بسم الله الرحمن الرحیم

# برنامه‌ریزی نیمه‌معین برای طراحی الگوریتم‌های تقریبی

## جلسه دهم: آیا برنامه‌ریزی هم‌مثبت الگوریتم سریع دارد؟

## Cone Programming

$$\text{(P)} \quad \begin{array}{ll} \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & \mathbf{x} \in K. \end{array}$$

الگوریتم سریع

## SDP

$$\begin{array}{ll} \text{maximize} & C \bullet X \\ \text{subject to} & A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m \\ & X \succeq 0. \end{array}$$

الگوریتم سریع

## LP

$$\begin{array}{ll} \text{maximize} & c^\top x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

الگوریتم سریع

# ماتریس هم‌مثبت و کاملا مثبت

# ماتریس هم‌مثبت

**7.1.1 Definition.** *A matrix $M \in \mathrm{SYM}_n$ is called copositive if*

$$\mathbf{x}^T M \mathbf{x} \geq 0 \textit{ for all } \mathbf{x} \geq 0.$$

$$\mathrm{COP}_n := \{M \in \mathrm{SYM}_n : \mathbf{x}^T M \mathbf{x} \geq 0 \textit{ for all } \mathbf{x} \geq 0\}$$

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$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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- بسته
- کنج
- محدب

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ماتریس M **کاملا مثبت**: اگر بتوان آن را به صورت زیر نوشت

$$M = \sum_{i=1}^{\ell} \mathbf{x}_i \mathbf{x}_i^T$$

که $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\ell \in \mathbb{R}^n_+$

$$M = \sum_{i=1}^{\ell} \mathbf{x}_i \mathbf{x}_i^T = AB$$

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$$\begin{array}{cc} & i \\ A = & \begin{pmatrix} \cdots \\ x_i[j] \\ \cdots \end{pmatrix} \end{array} \; j$$

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$$A = \begin{array}{cc} \mathrm{i} & \\ \left(\begin{array}{c} \cdots \\ x_i[j] \\ \cdots \end{array}\right) & \mathrm{j} \end{array}$$

$$A = (x_1 \quad x_2 \quad \cdots \quad x_t)$$

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$$A = \begin{array}{cc} i & \\ \left(\begin{array}{c} \cdots \\ x_i[j] \\ \cdots \end{array}\right) & j \end{array} \qquad B = \begin{array}{cc} k & \\ \left(\begin{array}{c} \cdots \\ x_i[k] \\ \cdots \end{array}\right) & i \end{array}$$

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$$A = \begin{pmatrix} x_1 & x_2 & \cdots & x_t \end{pmatrix} \qquad B = A^\top$$

**7.1.4 Definition.** *A matrix $M \in \mathrm{SYM}_n$ is called completely positive if for some $\ell$, there are $\ell$ nonnegative vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\ell \in \mathbb{R}^n_+$, such that*

$$M = \sum_{i=1}^{\ell} \mathbf{x}_i \mathbf{x}_i^T = AA^T, \tag{7.2}$$

*where $A \in \mathbb{R}^{n \times \ell}$ is the (nonnegative) matrix with columns $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\ell$.*

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برای کاملا مثبت بودن، تعداد ثابتی جمله کافی است.

**7.1.5 Lemma.** *$M$ is completely positive if and only if there are $\binom{n+1}{2}$ nonnegative vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{\binom{n+1}{2}} \in \mathbb{R}^n$ such that*

$$M = \sum_{i=1}^{\binom{n+1}{2}} \mathbf{x}_i \mathbf{x}_i^T.$$

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$$\mathrm{POS}_n \subseteq \mathrm{PSD}_n \subseteq \mathrm{COP}_n$$

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## کنج محدب <u>بسته</u> بودن ماتریس‌های کاملا مثبت

$$M^{(k)} = \sum_{i=1}^{\binom{n+1}{2}} \mathbf{x}_i^{(k)} {\mathbf{x}_i^{(k)}}^T = A^{(k)} {A^{(k)}}^T \in \mathrm{POS}_n$$

$$\lim_{k\to\infty} M^{(k)} = M \in \mathrm{SYM}_n$$

حکم: $M \in \mathrm{POS}_n$

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ستون i از $A^{(k)}$:

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$$m_{ii} = \lim_{k\to\infty} M_{ii}^{(k)} = \lim_{k\to\infty} {\mathbf{a}_i^{(k)}}^T \mathbf{a}_i^{(k)} = \lim_{k\to\infty} \|\mathbf{a}_i^{(k)}\|^2$$

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- حد بقیه درایه‌ها

$$m_{ij} = \lim_{k\to\infty} {\mathbf{a}_i^{(k)}}^T \mathbf{a}_j^{(k)} = \mathbf{a}_i^T \mathbf{a}_j$$

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- الف) $M \in \mathrm{COP}_n$ آنگاه $M \in \mathrm{POS}_n^*$
  - معادلا: $M \bullet X \geq 0$ برای هر $X \in \mathrm{POS}_n$

