CS 131 Problem Set 2

Catherine Angangco, Joshua Frankie Rayo

April 3, 2014

1 Value-at-Risk

1.1 Preliminaries

The value at risk *VaR* of a continuous loss distribution modeled by a probability distribution function, say $p(x)$ at a given risk level $(1-\alpha) \in [0,1]$ is defined as the value at which the cumulative probability of $p(x)$ from $-\infty$ is equal to α. Mathematically speaking, we say:

$$VaR_\alpha = z \Leftrightarrow \int_{-\infty}^{z} p(x)dx = \alpha$$

A value z is the value-at-risk at a if and only if the area under the curve of $p(x)$ in the interval $(-\infty, z]$ is equal to α. We first observe the properties of the probability functions

$$p_1(x) = \frac{1}{\sqrt{6\pi}} e^{-\frac{x^2}{6}}$$

and

$$p_2(x) = \frac{1}{\gamma\sqrt{3\pi}}\left(1+\frac{x^2}{3}\right)^{-2}, \gamma \approx 0.886226925453$$

where $p_1(x)$ is a Gaussian distribution function and $p_2(x)$ is a t distribution function.

- We know that these functions are symmetric to some line $x = \mu$, where μ is the mean of the distribution. When $\mu = 0$, its graph is symmetric to the line $x = 0$ or y-axis. It has the property

$$\int_m^n p(x)dx = \int_{-n}^{-m} p(x)dx$$

  for $m \le n$.

- The functions above are probability distribution functions. It follows that the sum of the probabilities in the sample space is equal to 1. Or equivalently,

$$\sum_{\forall x} p(x) = \int_{-\infty}^{\infty} p(x)dx = 1 \tag{1}$$

- We redefine the limits of integration (i.e.) split equation (2) into two as follows:

$$\int_{-\infty}^{\infty} p(x)dx = \int_{-\infty}^{0} p(x)dx + \int_{0}^{\infty} p(x)dx \tag{2}$$

- Since the graph of the function is symmetric to y-axis, the area under the curve in the interval $[0, \infty)$ is also equal to the area uder the curve in the interval $(-\infty, 0)$. Using equation (1) and (2),

$$\int_{-\infty}^{0} p(x)dx = \int_{0}^{\infty} p(x)dx = 0.5 \tag{3}$$

Now, we verify if $\int_0^\infty p(x)dx = 0.5$ using Gauss-Laguerre quadrature. For $p_1(x)$, the Gauss-Laguerre quadrature form of the integral is

$$I_1 = \int_0^\infty e^{-x}\left(\frac{e^x}{\sqrt{6\pi}} e^{-\frac{x^2}{6}}\right)dx = \int_0^\infty e^{-x}\left(\frac{1}{\sqrt{6\pi}} e^{x-\frac{x^2}{6}}\right)dx$$

where the weighting function $w(x) = e^{-x}$ and $g_1(x) = \frac{1}{\sqrt{6\pi}} e^{x-\frac{x^2}{6}}$. By Gaussian quadratures approximation,

$$I_1 \approx \sum_{i=1}^{n} A_i g_1(x_i)$$

where i is the number of nodes for approximation. We use $n = 6$ nodes. The nodal abscissas and corresponding weights for $n = 6$ is given below.

i	x_i	A_i
1	0.222 847	0.458 964
2	1.188 932	0.417 000
3	2.992 736	0.113 373
4	5.775 144	0.010 399 2
5	9.837 467	0.000 261 017
6	15.982 874	0.000 000 898 548

To solve the integral,

$$\begin{aligned}
I_1 &= A_1g_1(x_1) + A_2g_1(x_2) + A_3g_1(x_3) \\
&\quad + A_4g_1(x_4) + A_5g_1(x_5) + A_6g_1(x_6) \\
I_1 &= 0.458964 \cdot 0.285454 + 0.417000 \cdot 0.597557 \\
&\quad + 0.113373 \cdot 1.03226 + 0.0103992 \cdot 0.285985 \\
&\quad + 0.000261017 \cdot 0.000426427 \\
&\quad + 8.98548 \times 10^{-7} \cdot 6.50687 \times 10^{-13} \\
I_1 &= 0.500198 \approx 0.5
\end{aligned}$$

