بسم الله الرحمن الرحیم

نظریه علوم کامپیوتر

نظریه علوم کامپیوتر - بهار ۱۴۰۰-۱۴۰۱ - جلسه هفتم: محاسبه‌پذیری و محاسبه‌ناپذیری (۳)

Theory of computation - 002 - S07 - non-computability (3)

Mapping Reducibility

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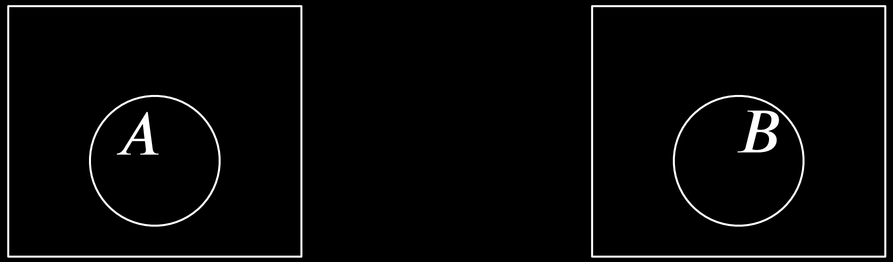

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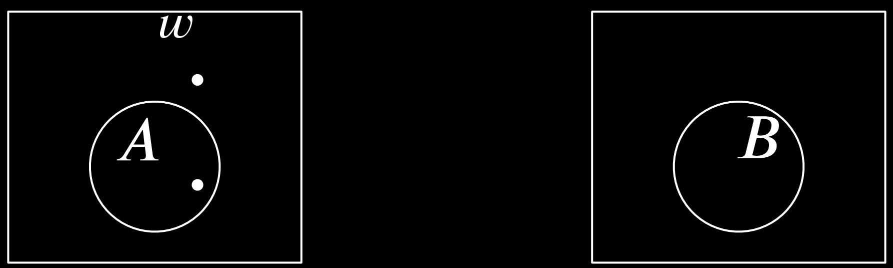

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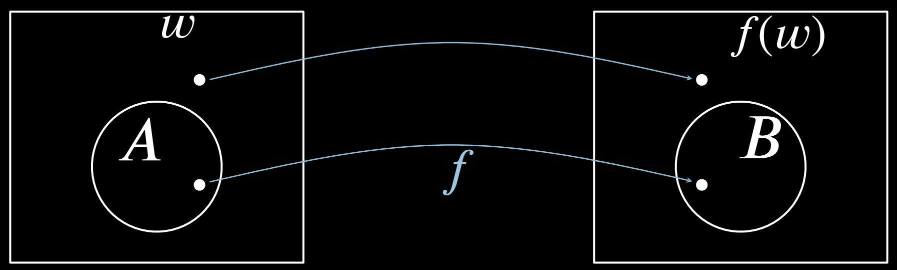

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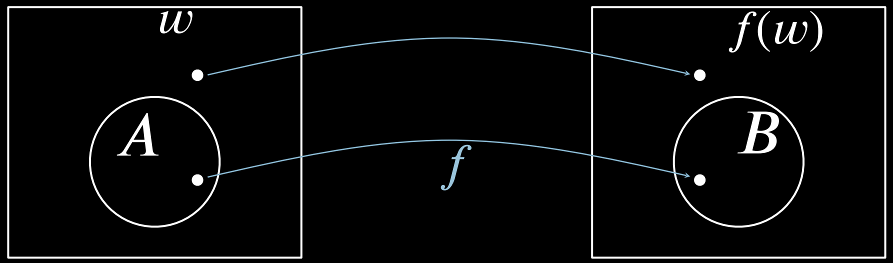

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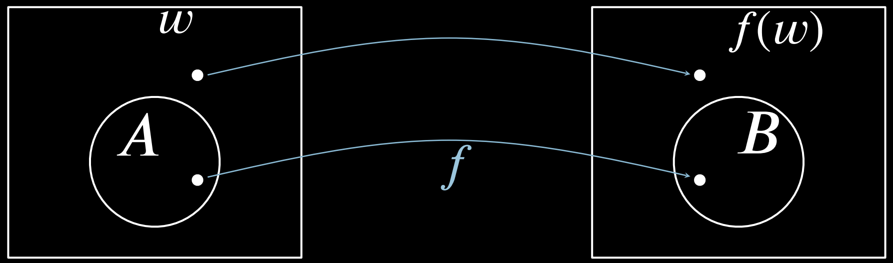