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  - معادلا: $M \bullet X \geq 0$ برای هر $X \in \mathrm{POS}_n$
  - $$\underbrace{M}_{\in \mathrm{COP}_n} \bullet \underbrace{\sum_{i=1}^{\ell} \mathbf{x}_i \mathbf{x}_i^T}_{\in \mathrm{POS}_n} = \sum_{i=1}^{\ell} M \bullet \mathbf{x}_i \mathbf{x}_i^T = \sum_{i=1}^{\ell} \underbrace{\mathbf{x}_i^T M \mathbf{x}_i}_{\geq 0}$$
- ب) $M \notin \mathrm{COP}_n$ آنگاه $M \notin \mathrm{POS}_n^*$

**7.1.7 Theorem.** $\mathrm{POS}_n^* = \mathrm{COP}_n$.

- الف) $M \in \mathrm{COP}_n$ آنگاه $M \in \mathrm{POS}_n^*$
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- ب) $M \notin \mathrm{COP}_n$ آنگاه $M \notin \mathrm{POS}_n^*$
  - بردار نامنفی x هست که $x^{\mathsf{T}} M x < 0$

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- ب) $M \notin \mathrm{COP}_n$ آنگاه $M \notin \mathrm{POS}_n^*$
  - بردار نامنفی x هست که $x^\top M x < 0$
  - بردار نامنفی x هست که $M \bullet x x^\top < 0$

**7.1.7 Theorem.** $\mathrm{POS}_n^* = \mathrm{COP}_n$.

$$\mathrm{POS}_n \subseteq \mathrm{PSD}_n \subseteq \mathrm{COP}_n$$

بسم الله الرحمن الرحیم

# برنامه‌ریزی نیمه‌معین برای طراحی الگوریتم‌های تقریبی

## جلسه یازدهم: آیا برنامه‌ریزی هم‌مثبت الگوریتم سریع دارد؟

Project: People

## Cone Programming

$$
\begin{array}{lll}
\text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\
& \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\
& & \mathbf{x} \in K.
\end{array}
$$

?

الگوریتم سریع

## SDP

$$
\begin{array}{ll}
\text{maximize} & C \bullet X \\
\text{subject to} & A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m \\
& X \succeq 0.
\end{array}
$$

الگوریتم سریع

## LP

$$
\begin{array}{ll}
\text{maximize} & c^\top x \\
\text{subject to} & Ax = b \\
& x \geq 0
\end{array}
$$

الگوریتم سریع

**7.1.1 Definition.** *A matrix $M \in \mathrm{SYM}_n$ is called copositive if*

$$\mathbf{x}^T M \mathbf{x} \geq 0 \text{ for all } \mathbf{x} \geq 0.$$

$$\mathrm{COP}_n := \{M \in \mathrm{SYM}_n : \mathbf{x}^T M \mathbf{x} \geq 0 \text{ for all } \mathbf{x} \geq 0\}$$

$$\mathrm{POS}_n \subseteq \mathrm{PSD}_n \subseteq \mathrm{COP}_n$$

**7.1.4 Definition.** *A matrix $M \in \mathrm{SYM}_n$ is called completely positive if for some $\ell$, there are $\ell$ nonnegative vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\ell \in \mathbb{R}^n_+$, such that*

$$M = \sum_{i=1}^{\ell} \mathbf{x}_i \mathbf{x}_i^T = AA^T, \tag{7.2}$$

*where $A \in \mathbb{R}^{n \times \ell}$ is the (nonnegative) matrix with columns $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\ell$.*

$$\mathrm{POS}_n := \{M \in \mathrm{SYM}_n : M \text{ is completely positive}\}$$

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# برنامه‌ریزی هم‌مثبت برای یک مسئله سخت!

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$$

الگوریتم سریع

## برنامه‌ریزی هم‌مثبت

$$
\begin{array}{ll}
\text{maximize} & C \bullet X \\
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 & X \in \mathrm{COP}_n
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\end{array}
$$

الگوریتم سریع

## برنامه‌ریزی کاملا مثبت

$$
\begin{array}{ll}
\text{maximize} & C \bullet X \\
\text{subject to} & A(X) = b \\
 & X \in \mathrm{POS}_n
\end{array}
$$

## LP

$$
\begin{array}{ll}
\text{maximize} & c^\top x \\
\text{subject to} & Ax = b \\
 & x \geq 0
\end{array}
$$

الگوریتم سریع

بیشترین نرخ ارسال با گراف G:

$$\sigma(G) = \sup\left\{\frac{1}{k}\log\alpha(G^k) : k \in \mathbb{N}\right\},$$

$$\Theta(G) = 2^{\sigma(G)} = \lim_{k\to\infty} \sqrt[k]{\alpha(G^k)}$$

$$\vartheta(G) := \min_{\mathcal{U}} \min_{\|\mathbf{c}\|=1} \max_{i=1}^{n} \frac{1}{(\mathbf{c}^T\mathbf{u}_i)^2}$$