For $p_2(x)$, the Gauss-Laguerre quadrature form of the integral is

$$I_2 = \int_0^\infty e^{-x}\left(\frac{e^x}{\gamma\sqrt{3\pi}}\left(1+\frac{x^2}{3}\right)^{-2}\right), \gamma \approx 0.886226925453$$

where the weighting function $w(x) = e^{-x}$ and $g_2(x) = \frac{e^x}{\gamma\sqrt{3\pi}}\left(1 + \frac{x^2}{3}\right)^{-2}$. By Gaussian quadratures approximation,

$$I_2 \approx \sum_{i=1}^{n} A_i g_2(x_i)$$

We used again $n = 6$ nodes, and the nodal abscissas and weights above.

To solve the integral,

$$\begin{aligned}
I_2 &= A_1 g_2(x_1) + A_2 g_2(x_2) + A_3 g_2(x_3) \\
&\quad + A_4 g_2(x_4) + A_5 g_2(x_5) + A_6 g_2(x_6) \\
I_2 &= 0.458964 \cdot 0.444468 + 0.417000 \cdot 0.55761 \\
&\quad + 0.113373 \cdot 0.461408 + 0.0103992 \cdot 0.806524 \\
&\quad + 0.000261017 \cdot 6.22113 \\
&\quad + 8.98548 \times 10^{-7} 432.589 \\
I_2 &= 0.499229 \approx 0.5
\end{aligned}$$

And now we established equation (3).

Going back to our original problem, take note that we are only estimating the value-at-risks for $\alpha \geq 0.5$, so we know that $z \geq 0$. Therefore we should use the relationship that

$$\begin{aligned}
\int_{-\infty}^{z} p(x)dx &= \int_{-\infty}^{0} p(x)dx + \int_{0}^{z} p(x)dx \\
&= 0.5 + \int_{0}^{z} p(x)dx
\end{aligned}$$

So it only remains to compute for $\int_0^z p(x)dx$. We use composite Simpson's 1/3 rule with 6 points also.

The general integral $\int_a^b f(x)dx$ can be numerically solved using the composite Simpson's 1/3 rule:

$$\begin{aligned}
\int_a^b f(x)dx = \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} \sum_{i\ even} f(x_i) \\
+ \frac{b-a}{3} \sum_{i\ odd} f(x_i) + \frac{b-a}{6} f(b)
\end{aligned} \tag{4}$$

The next step is to model our integral $\int_0^z p(x)dx$ as a function, say $P(z)$ so that we can use root-finding method for the equation $0.5 + P(z) - \alpha = 0$.

1.2 VaR estimates using Gaussian distribution

We first divide the interval $[0, z]$ into five panels with 6 points: $\{0, z/5, 2z/5, 3z/5, 4z/6, z\}$. Then the integral $\int_0^z \frac{1}{\sqrt{1\pi}} e^{-\frac{x^2}{6}} dx = \int_0^z p_1(x)dx$ is:

$$\begin{aligned}
\int_0^z p_1(x)dx &= \frac{z}{6} p_1(0) + \frac{2z}{3} p_1\left(\frac{z}{5}\right) + \frac{z}{3} p_1\left(\frac{2z}{5}\right) \\
&= \frac{2z}{3} p_1\left(\frac{3z}{5}\right) + \frac{z}{3} p_1\left(\frac{4z}{5}\right) + \frac{z}{6} p_1(z) \\
&= P_1(x)
\end{aligned} \tag{5}$$

And now we try to estimate the VaR with $\alpha = 0.8$. Then we use Regula-Falsi method for the equation $0.5 + P_1(z) - 0.8 = P_1(z) - 0.3 = 0 = Q_1(z)$, with $z \in [0, 1]$ and $tol \leq 10^{-6}$.

2 Naive Fourier Series Approximation

3 Halley's Comet

4 Yeast Growth Modelling