بسم الله الرحمن الرحیم

برنامه‌ریزی نیمه‌معین برای طراحی الگوریتم‌های تقریبی

جلسه هشتم: دوگانی (۳)

Project: People

مرور

کنج محدب بسته و دوگان کنج

4.2.1 Definition. *Let $K \subseteq V$ be a nonempty closed set.*[1] *K is called a closed convex cone if the following two conditions hold.*

(i) *For all $\mathbf{x} \in K$ and all nonnegative real numbers λ, we have $\lambda\mathbf{x} \in K$.*
(ii) *For all $\mathbf{x}, \mathbf{y} \in K$, we have $\mathbf{x} + \mathbf{y} \in K$.*

4.3.1 Definition. *Let $K \subseteq V$ be a closed convex cone. The set*

$$K^* := \{\mathbf{y} \in V : \langle \mathbf{y}, \mathbf{x} \rangle \geq 0 \text{ for all } \mathbf{x} \in K\}$$

is called the dual cone of K.

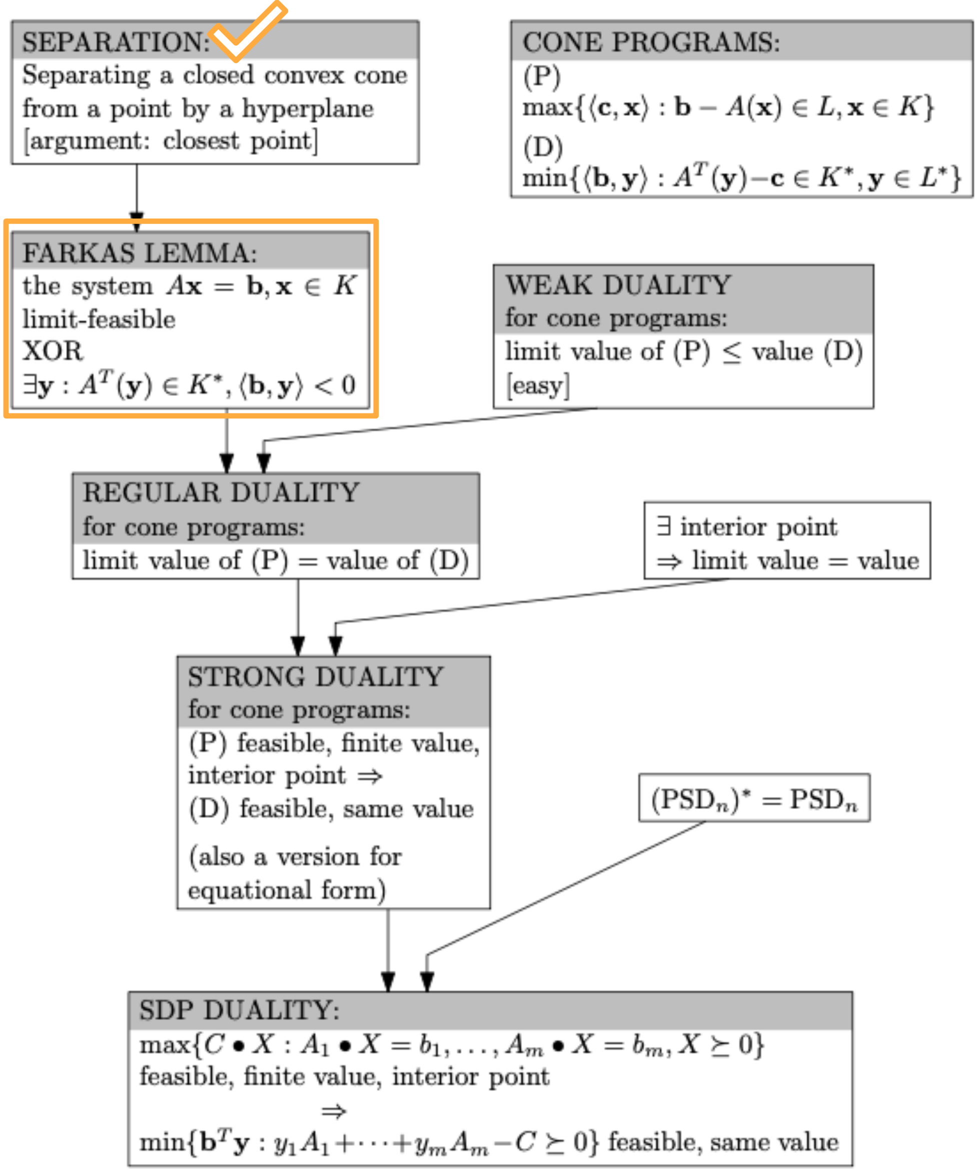
4.4.2 Theorem. *Let $K \subseteq V$ be a closed convex cone, and let $\mathbf{b} \in V \setminus K$. Then there exists a vector $\mathbf{y} \in V$ such that*

$$\langle \mathbf{y}, \mathbf{x} \rangle \geq 0 \textit{ for all } \mathbf{x} \in K, \textit{ and } \langle \mathbf{y}, \mathbf{b} \rangle < 0.$$

