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Obtaining form factors of resonances and bound states from lattice QCD

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This *master document* details all steps in the HadSpec numerical lattice QCD calculation of the ρ form factor. The novel aspect of this calculation is that the resonance nature of the ρ is rigorously accomodated. This is achieved by first determining the infinite-volume matrix element $\langle \pi\pi, \mathrm{out}|\mathcal{J}^\mu|\pi\pi, \mathrm{in}\rangle$ and then analytically continuing both the initial and final two-pion state energies into the complex plane, to the ρ pole.

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A. Toy analyses

Perform a toy-model analysis of the elastic form factor for a scalar system, i.e. the σ. Here there is only a single form factor. Then repeat the analysis for a vector system, i.e. the ρ. Here there are three form factors. In the case of the vector things are more complicated, not just because of the finite-volume effects, but also the Lorentz decomposition. Specifically we can no long disentangle the three form-factors independently because different irreps have different energies and virtualities.

B. General LQCD

C. π form factor

D. ρ form factor

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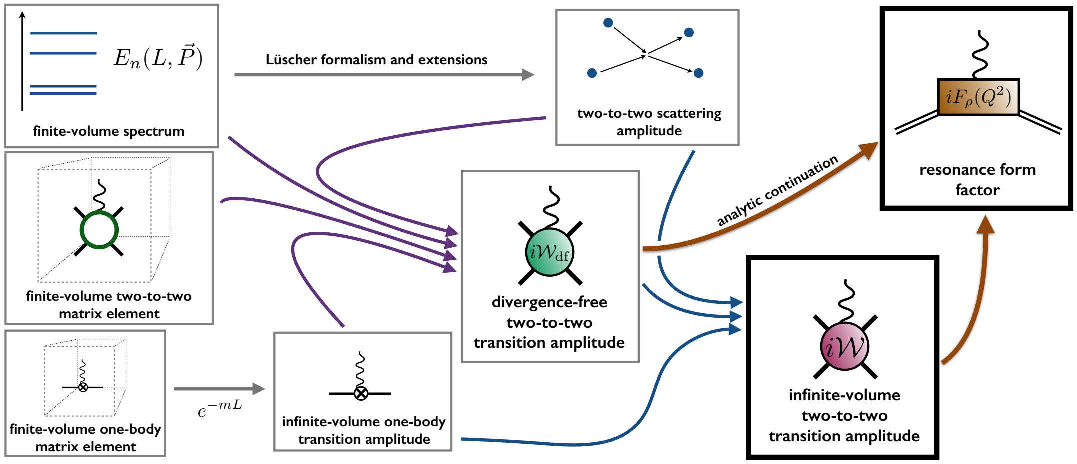

(a)

FIG. 1: Flowchart summarizing the process for extracting resonance form factors using the formalsim of Ref. [1]. **[Check second update, now two analytic continuation arrows and colors]**

II. FORM FACTORS, AMPLITUDES, AND LORENTZ DECOMPOSITION

Before describing how the desired observables can be extracted using lattice QCD, it is instructive to discuss the infinite-volume quantities that we aim to calculate. Our aim is two-fold:

1. Explain how different components of the $\pi\pi\gamma^\star \to \pi\pi$ amplitude can be extracted from lattice QCD. We will focus on the components where both the initial and final state have been projected onto an isotriplet.

2. Constrain the amplitude as much as possible, in the largest possible kinematic window. We can then analytically continue initial and final states to the ρ pole to obtain the four form factors of the ρ.

In order to take the first step, we will need the formalism introduced in Ref. [1]. The approach is sketched in Fig. 1 and is reviewed in detail in Sec. IV. The two key inputs, in addition to the standard matrix elements, are

- the $\pi\pi \to \pi\pi$ scattering amplitude,
- the single pion form factor, $\langle \pi | \mathcal{J}^\mu | \pi \rangle$.

In this section we begin by reviewing basic consequences of the symmetries of the infinite-volume system: isospin, parity, G-parity, angular momentum, and charge conservation.

A. Isospin and charge

The aim of this project is to explain how one can obtain the form factors of a resonance. In particular, we will be focusing on electromagnetic form factors and to resonances that couple to $\pi\pi$ states. With this in mind, it is useful to review the isospin projections of the $\pi\pi$ system, and the resonance content of each channel, beginning with the isostensor. This channel has no resonances; the individual states are given by

$$|I=2, m_I=2\rangle = |\pi^+\pi^+\rangle\,, \tag{1}$$

$$|I=2, m_I=1\rangle = \frac{1}{\sqrt{2}}\left(|\pi^+\pi^0\rangle + |\pi^0\pi^+\rangle\right)\,, \tag{2}$$

$$|I=2, m_I=0\rangle = \frac{1}{\sqrt{6}}\left(|\pi^+\pi^-\rangle + |\pi^-\pi^+\rangle + 2|\pi^0\pi^0\rangle\right)\,. \tag{3}$$

By contrast, the isovector channel contains the rho resonance, $\rho = (\rho^+, \rho^0, \rho^-)$. The ρ^0 is neutral and an eigenstate of charge conjugation. Furthermore, the electromagnetic current is odd under charge conjugation, $C\bar\psi\gamma^\mu\psi C = -\bar\psi\gamma^\mu\psi$.

This implies

$$\langle\rho^0|\bar{\psi}\gamma^\mu\psi|\rho^0\rangle = \langle\rho^0|C(C\bar{\psi}\gamma^\mu\psi C)C|\rho^0\rangle = -\langle\rho^0|\bar{\psi}\gamma^\mu\psi|\rho^0\rangle = 0. \tag{4}$$

Similarly, one concludes that the electromagnetic matrix elements of the π^0 vanishes. Note, since the ρ^0 is negative under charge conjugation and the π^0 is even, the $\langle\rho^0|\bar{\psi}\gamma^\mu\psi|\pi^0\rangle$ matrix element need not be zero, similarly the $\pi^0 \to \gamma\gamma$ is not zero.

Since the ρ^0 form factors are zero, we will focus on the ρ^+. The definition of $I = 1$ $\pi\pi$ states and the association to the ρ states is as follows:

$$\rho^+: \quad |I = 1, m_I = 1\rangle = \frac{1}{\sqrt{2}}\left(|\pi^+\pi^0\rangle - |\pi^0\pi^+\rangle\right), \tag{5}$$

$$\rho^0: \quad |I = 1, m_I = 0\rangle = \frac{1}{\sqrt{2}}\left(|\pi^+\pi^-\rangle - |\pi^-\pi^+\rangle\right). \tag{6}$$

Here we notice two things. As already stated above, we will need the form factors of the individual pions to calculate the rho form factor. Given that only that of the π^+ is nonzero, we will only need to calculate this one.

Finally, the isoscalar channel couples to the σ,

$$\sigma: \quad |I = 0, m_I = 0\rangle = \frac{1}{\sqrt{3}}\left(|\pi^+\pi^-\rangle + |\pi^-\pi^+\rangle - |\pi^0\pi^0\rangle\right). \tag{7}$$

This, of course, is neutral and is even under charge conjugation. As a result, similarly for the ρ^0 and π^0 its electromagnetic form factor vanishes.

In conclusion, of the $\pi\pi$ channels, the one that is most sensible to consider here is $(I, m_I) = (1, 1)$, since this is the only one that could have a resonance form factors. The ρ^+ is a vector, implying that it has three-independent form factors to determine, the charge, magnetic and quadrupole form factors. These are proportional to the the charge, magnetic moment, and quadrupole moment of the ρ^+. Furthermore, one can obtain the correspond radii of the ρ^+. As we argue below, in practice this *might* require to calculate four independent amplitudes from lattice QCD calculations. As one analytically continues these onto the ρ pole, one of them will vanish, resulting in three non-zero form factors.

After examining which matrix elements are allowed in the following subsection, we discuss the Lorentz decomposition of the matrix elements in terms of form factors in Sec. II C.

B. Allowed matrix elements

We are interested in evaluating matrix elements of the QED current

$$\mathcal{J}^\mu = q_u\bar{u}\gamma^\mu u + q_d\bar{d}\gamma^\mu d = \left(2\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d\right)/3. \tag{8}$$

It is convenient to decompose this into an isovector and isoscalar current,

$$\mathcal{J}^\mu = \frac{\rho^{0,\mu}}{\sqrt{2}} + \frac{\omega^\mu}{3\sqrt{2}}, \tag{9}$$

where

$$\rho^{0,\mu} = \left(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d\right)/\sqrt{2}, \tag{10}$$

$$\omega^\mu = \left(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d\right)/\sqrt{2}. \tag{11}$$

The $\rho^{0,\mu}$ component has the quantum number of the $I_z = 0$ component of the ρ meson, namely $I^G(J^P) = 1^+(1^-)$,[1] while the ω^μ piece has the quantum numbers of the ω meson, $0^-(1^-)$. Strictly speaking, these are the quantum numbers of the spatial piece of the current; the temporal components are related via Lorentz symmetry.

For the single-pion matrix elements, one can explicitly evaluate the contractions (I get that the disconnected piece of the $\rho^{0,\mu}$ current vanishes, but not for the ω piece, so not 100% sure about this), to find that only the component of

[1] I =isospin, G =G-parity, J=total angular momentum, P=parity.

the QED current that survives the ρ-like piece. **[I am not so sure if it is true that we can see the vanishing by evaluating contractions. Consider in particular**

$$\langle \pi | \omega^\mu | \pi \rangle \propto \langle 0 | (\bar{q}\tau^a \gamma_5 q)(\bar{q}\mathbb{I}\gamma^\mu q)(\bar{q}\tau^a\gamma_5 q)|0\rangle\,, \tag{12}$$

$$= -\mathbf{Tr}\big[S(i,f)\tau^a\gamma_5 S(f,c)\mathbb{I}\gamma^\mu S(c,i)\tau^a\gamma_5\big] + \mathbf{Tr}\big[S(i,f)\tau^a\gamma_5 S(f,i)\tau^a\gamma_5\big]\mathbf{Tr}\big[S(c,c)\mathbb{I}\gamma^\mu\big]\,. \tag{13}$$

I don't there is an easy way to see that this is zero. One approach to argue that it vanishes is to note that a field configuration and its G-parity conjugate must have the same weight in the path integral. So one can consider the sum of these terms with their G-parity conjugates. I would suspect these should cancel to zero. But in the end this is the same as just studying the effect of G-parity on the matrix element and I think the latter is more elegant. So, if this is true, I would vote to drop discussion about contractions here. Of course it is still important to state that the disconnected part of the $\rho^{0,\mu}$ matrix element vanishes.]

$$\langle \pi | \mathcal{J}^\mu | \pi \rangle = \langle \pi | \rho^{0,\mu} | \pi \rangle / \sqrt{2}. \tag{14}$$

One can more generally find which components vanish, by considering the quantum numbers of the initial and final states in conjunction with those of the current. It is convenient to consider this in terms of a scattering process, $\pi\gamma^\star \to \pi$. The advantage of doing this is that one is explicitly reminded of the additional degrees of freedom associated with the the relative angular momentum (ℓ) between the two initial particles. We also stress that this affects the total parity of the incoming state.

With this in mind, we tabulate all possible combinations of quantum numbers for $\pi\gamma^\star$ and check which configurations can overlap the final π

current $[I^G(J^P)]$	initial hadron	relative ℓ	possible final states	final hadron	overlap?
$\rho^{0,i}[1^+(1^-)]$	$\pi[1^-(0^-)]$	0	$[1^-(1^+)], \ldots$	$\pi[1^-(0^-)]$	No (violates P)
		1	$[1^-(0^-)], [1^-(1^-)], \ldots$		Yes
		≥ 2	$[1^-(J \geq 1)], \ldots$		No (violates J)
$\omega^i[0^-(1^-)]$	$\pi[1^-(0^-)]$	≥ 0	$[1^+(J \geq 0)]$	$\pi[1^-(0^-)]$	No (violates G)

(We conclude the same result found by studying contractions above, that only one component of the current contributes to this transition.) The ellipsis in the first two lines denotes other components of isospin that do not contribute.

Now, we turn to the target $\pi\pi\gamma^\star \to \pi\pi$ transitions. As already discussed above, we focus attention on the scenario where the $\pi\pi$ state has been projected onto the quantum numbers of the ρ. Therefore, we can equally well tabulate the allowed quantum numbers for $\rho\gamma^\star \to \rho$,

current $[I^G(J^P)]$	initial hadron	relative ℓ	possible final states	final hadron	overlap?
$\rho^{0,i}[1^+(1^-)]$	$\rho^{0,i}[1^+(1^-)]$	0	$[1^+(1^+)], \ldots$	$\rho^{0,i}[1^+(1^-)]$	No (violates P)
		1	$[1^+(0^-)], [1^+(1^-)], \ldots$		Yes
		2	$[1^+(0^+)], [1^+(1^+)], \ldots$		No (violates P)
		3	$[1^+(1^-)], \ldots$		Yes
		≥ 4	$[1^+(J \geq 2)], \ldots$		No (violates J)
$\omega^i[0^-(1^-)]$	$\rho^{0,i}[1^+(1^-)]$	≥ 0	$[1^-(J \geq 0)]$	$\rho^{0,i}[1^+(1^-)]$	No (violates G)

Again, we get that this transition is allowed and only the ρ component of the current contributes. **In a nut shell, because ω flips G-parity it can only mediate transitions where the initial- and final-state G-parity differ.**

[Please check changes to both tables]

[Question: We know with parity that, in addition to the intrinsic part, we must keep track of the parity induced by the ℓ value. Are we sure that nothing like this happens with G? For example it could be that the total G parity is not just the product of intrinsic parities but also some contribution that depends on the total isospin of the final state. Just wanna be sure we are not missing something...]

In summary, the following matrix elements are allowed

$$\overline{F}_{J=0}(p_f,p_i) \equiv \big\langle \pi[1^-(0^-)]\big|\Big[\rho^{0,\mu}\big|\pi[1^-(0^-)]\big\rangle\Big]_{[\ell=1,\ 1^-(0^-)]}\,, \tag{15}$$

$$\big\langle \rho^{0,i}[1^+(1^-)]\big|\Big[\rho^{0,\mu}\big|\rho^{0,i}[1^+(1^-)]\big\rangle\Big]_{[\ell=1,\ 1^+(1^-)]}\,, \tag{16}$$

$$\big\langle \rho^{0,i}[1^+(1^-)]\big|\Big[\rho^{0,\mu}\big|\rho^{0,i}[1^+(1^-)]\big\rangle\Big]_{[\ell=3,\ 1^+(1^-)]}\,. \tag{17}$$

The meaning of these is relatively obscure, consider for example $\overline{F}_{J=0}(p_f,p_i)$. To define this we first introduce

$$F^\mu(p_f,p_i) \equiv \langle \pi(p_f)|\rho^{0,\mu}(0)|\pi(p_i)\rangle\,. \tag{18}$$

Then, introducing $\Lambda^\mu{}_\nu$ as the boost to the $\mathbf{p}_f = 0$ frame we define

$$\overline{F}^\mu(\bar{p}_f^0,\bar{p}_i^0,\bar{\mathbf{p}}_i) \equiv \Lambda^\mu{}_\nu F^\nu(p_f,p_i)\,, \tag{19}$$

where $\bar{p}^\mu \equiv \Lambda^\mu{}_\nu p^\nu$. Note that in defining $\overline{F}$ we have done two things (1) boosted the original F^μ to the $\mathbf{p}_f = 0$ frame and (2) rexpressed the quantity as a function of momenta defined in that frame.

We are now in position to define the projection to definite λ and to definite m with $\ell = 1$

$$\overline{F}_{m;\lambda}(\bar{p}_f^0,\bar{p}_i^0,|\bar{\mathbf{p}}_i|) \equiv A\,\epsilon_\mu(\bar{p}_f-\bar{p}_i,\lambda)\int d\Omega_{\hat{\mathbf{p}}_i} Y_{1m}(\hat{\mathbf{p}}_i)\overline{F}^\mu(\bar{p}_f^0,\bar{p}_i^0,\bar{\mathbf{p}}_i)\,, \tag{20}$$

and to combine to $J=0$

$$\overline{F}_{J=0}(\bar{p}_f^0,\bar{p}_i^0,|\bar{\mathbf{p}}_i|) = \sum_{m,\lambda} C_{1m;1\lambda}^{J=0,m_J=0}\overline{F}_{m;\lambda}(\bar{p}_f^0,\bar{p}_i^0,|\bar{\mathbf{p}}_i|)\,. \tag{21}$$

Here A is a numerical prefactor to be chosen for convenience.

[At this point it is not clear whether it is really necessary to perform these projections. Perhaps thinking of the $\pi\gamma^*$ state is just a useful trick for deciding which matrix elements are allowed. Once we know these there is no need to project to them explicitly.]

C. Lorentz decomposition in terms of form factors

When considering QCD stable states, it is naturally to write the matrix elements of the QED current in terms of form factors up to overall factors which are completely constraints from kinematics. Here, we review this for matrix elements coupling pseudoscalar and vectors states. In this study, we are primarily interested on currents coupling to the π and ρ. In this section, we will ignore the fact that the ρ is unstable, and we address this in Sec. II D. Furthermore, for the sake of generality, when considering matrix elements coupling pseudoscalar-to-pseudoscalar states or vector-to-vector states, we will not assume that the initial and final states are degenerate. As we will see below, this is essential for later addressing the unstable nature of the ρ. To accommodate this added complexity, we will introduce another species of pseusocalar and vector mesons, which will refer to as π' and ρ' respectively, which will also be assumed to be stable in this section.

1. *Lorentz decomposition of $\pi\gamma^\star \to \pi'$*

We begin with the elastic pion matrix elements, $\langle \pi(p_f)|\mathcal{J}^\mu|\pi(p_i)\rangle$, for the general case where pions are not on shell. This matrix element is a Lorentz vector, therefore it can only be proportional to linear combinations of the two vectors in the problem (p_i^μ, p_f^μ),

$$\langle \pi(p_f)|\mathcal{J}^\mu|\pi(p_i)\rangle = F_\pi(p_f^2,p_i^2,Q^2)\,(p_i+p_f)^\mu + F_{\pi,-}(p_f^2,p_i^2,Q^2)\,(p_i-p_f)^\mu\,, \tag{22}$$

where the functions F_π and $F_{\pi,-}$ could *a priori* have non-vanishing imaginary parts. Time-reversal invariance implies

$$\begin{aligned}\langle \pi(p_f)|\mathcal{J}^\mu|\pi'(p_i)\rangle &= -\mathcal{T}^\mu{}_\nu\langle \pi(p_f^0,-\vec{p}_f)|J^\nu|\pi(p_i^0-\vec{p}_i)\rangle^*\,, &(23)\\ &= -\mathcal{T}^\mu{}_\nu\left[F_\pi^*(p_f^2,p_i^2,Q^2)(\tilde{p}_i+\tilde{p}_f)^\nu + F_{\pi,-}^*(p_f^2,p_i^2,Q^2)\,(\tilde{p}_i-\tilde{p}_f)^\nu\right]\,, &(24)\\ &= F_\pi^*(p_f^2,p_i^2,Q^2)\,(p_i+p_f)^\mu + F_{\pi,-}^*(p_f^2,p_i^2,Q^2)\,(p_i-p_f)^\mu\,, &(25)\end{aligned}$$

where in the second line we have defined $\tilde{p}_f = (p_f^0, -\vec{p}_f)$ and similarly for $\tilde{p}_i$. **[I think we can drop the following...]** **(Looking at $\mu = 0$ in Eq. (24) we obtain the following relation**

$$F_\pi(p_f^2, p_i^2, Q^2)\,(p_i^0 + p_f^0) + F_{\pi,-}(p_f^2, p_i^2, Q^2)\,(p_i^0 - p_f^0) \;=\; F_\pi^*(p_f^2, p_i^2, Q^2)\,(p_i^0 + p_f^0) + F_{\pi,-}^*(p_f^2, p_i^2, Q^2)\,(p_i^0 - p_f^0) \quad (26)$$

) We conclude that F_π and $F_{\pi,-}$ have vanishing imaginary parts for all p_f^2, p_i^2, Q^2. **(It is easy to see that when $\mu = i$ Eq. (24) is trivially satisfied.)**

We next take advantage of the hermiticity of the current to find

$$\langle \pi(p_f)|\mathcal{J}^\mu|\pi(p_i)\rangle = \langle \pi(p_i)|\mathcal{J}^\mu|\pi(p_f)\rangle^*\,, \quad (27)$$

that gives

$$F_\pi(p_f^2, p_i^2, Q^2)\,(p_i + p_f)^\mu + F_{\pi,-}(p_f^2, p_i^2, Q^2)\,(p_i - p_f)^\mu = F_\pi^*(p_i^2, p_f^2, Q^2)\,(p_i + p_f)^\mu - F_{\pi,-}^*(p_i^2, p_f^2, Q^2)\,(p_i - p_f)^\mu\,, \quad (28)$$

and since the form factors are real, we obtain the following constraints

$$F_\pi(p_f^2, p_i^2, Q^2) \;=\; F_\pi(p_i^2, p_f^2, Q^2) \quad (29)$$
$$F_{\pi,-}(p_f^2, p_i^2, Q^2) \;=\; -F_{\pi,-}(p_i^2, p_f^2, Q^2)\,. \quad (30)$$

The current $\mathcal{J}^\mu$ is conserved, $\partial_\mu \mathcal{J}^\mu = 0$, which in momentum space tells us that

$$(p_i - p_f)_\mu \langle \pi(p_f)|\mathcal{J}^\mu|\pi'(p_i)\rangle = F_\pi(p_f^2, p_i^2, Q^2)\,(p_i^2 - p_f^2) + F_{\pi,-}(p_f^2, p_i^2, Q^2)\,(p_i - p_f)^2 = 0, \quad (31)$$

$$\Rightarrow F_{\pi,-}(p_f^2, p_i^2, Q^2) = -F_\pi(p_f^2, p_i^2, Q^2)\,\frac{(p_i^2 - p_f^2)}{(p_i - p_f)^2} = -F_\pi(p_f^2, p_i^2, Q^2)\,\frac{(p_i^2 - p_f^2)}{Q^2}. \quad (32)$$

We conclude that this matrix element can be described using a single form factor

$$\langle \pi(p_f)|\mathcal{J}^\mu|\pi'(p_i)\rangle = \left((p_i + p_f)^\mu \; + (p_i - p_f)^\mu \frac{(p_i^2 - p_f^2)}{Q^2} \right) F_{\pi\pi}(p_f^2, p_i^2, Q^2). \quad (33)$$

It is straightforward to see that for pions on shell

$$\langle \pi(p_f)|\mathcal{J}^\mu|\pi'(p_i)\rangle = \left((p_i + p_f)^\mu \; + (p_i - p_f)^\mu \frac{(m_{\pi'}^2 - m_\pi^2)}{Q^2} \right) F_{\pi\pi}(Q^2). \quad (34)$$

Below we consider examples when the decomposition is less trivial.
[very nice!]

