بسم الله الرحمن الرحيم

نظریه علوم کامپیوتر

نظریه علوم کامپیوتر - بهار ۱۴۰۰-۱۴۰۱ - جلسه سیزدهم: پیچیدگی حافظه (۲)

Theory of computation - 002 - S13 - space complexity (2)

Review: SPACE Complexity

Defn: Let $f\colon \mathbb{N} \rightarrow \mathbb{N}$ where $f(n) \geq n$. Say TM M runs in space $f(n)$ if M always halts and uses at most $f(n)$ tape cells on all inputs of length n.

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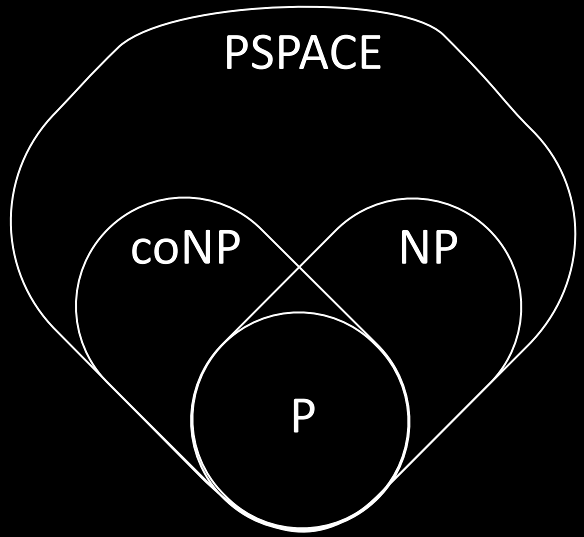
$\text{PSPACE} = \bigcup_k \text{SPACE}(n^k)$ "polynomial space"

$\text{NPSPACE} = \bigcup_k \text{NSPACE}(n^k)$ "nondeterministic polynomial space"



Or possibly: P = NP = coNP = PSPACE

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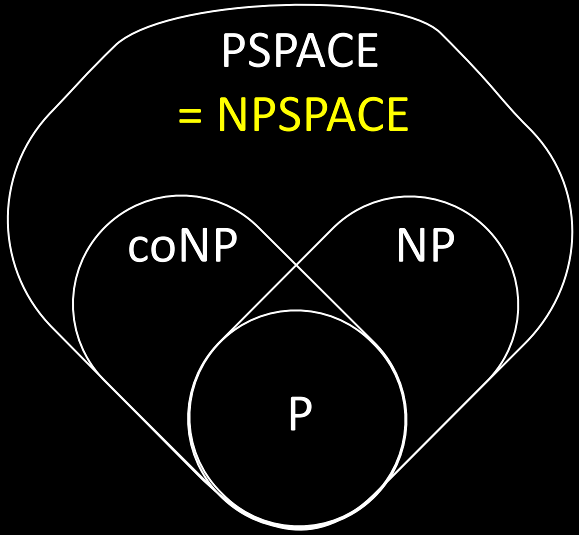

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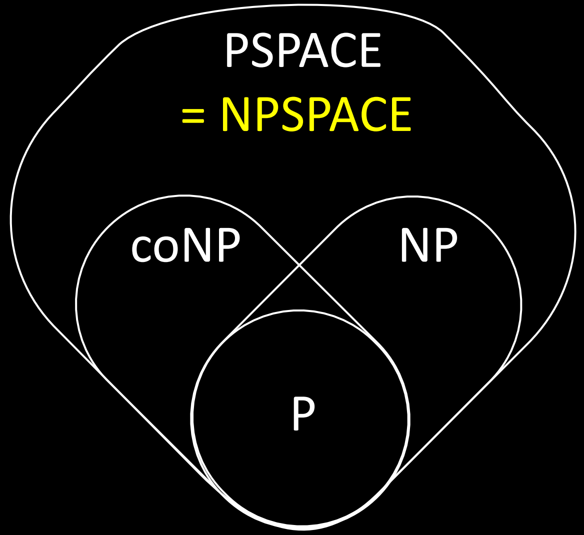

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Example: Ladder Problem

A <u>ladder</u> is a sequence of strings of a common length where consecutive strings differ in a single symbol.

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Defn: $LADDER_{\mathbf{DFA}} = \{\langle B, u, v\rangle \mid B$ is a DFA and $L(B)$ contains a ladder $y_1,\ y_2,\ \ldots,\ y_k$ where $y_1 = u$ and $y_k = v\}$.

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Theorem: $LADDER$**DFA** $\in$ NPSPACE

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$LADDER_{\text{DFA}} \in \text{NPSPACE}$

Theorem: $LADDER_{\text{DFA}} \in \text{NPSPACE}$

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$\leq t$

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Careful- (a) cannot store sequence, (b) must terminate.

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4. *Reject* if $y \notin L(B)$.

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Space used is for storing y and t.

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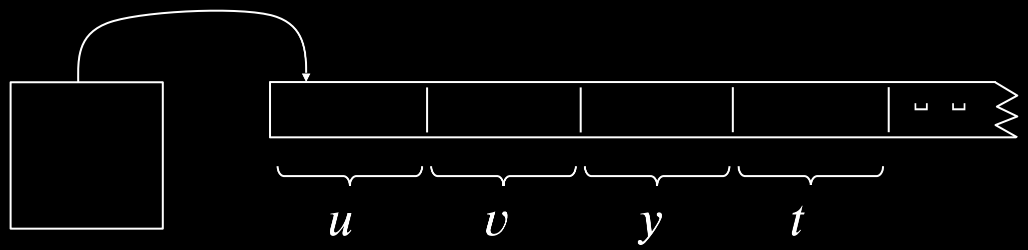

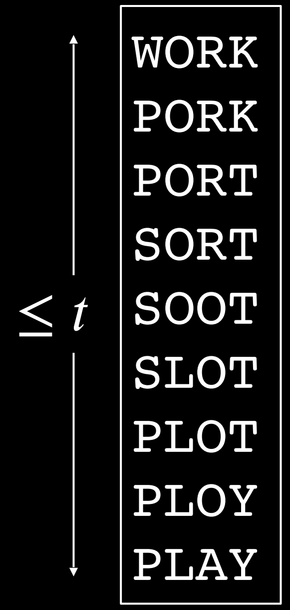

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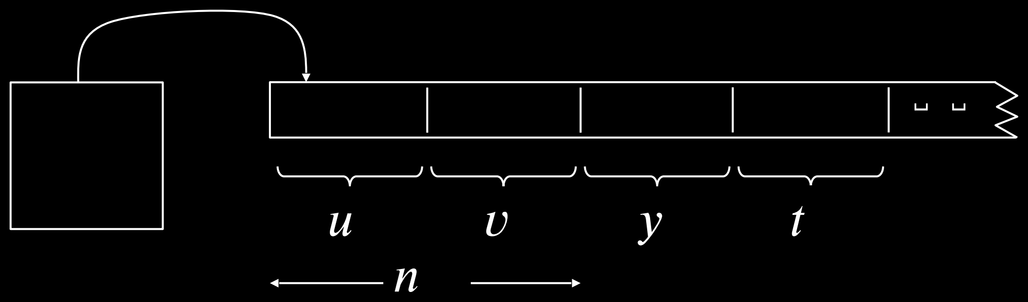

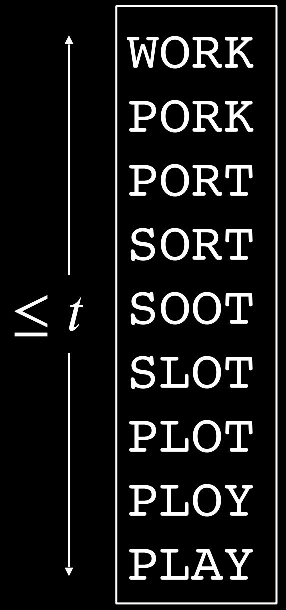

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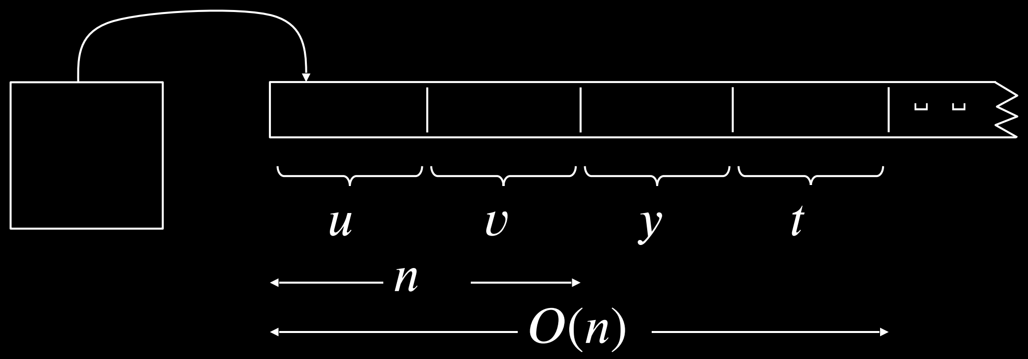

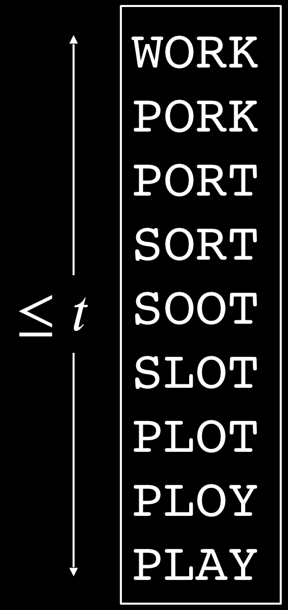

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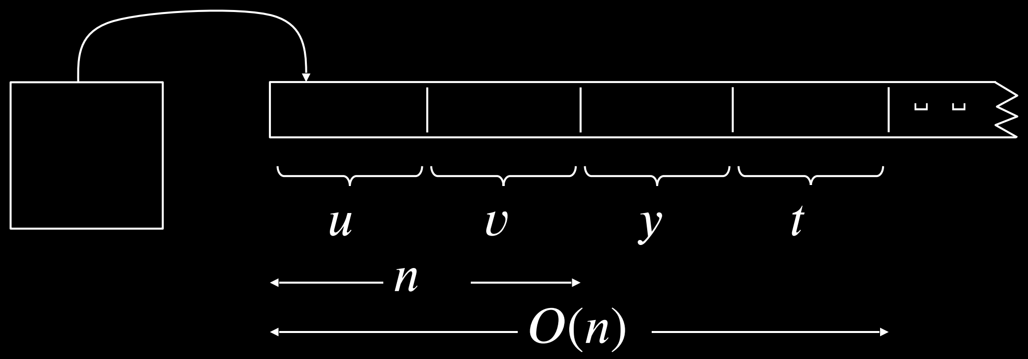

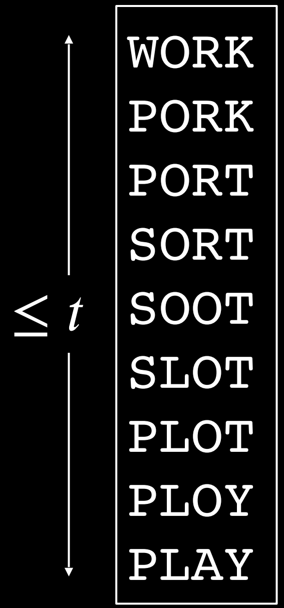

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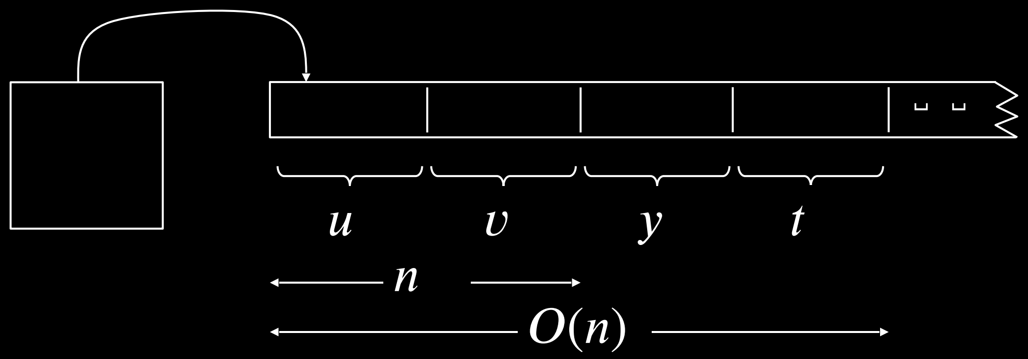

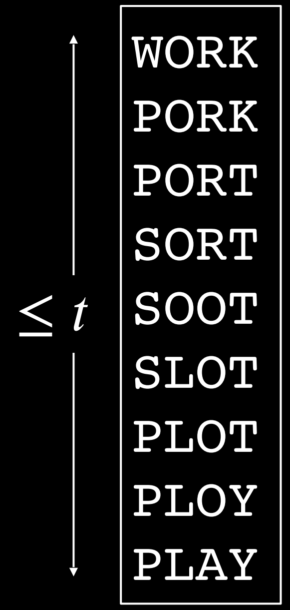

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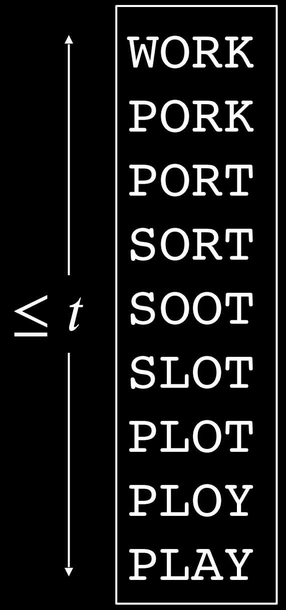

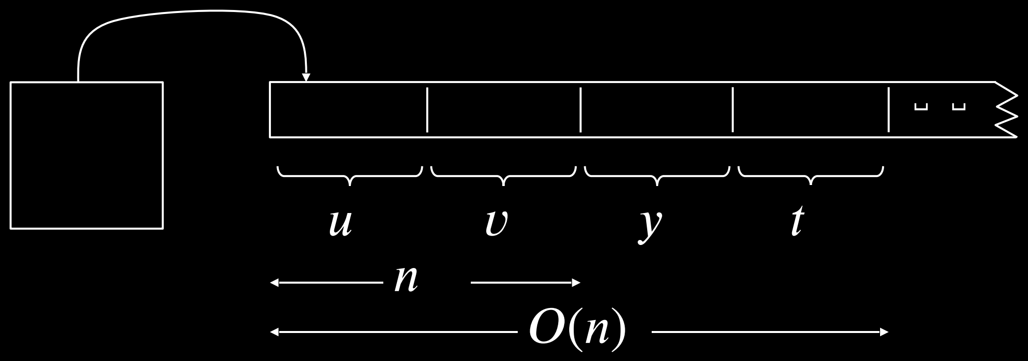

Space used is for storing y and t.

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Theorem: $LADDER$DFA $\in$ PSPACE (!)

$\mathit{LADDER}_{\mathrm{DFA}} \in \mathrm{PSPACE}$

$LADDER_{\text{DFA}} \in$ PSPACE

Theorem: $LADDER_{\text{DFA}} \in$ SPACE(n^2)

$LADDER$DFA $\in$ PSPACE

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Proof: Write $u \xrightarrow{b} v$ if there's a ladder from u to v of length $\leq b$.