4.5.3 Lemma. *Let* $V = \mathrm{SYM}_n, W = \mathbb{R}^m$, *and* $A\colon V \to W$ *defined by* $A(X) = (A_1 \bullet X, A_2 \bullet X, \ldots, A_m \bullet X)$. *Then*

$$A^T(\mathbf{y}) = \sum_{i=1}^{m} y_i A_i.$$

4.5.4 Lemma. *Let* $K \subseteq V$ *be a closed convex cone, and* $C = \{A(\mathbf{x}) : \mathbf{x} \in K\}$. *Then* $\overline{C}$, *the closure of* C, *is a closed convex cone.*


SEPARATION:
Separating a closed convex cone from a point by a hyperplane [argument: closest point]
CONE PROGRAMS:
(P) $\max\{\langle \mathbf{c}, \mathbf{x}\rangle : \mathbf{b} - A(\mathbf{x}) \in L, \mathbf{x} \in K\}$
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FARKAS LEMMA:
the system $A\mathbf{x} = \mathbf{b}, \mathbf{x} \in K$ limit-feasible
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WEAK DUALITY for cone programs:
limit value of (P) $\leq$ value (D)
[easy]
REGULAR DUALITY for cone programs:
limit value of (P) = value of (D)
$\exists$ interior point
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STRONG DUALITY for cone programs:
(P) feasible, finite value, interior point $\Rightarrow$ (D) feasible, same value
(also a version for equational form)
$(\mathrm{PSD}_n)^* = \mathrm{PSD}_n$
SDP DUALITY:
$\max\{C \bullet X : A_1 \bullet X = b_1, \ldots, A_m \bullet X = b_m, X \succeq 0\}$
feasible, finite value, interior point
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لم فارکاش

4.5.5 Definition. *Let $K \subseteq V$ be a closed convex cone. The system*

$$A(\mathbf{x}) = \mathbf{b}, \ \ \mathbf{x} \in K$$

is called limit-feasible if there exists a sequence $(\mathbf{x}_k)_{k \in \mathbb{N}}$ such that $\mathbf{x}_k \in K$ for all $k \in \mathbb{N}$ and

$$\lim_{k \to \infty} A(\mathbf{x}_k) = \mathbf{b}.$$

4.5.6 Lemma (Farkas lemma for cones). *Let* $K \subseteq V$ *be a closed convex cone, and* $\mathbf{b} \in W$. Either *the system*

$$A(\mathbf{x}) = \mathbf{b}, \mathbf{x} \in K$$

is limit-feasible, or *the system*

$$A^T(\mathbf{y}) \in K^*, \langle \mathbf{b}, \mathbf{y} \rangle < 0$$

has a solution, but not both.

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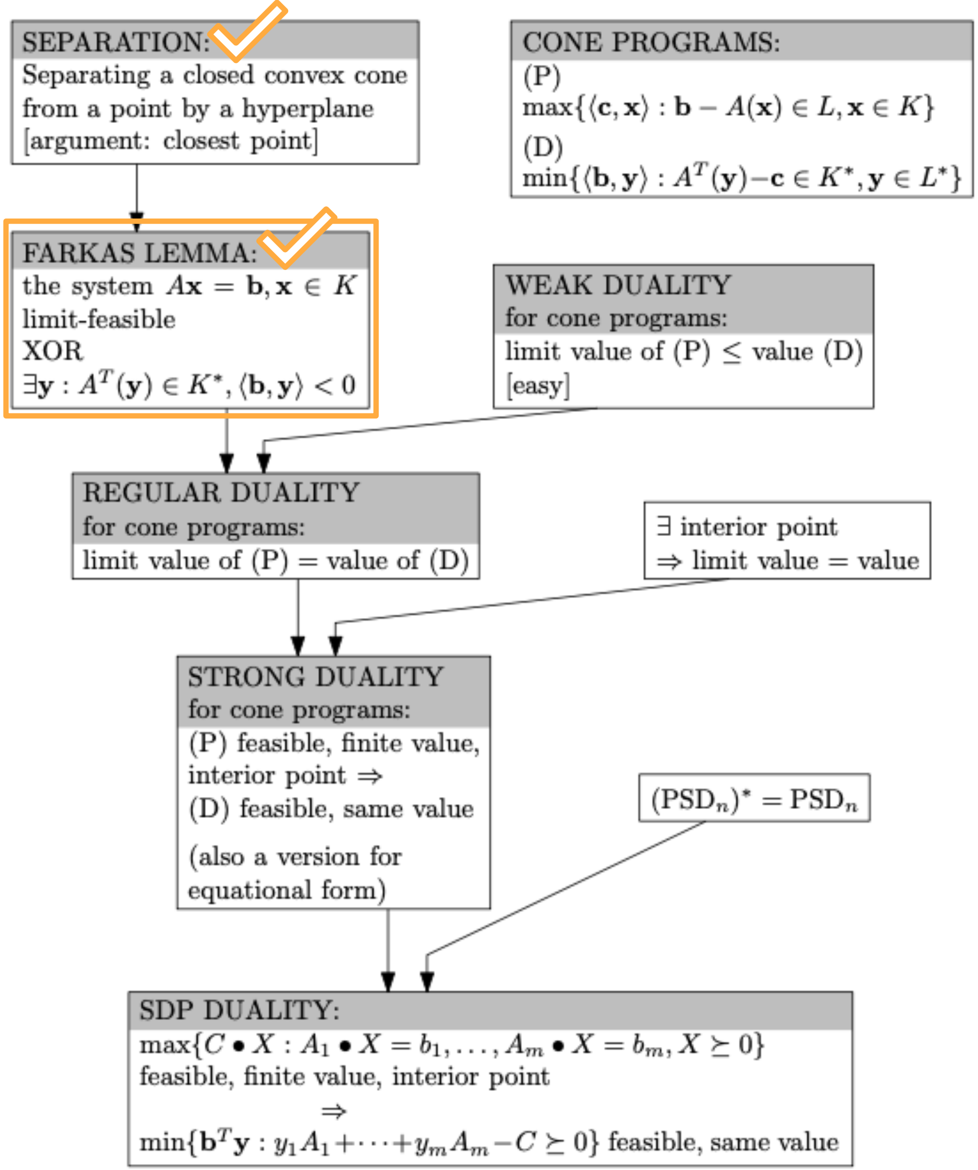
لم فارکاش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برنامه‌ریزی کنج