2. *Lorentz decomposition of $\rho\gamma^\star \to \pi$*

Now, we turn to a slightly more complicated case, where a stable vector meson couples to a pseudoscalar. In this case, the matrix element must be Lorentz vector. This is a bit difficult to see, since states in flight are not eigenstates of parity. This further complicated by the fact that the ρ has nonzero spin [2]. Nevertheless, we can derive the form of the Lorentz decomposition of the matrix element in a relatively straightforward manner.

First, the ρ state is now defined by its momentum and helicity, λ. In the limit that the momentum is taken to be zero, we choose to quantize the spin along the z axis. Next, we note that the matrix element we are after, must be proportional to the polarization of the polarization vector of the ρ,

$$\langle \pi(p_f)|\mathcal{J}^\mu|\rho(\lambda; p_i)\rangle \propto \epsilon^\nu(\lambda; p_i), \quad (35)$$

note the Lorentz index of ϵ^ν need not coincide with that of the current.

There are only three linearly independent components of a spin-1 vector meson. This is manifested by the fact that $\epsilon_\mu(\lambda, p)p^\mu = 0$. A standard choice for the polarization vectors that is normalized to -1 and satisfies this criterion is

$$\epsilon^\mu(\lambda, p) = \left(\frac{\vec{p}\cdot\hat{e}_\lambda}{\sqrt{p^2}},\; \hat{e}_\lambda + \frac{1}{\vec{p}^{\,2}}\left(\frac{p^0}{\sqrt{p^2}} - 1 \right) \vec{p}\cdot\hat{e}_\lambda\, \vec{p} \right). \quad (36)$$

This satisfies the orthogonality via

$$\epsilon^\mu(\lambda,p)p_\mu = p^0\frac{\vec{p}\cdot\hat{e}_\lambda}{\sqrt{p^2}} - \hat{e}_\lambda\cdot\vec{p} - \left(\frac{p^0}{\sqrt{p^2}}-1\right)\vec{p}\cdot\hat{e}_\lambda = 0\,, \tag{37}$$

and normalization via

$$\epsilon^\mu(\lambda,p)\epsilon_\mu(\lambda,p) = \frac{\vec{p}^2}{p^2}(\hat{p}\cdot\hat{e}_\lambda)^2 - \left[\hat{e}_\lambda + \left(\frac{p^0}{\sqrt{p^2}}-1\right)\hat{p}\cdot\hat{e}_\lambda\;\hat{p}\right]^2 \tag{38}$$

$$= \frac{\vec{p}^2}{p^2}(\hat{p}\cdot\hat{e}_\lambda)^2 - 1 - \left(\frac{p^0}{\sqrt{p^2}}-1\right)^2(\hat{p}\cdot\hat{e}_\lambda)^2 - 2\left(\frac{p^0}{\sqrt{p^2}}-1\right)(\hat{p}\cdot\hat{e}_\lambda)^2\,, \tag{39}$$

$$= -1 + (\hat{p}\cdot\hat{e}_\lambda)^2\left[\frac{\vec{p}^2}{p^2} - \frac{(p^0)^2}{p^2} - 1 + 2\frac{p^0}{\sqrt{p^2}} - 2\frac{p^0}{\sqrt{p^2}} + 2\right]\,, \tag{40}$$

$$= -1\,. \tag{41}$$

Here we use the four-dimensional units vectors

$$\hat{e}_\pm = \mp\frac{1}{\sqrt{2}}(0,1,\pm i,0), \tag{42}$$

$$\hat{e}_0 = (0,0,0,1). \tag{43}$$

It is important to note that $\epsilon^\mu(p,\lambda)$ is a pseudovector.

Under parity transformations, $\hat{P}$, the momentum of a state changes direction but its spin remains unchanged. Given that helicity is defined as the component of the spin along the direction of the momentum, under parity transformations the helicity changes sign. For example, the ρ state would transform as

$$\hat{P}|\rho(\lambda,p_i)\rangle = -|\rho(-\lambda,(p_{i,0},-\vec{p}_i)\rangle, \tag{44}$$

where the overall phase is due to the ρ intrinsic negative parity. If instead the ρ is at rest, defined by $p_i = (m_\rho,\vec{0})$, and denote the azimuthal component of its spin as m_z, then

$$\hat{P}|\rho(m_z,(m_\rho,\vec{0}))\rangle = -|\rho(m_z,(m_\rho,\vec{0})\rangle, \tag{45}$$

and the ρ would indeed be an eigenstate of parity.

With this in mind, let us consider the matrix element in the frame where the ρ is at rest. Under parity, this would transform as

$$\langle\pi(p_f)|\mathcal{J}^\mu|\rho(\lambda;p_i)\rangle = \langle\pi(p_f)|\hat{P}\hat{P}\mathcal{J}^\mu\hat{P}\hat{P}|\rho(\lambda,p_i)\rangle = (-1)^2\langle\pi(p_f^0,-\vec{p}_f)|\mathcal{J}_\mu|\rho(\lambda,p_i)\rangle = \langle\pi(p_f^0,-\vec{p}_f)|\mathcal{J}_\mu|\rho(\lambda,p_i)\rangle. \tag{46}$$

Immediately, we see that the Lorentz structure of the matrix does not coincide with that of the polarization vector. The only other two vector we have at our disposal are p_i and p_f. Requiring the matrix element to be proportional to the polarization vector, we find that the only allowed Lorentz structure is the following

$$\langle\pi(p_f)|\mathcal{J}^\mu|\rho(\lambda,p_i)\rangle = \epsilon^{\mu\alpha\beta\gamma}p_{f,\alpha}p_{i,\beta}\epsilon_\gamma F_1(Q^2,p_i^2,p_f^2) \tag{47}$$

note, a term of the form $p_f^\mu\,(p_i\cdot\epsilon)$ is exactly equal to zero. We also can rule out a term of the form $p_i^\mu\,(p_f\cdot\epsilon)$ because in the rest frame of the ρ, under parity this would trasnform as

$$p_i^\mu\,(p_f\cdot\epsilon) = (-m_\rho\,\vec{p}_f\cdot\vec{\epsilon},\vec{0}) \to (m_\rho\,\vec{p}_f\cdot\vec{\epsilon},\vec{0}). \tag{48}$$

Similarly, we can rule out a term of the form $p_i^\mu\,(p_i\cdot\epsilon)$. Note, Hemiticity also rules out terms proportional to $p_i^\mu\,(p_f\cdot\epsilon)$ or $p_f^\mu\,(p_f\cdot\epsilon)$. [need to think of this more...]. We note that a term of the type $p_i^\mu\,(p_f\cdot\epsilon)F_2(Q^2,p_i^2,p_f^2)$ cannot be present. This can be seen if we look for example at the time component $\mu=0$ that can be written as $p_i^0\,(p_f^0\epsilon^0-\vec{p}\cdot\vec{\epsilon})$, and under parity transformation goes into $p_i^0\,(p_f^0(-)\epsilon^0+\vec{p}\cdot\vec{\epsilon}) = -p_i^0\,(p_f^0\epsilon^0-\vec{p}\cdot\vec{\epsilon})$ and therefore implies $F_2(Q^2,p_i^2,p_f^2)=0$

3. Lorentz decomposition of $\rho\gamma^\star \to \rho'$

[I agree with Raul's suggestion here and have already implemented it.]

Now, we turn to the case where we have a stable spin-1 particle in the initial and final state. Once again, the matrix element must be proportional to the polarization vector of the initial state, $\epsilon^\mu(\lambda_i, p_i)$, and conjugate polarization vector of the final state, $\epsilon^{*\mu}(\lambda_f, p_f)$. To systematically ensure that all possible structures are included, we first list all possible building blocks that carry a Lorentz index

$$p_f^\mu,\ p_i^\mu,\ \epsilon_f^{*\mu},\ \epsilon_i^\mu\,. \tag{49}$$

We see that the following decomposition must hold:

$$\begin{aligned}\langle \rho'(\lambda_f, p_f)|\mathcal{J}^\mu|\rho(\lambda_i, p_i)\rangle = \left(\epsilon_f^* \cdot \epsilon_i\right)\ X^\mu\big(p_i^\alpha, p_f^\beta\big) + \epsilon_i^\mu \Big(\epsilon_f^* \cdot Y_f\big(p_i^\alpha, p_f^\beta\big)\Big) + \epsilon_f^{*\mu}\Big(\epsilon_i \cdot Y_i\big(p_i^\alpha, p_f^\beta\big)\Big) \\ + \epsilon_f^{*\nu}\epsilon_i^\sigma \sum_{a,b,c=i,f} p_a^\mu p_b^\nu p_c^\sigma Z_{abc}\big(Q^2, p_i^2, p_f^2\big)\,.\end{aligned} \tag{50}$$

Here we have separated out all ways in which the indices on the polarization vectors can enter the vector matrix element. The key point is that X^μ, Y_i^μ, Y_f^μ are vectors, and Z a scalar, that do not depend on the polarization. In particular, the decomposition of X^μ, Y_i^μ, Y_f^μ, must be identical to that of the pion vector matrix element, i.e.

$$X^\mu\big(p_i^\alpha, p_f^\beta\big) = (p_i + p_f)^\mu X_+\big(Q^2, p_i^2, p_f^2\big) + (p_i - p_f)^\mu X_-\big(Q^2, p_i^2, p_f^2\big)\,. \tag{51}$$

Next implementing the identities $p_i \cdot \epsilon_i = p_f \cdot \epsilon_f^* = 0$ we see that six terms survive, two from X^μ and Z and one each from Y_i^μ and Y_f^μ. After slight rearanging these can be written

$$\begin{aligned}\langle \rho'(\lambda_f, p_f)|\mathcal{J}^\mu|\rho(\lambda_i, p_i)\rangle &= (p_f + p_i)^\mu\, \epsilon_f^* \cdot \epsilon_i\, \widetilde{G}_1(Q^2, p_i^2, p_f^2) \\ &+ (p_i - p_f)^\mu\, \epsilon_f^* \cdot \epsilon_i\, \widetilde{G}_2(Q^2, p_i^2, p_f^2) \\ &+ \left(\epsilon_i^\mu\, p_i \cdot \epsilon_f^* + \epsilon_f^{*\mu}\, p_f \cdot \epsilon_i\right) \widetilde{G}_3(Q^2, p_i^2, p_f^2) \\ &+ \left(\epsilon_i^\mu\, p_i \cdot \epsilon_f^* - \epsilon_f^{*\mu}\, p_f \cdot \epsilon_i\right) \widetilde{G}_4(Q^2, p_i^2, p_f^2) \\ &+ (p_f + p_i)^\mu\, (\epsilon_f^* \cdot p_i)\,(\epsilon_i \cdot p_f)\, \widetilde{G}_5(Q^2, p_i^2, p_f^2) \\ &+ (p_i - p_f)^\mu\, (\epsilon_f^* \cdot p_i)\,(\epsilon_i \cdot p_f)\, \widetilde{G}_6(Q^2, p_i^2, p_f^2).\end{aligned} \tag{52}$$

Next we show that time-reversal invariance implies that all six scalar form factors are real functions of Q^2, p_i^2, p_f^2. We begin by considering a specific interpolator for the rho

$$|\rho^+(\lambda_{\mathbf{p}}, \mathbf{p})\rangle = \lim_{p^0 \to \omega_{\mathbf{p}}} \epsilon_\mu(\lambda_{\mathbf{p}}, p)(1/i)[-(p^0)^2 + \mathbf{p}^2 + m^2] \int d^4x e^{-ip^0x^0 + i\mathbf{p}\cdot\mathbf{x}}\ \overline{d}(x)\gamma^\mu u(x)|0\rangle\,, \tag{53}$$

where we introduced an index in the helicity to denote witch which axis it is defined. Note $\lambda_{\mathbf{p}} = (-1)^{\lambda_{\mathbf{p}}}\lambda_{\mathbf{-p}}$. Acting with the time-reversal operator on this state gives

$$T|\rho^+(\lambda_{\mathbf{p}}, \mathbf{p})\rangle = \lim_{p^0 \to \omega_{\mathbf{p}}} \epsilon^*_\mu(\lambda_{\mathbf{p}}, p) i[-(p^0)^2 + \mathbf{p}^2 + m^2] \int d^4x e^{ip^0x^0 - i\mathbf{p}\cdot\mathbf{x}}\ T\overline{d}(x)\gamma^\mu u(x)T^{-1}\ |0\rangle\,. \tag{54}$$

To simplify this we substitute $\epsilon^{\mu *}(\lambda_{\mathbf{p}}, p) = (-1)^\lambda \epsilon^\mu(-\lambda_{\mathbf{p}}, p)$ and[2]

$$T\overline{d}(x)\gamma^\mu u(x)T^{-1} = -\mathcal{T}^\mu{}_\nu \overline{d}(\mathcal{T}x)\gamma^\nu u(\mathcal{T}x)\,, \tag{56}$$

---

[2] The identify for $\epsilon^{\mu *}(p, \lambda)$ follows from the definition

$$\epsilon^\mu(p, \lambda) = \left(\frac{\vec{p}\cdot\hat{e}_\lambda}{\sqrt{p^2}},\ \hat{e}_\lambda + \frac{1}{\vec{p}^2}\left(\frac{p^0}{\sqrt{p^2}} - 1\right)\vec{p}\cdot\hat{e}_\lambda\ \vec{p}\right)\,, \tag{55}$$

together with $\hat{e}_\lambda^* = (-1)^\lambda \hat{e}_{-\lambda}$.

to reach

$$T|\rho^+(\lambda_{\mathbf{p}},\mathbf{p})\rangle = (-1)^{\lambda_{\mathbf{p}}} \lim_{p^0\to\omega_{\mathbf{p}}} \mathcal{T}^\mu{}_\nu \epsilon_\mu(-\lambda_{\mathbf{p}},p)(1/i)[-(p^0)^2+\mathbf{p}^2+m^2]\int d^4x e^{ip_\mu \mathcal{T}^\mu{}_\nu x^\nu}\ \overline{d}(x)\gamma^\nu u(x)|0\rangle\,, \tag{57}$$

Finally using $\mathcal{T}^\mu{}_\nu\epsilon_\mu(p,-\lambda_{\mathbf{p}}) = \epsilon_\nu(p^0,-\mathbf{p},-\lambda_{\mathbf{p}})$, we conclude

$$T|\rho^+(\lambda_{\mathbf{p}},\mathbf{p})\rangle = (-1)^{\lambda_{\mathbf{p}}}|\rho^+(-\lambda_{\mathbf{p}},-\mathbf{p})\rangle = (-1)^{\lambda_{\mathbf{p}}}|\rho^+(\lambda_{-\mathbf{p}},-\mathbf{p})\rangle\,. \tag{58}$$

Note, the spin and momentum change sign, as expected, which implies that that the helicity does not flip sign.

Next we use the identity

$$\langle\alpha|\mathcal{O}|\beta\rangle = \left[\Big(T|\alpha\rangle\Big)^\dagger T\mathcal{O}T^{-1}\Big(T|\beta\rangle\Big)\right]^*\,, \tag{59}$$

to write

$$\langle\rho'(\lambda_f,p_f)|\mathcal{J}^\mu|\rho(\lambda_i,p_i)\rangle = -(-1)^{\lambda_i+\lambda_f}\mathcal{T}^\mu{}_\nu\langle\rho'(-\lambda_f,-\mathbf{p}_f)|\mathcal{J}^\nu|\rho(-\lambda_i,-\mathbf{p}_i)\rangle^*\,. \tag{60}$$

[Not sure I understand this equation. **So Eq. (59) is indeed a bit strange but I think it is crucial to extracting the properties we want. I also think $T^2=1$ cannot be generally used because it does not hold for all states. One can also see that it leads to very weird predictions. For example suppose $T|\psi\rangle = |\psi\rangle$. Then one can argue**

$$i\langle\psi|\psi\rangle = \langle\psi|(i)|\psi\rangle = \langle\psi|T^2(i)T^2|\psi\rangle = \langle\psi|T(i)T|\psi\rangle = \langle\psi|(-i)|\psi\rangle = -i\langle\psi|\psi\rangle \Longrightarrow \langle\psi|\psi\rangle = 0\,. \tag{61}$$

Clearly this does not make sense and I think that inserting T^2 is the incorrect step. Using what we found above, we have

$$TT|\rho^+(\lambda_{\mathbf{p}},\mathbf{p})\rangle = (-1)^{\lambda_{\mathbf{p}}}T|\rho^+(-\lambda_{\mathbf{p}},-\mathbf{p})\rangle = (-1)^{\lambda_{\mathbf{p}}-\lambda_{\mathbf{p}}}|\rho^+(\lambda_{\mathbf{p}},\mathbf{p})\rangle = |\rho^+(\lambda_{\mathbf{p}},\mathbf{p})\rangle \tag{62}$$

meaning $T^2=1$ and $T=T^{-1}$. So then

$$\begin{aligned}\langle\rho^+(\lambda_{\mathbf{p}'},\mathbf{p}')|J^\mu|\rho^+(\lambda_{\mathbf{p}},\mathbf{p})\rangle &= \langle\rho^+(\lambda_{\mathbf{p}'},\mathbf{p}')|T^2J^\mu T^2|\rho^+(\lambda_{\mathbf{p}},\mathbf{p})\rangle\\ &= (-1)^{\lambda_{\mathbf{p}}+\lambda_{\mathbf{p}'}}\langle\rho^+(-\lambda_{\mathbf{p}'},-\mathbf{p}')|J_\mu|\rho^+(-\lambda_{\mathbf{p}},-\mathbf{p})\rangle\\ &= (-1)^{\lambda_{\mathbf{p}}+\lambda_{\mathbf{p}'}}\Big(\langle\rho^+(-\lambda_{\mathbf{p}},-\mathbf{p})|J_\mu^\dagger|\rho^+(-\lambda_{\mathbf{p}'},-\mathbf{p}')\rangle\Big)^*\\ &= (-1)^{\lambda_{\mathbf{p}}+\lambda_{\mathbf{p}'}}\Big(\langle\rho^+(-\lambda_{\mathbf{p}},-\mathbf{p})|J_\mu|\rho^+(-\lambda_{\mathbf{p}'},-\mathbf{p}')\rangle\Big)^* \qquad (63)\\ &= (-1)^{\lambda_{\mathbf{p}}+\lambda_{\mathbf{p}'}}\langle\rho^+(-\lambda_{\mathbf{p}'},-\mathbf{p}')|J_\mu|\rho^+(-\lambda_{\mathbf{p}},-\mathbf{p})\rangle \qquad (64)\end{aligned}$$

where in going from the third to fourth equality I used the fact that the current is Hermitian. **[It is important to note that our goal here is to find a relation between $\langle\alpha|\mathcal{O}|\beta\rangle$ and $\langle\alpha|\mathcal{O}|\beta\rangle^*$. If we find a relation between $\langle\alpha|\mathcal{O}|\beta\rangle$ and $\langle\beta|\mathcal{O}|\alpha\rangle^*$ then we just get a statement about operator hermiticity but not about the reality of the matrix element.]**]

