بسم الله الرحمن الرحیم

نظریه علوم کامپیوتر

نظریه علوم کامپیوتر - بهار ۱۴۰۰-۱۴۰۱ - جلسه ششم: محاسبه‌پذیری و محاسبه‌ناپذیری (۲)

Theory of computation - 002 - S06 - non-computability (2)

Recall

- Languages on machines which are decidable
  - $A\text{DFA} = \{\langle B, w\rangle \mid B$ is a DFA and B accepts $w\}$
  - $A\text{NFA} = \{\langle B, w\rangle \mid B$ is a NFA and B accepts $w\}$
  - $E\text{DFA} = \{\langle B\rangle \mid B$ is a DFA and $L(B) = \emptyset\}$
  - $EQ\text{DFA} = \{\langle A, B\rangle \mid A$ and B are DFAs and $L(A) = L(B)\}$
  - $A\text{CFG} = \{\langle G, w\rangle \mid G$ is a CFG and $w \in L(G)\}$
  - $E\text{CFG} = \{\langle G\rangle \mid G$ is a CFG and $L(G) = \emptyset\}$
- $EQ\text{CFG} = \{\langle G, H\rangle \mid G, H$ are CFGs and $L(G) = L(H)\}$
- $AMBIG\text{CFG} = \{\langle G\rangle \mid G$ is an ambiguous CFG $\}$

Acceptance Problem for TMs

Let ATM $= \{\langle M, w\rangle \mid M$ is a TM and M accepts $w\}$

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Theorem: A_{TM} is not decidable

Proof: Later ….

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Let $A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M$ is a TM and M accepts $w\}$

Theorem: A_{TM} is not decidable
Proof: Later ….

Theorem: A_{TM} is T-recognizable
Proof: The following TM U recognizes A_{TM}

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Von Neumann said U inspired the concept of a stored program computer.

Acceptance Problem for TMs

Let $ATM = \{\langle M, w\rangle \mid M$ is a TM and M accepts $w\}$

Theorem: ATM is not decidable

Proof: Later ....

Theorem: ATM is T-recognizable

Proof: The following TM U recognizes ATM

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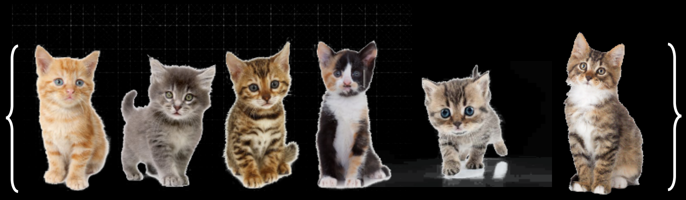

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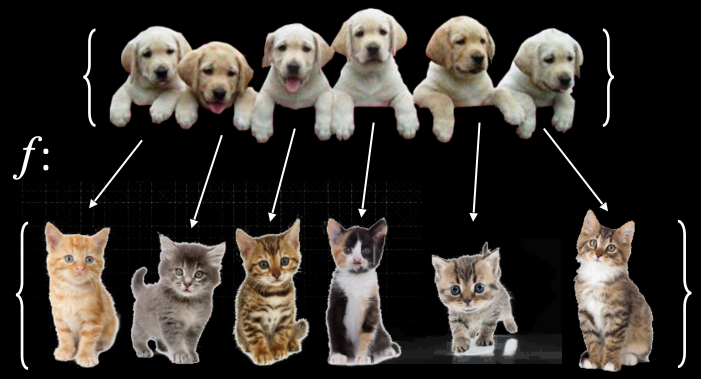

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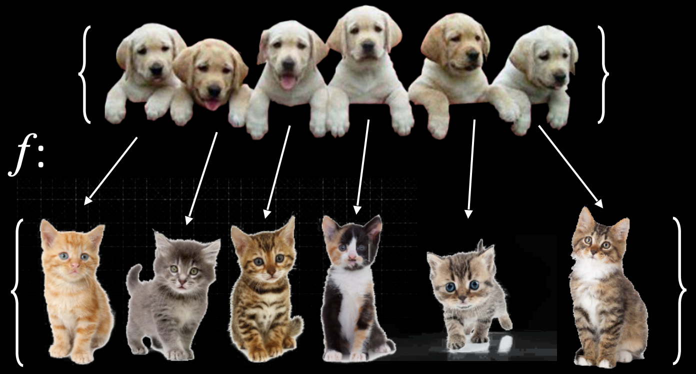