قضیه:

$$\Theta(G) \leq \vartheta(G)$$

قضیه: برنامه‌ریزی زیر $\vartheta(G)$ را محاسبه می‌کند

$$\begin{array}{lll} \textit{Minimize} & t & \\ \textit{subject to} & y_{ij} = -1 & \textit{if } \{i,j\} \in \overline{E} \\ & y_{ii} = t-1 & \textit{for all } i = 1, \ldots, n \\ & Y \succeq 0. & \end{array}$$

**7.2.1 Theorem.** *The copositive program*

$$\begin{array}{ll} \textit{minimize} & t \\ \textit{subject to} & y_{ij} = -1, \ \textit{if } \{i,j\} \in \overline{E} \\ & y_{ii} = t-1, \ \textit{for all } i = 1, 2, \ldots, n \\ & Y \in \mathrm{COP}_n \end{array}$$

*has value $\alpha(G)$, the size of a maximum independent set in $G$.*

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$$\begin{array}{lll} \text{(D)} & \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\ & \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\ & & \mathbf{y} \in L^*. \end{array}$$

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$$\begin{pmatrix} Y & 0 \\ 0 & t \end{pmatrix}$$

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$$
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$$

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$$\in \mathrm{POS}_n$$

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$$\sum_i x_{ii} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,i):1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

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$$\max \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \bullet \begin{pmatrix} Y & 0 \\ 0 & t \end{pmatrix}$$

$$\begin{pmatrix} Y & 0 \\ 0 & t \end{pmatrix} \bullet \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,i):1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -1$$

$$\begin{pmatrix} Y & 0 \\ 0 & t \end{pmatrix} \bullet \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,j):1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -1 \quad ij \in \bar{E}$$

$$Y \in \mathrm{COP}_n$$

$$\sum_i x_{ii} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,i):1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\sum_{ij \in \bar{E}} x_{ij} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,j):1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \in \mathrm{POS}_n$$

$$\begin{array}{lll}(\mathrm{P}) & \text{Maximize} & \langle \mathbf{c}, \mathbf{x}\rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K.\end{array}$$

$$\begin{array}{lll}(\mathrm{D}) & \text{Minimize} & \langle \mathbf{b}, \mathbf{y}\rangle \\ & \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\ & & \mathbf{y} \in L^*.\end{array}$$

$$\max \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \bullet \begin{pmatrix} Y & 0 \\ 0 & t \end{pmatrix}$$

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$$\begin{pmatrix} Y & 0 \\ 0 & t \end{pmatrix} \bullet \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,j):1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -1 \quad ij \in \bar{E}$$

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$$\sum_i x_{ii} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,i):1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\sum_{ij \in \bar{E}} x_{ij} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,j):1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{POS}_n$$

$$\begin{array}{llll} \text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K. \end{array}$$

$$\begin{array}{llll} \text{(D)} & \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\ & \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\ & & \mathbf{y} \in L^*. \end{array}$$

$$\max \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \bullet \begin{pmatrix} Y & 0 \\ 0 & t \end{pmatrix}$$

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$$Y \in \mathrm{COP}_n$$

$$\sum_i x_{ii} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,i):1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \sum_{ij \in \bar{E}} x_{ij} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,j):1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{POS}_n$$

$$\begin{array}{lll} \text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K. \end{array}$$

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$$\begin{pmatrix} Y & 0 \\ 0 & t \end{pmatrix} \bullet \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,j):1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -1$$

$$ij \in \bar{E}$$

$$Y \in \mathrm{COP}_n$$

$$\sum_{ij \in \bar{E}} -x_{ij} + \sum_i -x_{ii}$$

$$\sum_i x_{ii} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,i):1 & 0 \\ 0 & 0 & -1 \end{pmatrix} +$$

$$\sum_{ij \in \bar{E}} x_{ij} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,j):1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{POS}_n$$

$$
\begin{array}{llll}
\text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\
& \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\
& & \mathbf{x} \in K.
\end{array}
$$

$$
\begin{array}{llll}
\text{(D)} & \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\
& \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\
& & \mathbf{y} \in L^*.
\end{array}
$$

$$
\max \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \bullet \begin{pmatrix} Y & 0 \\ 0 & t \end{pmatrix}
$$

$$
\begin{pmatrix} Y & 0 \\ 0 & t \end{pmatrix} \bullet \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,i):1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -1
$$

$$
\begin{pmatrix} Y & 0 \\ 0 & t \end{pmatrix} \bullet \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,j):1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -1 \quad ij \in \bar{E}
$$

$$
Y \in \mathrm{COP}_n
$$

$$
\min \sum_{ij \in \bar{E}} -x_{ij} + \sum_i -x_{ii}
$$

$$
\sum_i x_{ii} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,i):1 & 0 \\ 0 & 0 & -1 \end{pmatrix} +
$$

$$
\sum_{ij \in \bar{E}} x_{ij} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,j):1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{POS}_n
$$

$$
\begin{array}{llll}
(\mathrm{P}) & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\
 & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\
 & & \mathbf{x} \in K.
\end{array}
$$

$$
\begin{array}{llll}
(\mathrm{D}) & \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\
 & \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\
 & & \mathbf{y} \in L^*.
\end{array}
$$

$$
\min \sum_{ij \in \bar{E}} - x_{ij} + \sum_{i} - x_{ii}
$$

$$
\sum_{i} x_{ii} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,i):1 & 0 \\ 0 & 0 & -1 \end{pmatrix} +
$$

$$
\sum_{ij \in \bar{E}} x_{ij} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,j):1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{POS}_n
$$