To understand the consequences of this identity, suppose that only the first form factor, $\widetilde{G}_1$ were nonzero. This would imply

$$\begin{aligned}(p_f+p_i)^\mu\,\epsilon_f^*\cdot\epsilon_i\,\widetilde{G}_1(Q^2,p_i^2,p_f^2) &= \Big[-(-1)^{\lambda_i+\lambda_f}\mathcal{T}^\mu{}_\nu\mathcal{P}^\nu{}_\alpha(p_f+p_i)^\alpha\,\epsilon^*(-\lambda_f,p_f)\cdot\epsilon(-\lambda_i,p_i)\,\widetilde{G}_1(Q^2,p_i^2,p_f^2)\Big]^*\,, \qquad (65)\\ &= \Big[-\mathcal{T}^\mu{}_\nu\mathcal{P}^\nu{}_\alpha(p_f+p_i)^\alpha\,\epsilon_f\cdot\epsilon_i^*\,\widetilde{G}_1(Q^2,p_i^2,p_f^2)\Big]^*\,, \qquad (66)\\ &= (p_f+p_i)^\mu\,\epsilon_f^*\cdot\epsilon_i\,\widetilde{G}_1^*(Q^2,p_i^2,p_f^2)\,, \qquad (67)\end{aligned}$$

where in the first line we used $\epsilon^*(-\lambda_f,(p_f^0,-\mathbf{p}_f))\cdot\epsilon(-\lambda_i,(p_f^0,-\mathbf{p}_i)) = \epsilon^*(-\lambda_f,p_f)\cdot\epsilon(-\lambda_i,p_i)$. We deduce

$$\widetilde{G}_1(Q^2,p_i^2,p_f^2) = \widetilde{G}_1^*(Q^2,p_i^2,p_f^2)\,. \tag{68}$$

A similar argument can be used to show that all six scalar form factors satisfy this condition. **[Check this!]**

We next turn to the parity invariance of the theory. Note that, while $\epsilon_f^{*\mu}$ is a pseudovector, the product $(p_f\cdot\epsilon_i)\,\epsilon_f^{*\mu}$ defines a regular (non-pseudo) vector. This is because the combination $p_f\cdot\epsilon_i$ is a pseudoscalar. In particular

$$(p_f\cdot\epsilon_i)\,\epsilon_f^{*\mu} \xrightarrow[P]{} (-\,p_f\cdot\epsilon_i)\left[-\,\mathcal{P}^{\mu}{}_{\nu}\epsilon_f^{*\nu}\right] = \mathcal{P}^{\mu}{}_{\nu}\,(p_f\cdot\epsilon_i)\,\epsilon_f^{*\nu}\,. \tag{69}$$

$$\begin{aligned}\epsilon_f^{*\mu}\,p_f\cdot\vec{\epsilon}_i &= -\epsilon_f^{*\mu}\,\vec{p}_f\cdot\vec{\epsilon}_i\\ &= \left(\frac{\vec{p}_f\cdot\hat{e}_{\lambda_f}}{\sqrt{p_f^2}},\hat{e}_{\lambda_f}+\frac{\vec{p}_f\cdot\hat{e}_{\lambda_f}}{\sqrt{p_f^2}\left(\sqrt{p_f^2}+p_f^0\right)}\vec{p}_f\right)\vec{p}_f\cdot\vec{\epsilon}_i\\ &\to -\left(-\frac{\vec{p}_f\cdot\hat{e}_{-\lambda_f}}{\sqrt{p_f^2}},\hat{e}_{-\lambda_f}+\frac{\vec{p}_f\cdot\hat{e}_{-\lambda_f}}{\sqrt{p_f^2}\left(\sqrt{p_f^2}+p_f^0\right)}\vec{p}_f\right)\vec{p}_f\cdot\vec{\epsilon}_i.\end{aligned} \tag{70}$$

The zero helicity component satisfies $\hat{e}_{-\lambda_f}=\hat{e}_{\lambda_f}$, while the other components satisfy $\hat{e}_{-\lambda_f}=-\hat{e}^*_{\lambda_f}$ ($\hat{e}_{-\lambda_f}=-\hat{e}_{\lambda_f}$). So, we find that the $\lambda_f=\pm1$ pieces transform like a vector under parity. **[How are we using parity here?]**

Finally, we make use of Hermiticity, i.e. $\mathcal{J}_\mu=\mathcal{J}_\mu^\dagger$, to find

$$\langle\rho'(\lambda_f,p_f)|\mathcal{J}^\mu|\rho(\lambda_i,p_i)\rangle = \langle\rho(\lambda_i,p_i)|\mathcal{J}^{\mu\dagger}|\rho'(\lambda_f,p_f)\rangle^* = \langle\rho(\lambda_i,p_i)|\mathcal{J}^{\mu}|\rho'(\lambda_f,p_f)\rangle^*. \tag{71}$$

Note, all kinematic terms in front $\widetilde{G}_1,\widetilde{G}_3,\widetilde{G}_5$ and $\widetilde{G}_2,\widetilde{G}_3,\widetilde{G}_6$ are respectively symmetric and antisymmetric with respect to the interchange of i and f together with complex conjugation. This, in conjunction with the fact that the kinematic terms are independent of each other, implies that $\widetilde{G}_1,\widetilde{G}_3,\widetilde{G}_5$ and $\widetilde{G}_2,\widetilde{G}_3,\widetilde{G}_6$ are respectively symmetric and antisymmetric under $(p_i^2,p_f^2)\to(p_f^2,p_i^2)$. **[To make this more clean one should define projectors.]**

To convince oneself of this, one can consider an example of this being implemented in practice. For example, let $\vec{p}_i=\vec{p}_f=0$, then all terms vanish except those appearing in the first two lines

$$\lim_{\vec{p}_i,\vec{p}_f=0}\langle\rho'(\lambda_f,p_f)|\mathcal{J}^\mu|\rho(\lambda_i,p_i)\rangle = (p_f+p_i)^\mu\,\epsilon_f^*\cdot\epsilon_i\,\widetilde{G}_1(Q^2,p_i^2,p_f^2)+(p_i-p_f)^\mu\,\epsilon_f^*\cdot\epsilon_i\,\widetilde{G}_2(Q^2,p_i^2,p_f^2)\,. \tag{72}$$

This then implies

$$\begin{aligned}&(p_f+p_i)^\mu\,\epsilon_f^*\cdot\epsilon_i\,\widetilde{G}_1(Q^2,p_i^2,p_f^2)+(p_i-p_f)^\mu\,\epsilon_f^*\cdot\epsilon_i\,\widetilde{G}_2(Q^2,p_i^2,p_f^2)\\ &\qquad= (p_f+p_i)^\mu\,\epsilon_f^*\cdot\epsilon_i\,\widetilde{G}_1(Q^2,p_f^2,p_i^2)-(p_i-p_f)^\mu\,\epsilon_f^*\cdot\epsilon_i\,\widetilde{G}_2(Q^2,p_f^2,p_i^2)\,,\end{aligned} \tag{73}$$

and thus

$$\widetilde{G}_1(Q^2,p_i^2,p_f^2)=\widetilde{G}_1(Q^2,p_f^2,p_i^2)\,,\qquad \widetilde{G}_2(Q^2,p_i^2,p_f^2)=-\widetilde{G}_2(Q^2,p_f^2,p_i^2)\,. \tag{74}$$

Next, we impose charge conservation

$$\begin{aligned}(p_i-p_f)_\mu\langle\rho(\lambda_i,p_i)|\mathcal{J}^\mu|\rho'(\lambda_f,p_f)\rangle &= (p_i^2-p_f^2)\,\epsilon_f^*\cdot\epsilon_i\,\widetilde{G}_1(Q^2,p_i^2,p_f^2)-Q^2\,\epsilon_f^*\cdot\epsilon_i\,\widetilde{G}_2(Q^2,p_i^2,p_f^2)\\ &\quad+\left(-p_f\cdot\epsilon_i\,p_i\cdot\epsilon_f^*+p_i\cdot\epsilon_f^*\,p_f\cdot\epsilon_i\right)\widetilde{G}_3(Q^2,p_i^2,p_f^2)\\ &\quad-2p_f\cdot\epsilon_i\,p_i\cdot\epsilon_f^*\,\widetilde{G}_4(Q^2,p_i^2,p_f^2)\\ &\quad+(p_i^2-p_f^2)\,(\epsilon_f^*\cdot p_i)\,(\epsilon_i\cdot p_f)\,\widetilde{G}_5(Q^2,p_i^2,p_f^2)-Q^2\,(\epsilon_f^*\cdot p_i)\,(\epsilon_i\cdot p_f)\,\widetilde{G}_6(Q^2,p_i^2,p_f^2)\\ &= (p_i^2-p_f^2)\,\epsilon_f^*\cdot\epsilon_i\,\widetilde{G}_1(Q^2,p_i^2,p_f^2)-Q^2\,\epsilon_f^*\cdot\epsilon_i\,\widetilde{G}_2(Q^2,p_i^2,p_f^2)\\ &\quad-2p_f\cdot\epsilon_i\,p_i\cdot\epsilon_f^*\,\widetilde{G}_4(Q^2,p_i^2,p_f^2)\\ &\quad+(p_i^2-p_f^2)\,(\epsilon_f^*\cdot p_i)\,(\epsilon_i\cdot p_f)\,\widetilde{G}_5(Q^2,p_i^2,p_f^2)-Q^2\,(\epsilon_f^*\cdot p_i)\,(\epsilon_i\cdot p_f)\,\widetilde{G}_6(Q^2,p_i^2,p_f^2)\\ &= 0.\end{aligned} \tag{75}$$

We see that we get two independent equations here. Note, this equation must be satisfied for all values of p_i, p_f, λ, and λ'. So if let $\vec{p}_i = \vec{p}_f = 0$, we get

$$(p_i^2 - p_f^2)\ \widetilde{G}_1(Q^2, p_i^2, p_f^2) - Q^2\ \widetilde{G}_2(Q^2, p_i^2, p_f^2) = 0 \tag{76}$$

$$\Rightarrow \widetilde{G}_2(Q^2, p_i^2, p_f^2) = \frac{(p_i^2 - p_f^2)}{Q^2}\ \widetilde{G}_1(Q^2, p_i^2, p_f^2). \tag{77}$$

Inserting this constraint into Eq. 75, we then get

$$-2\,\widetilde{G}_4(Q^2, p_i^2, p_f^2) + (p_i^2 - p_f^2)\ \widetilde{G}_5(Q^2, p_i^2, p_f^2) - Q^2\widetilde{G}_6(Q^2, p_i^2, p_f^2)\ = 0,$$

$$\Rightarrow \widetilde{G}_6(Q^2, p_i^2, p_f^2) = \frac{(p_i^2 - p_f^2)}{Q^2}\,\widetilde{G}_5(Q^2, p_i^2, p_f^2) - 2\,\widetilde{G}_4(Q^2, p_i^2, p_f^2). \tag{78}$$

Note, this is consistent with the observation made above that $\widetilde{G}_1, \widetilde{G}_3, \widetilde{G}_5$ and $\widetilde{G}_2, \widetilde{G}_4, \widetilde{G}_6$ are symmetric and antisymmetric.

Putting all of the constraints in place, we find that the matrix element can be described in terms of four form factors

$$\begin{aligned}
\langle \rho'(\lambda_f, p_f)|\mathcal{J}^\mu|\rho(\lambda_i, p_i)\rangle = & \left((p_f + p_i)^\mu\ + (p_i - p_f)^\mu \frac{(p_i^2 - p_f^2)}{Q^2}\right)\ (\epsilon_f^* \cdot \epsilon_i)\ \widetilde{G}_1(Q^2, p_i^2, p_f^2) \\
& + \left(\epsilon_i^\mu\, p_i \cdot \epsilon_f^*\ + \epsilon_f^{*\mu}\, p_f \cdot \epsilon_i\right)\ \widetilde{G}_3(Q^2, p_i^2, p_f^2) \\
& + \left(\epsilon_i^\mu\, p_i \cdot \epsilon_f^*\ - \epsilon_f^{*\mu}\, p_f \cdot \epsilon_i - 2(p_i - p_f)^\mu\,(\epsilon_f^* \cdot p_i)\,(\epsilon_i \cdot p_f)\right)\ \widetilde{G}_4(Q^2, p_i^2, p_f^2) \\
& + \left((p_f + p_i)^\mu\ + (p_i - p_f)^\mu \frac{(p_i^2 - p_f^2)}{Q^2}\right)(\epsilon_f^* \cdot p_i)\,(\epsilon_i \cdot p_f)\,\widetilde{G}_5(Q^2, p_i^2, p_f^2).
\end{aligned} \tag{79}$$

[I fixed a minor typo here.] For the case where $p_f^2 = p_i^2$ this is consistent with the standard Lorentz decomposition of the matrix elements of a stable spin-1 particle, e.g., Eq. (5) in Ref. [3]. This is because $\lim_{p_i^2 = p_f^2} \widetilde{G}_4(Q^2, p_i^2, p_f^2) = -\lim_{p_i^2 = p_f^2} \widetilde{G}_4(Q^2, p_f^2, p_i^2) = 0$. So one finds that indeed only three form factors are needed to describe the matrix element.

[not sure about this now...Also, Ref. [4] gives an expression, Eq. (2.1), which claims to be the most general form for a electroweak transition matrix element between two spin-1 particles with arbitrary masses. When the current is just the electromagnetic current, it states that only three form factors survive. This is reassuring but the relationship between their expression and this one is not immediately obvious.]

[compare with the equation appearing in page 18 of arXiv:0902.2241v1 [hep-ph]]

D. Lorentz decomposition in terms of amplitudes

In the previous section we reviewed the Lorentz decomposition of the matrix elements of the QED current involving stable states. We are ultimately interested in the matrix elements both the initial and final states are unstable under the strong interactions. In particular, we will consider two scenarios: $\pi\gamma^\star \to \pi\pi$ and $\pi\pi\gamma \to \pi\pi$. The aim is to show that by first writing the Lorentz decomposition of the amplitudes, proceeded by the partial wave projection, one finds the same Lorentz structure as was founding the previous section.

In this section we first write the general Lorentz decomposition of these amplitudes. While in the following section we confirm the equivalence of these with the matrix elements decomposition of the previous section.

1. *Lorentz decomposition of $\pi\gamma^\star \to \pi\pi$*

[A good reference on this is 1210.6793v2]

First, let us decompose

$$\gamma^\star(q)\pi^+(p_1) \to \pi^+(p_2)\pi^0(p_3), \tag{80}$$

in terms of the Mandelstam variables $s = (q + p_1)^2$, $t = (p_1 - p_2)^2$, $u = (p_1 - p_3)^2$. The amplitude, or the matrix element, must be a pseudo-vector. To see this, consider the matrix element

$$\langle \pi^+(p_2)\pi^0(p_3)|\mathcal{J}^\mu|\pi^+(p_1)\rangle. \tag{81}$$

Parity tells us

$$\begin{aligned}\langle \pi^+(p_2)\pi^0(p_3)|\mathcal{J}^\mu|\pi^+(p_1)\rangle &= (-1)^3\langle \pi^+(p_2^0, -\vec{p}_2)\pi^0(p_3^0, -\vec{p}_3)|\mathcal{J}_\mu|\pi^+(p_1^0, -\vec{p}_1)\rangle \\ &= -\langle \pi^+(p_2^0, -\vec{p}_2)\pi^0(p_3^0, -\vec{p}_3)|\mathcal{J}_\mu|\pi^+(p_1^0, -\vec{p}_1)\rangle.\end{aligned} \tag{82}$$

Here, we have only four vector to consider, namely the four momenta. Only three of these are linearly independent of each other. Furthermore, a dot product between any two of momenta can be written in terms of the Mandelstam variables. The only pseudovector we can construct in terms of these is

$$V^\mu = \epsilon^{\mu\nu\alpha\beta} p_{1,\nu} p_{2,\alpha} p_{3,\beta}. \tag{83}$$

Under parity

$$V^0 = \epsilon^{0ijk} p_{1,i} p_{2,j} p_{3,j} \to (-1)^3 \epsilon^{0ijk} p_{1,i} p_{2,j} p_{3,j} = -V_0 \tag{84}$$

$$V^i = \epsilon^{i0jk} p_{1,0} p_{2,j} p_{3,j} + \cdots \to (-1)^2 \epsilon^{i0jk} p_{1,0} p_{2,j} p_{3,j} + \cdots = V^i = -V_i. \tag{85}$$

where the "$\cdots$" denote the other components of the tensor which transform the same way.

Note, any other term can be rewritten in the same form above, e.g.,

$$\epsilon^{\mu\nu\alpha\beta} q_\nu p_{2,\alpha} p_{3,\beta} = \epsilon^{\mu\nu\alpha\beta}(p_2 + p_3 - p_1)_\nu p_{2,\alpha} p_{3,\beta} = -\epsilon^{\mu\nu\alpha\beta} p_{1,\nu} p_{2,\alpha} p_{3,\beta}. \tag{86}$$

We conclude that the amplitude can only have the following form

$$\mathcal{A}^\mu_{\pi\gamma^\star,\pi\pi}(s,t,u) = \epsilon^{\mu\nu\alpha\beta} p_{1,\nu} p_{2,\alpha} p_{3,\beta} \mathcal{F}_1(s,t,u). \tag{87}$$

Charge conservation implies

$$q_\mu \mathcal{A}^\mu_{\pi\gamma^\star,\pi\pi}(s,t,u) = 0, \tag{88}$$

which is clearly satisfied since $q = p_2 + p_3 - p_1$.

At this stage, we have found qualitative agreement with the expression for the $\rho\gamma^\star \to \pi$ matrix element, Eq. 47. Namely, we find that these two can be written in terms of a single scalar function. Equation 87 encodes more physics, since the final $\pi\pi$ need not have just the quantum numbers of the ρ.

2. *Lorentz decomposition of* $\pi\pi\gamma^\star \to \pi\pi$

see https://arxiv.org/pdf/hep-ph/0203075.pdf
https://arxiv.org/pdf/hep-ph/0209056.pdf

If instead, we consider the $\pi\pi\gamma \to \pi\pi$ matrix element, now we have have four linearly independent vectors, the four momenta of the system $p_1^\mu, p_2^{\mu'}, p_3^\nu, p_4^{\nu'}$. The matrix element is now a Lorentz vector,

$$\langle \pi^+(p_3); \pi^0(p_4)|\mathcal{J}^\mu|\pi^+(p_1)\pi^0(p_2)\rangle, \tag{89}$$

so we can, in principle, write the Lorentz decomposition in terms of these four vectors. Instead of doing this, let me first observe that this has more degrees of freedom that the previously example. With the ultimately goal of matching to the formalism presented in Ref. [1], we will use the following set of variables to described the on-shell scalar amplitudes,

$$Q^2 = -(P_i - P_f)^2, E_i^\star, \hat{k}_i^\star, E_f^\star, \hat{k}_f^\star, \tag{90}$$

and use the total and relative initial/final momenta (P_i, P_f, k_i, k_f) to describe the vector dependence in the moving frame, defined as

$$p_1 = \frac{P_i - k_i}{2}, \tag{91}$$

$$p_2 = \frac{P_i + k_i}{2}, \tag{92}$$

$$p_3 = \frac{P_f - k_f}{2}, \tag{93}$$

$$p_4 = \frac{P_f + k_f}{2}, \tag{94}$$

$$\tag{95}$$

With this, we have

$$\begin{aligned}\mathcal{W}^{\mu}_{\pi\pi\gamma^\star,\pi\pi}(p_1,p_2;p_3,p_4) = P^{+\mu}\mathcal{F}_1(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star) + q^\mu\mathcal{F}_2(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star)\\ + k^{+\mu}\mathcal{F}_3(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star) + k^{-\mu}\mathcal{F}_4(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star).\end{aligned} \tag{96}$$

$$P^+ = P_i + P_f = p_1 + p_2 + p_3 + p_4 \tag{97}$$

$$q = P^- = P_i - P_f = p_1 + p_2 - p_3 - p_4 \tag{98}$$

$$k^+ = k_i + k_f = p_2 - p_1 + p_4 - p_3 \tag{99}$$

$$k^- = k_i - k_f = p_2 - p_1 + p_3 - p_4. \tag{100}$$

The amplitude must be invariant under time-reversal, which means that it must satisfy

$$\mathcal{W}^{\mu}_{\pi\pi\gamma^\star,\pi\pi}(p_1,p_2;p_3,p_4) = \mathcal{W}_{\mu\,\pi\pi\gamma^\star,\pi\pi}(\tilde{p}_3,\tilde{p}_4;\tilde{p}_1,\tilde{p}_2) \tag{101}$$

where $\tilde{p}^\mu \equiv p_\mu$. This implies

$$\mathcal{F}_j(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star) = (-1)^{j+1}\mathcal{F}_j(Q^2;E_f^\star,\hat{k}_f^\star;E_i^\star,\hat{k}_i^\star). \tag{102}$$

Charge conservation implies that one of these amplitudes is not linearly independent

$$\begin{aligned}P^-_\mu\mathcal{W}^{\mu}_{\pi\pi\gamma^\star,\pi\pi}(p_1,p_2;p_3,p_4) &= (E_i^{*2} - E_f^{*2})\mathcal{F}_1(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star) - Q^2\mathcal{F}_2(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star)\\ &\quad + P^-_\mu k^{+\mu}\mathcal{F}_3(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star) + P^-_\mu k^{-\mu}\mathcal{F}_4(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star) \qquad (103)\\ &= 0 \qquad (104)\end{aligned}$$

$$\begin{aligned}\Rightarrow \mathcal{F}_2(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star) &= \frac{1}{Q^2}\Big[(E_i^{*2} - E_f^{*2})\mathcal{F}_1(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star)\\ &\quad + (P^-\cdot k^+)\mathcal{F}_3(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star) + (P^-\cdot k^-)\mathcal{F}_4(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star)\Big].\end{aligned} \tag{105}$$

Putting all of the pieces together, we get

$$\begin{aligned}\mathcal{W}^{\mu}_{\pi\pi\gamma^\star,\pi\pi}(p_1,p_2;p_3,p_4) &= \left(P^{+\mu} + \frac{(E_i^{*2} - E_f^{*2})}{Q^2}P^{-\mu}\right)\mathcal{F}_1(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star)\\ &\quad + \left(k^{+\mu} + \frac{(P^-\cdot k^+)}{Q^2}P^{-\mu}\right)\mathcal{F}_3(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star) + \left(k^{-\mu} + \frac{(P^-\cdot k^-)}{Q^2}P^{-\mu}\right)\mathcal{F}_4(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star).\end{aligned} \tag{106}$$

Because $\mathcal{F}_4(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star)$ is antisymmetric under the interchange of the initial and final states, it must vanish in the limit where the initial and final states are identical. We might want to make this explicit by redefining it in terms of

$$\mathcal{F}_4(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star) \equiv \frac{(E_i^{*2} - E_f^{*2})}{(E_i^{*2} + E_f^{*2})}\widetilde{\mathcal{F}}_4(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star) \tag{107}$$

Qualitative agreement between Eq. 106 and that for the matrix elements of the $\rho \to \rho'$, Eq. 79 is not so obvious. On one hand the latter seems to depend on four Lorentz scalars while the former depends on just three. This is somewhat misleading since the amplitudes appearing in Eq. 106 have a nontrivial angular dependence and they can consequently overlap with a large variety of different partial waves. One reassuring observation is that both have a term that exactly vanishes in the limit that the initial and final states are exactly degenerate.