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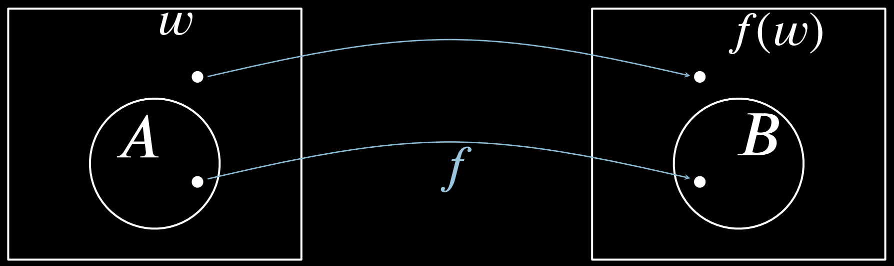

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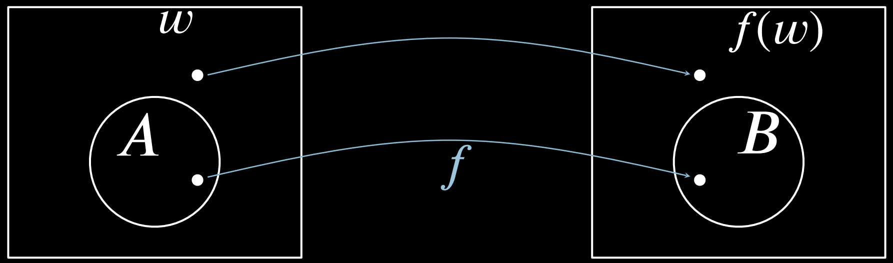

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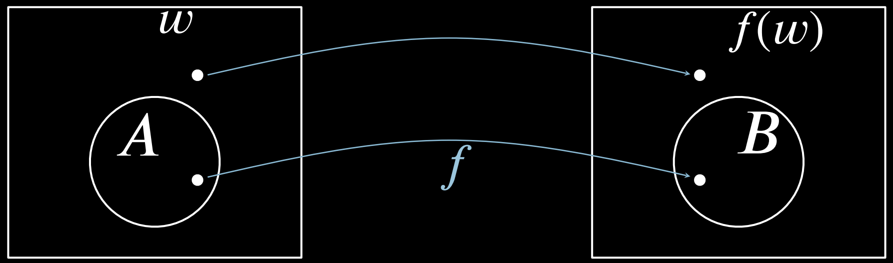

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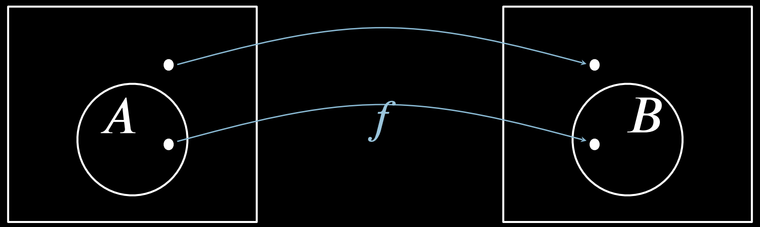

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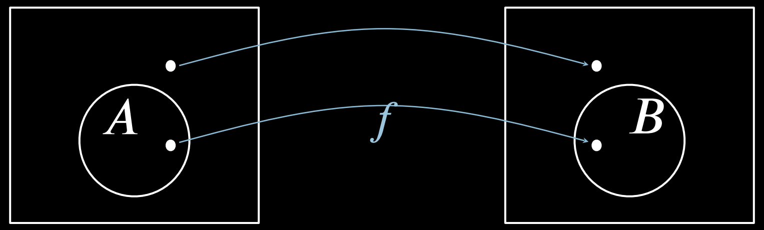

