بسم الله الرحمن الرحیم

یادگیری برخط

جلسه بیست و دوم:
بندیت ترکیبیاتی (۲)

کاهش آینه‌ای

$$a_1 = \operatorname{argmin}_{a \in \mathcal{A}} F(a)$$

$$a_{t+1} = \operatorname{argmin}_{a \in \mathcal{A}} \left(\eta \langle a, y_t \rangle + D_F(a, a_t)\right)$$

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قضیه:

$$R_n(a) \leq \frac{F(a) - F(a_1)}{\eta} + \sum_{t=1}^{n} \langle a_t - a_{t+1}, y_t \rangle - \frac{1}{\eta} \sum_{t=1}^{n} D(a_{t+1}, a_t).$$

$$R_n(a) \leq \frac{1}{\eta} \left(F(a) - F(a_1) + \sum_{t=1}^{n} D(a_t, \tilde{a}_{t+1}) \right)$$

الگوریتم کاهش آینه‌ای/پیروی از پیش‌روی منظم شده برای بندیت

Input Legendre potential F, action set $\mathcal{A}$ and learning rate $\eta > 0$

Choose $\bar{A}_1 = \operatorname{argmin}_{a \in \mathcal{A} \cap \mathrm{dom}(F)} F(a)$

for $t = 1, \ldots, n$ **do**

  Choose measure P_t on $\mathcal{A}$ with mean $\bar{A}_t$

  Sample action A_t from P_t and observe $\langle A_t, y_t \rangle$

  Compute estimate $\hat{Y}_t$ of the loss vector y_t

  Update:

$$\bar{A}_{t+1} = \operatorname{argmin}_{a \in \mathcal{A} \cap \mathrm{dom}(F)} \eta \langle a, \hat{Y}_t \rangle + D_F(a, \bar{A}_t) \qquad \text{(Mirror descent)}$$

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end for

THEOREM 28.10 (Regret of Mirror-Descent and FTRL with bandit feedback). *Suppose that Algorithm 16 is run with Legendre potential F, convex action set $\mathcal{A} \subset \mathbb{R}^d$ and learning rate $\eta > 0$ such that the loss estimators are unbiased: $\mathbb{E}[\hat{Y}_t \mid \bar{A}_t] = y_t$ for all $t \in [n]$. Then the regret for either variant of Algorithm 16, provided that they are well defined, is bounded by*

$$R_n(a) \leq \mathbb{E}\left[\frac{F(a) - F(\bar{A}_1)}{\eta} + \sum_{t=1}^{n} \langle \bar{A}_t - \bar{A}_{t+1}, \hat{Y}_t \rangle - \frac{1}{\eta} \sum_{t=1}^{n} D(\bar{A}_{t+1}, \bar{A}_t)\right].$$

مجموعه اعمال	بندیت ترکیبیاتی	بندیت معمولی
	$\mathcal{A} \subseteq \{a \in \{0,1\}^d : \|a\|_1 \leq m\}$	

بندیت ترکیبیاتی	بندیت معمولی	
$\mathcal{A} \subseteq \{a \in \{0,1\}^d : \|a\|_1 \leq m\}$		مجموعه اعمال
	$y_t \in \{y : \sup_{a \in \mathcal{A}} \lvert\langle a, y\rangle\rvert \leq 1\}$	

مجموعه اعمال

بندیت ترکیبیاتی

$$\mathcal{A} \subseteq \{a \in \{0,1\}^d : \|a\|_1 \leq m\}$$

$$y_t \in [0,1]^d$$

$$\Downarrow$$

$$|\langle A_t, y_t \rangle| \leq m$$

بندیت معمولی

$$y_t \in \{y : \sup_{a \in \mathcal{A}} |\langle a, y \rangle| \leq 1\}$$

	بندیت ترکیباتی	بندیت معمولی
مجموعه اعمال	$\mathcal{A} \subseteq \{a \in \{0,1\}^d : \|a\|_1 \leq m\}$ $y_t \in [0,1]^d$ $\Downarrow$ $\lvert\langle A_t, y_t\rangle\rvert \leq m$	$y_t \in \{y : \sup_{a\in\mathcal{A}} \lvert\langle a, y\rangle\rvert \leq 1\}$
بازخورد		$\langle A_t, y_t\rangle$

