بسم الله الرحمن الرحیم

نظریه علوم کامپیوتر

نظریه علوم کامپیوتر - بهار ۱۴۰۰-۱۴۰۱ - جلسه هفدهم: محاسبات تصادفی

Theory of computation - 002 - S17 - BPP

Probabilistic TMs

Defn: A probabilistic Turing machine (PTM) is a variant of a NTM where each computation step has 1 or 2 possible choices.

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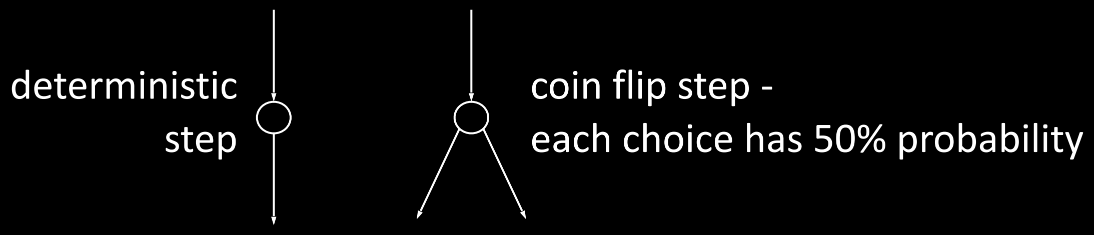

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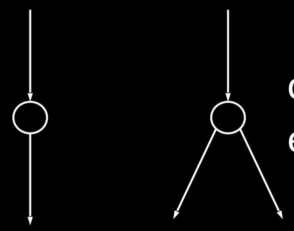
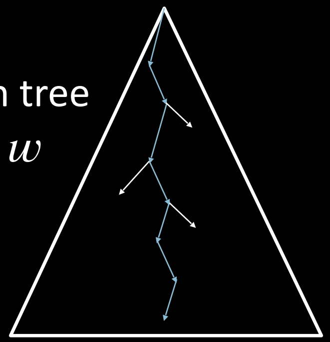
deterministic step



coin flip step - each choice has 50% probability

computation tree for M on w



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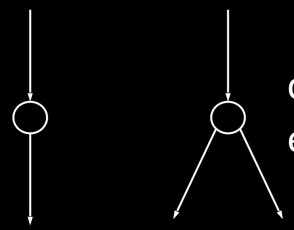

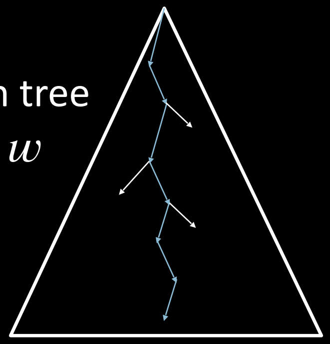

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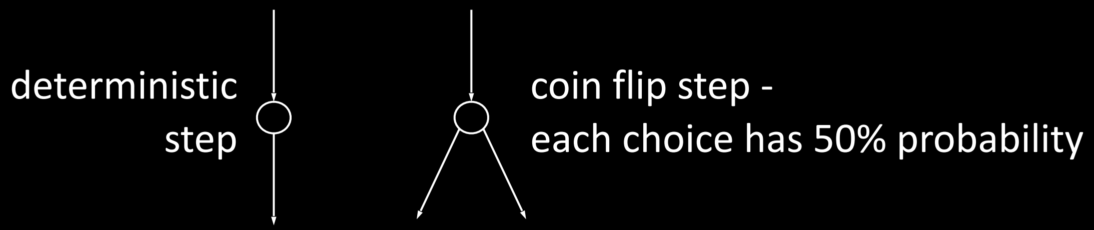

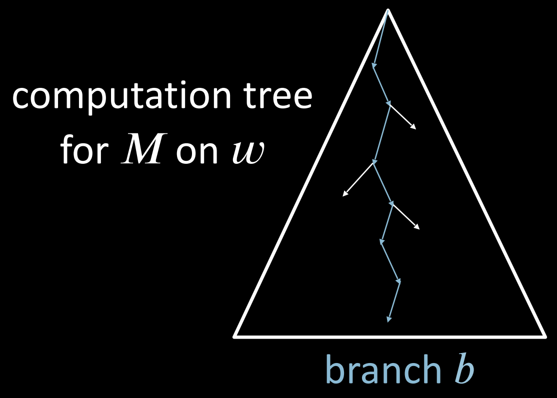

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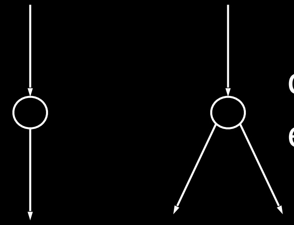