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4.6.1 Definition. *Let $K \subseteq V$, $L \subseteq W$ be closed convex cones, let $\mathbf{b} \in W$, $\mathbf{c} \in V$, and let $A: V \to W$ be a linear operator. A cone program is an optimization problem of the form*

$$\begin{array}{ll} \textit{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ \textit{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & \mathbf{x} \in K. \end{array} \tag{4.8}$$

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The *value* of a feasible cone program is defined as

$$\sup\{\langle \mathbf{c}, \mathbf{x} \rangle : \mathbf{b} - A(\mathbf{x}) \in L, \mathbf{x} \in K\}, \tag{4.9}$$

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4.6.2 Definition. *The cone program (4.8) is called limit-feasible if there exist sequences $(\mathbf{x}_k)_{k \in \mathbb{N}}$ and $(\mathbf{x}'_k)_{k \in \mathbb{N}}$ such that $\mathbf{x}_k \in K$ and $\mathbf{x}'_k \in L$ for all $k \in N$, and*

$$\lim_{k \to \infty} (A(\mathbf{x}_k) + \mathbf{x}'_k) = \mathbf{b}.$$

Such sequences $(\mathbf{x}_k)_{k \in \mathbb{N}}$ and $(\mathbf{x}'_k)_{k \in \mathbb{N}}$ are called feasible sequences of (4.8).

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4.6.3 Definition. *Given a feasible sequence* $(\mathbf{x}_k)_{k\in\mathbb{N}}$ *of a cone program (4.8), we define its value as*

$$\langle \mathbf{c}, (\mathbf{x}_k)_{k\in\mathbb{N}} \rangle := \limsup_{k\to\infty} \langle \mathbf{c}, \mathbf{x}_k \rangle.$$

The limit value of (4.8) is then defined as

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مقدار برنامه‌ریزی کنج < مقدار حدی برنامه‌ریزی کنج

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4.6.4 Definition. *An interior point (or* Slater point*) of the cone program (4.8) is a point* $\mathbf{x}$ *such that*

$$\mathbf{x} \in K, \quad \mathbf{b} - A(\mathbf{x}) \in L,$$

and the following additional requirement holds:

$$\begin{array}{rcll} \mathbf{x} & \in & \operatorname{int}(K) & \textit{if } L = \{\mathbf{0}\}, \textit{ and} \\ \mathbf{b} - A(\mathbf{x}) & \in & \operatorname{int}(L) & \textit{otherwise.} \end{array}$$

4.6.5 Theorem. *If the cone program (4.8) has an interior point (which, in particular, means that it is feasible), then the value equals the limit value.*

