CS 131 Problem Set 2

Catherine Angangco, Joshua Frankie Rayo

April 5, 2014

1 Value-at-Risk

1.1 Preliminaries

The value at risk *VaR* of a continuous loss distribution modeled by a probability distribution function, say $p(x)$ at a given risk level $(1-\alpha) \in [0,1]$ is defined as the value at which the cumulative probability of $p(x)$ from $-\infty$ is equal to α. Mathematically speaking, we say:

$$VaR_\alpha = z \Leftrightarrow \int_{-\infty}^{z} p(x)dx = \alpha$$

A value z is the value-at-risk at a if and only if the area under the curve of $p(x)$ in the interval $(-\infty, z]$ is equal to α. We first observe the properties of the probability functions

$$p_1(x) = \frac{1}{\sqrt{6\pi}} e^{-\frac{x^2}{6}}$$

and

$$p_2(x) = \frac{1}{\gamma\sqrt{3\pi}} \left(1 + \frac{x^2}{3}\right)^{-2}, \gamma \approx 0.886226925453$$

where $p_1(x)$ is a Gaussian distribution function and $p_2(x)$ is a t distribution function.

- We know that these functions are symmetric to some line $x = \mu$, where μ is the mean of the distribution. When $\mu = 0$, its graph is symmetric to the line $x = 0$ or y-axis. It has the property

$$\int_{m}^{n} p(x)dx = \int_{-n}^{-m} p(x)dx$$

  for $m \leq n$.

- The functions above are probability distribution functions. It follows that the sum of the probabilities in the sample space is equal to 1. Or equivalently,

$$\sum_{\forall x} p(x) = \int_{-\infty}^{\infty} p(x)dx = 1 \tag{1}$$

- We redefine the limits of integration (i.e.) split equation (2) into two as follows:

$$\int_{-\infty}^{\infty} p(x)dx = \int_{-\infty}^{0} p(x)dx + \int_{0}^{\infty} p(x)dx \tag{2}$$

- Since the graph of the function is symmetric to y-axis, the area under the curve in the interval $[0, \infty)$ is also equal to the area uder the curve in the interval $(-\infty, 0)$. Using equation (1) and (2),

$$\int_{-\infty}^{0} p(x)dx = \int_{0}^{\infty} p(x)dx = 0.5 \tag{3}$$

Now, we verify if $\int_0^\infty p(x)dx = 0.5$ using Gauss-Laguerre quadrature. For $p_1(x)$, the Gauss-Laguerre quadrature form of the integral is

$$I_1 = \int_0^\infty e^{-x}\left(\frac{e^x}{\sqrt{6\pi}} e^{-\frac{x^2}{6}}\right)dx = \int_0^\infty e^{-x}\left(\frac{1}{\sqrt{6\pi}} e^{x-\frac{x^2}{6}}\right)dx$$

where the weighting function $w(x) = e^{-x}$ and $g_1(x) = \frac{1}{\sqrt{6\pi}} e^{x-\frac{x^2}{6}}$. By Gaussian quadratures approximation,

$$I_1 \approx \sum_{i=1}^{n} A_i g_1(x_i)$$

where i is the number of nodes for approximation. We use $n = 6$ nodes. The nodal abscissas and corresponding weights for $n = 6$ is given below.

i	x_i	A_i
1	0.222 847	0.458 964
2	1.188 932	0.417 000
3	2.992 736	0.113 373
4	5.775 144	0.010 399 2
5	9.837 467	0.000 261 017
6	15.982 874	0.000 000 898 548

To solve the integral,

$$\begin{aligned}
I_1 &= A_1 g_1(x_1) + A_2 g_1(x_2) + A_3 g_1(x_3) \\
&\quad + A_4 g_1(x_4) + A_5 g_1(x_5) + A_6 g_1(x_6) \\
I_1 &= 0.458964 \cdot 0.285454 + 0.417000 \cdot 0.597557 \\
&\quad + 0.113373 \cdot 1.03226 + 0.0103992 \cdot 0.285985 \\
&\quad + 0.000261017 \cdot 0.000426427 \\
&\quad + 8.98548 \times 10^{-7} \cdot 6.50687 \times 10^{-13} \\
I_1 &= 0.500198 \approx 0.5
\end{aligned}$$