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Proof: Write $u \xrightarrow{b} v$ if there's a ladder from u to v of length $\leq b$.
Here's a recursive procedure to solve the bounded DFA ladder problem:

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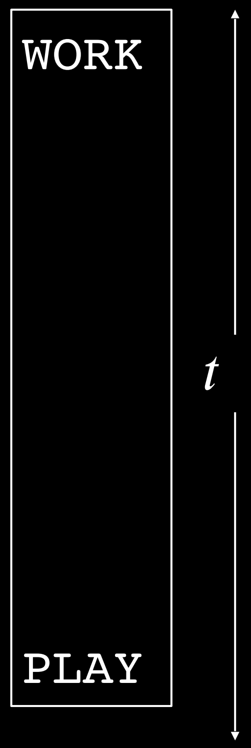
$BOUNDED\text{-}LADDER$DFA $= \left\{\langle B, u, v, b\rangle \middle|\ B \text{ a DFA and } u \xrightarrow{b} v \text{ by a ladder in } L(B)\right\}$



$LADDER$DFA $\in$ PSPACE

Theorem: $LADDER$DFA $\in$ SPACE(n^2)

Proof: Write $u \xrightarrow{b} v$ if there's a ladder from u to v of length $\leq b$.
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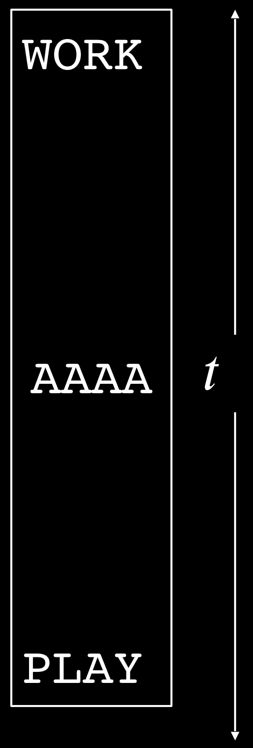

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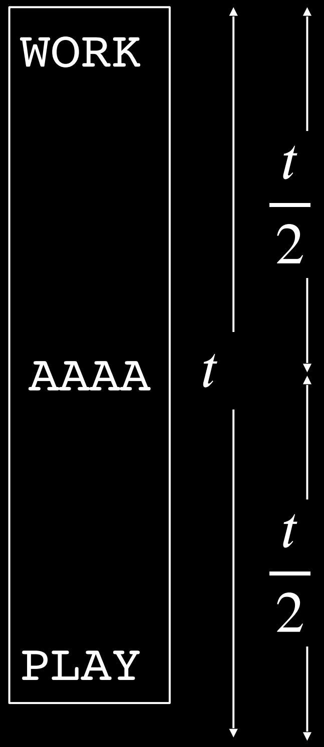

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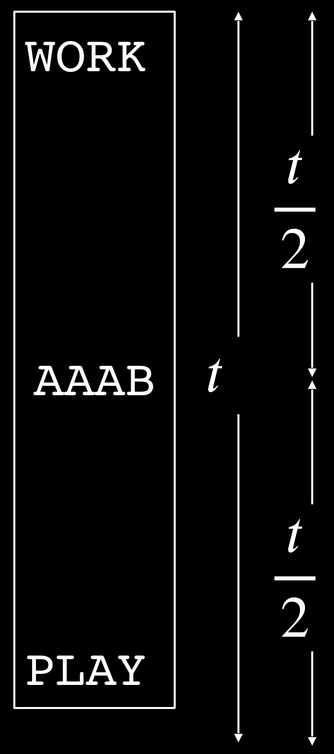

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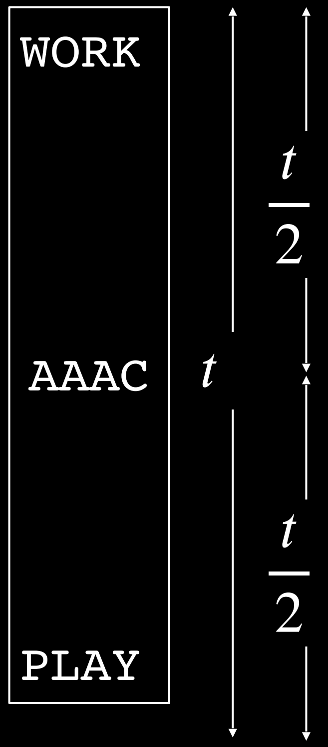

$LADDER_{\text{DFA}} \in$ PSPACE

Theorem: $LADDER_{\text{DFA}} \in$ SPACE(n^2)

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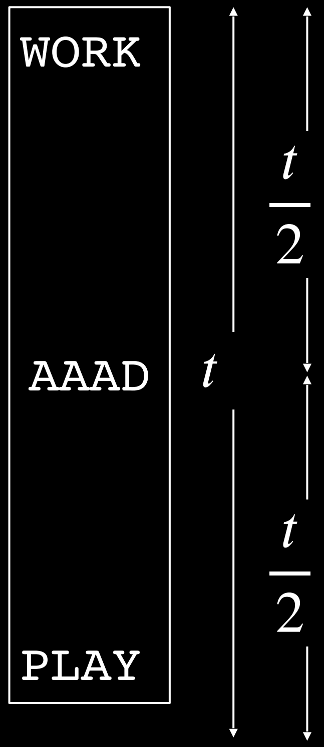

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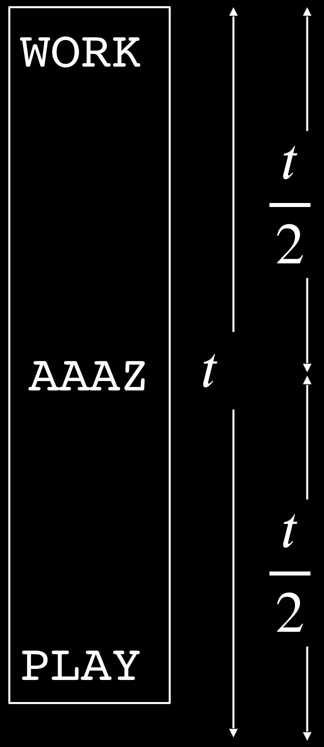

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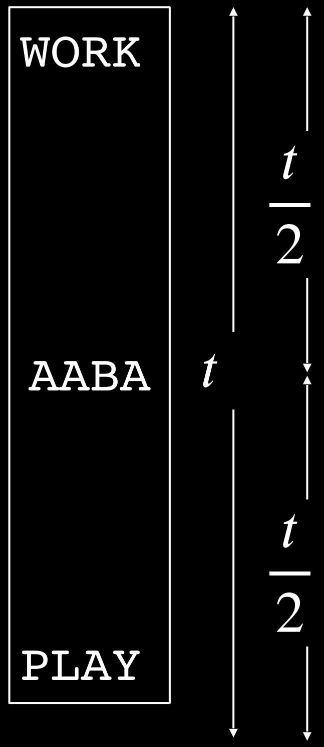

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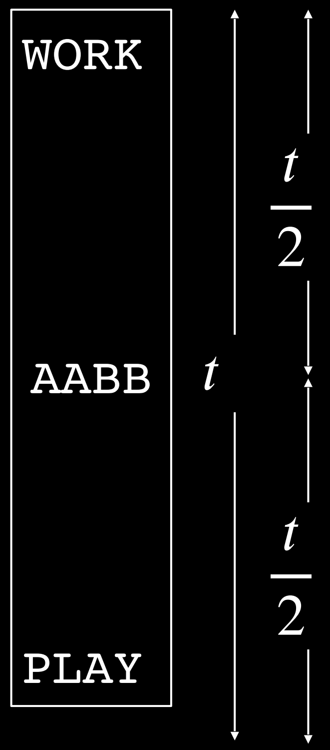

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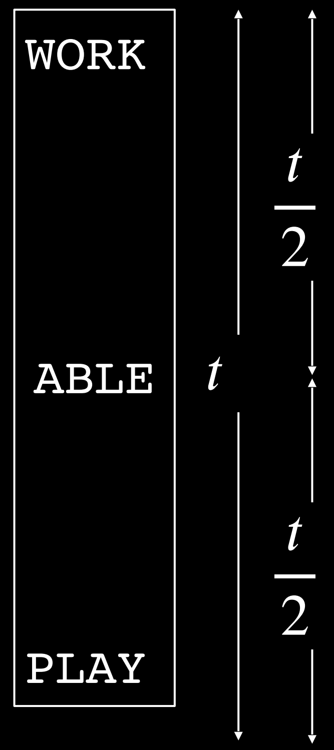

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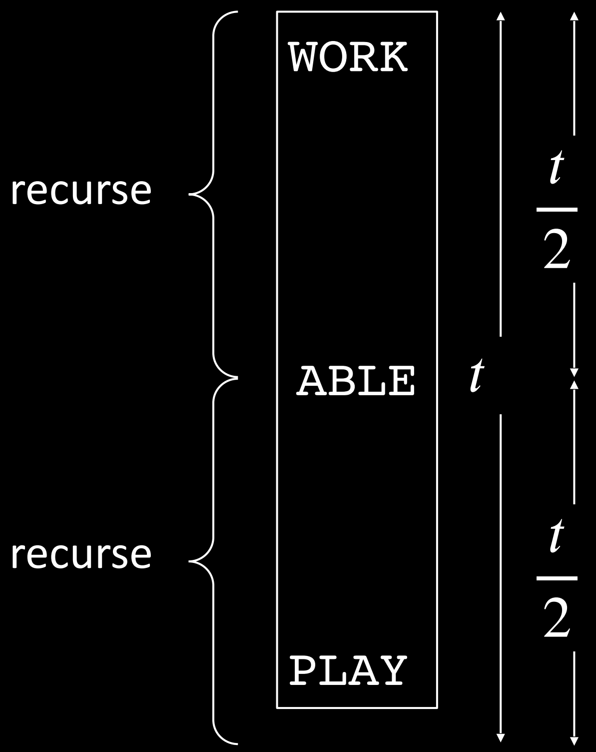

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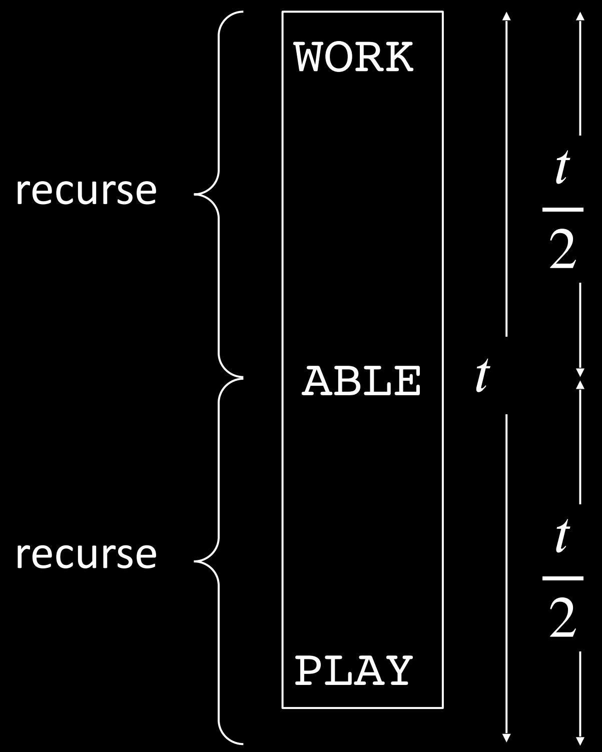

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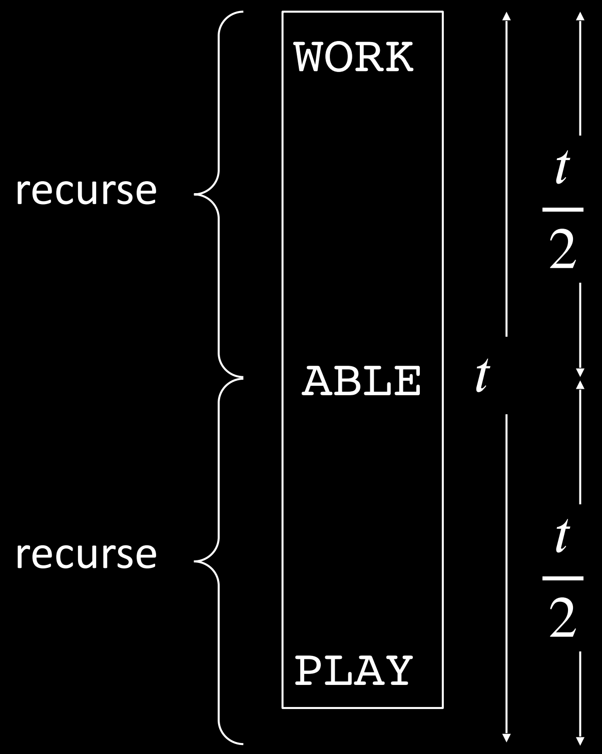

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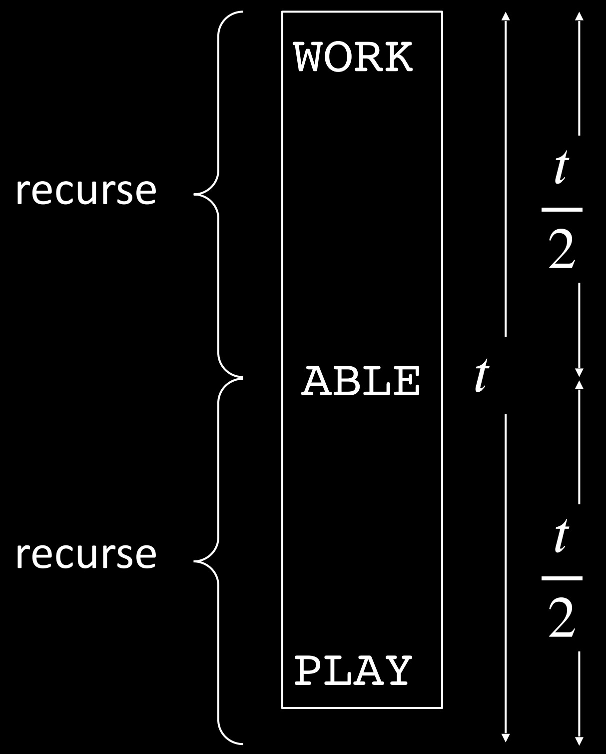

$LADDER_{\text{DFA}} \in$ PSPACE

Theorem: $LADDER_{\text{DFA}} \in$ SPACE(n^2)

Proof: Write $u \xrightarrow{b} v$ if there's a ladder from u to v of length $\leq b$.
Here's a recursive procedure to solve the bounded DFA ladder problem:

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1. For $b = 1$, *accept* if $u, v \in L(B)$ and differ in ≤ 1 place, else *reject*.
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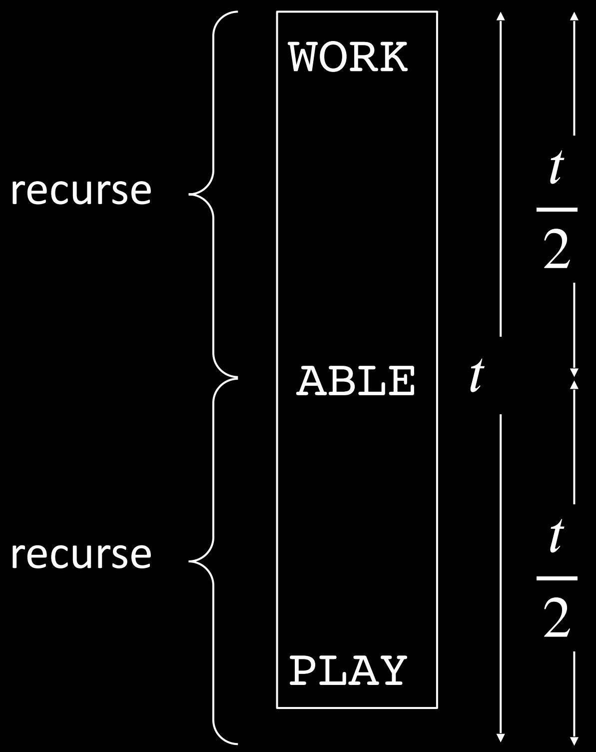

$LADDER$DFA $\in$ PSPACE

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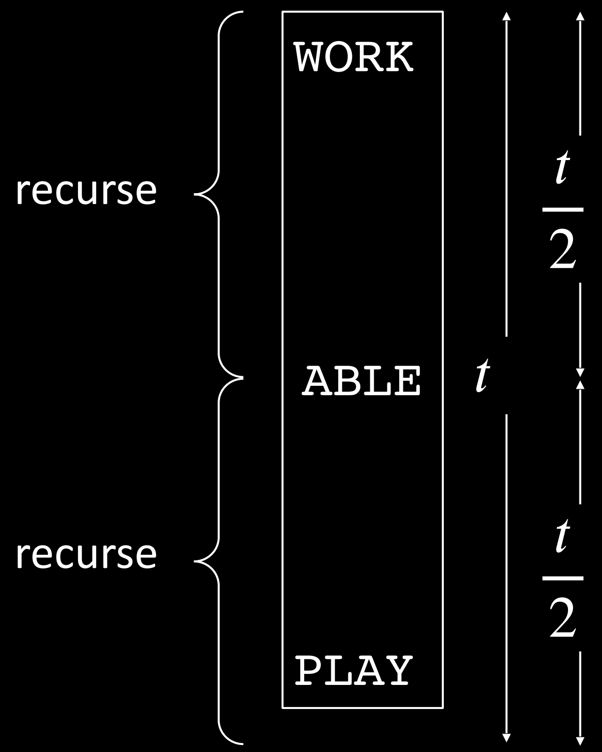

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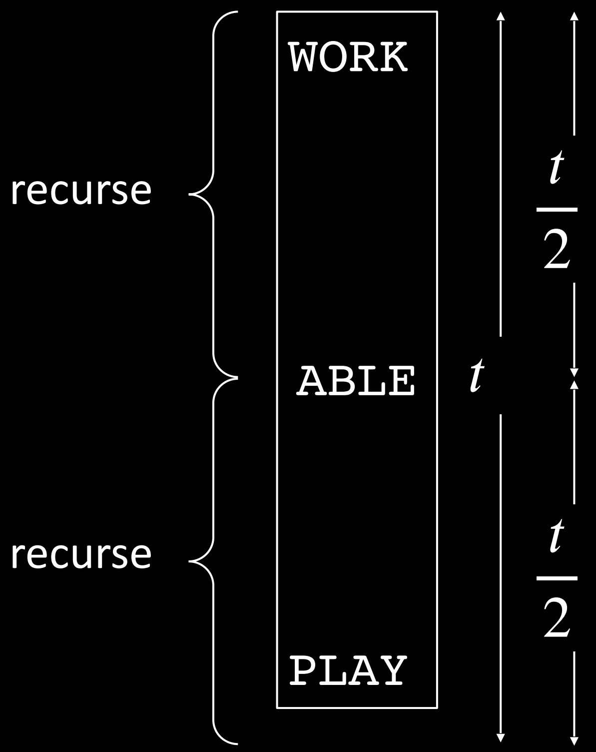

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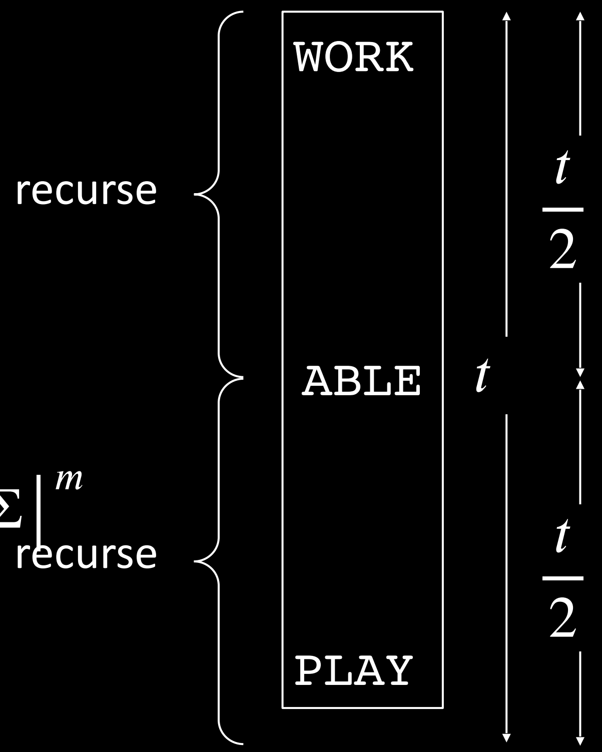
Test $\langle B, u, v \rangle \in LADDER$DFA with $B\text{-}L$ procedure on input $\langle B, u, v, t \rangle$ for $t = |\Sigma|^m$



