بسم الله الرحمن الرحیم

جلسه هشتم

درس تحقیق در عملیات

مرور روش سیمپلکس

$$\boxed{\begin{array}{ll} \text{Maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{array}}$$

simplex tableau $\mathcal{T}(B)$

$$\begin{array}{rcccl}
\mathbf{x}_B & = & \mathbf{p} & + & Q\,\mathbf{x}_N \\
\hline
z & = & z_0 & + & \mathbf{r}^T\mathbf{x}_N
\end{array}$$

$$\mathbf{x}_N = \mathbf{0}$$

$$\mathbf{x}_B = \mathbf{p}$$

$$z = z_0$$

گام لولا! (pivot)

$$\mathcal{T}(B) \longrightarrow \mathcal{T}(B')$$

$$B' = (B \setminus \{u\}) \cup \{v\}$$

x_v enters the basis

x_u leaves the basis

$$B = \{k_1, k_2, \ldots, k_m\},\ k_1 < k_2 < \cdots < k_m$$

$$N = \{\ell_1, \ell_2, \ldots, \ell_{n-m}\},\ \ell_1 < \ell_2 < \cdots < \ell_{n-m}$$

$$x_{k_i} = p_i + \sum_{j=1}^{n-m} q_{ij} x_{\ell_j}$$

$$q_{\alpha\beta} < 0 \quad \text{and} \quad -\frac{p_\alpha}{q_{\alpha\beta}} = \min\left\{-\frac{p_i}{q_{i\beta}} : q_{i\beta} < 0,\ i = 1, 2, \ldots, m\right\}$$

$$\mathbf{r} \leq \mathbf{0} \implies \text{optimal}\,!$$

اولین تابلو ..؟!

$$
\begin{array}{ll}
\text{Maximize} & \mathbf{c}^T\mathbf{x} \\
\text{subject to} & A\mathbf{x} = \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0}.
\end{array}
$$

$$\begin{aligned} \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

$$\begin{aligned} \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

$$\mathbf{b} \geq \mathbf{0}$$

$$
\begin{aligned}
\text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0}.
\end{aligned}
$$

$$\mathbf{b} \geq \mathbf{0}$$

$$\overline{\mathbf{x}} = (x_1, \ldots, x_{n+m})$$

$$\text{subject to} \quad \begin{aligned} A\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0}. \end{aligned}$$

$$\mathbf{b} \geq \mathbf{0}$$

$$\overline{\mathbf{x}} = (x_1, \ldots, x_{n+m})$$

$$\bar{A} = (A \,|\, I_m)$$

$$\begin{aligned} \bar{A}\,\overline{\mathbf{x}} &= \mathbf{b} \\ \overline{\mathbf{x}} &\geq \mathbf{0} \end{aligned}$$

$$
\begin{array}{ll}
\text{maximize} & -(x_{n+1} + x_{n+2} + \cdots + x_{n+m}) \\
\text{subject to} & \bar{A}\,\overline{\mathbf{x}} = \mathbf{b} \\
 & \overline{\mathbf{x}} \geq \mathbf{0},
\end{array}
$$

?

قانون لولا

Largest Coefficient

Largest Increase

Steepest Edge $\quad \dfrac{\mathbf{c}^T(\mathbf{x}_{\text{new}} - \mathbf{x}_{\text{old}})}{\|\mathbf{x}_{\text{new}} - \mathbf{x}_{\text{old}}\|}$

Bland's Rule

Random Edge

The simplex method with Bland’s rule is always finite!

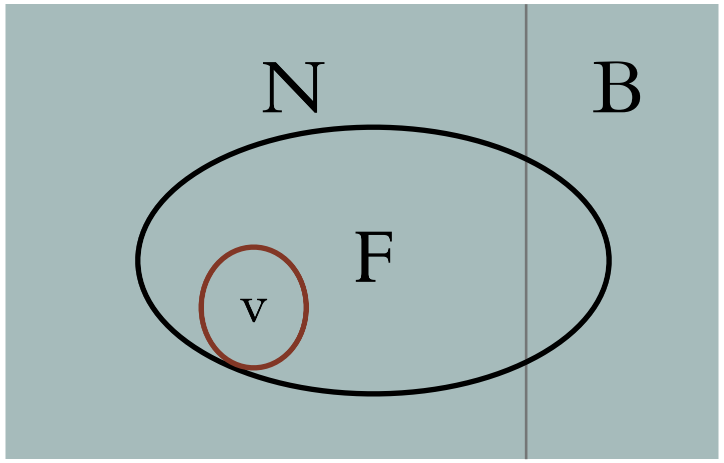
در انتخاب متغیر جدید و قدیم

اثبات

فرض: در دور افتاده‌ایم

$\textit{fickle}$ variables (F)