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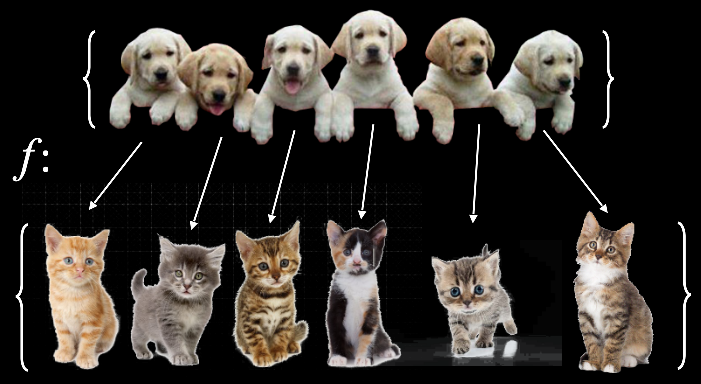

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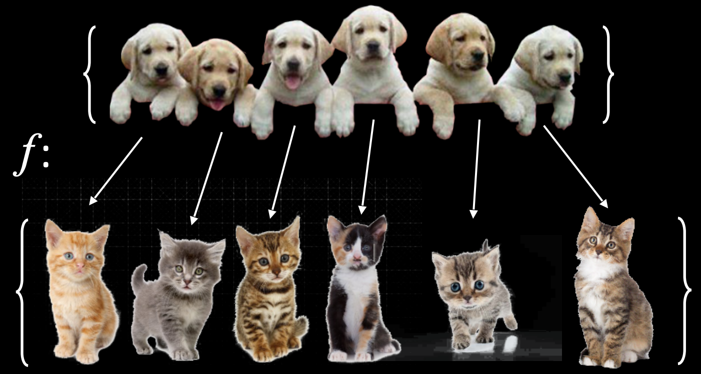

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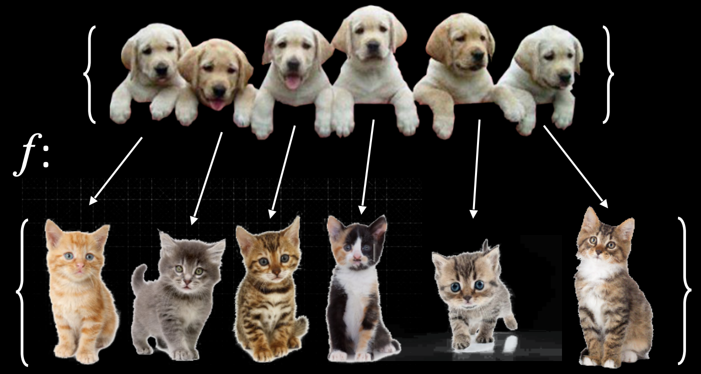

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	1	2	3	4	…
1	1/1	1/2	1/3	1/4	
2	2/1	2/2	2/3	2/4	…
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1	$\boxed{1/1}$	1/2	1/3	1/4	
2	$\boxed{2/1}$	2/2	2/3	2/4	…
3	3/1	3/2	3/3	3/4	
4	4/1	4/2	4/3	4/4	

$\mathbb{N}$	$\mathbb{Q}^+$
1	1/1
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Countable Sets

Let $\mathbb{N} = \{1,2,3,\ldots\}$ and let

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Defn: A set is <u>countable</u> if it is finite or it has the same size as $\mathbb{N}$.

Both $\mathbb{Z}$ and $\mathbb{Q}^+$ are countable.

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Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f : \mathbb{N} \to \mathbb{R}$

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

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1	
2	
3	
4	
5	
6	
7	

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n	$f(n)$
1	2.718281828…
2	
3	
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7	

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2	3.141592653…
3	0.000000000…
4	1.414213562…
5	0.142857242…
6	
7	

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4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$\mathbb{R}$ is Uncountable – Diagonalization

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Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f : \mathbb{N} \to \mathbb{R}$

n	$f(n)$
1	2.718281828…
2	3.141592653…
3	0.000000000…
4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$$x = 0.$$

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f: \mathbb{N} \to \mathbb{R}$

n	$f(n)$
1	2.718281828…
2	3.141592653…
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4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$$x = 0.$$