	بندیت ترکیبیاتی	بندیت معمولی
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بازخورد	$(A_{t1}y_{t1}, \ldots, A_{td}y_{td})$ نیمه‌ـ بندیت	$\langle A_t, y_t\rangle$

	بندیت ترکیبیاتی	بندیت معمولی
مجموعه اعمال	$\mathcal{A} \subseteq \{a \in \{0,1\}^d : \|a\|_1 \leq m\}$ $y_t \in [0,1]^d$ $\Downarrow$ $\lvert\langle A_t, y_t\rangle\rvert \leq m$	$y_t \in \{y : \sup_{a\in\mathcal{A}} \lvert\langle a, y\rangle\rvert \leq 1\}$
بازخورد	$(A_{t1}y_{t1}, \ldots, A_{td}y_{td})$ نیمه_ بندیت	$\langle A_t, y_t\rangle$

$$R_n = \max_{a\in\mathcal{A}} \mathbb{E}\left[\sum_{t=1}^{n} \langle A_t - a, y_t\rangle\right]$$

	بندیت ترکیباتی	بندیت معمولی
مجموعه اعمال	$\mathcal{A} \subseteq \{a \in \{0,1\}^d : \|a\|_1 \leq m\}$ $y_t \in [0,1]^d$ $\Downarrow$ $\lvert\langle A_t, y_t \rangle\rvert \leq m$	$y_t \in \{y : \sup_{a \in \mathcal{A}} \lvert\langle a, y \rangle\rvert \leq 1\}$
بازخورد	$(A_{t1}y_{t1}, \ldots, A_{td}y_{td})$ نیمه‌ـ بندیت	$\langle A_t, y_t \rangle$
	$R_n = \max_{a \in \mathcal{A}} \mathbb{E}\left[\sum_{t=1}^{n} \langle A_t - a, y_t \rangle\right]$	$R_n = \max_{a \in \mathcal{A}} \mathbb{E}\left[\sum_{t=1}^{n} \langle A_t - a, y_t \rangle\right]$

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پشیمانی	$R_n = \max_{a\in\mathcal{A}} \mathbb{E}\left[\sum_{t=1}^{n} \langle A_t - a, y_t\rangle\right]$	$R_n = \max_{a\in\mathcal{A}} \mathbb{E}\left[\sum_{t=1}^{n} \langle A_t - a, y_t\rangle\right]$

صورت ۲ : نیمه‌ـ بندیت + کاهش آینه‌ای

بازخورد: $(A_{t1}y_{t1}, \ldots, A_{td}y_{td})$

صورت ۲: نیمه_بندیت + کاهش آینه‌ای

بازخورد: $(A_{t1}y_{t1}, \ldots, A_{td}y_{td})$

$$\hat{Y}_{ti} = \frac{A_{ti}y_{ti}}{\bar{A}_{ti}}$$

$$\bar{A}_{ti} = \mathbb{E}[A_{ti} \,|\, \mathcal{F}_{t-1}]$$

صورت ۲ : نیمه‌ـ بندیت + کاهش آینه‌ای

بازخورد: $(A_{t1}y_{t1}, \ldots, A_{td}y_{td})$

$$\Leftarrow \quad \hat{Y}_{ti} = \frac{A_{ti}y_{ti}}{\bar{A}_{ti}}$$

$$\bar{A}_{ti} = \mathbb{E}[A_{ti} \mid \mathcal{F}_{t-1}]$$

صورت ۲ : نیمه_بندیت + کاهش آینه‌ای

بازخورد: $(A_{t1}y_{t1}, \dots, A_{td}y_{td})$

$$\hat{Y}_{ti} = \frac{A_{ti}y_{ti}}{\bar{A}_{ti}} \Rightarrow \mathbb{E}[\hat{Y}_{ti}] = \sum_{a \in A} P[a]\frac{a_i y_{ti}}{\bar{A}_{ti}}$$

$$\bar{A}_{ti} = \mathbb{E}[A_{ti} \mid \mathcal{F}_{t-1}]$$

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$$\hat{Y}_{ti} = \frac{A_{ti}y_{ti}}{\bar{A}_{ti}}$$

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$$= \frac{y_{ti}}{\bar{A}_{ti}} \sum_{a \in A} P[a] a_i$$