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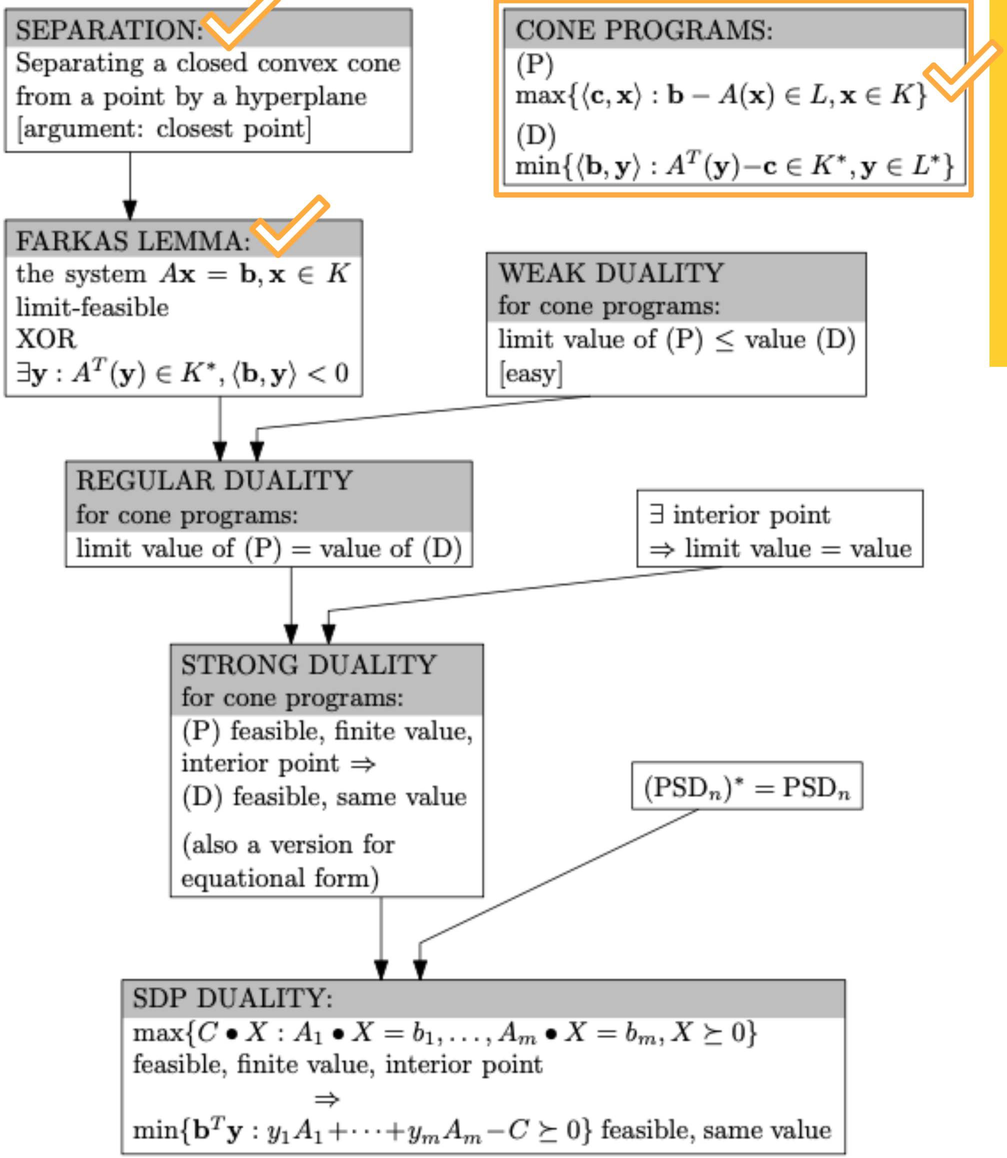
دوگان
برنامه‌ریزی کنج



برنامه‌ریزی کنج

$$
\begin{array}{lll}
\text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\
& \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\
& & \mathbf{x} \in K.
\end{array}
$$

(D)

$$\text{(P)} \quad \begin{array}{ll} \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & \mathbf{x} \in K. \end{array}$$

(D)

4.5.6 Lemma (Farkas lemma for cones). *Let $K \subseteq V$ be a closed convex cone, and $\mathbf{b} \in W$.* Either *the system*

$$A(\mathbf{x}) = \mathbf{b}, \mathbf{x} \in K$$

is limit-feasible, or *the system*

$$A^T(\mathbf{y}) \in K^*, \langle \mathbf{b}, \mathbf{y} \rangle < 0$$

has a solution, but not both.

برنامه‌ریزی کنج

$$\begin{array}{lll} \text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K. \end{array}$$

$$\begin{array}{lll} \text{(D)} & \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\ & \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\ & & \mathbf{y} \in L^*. \end{array}$$

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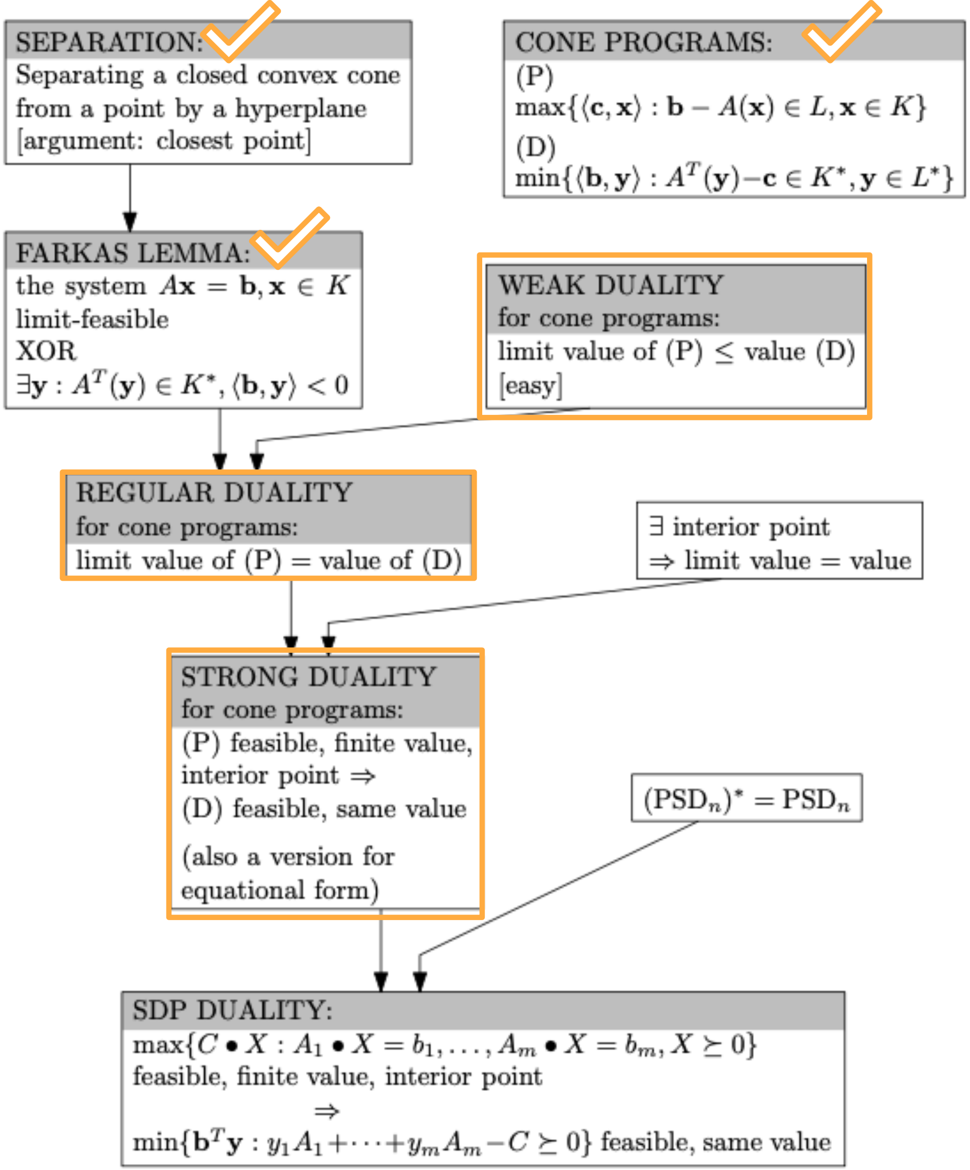
& & \mathbf{y} \in L^*.
\end{array}
$$

