies nothing about whether $\boldsymbol{x} \leftarrowtail \boldsymbol{y}$ for allocations $\boldsymbol{x}, \boldsymbol{y} \notin \mathcal{S}$.

Proof. Consider, without loss of generality, the three classes of allocations, $\boldsymbol{x} \in \mathcal{X}$:

1. $\boldsymbol{x} = (x, 1-x, 0)$ with $x \in \left(\frac{1}{2}, 1\right)$. Then, by axiom AN, $\boldsymbol{x} \in D\left(\boldsymbol{t}^1\right)$.

2. $\boldsymbol{x} > 0$ with $x_1 > \max\{x_2, x_3\}$ so that $x_1 > \frac{1}{3}$. Assume that $\boldsymbol{x} \notin D\left(\boldsymbol{t}^1\right)$. Then

$$\pi(\{1\}, \bar{\boldsymbol{p}}) < \pi(\{1\}, \boldsymbol{x}) \leq \pi(\{2,3\}, \boldsymbol{x}) \leq \pi(\{2,3\}, \bar{\boldsymbol{p}});$$

   where the inequalities follow from axiom SR, the assumption, and axiom WR, respectively. Thus, $\bar{\boldsymbol{p}} \notin \mathcal{K}$, a contradiction which, in turn, implies that $\boldsymbol{x} \in D\left(\boldsymbol{t}^1\right)$.

3. $\boldsymbol{x} = (x, x, 1-2x)$ $x \in \left(\frac{1}{3}, \frac{1}{2}\right)$. Then

$$\pi(\{1,2\}, \boldsymbol{x}) \geq \pi(\{1\}, \boldsymbol{x}) > \pi(\{3\}, \boldsymbol{x});$$

   by axioms WC, SR and AN, respectively. Thus, $\boldsymbol{x} \in D\left(\boldsymbol{s}^{12}\right)$.

As the three classes of allocations above cover all $\mathcal{X} \backslash \mathcal{K}$, the core is externally stable, and therefore the unique stable set. □

6 Discussion

The paper's main result is that, when a stable set exists in an $n = 3$ pillage game satisfying axioms AN, CX and RE, it is unique. None of the examples violating axioms CX and RE subsequently explored yield multiple stable sets. When anonymity does not hold, we expect that we shall still be interested in the B^i, and that (asymmetric) analogues of $\boldsymbol{q}$ and $\boldsymbol{r}$ shall play similar roles in the B^i that they do under anonymity. However, relaxing anonymity introduces dominance regions that do not exist in the anonymous cases considered here. With anonymity, either $\boldsymbol{x} \in D\left(\boldsymbol{t}^i\right)$ (so that the strongest agent is stronger than the other two), $\boldsymbol{x} \in B^i$ (so that the strongest agent is as strong as the other two), or no agent is as strong as the other two. Consider now a non-anonymous majority pillage game satisfying $\pi(\{2,3\}, \boldsymbol{x}) > \pi(\{1\}, \boldsymbol{x}) > \pi(\{2\}, \boldsymbol{x}) > \pi(\{3\}, \boldsymbol{x})$ for all $\boldsymbol{x} \in X$.[27] The SIN power function can generate this with $v_1 = 10, v_2 = 8$ and $v_3 = 6$. Although all the B^i are empty on X, the game has no stable set. In the anonymous case, the empty B^i implies a majority pillage game on all of X; this is not the case here.

We expect that stable sets in anonymous pillage games that violate continuity axiom CX will not be larger than those in their continuous counterparts: discontinuities may remove maximal elements from B^i but do not otherwise undermine domination's inheritance of a complete order. Finally, while the examples violating responsiveness axiom RE may seem extremely restrictive, they are responsible for two of the five types of core identified in Theorem 2.

For $n = 3$ games in characteristic function form, the best known class of cooperative games, "stable sets are typically not unique", but existence is guaranteed [Lucas, 1992, pp.562-3]. In contrast, $n = 3$ pillage games satisfying axioms AN, CX and RE have unique stable sets, but only when they exist. Given some $\boldsymbol{x}$ in a game in characteristic function form, an infinite subset of the allocations near it may be incomparable, in the sense that neither $\boldsymbol{x} \succ \boldsymbol{y}$ nor $\boldsymbol{y} \succ \boldsymbol{x}$ [q.v. Lucas, 1992, Examples 3, 4]. This allows construction of infinite stable sets from such incomparable allocations.