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f : \mathbb{N} \to \mathbb{R}$

1	2.718281828…
2	3.141592653…
3	0.000000000…
4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$$x = 0.\overset{\neq 7}{8}$$

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f: \mathbb{N} \to \mathbb{R}$

1	2.718281828…
2	3.141592653…
3	0.000000000…
4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$$x = 0.\overset{\neq 7}{8}$$

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f : \mathbb{N} \to \mathbb{R}$

1	2.718281828…
2	3.141592653…
3	0.000000000…
4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$$x = 0.\overset{\neq 7}{8}\overset{\neq 4}{5}$$

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f : \mathbb{N} \rightarrow \mathbb{R}$

1	2.718281828…
2	3.141592653…
3	0.000000000…
4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$$x = 0.\overset{\neq 7}{8}\overset{\neq 4}{5}$$

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f: \mathbb{N} \to \mathbb{R}$

n	$f(n)$
1	2.718281828…
2	3.141592653…
3	0.000000000…
4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$\neq 7$ $\neq 4$ $\neq 0$

$$x = 0.851$$

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f : \mathbb{N} \to \mathbb{R}$

n	$f(n)$
1	2.718281828…
2	3.141592653…
3	0.000000000…
4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$$x = 0.\underset{\neq 7}{8}\,\underset{\neq 4}{5}\,\underset{\neq 0}{1}$$

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f\colon \mathbb{N} \to \mathbb{R}$

1	2.718281828…
2	3.141592653…
3	0.000000000…
4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$$x = 0.\overset{\neq 7}{8}\overset{\neq 4}{5}\overset{\neq 0}{1}\overset{\neq 2}{6}$$

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f : \mathbb{N} \to \mathbb{R}$

1	2.718281828…
2	3.141592653…
3	0.000000000…
4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$\neq 7$ $\neq 4$ $\neq 0$ $\neq 2$

$$x = 0.8516$$

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f: \mathbb{N} \to \mathbb{R}$

1	2.718281828…
2	3.141592653…
3	0.000000000…
4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$$x = 0.\overset{\neq 7}{8}\overset{\neq 4}{5}\overset{\neq 0}{1}\overset{\neq 2}{6}\overset{\neq 5}{1}$$

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f : \mathbb{N} \rightarrow \mathbb{R}$

n	$f(n)$
1	2.718281828…
2	3.141592653…
3	0.000000000…
4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$\neq 7 \;\; \neq 4 \;\; \neq 0 \;\; \neq 2 \;\; \neq 5$

$$x = 0.85161$$

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f: \mathbb{N} \to \mathbb{R}$

n	$f(n)$
1	2.718281828…
2	3.141592653…
3	0.000000000…
4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$$x = 0.\overset{\neq 7}{8}\overset{\neq 4}{5}\overset{\neq 0}{1}\overset{\neq 2}{6}\overset{\neq 5}{1}\overset{\neq 9}{8}$$

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f : \mathbb{N} \to \mathbb{R}$

1	2.718281828…
2	3.141592653…
3	0.000000000…
4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$$x = 0.\overset{\neq 7}{8}\overset{\neq 4}{5}\overset{\neq 0}{1}\overset{\neq 2}{6}\overset{\neq 5}{1}\overset{\neq 9}{8}$$

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f: \mathbb{N} \to \mathbb{R}$

1	2.718281828…
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4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$\neq 7$ $\neq 4$ $\neq 0$ $\neq 2$ $\neq 5$ $\neq 9$ $\neq 8$

$$x = 0.8516182\ldots$$

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f\colon \mathbb{N} \to \mathbb{R}$

1	2.718281828…
2	3.141592653…
3	0.000000000…
4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$$x = 0.\overset{\neq 7}{8}\overset{\neq 4}{5}\overset{\neq 0}{1}\overset{\neq 2}{6}\overset{\neq 5}{1}\overset{\neq 9}{8}\overset{\neq 8}{2}\ldots$$

differs from the n^{th} number in the n^{th} digit
so cannot be the n^{th} number for any n.

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

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7	1.234567890…

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$$x = 0.\overset{\neq 7}{8}\overset{\neq 4}{5}\overset{\neq 0}{1}\overset{\neq 2}{6}\overset{\neq 5}{1}\overset{\neq 9}{8}\overset{\neq 8}{2}\ldots$$

differs from the n^{th} number in the n^{th} digit
so cannot be the n^{th} number for any n.

