بسم الله الرحمن الرحیم

تمرین سری اول درس بهینه‌سازی ترکیبیاتی – پاییز ۱۳۹٤–۹۵

لطفا ٤ تمرین از میان ٥ تمرین زیر را حل کنید.

Exercise 1. Let

$$S := \{x \in \{0,1\}^4 : 90x_1 + 35x_2 + 26x_3 + 25x_4 \le 138\}$$

(i) Show that

$$S = \{x \in \{0,1\}4 : 2x1 + x2 + x3 + x4 \le 3\},$$

and

$$S = \{x \in \{0,1\}^4 : 2x_1 + x_2 + x_3 + x_4 \le 3, x_1 + x_2 + x_3 \le 2, x_1 + x_2 + x_4 \le 2, x_1 + x_3 + x_4 \le 2\}$$

(ii) Can you rank these three formulations in terms of the tightness of their linear relaxations, when $x \in \{0,1\}^4$ is replaced by $x \in [0,1]^4$? Show any strict inclusion.

Exercise 2 (Combinatorial auctions). A company sets an auction for N objects. Bidders place their bids for some subsets of the N objects that they like. The auction house has received n bids, namely bids bj for subset Sj , for j = 1,...,n. The auction house is faced with the problem of choosing the winning bids so that profit is maximized and each of the N objects is given to at most one bidder. Formulate the optimization problem faced by the auction house as a set packing problem.

Exercise 3. For the following subsets of edges of an undirected graph G = (V,E), find an integer linear formulation and prove its correctness:

- The family of Hamiltonian paths of G with end nodes u, v. (A Hamiltonian path is a path that goes exactly once through each node of the graph.)
- The family of all Hamiltonian paths of G.
- The family of edge sets that induce a triangle of G.
- Assuming that G has 3n nodes, the family of n node-disjoint triangles.
- The family of odd cycles of G.

Exercise 4. Consider a connected undirected graph G = (V,E). For S ⊆ V , denote by E(S) the set of edges with both ends in S. For i ∈ V , denote by δ(i) the set of edges incident with i. Prove or disprove that the following formulation produces a spanning tree with maximum number of leaves.

$$max \sum_{i \in V} z_i$$

$$\sum_{e \in E} x_e = |V| - 1$$

$$\sum_{e \in E(s)} x_e \leq |S| - 1 \qquad \forall S \subset V, |S| \geq 2$$

$$\sum_{e \in \delta(i)} x_e + (|\delta(i)| - 1) z_i \leq |\delta(i)| \qquad \forall i \in V$$

$$x_e \in \{0, 1\} \qquad \forall e \in E$$

$$z_i \in \{0, 1\} \qquad \forall i \in V.$$

Exercise 5. Let $P = \{A_1 x \leq b_1\}$ be a polytope and $S = \{A_2 x < b_2\}$. Formulate the problem of maximizing a linear function over P \S as a mixed 0-1 program.