LADDERDFA $\in$ PSPACE

Theorem: LADDERDFA $\in$ SPACE(n^2)

Proof: Write $u \xrightarrow{b} v$ if there's a ladder from u to v of length $\leq b$.
Here's a recursive procedure to solve the bounded DFA ladder problem:

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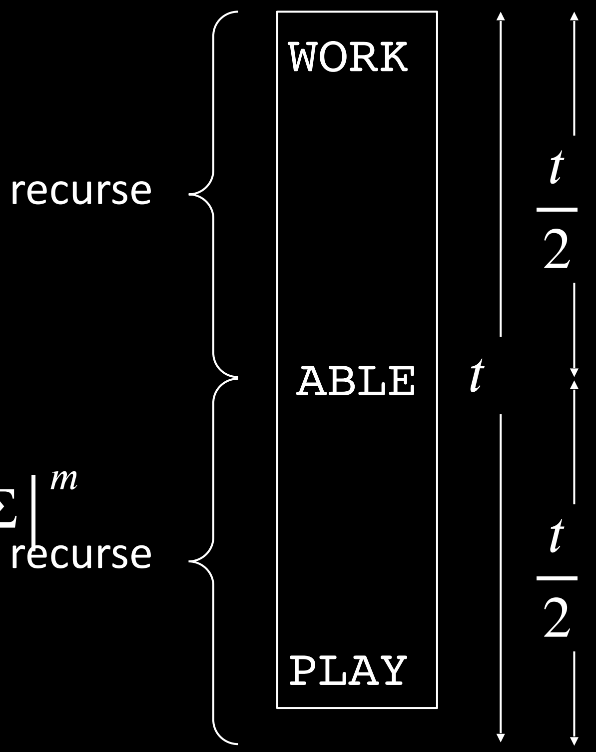
Test $\langle B, u, v \rangle \in \mathit{LADDER}$DFA with $B\text{-}L$ procedure on input $\langle B, u, v, t \rangle$ for $t = |\Sigma|^m$

Space analysis:



$LADDER_{\text{DFA}} \in$ PSPACE

Theorem: $LADDER_{\text{DFA}} \in$ SPACE(n^2)

Proof: Write $u \xrightarrow{b} v$ if there's a ladder from u to v of length $\leq b$.
Here's a recursive procedure to solve the bounded DFA ladder problem:

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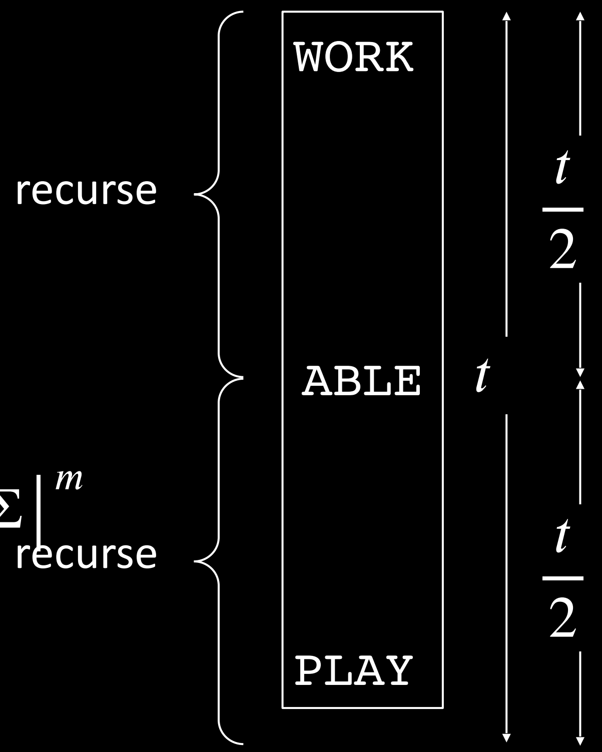

Test $\langle B, u, v\rangle \in LADDER_{\text{DFA}}$ with $B\text{-}L$ procedure on input $\langle B, u, v, t\rangle$ for $t = |\Sigma|^m$

Space analysis:

Each recursive level uses space $O(n)$ (to record w).

$LADDER$DFA $\in$ PSPACE

Theorem: $LADDER$DFA $\in$ SPACE(n^2)

Proof: Write $u \xrightarrow{b} v$ if there's a ladder from u to v of length $\leq b$.
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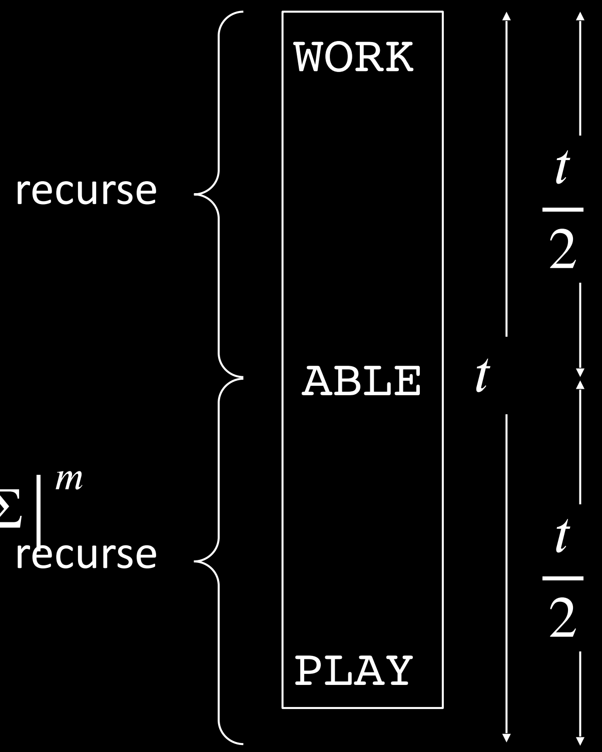

Space analysis:

Each recursive level uses space $O(n)$ (to record w).

Recursion depth is $\log t = O(m) = O(n)$.

$LADDER$DFA $\in$ PSPACE

Theorem: $LADDER$DFA $\in$ SPACE(n^2)

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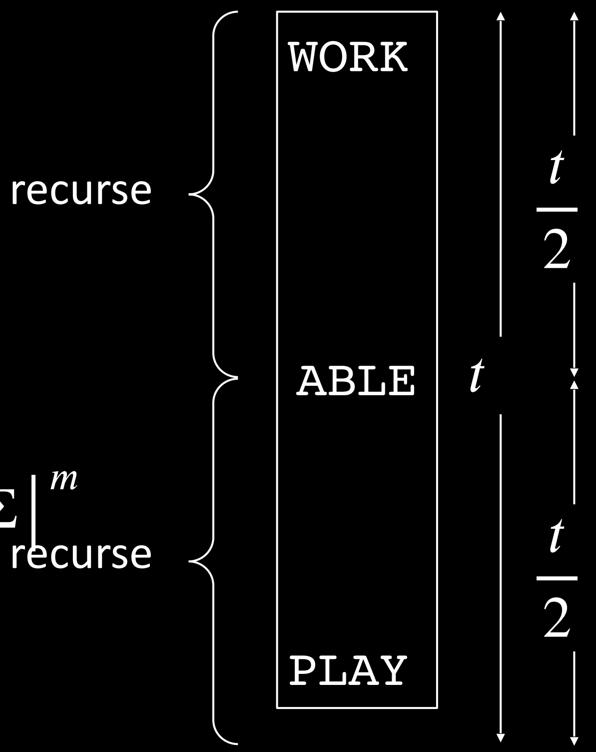

Test $\langle B, u, v\rangle \in LADDER$DFA with $B\text{-}L$ procedure on input $\langle B, u, v, t\rangle$ for $t = |\Sigma|^m$

Space analysis:

Each recursive level uses space $O(n)$ (to record w).

Recursion depth is $\log t = O(m) = O(n)$.

Total space used is $O(n^2)$.

$LADDER_{\text{DFA}} \in \text{PSPACE}$

Theorem: $LADDER_{\text{DFA}} \in \text{SPACE}(n^2)$

Proof: Write $u \xrightarrow{b} v$ if there's a ladder from u to v of length $\leq b$.
Here's a recursive procedure to solve the bounded DFA ladder problem:

$BOUNDED\text{-}LADDER_{\text{DFA}} = \left\{ \langle B, u, v, b \rangle \,\middle|\, B \text{ a DFA and } u \xrightarrow{b} v \text{ by a ladder in } L(B) \right\}$

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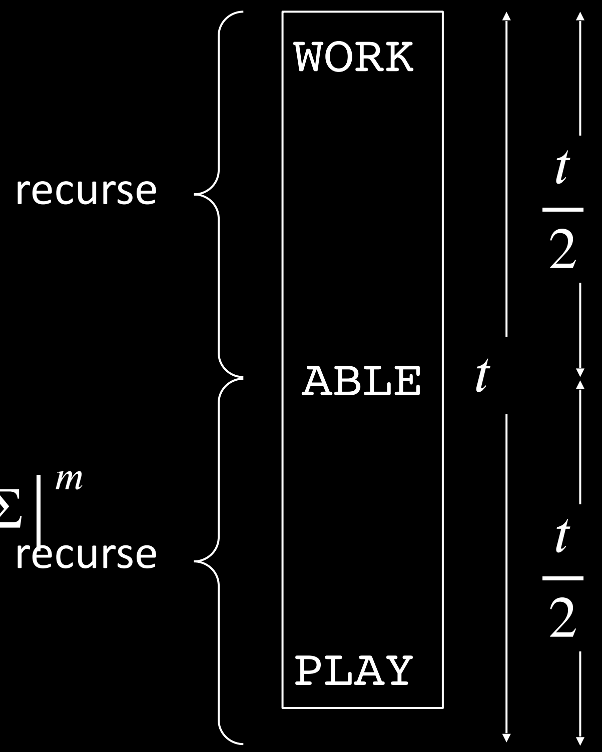

Test $\langle B, u, v \rangle \in LADDER_{\text{DFA}}$ with $B\text{-}L$ procedure on input $\langle B, u, v, t \rangle$ for $t = |\Sigma|^m$

Space analysis:

Each recursive level uses space $O(n)$ (to record w).

Recursion depth is $\log t = O(m) = O(n)$.

Total space used is $O(n^2)$.

Check-in 17.3

$\mathit{LADDER}_{\text{DFA}} \in$ PSPACE

Theorem: $\mathit{LADDER}_{\text{DFA}} \in$ SPACE(n^2)

Proof: Write $u \xrightarrow{b} v$ if there's a ladder from u to v of length $\leq b$.
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4. *Accept* both accept.
5. *Reject* [if all fail]."

Test $\langle B, u, v\rangle \in \mathit{LADDER}_{\text{DFA}}$ with $B\text{-}L$ procedure on input $\langle B, u, v, t\rangle$ for t

Space analysis:

Each recursive level uses space $O(n)$ (to record w).

Recursion depth is $\log t = O(m) = O(n)$.

Total space used is $O(n^2)$.

Check-in 17.3

Find an English word ladder connecting MUST and VOTE.

(a) Already did it.

(b) I will.

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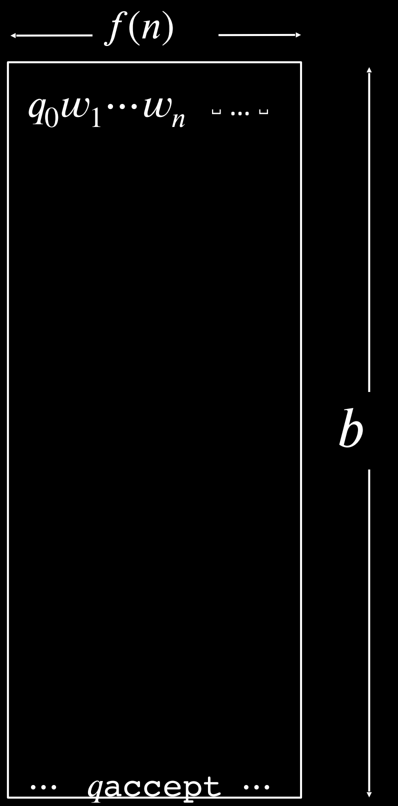

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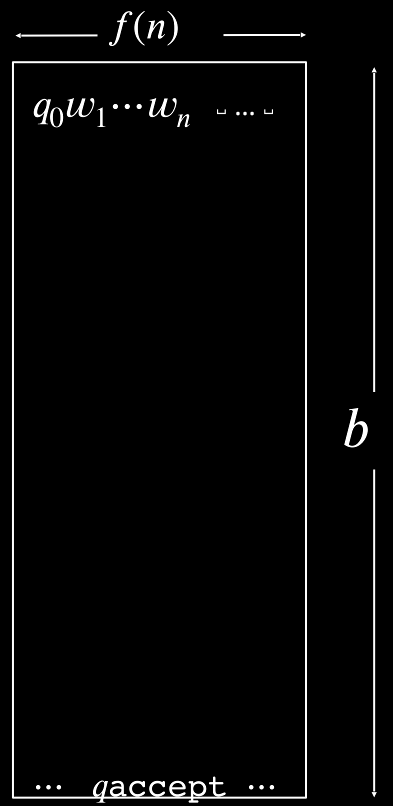

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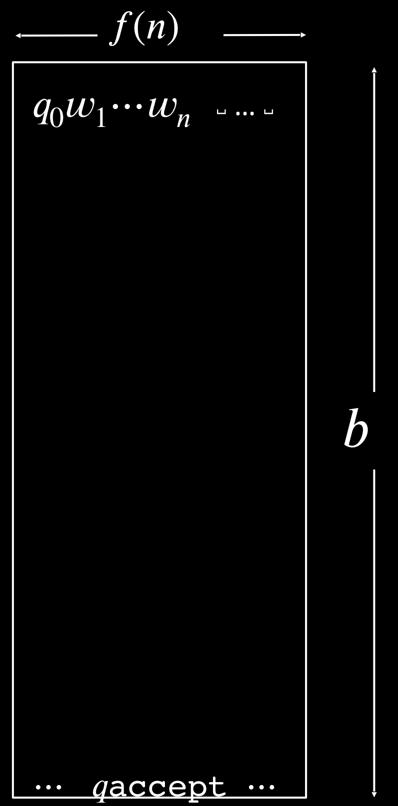

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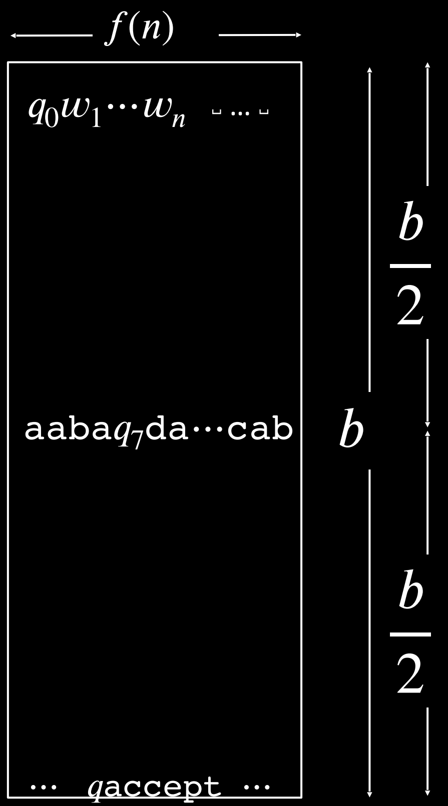

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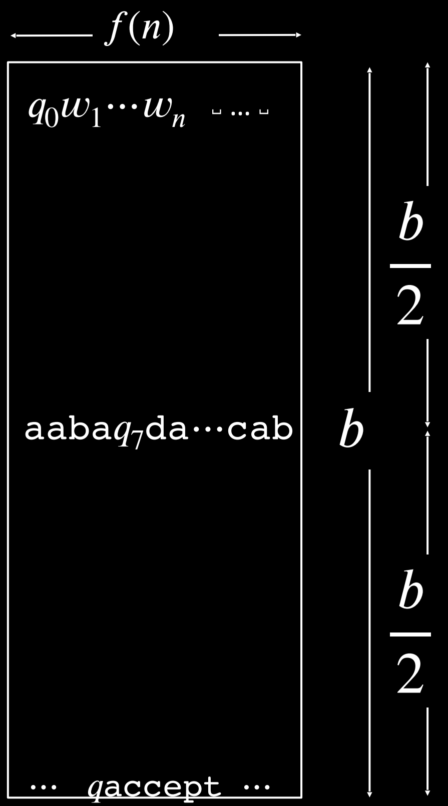

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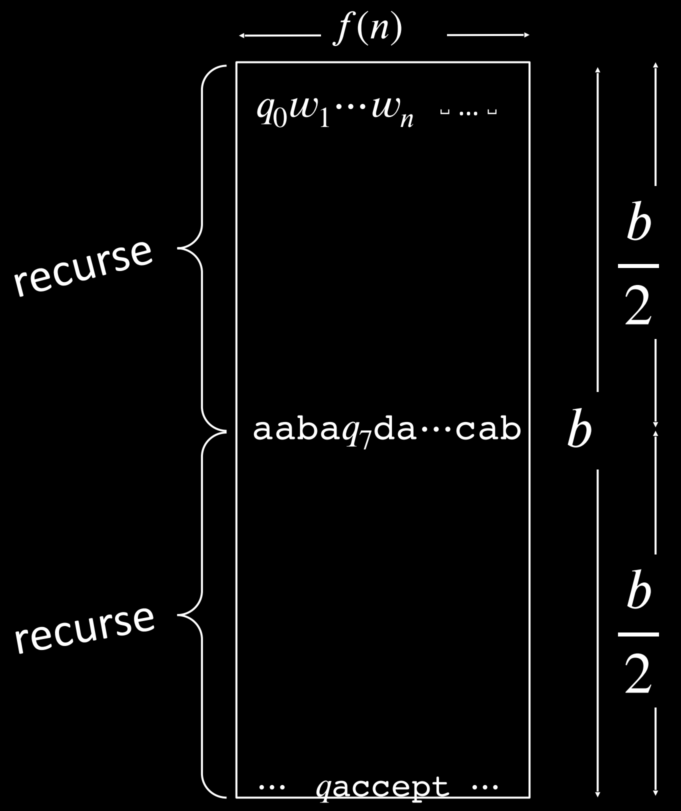