(P) Maximize $\langle \mathbf{c}, \mathbf{x} \rangle$
subject to $\mathbf{b} - A(\mathbf{x}) \in L$
$\mathbf{x} \in K.$

(D) Minimize $\langle \mathbf{b}, \mathbf{y} \rangle$
subject to $A^T(\mathbf{y}) - \mathbf{c} \in K^*$
$\mathbf{y} \in L^*.$

قید آخر:

$$\min \sum_{ij \in \bar{E}} -x_{ij} + \sum_{i} -x_{ii}$$

$$\sum_{i} x_{ii} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,i):1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \sum_{ij \in \bar{E}} x_{ij} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,j):1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{POS}_n$$

$$\begin{array}{lll} \text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K. \end{array}$$

$$\begin{array}{lll} \text{(D)} & \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\ & \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\ & & \mathbf{y} \in L^*. \end{array}$$

قید آخر:

$$\sum_i - x_{ii} + 1 \geq 0$$

$$\min \sum_{ij \in \bar{E}} - x_{ij} + \sum_i - x_{ii}$$

$$\sum_i x_{ii} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,i):1 & 0 \\ 0 & 0 & -1 \end{pmatrix} +$$

$$\sum_{ij \in \bar{E}} x_{ij} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,j):1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{POS}_n$$

$$\begin{array}{lll} \text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K. \end{array}$$

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$$\sum_{ij \in \bar{E}} x_{ij} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (i,j):1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{POS}_n$$

$$
\begin{array}{llll}
(\mathrm{P}) & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\
& \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\
& & \mathbf{x} \in K.
\end{array}
$$

$$
\begin{array}{llll}
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$$\sum_{ij \in \bar{E}} x_{ij} \begin{pmatrix} 0 & 0 \\ 0 & (i,j):1 \end{pmatrix} \quad - \quad (0) \qquad \in \mathrm{POS}_n$$

$$\sum_i x_{ii} \leq 1$$

$$\begin{array}{lll} \text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K. \end{array}$$

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$$\max \sum_{ij \in \bar{E}} x_{ij} + \sum_i x_{ii}$$

$$\sum_i x_{ii} \begin{pmatrix} 0 & 0 \\ 0 & (i,i):1 \end{pmatrix} \quad + $$

$$\sum_{ij \in \bar{E}} x_{ij} \begin{pmatrix} 0 & 0 \\ 0 & (i,j):1 \end{pmatrix} \quad - \quad (0) \qquad \in \mathrm{POS}_n$$

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$$\begin{array}{lll} \text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K. \end{array}$$

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$$\max \quad J_n \bullet X$$

$$X \in \mathrm{POS}_n$$

$$\mathrm{Tr}(X) \leq 1$$

$$\max \sum_{ij \in \bar{E}} x_{ij} + \sum_i x_{ii}$$

$$\sum_i x_{ii} \begin{pmatrix} 0 & 0 \\ 0 & (i,i):1 \end{pmatrix} \quad +$$

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$$x_{ij} = 0 \qquad ij \in E$$

$$\text{(P)} \quad \begin{array}{ll} \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & \mathbf{x} \in K. \end{array}$$

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$$X \in \mathrm{POS}_n$$

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$$\max \sum_{ij \in \bar{E}} x_{ij} + \sum_i x_{ii} + \sum_{ij \in E} x_{ij}$$

$$\sum_i x_{ii} \begin{pmatrix} 0 & 0 \\ 0 & (i,i):1 \end{pmatrix} \quad +$$

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$$\sum_i x_{ii} \begin{pmatrix} 0 & 0 \\ 0 & (i,i):1 \end{pmatrix} + \sum_{ij \in E} x_{ij} \begin{pmatrix} 0 & 0 \\ 0 & (i,j):1 \end{pmatrix}$$

$$\sum_{ij \in \bar{E}} x_{ij} \begin{pmatrix} 0 & 0 \\ 0 & (i,j):1 \end{pmatrix} - (0) \quad \in \mathrm{POS}_n$$

$$\sum_i x_{ii} \leq 1$$

$$x_{ij} = 0 \qquad ij \in E$$

$$\begin{array}{lll} \text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K. \end{array}$$

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$$\sum_{ij \in \bar{E}} x_{ij} \begin{pmatrix} 0 & 0 \\ 0 & (i,j):1 \end{pmatrix} - (0) \qquad \in \mathrm{POS}_n$$