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$$\mathbb{E}[\hat{Y}_t \mid \mathcal{F}_{t-1}] = y_t$$

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end for

Input $\mathcal{A}, \eta, F$

$\bar{A}_1 = \operatorname{argmin}_{a \in \operatorname{co}(\mathcal{A})} F(a)$

for $t = 1, \dots, n$ **do**

Choose distribution P_t on $\mathcal{A}$ such that $\sum_{a \in \mathcal{A}} P_t(a) a = \bar{A}_t$

Sample $A_t \sim P_t$ and observe $A_{t1} y_{t1}, \dots, A_{td} y_{td}$

Compute $\hat{Y}_{ti} = A_{ti} y_{ti} / \bar{A}_{ti}$ for all $i \in [d]$

Update $\bar{A}_{t+1} = \operatorname{argmin}_{a \in \operatorname{co}(\mathcal{A})} \eta \langle a, \hat{Y}_t \rangle + D_F(a, \bar{A}_t)$

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$$F(a) = \sum_{i=1}^{d} (a_i \log(a_i) - a_i)$$

الگوریتم کاهش آینه‌ای

$$R_n \leq \frac{\mathrm{diam}_F(\mathrm{co}(\mathcal{A}))}{\eta} + \mathbb{E}\left[\sum_{t=1}^{n} \langle \bar{A}_t - \bar{A}_{t+1}, \hat{Y}_t \rangle - \frac{1}{\eta} D_F(\bar{A}_{t+1}, \bar{A}_t)\right]$$

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$$\operatorname{diam}_F(\operatorname{co}(\mathcal{A})) \leq \sup_{a \in \operatorname{co}(\mathcal{A})} \sum_{i=1}^{d} \left(a_i + a_i \log\left(\frac{1}{a_i}\right)\right)$$

$$F(a) = \sum_{i=1}^{d} (a_i \log(a_i) - a_i)$$

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$$\leq m(1 + \log(d/m)).$$

$$F(a) = \sum_{i=1}^{d} (a_i \log(a_i) - a_i)$$

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$$\leq m(1 + \log(d/m))/\eta$$

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$\leq m(1 + \log(d/m))/\eta$

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$$\hat{Y}'_{ti} = \hat{Y}_{ti} \mathbb{I}\left\{\bar{A}_{t+1,i} \leq \bar{A}_{ti}\right\}$$

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$$\le \langle \bar{A}_t - \bar{A}_{t+1}, \hat{Y}'_t \rangle - \frac{1}{\eta} D_F(\bar{A}_{t+1}, \bar{A}_t)$$

$$\le \frac{\eta}{2} \|\hat{Y}'_t\|^2_{\nabla^2 F(Z_t)^{-1}}$$

$$F(a) = \sum_{i=1}^{d} (a_i \log(a_i) - a_i)$$

الگوریتم کاهش آینه‌ای

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$$\le \langle \bar{A}_t - \bar{A}_{t+1}, \hat{Y}'_t \rangle - \frac{1}{\eta} D_F(\bar{A}_{t+1}, \bar{A}_t)$$

$$\le \frac{\eta}{2} \|\hat{Y}'_t\|^2_{\nabla^2 F(Z_t)^{-1}} = \frac{\eta}{2} \hat{Y}_t^{\prime\top} \nabla^2 F(Z)^{-1} \hat{Y}'_t$$

$$F(a) = \sum_{i=1}^{d} (a_i \log(a_i) - a_i)$$

الگوریتم کاهش آینه‌ای

$$R_n \leq \frac{\operatorname{diam}_F(\operatorname{co}(\mathcal{A}))}{\eta} + \mathbb{E}\left[\sum_{t=1}^{n} \langle \bar{A}_t - \bar{A}_{t+1}, \hat{Y}_t \rangle - \frac{1}{\eta} D_F(\bar{A}_{t+1}, \bar{A}_t)\right]$$