For $p_2(x)$, the Gauss-Laguerre quadrature form of the integral is

$$I_2 = \int_0^\infty e^{-x}\left(\frac{e^x}{\gamma\sqrt{3\pi}}\left(1 + \frac{x^2}{3}\right)^{-2}\right), \gamma \approx 0.886226925453$$

where the weighting function $w(x) = e^{-x}$ and $g_2(x) = \frac{e^x}{\gamma\sqrt{3\pi}}\left(1 + \frac{x^2}{3}\right)^{-2}$. By Gaussian quadratures approximation,

$$I_2 \approx \sum_{i=1}^{n} A_i g_2(x_i)$$

We used again $n = 6$ nodes, and the nodal abscissas and weights above.

To solve the integral,

$$\begin{aligned}
I_2 &= A_1 g_2(x_1) + A_2 g_2(x_2) + A_3 g_2(x_3) \\
&\quad + A_4 g_2(x_4) + A_5 g_2(x_5) + A_6 g_2(x_6) \\
I_2 &= 0.458964 \cdot 0.444468 + 0.417000 \cdot 0.55761 \\
&\quad + 0.113373 \cdot 0.461408 + 0.0103992 \cdot 0.806524 \\
&\quad + 0.000261017 \cdot 6.22113 \\
&\quad + 8.98548 \times 10^{-7} \cdot 432.589 \\
I_2 &= 0.499229 \approx 0.5
\end{aligned}$$

And now we established equation (3), (2), and (1).

Going back to our original problem, take note that we are only estimating the value-at-risks for $\alpha \geq 0.5$, so we know that $z \geq 0$. Therefore we should use the relationship that

$$\begin{aligned}
\int_{-\infty}^{z} p(x)dx &= \int_{-\infty}^{0} p(x)dx + \int_{0}^{z} p(x)dx \\
&= 0.5 + \int_{0}^{z} p(x)dx \\
&= \alpha
\end{aligned}$$

So it only remains to compute for $\int_0^z p(x)dx$. We use composite Simpson's 1/3 rule with 6 points also..

The general integral $\int_a^b f(x)dx$ can be numerically solved using the composite Simpson's 1/3 rule:

$$\begin{aligned}
\int_{a}^{b} f(x)dx &= \frac{h}{3}f(a) + \frac{4h}{3}\sum_{i\ even} f(x_i) \\
&\quad + \frac{2h}{3}\sum_{i\ odd} f(x_i) + \frac{h}{3}f(b) \\
h &= \frac{b-a}{n}
\end{aligned} \tag{4}$$

where n is the number of panels. The next step is to model our integral $\int_0^z p(x)dx$ as a function, say $P(z)$ so that we can use root-finding method for the equation $0.5 + P(z) - \alpha = 0$.

1.2 VaR estimates using Gaussian distribution

We first divide the interval $[0, z]$ into five panels with 6 points with 5 panels: $\{0, z/5, 2z/5, 3z/5, 4z/5, 5z/5, z\}$. Using equation (4), the integral $\int_0^z \frac{1}{\sqrt{1\pi}} e^{-\frac{x^2}{6}} dx = \int_0^z p_1(x)dx$ is:

$$\begin{aligned}
\int_{0}^{z} p_1(x)dx &= \frac{h}{3}p_1(0) + \frac{4h}{3}p_1\left(\frac{z}{5}\right) + \frac{2h}{3}p_1\left(\frac{2z}{5}\right) \\
&= \frac{4h}{3}p_1\left(\frac{3z}{5}\right) + \frac{2h}{3}p_1\left(\frac{4z}{5}\right) + \frac{h}{6}p_1(z) \\
&= P_1(z) \\
h &= \frac{z}{5}
\end{aligned}$$

And now we try to estimate the VaR with $\alpha = 0.8$. Then we use Regula-Falsi method for the equation $0.5 + P_1(z) - 0.8 = P_1(z) - 0.3 = 0 = Q_1(z)$, with $z \in [0, 1]$ and $tol = 10^{-9}$. The file `PS2_1.sce` implements the root-finding method, the root converges to 1.567 692 7.