$$
\begin{array}{lll}
\text{(D}'\text{)} & \text{Maximize} & -\langle \mathbf{b}, \mathbf{y} \rangle \\
& \text{subject to} & -\mathbf{c} + A^T(\mathbf{y}) \in K^* \\
& & \mathbf{y} \in L^*.
\end{array}
$$



دوگانی برای برنامه‌ریزی کنج

$$
\begin{array}{lll}
(\mathrm{P}) & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\
& \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\
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$$

4.7.1 Theorem. *If the primal program (P) is feasible, has a finite value* γ and *has an interior point* $\tilde{\mathbf{x}}$, *then the dual program (D) is also feasible and has the same value* γ.

$$\begin{array}{lll}(\mathrm{P}) & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K.\end{array}$$

$$\begin{array}{lll}(\mathrm{D}) & \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\ & \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\ & & \mathbf{y} \in L^*.\end{array}$$

4.7.1 Theorem. *If the primal program (P) is feasible, has a finite value γ and has an interior point $\tilde{\mathbf{x}}$, then the dual program (D) is also feasible and has the same value γ.*

- دوگانی ضعیف: اگر (P) شدنی حدی باشد، اگر D شدنی باشد، مقدار D => مقدار حدی P
- دوگانی معمولی: اگر (P) شدنی حدی باشد، D شدنی است و مقدار D = مقدار حدی P
  - + لم برابری مقدار حدی و مقدار برنامه‌ریزی با جواب درونی ==>> اثبات قضیه

$$
\begin{array}{lll}
\text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\
 & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\
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$$

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\begin{array}{lll}
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\end{array}
$$

4.7.2 Theorem. *If the dual program (D) is feasible, and if the primal program (P) is limit-feasible, then the limit value of (P) is bounded above by the value of (D).*

$$\text{(P)} \quad \text{Maximize} \quad \langle \mathbf{c}, \mathbf{x} \rangle \quad \text{subject to} \quad \mathbf{b} - A(\mathbf{x}) \in L, \quad \mathbf{x} \in K.$$

$$\text{(D)} \quad \text{Minimize} \quad \langle \mathbf{b}, \mathbf{y} \rangle \quad \text{subject to} \quad A^T(\mathbf{y}) - \mathbf{c} \in K^*, \quad \mathbf{y} \in L^*.$$

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y از D:

$$
\begin{array}{llr}
\text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\
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از P: $(\mathbf{x}_k)_{k\in\mathbb{N}}, (\mathbf{x}'_k)_{k\in\mathbb{N}}$ که جواب بهینه را می‌سازد

$$\begin{array}{lll}(P) & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K.\end{array}$$

$$\begin{array}{lll}(D) & \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\ & \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\ & & \mathbf{y} \in L^*.\end{array}$$

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$$0 \leq \langle \underbrace{A^T(\mathbf{y}) - \mathbf{c}}_{\in K^*}, \underbrace{\mathbf{x}_k}_{\in K} \rangle + \langle \underbrace{\mathbf{y}}_{\in L^*}, \underbrace{\mathbf{x}'_k}_{\in L} \rangle$$

$$
\begin{array}{lll}
\text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\
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$$\begin{array}{llll} (P) & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K. \end{array}$$

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$$\limsup_{k\to\infty} \langle \mathbf{c}, \mathbf{x}_k \rangle \le \limsup_{k\to\infty} \langle \mathbf{y}, A(\mathbf{x}_k) + \mathbf{x}'_k \rangle$$

$$\text{(P)}\quad \begin{array}{ll} \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & \mathbf{x} \in K. \end{array}$$

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$$
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4.7.3 Theorem. *The dual program (D) is feasible and has a finite value β if and only if the primal program (P) is limit-feasible and has a finite limit value γ. Moreover, $\beta = \gamma$.*