As, in pillage games, power depends monotonically on resource holdings, the domain of this incomparability is reduced.[28] This reduced domain also seems responsible for the finitude of stable sets in pillage games. Indeed, as Lucas [1992] notes, only one $n = 3$ game in characteristic function form has a finite stable set, namely the majority game in which $\boldsymbol{y} \succ \boldsymbol{x}$ iff two agents prefer $\boldsymbol{y}$ to $\boldsymbol{x}$. In this case, the regions of incomparability are 1-dimensional curves in the simplex, which still admit a family of infinite stable sets.[29]

A Other results

Lemma 15 (convex transitivity)**.** *If* $\boldsymbol{x} \succ \boldsymbol{y}$ *then* $\boldsymbol{x} \succ \alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} \succ \boldsymbol{y} \forall \alpha \in (0,1)$.

Proof. $\boldsymbol{x} \succ \boldsymbol{y} \Leftrightarrow \pi(W_1, \boldsymbol{y}) > \pi(L_1, \boldsymbol{y})$, where $W_1 \equiv \{i \,|x_i > y_i\}$ and $L_1 \equiv \{i \,|y_i > x_i\}$. Now define

$$W_\alpha \equiv \{i \,|\alpha x_i + (1-\alpha) y_i > y_i\};$$

[27] We are grateful to Jim Jordan for this suggestion.

[28] Jordan [2006a] proved that stable sets in pillage games are finite by showing the impossibility of a sequence of four allocations over which W and L remain constant.

[29] See Jordan and Obadia [2004] for a discussion of the majority pillage game's use of resource holdings to break ties in the majority game.

and L_α similarly for $\alpha \in (0,1)$. It is then immediate that $W_\alpha = W_1$ (*resp.* $L_\alpha = L_1$) so that

$$\alpha \boldsymbol{x} + (1-\alpha)\,\boldsymbol{y} \mathrel{\varepsilon\!\!-} \boldsymbol{y}.$$

The second dominance operation follows in the same way. □

Lemma 16. *When $n = 3$, π satisfies axioms AN and CX, and $\boldsymbol{r}^i_j \in B^i \backslash B^i_+$, then $\boldsymbol{r}^i_j = \boldsymbol{s}^{ij}$.*

Proof. Suppose otherwise, so that, setting $(i, j) = (1, 2)$ without loss of generality, $\boldsymbol{r}^i_j = (r, r, 1-2r)$, with $r \in \left[\frac{1}{3}, \frac{1}{2}\right)$. By Lemma 13, there is a $\boldsymbol{y} \in B^i$ with $y_3 < 1-2r$; by Lemma 1, this may be written as $\pi(r+\delta) = \pi(r+\varepsilon-\delta, 1-2r-\varepsilon)$ for some $\delta \in [0, \varepsilon]$. Axiom AN implies, by Lemma 6, that $\delta > \varepsilon - \delta$ so that $\delta > \frac{1}{2}\varepsilon > 0$, so that $\pi(r) < \pi(r+\delta)$, contradicting the requirement that $\boldsymbol{r}^-_j \in M^i$. □

Lemma 17. *When $n = 3$, and π satisfies axioms AN and CX, M^i is non-empty and $\boldsymbol{q}^i$ exists.*

Proof. By Corollary 2, B^i may be represented by a curve indexed by $\frac{y_j}{y_k}$, which defines the Corollary's rays; the Lemma establishes that $\frac{y_j}{y_k} \in [0, \infty)$. As B^i is connected, the curve exists at $\frac{y_j}{y_k} = 1$, so $\boldsymbol{q}^i$ does as well. As B^i is closed and bounded, M^i is non-empty. □

Lemma 18. *When $\mathcal{D}_1 \subseteq \mathcal{K}$, all $\boldsymbol{x} \in B^i \backslash B^i_+$ either belong to the core, or are dominated by a core allocation.*

Proof. Any $\boldsymbol{x} \in B^i \backslash B^i_+$ may be written as a permutation of $(x, x, 1-2x)$, where $x \in \left[\frac{1}{3}, \frac{1}{2}\right]$. As $\mathcal{D}_1 \subseteq \mathcal{K}$ the core allocation $\boldsymbol{s}^{12}$, which corresponds to $x = \frac{1}{2}$, dominates $\boldsymbol{x}$ for all $x \in \left(\frac{1}{3}, \frac{1}{2}\right)$; when $\boldsymbol{s}^{12}$ does not also dominate $\bar{\boldsymbol{p}}$, which corresponds to $x = \frac{1}{3}$, then $\bar{\boldsymbol{p}}$ is undominated, and so belongs to the core. □

When axiom RE holds, the lemma is less important, as the $\boldsymbol{s}^{jk}$ are the only possible allocations in $B^i \backslash B^i_+$.

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