Because Eq. 106 describes any allowed process where the initial and final states are projected to any partial wave, not just $\ell = 1$, one should be able to find that when the initial and final states are projected onto an S-wave, only one Lorentz scalar amplitude survives, leading to a similar expression as in Eq. 34. This already tells us that likely that in this limit, the terms proportional to $\mathcal{F}_3$ and $\mathcal{F}_4$ must vanish.

Questions that I do not yet have an answer to:

1) If we project the initial and final states onto S-wave, should only one of these survive? I think the answer is yes and it's probably just $\mathcal{F}_1$, since the others depend linearly on k^+ or k^-. but how do I show this? I would have assumed this expression holds for any partial wave, and then you

2) Do have the arguments of the $\mathcal{F}$'s right? I think so...but getting a bit confused.

3) how do I do the partial wave projection of this thing?

A reference that might be useful in all of this discussion is Ref. [5], although I am still deciphering it.

Relativistic quark model calculation of the ρ form factors [6]

E. Equivalence between Lorentz decompositions

1. Partial-wave projected states

In order to match the amplitudes with the form-factor decomposition, we will need to do a partial-wave projection of the amplitude. To do this, we follow steps presented in Ref. [7], which requires defining the partial-wave projected $\pi\pi$ states. In the cm-frame we can define these using spherical harmonics,

$$\big||\mathbf{k}^\star|;\ell\lambda\big\rangle = \int d\hat{\mathbf{k}}^\star\, \frac{Y_{\ell\lambda}(\hat{\mathbf{k}}^\star)}{\sqrt{4\pi}}\,\big|\pi(\mathbf{k}^\star)\pi(-\mathbf{k}^\star)\big\rangle, \tag{108}$$

For systems with arbitrary momentum $\mathbf{P}$, we define helicity states by first boosting along the $\hat{z}$-axis followed by a rotation to the momentum axis,

$$\big|\mathbf{P};|\mathbf{k}^\star|;\ell\lambda\big\rangle = U[R(\hat{P})]\,U[Z_P]\,\big||\mathbf{k}^\star|;\ell\lambda\big\rangle. \tag{109}$$

where $U[Z_P]$ and $U[R(\hat{P})]$ are the unitary transformations for performing the boost and rotation respectively.

The $\hat{z}$-axis boost acting on four-vectors can be expressed as

$$\big[Z_P\big]^{\mu}_{\ \nu} = \begin{bmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{bmatrix} = \frac{1}{E_i^\star}\begin{bmatrix} E_i & 0 & 0 & |\mathbf{P}| \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ |\mathbf{P}| & 0 & 0 & E_i \end{bmatrix} \tag{110}$$

since $\gamma = \frac{E_i}{E_i^\star}$ and $\beta\gamma = \frac{|\mathbf{P}|}{E_i^\star}$ where $E_i^\star = 2\omega_\pi = 2\sqrt{m_\pi^2 + k^{\star 2}}$. Then the action of the boost on $k^\mu = \big(\omega_\pi, \mathbf{k}^\star\big)$ and $\bar{k}^\mu = \big(\omega_\pi, -\mathbf{k}^\star\big)$ is

$$k'^{\mu} = \big[Z_P\big]^{\mu}_{\ \nu}\, k^\nu = \begin{bmatrix} \frac{E_i}{2} + \frac{|\mathbf{P}|}{2\omega_\pi}k_z^\star \\ k_x^\star \\ k_y^\star \\ \frac{|\mathbf{P}|}{2} + \frac{E_i}{E_i^\star}k_z^\star \end{bmatrix},$$
$$\bar{k}'^{\mu} = \big[Z_P\big]^{\mu}_{\ \nu}\, \bar{k}^\nu = \begin{bmatrix} \frac{E_i}{2} - \frac{|\mathbf{P}|}{2\omega_\pi}k_z^\star \\ -k_x^\star \\ -k_y^\star \\ \frac{|\mathbf{P}|}{2} - \frac{E_i}{E_i^\star}k_z^\star \end{bmatrix}, \tag{111}$$

and as expected, $\bar{k}'^{\mu} = \widetilde{P}^{\mu} - k'^{\mu}$, where

$$\widetilde{P}^{\mu} = \begin{bmatrix} E_i \\ 0 \\ 0 \\ |\mathbf{P}|\,. \end{bmatrix} \tag{112}$$

We point the reader to Ref. [7] for details for how this can then be used to find the equivalence between Eq. 47 and Eq. 87.

2. Equivalence between $\gamma^\star\sigma \to \sigma$ and $\gamma^\star(\pi\pi)_{\ell=0} \to (\pi\pi)_{\ell=0}$

The Lorentz decomposition of the matrix element $\gamma^\star\sigma \to \sigma$ is the same as that of $\gamma^\star\pi \to \pi$, and therefore will not be repeated here. This must also be the same as Eq. 106 when the initial and final state are projected onto an $\ell = 0$ partial wave,

$$\begin{aligned}
&\langle \mathbf{P}_f; |\mathbf{k}^\star_f|; \ell = 0 | \mathcal{J}^\mu(0) | \mathbf{P}_i; |\mathbf{k}^\star_i|; \ell = 0\rangle \\
&\quad = \frac{1}{(4\pi)^2} \int d\hat{\mathbf{k}}^\star_i \int d\hat{\mathbf{k}}^\star_f \langle \pi(p_3)\pi(p_4) | \mathcal{J}^\mu(0) | \pi(p_1)\pi(p_2)\rangle, \\
&\quad = \frac{1}{(4\pi)^2} \int d\hat{\mathbf{k}}^\star_i \int d\hat{\mathbf{k}}^\star_f \Bigg[\left(P^{+\mu} + \frac{(E^{\star 2}_i - E^{\star 2}_f)}{Q^2} P^{-\mu} \right) \mathcal{F}_1(Q^2; E^\star_i, \hat{k}^\star_i; E^\star_f, \hat{k}^\star_f) \\
&\quad + \left(k^{+\mu} + \frac{(P^- \cdot k^+)}{Q^2} P^{-\mu} \right) \mathcal{F}_3(Q^2; E^\star_i, \hat{k}^\star_i; E^\star_f, \hat{k}^\star_f) + \left(k^{-\mu} + \frac{(P^- \cdot k^-)}{Q^2} P^{-\mu} \right) \mathcal{F}_4(Q^2; E^\star_i, \hat{k}^\star_i; E^\star_f, \hat{k}^\star_f) \Bigg].
\end{aligned} \tag{113}$$

We can simplify this by first writing the angular dependence of the amplitudes in terms of spherical harmonics,

$$\mathcal{F}_j(Q^2; E^\star_i, \hat{k}_i; E^\star_f, \hat{k}^\star_f) = 4\pi Y_{l'm'}(\hat{k}^\star_f) \mathcal{F}_{j;l'm';lm}(Q^2; E^\star_i; E^\star_f) Y_{lm}(\hat{k}^\star_i)\,. \tag{114}$$

Next, we note that these amplitudes must be invariant under parity. This tells us that

$$\mathcal{F}_{j;1m;00} = 0. \tag{115}$$

This is sufficient to show that

$$\langle \mathbf{P}_f; |\mathbf{k}^\star_f|; \ell = 0 | \mathcal{J}^\mu(0) | \mathbf{P}_i; |\mathbf{k}^\star_i|; \ell = 0\rangle = \left(P^{+\mu} + \frac{(E^{\star 2}_i - E^{\star 2}_f)}{Q^2} P^{-\mu} \right) \mathcal{F}_s(Q^2; E^\star_i; E^\star_f) \tag{116}$$

where

$$\mathcal{F}_s(Q^2; E^\star_i; E^\star_f) = \mathcal{F}_{1;00;00}(Q^2; E^\star_i; E^\star_f) + \frac{1}{2} \mathcal{F}_{3;00;00}(Q^2; E^\star_i; E^\star_f). \tag{117}$$

Now, we proceed to derive this expression. Using the partial wave projection of $\mathcal{F}_1$ and noting that the kinematic term appearing in front of it in Eq. 113 carries no angular dependence, it is a straightforward exercise to show that $\mathcal{F}_s$ should depend on $\mathcal{F}_1$ as shown.

To see the dependence on $\mathcal{F}_3$ and $\mathcal{F}_4$, we note that the angular dependence is only more complicated by the fact that the kinematic terms in front of these functions depend on k^+ and k^-. Because these depend at most on the $\ell = 1$ partial waves. Thanks to this and Eq. 115, we can safely conclude that only the $\ell = \ell' = 0$ components of these amplitudes contribute. This is to say that we can consider the integral over the kinematic pieces in isolation. For example, to understand the factor in front of $\mathcal{F}_{3;00;00}$

$$\frac{1}{(4\pi)^2} \int d\hat{\mathbf{k}}^\star_i \int d\hat{\mathbf{k}}^\star_f \left(k^{+\mu} + \frac{(P^- \cdot k^+)}{Q^2} P^{-\mu} \right) = \frac{1}{4\pi} \int d\hat{\mathbf{k}}^\star_i \left(k^\mu_i + \frac{(P^- \cdot k_i)}{Q^2} P^{-\mu} \right) + \frac{1}{4\pi} \int d\hat{\mathbf{k}}^\star_f \left(k^\mu_f + \frac{(P^- \cdot k_f)}{Q^2} P^{-\mu} \right). \tag{118}$$

Now, let us consider the first term in the right by rewriting it in terms of the center of mass coordinates,

$$
\begin{aligned}
&\frac{1}{4\pi}\int d\hat{\mathbf{k}}_i^\star \left(k_i^\mu + \frac{(P^-\cdot k_i)}{Q^2}P^{-\mu}\right) \\
&\quad = \frac{1}{4\pi}\int d\hat{\mathbf{k}}_i^\star \left(\begin{bmatrix} \gamma_i\left(\omega_{k_i}^\star + \beta_i k_{i||}^\star\right) \\ \gamma_i\left(\beta_i\omega_{k_i}^\star + k_{i||}^\star\right) \\ k_{i\perp}^\star \end{bmatrix}^\mu + \frac{\left[P_0^-\gamma_i\left(\omega_{k_i}^\star + \beta_i k_{i||}^\star\right) - \vec{P}_{||}^-\cdot\hat{P}_i\gamma_i\left(\beta_i\omega_{k_i}^\star + k_{i||}^\star\right) - \vec{P}_{||}^-\cdot\vec{k}_\perp^\star\right]}{Q^2}P^{-\mu}\right) \\
&\quad = \frac{1}{4\pi}\int d\hat{\mathbf{k}}_i^\star \left(\begin{bmatrix} \gamma_i\omega_{k_i}^\star \\ \gamma_i\beta_i\omega_{k_i}^\star \\ 0 \end{bmatrix}^\mu + \frac{\left[P_0^-\gamma_i\omega_{k_i}^\star - \vec{P}_{||}^-\cdot\hat{P}_i\gamma_i\beta_i\omega_{k_i}^\star\right]}{Q^2}P^{-\mu}\right) \\
&\quad = \frac{1}{2}\left(P_i^\mu + \frac{P^-\cdot P_i}{Q^2}P^{-\mu}\right).
\end{aligned}
\tag{119}
$$

The last equality follows from the fact that $\gamma_i\omega_{k_i} = \frac{1}{2}\gamma_i E_i^\star = \frac{1}{2}E_i$ and $\beta_i\gamma_i\omega_{k_i} = \frac{1}{2}\beta_i E_i = \frac{1}{2}P_i$. It is easy to see that a similar decomposition holds for the term proportional to k_f.

Putting all of the pieces together we find

$$
\begin{aligned}
&\frac{1}{(4\pi)^2}\int d\hat{\mathbf{k}}_i^\star\int d\hat{\mathbf{k}}_f^\star\Bigg[\left(k^{+\mu} + \frac{(P^-\cdot k^+)}{Q^2}P^{-\mu}\right)\mathcal{F}_3(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star) + \left(k^{-\mu} + \frac{(P^-\cdot k^-)}{Q^2}P^{-\mu}\right)\mathcal{F}_4(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star)\Bigg] \\
&\qquad = \left(P^{+\mu} + \frac{(P^-\cdot P^+)}{Q^2}P^{-\mu}\right)\mathcal{F}_3(Q^2;E_i^\star;E_f^\star) + \left(P^{-\mu} + \frac{(P^-\cdot P^-)}{Q^2}P^{-\mu}\right)\mathcal{F}_4(Q^2;E_i^\star;E_f^\star) \\
&\qquad = \left(P^{+\mu} + \frac{(P^-\cdot P^+)}{Q^2}P^{-\mu}\right)\mathcal{F}_3(Q^2;E_i^\star;E_f^\star) + \left(P^{-\mu} - \frac{Q^2}{Q^2}P^{-\mu}\right)\mathcal{F}_4(Q^2;E_i^\star;E_f^\star) \\
&\qquad = \left(P^{+\mu} + \frac{(P^-\cdot P^+)}{Q^2}P^{-\mu}\right)\mathcal{F}_3(Q^2;E_i^\star;E_f^\star),
\end{aligned}
\tag{120}
$$

confirming Eq. 116.

This is a first important check on Eq. 106. Naively, one might have expected that three or more amplitudes would contributed to the S-wave $\gamma^\star(\pi\pi)_{\ell=0}\to(\pi\pi)_{\ell=0}$ amplitude. Here we see that only one linear combination contributes, as one could have argued from the fact that a scalar meson can only have one electromagnetic form factor, as shown in Eq. 34.

3. *Checking equivalence between $\gamma^\star\rho\to\rho'$ and $\gamma^\star(\pi\pi)_{\ell=1}\to(\pi\pi)_{\ell=1}$*

Here we follow similar steps to the ones above to show that when the initial and final states appearing in Eq. 106 are projected onto an an $\ell=1$ partial wave, the subsequent amplitude has the same Lorentz decomposition as that of the $\gamma^\star\rho\to\rho$ electromagnetic matrix elements, Eq. 79. We begin by writing the $\ell=1$ partial wave projection of the amplitude,

$$
\begin{aligned}
&\langle\mathbf{P}_f;|\mathbf{k}_f^\star|;\ell=1\lambda_f|\mathcal{J}^\mu(0)|\mathbf{P}_i;|\mathbf{k}_i^\star|;\ell=1\lambda_i\rangle \\
&\qquad = \int d\hat{\mathbf{k}}_i^\star\int d\hat{\mathbf{k}}_f^\star\frac{Y_{1\lambda_i}(\hat{k}_i^\star)}{\sqrt{4\pi}}\frac{Y_{1\lambda_f}^*(\hat{k}_f^\star)}{\sqrt{4\pi}}\langle\pi(p_3)\pi(p_4)|\mathcal{J}^\mu(0)|\pi(p_1)\pi(p_2)\rangle, \\
&\qquad = \int d\hat{\mathbf{k}}_i^\star\int d\hat{\mathbf{k}}_f^\star\frac{Y_{1\lambda_i}(\hat{k}_i^\star)}{\sqrt{4\pi}}\frac{Y_{1\lambda_f}^*(\hat{k}_f^\star)}{\sqrt{4\pi}}\Bigg[\left(P^{+\mu} + \frac{(E_i^{\star2}-E_f^{\star2})}{Q^2}P^{-\mu}\right)\mathcal{F}_1(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star) \\
&\qquad + \left(k^{+\mu} + \frac{(P^-\cdot k^+)}{Q^2}P^{-\mu}\right)\mathcal{F}_3(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star) + \left(k^{-\mu} + \frac{(P^-\cdot k^-)}{Q^2}P^{-\mu}\right)\mathcal{F}_4(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star)\Bigg].
\end{aligned}
\tag{121}
$$

[I would like to first write down the final result and then derive it...but I don't know it yet]

To arrive at this equation, we

Looking at the first term, it is to see that

$$
\int d\hat{\mathbf{k}}_i^\star\int d\hat{\mathbf{k}}_f^\star\frac{Y_{1\lambda_i}(\hat{k}_i^\star)}{\sqrt{4\pi}}\frac{Y_{1\lambda_f}^*(\hat{k}_f^\star)}{\sqrt{4\pi}}(4\pi)Y_{l'm'}(\hat{k}_f^\star)Y_{lm}^*(\hat{k}_i^\star)\mathcal{F}_{1;lm;l'm'}(Q^2;E_i^\star;E_f^\star) = \mathcal{F}_{1;1\lambda_i;1\lambda_f}(Q^2;E_i^\star;E_f^\star)\,. \tag{122}
$$

Note, this has the same Lorentz structure as the first and last terms in Eq. 79,

$$\langle \rho'(\lambda_f,P_f)|\mathcal{J}^\mu|\rho(\lambda_i,P_i)\rangle \supset \left((P_f+P_i)^\mu \, +(P_i-P_f)^\mu\frac{(E_i^{\star 2}-E_f^{\star 2})}{Q^2}\right)\left(\epsilon_f^*\cdot\epsilon_i\right)\,\widetilde{G}_1(Q^2,P_i^2,P_f^2)$$
$$+\left((P_f+P_i)^\mu \, +(P_i-P_f)^\mu\frac{(P_i^2-P_f^2)}{Q^2}\right)(\epsilon_f^*\cdot P_i)\,(\epsilon_i\cdot P_f)\,\widetilde{G}_5(Q^2,P_i^2,P_f^2).,\tag{123}$$

but the latter depend on only two amplitudes, while the former could, in principle depend on 9 ($\lambda_f,\lambda_i=-1,0,1$) or 18...?. [need to sort this out...since $\mathcal{F}_{1;1\lambda_i;1\lambda_f}(Q^2;E_i^\star;E_f^\star)$ is a Lorentz scalar and the barrier factors..., can we write it as....????

$$\mathcal{F}_{1;1\lambda_i;1\lambda_f}(Q^2;E_i^\star;E_f^\star) = P_i\cdot P_f\widetilde{\mathcal{F}}_{1;1\lambda_i;1\lambda_f}(Q^2;E_i^\star;E_f^\star)\,.\tag{124}$$

]

Next, let us consider part of the second term

$$\int d\hat{\mathbf{k}}_i^\star\int d\hat{\mathbf{k}}_f^\star\frac{Y_{1\lambda_i}(\hat{k}_i^\star)}{\sqrt{4\pi}}\frac{Y^*_{1\lambda_f}(\hat{k}_f^\star)}{\sqrt{4\pi}}\mathcal{F}_3(Q^2;E_i^\star,\hat{k}_i^\star;E_f^\star,\hat{k}_f^\star)\left(k_i^\mu+\frac{(P^-\cdot k_i)}{Q^2}P^{-\mu}\right)$$
$$=\int d\hat{\mathbf{k}}_i^\star\int d\hat{\mathbf{k}}_f^\star Y_{1\lambda_i}(\hat{k}_i^\star)Y^*_{1\lambda_f}(\hat{k}_f^\star)Y_{l'm'}(\hat{k}_f^\star)Y^*_{lm}(\hat{k}_i^\star)\mathcal{F}_{1;lm;l'm'}(Q^2;E_i^\star;E_f^\star)$$
$$\times\left(\begin{bmatrix}\gamma_i\left(\omega_{k_i}^\star+\beta_i k_{i||}^\star\right)\\ \gamma_i\left(\beta_i\omega_{k_i}^\star+k_{i||}^\star\right)\\ k_{i\perp}^\star\end{bmatrix}^\mu+\frac{\left[P_0^-\gamma_i\left(\omega_{k_i}^\star+\beta_i k_{i||}^\star\right)-\vec{P}_{||}^-\cdot\hat{P}_i\gamma_i\left(\beta_i\omega_{k_i}^\star+k_{i||}^\star\right)-\vec{P}_{||}^-\cdot\vec{k}_\perp^\star\right]}{Q^2}P^{-\mu}\right)$$
$$=\frac{1}{2}\left(P_i^\mu+\frac{P^-\cdot P_i}{Q^2}P^{-\mu}\right)\mathcal{F}_{1;\lambda_i;1\lambda_f}(Q^2;E_i^\star;E_f^\star)$$
$$+\int d\hat{\mathbf{k}}_i^\star\int d\hat{\mathbf{k}}_f^\star Y_{1\lambda_i}(\hat{k}_i^\star)Y^*_{1\lambda_f}(\hat{k}_f^\star)Y_{l'm'}(\hat{k}_f^\star)Y^*_{lm}(\hat{k}_i^\star)\mathcal{F}_{1;lm;l'm'}(Q^2;E_i^\star;E_f^\star)$$
$$\times\left(\begin{bmatrix}\gamma_i\beta_i k_{i||}^\star\\ \gamma_i k_{i||}^\star\\ k_{i\perp}^\star\end{bmatrix}^\mu+\frac{\left[P_0^-\gamma_i\beta_i k_{i||}^\star-\vec{P}_{||}^-\cdot\hat{P}_i\gamma_i k_{i||}^\star-\vec{P}_{||}^-\cdot\vec{k}_\perp^\star\right]}{Q^2}P^{-\mu}\right)$$
$$=\frac{1}{2}\left(P_i^\mu+\frac{P^-\cdot P_i}{Q^2}P^{-\mu}\right)\mathcal{F}_{1;\lambda_i;1\lambda_f}(Q^2;E_i^\star;E_f^\star)+\int d\hat{\mathbf{k}}_i^\star Y_{1\lambda_i}(\hat{k}_i^\star)Y^*_{lm}(\hat{k}_i^\star)\mathcal{F}_{1;lm;1\lambda_f}(Q^2;E_i^\star;E_f^\star)$$
$$\times\left(\begin{bmatrix}\gamma_i\beta_i k_{i||}^\star\\ \gamma_i k_{i||}^\star\\ k_{i\perp}^\star\end{bmatrix}^\mu+\frac{\left[P_0^-\gamma_i\beta_i k_{i||}^\star-\vec{P}_{||}^-\cdot\hat{P}_i\gamma_i k_{i||}^\star-\vec{P}_{||}^-\cdot\vec{k}_\perp^\star\right]}{Q^2}P^{-\mu}\right).\tag{125}$$