1.3 VaR estimates using t distribution

Using equation (4), the integral $\int_0^z \frac{1}{\gamma\sqrt{3\pi}}\left(1 + \frac{x^2}{3}\right)^{-2} = \int_0^z p_2(x)dx$ is:

$$\begin{aligned}
\int_{0}^{z} p_2(x)dx &= \frac{h}{3}p_2(0) + \frac{4h}{3}p_2\left(\frac{z}{5}\right) + \frac{2h}{3}p_2\left(\frac{2z}{5}\right) \\
&= \frac{4h}{3}p_2\left(\frac{3z}{5}\right) + \frac{2h}{3}p_2\left(\frac{4z}{5}\right) + \frac{h}{6}p_2(z) \\
&= P_2(z) \\
h &= \frac{z}{5}
\end{aligned}$$

And now we try to estimate the VaR with $\alpha = 0.8$. Then we use Regula-Falsi method for the equation $0.5 + P_2(z) - 0.8 = P_2(z) - 0.3 = 0 = Q_2(z)$, with $z \in [0, 1]$ and $tol = 10^{-9}$. The root converges to 1. 053 385 1. We estimate the VaR with $\alpha \in [0.8, 0.99]$ at intervals of 0.01 for the two models. Results are displayed in a tabular and graphical form when the `PS2_1.sce` is run.

2 Naive Fourier Series Approximation

3 Halley's Comet

4 Yeast Growth Modelling

4.1 Preliminaries

Recall the Runge-Kutta order 4 method for numerical solution of first-order differential equation. Given a differential equation

$$\frac{dy}{dt} = f(t, y)$$

the numerical solution of the differential equation with RK4 is

$$\begin{aligned}
x_1 &= f(t_n, y_n) \\
x_2 &= f\left(t_n + \frac{\Delta t}{2}, y_n + x_1\frac{\Delta t}{2}\right) \\
x_3 &= f\left(t_n + \frac{\Delta t}{2}, y_n + x_2\frac{\Delta t}{2}\right) \\
x_4 &= f(t_n + \Delta t, y_n + x_3\Delta t) \\
x &= \frac{x_1 + 2x_2 + 2x_3 + x_4}{6} \\
y_{n+1} &= y_n + x\Delta t
\end{aligned} \tag{5}$$

Where f is the differential equation, y is the dependent variable, t is the dependent variable, x is the approximated slope and Δt is the increment for the independent variable.
Also recall the Gauss-Newton optimization for nonlinear curve fitting. The optimum parameter vector $\mathbf{P} = [p_1\ p_2\ ...p_n]$ for a parameterized function $f_p(x)$ to fit with n points (x_n, y_n) can be obtained using the iteration

$$\begin{aligned}
\mathbf{P_{new}} &= \mathbf{P_{old}} - \delta \\
\delta &= \mathrm{GE}\left(\mathbf{J_r^T J_r}, \mathbf{J_r^T r}\right) \\
\mathbf{r} &= \begin{bmatrix} f_p(x_1) - y_i \\ \vdots \\ f_p(x_n) - y_n \end{bmatrix}
\end{aligned} \tag{6}$$

where $\mathbf{r}$ is the residual matrix and $\mathbf{J_r}$ is the Jacobian of the residual matrix.