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$$\begin{array}{lll} \text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K. \end{array}$$

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$$\begin{array}{ll} \max & J_n \bullet X \\ & X \in \mathrm{POS}_n \\ & \mathrm{Tr}(X) = 1 \\ & x_{ij} = 0 \qquad ij \in E \end{array}$$

$\mathrm{Tr}(X) \le 1$

$\alpha(G) \leq$

**7.2.1 Theorem.** *The copositive program*

$$\begin{array}{ll} \textit{minimize} & t \\ \textit{subject to} & y_{ij} = -1, \ \textit{if } \{i,j\} \in \overline{E} \\ & y_{ii} = t-1, \ \textit{for all } i = 1,2,\ldots,n \\ & Y \in \mathrm{COP}_n \end{array}$$

*has value $\alpha(G)$, the size of a maximum independent set in G.*

- روش: یک جواب شدنی برای دوگان با مقدار $\alpha(G)$
- I: یکی از بزرگ‌ترین مجموعه‌های مستقل

$$\begin{array}{lll} \max & J_n \bullet X & \\ & X \in \mathrm{POS}_n & \\ & \mathrm{Tr}(X) = 1 & \\ & x_{ij} = 0 & ij \in E \end{array}$$

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- روش: یک جواب شدنی برای دوگان با مقدار $\alpha(G)$
- I: یکی از بزرگ‌ترین مجموعه‌های مستقل

$\tilde{x}_i = 1_{[i \in I]}$

$$\tilde{X} = \frac{1}{\alpha(G)} \tilde{\mathbf{x}} \tilde{\mathbf{x}}^T$$

$$\begin{array}{lll} \max & J_n \bullet X & \\ & X \in \mathrm{POS}_n & \\ & \mathrm{Tr}(X) = 1 & \\ & x_{ij} = 0 & ij \in E \end{array}$$

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- روش: یک جواب شدنی برای دوگان با مقدار $\alpha(G)$
- I: یکی از بزرگ‌ترین مجموعه‌های مستقل

$$\tilde{x}_i = 1_{[i \in I]}$$

$$\tilde{X} = \frac{1}{\alpha(G)} \tilde{\mathbf{x}} \tilde{\mathbf{x}}^T \quad \tilde{x}_{ij} = \frac{1}{\alpha(G)} \tilde{x}_i \tilde{x}_j$$

$$\begin{array}{lll} \max & J_n \bullet X & \\ & X \in \mathrm{POS}_n & \\ & \mathrm{Tr}(X) = 1 & \\ & x_{ij} = 0 & ij \in E \end{array}$$

$\alpha(G) \leq$

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- روش: یک جواب شدنی برای دوگان با مقدار $\alpha(G)$
- I: یکی از بزرگ‌ترین مجموعه‌های مستقل

$$\tilde{x}_i = 1_{[i \in I]}$$

$$\tilde{X} = \frac{1}{\alpha(G)} \tilde{\mathbf{x}}\tilde{\mathbf{x}}^T \quad \tilde{x}_{ij} = \frac{1}{\alpha(G)} \tilde{x}_i \tilde{x}_j$$

$$\begin{array}{lll} \max & J_n \bullet X & \\ & X \in \mathrm{POS}_n & \\ & \mathrm{Tr}(X) = 1 & \\ & x_{ij} = 0 & ij \in E \end{array}$$

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$$Y = tI_n + Z - J_n$$

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صفر روی درایه‌های بدون یال

$$Y = tI_n + Z - J_n$$

$$z = \max_{i,j}(Z_{i,j})$$

$$Y' = tI_n + zA_G - J_n$$

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**7.2.5 Lemma.** *The copositive program*

$$\begin{array}{ll} \text{Minimize} & t \\ \text{subject to} & tI_n + zA_G - J_n \in \mathrm{COP}_n \\ & t, z \in \mathbb{R} \end{array}$$

جوابش برابر با جواب برنامه‌ریزی بالاست.

## قضیه Motzkin–Straus:

**7.2.6 Theorem.** *For every graph $G$,*

$$\frac{1}{\alpha(G)} = \min\{\mathbf{x}^T(A_G + I_n)\mathbf{x} : \mathbf{x} \geq \mathbf{0}, \sum_{i=1}^{n} x_i = 1\}.$$

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- نرم ۱ = ۱، کنج مثبت، $f(\dfrac{1}{\alpha(G)}\tilde{x}) = \dfrac{1}{\alpha(G)}$

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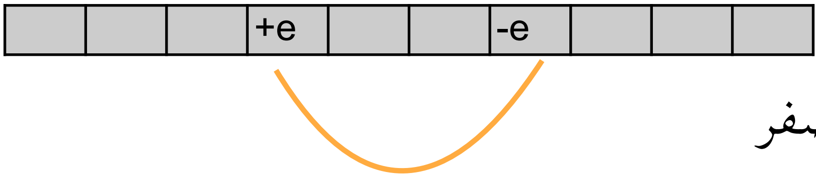