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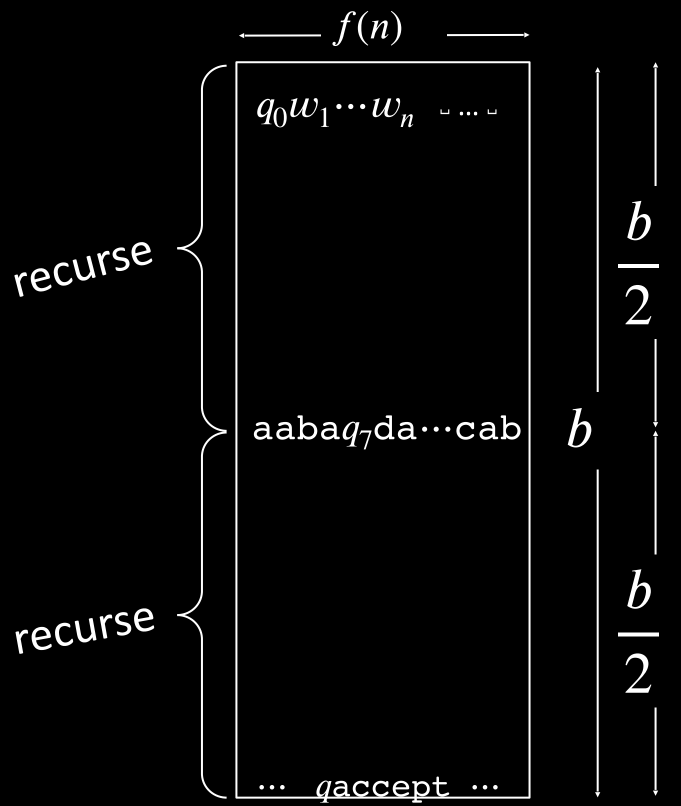

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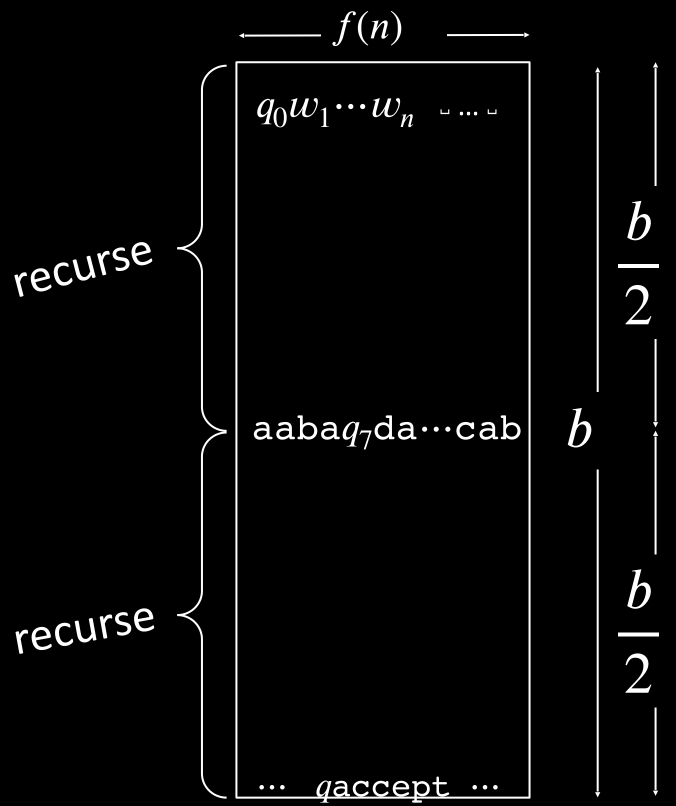

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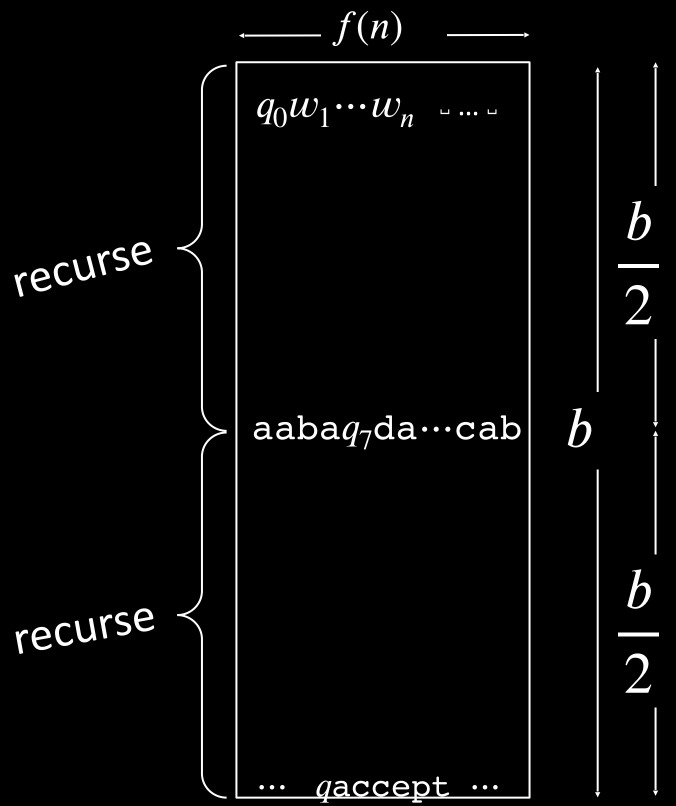

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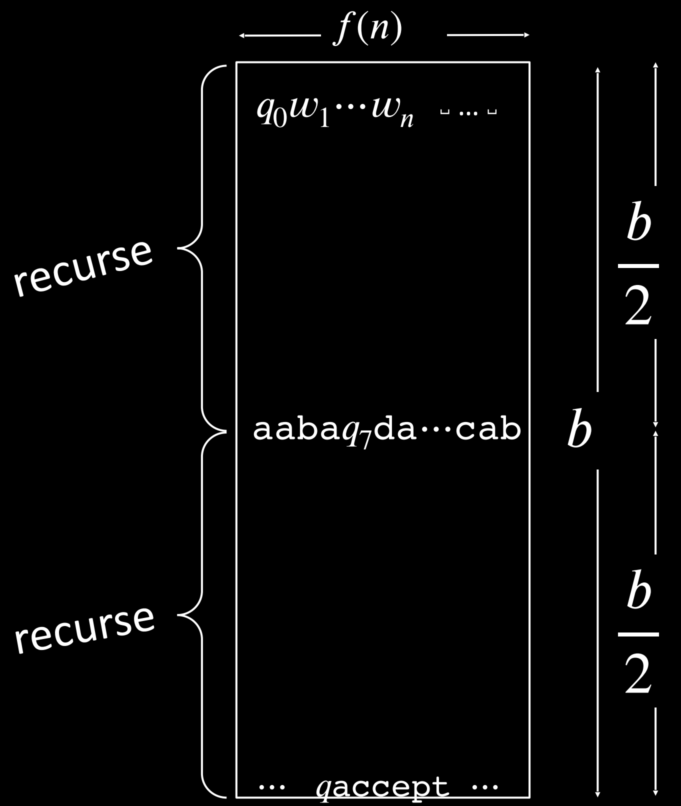

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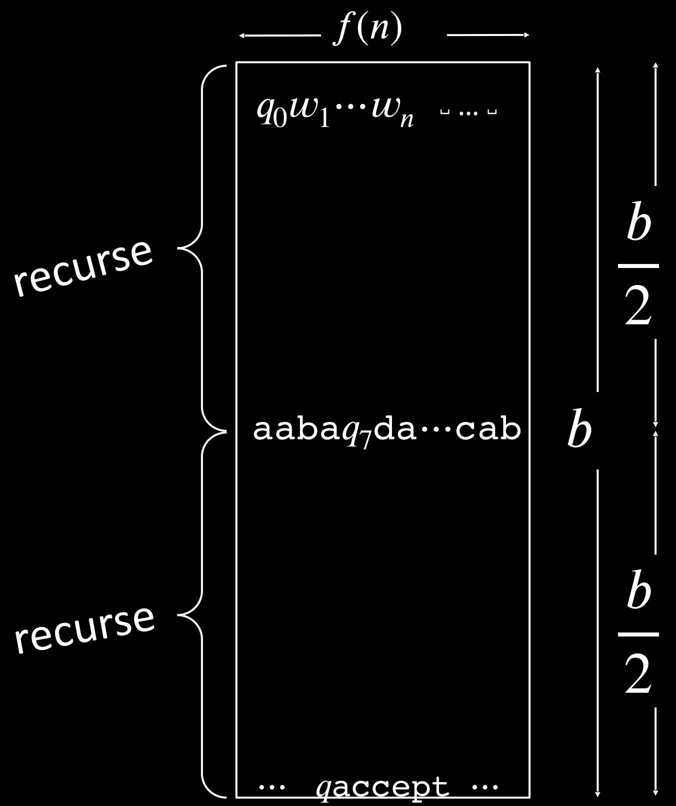

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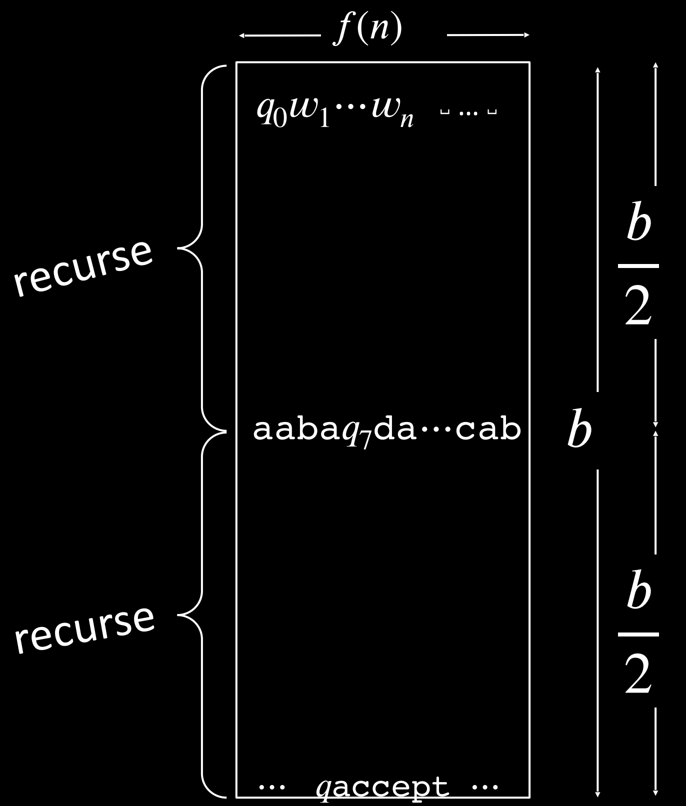

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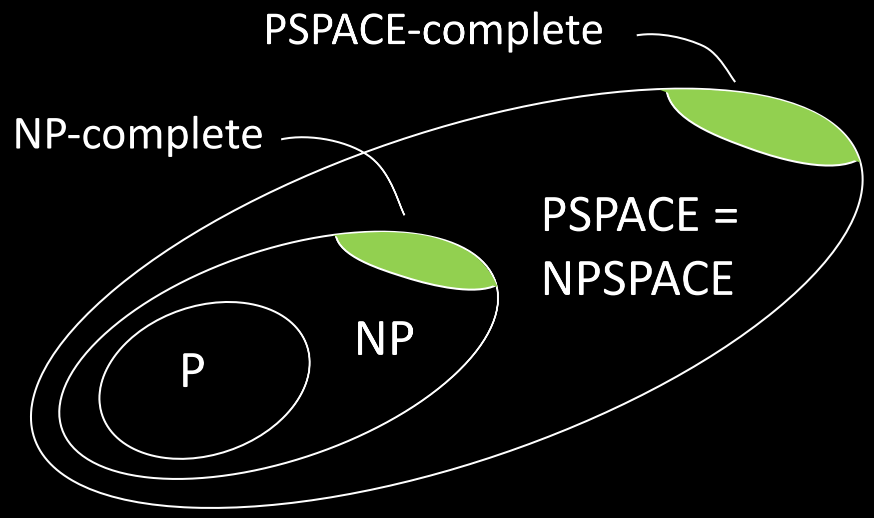

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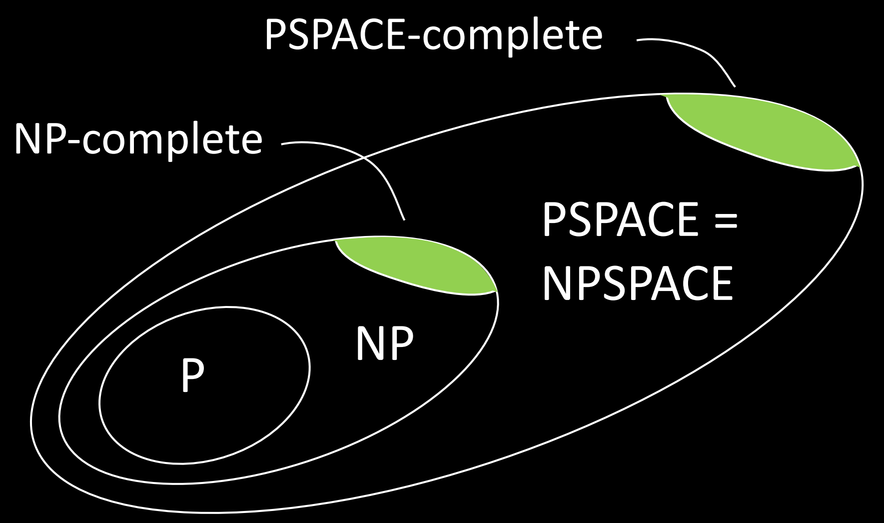

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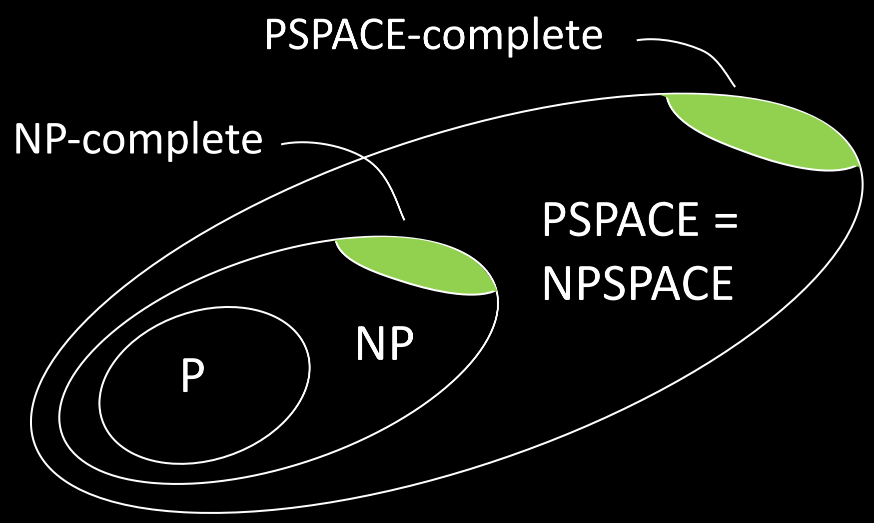

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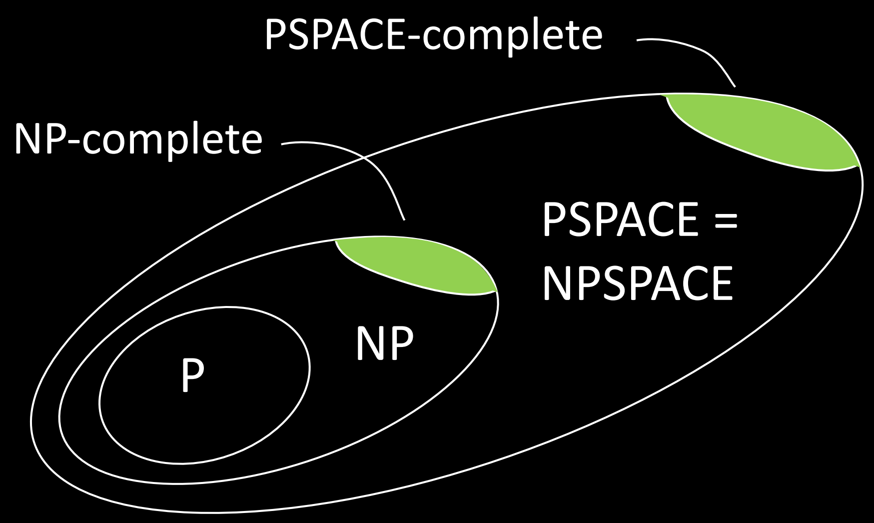