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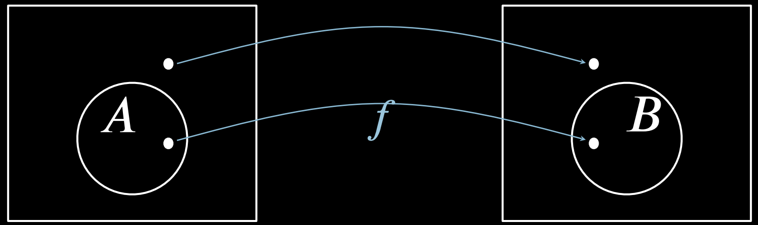

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Suppose $A \leq_{\mathrm{m}} B$.

What can we conclude?

Check all that apply.

(a) $B \leq_{\mathrm{m}} A$

(b) $\overline{A} \leq_{\mathrm{m}} \overline{B}$

(c) None of the above

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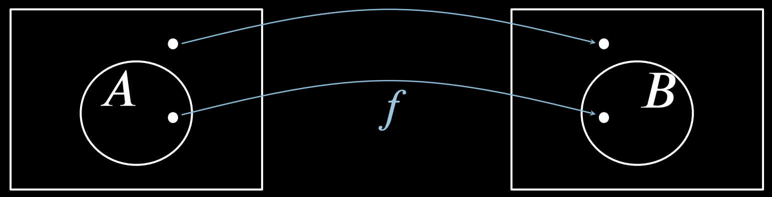

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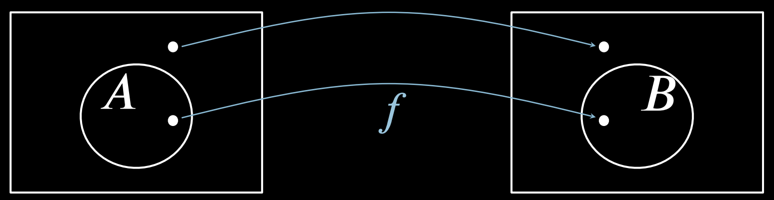

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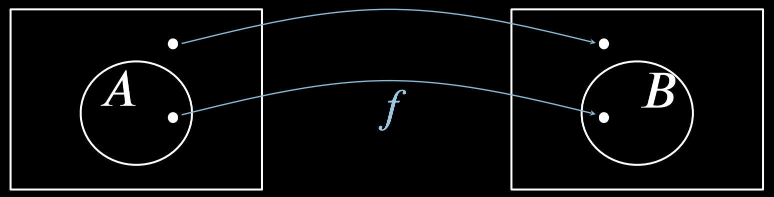

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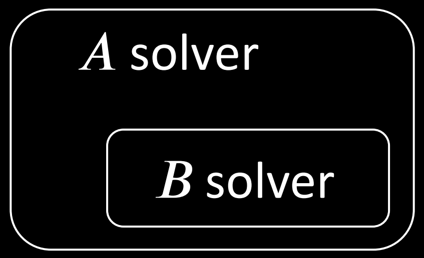

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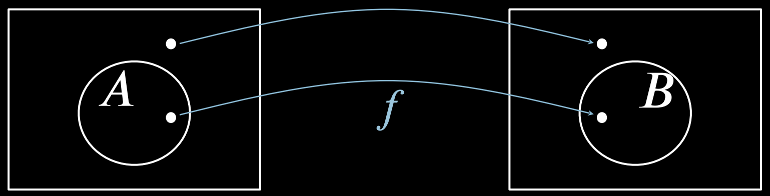

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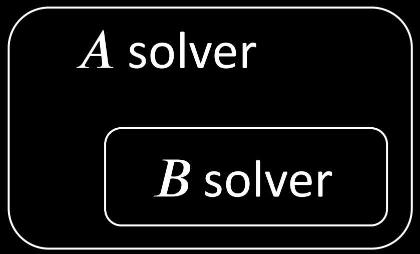