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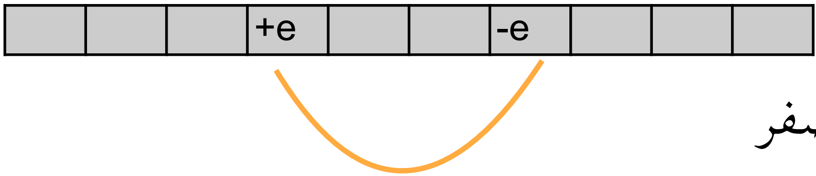


$$z_k = \begin{cases} x_k^* + \varepsilon & \text{if } k = i \\ x_k^* - \varepsilon & \text{if } k = j \\ x_k^* & \text{otherwise,} \end{cases}$$

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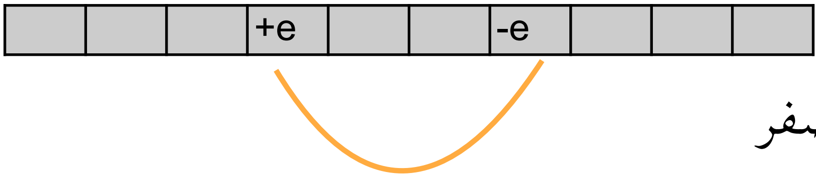


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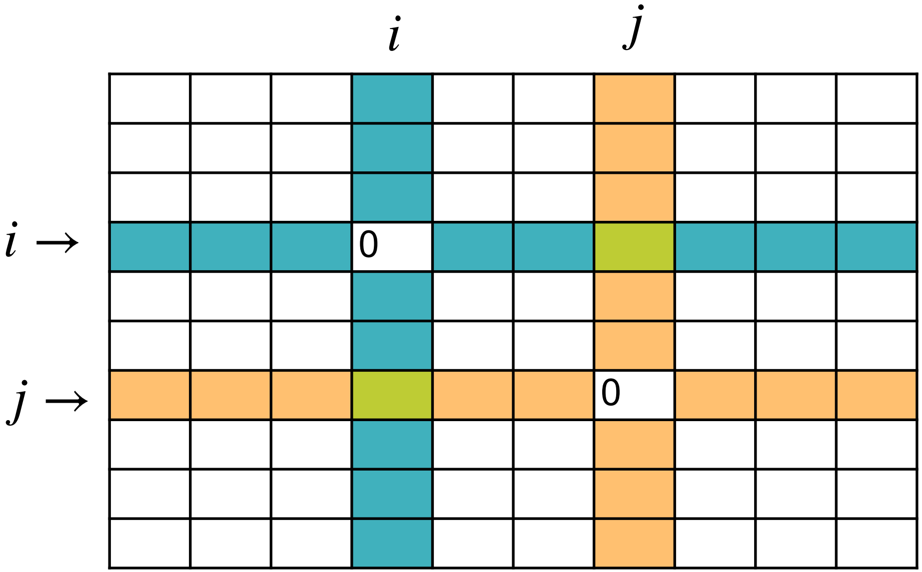

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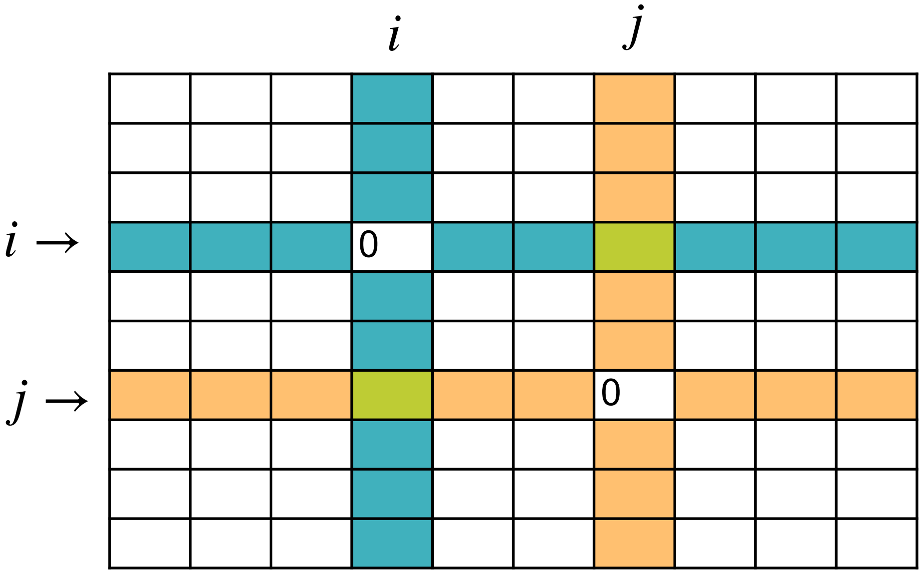


$$f(z) = \epsilon \mathrm{B} - \epsilon \mathrm{O} + 2(x_i + \epsilon)(x_j - \epsilon) +$$