$x_v = 0$: برای vهای در F



v: راس با بزرگترین اندیس در F

ارزش جای Q
x_v ولدرو
کرد !

$$B = \{k_1, k_2, \ldots, k_m\}$$

$$N = \{\ell_1, \ell_2, \ldots, \ell_{n-m}\}$$

$$v = \ell_\beta$$

$r_\beta > 0$ and $r_j \leq 0$ for all j such that $\ell_j \in F \setminus \{v\} \setminus \mathrm{B}$

$$\begin{array}{lcccc} \mathbf{x}_B & = & \mathbf{p} & + & Q\,\mathbf{x}_N \\ \hline z & = & z_0 & + & \mathbf{r}^T\mathbf{x}_N \end{array}$$

ارزش‌های Q
x_v ضرب کرد.

$$B' = \{k'_1, k'_2, \ldots, k'_m\}$$

$$N' = \{\ell'_1, \ell'_2, \ldots, \ell'_{n-m}\}$$

$$k'_{\alpha'} = v$$

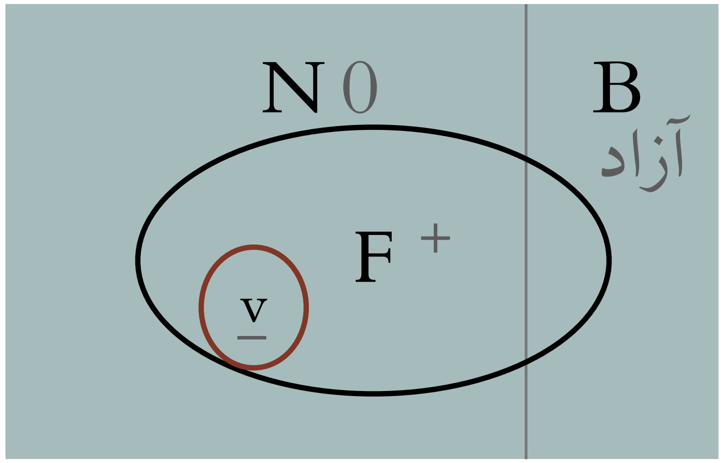
$$\ell'_{\beta'} = u$$

$q'_{\alpha'\beta'} < 0$ and $q'_{i\beta'} \geq 0$ for all i such that $k'_i \in F \setminus \{v\} \setminus \mathrm{N}$

$$\begin{array}{ll} \text{Maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x}_{F\setminus\{v\}} \geq \mathbf{0} \\ & x_v \leq 0 \\ & \mathbf{x}_{N\setminus F} = \mathbf{0}. \end{array}$$

ادعا ۱: جواب مربوط به B بهینه است:

الف) x شدنی است



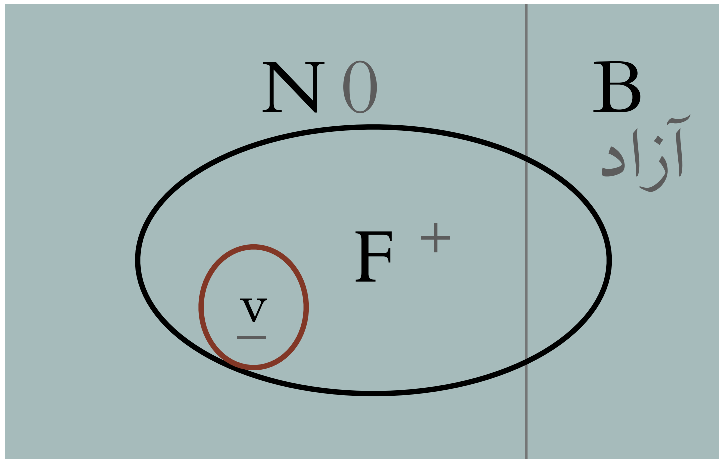
$$\begin{array}{llll} \mathbf{x}_B & = \mathbf{p} & + & Q\,\mathbf{x}_N \\ \hline z & = z_0 & + & \mathbf{r}^T\mathbf{x}_N \end{array}$$