To evaluate the the last term, we note that parity dictates that only odd values of l will contribute. Next, we can write the product of two spherical harmonics in terms of a single spherical harmonic using Wigner 3j-symbols. Generically, we can write this product as

$$Y_{1\lambda_i}(\hat{k}_i^\star)Y^*_{lm}(\hat{k}_i^\star)=\sum_{\ell_f m_f}c_{\ell_f m_f;1\lambda_i;lm}Y_{\ell_f m_f}(\hat{k}_i^\star)\tag{126}$$

where

$$c_{\ell_f m_f;1\lambda_i;lm}=\int d\hat{\mathbf{k}}_i^\star\,Y^*_{\ell_f m_f}(\hat{k}_i^\star)\,Y_{1\lambda_i}(\hat{k}_i^\star)\,Y^*_{lm}(\hat{k}_i^\star)\tag{127}$$

and it can of course be written in terms of the standard Wigner 3j-symbols. Next, we note that if we insert this identify into Eq. 125, we will have to integrate $\int Y_{\ell_f m_f}(\hat{k}_i^\star)k_i^\star$. Since $k_i^\star$ can be written in terms of the $\ell=1$ spherical harmonic, this tells us that only $\ell_f=1$ contributes. Given that only l contribute and the constraints of the Wigner 3j-symbols symbols, this tells that only the $l=1$ components have nonzero contributions.

$$\int d\hat{\mathbf{k}}_i^\star Y_{1\lambda_i}(\hat{k}_i^\star)Y_{lm}^*(\hat{k}_i^\star)\mathcal{F}_{1;lm;1\lambda_f}(Q^2;E_i^\star;E_f^\star)\left(\begin{bmatrix}\gamma_i\beta_i k_{i||}^\star\\ \gamma_i k_{i||}^\star\\ k_{i\perp}^\star\end{bmatrix}^\mu+\frac{\left[P_0^-\gamma_i\beta_i k_{i||}^\star-\vec{P}_{||}^-\cdot\hat{P}_i\gamma_i k_{i||}^\star-\vec{P}_{||}^-\cdot\vec{k}_\perp^\star\right]}{Q^2}P^{-\mu}\right).$$

$$=c_{1m_f;1\lambda_i;1m}\int d\hat{\mathbf{k}}_i^\star Y_{1m_f}(\hat{k}_i^\star)\mathcal{F}_{1;1m;1\lambda_f}(Q^2;E_i^\star;E_f^\star)\left(\begin{bmatrix}\gamma_i\beta_i k_{i||}^\star\\ \gamma_i k_{i||}^\star\\ k_{i\perp}^\star\end{bmatrix}^\mu+\frac{\left[P_0^-\gamma_i\beta_i k_{i||}^\star-\vec{P}_{||}^-\cdot\hat{P}_i\gamma_i k_{i||}^\star-\vec{P}_{||}^-\cdot\vec{k}_\perp^\star\right]}{Q^2}P^{-\mu}\right) \tag{128}$$

Next, we make use of the observation made in Ref. [7], that $X^\sigma(\lambda)=\int d\hat{\mathbf{k}}\,Y_{1\lambda}(\hat{\mathbf{k}})\,k'^\sigma$ transforms like $\epsilon^\sigma(P_z,\lambda)$. seems to suggest the following structure

$$c_{1m_f;1\lambda_i;1m}\,\mathcal{F}_{1;1m;1\lambda_f}(Q^2;E_i^\star;E_f^\star)\left(\epsilon_i^\mu+\frac{P_0^-\cdot\epsilon_i}{Q^2}P^{-\mu}\right) \tag{129}$$

this seems to suggest that the first term will contribute to $\widetilde{G}_3$ and the second to $\widetilde{G}_5$ in Eq. 79, [continue here]

First we establish that $P_\mu X^\mu=0$,

$$P_\mu X^\mu=\int d\hat{\mathbf{k}}\,Y_{1\lambda}(\hat{\mathbf{k}})\left[\frac{E_{\pi\pi}^2}{2}-\frac{P^2}{2}\right]=0, \tag{130}$$

and then we may check that the $\lambda=\pm1$ components are what is expected, e.g.,

$$\begin{aligned}X^\sigma(\lambda=+1)&=\int d\hat{\mathbf{k}}\,Y_{1,+1}(\hat{\mathbf{k}})\,k'^\sigma\\ &=-\sqrt{\frac{4\pi}{3}}|\mathbf{k}|\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\i\\0\end{bmatrix}\\ &=\sqrt{\frac{4\pi}{3}}|\mathbf{k}|\epsilon^\sigma(P_z,\lambda=+1),\end{aligned} \tag{131}$$

and indeed the forms are equivalent.

F. Unitarity and Watson's theorem

G. Defining resonance pole

I believe that in order to extract the form factor of a resonance or bound state, we only need to consider $\mathcal{W}_{\rm df}$, not the full amplitude $\mathcal{W}$. To understand this, we need to remember the definition of $\mathcal{W}$, which is the sum over all diagrams with two-particle in the initial and final states and a single insertion of an external current. $\mathcal{W}_{\rm df}$ has the same definition but it exclude diagrams where the current couples to the external legs. An example of such diagrams are shown in Fig. 2(a).

Now, the point is that the diagrams that are absent in $\mathcal{W}_{\rm df}$ will not give a pole in both the initial and final two-particle states. These only arise from the reescattering of two-particles states. In other words, the limit of $\mathcal{W}$ where both the energy of the initial and final states are taken to their corresponding resonant poles coincides with that of $\mathcal{W}_{\rm df}$. This limit, being depicted in Fig. 2(b), these amplitudes are equal to,

$$\lim_{E_i^\star,E_f^\star\to E_R} i\mathcal{W}=\lim_{E_i^\star,E_f^\star\to E_R} i\mathcal{W}_{\rm df}=ig\,\frac{i}{E_i^{*2}-E_R^2}iw_R(Q^2)\frac{i}{E_f^{*2}-E_R^2}\,ig, \tag{132}$$

where g is the coupling of the resonance to the two-particle states, $w_R(Q^2)$ is the electromagnetic matrix element of the resonance. From the Lorentz decomposition of this, we can define its form factors at $Q^2=-(P_i-P_f)^2$.

The power of this observation, is that, unlike $\mathcal{W}$, $\mathcal{W}_{\rm df}$ does not have kinematic divergences. This implies that we can do all of our analysis on a function that has no kinematic divergences to get a the form factor.

Here my argument for why I think that in the narrow width limit,

$$\mathcal{W}_{L,\rm df}(P_f,P_i,L)\equiv\mathcal{W}_{\rm df}(P_f,P_i)+\cdots \tag{133}$$

where ellipses denote corrections that depend on the volume but are suppressed by the width.

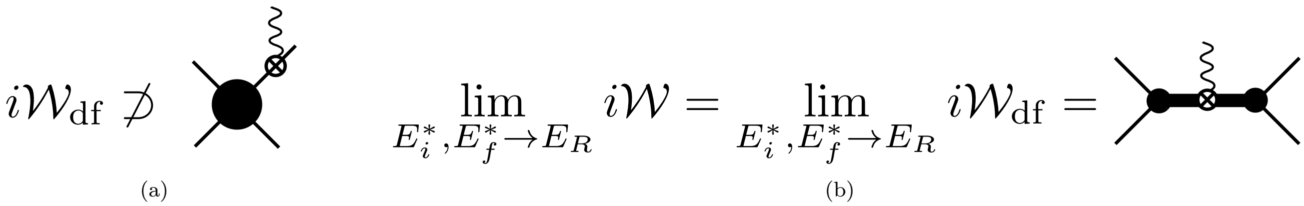

FIG. 2: (a) Shown is an example of diagrams absent in $\mathcal{W}_{\rm df}$. (b) Shown is the postulate behavior of the $\mathcal{W}_{\rm df}$ near the resonance pole.

1. *Narrow-width limit*

In the narrow width limit and in vicinity of the resonance,

$$i\mathcal{W}_{\rm df}(P_f, P_i) \sim \; ig \,\frac{i}{E_i^{*2} - E_R^2} iw_R(Q^2) \frac{i}{E_f^{*2} - E_R^2}\, ig, \tag{134}$$

$$i\mathcal{M}(P_i) \sim \; ig \,\frac{i}{E_i^{*2} - E_R^2}\, ig, \tag{135}$$

$$i\mathcal{M}(P_f) \sim \; ig \,\frac{i}{E_f^{*2} - E_R^2}\, ig, \tag{136}$$

This implies that

$$\mathcal{W}_{L,{\rm df}}(P_f, P_i, L) \sim \; ig \,\frac{i}{E_i^{*2} - E_R^2} \left[w_R(Q^2) + g^2 [G(L) \cdot w](P_f, P_i)\right] \frac{i}{E_f^{*2} - E_R^2}\, ig. \tag{137}$$

Since $\Gamma \sim g^2$, we see that the residue of this term is equal to $w_R(Q^2)$ up to finite-volume corrections that are suppressed by the width. Note, the resonance poles will be canceled by the $\mathcal{R}$ appearing adjacent to them.

III. QUANTIZATION CONDITIONS

The relativistic scattering amplitude, $\mathcal{M}$, can be written in terms of the S-matrix. When there are N-channels open, both the scattering amplitude and the S-matrix are $N \times N$ size matrices. It is convenient to introduce a diagonal matrix, $\mathbb{P} = \text{diag}(\sqrt{\xi_1 q_1^*}, \sqrt{\xi_2 q_2^*}, \ldots, \sqrt{\xi_N q_N^*})/\sqrt{4\pi E^*}$, where ξ_a is 1/2 if the two particles in the ath channel are identical and 1 otherwise. The CM energy E^* is defined in terms of total laboratory frame energy and momentum, E and $\mathbf{P}$, as $E^* = \sqrt{E^2 - \mathbf{P}^2}$. For the ath channel with two mesons each having masses $m_{i,1}$ and $m_{i,2}$, the CM relative momentum is

$$q_a^{*2} = \frac{1}{4}\left(E^{*2} - 2(m_{a,1}^2 + m_{a,2}^2) + \frac{(m_{,1}^2 - m_{a,2}^2)^2}{E^{*2}}\right), \tag{138}$$

which simplifies to $\frac{E^{*2}}{4} - m_a^2$ when $m_{a,1} = m_{a,2} = m_a$. With this, the scattering amplitude can be written as

$$i\mathcal{M} = \mathbb{P}^{-1}\,(S - \mathbb{I}_N)\,\mathbb{P}^{-1}, \tag{139}$$

$$\tag{140}$$

where $\mathbb{I}_N$ is the N-dimensional identity.

When a single channel is open, composed of a spinless particles, the S-matrix is a diagonal unitary matrix. The diagonal elements correspond to the different angular momenta that the two particles can carry. Conservation of angular momentum requires the S-matrix to be diagonal in ℓ. Azimuthal symmetry tells us that all m components are identical. In summary, the S-matrix satisfies the following expression,

$$\mathcal{S}_{\ell,m;\ell',m'} = \delta_{\ell\ell'}\,\delta_{mm'}\,\mathcal{S}_\ell. \tag{141}$$

which can be parametrized by a single function, the scattering phase shift,

$$S_\ell = e^{2i\delta_\ell}. \tag{142}$$

From this and Eq. 139, one can arrives at

$$\mathcal{M}_\ell = \frac{8\pi E^*}{\xi}\frac{1}{q^*\cot\delta_\ell - iq^*}\,. \tag{143}$$

[Task #1: Proof this]

The master equation that defines the relationship between the finite volume spectrum and the infinite volume scattering parameters is [8–13]

$$\det[F^{-1}(P,L) + \mathcal{M}(P)] = \det{}_{\text{oc}}\left[\det{}_{\text{pw}}\left[\mathrm{F}^{-1}(\mathrm{P},\mathrm{L}) + \mathcal{M}(\mathrm{P})\right]\right] = 0\,. \tag{144}$$

where the determinant $\det_{\text{oc}}$ is over the N open channels and the determinant $\det_{\text{pw}}$ is over the partial waves, and both $\mathcal{M}$ and $\delta\mathcal{G}^V$ functions are evaluated on the on-shell value of the momenta. F is a diagonal matrix in the number of open channels, but it is non-diagonal in angular momentum,

$$F_{a\ell m;a'\ell'm'}(P,L) \equiv \delta_{aa'}\xi_a\left[\frac{1}{L^3}\sum\!\!\!\!\!\!\int_{\mathbf{k}}\right]\frac{4\pi Y_{\ell m}(\hat{\mathbf{k}}_a^*)Y^*_{\ell' m'}(\hat{\mathbf{k}}_a^*)}{2\omega_{a1}2\omega_{a2}(E - \omega_{a1} - \omega_{a2} + i\epsilon)}\left(\frac{k_a^*}{q_a^*}\right)^{\ell+\ell'}, \tag{145}$$

where we are adopting the compact notation of Ref. [1]

$$\left[\frac{1}{L^3}\sum\!\!\!\!\!\!\int_{\mathbf{k}}\right] f(\mathbf{k}) \equiv \left[\frac{1}{L^3}\sum_{\mathbf{k}\in(2\pi/L)\mathbb{Z}^3} - \int\frac{d\mathbf{k}}{(2\pi)^3}\right] f(\mathbf{k})\,. \tag{146}$$

In Sec. V A we show that this can be written as

$$F_{a\ell m,a'\ell'm'}(P,L) = \delta_{aa'}\frac{iq_a^*}{8\pi E^*}\xi_a\left[\delta_{\ell\ell'}\delta_{mm'} + i\sum_{\ell''m''}\frac{(4\pi)^{3/2}}{q_a^{*(\ell''+1)}}c^{\mathbf{\Delta}}_{a\ell''m''}(q_a^{*2};L)\int d\Omega\; Y^*_{\ell m}(\hat{\mathbf{k}}_a^*)Y^*_{\ell''m''}(\hat{\mathbf{k}}_a^*)Y_{\ell'm'}(\hat{\mathbf{k}}_a^*)\right], \tag{147}$$

where $\Delta \equiv \frac{\mathbf{P}L}{2\pi}\left(1 + \frac{m_{a1}^2 - m_2^{a2}}{E^{*2}}\right)$ and $c^{\mathbf{\Delta}}_{a\ell''m''}$ are defined in Sec. V A.

In this work, we will primarily be focused on the case where there is a single channel open composed of two spinless particles that are degenerate. In particular, we will only consider $\pi\pi$ systems, with emphasis on the ρ channel. This means that $\Delta \equiv \frac{\mathbf{P}L}{2\pi} = \mathbf{d}$. Given that it will be surpperflous will drop the channel index "a". Finally, in this limit the determinant appearing in Eq. 144 reduces to one over partial waves alone

$$\det[F^{-1}(P,L) + \mathcal{M}(P)] = \det{}_{\text{pw}}\left[\mathrm{F}^{-1}(\mathrm{P},\mathrm{L}) + \mathcal{M}(\mathrm{P})\right] = 0\,. \tag{148}$$

$\mathbf{d}$	$(00n)$	$(nn0)$		(nnn)
	$\alpha^{\mathbf{d}}_{20,\mathbb{A}_1} = \frac{2}{\sqrt{5}}$	$\alpha^{\mathbf{d}}_{20,\mathbb{A}_1} = -\frac{1}{\sqrt{5}}$,	$\alpha^{\mathbf{d}}_{22,\mathbb{A}_1} = -i\sqrt{\frac{6}{5}}$	$\alpha^{\mathbf{d}}_{22,\mathbb{A}_1} = -2i\sqrt{\frac{6}{5}}$
	$\alpha^{\mathbf{d}}_{20,\mathbb{E}} = -\frac{1}{\sqrt{5}}$	$\alpha^{\mathbf{d}}_{20,\mathbb{B}_1} = -\frac{1}{\sqrt{5}}$,	$\alpha^{\mathbf{d}}_{22,\mathbb{B}_1} = i\sqrt{\frac{6}{5}}$	$\alpha^{\mathbf{d}}_{22,\mathbb{E}} = i\sqrt{\frac{6}{5}}$
		$\alpha^{\mathbf{d}}_{20,\mathbb{B}_2} = \frac{2}{\sqrt{5}}$		

TABLE I: Nonzero values of $\alpha^{\mathbf{d}}_{20,\Lambda}$ and $\alpha^{\mathbf{d}}_{22,\Lambda}$ for $\mathbf{d}^2 \leq 3$. For the $\mathbb{T}_1^-$ irrep of O_h^D, the $c^{\mathbf{d}}_{2m}$ vanish, therefore there is no need to define $\alpha^{\mathbf{d}}_{2m,\Lambda}$ for this irrep.

A. Simplification for $\pi\pi$ in the S- and P-wave channels

Let us consider the low-energy behavior of the quantization condition. In particular, we will consider the case where the the scattering amplitude is dominated by a single angular momentum. Furthermore, given that the $\pi\pi$ must satisfy Bose statistics, they must have a totally symmetric wavefunction. If we consider the isoscalar ($I = 0$) or isotensor ($I = 2$) channels, where the isospin component of the wavefunction is symmetric, the orbital angular momenta must be even ($\ell = 0, 2, 4, \ldots$). Equivalently, for the isotriplet channel ($I = 1$), the angular momenta must be odd even ($\ell = 1, 3, 5, \ldots$). Some particularly interesting examples, are the σ resonance, which couples to the couples to the $(I, \ell) = (0, 0)$ channel [14], and the ρ resonance which couples to the $(I, \ell) = (1, 1)$ channel [15, 16].

First, let us consider the case where the scattering amplitude is dominated by the S-wave, in other words,

$$\mathcal{M} \approx \begin{pmatrix} \mathcal{M}_0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}. \tag{149}$$

Inserting this into Eq. 144, one can show that the quantization condition is fairly simple

$$q^* \cot \delta_S - 4\pi c^{\mathbf{d}}_{00}(q^{*2}; L) = 0. \tag{150}$$

[Task #2: Proof this]

The P-waves quantization conditions are only slightly more complicated. First, we need to remember that the $\ell = 1$ angular momentum is a three-dimensional irrep of the $\mathcal{O}(3)$ group. This implies that the scattering amplitude has three identical components

$$\mathcal{M} \approx \begin{pmatrix} 0 & & & & \\ & \mathcal{M}_1 & & & \\ & & \mathcal{M}_1 & & \\ & & & \mathcal{M}_1 & \\ & & & & \ddots \end{pmatrix}. \tag{151}$$

Inserting these into Eq. 144 and block-diagonalizing the matrix inside the determinant, one finds that for each boost vector, $\mathbf{d}$, there are three quantization conditions describing the spectrum. For the symmetry group we will be considering, these can be compactly written as,

$$q^* \cot \delta_P - 4\pi \left(c^{\mathbf{d}}_{00}(q^{*2}; L) + \frac{\alpha^{\mathbf{d}}_{20,\Lambda}}{q^{*2}} c^{\mathbf{d}}_{20}(q^{*2}; L) + \frac{\alpha^{\mathbf{d}}_{22,\Lambda}}{q^{*2}} c^{\mathbf{d}}_{22}(q^{*2}; L) \right) = 0 \tag{152}$$

where the values of $\alpha^{\mathbf{d}}_{2m,\Lambda}$ are shown in Table I.