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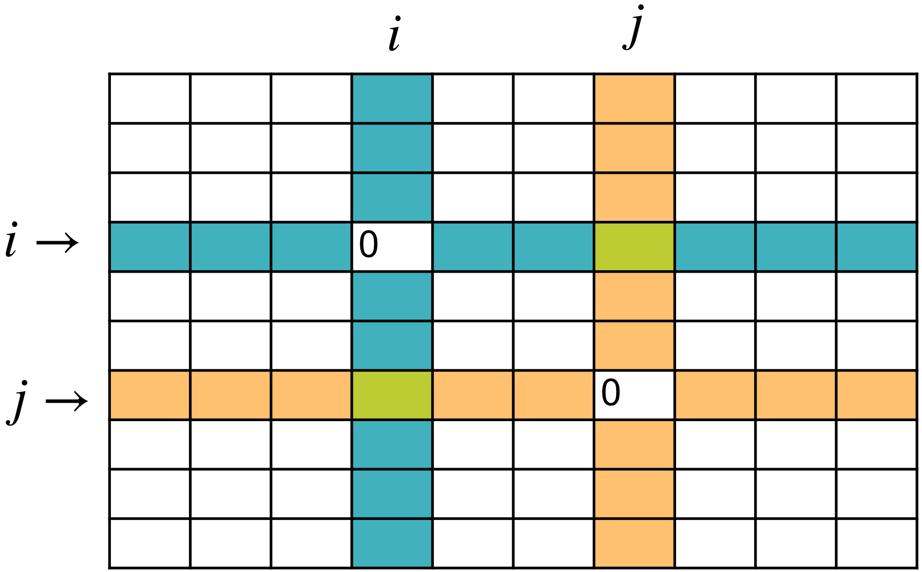


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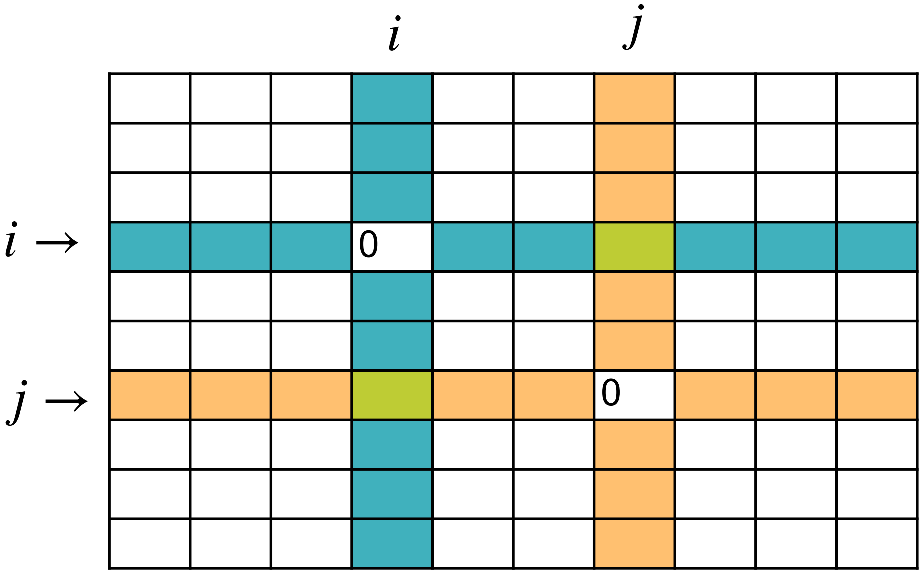


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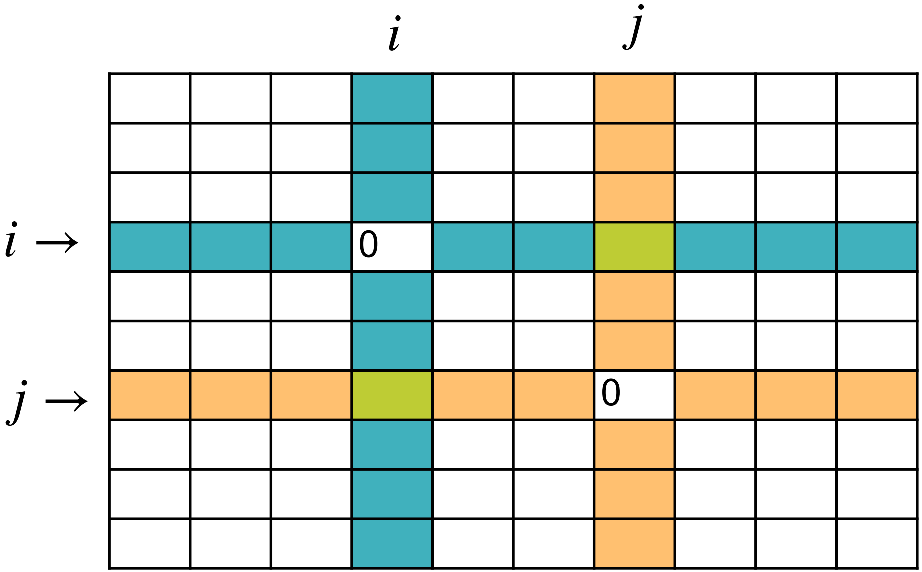


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می‌توان یکی از درایه‌های $\mathrm{x}^*$ را صفر کرد

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می‌توان یکی از درایه‌های x* را صفر کرد

تناقض !