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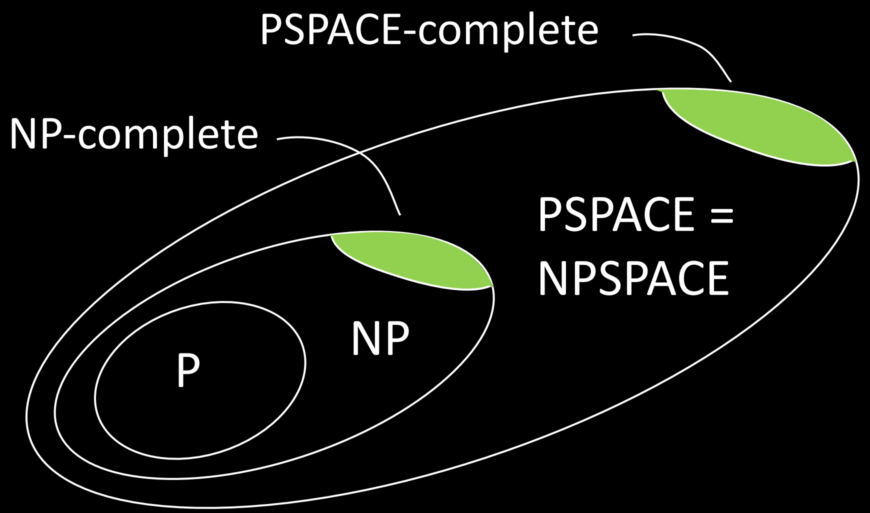
Theorem: $TQBF$ is PSPACE-complete



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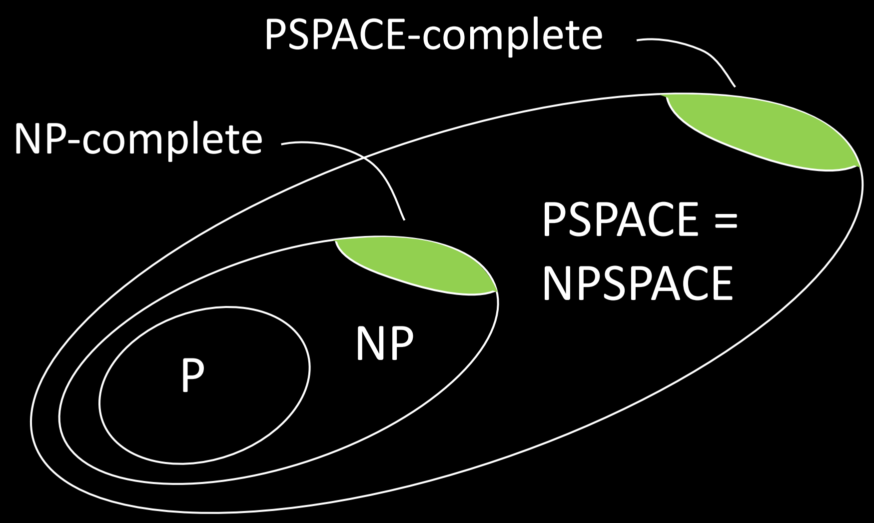

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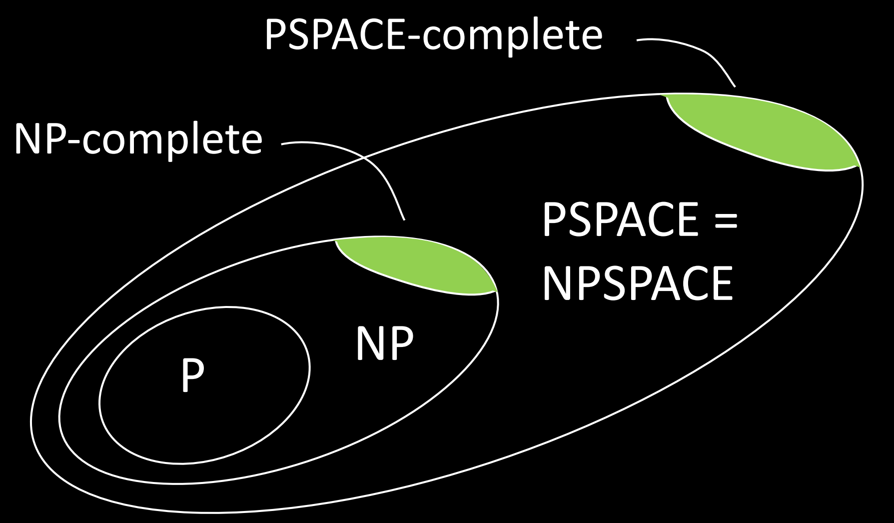

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Check-in 18.1

Knowing that $TQBF$ is PSPACE-complete, what can we conclude if $TQBF \in$ NP? Check all that apply.

(a) P = PSPACE

(b) NP = PSPACE

(c) P = NP

(d) NP = coNP

Check-in 18.1

$TQBF$ is PSPACE-complete

Recall: $TQBF = \left\{ \langle \phi \rangle \,\middle|\, \phi \text{ is a QBF that is } \textsc{True} \right\}$

Examples: $\phi_1 = \forall x \ \exists y \left[\left(x \vee y \right) \wedge \left(\overline{x} \vee \overline{y} \right) \right] \in TQBF$ [True]

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2) For all $A \in$ PSPACE, $A \leq_{\mathrm{P}} TQBF$

Let $A \in$ PSPACE be decided by TM M in space n^k.

Give a polynomial-time reduction f mapping A to $TQBF$.

$TQBF$ is PSPACE-complete

Recall: $TQBF = \{\langle\phi\rangle \mid \phi \text{ is a QBF that is } \textsc{True}\}$

Examples: $\phi_1 = \forall x\ \exists y\ \left[\left(x \vee y\right) \wedge \left(\overline{x} \vee \overline{y}\right)\right] \in TQBF$ [TRUE]

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$f\colon \Sigma^* \rightarrow$ QBFs

$f(w) = \langle\phi_{M,w}\rangle$

$w \in A$ iff $\phi_{M,w}$ is TRUE

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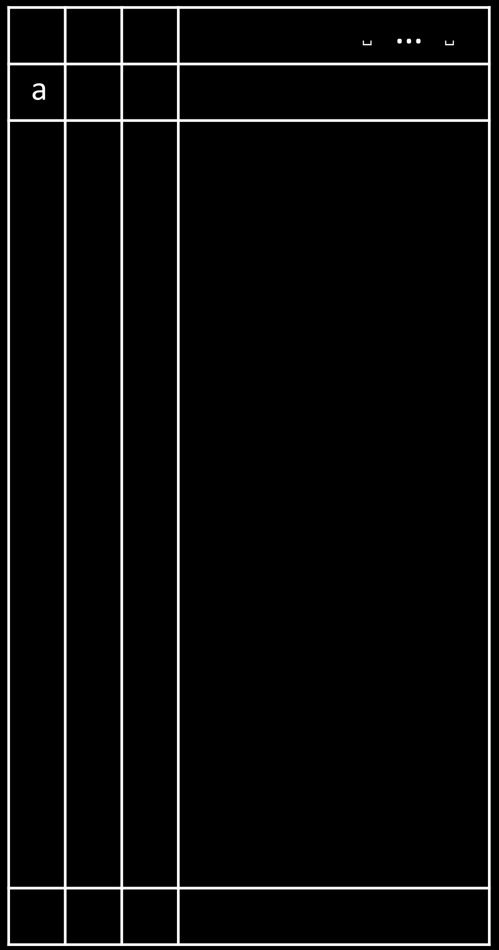
Constructing $\phi_{M,w}$: 1st try

Tableau for M on w



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Rows of that tableau are configurations.

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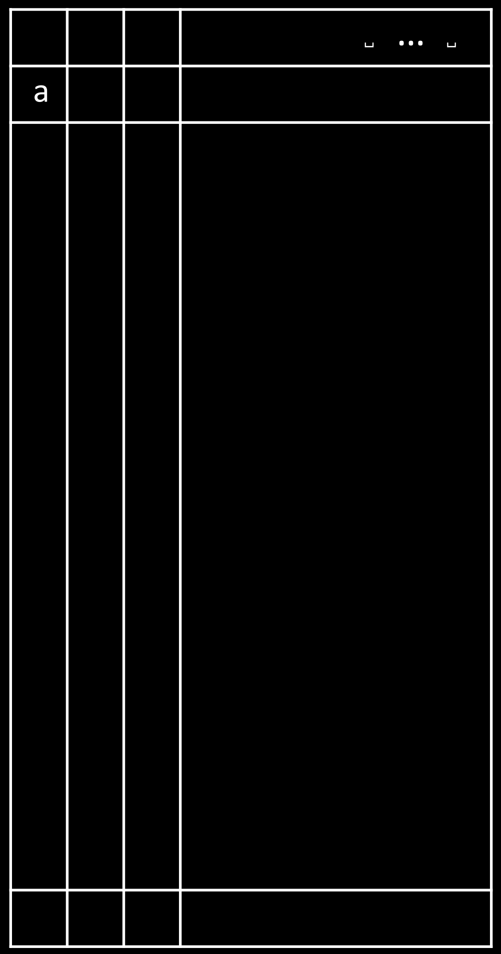

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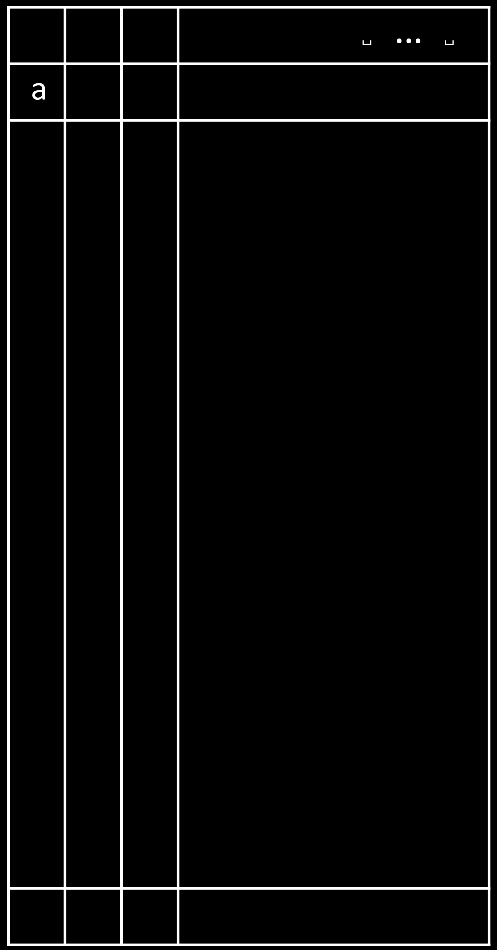

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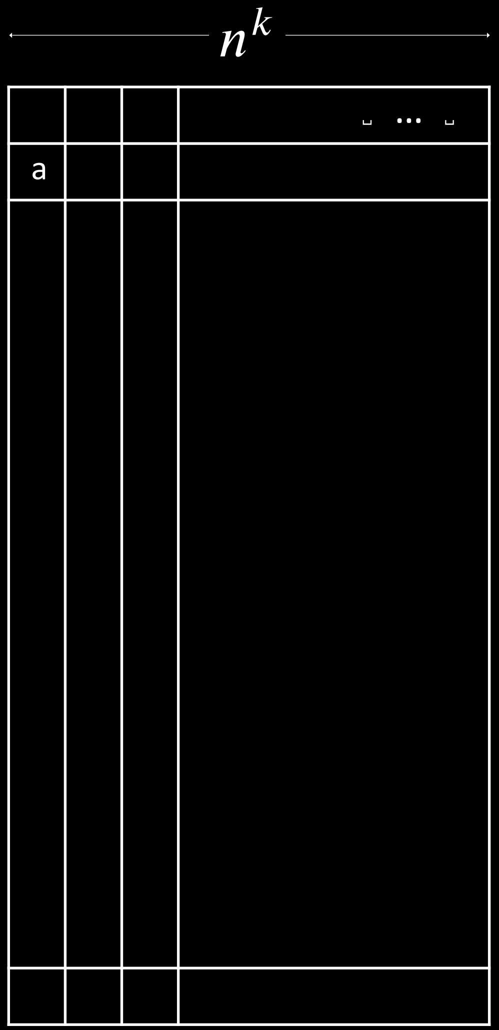

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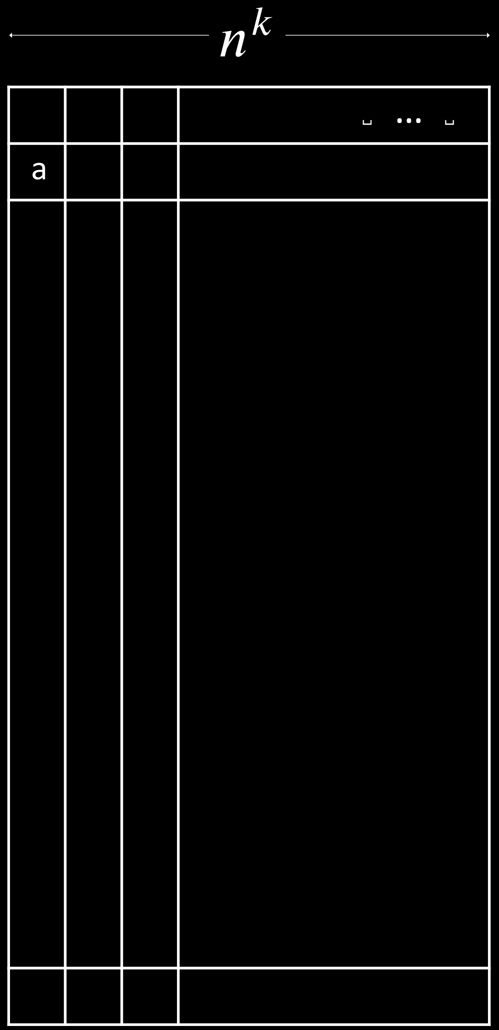

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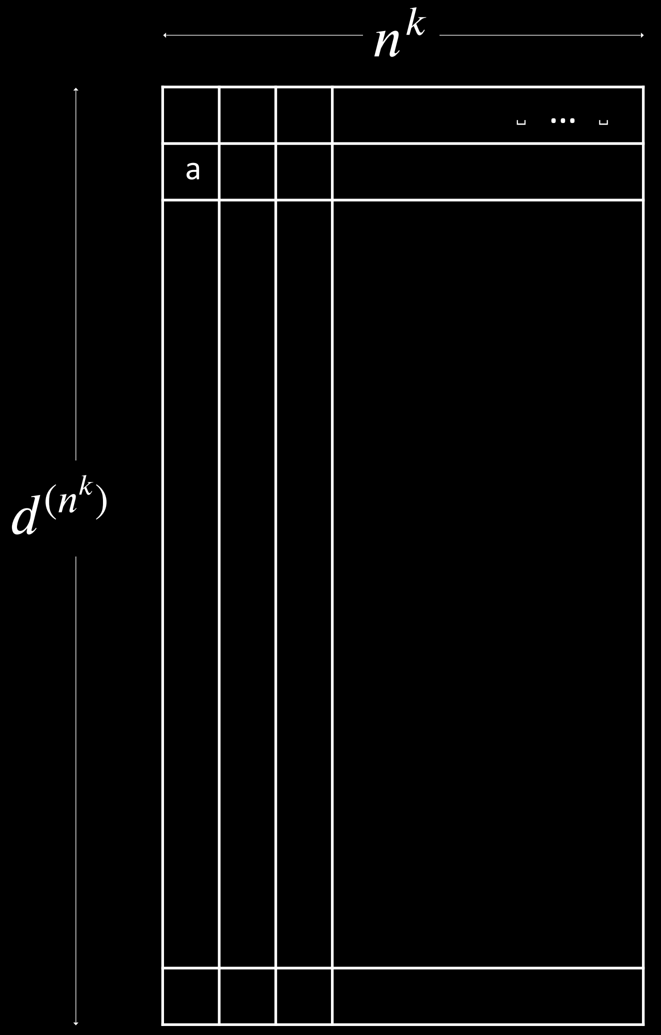

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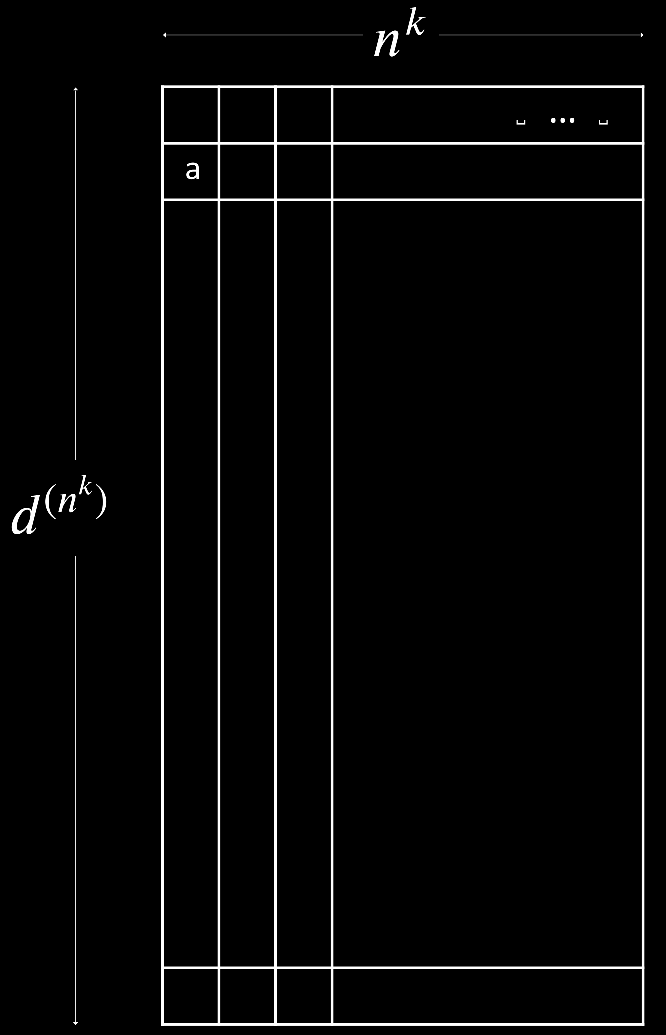