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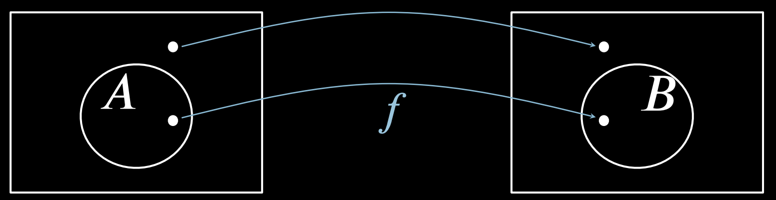

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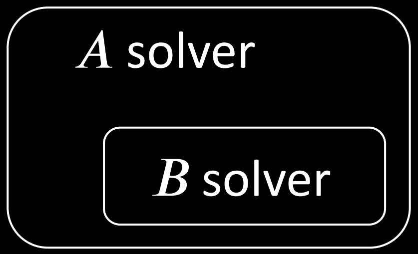

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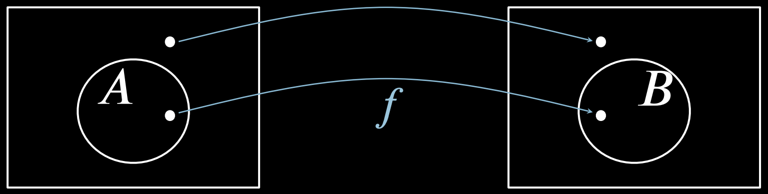

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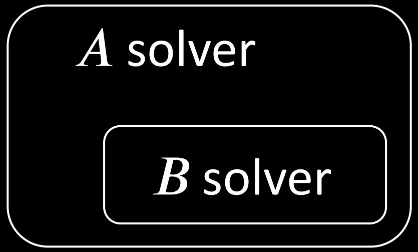

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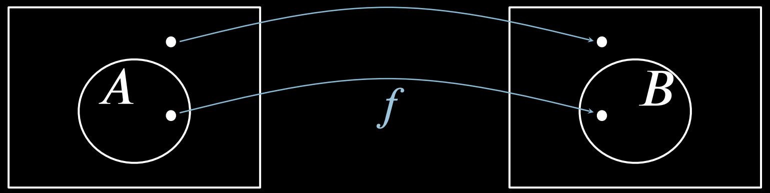

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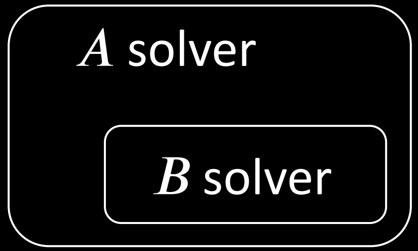

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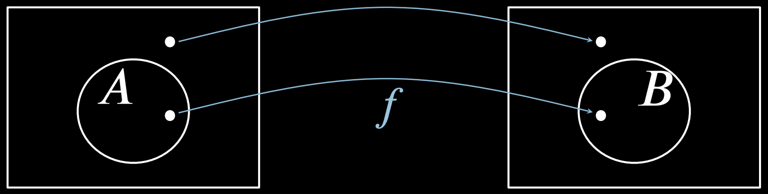

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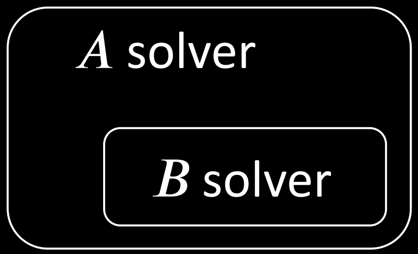

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We showed that if $A \leq_{\text{m}} B$ and B is T-recognizable then so is A.

Is the same true if we use general reducibility instead of mapping reducibility?