## قضیه Motzkin–Straus:

**7.2.6 Theorem.** *For every graph $G$,*

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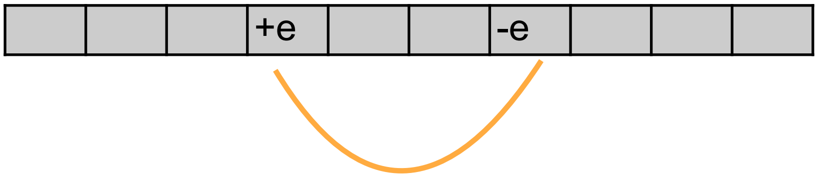


- $x^*$: جواب بهینه
- با بیشترین صفر
- یال i و j که دو سرشان مثبت است

$$f(z) = f(x^*) + l(\epsilon) \qquad z_k = \begin{cases} x_k^* + \varepsilon & \text{if } k = i \\ x_k^* - \varepsilon & \text{if } k = j \\ x_k^* & \text{otherwise,} \end{cases}$$

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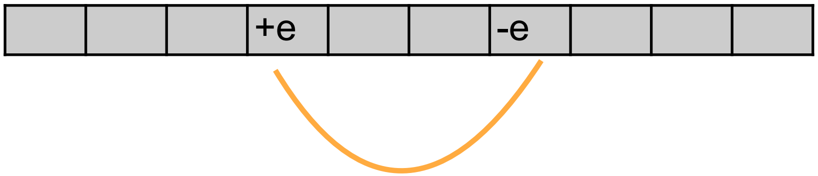


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نداریم

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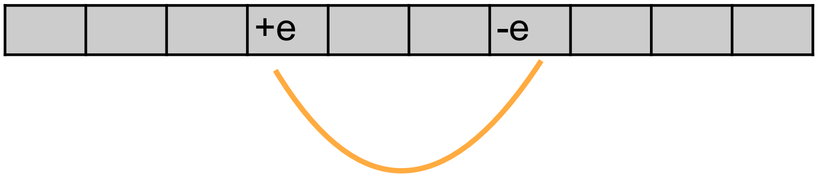

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  - ...

**7.2.5 Lemma.** *The copositive program*

$$\begin{array}{ll} \text{Minimize} & t \\ \text{subject to} & tI_n + zA_G - J_n \in \mathrm{COP}_n \\ & t, z \in \mathbb{R} \end{array}$$

جوابش برابر با جواب برنامه‌ریزی بالاست.

$\alpha$ $\leq \alpha(G)$

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برای هر x روی سادک:

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هم‌مثبت: $\tilde{Y} := \alpha(G)I_n + \alpha(G)A_G - J_n$

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جوابش برابر با جواب برنامه‌ریزی بالاست.

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## برنامه‌ریزی کاملا مثبت سخت است!

$$\begin{array}{lll} \max & J_n \bullet X & \\ & X \in \mathrm{POS}_n & \\ & \mathrm{Tr}(X) = 1 & \\ & x_{ij} = 0 & ij \in E \end{array}$$

# تقریب‌ناپذیری مسئله مجموعه مستقل

- تحت فرض‌های خوبی $(\mathrm{NP} \not\subseteq \mathrm{ZPP})$
- هیچ الگوریتم تقریبی برای مسئله بزرگ‌تری مجموعه مستقل
- با ضریب تقریب $n^{1-\epsilon}$ برای هیچ $\epsilon > 0$ وجود ندارد.

**Cone Programming**

$$\begin{array}{lll} (\mathrm{P}) & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K. \end{array}$$

الگوریتم سریع ?

**برنامه‌ریزی هم‌مثبت**

$$\begin{array}{ll} \text{maximize} & C \bullet X \\ \text{subject to} & A(X) = b \\ & X \in \mathrm{COP}_n \end{array}$$

**SDP**

$$\begin{array}{lll} \text{maximize} & C \bullet X & \\ \text{subject to} & A_i \bullet X = b_i, & i = 1, 2, \ldots, m \\ & X \succeq 0. & \end{array}$$

الگوریتم سریع

**برنامه‌ریزی کاملا مثبت**

$$\begin{array}{ll} \text{maximize} & C \bullet X \\ \text{subject to} & A(X) = b \\ & X \in \mathrm{POS}_n \end{array}$$

**LP**

$$\begin{array}{ll} \text{maximize} & c^\top x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

الگوریتم سریع

پایان