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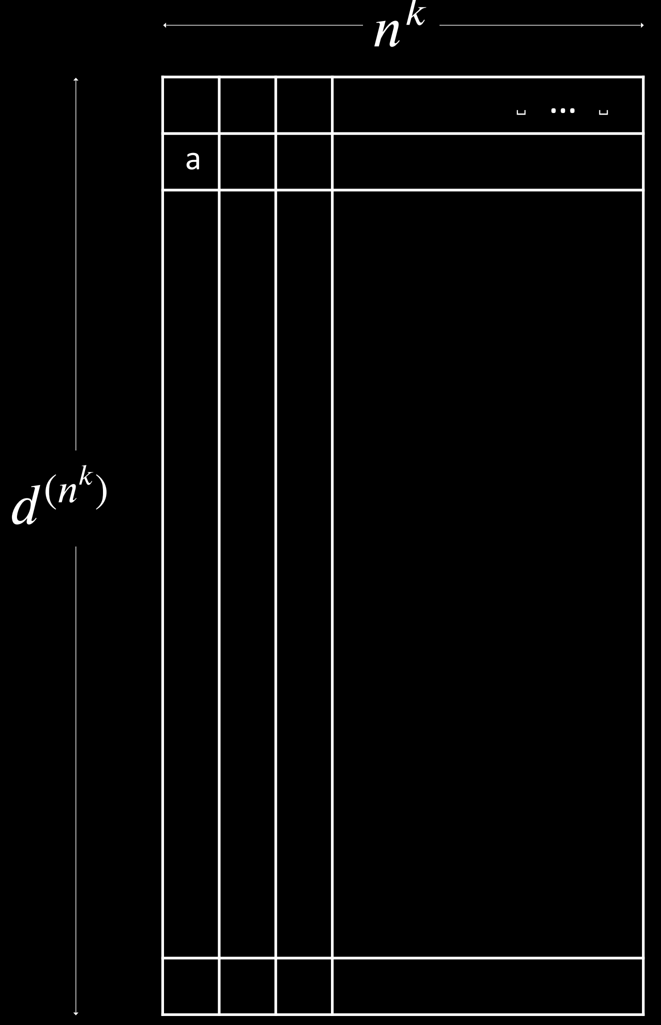

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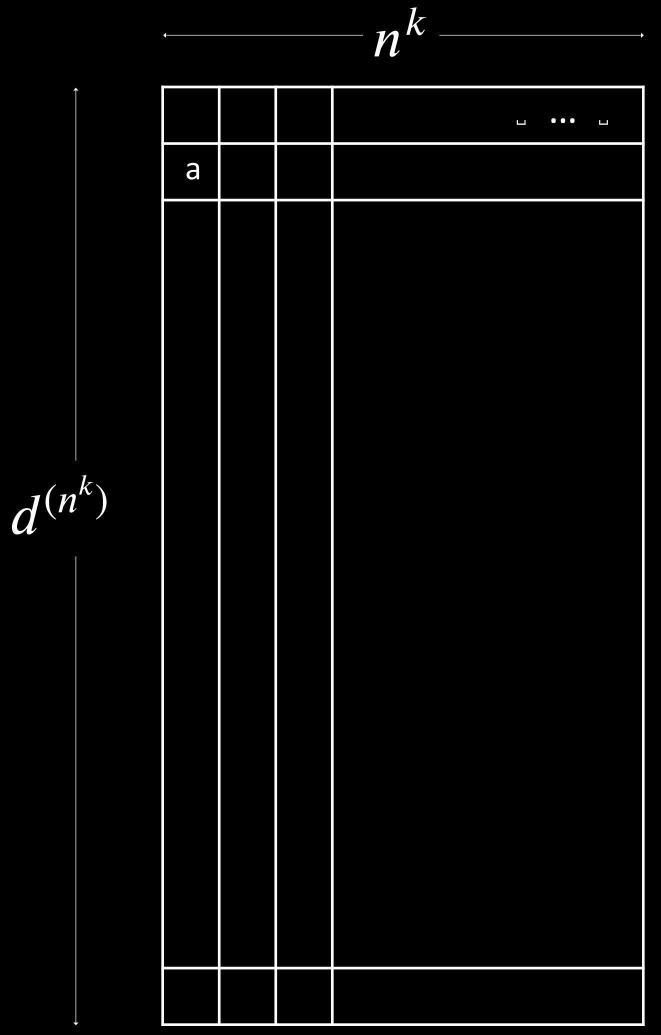

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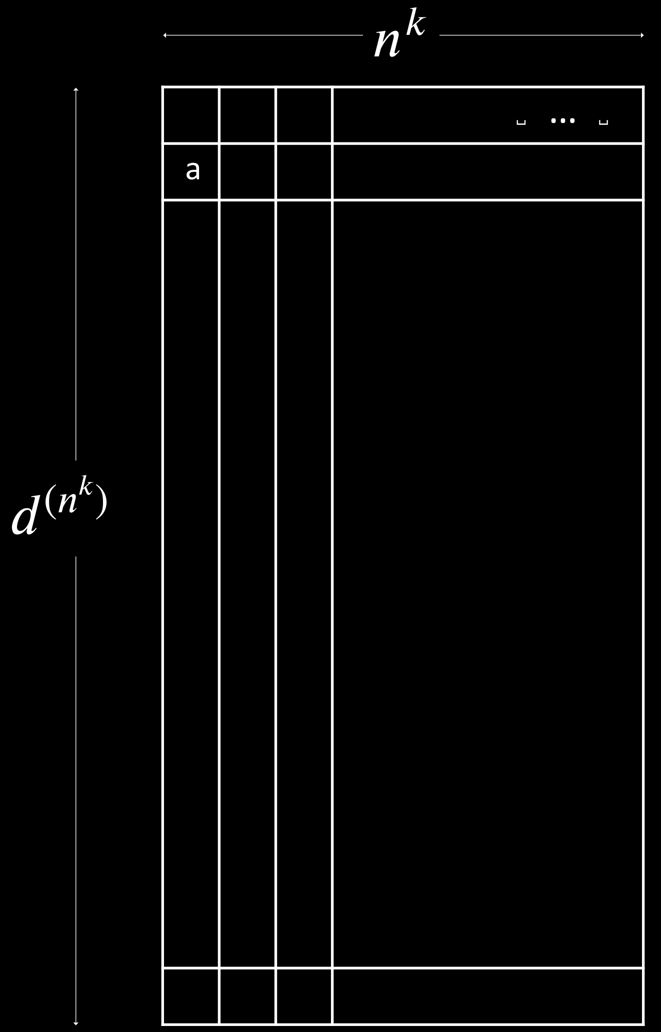

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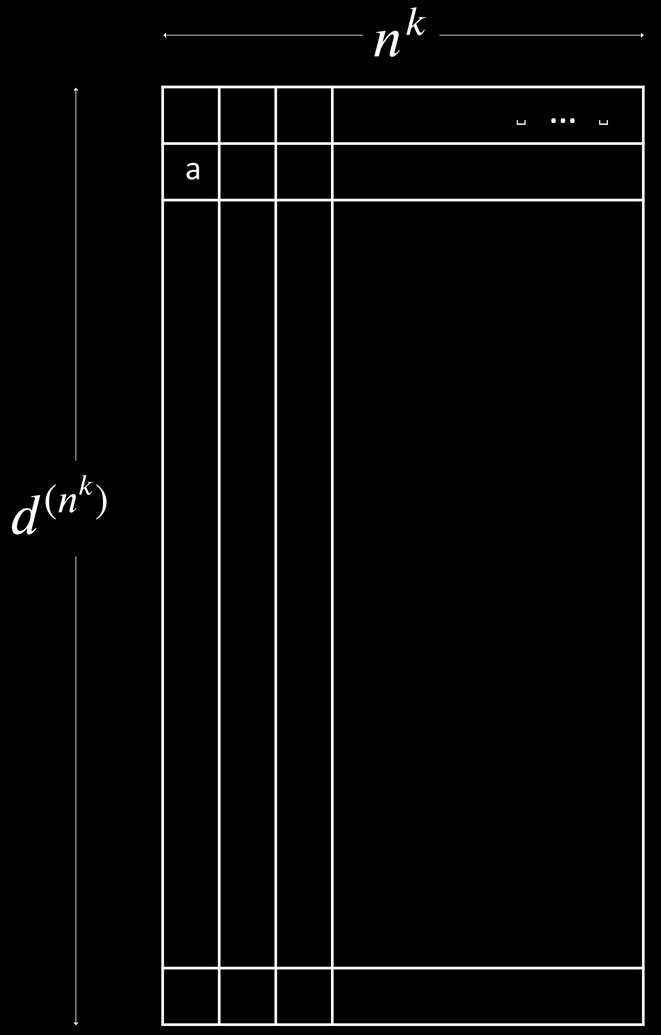

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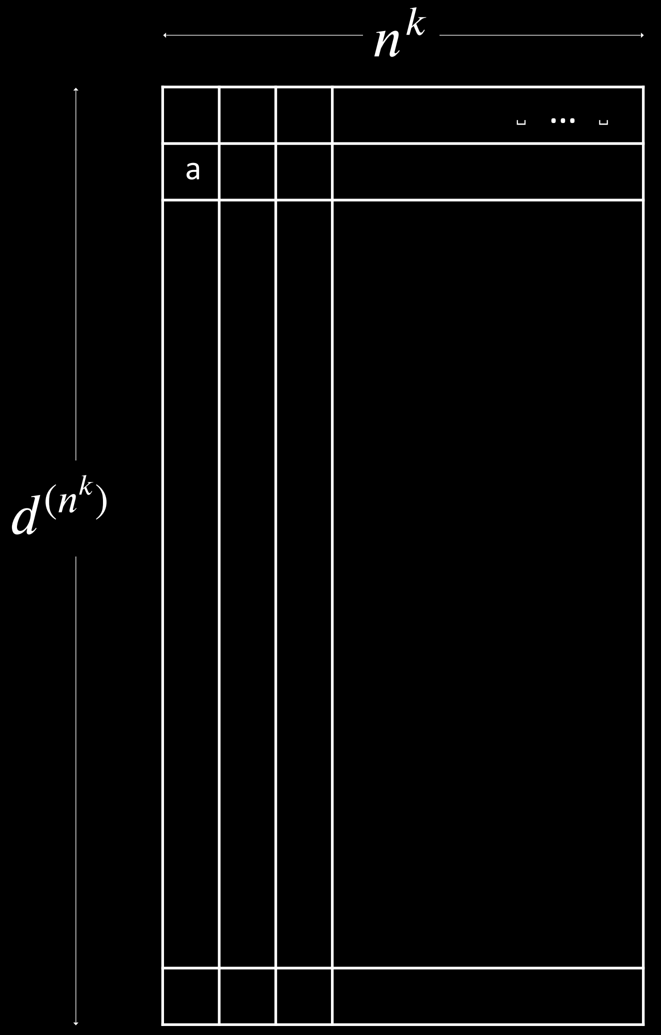
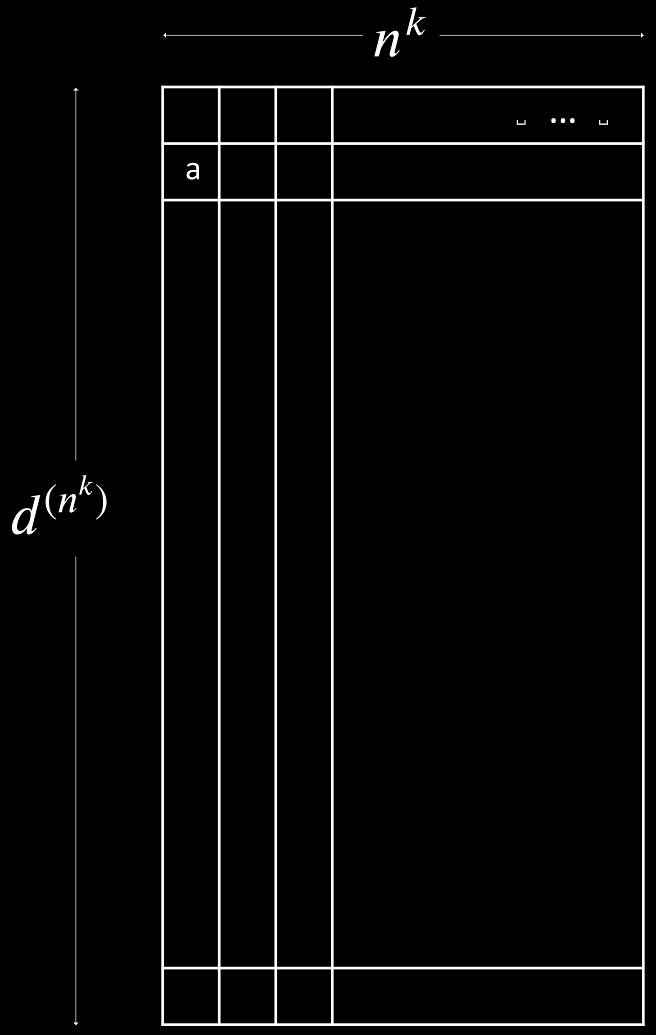


Constructing $\phi_{M,w}$: 2nd try

hide →

Tableau for M on w



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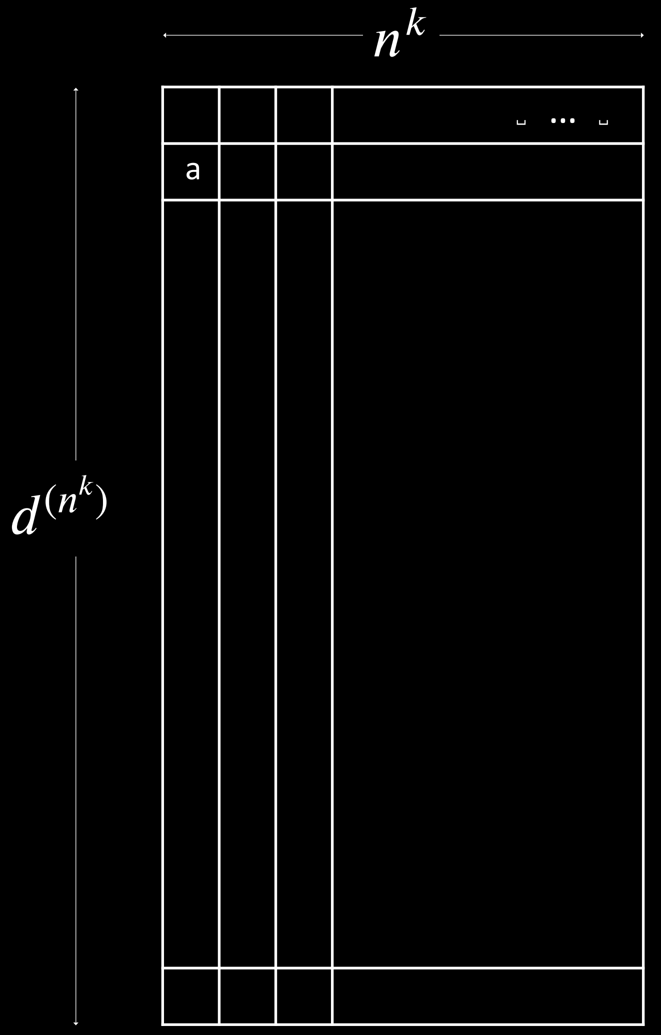

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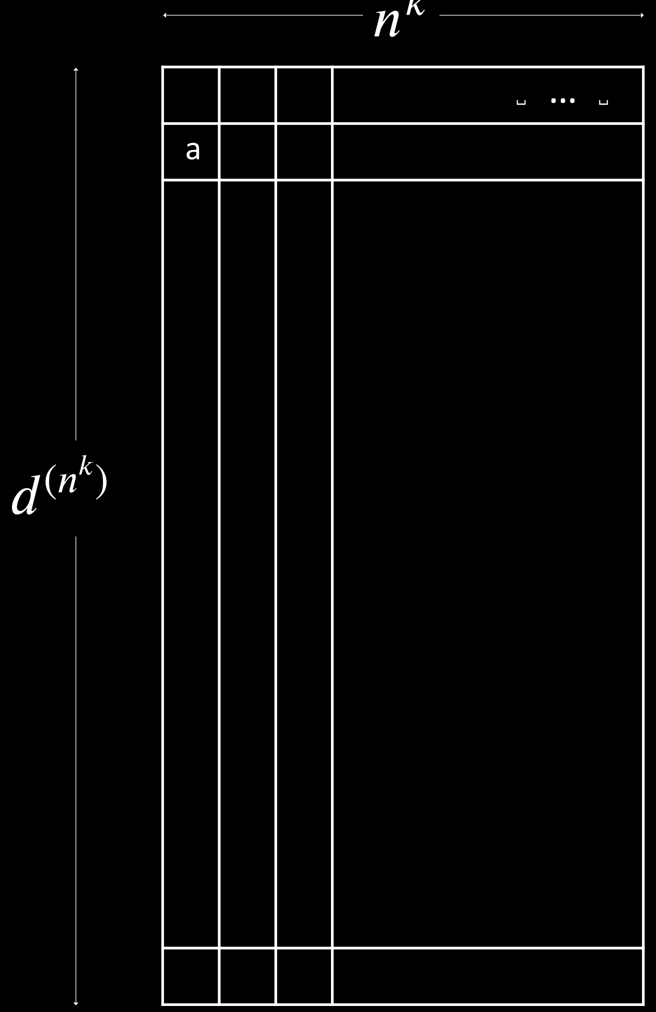

For configs c_i and c_j construct $\phi_{c_i,\ c_j,\ b}$ which "says" $c_i \xrightarrow{b} c_j$ recursively.