When the system is at rest $\alpha^{\mathbf{d}}_{2m,\Lambda} = 0$, there is a single irrep of the cubic group which couples to P-wave channel. This is the $\mathbb{T}_1^-$ irrep, which is three-dimensional. Given that the three-components of the $\mathbb{T}_1^-$ irrep are necessarily degenerate, only one QC is needed to describe its spectrum, which resembles that of the S-wave,

$$\mathbb{T}_1^- : \qquad q^* \cot \delta_P - 4\pi c^{\mathbf{d}}_{00}(q^{*2}; L) = 0. \tag{153}$$

When you boost the system along the z-axis, you get two different irreps, the $\mathbb{A}_1$ and $\mathbb{E}$. Note, the $\mathbb{A}_1$ is one-dimensional and $\mathbb{E}$ is two-dimensional. More technically, the $\mathbb{A}_1$ couples only to the helicity-0 components of the P-wave amplitude, while $\mathbb{E}$ couples to the helicity 1 and -1 components. The two irreps have distinct spectra satisfying,

$$\mathbb{A}_1: \qquad q^* \cot\delta_P - 4\pi c_{00}^{\mathbf{d}}(q^{*2}; L) - \frac{2}{\sqrt{5}} \frac{4\pi}{q^{*2}} c_{20}^{\mathbf{d}}(q^{*2}; L) = 0 \tag{154}$$

$$\mathbb{E}: \qquad q^* \cot\delta_P - 4\pi c_{00}^{\mathbf{d}}(q^{*2}; L) + \frac{1}{\sqrt{5}} \frac{4\pi}{q^{*2}} c_{20}^{\mathbf{d}}(q^{*2}; L) = 0. \tag{155}$$

Note, the two components of the $\mathbb{E}$ are degenerate.
Similarly, for the other irreps.

IV. RELATING THE FINITE-VOLUME MATRIX ELEMENT TO $\pi\pi\gamma^\star \to \pi\pi$ AMPLITUDE

$$\left|\langle E_{n_f}, \mathbf{P}_f, L|\mathcal{J}(0)|E_{n_i}, \mathbf{P}_i, L\rangle\right|_L^2 = \frac{1}{L^6} \mathrm{Tr}\left[\mathcal{R}(E_{n_i}, \mathbf{P}_i) \mathcal{W}_{L,\mathrm{df}}(P_i, P_f, L) \mathcal{R}(E_{n_f}, \mathbf{P}_f) \mathcal{W}_{L,\mathrm{df}}(P_f, P_i, L)\right]. \tag{156}$$

$$\mathcal{W}_{L,\mathrm{df}}(P_f, P_i, L) \equiv \mathcal{W}_{\mathrm{df}}(P_f, P_i) + \mathcal{M}(P_f)\ [G(L)\cdot w](P_f, P_i)\ \mathcal{M}(P_i)\,. \tag{157}$$

where ω is the one-body amplitude (closely related to the pion form factor)

$$\mathcal{R}(E_n, \mathbf{P}) \equiv \lim_{P_4 \to iE_n} \left[-(iP_4 + E_n) \frac{1}{F^{-1}(P, L) + \mathcal{M}(P)}\right]. \tag{158}$$

$$[G(L)\cdot w]_{a\ell_f m_f; b\ell_i m_i}(P_f, P_i) \equiv \sum_{s,t=1,2} \xi_a \xi_b G^{st}_{a\ell_f m_f, a'\ell'_f m'_f; b'\ell'_i m'_i, b\ell_i m_i}(P_f, P_i, L)\ w_{a'sb't;\ell'_f m'_f;\ell'_i m'_i}(P_f; P_i)\,. \tag{159}$$

and

$$G^{st}_{a\ell_f m_f, a'\ell'_f m'_f; b'\ell'_i m'_i, b\ell_i m_i}(P_f, P_i, L) \equiv$$
$$\delta_{aa'}\delta_{bb'} \left[\frac{1}{L^3} \sum\!\!\!\!\!\!\int_{\mathbf{k}}\right] \frac{1}{2\omega_{a\not{s}}} \frac{4\pi Y_{\ell_f m_f}(\hat{\mathbf{k}}^*_{af}) Y^*_{\ell'_f m'_f}(\hat{\mathbf{k}}^*_{af})}{2\omega_{asf}(E_f - \omega_{a\not{s}} - \omega_{asf} + i\epsilon)} \left(\frac{k^*_{af}}{q^*_{af}}\right)^{\ell_f + \ell'_f} \frac{4\pi Y_{\ell'_i m'_i}(\hat{\mathbf{k}}^*_{bi}) Y^*_{\ell_i m_i}(\hat{\mathbf{k}}^*_{bi})}{2\omega_{bti}(E_i - \omega_{b\not{t}} - \omega_{bti} + i\epsilon)} \left(\frac{k^*_{bi}}{q^*_{bi}}\right)^{\ell_i + \ell'_i}. \tag{160}$$

Here we have a list of things to do,

- Simplify these expressions for the single channel case, degenerate masses, etc
- I think we might want to write things in terms of $G(L)\cdot w$ in the first place. This avoid introducing a weird spherical harmonic decomposition of w.

A. Subduction

B. Simplifying limits

In this section we consider various simplifying limits of the general result, derived in the last section. We begin by taking the energies considered to be very close to the lowest two-particle threshold. In this case, the infinite-volume quantities w, $\mathcal{M}$ and $\mathcal{W}_{\mathrm{df}}$ are all dominated by their S-wave values. We thus drop all higher partial waves in the matrices $w_{a1b1;\ell'm';\ell m}(P_f, P_i)$, $\mathcal{M}_{ab;\ell'm';\ell m}(P)$ and $\mathcal{W}_{\mathrm{df};ab;\ell'm';\ell m}(P_f, P_i)$. The second consequence of near-threshold energies is that only the lowest two-particle channel is open. In discussing this system it is convenient to introduce the shorthand

$$w_{11}(P_f, P_i) \equiv w_{a1b1;00;00}(P_f, P_i)\,, \tag{161}$$
$$\mathcal{M}(P) \equiv \mathcal{M}_{ab;00;00}(P)\,, \tag{162}$$
$$\mathcal{W}_{\mathrm{df}}(P_f, P_i) \equiv \mathcal{W}_{\mathrm{df};ab;00;00}(P_f, P_i)\,. \tag{163}$$

We comment here that, for a scalar form factor, symmetry and on-shell constraints guarantee that w only depends on $(P_f - P_i)^2$ and thus not on $\mathbf{k}$. In this case, the truncation of w to the S-wave is exact. Since all matrices have reduced to one dimensional, the trace may be dropped from Eq. (156)

$$\left|\langle E_{n_f}, \mathbf{P}_f, L|\mathcal{J}(0)|E_{n_i}, \mathbf{P}_i, L\rangle\right|_L^2 = \frac{1}{L^6}\mathcal{R}(E_{n_i}, \mathbf{P}_i)\mathcal{W}_{L,\mathrm{df}}(P_i, P_f, L)\mathcal{R}(E_{n_f}, \mathbf{P}_f)\mathcal{W}_{L,\mathrm{df}}(P_f, P_i, L)\,. \tag{164}$$

In addition, the residue matrix $\mathcal{R}$ simplifies significantly

$$\mathcal{R}(E_n, \mathbf{P}) = \left[\frac{\partial}{\partial E}\left(F^{-1}(P, L) + \mathcal{M}(P)\right)\right]^{-1}_{E=E_n} = -\left[\mathcal{M}^2(P)\frac{\partial}{\partial E}\left(F(P, L) + \mathcal{M}^{-1}(P)\right)\right]^{-1}_{E=E_n}\,, \tag{165}$$

$$= -\xi\frac{q^*}{8\pi E^*}\left[\sin^2\delta\ e^{2i\delta}\ \frac{\partial}{\partial E}\left(\cot\phi^{\mathbf{d}} + \cot\delta\right)\right]^{-1}_{E=E_n}\,, \tag{166}$$

$$= \xi\frac{q^*}{8\pi E^*}e^{-2i\delta}\left[\frac{\partial}{\partial E}\left(\phi^{\mathbf{d}} + \delta\right)\right]^{-1}_{E=E_n}\,, \tag{167}$$

where $F = F_{a00;b00}$ is understood and where we have introduced the S-wave Lüscher pseudophase

$$\cot\phi^{\mathbf{d}} = \xi\frac{q^*}{8\pi E^*}\mathrm{Re}F(P, L)\,. \tag{168}$$

Here we have also used the relation between scattering amplitude $\mathcal{M}$ and scattering phase shift δ, given in Eq. (143) above. Substituting this result for $\mathcal{R}$ into Eq. (164) and rearranging gives

$$\begin{aligned}&\left[e^{-i\delta_i}\mathcal{W}_{L,\mathrm{df}}(P_i, P_f, L)e^{-i\delta_f}\right]\left[e^{-i\delta_f}\mathcal{W}_{L,\mathrm{df}}(P_f, P_i, L)e^{-i\delta_i}\right]\\ &\quad= \frac{8\pi E_f^*}{q_f^*\xi}\frac{8\pi E_i^*}{q_i^*\xi}\left[\frac{\partial}{\partial E_f}\left(\phi^{\mathbf{d}} + \delta\right)\right]_{E_f=E_{f,n}}\left[\frac{\partial}{\partial E_i}\left(\phi^{\mathbf{d}} + \delta\right)\right]_{E_i=E_{i,n}} L^6\left|\langle E_{n_f}, \mathbf{P}_f, L|\mathcal{J}(0)|E_{n_i}, \mathbf{P}_i, L\rangle\right|_L^2\,.\end{aligned} \tag{169}$$

We thus see that a naive Lellouch-Lüscher-like proportionality factor arises between the finite- and infinite-volume quantities. Since the right-hand side of this expression is manifestly pure real, this result also suggest a Watson-like theorem for $\mathcal{W}_{L,\mathrm{df}}$, namely that its complex phases are the strong scattering phases associated with the incoming and outgoing two-particle states.

V. NUMERICAL EVALUATION OF THE RELEVANT GEOMETRIC FUNCTIONS

A. Numerically evaluating F

Here I am stealing content from Ref. [1], where we give some detail between the different expressions for the F-function. First, let us equate the expressions give in Eq. 145 and Eq. 147. To do this, we will explicitly separate the real and imaginary parts of the propagator. The imaginary component is given by the $i\epsilon$ prescription. We do this using

$$\frac{1}{2\omega_{a2}(E - \omega_{a1} - \omega_{a2} + i\epsilon)} = \frac{\omega_{a1}^*}{E^*(q_a^{*2} - k_a^{*2} + i\epsilon)} + \mathcal{S}_2 = \frac{\omega_{a1}^*}{E^*}\left[\mathrm{P.V.}\frac{1}{(q_a^{*2} - k_a^{*2})} - i\pi\delta(q_a^{*2} - k_a^{*2})\right] + \mathcal{S}_2, \tag{170}$$

where $\mathcal{S}_2$ is a smooth function that will be annihilated by the sum-integral difference. Here P.V. denotes the principal-value pole prescription. We further reduce the expression by combining the two spherical harmonics into one

$$4\pi Y_{\ell m}(\hat{\mathbf{k}}_a^*)Y_{\ell' m'}^*(\hat{\mathbf{k}}_a^*) = 4\pi\sum_{\ell'' m''} Y_{\ell'' m''}(\hat{\mathbf{k}}_a^*)\int d\Omega_{\mathbf{p}}\ Y_{\ell m}^*(\hat{\mathbf{p}}_a^*)Y_{\ell'' m''}^*(\hat{\mathbf{p}}_a^*)Y_{\ell' m'}(\hat{\mathbf{p}}_a^*). \tag{171}$$

Putting all the pieces together, we can rewrite Eq. (145) as Eq. 147, where $c^{\mathbf{\Delta}}_{a\ell m}(q_a^{*2}; L)$ is defined as

$$c^{\mathbf{\Delta}}_{a\ell m}(q_a^{*2}; L) = \left[\frac{1}{L^3}\sum\!\!\!\!\!\!\!\int_{\mathbf{k}}\right]\frac{\omega_{a1}^* k_a^{*\ell}}{\omega_{a1}}\mathrm{P.V.}\frac{\sqrt{4\pi}Y_{\ell m}(\hat{\mathbf{k}}_a^*)}{k_a^{*2} - q_a^{*2}}. \tag{172}$$

Alternatively, this function can be written in terms of the generalized Zeta functions [10]

$$c^{\boldsymbol{\Delta}}_{a\ell m}(q_a^{*2};L) = \frac{\sqrt{4\pi}}{\gamma L^3}\left(\frac{2\pi}{L}\right)^{\ell-2}\mathcal{Z}^{\boldsymbol{\Delta}}_{a\ell m}[1;(q_a^* L/2\pi)^2], \qquad \mathcal{Z}^{\boldsymbol{\Delta}}_{a\ell m}[s;x^2] = \sum_{\mathbf{r}\in\mathcal{P}_{\boldsymbol{\Delta}}}\frac{r^\ell Y_{\ell m}(\hat{\mathbf{r}})}{(r^2-x^2)^s}\,, \tag{173}$$

where

$$P_{\boldsymbol{\Delta}} \equiv \left\{ \mathbf{r} \,\middle|\, \mathbf{r} = \boldsymbol{\gamma}^{-1}\left(\mathbf{n} - \frac{1}{2}\boldsymbol{\Delta}\right)\,,\ \mathbf{n}\in\mathbb{Z}^3 \right\}, \tag{174}$$

$$\boldsymbol{\Delta} \equiv \frac{\mathbf{P}L}{2\pi}\left(1 + \frac{m_{a1}^2 - m_{a2}^2}{E^{*2}}\right)\,, \tag{175}$$

$$\boldsymbol{\gamma}^{-1}\mathbf{p} \equiv \gamma^{-1}\mathbf{p}_\parallel + \mathbf{p}_\perp = (E/E^*)^{-1}\mathbf{p}_\parallel + \mathbf{p}_\perp\,, \tag{176}$$

and where $\mathbf{p}_\parallel$ and $\mathbf{p}_\perp$ are the parallel and perpendicular components of $\mathbf{p}$ with respect to the fixed total momentum $\mathbf{P}$. We close by giving a particularly efficient form for evaluating these quantities [17],

$$\begin{aligned} Z^{\boldsymbol{\Delta}}_{a\ell m}(1,x^2) = &\sum_{\mathbf{r}\in P_{\boldsymbol{\Delta}}}\frac{r^\ell Y_{\ell m}(\hat{\mathbf{r}})}{r^2-x^2}e^{-(r^2-x^2)} + \gamma\frac{\pi}{2}\delta_{\ell 0}\delta_{m0}G(x) \\ &+ \gamma\pi^{3/2}\int_0^1 dt\frac{e^{tx^2}}{t^{3/2}}\left(\frac{\pi i}{t}\right)^\ell \sum_{\mathbf{n}\neq 0} e^{-i\pi\mathbf{n}\cdot\boldsymbol{\Delta}}|\boldsymbol{\gamma}\mathbf{n}|^\ell Y_{\ell m}(\widehat{\boldsymbol{\gamma}\mathbf{n}})e^{-(\pi\boldsymbol{\gamma}\mathbf{n})^2/t}\,, \end{aligned} \tag{177}$$

where $\boldsymbol{\gamma}\mathbf{p} \equiv \gamma\mathbf{p}_\parallel + \mathbf{p}_\perp$ and

$$G(x) \equiv \int_0^1 dt\frac{e^{tx^2}-1}{t^{3/2}} - 2\,. \tag{178}$$

where $\alpha = \frac{1}{2}\left[1 + \frac{m_1^2 - m_2^2}{E^{*2}}\right]$.

[Task #3: Write code for $\mathcal{Z}^{\mathbf{d}}_{lm}$. In other words, drop all of the channel indices. Also, we will only need to think about the case where the masses are degenerate.]

[Task #4: Compare to Eq. 70 in https://arxiv.org/pdf/1202.2145.pdf]

[Task #5: Derive Eq. 70 starting from Eq. 59 in https://arxiv.org/pdf/1202.2145.pdf]

B. Numerically evaluating G

1. *Introduction and general definition*

The goal of this project is to calculate infinite-volume matrix elements of the from

$$\langle \pi\pi;\text{out};P_f|\mathcal{J}_\mu(0)|\pi\pi;\text{in};P_i\rangle\,, \tag{179}$$

where $\mathcal{J}_\mu$ is a vector current and the states on either side are two-particle asymptotic states with the momentum as indicated. As is described in detail in Ref. [**?**], one cannot directly measure these matrix elements in a numerical lattice calculation, due to the restriction to finite volume and Euclidean time. It is however possible to extract the matrix elements by combining the following ingredients:

- The pion vector form factor, $F_\pi(Q^2)$, defined via:

$$\langle \pi,p_f|\mathcal{J}_\mu(0)|\pi,p_i\rangle = (p_f+p_i)_\mu F_\pi(Q^2)\Big|_{Q^2=(p_f-p_i)^2}\,. \tag{180}$$

- The finite-volume energies with two-pion quantum numbers in various moving frames: $E_n(L,\mathbf{P})$.
- The finite-volume matrix elements of states with two-pion quantum numbers in various moving frames:

$$\langle E_{n_f},\mathbf{P}_f,L|\mathcal{J}_\mu(0)|E_{n_i},\mathbf{P}_i,L\rangle\,. \tag{181}$$

In order to combine these ingredients and extract the desired quantity, one must calculate a new kinematic function called G and defined by

$$G_{a\ell_f m_f,a'\ell'_f m'_f;b'\ell'_i m'_i,b\ell_i m_i}(P_f,P_i,L) \equiv$$
$$\delta_{aa'}\delta_{bb'}\left[\frac{1}{L^3}\sum\!\!\!\!\!\!\int_{\mathbf{k}}\right]\frac{1}{2\omega_{a1}}\frac{4\pi Y_{\ell_f m_f}(\hat{\mathbf{k}}^*_{af})Y^*_{\ell'_f m'_f}(\hat{\mathbf{k}}^*_{af})}{2\omega_{a2f}(E_f-\omega_{a1}-\omega_{a2f}+i\epsilon)}\left(\frac{k^*_{af}}{q^*_{af}}\right)^{\ell_f+\ell'_f}\frac{4\pi Y_{\ell'_i m'_i}(\hat{\mathbf{k}}^*_{bi})Y^*_{\ell_i m_i}(\hat{\mathbf{k}}^*_{bi})}{2\omega_{b2i}(E_i-\omega_{b1}-\omega_{b2i}+i\epsilon)}\left(\frac{k^*_{bi}}{q^*_{bi}}\right)^{\ell_i+\ell'_i}. \tag{182}$$

To evaluate this one takes $E_i,\mathbf{P}_i,E_f,\mathbf{P}_f,L$ and the masses as inputs. For a given point in the summation and integration space, $\mathbf{k}$ should also be viewed as an input. Given these quantities one must first determine q^*_{bi}, $q^*_{af}, k^*_{bi}, k^*_{af}, \hat{\mathbf{k}}^*_{bi}, \hat{\mathbf{k}}^*_{af}$ and the ωs via

$$\sqrt{E_i^2-\mathbf{P}_i^2}=\sqrt{m_{b1}^2+q^{*2}_{bi}}+\sqrt{m_{b2}^2+q^{*2}_{bi}}\,,\qquad \sqrt{E_f^2-\mathbf{P}_f^2}=\sqrt{m_{a1}^2+q^{*2}_{af}}+\sqrt{m_{a2}^2+q^{*2}_{af}}\,, \tag{183}$$

$$\begin{pmatrix}\omega^*_{b1}\\ \mathbf{k}^*_{bi}\end{pmatrix}^{\mu}=\Lambda^{\mu}{}_{\nu}(-\mathbf{P}_i/E_i)\begin{pmatrix}\omega_{b1}\\ \mathbf{k}\end{pmatrix}^{\nu},\qquad \begin{pmatrix}\omega^*_{a1}\\ \mathbf{k}^*_{af}\end{pmatrix}^{\mu}=\Lambda^{\mu}{}_{\nu}(-\mathbf{P}_f/E_f)\begin{pmatrix}\omega_{a1}\\ \mathbf{k}\end{pmatrix}^{\nu}, \tag{184}$$

$$\omega_{b1}=\sqrt{m_{b1}^2+\mathbf{k}^2}\,,\ \ \omega_{b2i}=\sqrt{m_{b2}^2+(\mathbf{P}_i-\mathbf{k})^2}\,,\qquad \omega_{a1}=\sqrt{m_{a1}^2+\mathbf{k}^2}\,,\ \ \omega_{a2f}=\sqrt{m_{a2}^2+(\mathbf{P}_f-\mathbf{k})^2}\,, \tag{185}$$

where $\Lambda^{\mu}{}_{\nu}(-\mathbf{P}_i/E_i)$ is the boost matrix that takes the four-vector $(E_i,\mathbf{P}_i)$ to its zero-momentum frame. From here one has all building blocks for the integrand and it remains only to sum and integrate.

[For $M_\pi L=4$, $\mathbf{P}_i=2\pi\hat{\mathbf{z}}/L$, $\mathbf{k}=2\pi(1,1,0)/L$ calculate $\mathbf{k}^*_i/M_\pi$ as a function of E_i.]

