

CS 131 Problem Set 2

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1 Value-at-Risk

1.1 Preliminaries

The value at risk VaR of a continuous loss distribution modeled by a probability distribution function, say $p(x)$ at a given risk level $(1 - \alpha) \in [0, 1]$ is defined as the value at which the cumulative probability of $p(x)$ from $-\infty$ is equal to α . Mathematically speaking, we say:

$$VaR_\alpha = z \Leftrightarrow \int_{-\infty}^z p(x)dx = \alpha$$

A value z is the value-at-risk at α if and only if the area under the curve of $p(x)$ in the interval $(-\infty, z]$ is equal to α . We first observe the properties of the probability functions

$$p_1(x) = \frac{1}{\sqrt{6\pi}} e^{-\frac{x^2}{6}}$$

and

$$p_2(x) = \frac{1}{\gamma\sqrt{3\pi}} \left(1 + \frac{x^2}{3}\right)^{-2}, \gamma \approx 0.886226925453$$

where $p_1(x)$ is a Gaussian distribution function and $p_2(x)$ is a t distribution function.

- We know that these functions are symmetric to some line $x = \mu$, where μ is the mean of the distribution. When $\mu = 0$, its graph is symmetric to the line $x = 0$ or y-axis. It has the property

$$\int_m^n p(x)dx = \int_{-n}^{-m} p(x)dx$$

for $m \leq n$.

- The functions above are probability distribution functions. It follows that the sum of the probabilities in the sample space is equal to 1. Or equivalently,

$$\sum_{\forall x} p(x) = \int_{-\infty}^{\infty} p(x)dx = 1 \quad (1)$$

- We redefine the limits of integration (i.e.) split equation (2) into two as follows:

$$\int_{-\infty}^{\infty} p(x)dx = \int_{-\infty}^0 p(x)dx + \int_0^{\infty} p(x)dx \quad (2)$$

- Since the graph of the function is symmetric to y-axis, the area under the curve in the interval $[0, \infty)$ is also equal to the area under the curve in the interval $(-\infty, 0)$. Using equation (1) and (2),

$$\int_{-\infty}^0 p(x)dx = \int_0^{\infty} p(x)dx = 0.5 \quad (3)$$

Now, we verify if $\int_0^{\infty} p(x)dx = 0.5$ using Gauss-Laguerre quadrature. For $p_1(x)$, the Gauss-Laguerre quadrature form of the integral is

$$I_1 = \int_0^{\infty} e^{-x} \left(\frac{e^x}{\sqrt{6\pi}} e^{-\frac{x^2}{6}} \right) dx = \int_0^{\infty} e^{-x} \left(\frac{1}{\sqrt{6\pi}} e^{x - \frac{x^2}{6}} \right) dx$$

where the weighting function $w(x) = e^{-x}$ and $g_1(x) = \frac{1}{\sqrt{6\pi}} e^{x - \frac{x^2}{6}}$. By Gaussian quadratures approximation,

$$I_1 \approx \sum_{i=1}^n A_i g_1(x_i)$$

where i is the number of nodes for approximation. We use $n = 6$ nodes. The nodal abscissas and corresponding weights for $n = 6$ is given below.

i	x_i	A_i
1	0.222 847	0.458 964
2	1.188 932	0.417 000
3	2.992 736	0.113 373
4	5.775 144	0.010 399 2
5	9.837 467	0.000 261 017
6	15.982 874	0.000 000 898 548

To solve the integral,

$$\begin{aligned} I_1 &= A_1 g_1(x_1) + A_2 g_1(x_2) + A_3 g_1(x_3) \\ &\quad + A_4 g_1(x_4) + A_5 g_1(x_5) + A_6 g_1(x_6) \\ I_1 &= 0.458964 \cdot 0.285454 + 0.417000 \cdot 0.597557 \\ &\quad + 0.113373 \cdot 1.03226 + 0.0103992 \cdot 0.285985 \\ &\quad + 0.000261017 \cdot 0.000426427 \\ &\quad + 8.98548 \times 10^{-7} \cdot 6.50687 \times 10^{-13} \\ I_1 &= 0.500198 \approx 0.5 \end{aligned}$$

For $p_2(x)$, the Gauss-Laguerre quadrature form of the integral is

$$I_2 = \int_0^{\infty} e^{-x} \left(\frac{e^x}{\gamma\sqrt{3\pi}} \left(1 + \frac{x^2}{3}\right)^{-2} \right) dx, \gamma \approx 0.886226925453$$

where the weighting function $w(x) = e^{-x}$ and $g_2(x) = \frac{e^x}{\gamma\sqrt{3\pi}} \left(1 + \frac{x^2}{3}\right)^{-2}$. By Gaussian quadratures approximation,

$$I_2 \approx \sum_{i=1}^n A_i g_2(x_i)$$

We used again $n = 6$ nodes, and the nodal abscissas and weights above.