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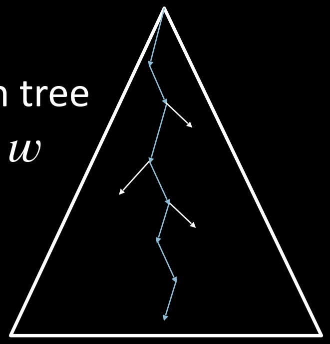

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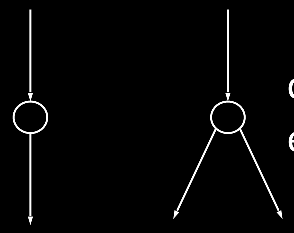

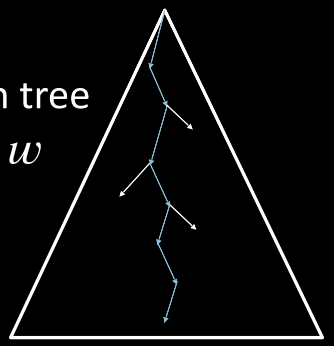

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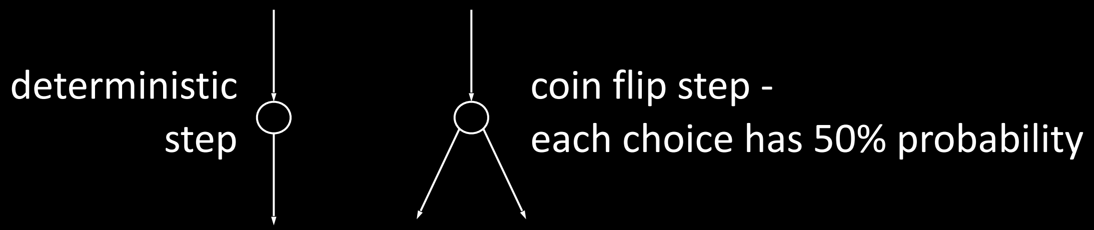

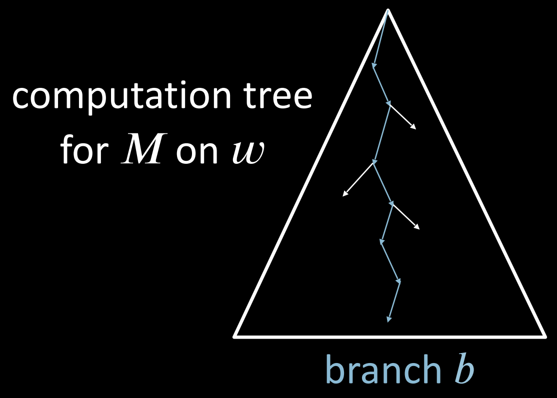

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The Class BPP

Defn: BPP $= \{A \,|\,$ some poly-time PTM decides A with error $\epsilon = \frac{1}{3}\}$

Amplification lemma: If M_1 is a poly-time PTM with error $\epsilon_1 < \frac{1}{2}$ then,

for any $0 < \epsilon_2 < \frac{1}{2}$, there is an equivalent poly-time PTM M_2 with error ϵ_2.

Can strengthen to make $\epsilon_2 < 2^{-\text{poly}(n)}$.

Proof idea: $M_2 =$ "On input w

1. Run M_1 on w for k times and output the majority response."

Details: Calculation to obtain k and the improved error probability.