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$$\leq \langle \bar{A}_t - \bar{A}_{t+1}, \hat{Y}'_t \rangle - \frac{1}{\eta} D_F(\bar{A}_{t+1}, \bar{A}_t)$$

$$\leq \frac{\eta}{2} \|\hat{Y}'_t\|^2_{\nabla^2 F(Z_t)^{-1}} = \frac{\eta}{2} \hat{Y}'^{\top}_t \nabla^2 F(Z)^{-1} \hat{Y}'_t$$

$$= \frac{\eta}{2} \sum_i \hat{Y}'_{t,i} Z_i \hat{Y}'_{t,i}$$

$$\nabla F(x) = \log(x) - 1$$

$$\nabla^2 F(x) = \operatorname{diam}(1/x)$$

$$F(a) = \sum_{i=1}^{d} \left(a_i \log(a_i) - a_i\right)$$

الگوریتم کاهش آینه‌ای

$$R_n \leq \frac{\operatorname{diam}_F(\operatorname{co}(\mathcal{A}))}{\eta} + \mathbb{E}\left[\sum_{t=1}^{n} \langle \bar{A}_t - \bar{A}_{t+1}, \hat{Y}_t \rangle - \frac{1}{\eta} D_F(\bar{A}_{t+1}, \bar{A}_t)\right]$$

$$\leq m(1 + \log(d/m))/\eta$$

$$\bar{A}_{ti} = \mathbb{E}[A_{ti} \,|\, \mathcal{F}_{t-1}]$$

$$\hat{Y}'_{ti} = \hat{Y}_{ti} \mathbb{I}\left\{\bar{A}_{t+1,i} \leq \bar{A}_{ti}\right\}$$

$$\leq \langle \bar{A}_t - \bar{A}_{t+1}, \hat{Y}'_t \rangle - \frac{1}{\eta} D_F(\bar{A}_{t+1}, \bar{A}_t)$$

$$\leq \frac{\eta}{2} \|\hat{Y}'_t\|^2_{\nabla^2 F(Z_t)^{-1}} = \frac{\eta}{2} \hat{Y}'^{\top}_t \nabla^2 F(Z)^{-1} \hat{Y}'_t$$

$$\nabla F(x) = \log(x) - 1$$

$$\nabla^2 F(x) = \operatorname{diam}(1/x)$$

$$= \frac{\eta}{2} \sum_i \hat{Y}'_{t,i} Z_i \hat{Y}'_{t,i}$$

$Z_t = \bar{A}_t$: بیشینه

$$F(a) = \sum_{i=1}^{d} (a_i \log(a_i) - a_i)$$

الگوریتم کاهش آینه‌ای

$$R_n \leq \frac{\mathrm{diam}_F(\mathrm{co}(\mathcal{A}))}{\eta} + \mathbb{E}\left[\sum_{t=1}^{n} \langle \bar{A}_t - \bar{A}_{t+1}, \hat{Y}_t \rangle - \frac{1}{\eta} D_F(\bar{A}_{t+1}, \bar{A}_t)\right]$$

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$$= \frac{\eta}{2} \sum_i \hat{Y}'_{t,i} Z_i \hat{Y}'_{t,i} \leq \frac{\eta}{2} \sum_{i=1}^{d} \frac{A_{ti}}{\bar{A}_{ti}}$$

$Z_t = \bar{A}_t$:بیشینه

$$\hat{Y}_{ti} = \frac{A_{ti} y_{ti}}{\bar{A}_{ti}}$$

$$\hat{Y}_{ti} = \frac{A_{ti} y_{ti}}{\bar{A}_{ti}}$$

$$F(a) = \sum_{i=1}^{d} (a_i \log(a_i) - a_i)$$

الگوریتم کاهش آینه‌ای

$$R_n \leq \frac{\operatorname{diam}_F(\operatorname{co}(\mathcal{A}))}{\eta} + \mathbb{E}\left[\sum_{t=1}^{n} \langle \bar{A}_t - \bar{A}_{t+1}, \hat{Y}_t \rangle - \frac{1}{\eta} D_F(\bar{A}_{t+1}, \bar{A}_t)\right]$$

$$\operatorname{diam}_F(\operatorname{co}(\mathcal{A})) \leq m(1 + \log(d/m)).$$

$$\leq \frac{\eta}{2} \sum_{i=1}^{d} \frac{A_{ti}}{\bar{A}_{ti}}$$

$$\hat{Y}_{ti} = \frac{A_{ti} y_{ti}}{\bar{A}_{ti}}$$

$$F(a) = \sum_{i=1}^{d} (a_i \log(a_i) - a_i)$$

الگوریتم کاهش آینه‌ای

$$R_n \leq \frac{\mathrm{diam}_F(\mathrm{co}(\mathcal{A}))}{\eta} + \mathbb{E}\left[\sum_{t=1}^{n} \langle \bar{A}_t - \bar{A}_{t+1}, \hat{Y}_t \rangle - \frac{1}{\eta} D_F(\bar{A}_{t+1}, \bar{A}_t)\right]$$