Hence x is not paired with any n. It is missing from the list.

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R} =$ all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

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Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$\neq 7 \quad \neq 4 \quad \neq 0 \quad \neq 2 \quad \neq 5 \quad \neq 9 \quad \neq 8$

$$x = 0.8516182\ldots$$

differs from the n^{th} number in the n^{th} digit so cannot be the n^{th} number for any n.

Hence x is not paired with any n. It is missing from the list.

Therefore f is not a 1-1 correspondence.

$\mathbb{R}$ is Uncountable – Diagonalization

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Diagonalization

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Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$$x = 0.\overset{\neq 7}{8}\overset{\neq 4}{5}\overset{\neq 0}{1}\overset{\neq 2}{6}\overset{\neq 5}{1}\overset{\neq 9}{8}\overset{\neq 8}{2}\ldots$$

differs from the n^{th} number in the n^{th} digit
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Diagonalization

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Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$\neq 7 \quad \neq 4 \quad \neq 0 \quad \neq 2 \quad \neq 5 \quad \neq 9 \quad \neq 8$

$$x = 0.8516182\ldots$$

differs from the n^{th} number in the n^{th} digit
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Hence x is not paired with any n. It is missing from the list.

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$\mathbb{R}$ is Uncountable – Corollaries

Let $\mathscr{L} =$ all languages

$\mathbb{R}$ is Uncountable – Corollaries

Let $\mathscr{L} =$ all languages

Corollary 1: $\mathscr{L}$ is uncountable

$\mathbb{R}$ is Uncountable – Corollaries

Let $\mathscr{L} =$ all languages

Corollary 1: $\mathscr{L}$ is uncountable

Proof: There's a 1-1 correspondence from $\mathscr{L}$ to $\mathbb{R}$ so they are the same size.

	{,	0,	1,	00,	01,	10,	11,	000,	…
	{	0,		00,	01,				…
	.0	1	0	1	1	0	0	0	…

$\mathbb{R}$ is Uncountable – Corollaries

Let $\mathscr{L} =$ all languages

Corollary 1: $\mathscr{L}$ is uncountable

Proof: There's a 1-1 correspondence from $\mathscr{L}$ to $\mathbb{R}$ so they are the same size.

Observation: $\Sigma^* = \{\varepsilon,\ 0{,}1, 00{,}01, 10{,}11, 000,\ \ldots\}$ is countable.

	{,	0,	1,	00,	01,	10,	11,	000,	…
	{	0,		00,	01,				…
	.0	1	0	1	1	0	0	0	…

$\mathbb{R}$ is Uncountable – Corollaries

Let $\mathscr{L} =$ all languages

Corollary 1: $\mathscr{L}$ is uncountable

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	{,	0,	1,	00,	01,	10,	11,	000,	…
	{	0,		00,	01,				…
	.0	1	0	1	1	0	0	0	…

Let $\mathscr{M} =$ all Turing machines

$\mathbb{R}$ is Uncountable – Corollaries

Let $\mathscr{L} =$ all languages

Corollary 1: $\mathscr{L}$ is uncountable

Proof: There's a 1-1 correspondence from $\mathscr{L}$ to $\mathbb{R}$ so they are the same size.

Observation: $\Sigma^* = \{\varepsilon,\ 0,1, 00,01, 10,11, 000, \ldots\}$ is countable.

	{,	0,	1,	00,	01,	10,	11,	000,	…
	{	0,		00,	01,				…
	.0	1	0	1	1	0	0	0	…

Let $\mathscr{M} =$ all Turing machines

Observation: $\mathscr{M}$ is countable.

$\mathbb{R}$ is Uncountable – Corollaries

Let $\mathscr{L} =$ all languages

Corollary 1: $\mathscr{L}$ is uncountable

Proof: There's a 1-1 correspondence from $\mathscr{L}$ to $\mathbb{R}$ so they are the same size.

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	{,	0,	1,	00,	01,	10,	11,	000,	…
	{	0,		00,	01,				…
	.0	1	0	1	1	0	0	0	…

Let $\mathscr{M} =$ all Turing machines

Observation: $\mathscr{M}$ is countable.

Because $\{\langle M \rangle \mid M$ is a TM$\} \subseteq \Sigma^*$.