ب) x بهینه است

$r_\beta > 0$ and $r_j \leq 0$ for all j such that $\ell_j \in F \setminus \{v\} \setminus \mathrm{B}$

$$
\begin{array}{ll}
\text{Maximize} & \mathbf{c}^T\mathbf{x} \\
\text{subject to} & A\mathbf{x} = \mathbf{b} \\
 & \mathbf{x}_{F\setminus\{v\}} \geq \mathbf{0} \\
 & x_v \leq 0 \\
 & \mathbf{x}_{N\setminus F} = \mathbf{0}.
\end{array}
$$

ادعا ۲: جواب بی‌کران دارد:



$x_u(t) = t$

$$
\begin{array}{lcl}
\mathbf{x}_{B'} & = & \mathbf{p}' + Q'\mathbf{x}_{N'} \\
\hline
z' & = & z_0' + \mathbf{r}'^T\mathbf{x}_{N'}
\end{array}
$$

$x_{B'}(t) = \ldots$

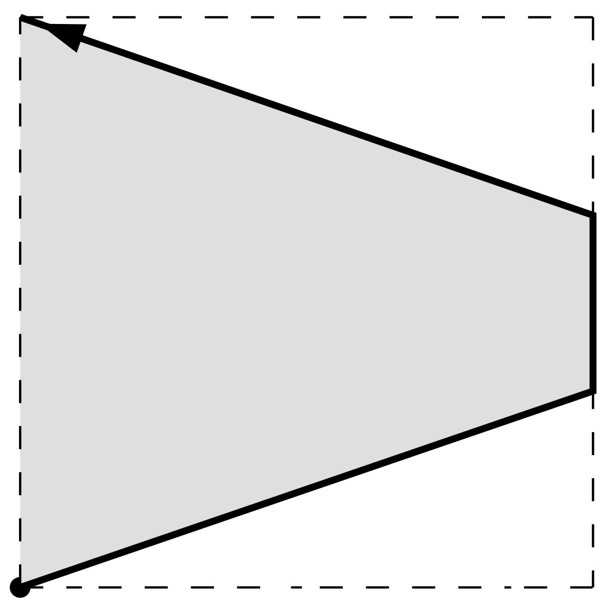
$$
\tilde{x}_{k_i'}(t) = \tilde{x}_{k_i'} + tq'_{i\beta'} \begin{cases} \geq 0 & \text{if } k_i' \in F \setminus \{v\} \\ < 0 & \text{if } k_i' = k'_{\alpha'} = v. \end{cases}
$$

$q'_{\alpha'\beta'} < 0$ and $q'_{i\beta'} \geq 0$ for all i such that $k_i' \in F \setminus \{v\}$

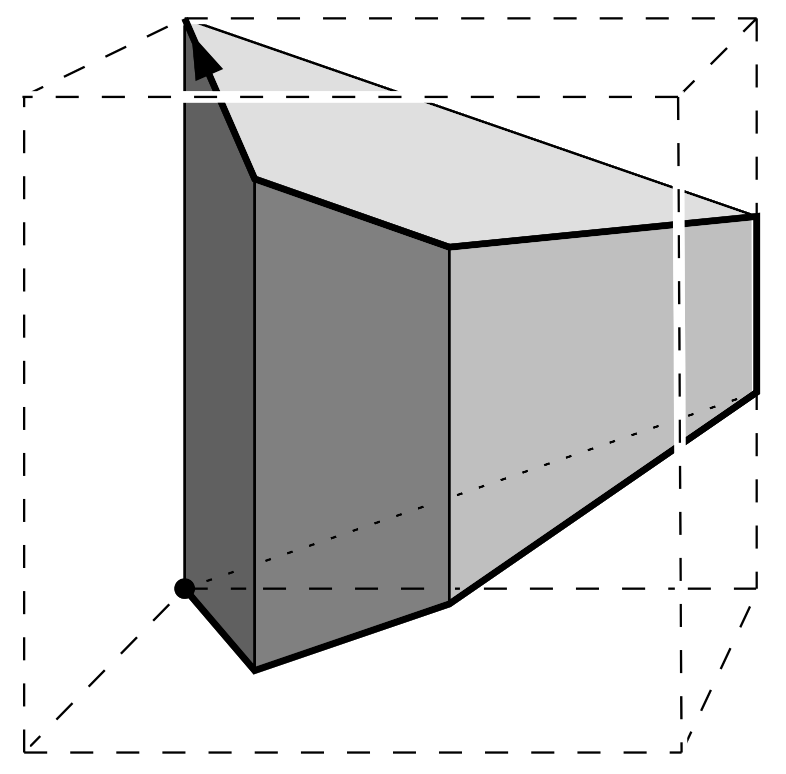
Efficiency of the Simplex Method

?