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- اگر D شدنی باشد: مقدار = β،

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- اگر D شدنی باشد: مقدار = β،

$$A^T(\mathbf{y}) - \mathbf{c} \in K^*, \ \mathbf{y} \in L^* \quad \Rightarrow \quad \langle \mathbf{b}, \mathbf{y} \rangle \geq \beta$$

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$$\boxed{A^T(\mathbf{y}) - \mathbf{c} \in K^*, \ \mathbf{y} \in L^* \quad \Rightarrow \quad \langle \mathbf{b}, \mathbf{y} \rangle \geq \beta}$$

$$A^T(\mathbf{y}) - z\mathbf{c} \in K^*, \ \mathbf{y} \in L^*, \ z \geq 0 \quad \Rightarrow \quad \langle \mathbf{b}, \mathbf{y} \rangle \geq z\beta.$$

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$$\left(\begin{array}{c|c} A^T & -\mathbf{c} \\ \hline \mathrm{id} & \mathbf{0} \\ \hline 0 & 1 \end{array}\right)(\mathbf{y}, z) \in K^* \oplus L^* \oplus \mathbb{R}_+ \quad \Rightarrow \quad \langle (\mathbf{b}, -\beta), (\mathbf{y}, z) \rangle \geq 0$$

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is limit-feasible, or *the system*

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شدنی حدی است:

$$\left(\begin{array}{c|c|c} A & \mathrm{id} & 0 \\ \hline -\mathbf{c}^T & \mathbf{0}^T & 1 \end{array}\right)(\mathbf{x}, \mathbf{x}', z) = (\mathbf{b}, -\beta), \ (\mathbf{x}, \mathbf{x}', z) \in (K^* \oplus L^* \oplus \mathbb{R}_+)^* = K \oplus L \oplus \mathbb{R}_+$$

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4.5.5 Definition. *Let $K \subseteq V$ be a closed convex cone. The system*

$$A(\mathbf{x}) = \mathbf{b},\ \ \mathbf{x} \in K$$

is called limit-feasible if there exists a sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ such that $\mathbf{x}_k \in K$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} A(\mathbf{x}_k) = \mathbf{b}.$$

شدنی حدی است:

$$\left(\begin{array}{c|c|c} A & \mathrm{id} & 0 \\ \hline -\,\mathbf{c}^T & \mathbf{0}^T & 1 \end{array}\right)(\mathbf{x}, \mathbf{x}', z) = (\mathbf{b}, -\beta),\ \ (\mathbf{x}, \mathbf{x}', z) \in (K^* \oplus L^* \oplus \mathbb{R}_+)^* = K \oplus L \oplus \mathbb{R}_+$$

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وجود دارد:

$$\mathbf{x}_k \in K,\ \mathbf{x}'_k \in L,\ z_k \geq 0 \qquad \lim_{k \to \infty} A(\mathbf{x}_k) + \mathbf{x}'_k = \mathbf{b} \qquad \lim_{k \to \infty} \langle \mathbf{c}, \mathbf{x}_k \rangle - z_k = \beta.$$

شدنی حدی است:

$$\left(\begin{array}{c|c|c} A & \mathrm{id} & 0 \\ \hline -\mathbf{c}^T & \mathbf{0}^T & 1 \end{array}\right)(\mathbf{x}, \mathbf{x}', z) = (\mathbf{b}, -\beta), \ (\mathbf{x}, \mathbf{x}', z) \in (K^* \oplus L^* \oplus \mathbb{R}_+)^* = K \oplus L \oplus \mathbb{R}_+$$

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$$\begin{array}{lll} \text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K. \end{array}$$

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وجود دارد:

$$\mathbf{x}_k \in K,\ \mathbf{x}'_k \in L,\ z_k \geq 0 \qquad \lim_{k \to \infty} A(\mathbf{x}_k) + \mathbf{x}'_k = \mathbf{b} \qquad \lim_{k \to \infty} \langle \mathbf{c}, \mathbf{x}_k \rangle - z_k = \beta.$$

پس (P) شدنی حدی است با مقداری بیشتر مساوی β

$$\begin{array}{lll} \text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K. \end{array}$$

شدنی حدی است:

$$\left(\begin{array}{c|c|c} A & \mathrm{id} & 0 \\ \hline -\mathbf{c}^T & \mathbf{0}^T & 1 \end{array}\right)(\mathbf{x}, \mathbf{x}', z) = (\mathbf{b}, -\beta),\ (\mathbf{x}, \mathbf{x}', z) \in (K^* \oplus L^* \oplus \mathbb{R}_+)^* = K \oplus L \oplus \mathbb{R}_+$$

4.5.5 Definition. *Let $K \subseteq V$ be a closed convex cone. The system*

$$A(\mathbf{x}) = \mathbf{b},\ \mathbf{x} \in K$$

is called limit-feasible if there exists a sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ such that $\mathbf{x}_k \in K$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} A(\mathbf{x}_k) = \mathbf{b}.$$

وجود دارد:

$$\mathbf{x}_k \in K,\ \mathbf{x}'_k \in L,\ z_k \geq 0 \qquad \lim_{k\to\infty} A(\mathbf{x}_k) + \mathbf{x}'_k = \mathbf{b} \qquad \lim_{k\to\infty} \langle \mathbf{c}, \mathbf{x}_k\rangle - z_k = \beta.$$