Constructing $\phi_{M,w}$: 2nd try

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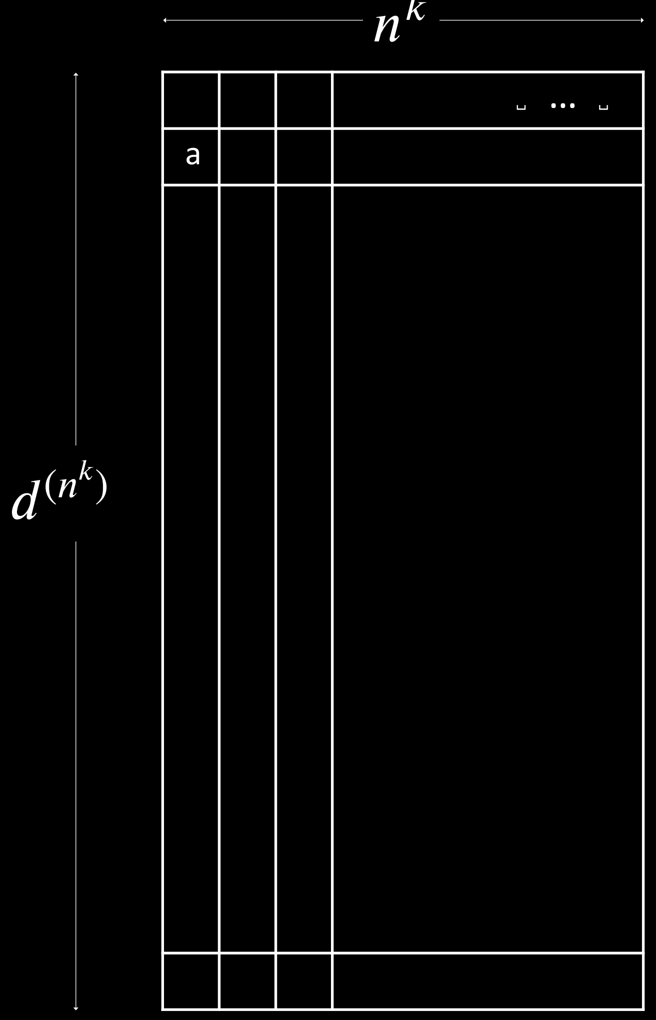

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Constructing $\phi_{M,w}$: 2nd try

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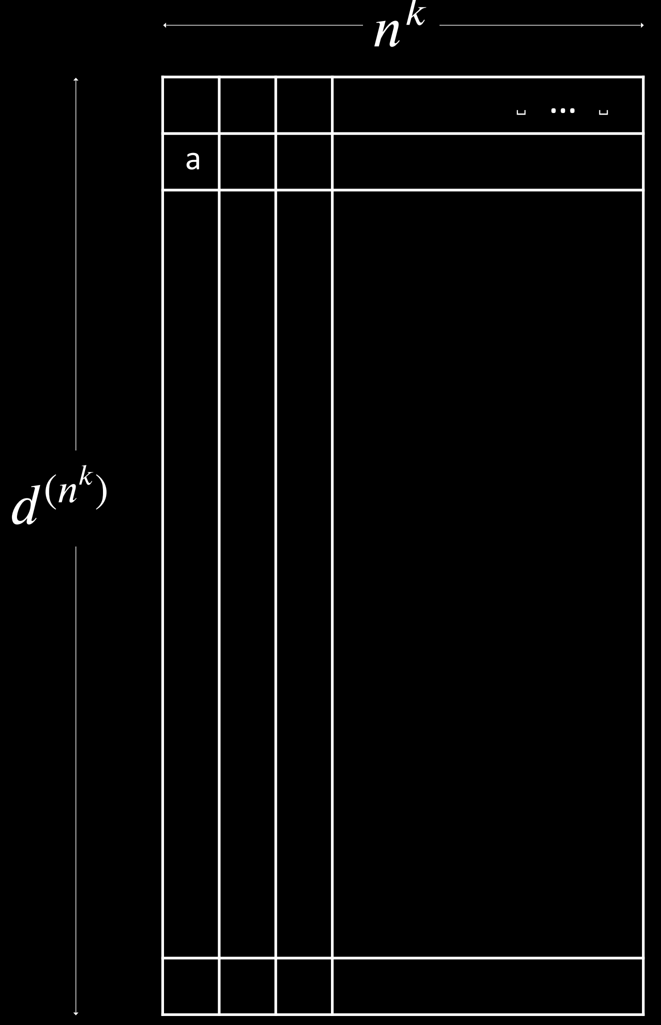

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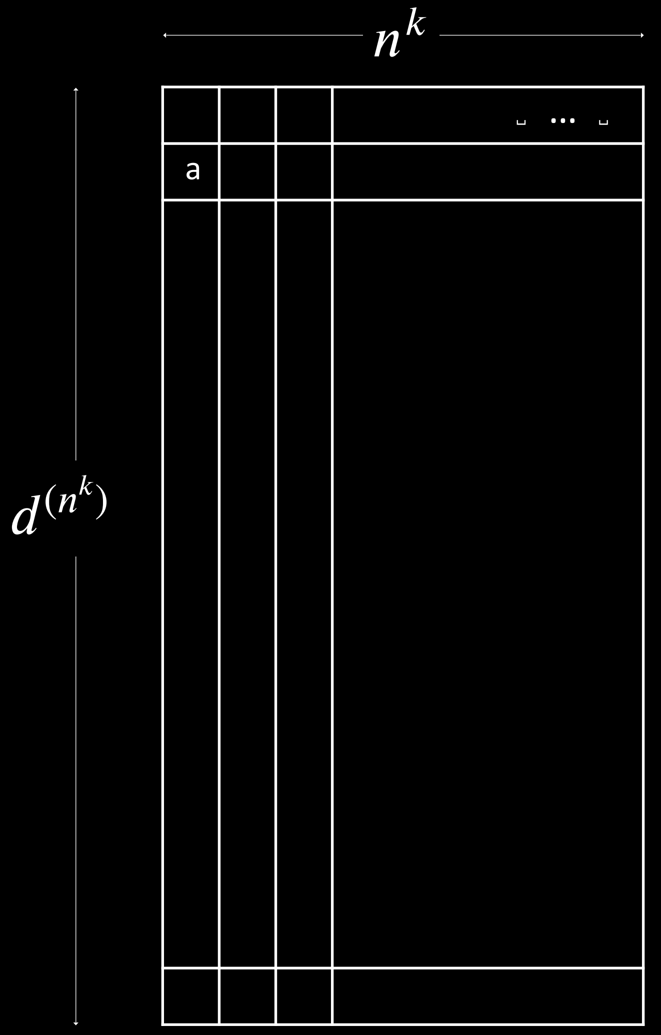

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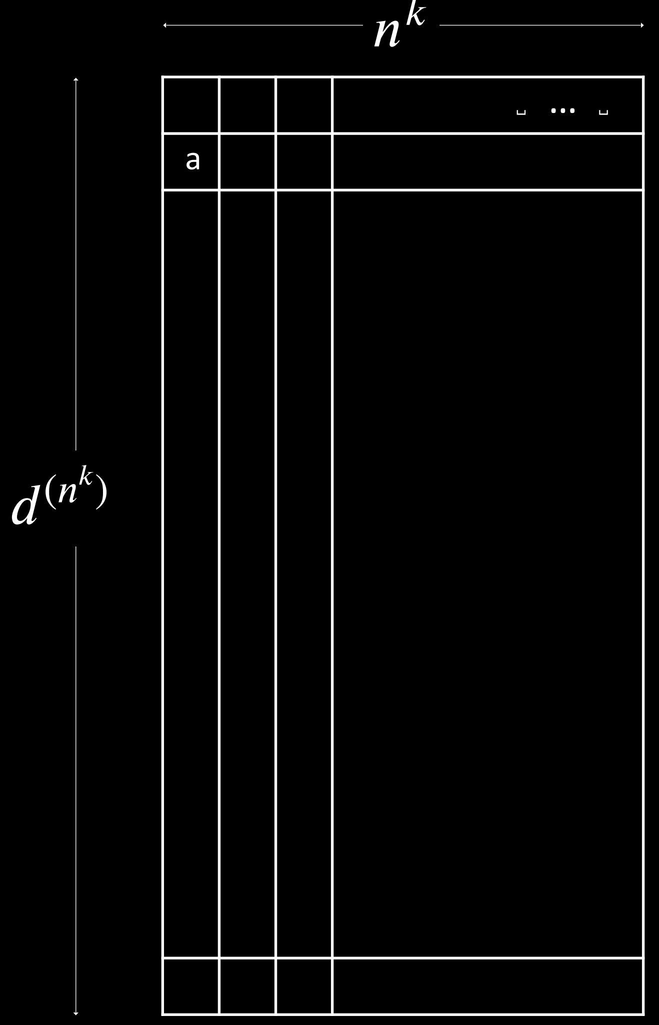

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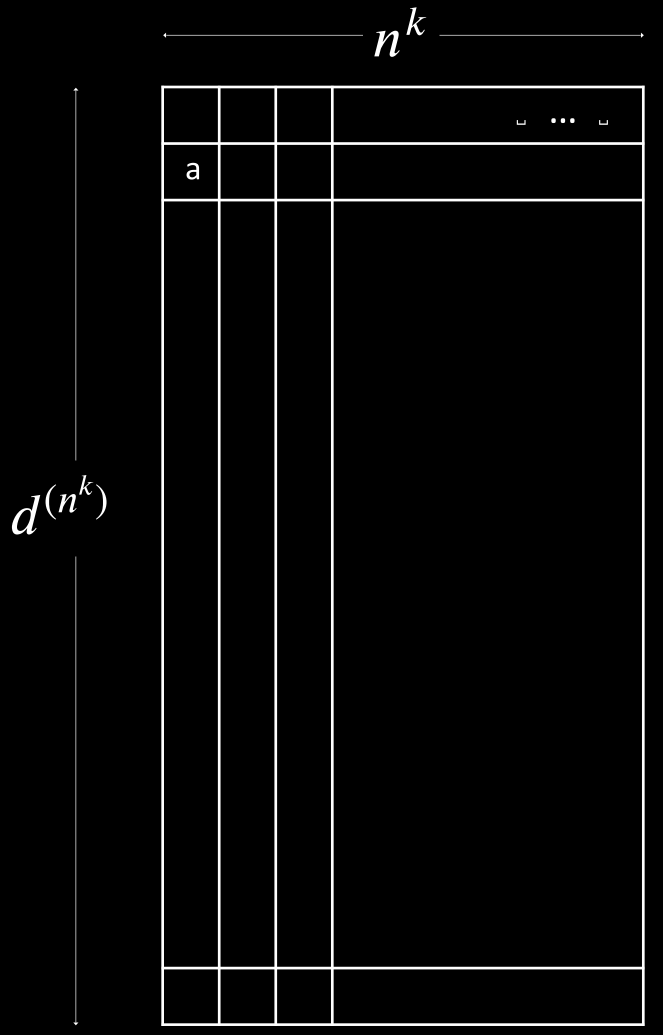

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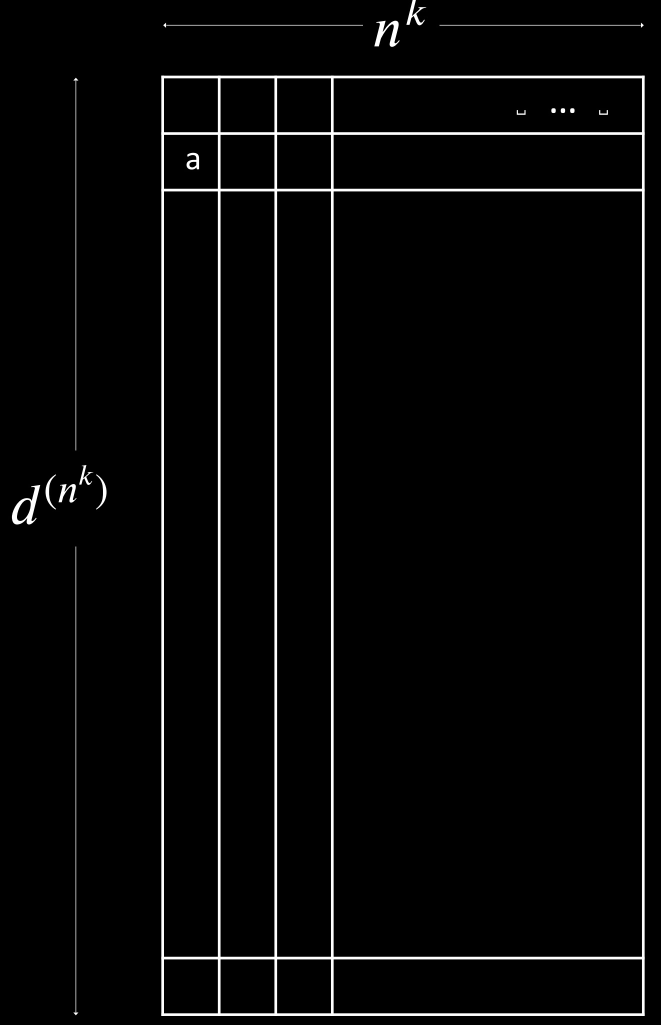

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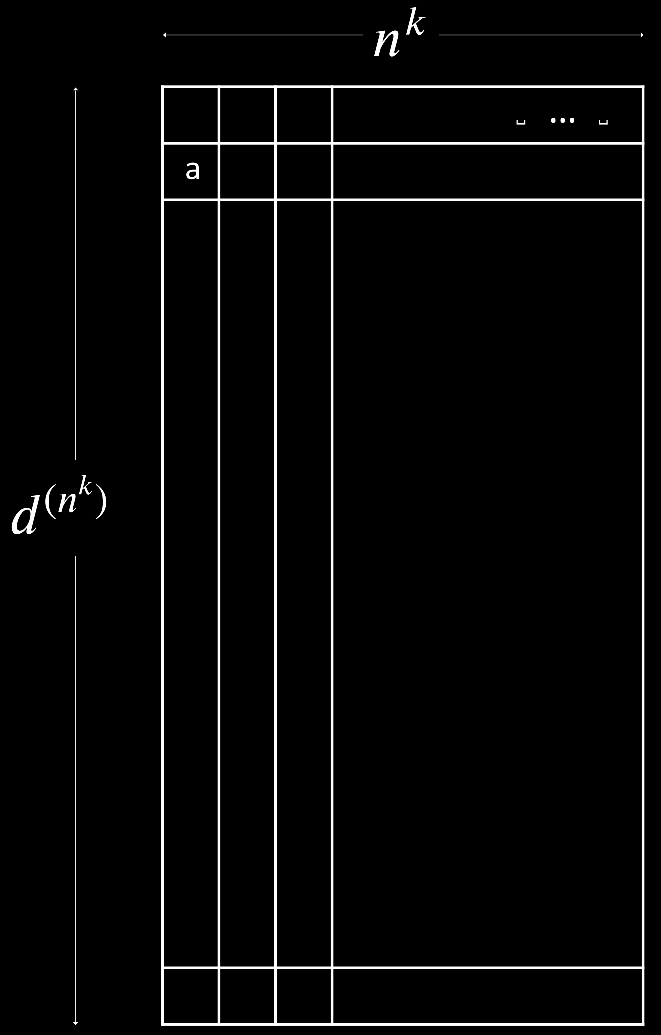

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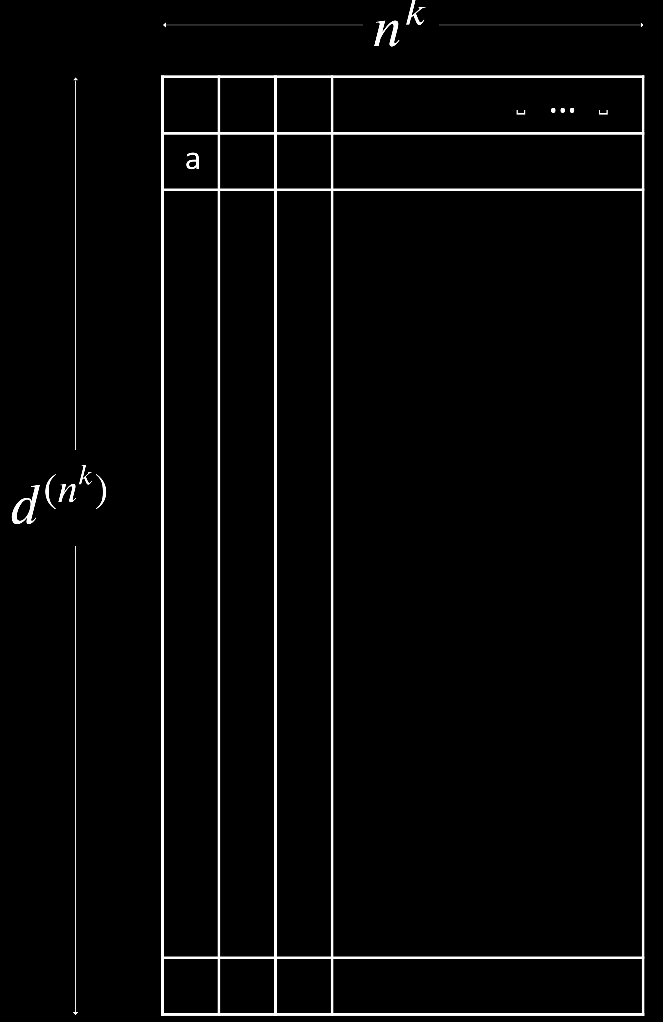

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$$\boxed{\begin{array}{c}\phi_{M,w} = \phi_{c_{\text{start}},\ c_{\text{accept}},\ t} \\ t = d^{(n^k)}\end{array}}$$

Constructing $\phi_{M,w}$: 2nd try

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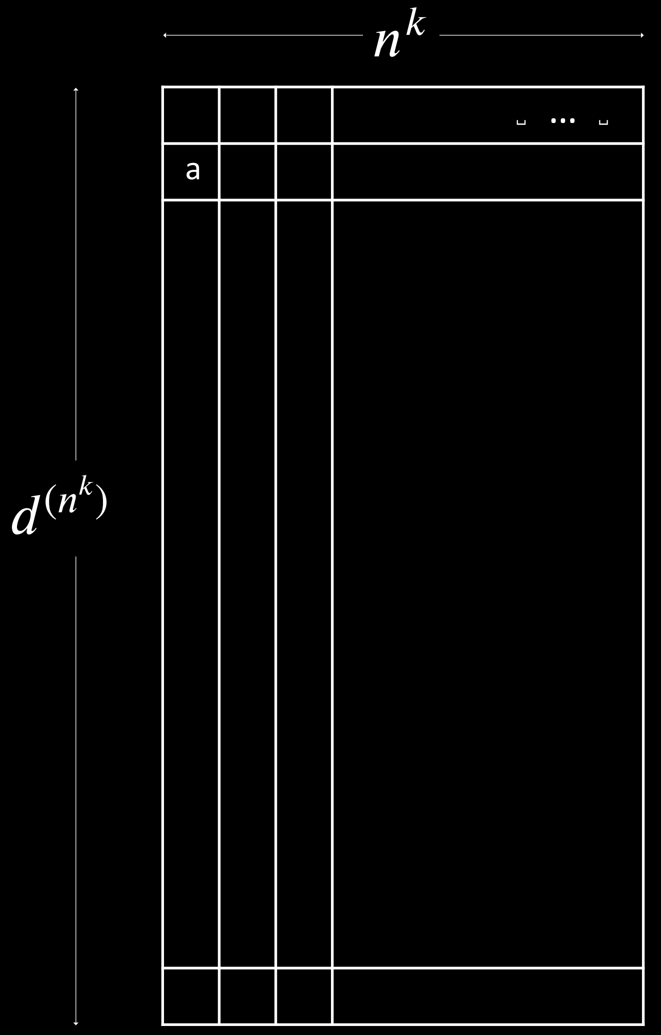

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Size analysis:
Each recursive level doubles number of QBFs.