4.2 Logistic Model

The logistic growth model of the yeast is given by

$$\frac{dP}{dt} = kP(C-P) = f(t,P)$$

where P is the current population. Variables k and C are the parameters of the differential equation. Now, we obtain the solution of this model using RK4 with $\Delta t = 1$ using equation (5).

$$\begin{aligned}
x_1 &= kP_n(C-P_n)\\
x_2 &= k\Big(P_n+\frac{1}{2}kP_n(C-P_n)\Big)\Big(C-\Big(P_n+\frac{1}{2}kP_n(C-P_n)\Big)\Big)\\
x_3 &= k\Big(C-P_n-\frac{1}{2}k\Big(C-P_n-\frac{1}{2}kP_n(C-P_n)\Big)\Big(P_n+\frac{1}{2}kP_n(C-P_n)\Big)\Big)\Big(P_n+\frac{1}{2}k\Big(C-P_n-\frac{1}{2}kP_n(C-P_n)\Big)\Big(P_n+\frac{1}{2}kP_n(C-P_n)\Big)\Big)\\
x_4 &= k\Big(C-P_n-k\Big(C-P_n-\frac{1}{2}k\Big(C-P_n-\frac{1}{2}kP_n(C-P_n)\Big)\Big(P_n+\frac{1}{2}kP_n(C-P_n)\Big)\Big)\Big(P_n+\frac{1}{2}k\Big(C-P_n-\frac{1}{2}kP_n(C-P_n)\Big(P_n+\frac{1}{2}kP_n(C-P_n)\Big)\Big)\Big)\Big)\\
&\quad\Big(P_n+k\Big(C-P_n-\frac{1}{2}k\Big(C-P_n-\frac{1}{2}kP_n(C-P_n)\Big)\Big(P_n+\frac{1}{2}kP_n(C-P_n)\Big)\Big)\Big(P_n+\frac{1}{2}k\Big(C-P_n-\frac{1}{2}kP_n(C-P_n)\Big(P_n+\frac{1}{2}kP_n(C-P_n)\Big)\Big)\Big)\Big)\\
x &= \frac{x_1+2x_2+2x_3+x_4}{6}
\end{aligned}$$

$$\begin{aligned}
P_{n+1} = P_n &+ \frac{k}{6}P_n(C-P_n)+\frac{k}{6}\Big(P_n+\frac{1}{2}kP_n(C-P_n)\Big)\Big(C-\Big(P_n+\frac{1}{2}kP_n(C-P_n)\Big)\Big)\\
&+\frac{k}{6}\Big(C-P_n-\frac{1}{2}k\Big(C-P_n-\frac{1}{2}kP_n(C-P_n)\Big)\Big(P_n+\frac{1}{2}kP_n(C-P_n)\Big)\Big)\Big(P_n+\frac{1}{2}k\Big(C-P_n-\frac{1}{2}kP_n(C-P_n)\Big)\Big(P_n+\frac{1}{2}kP_n(C-P_n)\Big)\Big)\\
&+\frac{k}{6}\Big(C-P_n-k\Big(C-P_n-\frac{1}{2}k\Big(C-P_n-\frac{1}{2}kP_n(C-P_n)\Big)\Big(P_n+\frac{1}{2}kP_n(C-P_n)\Big)\Big)\Big(P_n+\frac{1}{2}k\Big(C-P_n-\frac{1}{2}kP_n(C-P_n)\Big(P_n+\frac{1}{2}kP_n(C-P_n)\Big)\Big)\\
&\quad\Big(P_n+k\Big(C-P_n-\frac{1}{2}k\Big(C-P_n-\frac{1}{2}kP_n(C-P_n)\Big)\Big(P_n+\frac{1}{2}kP_n(C-P_n)\Big)\Big)\Big(P_n+\frac{1}{2}k\Big(C-P_n-\frac{1}{2}kP_n(C-P_n)\Big(P_n+\frac{1}{2}kP_n(C-P_n)\Big)\Big)\Big)\Big)
\end{aligned}$$

And this is the function we want to optimize, given different values of P_n and P_{n+1}.