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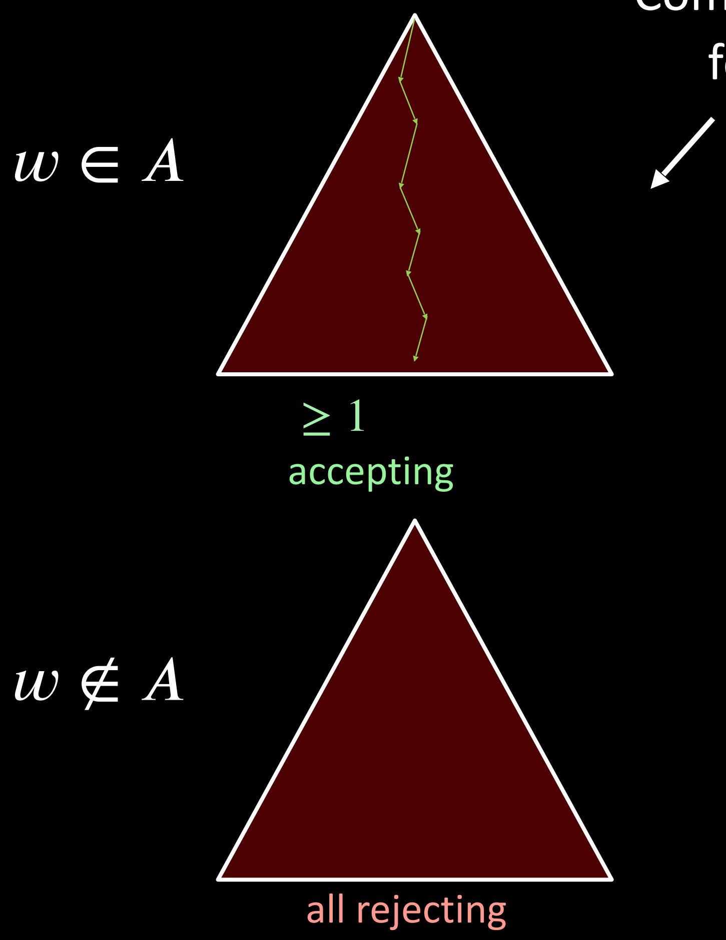
NP and BPP

NP

Computation trees for M on w



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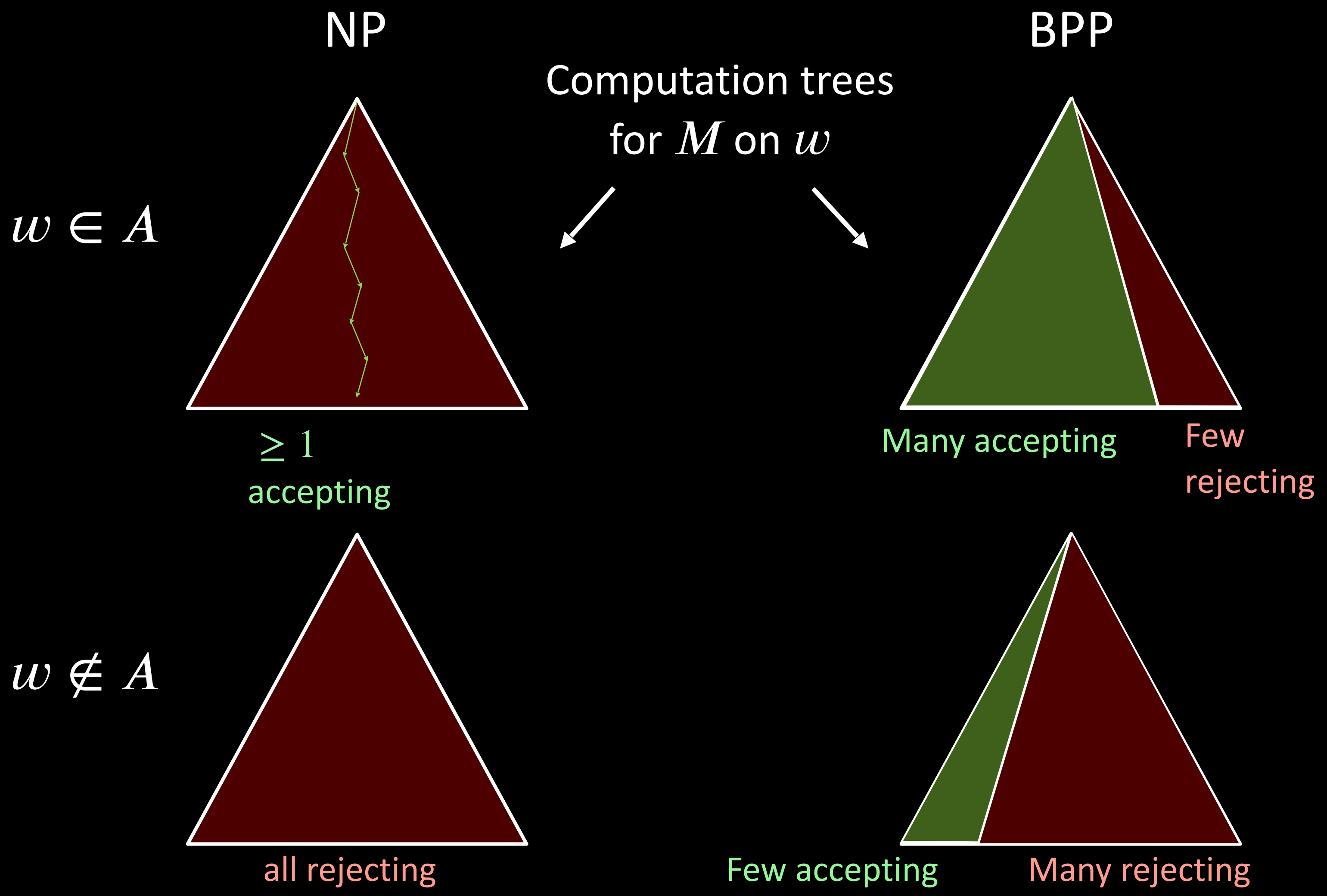