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$$\leq \frac{\eta}{2} \sum_{i=1}^{d} \frac{A_{ti}}{\bar{A}_{ti}}$$

$$\frac{\eta}{2} \mathbb{E}\left[\sum_{t=1}^{n} \sum_{i=1}^{d} \frac{A_{ti}}{\bar{A}_{ti}}\right] = \frac{\eta n d}{2}$$

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الگوریتم کاهش آینه‌ای

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$$\frac{\eta}{2} \mathbb{E}\left[\sum_{t=1}^{n} \sum_{i=1}^{d} \frac{A_{ti}}{\bar{A}_{ti}}\right] = \frac{\eta n d}{2}$$

$$\leq \frac{m(1 + \log(d/m))}{\eta} + \frac{\eta n d}{2}$$

الگوریتم کاهش آینه‌ای

$$\hat{Y}_{ti} = \frac{A_{ti} y_{ti}}{\bar{A}_{ti}}$$

$$F(a) = \sum_{i=1}^{d} (a_i \log(a_i) - a_i)$$

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$$\operatorname{diam}_F(\operatorname{co}(\mathcal{A})) \leq m(1 + \log(d/m)).$$

$$\leq \frac{\eta}{2} \sum_{i=1}^{d} \frac{A_{ti}}{\bar{A}_{ti}}$$

$$\frac{\eta}{2} \mathbb{E}\left[\sum_{t=1}^{n} \sum_{i=1}^{d} \frac{A_{ti}}{\bar{A}_{ti}}\right] = \frac{\eta n d}{2}$$

$$\leq \frac{m(1 + \log(d/m))}{\eta} + \frac{\eta n d}{2} \quad = \sqrt{2nmd(1 + \log(d/m))}$$

قضیه:

$$F(a) = \sum_{i=1}^{d} (a_i \log(a_i) - a_i) \qquad a \in [0, \infty)^d$$

$$F(a) = \infty \qquad \textit{otherwise.}$$

$$\eta = \sqrt{2m(1 + \log(d/m))/(nd)},$$

$$\Downarrow$$

$$R_n \leq \sqrt{2nmd(1 + \log(d/m))}$$

از لحاظ محاسبات

Input $\mathcal{A}, \eta, F$

$\bar{A}_1 = \operatorname{argmin}_{a \in \text{co}(\mathcal{A})} F(a)$

for $t = 1, \ldots, n$ **do**

  Choose distribution P_t on $\mathcal{A}$ such that $\sum_{a \in \mathcal{A}} P_t(a)a = \bar{A}_t$

  Sample $A_t \sim P_t$ and observe $A_{t1}y_{t1}, \ldots, A_{td}y_{td}$

  Compute $\hat{Y}_{ti} = A_{ti}y_{ti}/\bar{A}_{ti}$ for all $i \in [d]$

  Update $\bar{A}_{t+1} = \operatorname{argmin}_{a \in \text{co}(\mathcal{A})} \eta\langle a, \hat{Y}_t\rangle + D_F(a, \bar{A}_t)$

end for

برای مسئله کوتاه‌ترین مسیر نیمه‌ـ بندیتی

Input $\mathcal{A}, \eta, F$

$\bar{A}_1 = \operatorname{argmin}_{a \in \mathrm{co}(\mathcal{A})} F(a)$

for $t = 1, \ldots, n$ **do**

  Choose distribution P_t on $\mathcal{A}$ such that $\sum_{a \in \mathcal{A}} P_t(a) a = \bar{A}_t$

  Sample $A_t \sim P_t$ and observe $A_{t1} y_{t1}, \ldots, A_{td} y_{td}$

  Compute $\hat{Y}_{ti} = A_{ti} y_{ti} / \bar{A}_{ti}$ for all $i \in [d]$

  Update $\bar{A}_{t+1} = \operatorname{argmin}_{a \in \mathrm{co}(\mathcal{A})} \eta \langle a, \hat{Y}_t \rangle + D_F(a, \bar{A}_t)$

end for