$\mathbb{R}$ is Uncountable – Corollaries

Let $\mathscr{L} =$ all languages

Corollary 1: $\mathscr{L}$ is uncountable

Proof: There's a 1-1 correspondence from $\mathscr{L}$ to $\mathbb{R}$ so they are the same size.

Observation: $\Sigma^* = \{\varepsilon,\ 0,1, 00,01, 10,11, 000, \ldots\}$ is countable.

	{,	0,	1,	00,	01,	10,	11,	000,	…
	{	0,		00,	01,				…
	.0	1	0	1	1	0	0	0	…

Let $\mathscr{M} =$ all Turing machines

Observation: $\mathscr{M}$ is countable.

Because $\{\langle M \rangle \mid M \text{ is a TM}\} \subseteq \Sigma^*$.

Corollary 2: Some language is not decidable.

$\mathbb{R}$ is Uncountable – Corollaries

Let $\mathscr{L} =$ all languages

Corollary 1: $\mathscr{L}$ is uncountable

Proof: There's a 1-1 correspondence from $\mathscr{L}$ to $\mathbb{R}$ so they are the same size.

Observation: $\Sigma^* = \{\varepsilon,\ 0,1, 00,01, 10,11, 000, \ldots\}$ is countable.

	{,	0,	1,	00,	01,	10,	11,	000,	…
	{	0,		00,	01,				…
	.0	1	0	1	1	0	0	0	…

Let $\mathscr{M} =$ all Turing machines

Observation: $\mathscr{M}$ is countable.

Because $\left\{\langle M \rangle \middle|\ M \text{ is a TM}\right\} \subseteq \Sigma^*$.

Corollary 2: Some language is not decidable.

Because there are more languages than TMs.

$\mathbb{R}$ is Uncountable – Corollaries

Let $\mathscr{L} =$ all languages

Corollary 1: $\mathscr{L}$ is uncountable

Proof: There's a 1-1 correspondence from $\mathscr{L}$ to $\mathbb{R}$ so they are the same size.

Observation: $\Sigma^* = \{\varepsilon,\ 0{,}1, 00{,}01, 10{,}11, 000, \ldots\}$ is countable.

	{,	0,	1,	00,	01,	10,	11,	000,	…
	{	0,		00,	01,				…
	.0	1	0	1	1	0	0	0	…

Let $\mathscr{M} =$ all Turing machines

Observation: $\mathscr{M}$ is countable.

Because $\left\{\langle M \rangle \,\middle|\, M \text{ is a TM}\right\} \subseteq \Sigma^*$.

Corollary 2: Some language is not decidable.

Because there are more languages than TMs.

We will show some specific language ATM is not decidable.

$\mathbb{R}$ is Uncountable – Corollaries

Let $\mathscr{L} =$ all languages

Corollary 1: $\mathscr{L}$ is uncountable

Proof: There's a 1-1 correspondence from $\mathscr{L}$ to $\mathbb{R}$ so they are the same size.

Observation: $\Sigma^* = \{\varepsilon,\ 0{,}1, 00{,}01, 10{,}11, 000, \ldots\}$ is countable.

	{,	0,	1,	00,	01,	10,	11,	000,	…
	{	0,		00,	01,				…
	.0	1	0	1	1	0	0	0	…

Let $\mathscr{M} =$ all Turing machines

Observation: $\mathscr{M}$ is countable.

Because $\{\langle M \rangle \mid M$ is a TM$\} \subseteq \Sigma^*$.

Corollary 2: Some language is not decidable.

Because there are more languages than TMs.

We will show some specific language ATM is not decidable.

Check-in 8.1

$\mathbb{R}$ is Uncountable – Corollaries

Let $\mathscr{L} =$ all languages

Corollary 1: $\mathscr{L}$ is uncountable

Proof: There's a 1-1 correspondence from $\mathscr{L}$ to $\mathbb{R}$ so they are the same size.

Observation: $\Sigma^* = \{\varepsilon,\ 0,1,00,01,10,11,000,\ \ldots\}$ is countable.

Let $\mathscr{M} =$ all Turing machines

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Hilbert's 1st question asked if there is a set of intermediate size between $\mathbb{N}$ and $\mathbb{R}$. Gödel and Cohen showed that we cannot answer this question by using the standard axioms of mathematics. How can we interpret their conclusion?

