گسترش، برش تُنُک، و نظریه طیفی گراف (پاییز ۹۶-۹۵)

تمرین اول

Note that combinatorial Laplacian is defined as $L = dI - A$ while normalized Laplacian is $\mathcal{L} = I - \frac{1}{d}A$.

Problem 1

We proved following theorem in the class.

$$\frac{\lambda_2}{2} \leq \Phi_G \leq 2\sqrt{\lambda_2}$$

(a) Prove that for the normalized Laplacian, just as for the combinatorial Laplacian, the number of connected components in a graph G is precisely the multiplicity of the smallest eigenvalue.

(b) Show that right-hand inequality in above equation is asymptotically tight.

(c) Prove that the left-hand side is asymptotically tight for the complete graph on n nodes.

Problem 2: Path's Spectrum

Recall that for a function $f : V \to \mathbb{R}$, we define its Rayleigh quotient by

$$\mathcal{R}_G(f) = \frac{\sum_{x\sim y}(f(x) - f(y))^2}{\sum_{x\in V} f(x)^2}.$$

If P_n is the path graph on $n$vertices, then as we argued today, we have $\lambda_k(L) = \Theta(\left(\frac{k}{n}\right)^2)$.

(a) Prove that $\lambda_2(L) = \Theta(\frac{1}{n^2})$ by showing that (a) there exists a map $f : V \to \mathbb{R}$ with $f \perp \mathbf{1}$ and $\mathcal{R}_G(f) = O(\frac{1}{n^2})$ and (b) for any such map with $f \perp \mathbf{1}$, we have $\mathcal{R}_G(f) = \Omega(\frac{1}{n^2})$.

(b) Try to prove that $\lambda_k(L) = O(\left(\frac{k}{n}\right)^2)$ by exhibiting an explicit subspace of test functions that achieves this bound. It may help to use the Courant-Fischer min-max principle which says that

$$\lambda_k(L) = \min_{S\subseteq\mathbb{R}^V} \max_{0\neq f\in S} \mathcal{R}_G(f),$$

where the minimum is over all k-dimensional subspaces S.

[Hint: Have your subspace be the span of k functions with disjoint supports, where the support of a function $f : V \to \mathbb{R}$ is $\mathrm{supp}(f) = \{x \in V : f(x) \neq 0\}$.]

Problem 3: C_n Spectrum

Compute the Laplacian spectrum of C_n (the cycle with n vertices).

Problem 4

A hypercube of n-dimension is an undirected graph with 2^n vertices. Each vertex corresponds to a string of n bits. Two vertices have an edge if and only if their corresponding strings differ by exactly one bit.

(a) Given two undirected graphs $G = (V, E)$ and $H = (U, F)$, we define $G \times H$ as the undirected graph with vertex set $V \times U$ and two vertices (v_1, u_1), (v_2, u_2) have an edge if and only if either (1) $v_1 = v_2$ and $\{u_1, u_2\} \in F$ or (2) $u_1 = u_2$ and $\{v_1, v_2\} \in E$. Let x be an eigenvector of the Laplacian of G with eigenvalue α, and let y be an eigenvector of the Laplacian of H with eigenvalue β. Prove that we can use x and y to construct an eigenvector of the Laplacian of $G \times H$ with eigenvalue $\alpha + \beta$.

(b) Use (a) to compute the spectrum of the hypercube of n dimension.

(c) Show that the spectral algorithm for conductance may return a set S of conductance $\Omega(\sqrt{\phi(S)})$ in a hypercube, where S is the set with minimum conductance in the hypercube and $\phi(S) = \frac{E(S, V-S)}{\min\{|S|, |V-S|\}}$ is the conductance of S.

Problem 5: K_n, and S_n Spectrum

Compute the spectrum of complete graphs, and stars.

Problem 6

Let $G = (V, E)$ be an undirected d-regular graph. The line graph H of G has vertex set E and two vertices e_1, e_2 are adjacent if and only if their corresponding edges in G have a common vertex.

(a) Let $V = \{1, \ldots, n\}$, $e = ij$, and B_e be the n-dimensional vector with $+1$ in the i-th entry and -1 in the j-th entry and 0 otherwise. Let B be an $n \times m$ matrix where the columns are b_e and m is the number of edges in G. Prove that the adjacency matrix of H is $B^T B - 2I$. Conclude that the smallest eigenvalue of the adjacency matrix of H is at least -2.

(b) Use (a) to compute the spectrum of the adjacency matrix of H in term of the spectrum of the adjacency matrix of G.
Hint: (1) Write the adjacency matrix of G in terms of B, (2) Show that $det(I - XY) = det(I - YX)$ for any X, Y with appropriate dimensions (not necessary square matrices).

Problem 7: Laplacian of Spanning Tree

Let $G = (V, E)$ be an undirected graph.

(a) Let $V = \{1, \ldots, n\}$, $e = ij$, and B_e be the n-dimensional vector with $+1$ in the i-th entry and -1 in the j-th entry and 0 otherwise. Let B be an $n \times m$ matrix where the columns are b_e and m is the number of edges in G. Prove that the determinant of any $(n-1) \times (n-1)$ submatrix of B is in $\{-1, +1\}$ if and only if the $n-1$ edges corresponding to the columns form a spanning tree of G.

(b) Let L be the Laplacian matrix of G and let L' be the matrix obtained from L by deleting the last row and last column. Use (a) to prove that $det(L')$ is equal to the number of spanning trees in G.
Hint: Look up the Cauchy-Binet formula in wikipedia.

Problem 8: Grid

Consider the $k \times k$ two-dimensional grid graph $G_{k,k}$. Let $n = k^2$. Prove that

$$\lambda_2(L_{G_{k,k}}) = \Theta(\frac{1}{n}),$$

where λ_2 refers to the second eigenvalue of the combinatorial Laplacian. (Prove this without using the Spielman-Teng theorem for planar graphs.) For the lower bound, it may help to remember that $\lambda_2(L_{P_k}) = \Theta(\frac{1}{k^2})$, where P_k is the path on k vertices.

Problem 9: Binary Tree

Let $n = 2^h - 1$ for some integer $h \geq 1$. Prove that if T_h is the complete binary tree of height h, we have

$$\lambda_2(T_h) = \Theta(\frac{1}{n}),$$

where again λ_2 refers to the combinatorial Laplacian.

Problem 10

Let G be an arbitrary graph. Recall that

$$h_G = \min_{|S| \leq n/2} \frac{|E(S)|}{|S|}.$$

Let λ_2 be the second eigenvalue of the combinatorial Laplacian on G. Prove that,

$$\lambda_2 = \min_{f:V\to\mathbb{R}} \max_{c\in\mathbb{R}} \frac{\sum_{u\sim v} |f(u) - f(v)|^2}{\sum_{u\in V} |f(u) - c|^2},$$

and

$$h_G = \min_{f:V\to\mathbb{R}} \max_{c\in\mathbb{R}} \frac{\sum_{u\sim v} |f(u) - f(v)|}{\sum_{u\in V} |f(u) - c|},$$

where both minimums are over non-constant functions.

موفق باشید