پس (P) شدنی حدی است با مقداری بیشتر مساوی β

$$\begin{array}{lll} (\mathrm{P}) & \text{Maximize} & \langle \mathbf{c}, \mathbf{x}\rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K. \end{array}$$

(بنابر دوگانی ضعیف:) برنامه‌ریزی کنج (P) شدنی است با مقداری برابر با β

$$\text{(P)} \quad \text{Maximize} \quad \langle \mathbf{c}, \mathbf{x} \rangle \quad \text{subject to} \quad \mathbf{b} - A(\mathbf{x}) \in L, \quad \mathbf{x} \in K.$$

$$\text{(D)} \quad \text{Minimize} \quad \langle \mathbf{b}, \mathbf{y} \rangle \quad \text{subject to} \quad A^T(\mathbf{y}) - \mathbf{c} \in K^*, \quad \mathbf{y} \in L^*.$$

4.7.3 Theorem. *The dual program (D) is feasible and has a finite value β if and only if the primal program (P) is limit-feasible and has a finite limit value γ. Moreover, $\beta = \gamma$.*

اگر D شدنی باشد:

$$
\text{(P)} \quad \begin{array}{ll} \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & \mathbf{x} \in K. \end{array}
$$

$$
\text{(D)} \quad \begin{array}{ll} \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\ \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\ & \mathbf{y} \in L^*. \end{array}
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اگر D شدنی باشد:

- اگر P شدنی حدی باشد:
  - فرض خلف (D) شدنی نیست.

$$\text{(P)} \quad \begin{array}{ll} \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & \mathbf{x} \in K. \end{array}$$

$$\text{(D)} \quad \begin{array}{ll} \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\ \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\ & \mathbf{y} \in L^*. \end{array}$$

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اگر D شدنی باشد:

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  - فرض خلف (D) شدنی نیست.

$$A^T(\mathbf{y}) - z\mathbf{c} \in K^*, \ \mathbf{y} \in L^*, \quad \Rightarrow \quad z \leq 0,$$

$$
\begin{array}{lll}
\text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\
& \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\
& & \mathbf{x} \in K.
\end{array}
$$

$$
\begin{array}{lll}
\text{(D)} & \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\
& \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\
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$$

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- اگر D شدنی باشد:
- اگر P شدنی حدی باشد:
    - فرض خلف (D) شدنی نیست.

(وگرنه 1/z سمت چپ، جواب شدنی (D) است.)

$$A^T(\mathbf{y}) - z\mathbf{c} \in K^*, \; \mathbf{y} \in L^*, \quad \Rightarrow \quad z \leq 0,$$

(P) Maximize $\langle \mathbf{c}, \mathbf{x} \rangle$
subject to $\mathbf{b} - A(\mathbf{x}) \in L$
$\mathbf{x} \in K.$

(D) Minimize $\langle \mathbf{b}, \mathbf{y} \rangle$
subject to $A^T(\mathbf{y}) - \mathbf{c} \in K^*$
$\mathbf{y} \in L^*.$

4.7.3 Theorem. *The dual program (D) is feasible and has a finite value β if and only if the primal program (P) is limit-feasible and has a finite limit value γ. Moreover, $\beta = \gamma$.*

اگر D شدنی باشد:

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(وگرنه 1/z سمت چپ، جواب شدنی (D) است.)

$$A^T(\mathbf{y}) - z\mathbf{c} \in K^*, \ \mathbf{y} \in L^*, \quad \Rightarrow \quad z \leq 0,$$

$$\left(\begin{array}{c|c} A^T & -\mathbf{c} \\ \hline \mathrm{id} & 0 \end{array}\right)(\mathbf{y}, z) \in K^* \oplus L^* \quad \Rightarrow \quad \langle (\mathbf{0}, -1), (\mathbf{y}, z) \rangle \geq 0.$$

$$\left(\begin{array}{c|c} A^T & -\mathbf{c} \\ \hline \mathrm{id} & 0 \end{array}\right)(\mathbf{y}, z) \in K^* \oplus L^* \quad \Rightarrow \quad \langle(\mathbf{0}, -1), (\mathbf{y}, z)\rangle \geq 0.$$

4.5.6 Lemma (Farkas lemma for cones). *Let $K \subseteq V$ be a closed convex cone, and $\mathbf{b} \in W$. Either the system*

$$A(\mathbf{x}) = \mathbf{b}, \mathbf{x} \in K$$

is limit-feasible, or the system

$$A^T(\mathbf{y}) \in K^*, \langle \mathbf{b}, \mathbf{y}\rangle < 0$$

has a solution, but not both.

$$\left(\begin{array}{c|c} A^T & -\mathbf{c} \\ \hline \mathrm{id} & 0 \end{array}\right)(\mathbf{y}, z) \in K^* \oplus L^* \quad \Rightarrow \quad \langle(\mathbf{0}, -1), (\mathbf{y}, z)\rangle \geq 0.$$

4.5.6 Lemma (Farkas lemma for cones). *Let $K \subseteq V$ be a closed convex cone, and $\mathbf{b} \in W$.* Either *the system*

$$A(\mathbf{x}) = \mathbf{b}, \mathbf{x} \in K$$

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$$A^T(\mathbf{y}) \in K^*, \langle \mathbf{b}, \mathbf{y}\rangle < 0$$

has a solution, but not both.