2. *Reduction for two-pion states*

Here we consider the simplifications that arise for two-pion states. We find it useful to work with the target quantity $[G(L)\cdot w]$, where w is the matrix element of the vector current with single-pion states

$$w_\mu(P_f-k,P_i-k)\equiv\langle\pi;P_f-k|\mathcal{J}_\mu(0)|\pi;P_i-k\rangle\,, \tag{186}$$

and the $\cdot$ indicates that w is decomposed in spherical harmonics and its indices are contracted with those in $G(L)$. Using Lorentz invariance and current conservation we can rewrite

$$w_\mu(P_f-k,P_i-k)=(P_f-k+P_i-k)_\mu\ F_\pi(Q^2)\,, \tag{187}$$

where $Q^2=(P_f-P_i)^2$. **[Prove this result.]** Note that the pions are on shell, meaning that $(P_f-k)^2=(P_i-k)^2=M_\pi^2$. The scalar quantity F_π is called the pion vector form factor.

Next, following the recipe outlined in Ref. [?], we substitute $\mathbf{k}(\mathbf{k}^*_f,P_f)$ into P_f-k and we substitute $\mathbf{k}(\mathbf{k}^*_i,P_i)$ into P_i-k. Here we have dropped the channel indices since we are only considering single-pion states. This substitution defines a new coordinate system for w_μ

$$w^*_\mu(P_f,\mathbf{k}^*_f;P_i,\mathbf{k}^*_i)\equiv\Big[\big[P_f-k\big](\mathbf{k}^*_f)+\big[P_i-k\big](\mathbf{k}^*_i)\Big]_\mu\ F_\pi\big[(P_f-P_i)^2\big]\,. \tag{188}$$

We are thinking of $\big[P_f-k\big]$ as a function of $\mathbf{k}^*_f$, with the exact definition given by

$$\big[P_f-k\big]^{\mu}(\mathbf{k}^*_f)=\Lambda^{\mu}{}_{\nu}(\mathbf{P}_f/E_f)\begin{pmatrix}E^*_f-\omega^*_f\\ -\mathbf{k}^*_f\end{pmatrix}^{\nu}. \tag{189}$$

Since we take these four-vectors to be on shell we know that $|\mathbf{k}^*_f|=q^*_f$. Starting with w^* as defined in Eq. (188), the procedure outlined in Ref. [?] is as follows

1. We decompose $w^*_\mu(P_f,\mathbf{k}^*_f;P_i,\mathbf{k}^*_i)$ in spherical harmonics via

$$w^*_\mu(P_f,\mathbf{k}^*_f;P_i,\mathbf{k}^*_i)=4\pi\,Y^*_{\ell'_f m'_f}(\hat{\mathbf{k}}^*_f)\ w^*_{\mu;\ell'_f m'_f;\ell'_i m'_i}(P_f,q^*_f;P_i,q^*_i)\ Y_{\ell'_i m'_i}(\hat{\mathbf{k}}^*_i)\,. \tag{190}$$

2. In evaluating G we do not directly use the on-shell value of $w^*_\mu(P_f, \mathbf{k}^*_f; P_i, \mathbf{k}^*_i)$ but instead

$$\overline{w}^*_\mu(P_f, \mathbf{k}^*_f; P_i, \mathbf{k}^*_i) \equiv 4\pi\, Y^*_{\ell'_f m'_f}(\hat{\mathbf{k}}^*_f)\left(\frac{k^*_f}{q^*_f}\right)^{\ell'_f} w^*_{\mu;\ell'_f m'_f;\ell'_i m'_i}(P_f, q^*_f; P_i, q^*_i)\; Y_{\ell'_i m'_i}(\hat{\mathbf{k}}^*_i)\left(\frac{k^*_i}{q^*_i}\right)^{\ell'_i}. \tag{191}$$

[Suppose $w^* = C(\hat{\mathbf{k}}^{*2}_{i\,x} + \hat{\mathbf{k}}^{*2}_{f\,x})$, then what is the value of $\overline{w}^*$?]

This complicated recipe is required to define an on-shell projection that does not introduce singularities at $\mathbf{k}^*_f = 0$. However, in the present case in turns out that the procedure can be rephrased in a much more straight-forward way. In particular one can show that, for the pion vector form-factor, the corresponding "barred" version defined in Eq. (191) reduces to

$$\overline{w}^*_\mu(P_f, \mathbf{k}^*_f; P_i, \mathbf{k}^*_i) = (P_f - k + P_i - k)_\mu\; F_\pi\big[(P_f - P_i)^2\big]\,, \tag{192}$$

with $k^\mu = (\omega_\mathbf{k}, \mathbf{k})^\mu$ defined by the summation and integration coordinate. **[Prove this.]** In other words the steps of re-expressing in terms $\mathbf{k}^*_f$ and $\mathbf{k}^*_i$, decomposing in harmonics, and inserting factors of k^*_f/q^*_f and k^*_i/q^*_i are all achieved with a simple redefinition of k^μ.

Given this observation, the target quantity, $G(L)\cdot w_\mu$, can be written

$$[G(L)\cdot w_\mu]_{\ell_f m_f,\ell_i m_i}(P_f, P_i, L) \equiv F_\pi[(P_f - P_i)^2]\Big[(P_f + P_i)_\mu\; G^S_{\ell_f m_f,\ell_i m_i}(P_f, P_i, L) - 2G^V_{\mu;\ell_f m_f,\ell_i m_i}(P_f, P_i, L)\Big]\,, \tag{193}$$

where

$$G^S_{\ell_f m_f,\ell_i m_i} \equiv \left[\frac{1}{L^3}\sum\!\!\!\!\!\!\!\int_{\mathbf{k}}\right] I_{\ell_f m_f,\ell_i m_i}(P_f, P_i, \mathbf{k})\,, \tag{194}$$

$$G^V_{\mu;\ell_f m_f,\ell_i m_i} \equiv \left[\frac{1}{L^3}\sum\!\!\!\!\!\!\!\int_{\mathbf{k}}\right] k_\mu I_{\ell_f m_f,\ell_i m_i}(P_f, P_i, \mathbf{k})\,, \tag{195}$$

$$I_{\ell_f m_f,\ell_i m_i}(P_f, P_i, \mathbf{k}) = \frac{1}{2\omega_\mathbf{k}}\frac{\sqrt{4\pi}Y_{\ell_f m_f}(\hat{\mathbf{k}}^*_f)}{2\omega_{\mathbf{P}_f-\mathbf{k}}(E_f - \omega_\mathbf{k} - \omega_{\mathbf{P}_f-\mathbf{k}} + i\epsilon)}\left(\frac{k^*_f}{q^*_f}\right)^{\ell_f}\frac{\sqrt{4\pi}Y^*_{\ell_i m_i}(\hat{\mathbf{k}}^*_i)}{2\omega_{\mathbf{P}_i-\mathbf{k}}(E_i - \omega_\mathbf{k} - \omega_{\mathbf{P}_i-\mathbf{k}} + i\epsilon)}\left(\frac{k^*_i}{q^*_i}\right)^{\ell_i}. \tag{196}$$

[Using the definitions above, Prove Eq. (193)]

3. Reducing G using the decomposition of the ρ form factor

In Eqs. (79) and () above we have shown that

$$\begin{aligned}
\langle \pi\pi, P_f, \lambda_f|\mathcal{J}^\mu|\pi\pi, P_i, \lambda_i\rangle = &\left((P_f + P_i)^\mu + (P_i - P_f)^\mu\frac{(P_f^2 - P_i^2)}{Q^2}\right)(\epsilon^*_f\cdot\epsilon_i)\;\mathcal{F}_1(Q^2, P_i^2, P_f^2)\\
&+ \left(\epsilon^\mu_i\, P_i\cdot\epsilon^*_f + \epsilon^{*\mu}_f\, P_f\cdot\epsilon_i\right)\mathcal{F}_2(Q^2, P_i^2, P_f^2)\\
&+ \left(\epsilon^\mu_i\, P_i\cdot\epsilon^*_f - \epsilon^{*\mu}_f\, P_f\cdot\epsilon_i - 2(P_i - P_f)^\mu\,(\epsilon^*_f\cdot P_i)\,(\epsilon_i\cdot P_f)\right)\mathcal{F}_3(Q^2, P_i^2, P_f^2)\\
&+ \left((P_f + P_i)^\mu + (P_i - P_f)^\mu\frac{(P_f^2 - P_i^2)}{Q^2}\right)(\epsilon^*_f\cdot P_i)\,(\epsilon_i\cdot P_f)\,\mathcal{F}_4(Q^2, P_i^2, P_f^2)\,,
\end{aligned} \tag{197}$$

where we have renamed $\widetilde{G}_1 \to \mathcal{F}_1$, $\widetilde{G}_3 \to \mathcal{F}_2$, $\widetilde{G}_4 \to \mathcal{F}_3$, $\widetilde{G}_5 \to \mathcal{F}_4$.

In this subsection we use this decomposition to define scalar projections of G. The basic idea here is to start with

$$\mathcal{W}_{\rm df}(P_f, P_i) \equiv \mathcal{W}_{L,{\rm df}}(P_f, P_i, L) - \mathcal{M}(P_f)\;[G(L)\cdot w](P_f, P_i)\;\mathcal{M}(P_i)\,, \tag{198}$$

and rewrite this at the level of the scalar form factors $\mathcal{F}_i$. Many issues must be addressed to do this property. To name a few, we must

1. **Understand the subduction of the Lellouch-Lüscher-like equation that is used to extract $\mathcal{W}_{L,\mathrm{df}}(P_f, P_i, L)$ from the finite-volume matrix element.**
2. **Understand how the mixing of partial waves enters the subtraction $\mathcal{M}(P_f)\ [G(L)\cdot w](P_f, P_i)\ \mathcal{M}(P_i)$.**
3. **Understand the scalar decomposition of $\mathcal{W}_{\mathrm{df}}(P_f, P_i)$ and how it is related to the decomposition of the full matrix element given above.**

To get started we circumvent these issues by assuming that the symmetry group of the infinite-volume gives a leading-order approximation and assuming that $\mathcal{W}_{\mathrm{df}}(P_f, P_i)$ admits the same decomposition as the full matrix element. In equations, the latter implies

$$\begin{aligned}\mathcal{W}^{\mu}_{\mathrm{df},\ell'=1,m'=\lambda_f;\ell=1,m=\lambda_i}(P_f, P_i) = &\left((P_f + P_i)^\mu + (P_i - P_f)^\mu \frac{(P_f^2 - P_i^2)}{Q^2}\right)(\epsilon_f^* \cdot \epsilon_i)\ \mathcal{F}_1^{\mathrm{df}}(Q^2, P_i^2, P_f^2)\\ &+ \left(\epsilon_i^\mu\, P_i \cdot \epsilon_f^* + \epsilon_f^{*\mu}\, P_f \cdot \epsilon_i\right) \mathcal{F}_2^{\mathrm{df}}(Q^2, P_i^2, P_f^2)\\ &+ \left(\epsilon_i^\mu\, P_i \cdot \epsilon_f^* - \epsilon_f^{*\mu}\, P_f \cdot \epsilon_i - 2(P_i - P_f)^\mu\,(\epsilon_f^* \cdot P_i)\,(\epsilon_i \cdot P_f)\right) \mathcal{F}_3^{\mathrm{df}}(Q^2, P_i^2, P_f^2)\\ &+ \left((P_f + P_i)^\mu + (P_i - P_f)^\mu \frac{(P_f^2 - P_i^2)}{Q^2}\right)(\epsilon_f^* \cdot P_i)\,(\epsilon_i \cdot P_f)\,\mathcal{F}_4^{\mathrm{df}}(Q^2, P_i^2, P_f^2)\,. \end{aligned}\tag{199}$$

Next suppose that we are able to identify projectors, $\mathcal{P}^k_{\lambda_i,\lambda_f,\mu}$ such that

$$\mathcal{P}^k_{\lambda_i,\lambda_f,\mu} \mathcal{W}^\mu_{\mathrm{df},\lambda_f,\lambda_i}(P_f, P_i) = \mathcal{F}_k^{\mathrm{df}}(Q^2, P_i^2, P_f^2)\,. \tag{200}$$

Then, applying these to Eq. (198), we find

$$\mathcal{F}_k^{\mathrm{df}}(Q^2, P_i^2, P_f^2) \equiv \mathcal{P}^k_{\lambda_i,\lambda_f,\mu} \mathcal{W}^\mu_{L,\mathrm{df}}(P_f, P_i, L) - F_\pi[(P_f - P_i)^2]\mathcal{M}_p(P_f)\mathcal{G}^k(P_f, P_i, L)\mathcal{M}_p(P_i)\,, \tag{201}$$

where $\mathcal{M}_p(P)$ is the p-wave scattering amplitude and where

$$\mathcal{G}^k(P_f, P_i, L) \equiv \mathcal{P}^k_{\lambda_i,\lambda_f,\mu}\left[(P_f + P_i)^\mu\ G^S_{1\lambda_f,1\lambda_i}(P_f, P_i, L) - 2G^{V,\mu}_{1\lambda_f,1\lambda_i}(P_f, P_i, L)\right]. \tag{202}$$

Something like this is likely the final form of the kinematic function that we will wish to code in this project.

4. *Evaluating G for $P_i^\mu = P_f^\mu$*

[This section is written for general G. Simplify it for the case of pions and a vector current.]

The approach for evaluating G differs depending on whether or not the two momenta are equal. In this section we describe the approach for the case of $P_i^\mu = P_f^\mu = P^\mu$, equivalently for $Q^2 = 0$. We focus here on the integral part of G, denoted by G_I

$$G_{I;\ell_f m_f,\ell'_f m'_f;\ell'_i m'_i,\ell_i m_i}(P) \equiv \int \frac{d\mathbf{k}}{(2\pi)^3}\, \frac{1}{8\omega_1\omega_2^2} 4\pi Y_{\ell_f m_f}(\hat{\mathbf{k}}^*) Y^*_{\ell'_f m'_f}(\hat{\mathbf{k}}^*) 4\pi Y_{\ell'_i m'_i}(\hat{\mathbf{k}}^*) Y^*_{\ell_i m_i}(\hat{\mathbf{k}}^*) \left(\frac{k^*}{q^*}\right)^{\ell_f+\ell'_f+\ell_i+\ell'_i} \left[\frac{1}{(E - \omega_1 - \omega_2 + i\epsilon)}\right]^2. \tag{203}$$

We have also already restricted attention to a single channel and dropped the i and f labels since there is only one frame for $P_i^\mu = P_f^\mu = P^\mu$.

We begin by rewriting the integral as

$$G_I(P) = \int \frac{d\mathbf{k}^*}{(2\pi)^3} \frac{1}{2\omega_1^*} \mathcal{F}(\mathbf{k}^*)(E^* - \omega_1^* + \omega_2^*)^2 \left[\frac{1}{(E - \omega_1)^2 - \omega_2^2 + i\epsilon}\right]^2, \tag{204}$$

where we have left the harmonic indices on G_I implicit and where

$$\mathcal{F}(\mathbf{k}^*) \equiv \frac{(E - \omega_1 + \omega_2)^2}{4\omega_2^2(E^* - \omega_1^* + \omega_2^*)^2} 4\pi Y_{\ell_f m_f}(\hat{\mathbf{k}}^*) Y^*_{\ell'_f m'_f}(\hat{\mathbf{k}}^*) 4\pi Y_{\ell'_i m'_i}(\hat{\mathbf{k}}^*) Y^*_{\ell_i m_i}(\hat{\mathbf{k}}^*) \left(\frac{k^*}{q^*}\right)^{\ell_f+\ell'_f+\ell_i+\ell'_i}. \tag{205}$$

Here we have also used the fact that $d\mathbf{k}/\omega_1 = d\mathbf{k}^*/\omega_1^*$. The next step is to rewrite the double pole in CM frame variables

$$(E-\omega_1)^2 - \omega_2^2 = [(E,\mathbf{P}) - (\omega_1,\mathbf{k})]^2 - m_2^2 = [(E^*,0) - (\omega_1^*,\mathbf{k}^*)]^2 - m_2^2 = (E^*-\omega_1^*)^2 - \omega_2^{*2}\,. \tag{206}$$

Substituting this into Eq. (204) and also substituting

$$\mathcal{F}(k^*) \equiv \int d\Omega\ \mathcal{F}(\mathbf{k}^*)\,, \tag{207}$$

then gives

$$G_I(P) = \int_0^\infty \frac{dk^* k^{*2}}{(2\pi)^3}\frac{1}{2\omega_1^*}\frac{\mathcal{F}(k^*)}{(E^*-\omega_1^*-\omega_2^*+i\epsilon)^2}\,. \tag{208}$$

The final step is to observe

$$\frac{1}{E^*-\omega_1^*-\omega_2^*+i\epsilon} = \frac{H(k^*)}{q^*-k^*+i\epsilon}\,, \tag{209}$$

where $E^* = \sqrt{m_1^2+q^{*2}} + \sqrt{m_2^2+q^{*2}}$ and

$$H(k^*) = \frac{(E^*+\omega_1^*+\omega_2^*)(E^{*2}-\omega_1^{*2}-\omega_2^{*2}+2\omega_1^*\omega_2^*)}{4E^{*2}(q^*+k^*)}\,. \tag{210}$$

This equality follows from

$$\begin{aligned}
&(E^{*2}-\omega_1^{*2}-\omega_2^{*2}+2\omega_1^*\omega_2^*)(E^{*2}-\omega_1^{*2}-\omega_2^{*2}-2\omega_1^*\omega_2^*)\\
&\quad= E^{*4} + (2k^{*2}+m_1^2+m_2^2)^2 - 2E^{*2}(2k^{*2}+m_1^2+m_2^2) - 4(k^{*2}+m_1^2)(k^{*2}+m_2^2)\\
&\quad= E^{*4} - 2E^{*2}(m_1^2+m_2^2) + m_1^4 + m_2^4 - 2m_1^2m_2^2 - 4E^{*2}k^{*2}\\
&\quad= E^{*2}\left(E^{*2} - 2(m_1^2+m_2^2) + \frac{\left(m_1^2-m_2^2\right)^2}{E^{*2}}\right) - 4E^{*2}k^{*2}\\
&\quad= 4E^{*2}(q^{*2}-k^{*2}).
\end{aligned} \tag{211}$$

Finally, we can rewrite the integral as

$$G_I(P) = \int_{-q^*}^\infty dx f(x)\frac{1}{(x-i\epsilon)^2}\,, \tag{212}$$

where

$$f(x) \equiv \frac{(x+q^*)^2}{(2\pi)^3}\frac{1}{2\sqrt{(x+q^*)^2+m_1^2}}\mathcal{F}(x+q^*)H(x+q^*)^2\,. \tag{213}$$

To reduce further we substitute $f(x) = f(x-i\epsilon) - f(0) + f(0) = g(x)(x-i\epsilon) + f(0)$ where $g(x) \equiv [f(x-i\epsilon) - f(0)]/(x-i\epsilon)$ has the same analytic properties as $f(x)$

$$G_I(E) \equiv \int_{-a}^\infty dx g(x)\frac{1}{(x-i\epsilon)} + f(0)\int_{-a}^\infty dx\frac{1}{(x-i\epsilon)^2} = \mathcal{P}\int_{-a}^\infty dx g(x)\frac{1}{x} + i\pi g(0) - \frac{f(0)}{a}\,. \tag{214}$$

Substituting for g we conclude

$$G_I(E) \equiv \mathcal{P}\int_{-a}^\infty dx\frac{f(x)-f(0)}{x^2} + i\pi f'(0) - \frac{f(0)}{a}\,. \tag{215}$$

5. Evaluating G for $\boldsymbol{P}_i = \boldsymbol{P}_f = 0$

The simplest case is given by $E_i \neq E_f$ but $\mathbf{P}_i = \mathbf{P}_f = 0$. Then the quantities defined above reduce to

$$G^S_{\ell_f m_f,\ell_i m_i} = \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int_{\mathbf{k}}\right] \frac{1}{(2\omega_{\mathbf{k}})^3} \mathcal{P}\frac{\sqrt{4\pi}Y_{\ell_f m_f}(\hat{\mathbf{k}})}{E_f - 2\omega_{\mathbf{k}}}\left(\frac{k}{q_f^*}\right)^{\ell_f} \mathcal{P}\frac{\sqrt{4\pi}Y^*_{\ell_i m_i}(\hat{\mathbf{k}})}{E_i - 2\omega_{\mathbf{k}}}\left(\frac{k}{q_i^*}\right)^{\ell_i} + \delta G^{S,i\epsilon}_{\ell_f m_f,\ell_i m_i}(P_f, P_i)\,, \tag{216}$$

$$G^V_{\mu;\ell_f m_f,\ell_i m_i} = \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int_{\mathbf{k}}\right] \frac{k_\mu}{(2\omega_{\mathbf{k}})^3} \mathcal{P}\frac{\sqrt{4\pi}Y_{\ell_f m_f}(\hat{\mathbf{k}})}{E_f - 2\omega_{\mathbf{k}}}\left(\frac{k}{q_f^*}\right)^{\ell_f} \mathcal{P}\frac{\sqrt{4\pi}Y^*_{\ell_i m_i}(\hat{\mathbf{k}})}{E_i - 2\omega_{\mathbf{k}}}\left(\frac{k}{q_i^*}\right)^{\ell_i} + \delta G^{V,i\epsilon}_{\mu;\ell_f m_f,\ell_i m_i}(P_f, P_i)\,, \tag{217}$$

where in the expressions for G^S and G^V here both poles are regulated by a principal-value prescription. The second term, $\delta G^{i\epsilon}$, corrects this to the proper $i\epsilon$-definition of the function.