Constructing $\phi_{M,w}$: 2nd try

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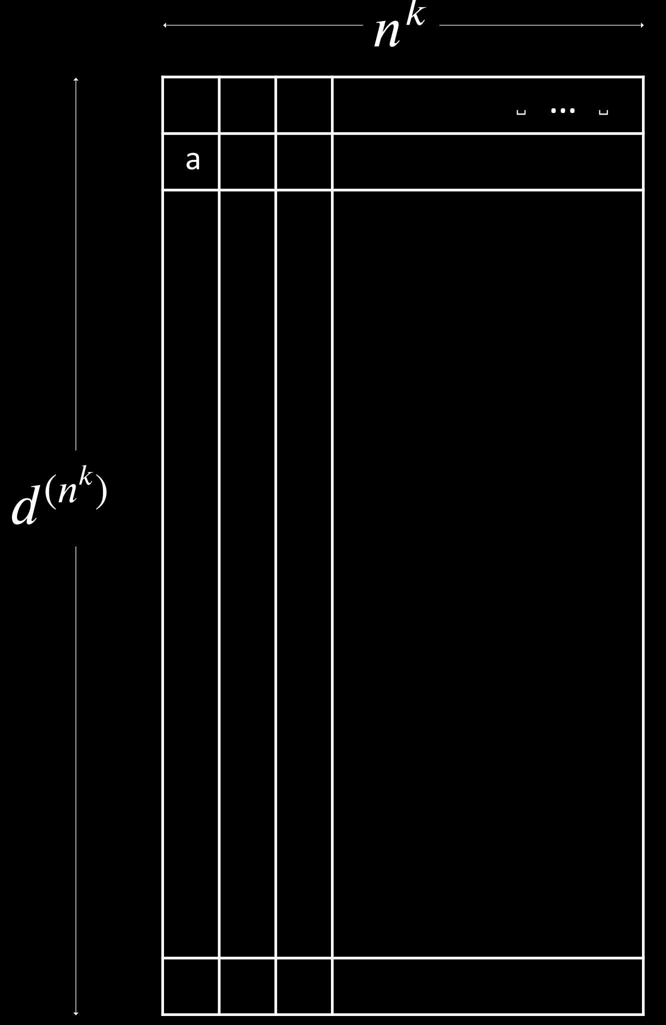

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Number of levels is $\log d^{(n^k)} = O(n^k)$.

Constructing $\phi_{M,w}$: 2nd try

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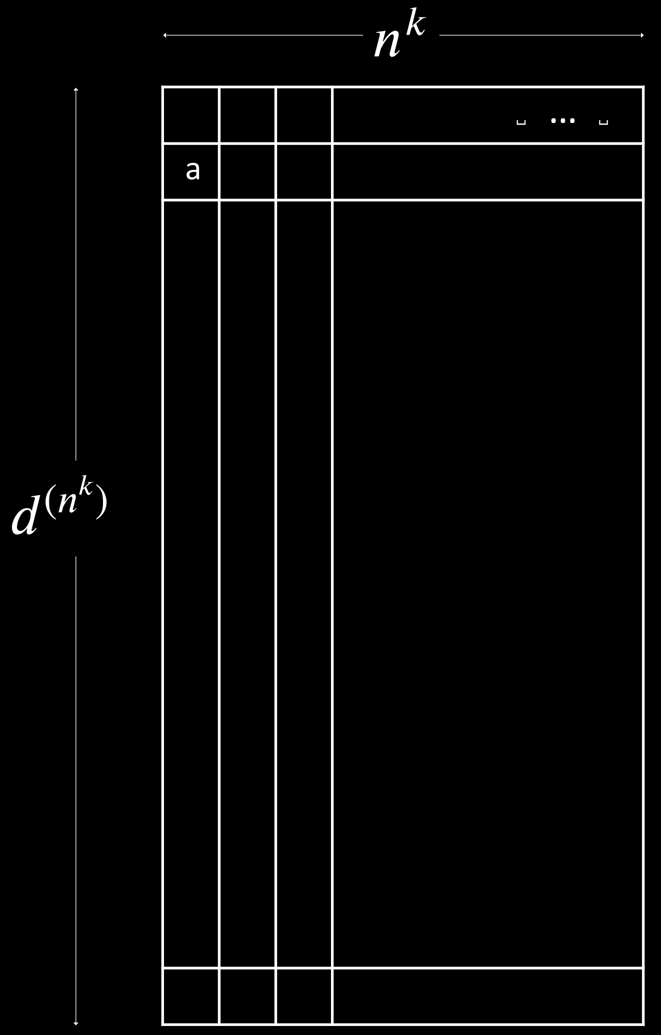

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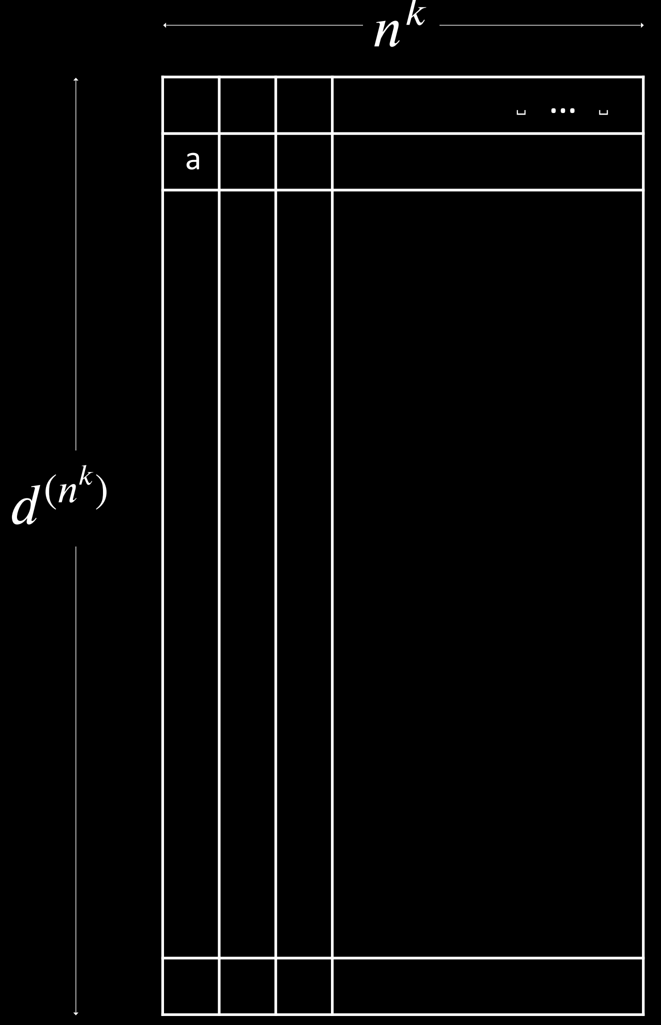
$\phi_{\ ,\ ,\ 1}$ defined as in Cook-Levin

$$\boxed{\begin{gathered}\phi_{M,w} = \phi_{c_{\text{start}},\ c_{\text{accept}},\ t} \\ t = d^{(n^k)}\end{gathered}}$$

Size analysis:
Each recursive level doubles number of QBFs.
Number of levels is $\log d^{(n^k)} = O(n^k)$.
→ Size is exponential. ☹

Constructing $\phi_{M,w}$: 2nd try

Tableau for M on w



For configs c_i and c_j construct $\phi_{c_i,\ c_j,\ b}$ which "says" $c_i \xrightarrow{b} c_j$ recursively.

$$\phi_{c_i,\ c_j,\ b} = \underbrace{\exists c_{\text{mid}}}\left[\ \phi_{c_i,\ c_{\text{mid}},\ b/2} \ \wedge\ \phi_{c_{\text{mid}},\ c_j,\ b/2}\ \right]$$

$\exists x_1, x_2, \cdots, c_l$
as in Cook-Levin

$$\exists c_{\text{mid}}\left[\phi_{\ ,\ ,\ b/4} \ \wedge\ \phi_{\ ,\ ,\ b/4}\right] \qquad \exists c_{\text{mid}}\left[\phi_{\ ,\ ,\ b/4} \ \wedge\ \phi_{\ ,\ ,\ b/4}\right]$$

$\vdots \qquad \vdots \qquad \exists c_{\text{mid}}[\phi_{\ ,\ ,\ b/8} \cdots]$

$\phi_{\ ,\ ,\ 1}$ defined as in Cook-Levin

$$\boxed{\begin{aligned}\phi_{M,w} &= \phi_{c_{\text{start}},\ c_{\text{accept}},\ t} \\ t &= d^{(n^k)}\end{aligned}}$$

Size analysis:
Each recursive level doubles number of QBFs.
Number of levels is $\log d^{(n^k)} = O(n^k)$.
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Check-in 18.2

Constructing $\phi_{M,w}$: 2nd try

configs c_i and c_j construct $\phi_{c_i,\ c_j,\ b}$ which "says" $c_i \xrightarrow{b} c_j$ recursively.

$$\phi_{c_i,\ c_j,\ b} = \underbrace{\exists c_{\text{mid}}}\left[\ \phi_{c_i,\ c_{\text{mid}},\ b/2}\ \wedge\ \phi_{c_{\text{mid}},\ c_j,\ b/2}\ \right]$$

$\exists x_1, x_2, \cdots, c_l$
as in Cook-Levin

$$\exists c_{\text{mid}}\left[\phi_{\ ,\ ,\ b/4} \wedge \phi_{\ ,\ ,\ b/4}\right] \qquad \exists c_{\text{mid}}\left[\phi_{\ ,\ ,\ b/4} \wedge \phi_{\ ,\ ,\ b/4}\right]$$

$\vdots \qquad \vdots \qquad \exists c_{\text{mid}}[\phi_{\ ,\ ,\ b/8} \cdots]$

$\phi_{\ ,\ ,\ 1}$ defined as in Cook-Levin

$$\phi_{M,w} = \phi_{c_{\text{start}},\ c_{\text{accept}},\ t}$$

$$t = d^{(n^k)}$$

Size analysis:
Each recursive level doubles number of QBFs.
Number of levels is $\log d^{(n^k)} = O(n^k)$.
→ Size is exponential. ☹

Check-in 18.2

Check-in 18.2

Why shouldn't we be surprised that this construction fails?

(a) We can't define a QBF by using recursion.

(b) It doesn't use $\forall$ anywhere.

(c) We know that $TQBF \notin$ P.

Constructing $\phi_{M,w}$: 3rd try

$$\phi_{c_i,\ \ c_j,\ \ b} = \exists c_{\text{mid}}\left[\ \phi_{c_i,\, c_{\text{mid}},\ \ b/2}\ \ \wedge\ \ \phi_{c_{\text{mid}},\ \ c_j,\ \ b/2}\ \right]$$

Constructing $\phi_{M,w}$: 3rd try

$$\phi_{c_i,\ c_j,\ b} = \exists c_{\text{mid}} \underbrace{\left[\ \phi_{c_i,\, c_{\text{mid}},\ b/2} \ \wedge\ \phi_{c_{\text{mid}},\ c_j,\ b/2}\ \right]}_{\forall (c_g,\ c_h) \in \left\{ (c_i,\ c_{\text{mid}}),\ (c_{\text{mid}},\ c_j) \right\} \left[\phi_{c_g,\, c_h,\ b/2} \right]}$$

Constructing $\phi_{M,w}$: 3rd try

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$$\vdots$$

Constructing $\phi_{M,w}$: 3rd try

$$\phi_{c_i,\ c_j,\ b} = \exists c_{\mathrm{mid}} \underbrace{\left[\ \phi_{c_i,\ c_{\mathrm{mid}},\ b/2}\ \wedge\ \phi_{c_{\mathrm{mid}},\ c_j,\ b/2}\ \right]}_{\forall\left(c_g,\ c_h\right) \in \left\{\left(c_i,\ c_{\mathrm{mid}}\right),\ \left(c_{\mathrm{mid}},\ c_j\right)\right\}\left[\phi_{c_g,\ c_h,\ b/2}\right]}$$

$$\vdots$$

$\phi_{\ ,\ ,\ 1}$ defined as in Cook-Levin

Constructing $\phi_{M,w}$: 3rd try

$$\phi_{c_i,\ c_j,\ b} = \exists c_{\text{mid}} \underbrace{\left[\phi_{c_i,\, c_{\text{mid}},\ b/2} \ \wedge\ \phi_{c_{\text{mid}},\ c_j,\ b/2} \right]}_{\forall \left(c_g,\ c_h\right) \in \left\{ \left(c_i,\ c_{\text{mid}}\right),\ \left(c_{\text{mid}},\ c_j\right) \right\} \left[\phi_{c_g,\, c_h,\ b/2}\right]}$$

$\forall (x \in S)\ [\ \psi\ ]$
is equivalent to
$\forall x\ [(x \in S) \longrightarrow \psi]$

$\vdots$

$\phi_{\ ,\ ,\ 1}$ defined as in Cook-Levin

Constructing $\phi_{M,w}$: 3rd try

$$\phi_{c_i,\ c_j,\ b} = \exists c_{\text{mid}} \Big[\phi_{c_i,\, c_{\text{mid}},\ b/2} \ \wedge\ \phi_{c_{\text{mid}},\ c_j,\ b/2} \Big]$$

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$\phi_{M,w} = \phi_{c_{\text{start}},\ c_{\text{accept}},\ t}$

$t = d^{(n^k)}$

$\phi_{\ ,\ ,\ 1}$ defined as in Cook-Levin

Constructing $\phi_{M,w}$: 3rd try

$$\phi_{c_i,\ c_j,\ b} = \exists c_{\text{mid}} \underbrace{\left[\phi_{c_i,\, c_{\text{mid}},\ b/2} \ \wedge\ \phi_{c_{\text{mid}},\ c_j,\ b/2} \right]}$$

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$\vdots$

$\forall (x \in S)\ [\,\psi\,]$ is equivalent to $\forall x\ \left[(x \in S) \longrightarrow \psi\right]$

$\phi_{M,w} = \phi_{c_{\text{start}},\ c_{\text{accept}},\ t}$

$t = d^{\left(n^k\right)}$

$\phi_{\ ,\ ,\ 1}$ defined as in Cook-Levin

Size analysis:

Each recursive level <u>adds</u> $O(n^k)$ to the QBF.

Number of levels is $\log d^{\left(n^k\right)} = O\left(n^k\right)$.

→ Size is $O\left(n^k \times n^k\right) = O\left(n^{2k}\right)$ ☺

Constructing $\phi_{M,w}$: 3rd try

$$\phi_{c_i,\ c_j,\ b} = \exists c_{\text{mid}} \underbrace{\left[\ \phi_{c_i,\, c_{\text{mid}},\ b/2}\ \wedge\ \phi_{c_{\text{mid}},\ c_j,\ b/2}\ \right]}$$

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Check-in 18.3

Constructing $\phi_{M,w}$: 3rd try

$$\phi_{c_i,\ c_j,\ b} = \exists c_{\text{mid}} \underbrace{\left[\ \phi_{c_i,\, c_{\text{mid}},\ b/2} \ \wedge\ \phi_{c_{\text{mid}},\ c_j,\ b/2}\ \right]}$$

$$\forall\left(c_g,\ c_h\right) \in \left\{\ \left(c_i,\ c_{\text{mid}}\right),\ \left(c_{\text{mid}},\ c_j\right)\right\}\left[\phi_{c_g,\, c_h,\ b/2}\right]$$

$\forall(x \in S)\ [\ \psi\ ]$
is equivalent to
$\forall x\ \left[(x \in S)\ \longrightarrow\ \psi\right]$

$\vdots$

$\phi_{\ ,\ ,\ 1}$ defined as in Cook-Levin

$$\phi_{M,w} = \phi_{c_{\text{start}},\ c_{\text{accept}},\ t}$$
$$t = d^{\left(n^k\right)}$$

Size analysis:

Each recursive level <u>adds</u> $O(n^k)$ to the QBF.

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→ Size is $O\left(n^k \times n^k\right) = O\left(n^{2k}\right)$ ☺

Check-in 18.3

Would this construction still work if M were nondeterministic?

(a) Yes.

(b) No.

Check-in 18.3

Quick review of today

1. Space complexity
2. SPACE$\big(f(n)\big)$, NSPACE$\big(f(n)\big)$
3. PSPACE, NPSPACE
4. Relationship with TIME classes
5. $TQBF \in$ PSPACE
6. $LADDER$DFA $\in$ NSPACE(n)
7. $LADDER$DFA $\in$ SPACE(n^2)
8. Savitch's Theorem: NSPACE$\big(f(n)\big) \subseteq$ SPACE$\big(f^2(n)\big)$
9. $TQBF$ is PSPACE-complete

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