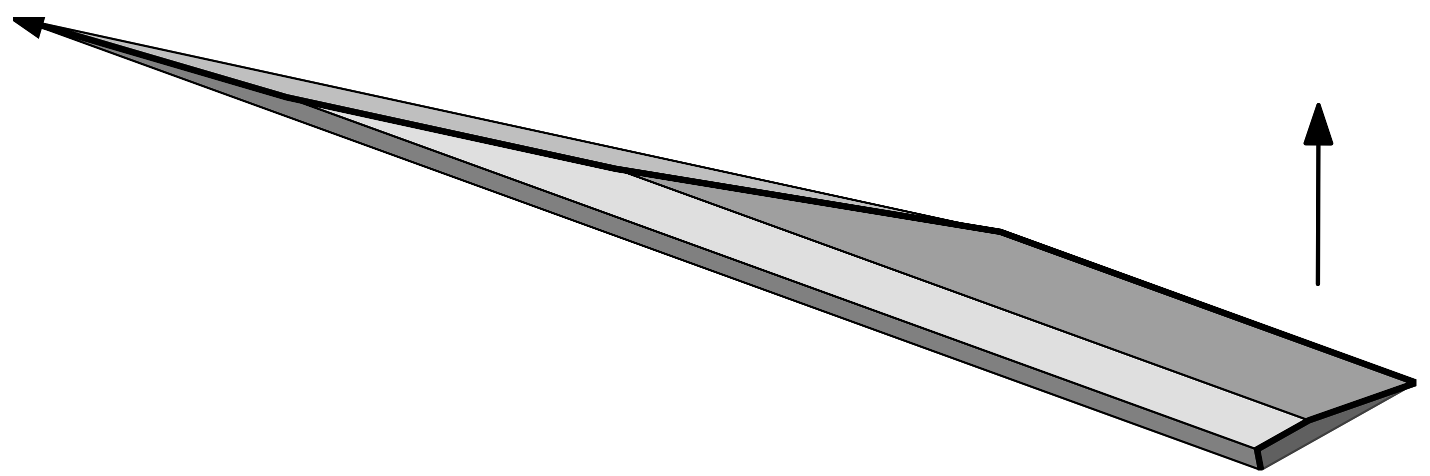
between $2m$ and $3m$ pivot steps.



$n = 2$



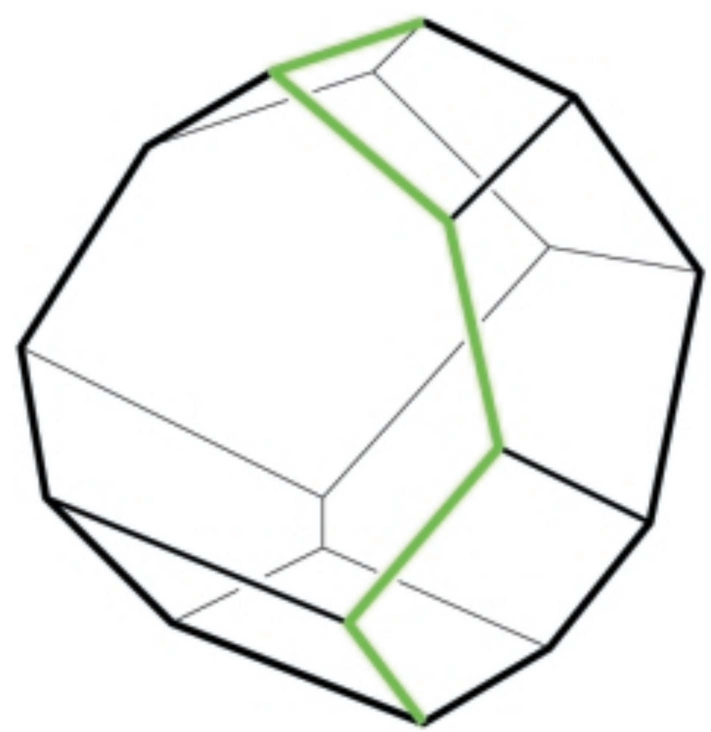
$n = 3$



Best theoretical result ...

$$e^{C\sqrt{n \ln n}}$$

Hirsch Conjecture



$$n^{1+\ln n}$$