(a) Yes

(b) No

Check-in 9.3

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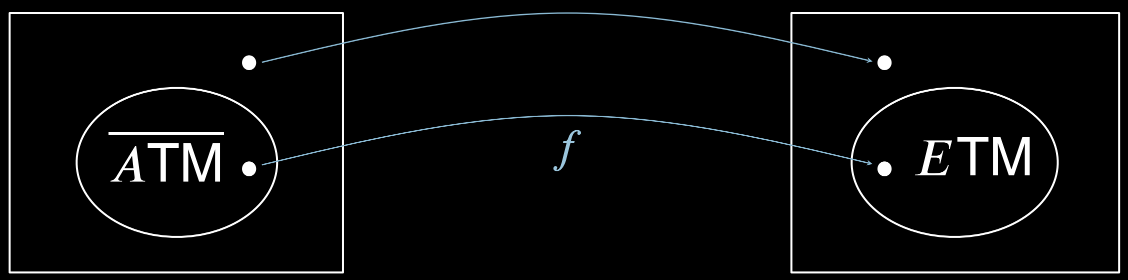

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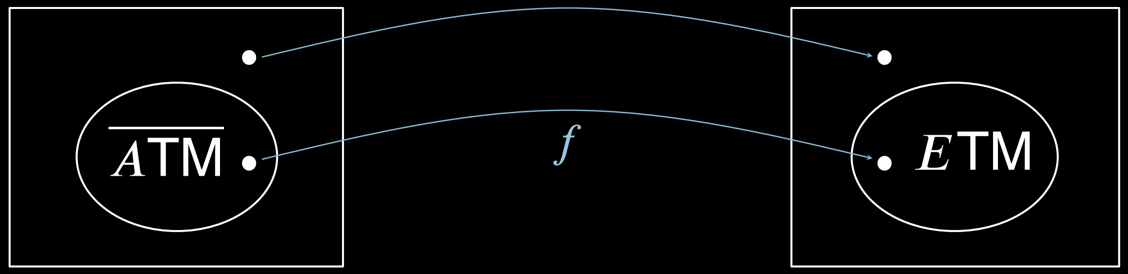

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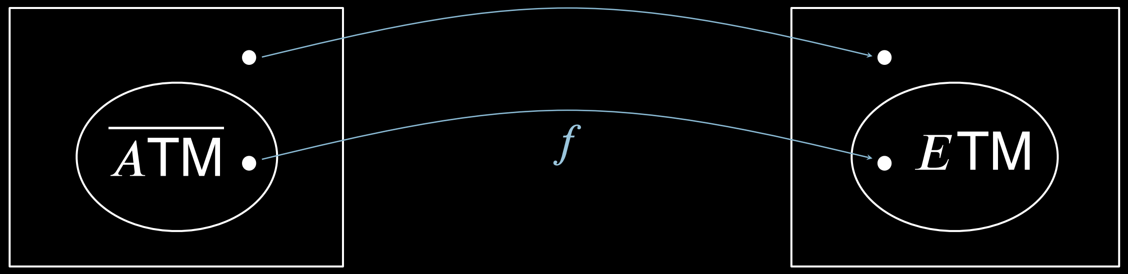

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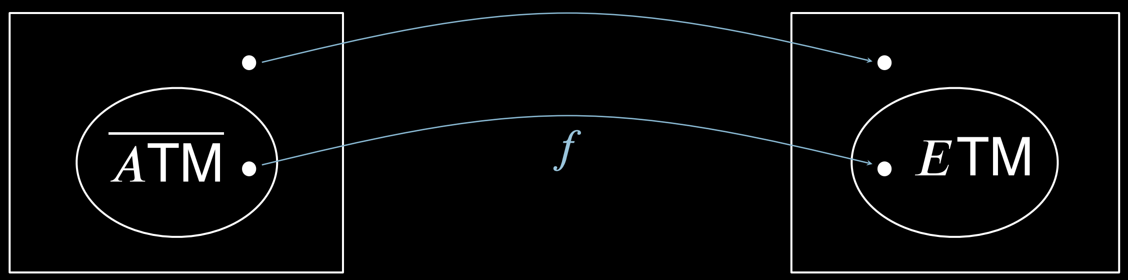

ETM is T-unrecognizable

Recall ETM $= \{\langle M \rangle \mid M$ is a TM and $L(M) = \varnothing\ \}$

Theorem: E**TM** is T-unrecognizable

Proof: Show $\overline{A\text{TM}} \leq_{\mathrm{m}} E$TM

Reduction function: $f(\langle M, w \rangle) = \langle M_w \rangle$

Explanation: $\langle M, w \rangle \in \overline{A\text{TM}}$ iff $\langle M_w \rangle \in E$TM