(a) We need better axioms to describe reality.

(b) Infinite sets have no mathematical reality so Hilbert's 1st question has no answer.

Check-in 8.1

ATM is undecidable

Recall ATM $= \{\langle M, w\rangle \mid M$ is a TM and M accepts $w\}$

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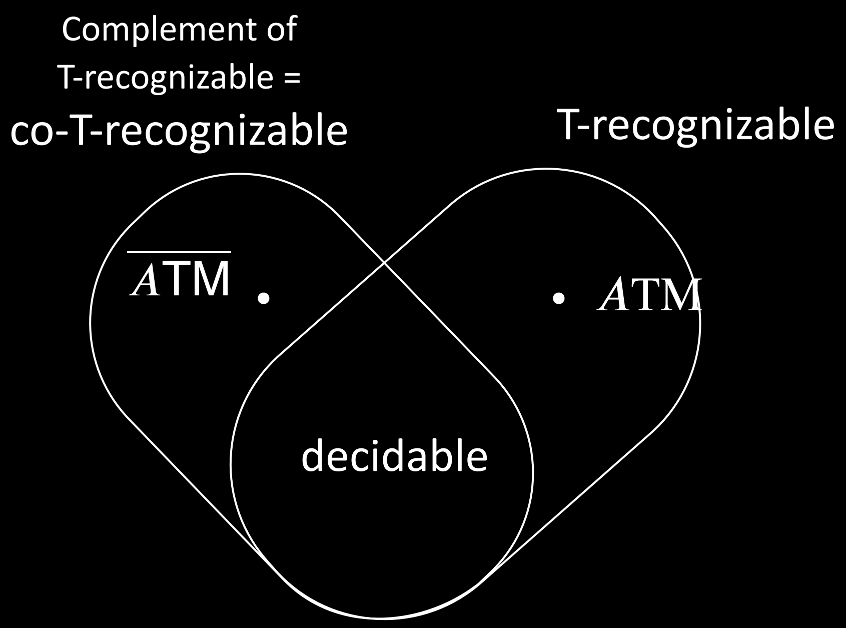

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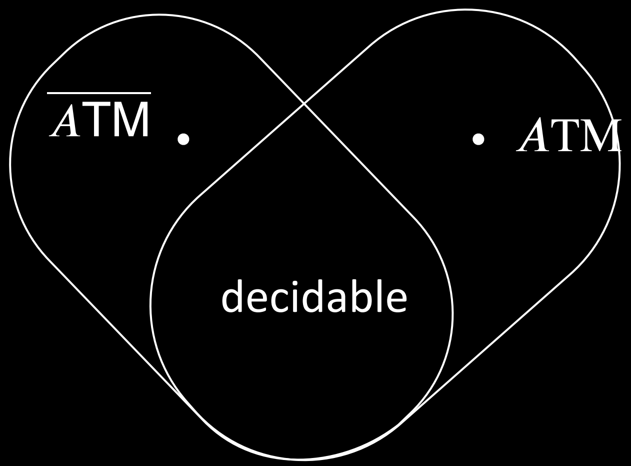

Corollary: $\overline{ATM}$ is T-unrecognizable

Proof: ATM is T-recognizable but also undecidable

Check-in 8.3

$\overline{ATM}$ is T-unrecognizable

Theorem: If A and $\overline{A}$ are T-recognizable then A is decidable

Proof: Let TM M_1 and M_2 recognize A and $\overline{A}$.

Construct TM T deciding A.

$T =$ "On input w

1. Run M_1 and M_2 on w in parallel until one accepts
2. If M_1 accepts then *accept*.
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From what we've learned, which closure properties can we prove for the class of T-recognizable languages?

Choose all that apply.

(a) Closed under union.

(b) Closed under intersection.

(c) Closed under complement.

(d) Closed under concatenation.

(e) Closed under star.

Check-in 8.3

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Is Biology reducible to Physics?

(a) Yes, all aspects of the physical world may be explained in terms of Physics, at least in principle.

(b) No, some things in the world, maybe life, the brain, or consciousness, are beyond the realm pf Physics.

(c) I'm on the fence on this question!

Check-in 9.1

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Let ETM $= \{\langle M \rangle \mid M$ is a TM and $L(M) = \emptyset\}$

Theorem: E**TM** is undecidable

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