The separation of pole prescriptions is based on the identity

$$\frac{1}{E - 2\omega_{\mathbf{k}} + i\epsilon} = \frac{E - 2\omega_{\mathbf{k}}}{(E - 2\omega_{\mathbf{k}})^2 + \epsilon^2} - \frac{i\epsilon}{(E - 2\omega_{\mathbf{k}})^2 + \epsilon^2}\,. \tag{218}$$

Note that, for very small ϵ, integrating the first term is the same as integrating $\omega_{\mathbf{k}}$ from m up to $E/2 - \delta$ and then from $E/2 + \delta$ to ∞, with δ taken arbitrarily small. The second term corresponds to a delta function in the limit of small ϵ.

In fact we see that, given the assumption $E_i \neq E_f$, the conditions $E_f = 2\omega_{\mathbf{k}}$ and $E_f = 2\omega_{\mathbf{k}}$ cannot be fulfilled simultaneously. This means that $\delta G^{S,i\epsilon}_{\ell_f m_f,\ell_i m_i}(P_f, P_i, L)$ only depends on the cross terms of one δ-function with one principal value. (The double δ-function term vanishes.) The cross terms can be significantly simplifed

$$\delta G^{S,i\epsilon}_{\ell_f m_f,\ell_i m_i}(P_f, P_i) = i\pi\int_{\mathbf{k}} \frac{1}{(2\omega_{\mathbf{k}})^3}\sqrt{4\pi}Y_{\ell_f m_f}(\hat{\mathbf{k}})\delta(E_f - 2\omega_{\mathbf{k}})\left(\frac{k}{q_f^*}\right)^{\ell_f}\frac{\sqrt{4\pi}Y^*_{\ell_i m_i}(\hat{\mathbf{k}})}{E_i - 2\omega_{\mathbf{k}}}\left(\frac{k}{q_i^*}\right)^{\ell_i} + (i \leftrightarrow f)\,, \tag{219}$$

$$= \frac{i\pi}{8\pi^3}\int d\Omega_{\hat{\mathbf{k}}}\int_0^\infty \frac{dk k^2}{(2\omega_{\mathbf{k}})^3}\sqrt{4\pi}Y_{\ell_f m_f}(\hat{\mathbf{k}})\delta(E_f - 2\omega_{\mathbf{k}})\frac{\sqrt{4\pi}Y^*_{\ell_i m_i}(\hat{\mathbf{k}})}{E_i - 2\omega_{\mathbf{k}}}\left(\frac{k}{q_i^*}\right)^{\ell_i} + (i \leftrightarrow f)\,, \tag{220}$$

$$= \frac{i}{8\pi}\delta_{\ell_i\ell_f}\delta_{m_i m_f}\int_{2M_\pi}^\infty \frac{d(2\omega)k}{(2\omega)^2}\delta(E_f - 2\omega)\frac{1}{E_i - 2\omega}\left(\frac{k}{q_i^*}\right)^{\ell_i} + (i \leftrightarrow f)\,, \tag{221}$$

$$= \frac{i}{8\pi}\delta_{\ell_i\ell_f}\delta_{m_i m_f}\frac{1}{E_i - E_f}\left[\frac{q_f^*}{E_f^2}\left(\frac{q_f^*}{q_i^*}\right)^{\ell_i} - \frac{q_i^*}{E_i^2}\left(\frac{q_i^*}{q_f^*}\right)^{\ell_i}\right]. \tag{222}$$

[Prove this and derive the analogous result for G^V.]

Focusing now on a particular spherical harmonic ($\ell = 1, m = 0$) we write

$$\mathcal{G}^S_{zz}(E_f, E_i, L) \equiv 3\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int_{\mathbf{k}}\right] \frac{1}{(2\omega_{\mathbf{k}})^3}\frac{k_z^2}{q_f^* q_i^*}\mathcal{P}\frac{1}{E_f - 2\omega_{\mathbf{k}}}\mathcal{P}\frac{1}{E_i - 2\omega_{\mathbf{k}}}e^{-\alpha^2(E_f - 2\omega_{\mathbf{k}})(E_i - 2\omega_{\mathbf{k}})}\,, \tag{223}$$

$$\delta\mathcal{G}^{S,i\epsilon}_{zz}(E_f, E_i) \equiv \frac{i}{8\pi}\frac{1}{E_i - E_f}\left[\frac{(q_f^*)^2}{q_i^* E_f^2} - \frac{(q_i^*)^2}{q_f^* E_i^2}\right]. \tag{224}$$

Here we have also introduced our approach for evaluating $\mathcal{G}_{zz}$. Following Ref. [?] we introduce exponential damping factors with some constant α that has units of length. This damping is expected to introduce artifacts that scale as $e^{-L/\alpha}$ and vanish for $\alpha \to 0$. The function $\mathcal{G}^S_{zz}(E_f, E_i, L)$ is plotted in Figure 3 for $\alpha = 0.5$ and $M_\pi L = 6$. **[Write code to check my code for G^S and also to calculate G^V]**

Appendix A: Accelerated form of the geometric Z function

We describe here how to derive a form of the generalized $Z^{\boldsymbol{\Delta}}_{alm}[s, x^2]$ in particular how to obtain Eq. (**??**) from Eq. (173).

We will follow Ref. [17]. The generalized Zeta function is defined as

$$Z^{\boldsymbol{\Delta}}_{lm}[s, x^2] \;=\; \sum_{\mathbf{r}\in P_{\boldsymbol{\Delta}}} \frac{r^l\, Y_{lm}(\hat{\mathbf{r}})}{(r^2 - x^2)^s}\,, \tag{A1}$$

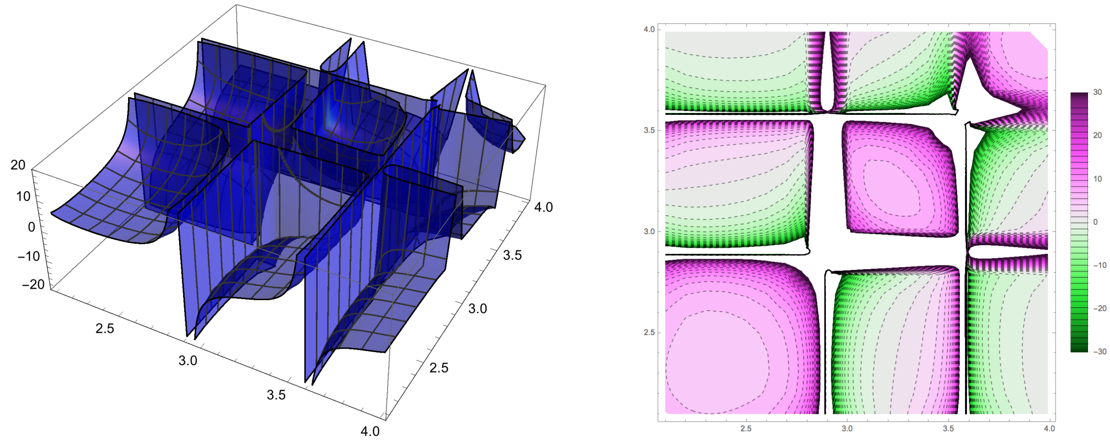

FIG. 3: $\mathcal{G}_{zz}(E_i, E_f, L)$ plotted as a function of E_i, E_f in units of M_π, with L fixed at $M_\pi L = 6$. The singularities arise when either E_i or E_f (or both) coincide with one of the non-interacting two-pion energies. For interacting pions one must sample this function at the finite-volume energies (away from the poles) to determine $\mathcal{G}_{zz}$. Multiplying this with the pion vector form factor then gives the term needed to extract the $\mathbf{2} + \mathcal{J} \to \mathbf{2}$ matrix element.

where the domain $P_{\mathbf{\Delta}}$ is given by

$$P_\Delta = \left\{ \mathbf{r} | \mathbf{r} = \boldsymbol{\gamma}^{-1}(\mathbf{n} - \frac{1}{2}\mathbf{\Delta}), \mathbf{n} \in Z^3 \right\}, \tag{A2}$$

with

$$\mathbf{\Delta} \;=\; \frac{\mathbf{P}L}{2\pi}\left(1 + \frac{m_{a1}^1 - m_{a2}^2}{E^{*2}}\right). \tag{A3}$$

Using the following integral definition of the Gamma function (valid only if $Re(s) > 0$)

$$\Gamma(s) \;=\; a^s \int_0^\infty dt t^{s-1} e^{-ta}\,, \tag{A4}$$

for $a = r^2 - x^2$ we can rewrite Eq. (A1) as

$$\begin{aligned} Z_{lm}^{\mathbf{\Delta}}[s, x^2] \;&=\; \frac{1}{\Gamma(s)} \sum_{\mathbf{r} \in P_{\mathbf{\Delta}}} r^l Y_{lm}(\hat{\mathbf{r}}) \int_0^\infty dt t^{s-1} e^{-t(r^2 - x^2)} \qquad (A5) \\ &=\; \frac{1}{\Gamma(s)} \sum_{\mathbf{r} \in P_{\mathbf{\Delta}}} r^l Y_{lm}(\hat{\mathbf{r}}) \left[\int_0^1 dt t^{s-1} e^{-t(r^2 - x^2)} + \int_1^\infty dt t^{s-1} e^{-t(r^2 - x^2)} \right]. \\ &=\; I_{lm}^{\mathbf{\Delta}}[s, x^2] + J_{lm}^{\mathbf{\Delta}}[s, x^2]\,, \qquad (A6) \end{aligned}$$

with

$$J_{lm}^{\mathbf{\Delta}}[s, x^2] \;=\; \frac{1}{\Gamma(s)} \sum_{\mathbf{r} \in P_{\mathbf{\Delta}}} r^l Y_{lm}(\hat{\mathbf{r}}) \int_1^\infty dt t^{s-1} e^{-t(r^2 - x^2)}\,, \tag{A7}$$

and similarly for $I_{lm}^{\mathbf{\Delta}}[s; x^2]$. For $s = 1$ the function $J_{lm}^{\mathbf{\Delta}}[s; x^2]$ is easily evaluated

$$J_{lm}^{\mathbf{\Delta}}[s, x^2] \;=\; \sum_{\mathbf{r} \in P_{\mathbf{\Delta}}} r^l Y_{lm}(\hat{\mathbf{r}}) \frac{e^{-(r^2 - x^2)}}{r^2 - x^2}\,. \tag{A8}$$

For reasons that will be clear later it is convenient to retain the general s dependence in $I_{lm}^{\mathbf{\Delta}}[s, x^2]$. We also define, for convenience, the following function

$$F(\mathbf{r}) \;=\; r^l Y_{lm}(\hat{\mathbf{r}})\, e^{-tr^2}\,, \tag{A9}$$

and therefore we can write

$$I^{\boldsymbol{\Delta}}_{lm}[s,x^2] \;=\; \frac{1}{\Gamma(s)}\int_0^1 dt\, t^{s-1}e^{tx^2}\sum_{\mathbf{r}\in P_\Delta} F(\mathbf{r})\,. \tag{A10}$$

We recall that $\mathbf{r}\in P_\Delta$ is a function of $\mathbf{n}\in\mathbb{Z}^3$ (as it can be seen from Eq. (A2)) equivalent to $\sum_{\mathbf{n}\in\mathbb{Z}^3} F(\mathbf{r}(\mathbf{n}))$ and therefore

$$\sum_{\mathbf{r}\in P_\Delta} F(\mathbf{r}) \;=\; \sum_{\mathbf{n}\in\mathbb{Z}^3} F(\mathbf{r}(\mathbf{n})) = \sum_{\mathbf{n}\in\mathbb{Z}^3} f(\mathbf{n})\,, \tag{A11}$$

where we have defined the function $f(\mathbf{n})$. Using Poisson summation formula we can rewrite the sum of Eq. (A11) as

$$\sum_{\mathbf{n}\in\mathbb{Z}^3} F(\mathbf{r}(\mathbf{n})) = \sum_{\mathbf{n}\in\mathbb{Z}^3} f(\mathbf{n}) \;=\; \sum_{\mathbf{n}\in\mathbb{Z}^3}\int d\mathbf{y}\, f(\mathbf{y})e^{i2\pi\mathbf{n}\cdot\mathbf{y}} = \sum_{\mathbf{n}\in\mathbb{Z}^3}\int d\mathbf{y} F(\mathbf{r}(\mathbf{y}))e^{i2\pi\mathbf{n}\cdot\mathbf{y}}\,. \tag{A12}$$

where in the last passage is $\mathbf{r} = \boldsymbol{\gamma}^{-1}(\mathbf{y}-\frac{1}{2}\boldsymbol{\Delta})$. Remembering Eq. (A9) we can rewrite the previous expression as

$$\sum_{\mathbf{n}\in\mathbb{Z}^3} f(\mathbf{n}) \;=\; \sum_{\mathbf{n}\in\mathbb{Z}^3}\int d\mathbf{y}|\mathbf{r}(\mathbf{y})|^l Y_{lm}(\hat{\mathbf{r}}(\mathbf{y}))\, e^{-t|\mathbf{r}(\mathbf{y})|^2+i2\pi\mathbf{n}\cdot\mathbf{y}}\,. \tag{A13}$$

It is convenient to perform the change of variables

$$\mathbf{y} \;=\; \boldsymbol{\gamma}\mathbf{r} + \frac{1}{2}\boldsymbol{\Delta}\,, \tag{A14}$$

that leads to the following change in the integration measure $d\mathbf{y} = \gamma d\mathbf{r}$. Therefore we have

$$\begin{aligned} f(\mathbf{n}) \;&=\; e^{i\pi\mathbf{n}\cdot\boldsymbol{\Delta}}\gamma\int d\mathbf{r} r^l Y_{lm}(\hat{\mathbf{r}})e^{-tr^2+i2\,\pi\mathbf{n}\cdot(\boldsymbol{\gamma}\mathbf{r})} \\ &=\; e^{i\pi\mathbf{n}\cdot\boldsymbol{\Delta}}\gamma\int_0^\infty dr r^{2+l}e^{-tr^2}\int d\hat{\mathbf{r}} Y_{lm}(\hat{\mathbf{r}})e^{-i\mathbf{k}\cdot\mathbf{r}}\,, \end{aligned} \tag{A15}$$

where in the last passage we have defined, for convenience, $\mathbf{k} = -2\pi\boldsymbol{\gamma}^T\mathbf{n}$. Using the well known relation for $e^{-i\mathbf{k}\cdot\mathbf{r}}$

$$e^{-i\mathbf{k}\cdot\mathbf{r}} \;=\; 4\pi\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'}(-i)^{l'}Y_{l'm'}(\hat{\mathbf{k}})Y_{l'm'}(\hat{\mathbf{r}})^* j_{l'}(kr)\,, \tag{A16}$$

we obtain

$$f(\mathbf{n}) \;=\; 4\pi(-i)^l e^{i\pi\mathbf{n}\cdot\boldsymbol{\Delta}}\gamma\, Y_{lm}(\hat{\mathbf{k}})\underbrace{\int_0^\infty dr r^{2+l}e^{-tr^2}j_l(kr)}_{f_l(t)}\,. \tag{A17}$$

The Hankel transform can be evaluated using MATHEMATICA (as Felipe did) or SYMPY (as I did)

$$f_l(t) = \frac{k^l}{2^{l+1}t^{l+3/2}}\Gamma(l+3/2)\frac{{}_1F_1(l+3/2,l+1,-k^2/(4t))}{\Gamma(l+1)}\,, \tag{A18}$$

where ${}_1F_1(a,b,c)$ is the hypergeometric function of the first kind and can be written as

$$\begin{aligned} J = \Gamma(l+3/2)\frac{{}_1F_1(l+3/2,l+1,-k^2/(4t))}{\Gamma(l+1)} \;&=\; \frac{\Gamma(l+3/2)}{\Gamma(l+1)}\sum_{i=0}^{\infty}\frac{(l+3/2)_i}{(l+1)_i}\frac{z^i}{i!} \\ &=\; \sum_{i=0}^{\infty}\frac{\Gamma(l+i+3/2)}{\Gamma(l+i+1)}\frac{z^i}{i!}\,, \end{aligned} \tag{A19}$$

where we have defined $z = -k^2/(4t)$, and we have used the definition of the Pochhammer symbols i. e. $(l+3/2)_i = \Gamma(l+3/2+i)/\Gamma(l+3/2)$, and similar for $(l+1)_i$.
In Ref. [17] the Hankel transform $f_l(t)$ is stated to be

$$f_l(t) \;=\; \frac{e^{-k^2/(4\,t)}}{4\,\pi}\left(\frac{k}{2t}\right)^l\left(\frac{\pi}{t}\right)^{3/2}\,. \tag{A20}$$

The expressions in Eq. (A18) and Eq. (A20) are the same if and only if

$$J \;=\; \frac{\sqrt{\pi}}{2} e^{-z} \,, \tag{A21}$$

and since $\Gamma(3/2)/\Gamma(1) = \sqrt{\pi}/2$ that would imply

$$\sum_{i=0}^{\infty} \frac{\Gamma(l+i+3/2)}{\Gamma(l+i+1)} \frac{z^i}{i!} \;=\; \Gamma(3/2) e^{-z} \,, \tag{A22}$$

and SEEMS to imply the following identity

$$\frac{\Gamma(l+i+3/2)}{\Gamma(l+i+1)} = \frac{\Gamma(3/2)}{\Gamma(1)} \quad \forall l, i \in \mathbb{N} \tag{A23}$$

this cannot be true. Indeed we have a counterexample if first we consider the case $l+i=0$ the l.h.s of Eq. (A23) is $\sqrt{\pi}/2$ and if then we consider the case $l+i=1$ the l.h.s of Eq (A23) is $3\sqrt{\pi}/8$.
Remains to see if the two expressions give significant numerical differences.

So proceeding with Eq. (A20) we finally obtain

$$f(\mathbf{n}) \;=\; (-i)^l e^{i\pi \mathbf{n}\cdot\mathbf{\Delta}} \gamma \, Y_{lm}(\hat{\mathbf{k}}) e^{-k^2/(4\,t)} \left(\frac{k}{2t}\right)^l \left(\frac{\pi}{t}\right)^{3/2} \,. \tag{A24}$$

Therefore $I^{\mathbf{\Delta}}_{lm}[s,x^2]$ can now be written as

$$I^{\mathbf{\Delta}}_{lm}[s,x^2] \;=\; \frac{(-i)^l \pi^{3/2}\gamma}{\Gamma(s)} \int_0^1 dt\, t^{s-5/2} e^{tx^2} \sum_{\mathbf{n}\in\mathbb{Z}^3} e^{i\pi \mathbf{n}\cdot\mathbf{\Delta}} Y_{lm}(\hat{\mathbf{k}}) e^{-k^2/(4\,t)} \left(\frac{k}{2t}\right)^l \,. \tag{A25}$$

We note that the term $\mathbf{n} = \mathbf{0}$ in the above sum is different from zero only for $l = m = 0$. It is important to note now that for the case of interest here ($s = 1$) there is a diverrgence for $\mathbf{n} = 0$ and $l = m = 0$. Indeed we have

$$\begin{aligned}
(term\ in\ I^{\mathbf{\Delta}}_{00}[s,x^2]\ with\ n=0) \;&=\; \frac{\pi\gamma}{2\Gamma(s)} \int_0^1 dt t^{s-5/2} e^{tq^2} \\
&=\; \frac{\pi\gamma}{2\Gamma(s)} \int_0^1 dt \left[t^{s-5/2}(e^{tq^2}-1) + t^{s-5/2} \right] \,,
\end{aligned} \tag{A26}$$

The first integral for $s = 1$ becomes

$$\int_0^1 dt \frac{e^{tq^2}-1}{t^{3/2}} \,, \tag{A27}$$

which is finite. The second integral in Eq. A26 is divergent for $s = 1$ (trivially $\int_0^1 dt/t^{3/2} = \infty$). Following Ref. [17] we consider the function

$$g(s) \;=\; \int_0^1 dt t^{s-5/2} = \frac{1}{s-3/2} \,, \tag{A28}$$

for $s > 3/2$.which is finite. We can have a finite value of $g(s)$ for $s = 1$ if we analitically continue Eq. (A28) to $s = 1$. Using this procedure (and evaluating the r.h.s. of Eq. (A28)) we have $g(s=1) = -2$.
Finally collecting everything we have

$$\begin{aligned}
Z^{\mathbf{\Delta}}_{lm}(s=1,x^2) \;&=\; \sum_{\mathbf{r}\in P_\Delta} \frac{e^{-(r^2-x^2)}}{r^2-x^2} r^l Y_{lm}(\hat{\mathbf{r}}) + \gamma(-i)^l \pi^{3/2} \int_0^1 dt \frac{e^{tx^2}}{t^{3/2}} \sum_{\mathbf{n}\in\mathbb{Z}^3/\{0\}} e^{i\pi \mathbf{n}\cdot\mathbf{\Delta}} Y_{lm}(\hat{\mathbf{k}}) e^{-k^2/(4t)} \left(\frac{k}{2t}\right)^l \\
&+\; \frac{\pi\gamma}{2} \delta_{l,0}\delta_{m,0} \left[\int_0^1 dt \frac{(e^{tx^2}-1)}{t^{3/2}} - 2 \right] \,,
\end{aligned} \tag{A29}$$

which agrees with Eq. (**??**) except for the sign in the exponent $e^{i\pi \mathbf{n}\cdot\mathbf{\Delta}}$.

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