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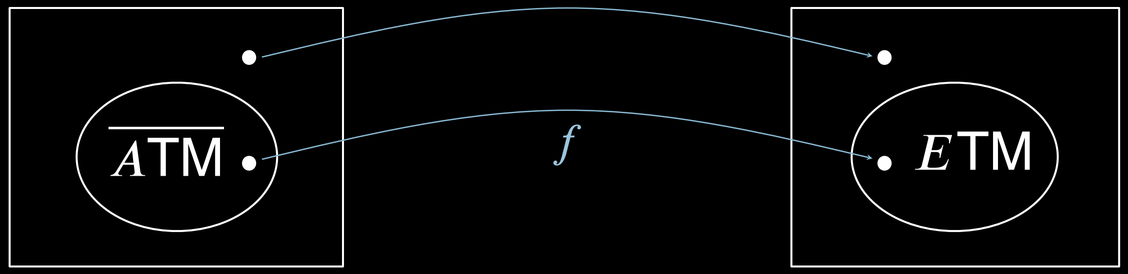

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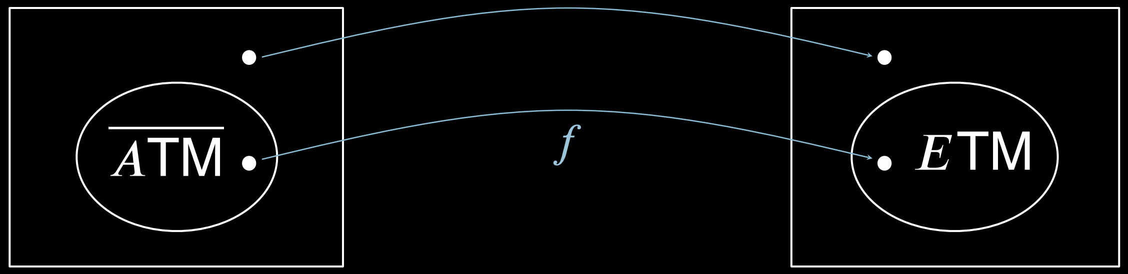

EQTM and $\overline{EQ\text{TM}}$ are T-unrecognizable

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Given a collection of pairs of strings as dominoes:

$$P = \left\{ \begin{bmatrix} t_1 \\ b_1 \end{bmatrix}, \quad \begin{bmatrix} t_2 \\ b_2 \end{bmatrix}, \quad \dots \quad , \begin{bmatrix} t_k \\ b_k \end{bmatrix} \right\}$$

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Match:

a b | a a | b a | a a |

a b a | a b a | a a | a b a |

Problem: Given P, is there a match?
Theorem: Undecidable!

Let $PCP = \{ \langle P \rangle \mid P \text{ has a match } \}$

Proof: Show ATM is reducible to PCP.

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Check-in 10.1

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Match:

a b	a a	b a	a a	a b a b

a b a	a b a	a a	a b a	b

✓

Check-in 10.1

Let

$$P_1 = \left\{ \begin{bmatrix} \text{aa} \\ \text{aaba} \end{bmatrix}, \begin{bmatrix} \text{ba} \\ \text{ab} \end{bmatrix}, \begin{bmatrix} \text{ab} \\ \text{ba} \end{bmatrix} \right\}$$

Does P_1 have a match?

(a) Yes.

(b) No.

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TM Configurations

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Defn: A configuration of a TM is a triple (q, p, t) where

q = the state,

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representing a snapshot of the TM at a point in time.