شدنی حدی است:

$$\left(\begin{array}{c|c} A & \mathrm{id} \\ \hline -\mathbf{c}^T & 0 \end{array}\right)(\mathbf{x}, \mathbf{x}') = (\mathbf{0}, -1), \ (\mathbf{x}, \mathbf{x}') \in (K^* \oplus L^*)^* = K \oplus L$$

شدنی حدی است:

$$\left(\begin{array}{c|c} A & \mathrm{id} \\ \hline -\mathbf{c}^T & 0 \end{array}\right)(\mathbf{x}, \mathbf{x}') = (\mathbf{0}, -1), \ (\mathbf{x}, \mathbf{x}') \in (K^* \oplus L^*)^* = K \oplus L$$

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$$\left(\begin{array}{c|c} A & \mathrm{id} \\ \hline -\mathbf{c}^T & 0 \end{array}\right)(\mathbf{x}, \mathbf{x}') = (\mathbf{0}, -1), \; (\mathbf{x}, \mathbf{x}') \in (K^* \oplus L^*)^* = K \oplus L$$

$$\mathbf{x}_k \in K, \, \mathbf{x}'_k \in L \qquad \lim_{k\to\infty} A(\mathbf{x}_k) + \mathbf{x}'_k = \mathbf{0} \qquad \lim_{k\to\infty} \langle \mathbf{c}, \mathbf{x}_k \rangle = 1.$$

شدنی حدی است:

$$\left(\begin{array}{c|c} A & \mathrm{id} \\ \hline -\mathbf{c}^T & 0 \end{array}\right)(\mathbf{x}, \mathbf{x}') = (\mathbf{0}, -1), \ (\mathbf{x}, \mathbf{x}') \in (K^* \oplus L^*)^* = K \oplus L$$

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این جواب + یک جواب حدی برای (P): جوابی بهتر برای (P)

شدنی حدی است:

$$\left(\begin{array}{c|c} A & \mathrm{id} \\ \hline -\mathbf{c}^T & 0 \end{array}\right)(\mathbf{x}, \mathbf{x}') = (\mathbf{0}, -1), \ (\mathbf{x}, \mathbf{x}') \in (K^* \oplus L^*)^* = K \oplus L$$

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این جواب + یک جواب حدی برای (P): جوابی بهتر برای (P)

$$\begin{array}{llll} \text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & & \mathbf{x} \in K. \end{array}$$

$$\begin{array}{llll} \text{(D)} & \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\ & \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\ & & \mathbf{y} \in L^*. \end{array}$$

4.7.3 Theorem. *The dual program (D) is feasible and has a finite value β if and only if the primal program (P) is limit-feasible and has a finite limit value γ. Moreover, $\beta = \gamma$.*

اگر D شدنی باشد:

اگر P شدنی حدی باشد:

- فرض خلف (D) شدنی نیست.

(وگرنه 1/z سمت چپ، جواب شدنی (D) است.)

$$A^T(\mathbf{y}) - z\mathbf{c} \in K^*,\ \mathbf{y} \in L^*, \quad \Rightarrow \quad z \leq 0,$$

$$\left(\begin{array}{c|c} A^T & -\mathbf{c} \\ \hline \mathrm{id} & 0 \end{array}\right)(\mathbf{y}, z) \in K^* \oplus L^* \quad \Rightarrow \quad \langle (\mathbf{0}, -1), (\mathbf{y}, z) \rangle \geq 0.$$

SEPARATION:
Separating a closed convex cone from a point by a hyperplane [argument: closest point]

CONE PROGRAMS:
(P)
$\max\{\langle \mathbf{c},\mathbf{x}\rangle : \mathbf{b}-A(\mathbf{x})\in L, \mathbf{x}\in K\}$
(D)
$\min\{\langle \mathbf{b},\mathbf{y}\rangle : A^T(\mathbf{y})-\mathbf{c}\in K^*, \mathbf{y}\in L^*\}$

FARKAS LEMMA:
the system $A\mathbf{x}=\mathbf{b}, \mathbf{x}\in K$
limit-feasible
XOR
$\exists \mathbf{y} : A^T(\mathbf{y})\in K^*, \langle \mathbf{b},\mathbf{y}\rangle < 0$

WEAK DUALITY
for cone programs:
limit value of (P) $\leq$ value (D)
[easy]

REGULAR DUALITY
for cone programs:
limit value of (P) $=$ value of (D)

$\exists$ interior point
$\Rightarrow$ limit value $=$ value

STRONG DUALITY
for cone programs:
(P) feasible, finite value, interior point $\Rightarrow$
(D) feasible, same value

(also a version for equational form)

$(\mathrm{PSD}_n)^* = \mathrm{PSD}_n$

SDP DUALITY:
$\max\{C\bullet X : A_1\bullet X = b_1,\ldots,A_m\bullet X = b_m, X\succeq 0\}$
feasible, finite value, interior point
$\Rightarrow$
$\min\{\mathbf{b}^T\mathbf{y} : y_1A_1+\cdots+y_mA_m-C\succeq 0\}$ feasible, same value

دوگانی برای برنامه‌ریزی کنج