To solve the integral,

$$\begin{aligned} I_2 &= A_1 g_2(x_1) + A_2 g_2(x_2) + A_3 g_2(x_3) \\ &\quad + A_4 g_2(x_4) + A_5 g_2(x_5) + A_6 g_2(x_6) \\ I_2 &= 0.458964 \cdot 0.444468 + 0.417000 \cdot 0.55761 \\ &\quad + 0.113373 \cdot 0.461408 + 0.0103992 \cdot 0.806524 \\ &\quad + 0.000261017 \cdot 6.22113 \\ &\quad + 8.98548 \times 10^{-7} \cdot 432.589 \\ I_2 &= 0.499229 \approx 0.5 \end{aligned}$$

And now we established equation (3).

Going back to our original problem, take note that we are only estimating the value-at-risks for $\alpha \geq 0.5$, so we know that $z \geq 0$. Therefore we should use the relationship that

$$\begin{aligned} \int_{-\infty}^z p(x)dx &= \int_{-\infty}^0 p(x)dx + \int_0^z p(x)dx \\ &= 0.5 + \int_0^z p(x)dx \end{aligned}$$

So it only remains to compute for $\int_0^z p(x)dx$. We use composite Simpson's 1/3 rule with 6 points also.

The general integral $\int_a^b f(x)dx$ can be numerically solved using the composite Simpson's 1/3 rule:

$$\begin{aligned} \int_a^b f(x)dx &= \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} \sum_{i \text{ even}} f(x_i) \\ &\quad + \frac{b-a}{3} \sum_{i \text{ odd}} f(x_i) + \frac{b-a}{6} f(b) \end{aligned} \quad (4)$$

The next step is to model our integral $\int_0^z p(x)dx$ as a function, say $P(z)$ so that we can use root-finding method for the equation $0.5 + P(z) - \alpha = 0$.

1.2 VaR estimates using Gaussian distribution

We first divide the interval $[0, z]$ into five panels with 6 points: $\{0, z/5, 2z/5, 3z/5, 4z/6, z\}$. Then the integral $\int_0^z \frac{1}{\sqrt{1\pi}} e^{-\frac{x^2}{6}} dx = \int_0^z p_1(x)dx$ is:

$$\begin{aligned} \int_0^z p_1(x)dx &= \frac{z}{6} p_1(0) + \frac{2z}{3} p_1\left(\frac{z}{5}\right) + \frac{z}{3} p_1\left(\frac{2z}{5}\right) \\ &= \frac{2z}{3} p_1\left(\frac{3z}{5}\right) + \frac{z}{3} p_1\left(\frac{4z}{5}\right) + \frac{z}{6} p_1(z) \\ &= P_1(z) \end{aligned} \quad (5)$$

And now we try to estimate the VaR with $\alpha = 0.8$. Then we use Regula-Falsi method for the equation $0.5 + P_1(z) - 0.8 = P_1(z) - 0.3 = 0 = Q_1(z)$, with $z \in [0, 1]$ and $tol = 10^{-7}$. The file `PS2.1.sce` implements the root-finding method, the root converges to 0.567 192 5.

1.3 VaR estimates using t distribution

The integral $\int_0^z \frac{1}{\gamma\sqrt{3\pi}} \left(1 + \frac{x^2}{3}\right)^{-2} = \int_0^z p_2(x)dx$ is:

$$\begin{aligned} \int_0^z p_2(x)dx &= \frac{z}{6} p_2(0) + \frac{2z}{3} p_2\left(\frac{z}{5}\right) + \frac{z}{3} p_2\left(\frac{2z}{5}\right) \\ &= \frac{2z}{3} p_2\left(\frac{3z}{5}\right) + \frac{z}{3} p_2\left(\frac{4z}{5}\right) + \frac{z}{6} p_2(z) \\ &= P_2(z) \end{aligned} \quad (6)$$

And now we try to estimate the VaR with $\alpha = 0.8$. Then we use Regula-Falsi method for the equation $0.5 + P_2(z) - 0.8 = P_2(z) - 0.3 = 0 = Q_2(z)$, with $z \in [0, 1]$ and $tol = 10^{-7}$. The root converges to 0.358 710 1. We estimate the VaR with $\alpha \in [0.8, 0.99]$ at intervals of 0.1 for the two models. Results are displayed in a tabular and graphical form when the `PS2.1.sce` is run.

2 Naive Fourier Series Approximation

3 Halley's Comet

4 Yeast Growth Modelling