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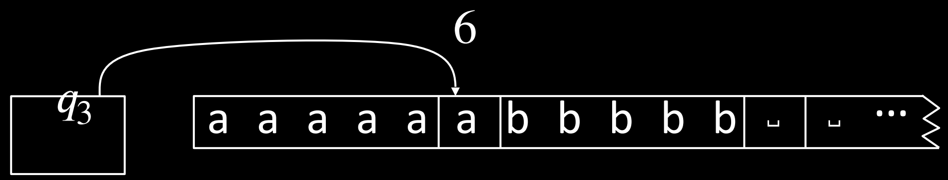

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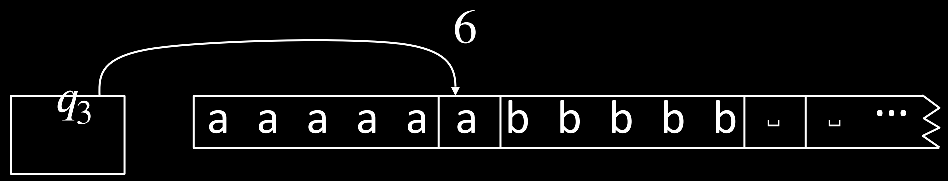

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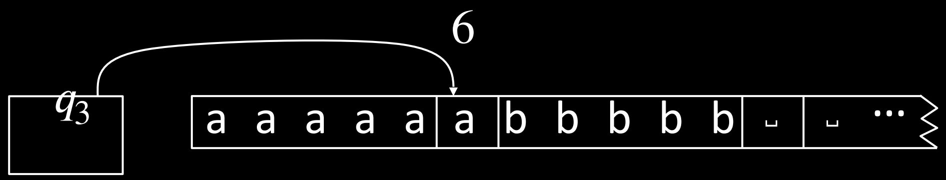

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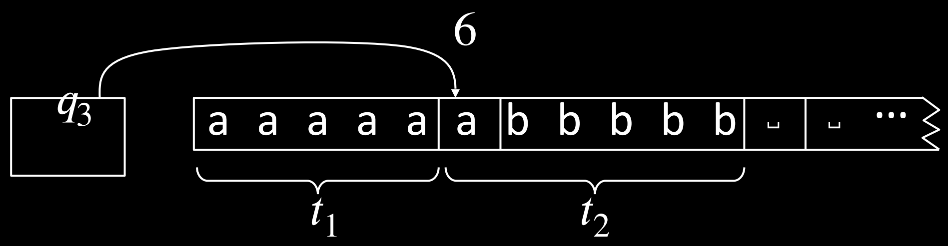

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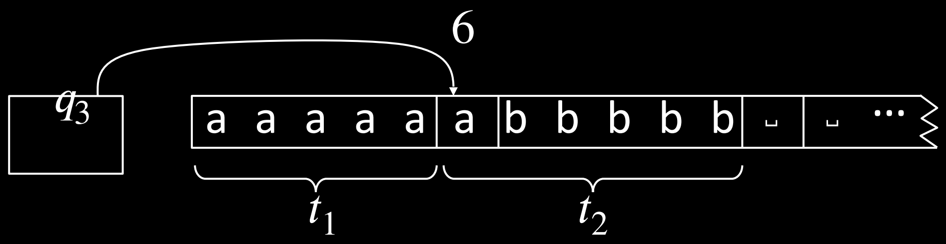

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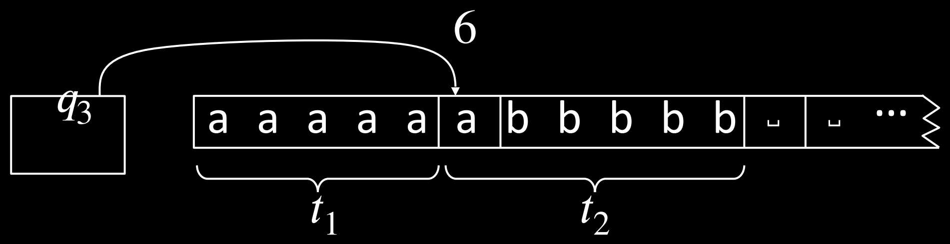

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A computation history for M on $w = w_1 w_2 \cdots w_n$.

Here say $\delta(q_0, w_1) = (q_7, \text{a}, \text{R})$ and $\delta(q_7, w_2) = (q_8, \text{c}, R)$.

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