NP
BPP
Computation trees
for M on w
$w \in A$
$w \notin A$
≥ 1
accepting
Many accepting
Few
rejecting
all rejecting
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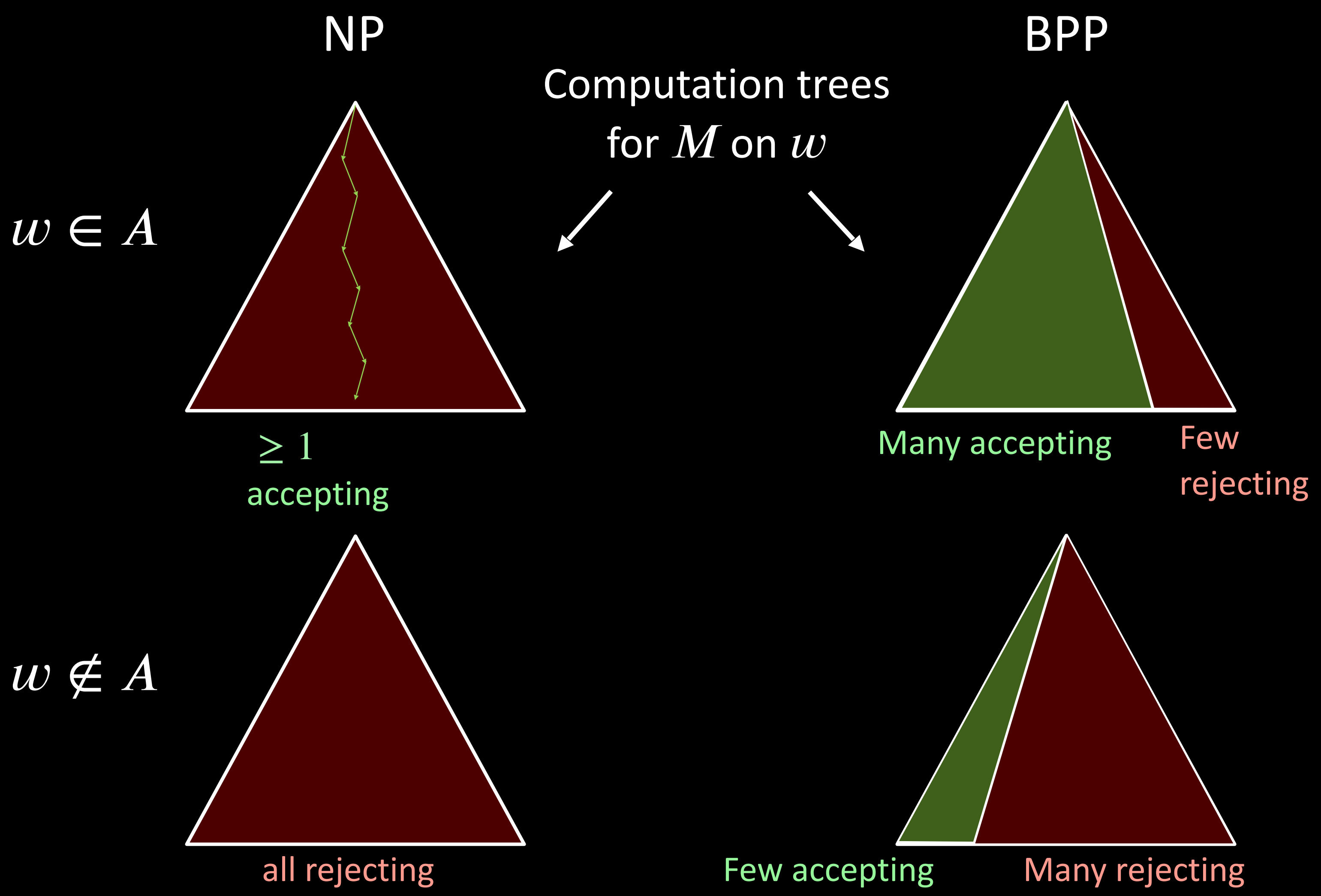

Check-in 23.1

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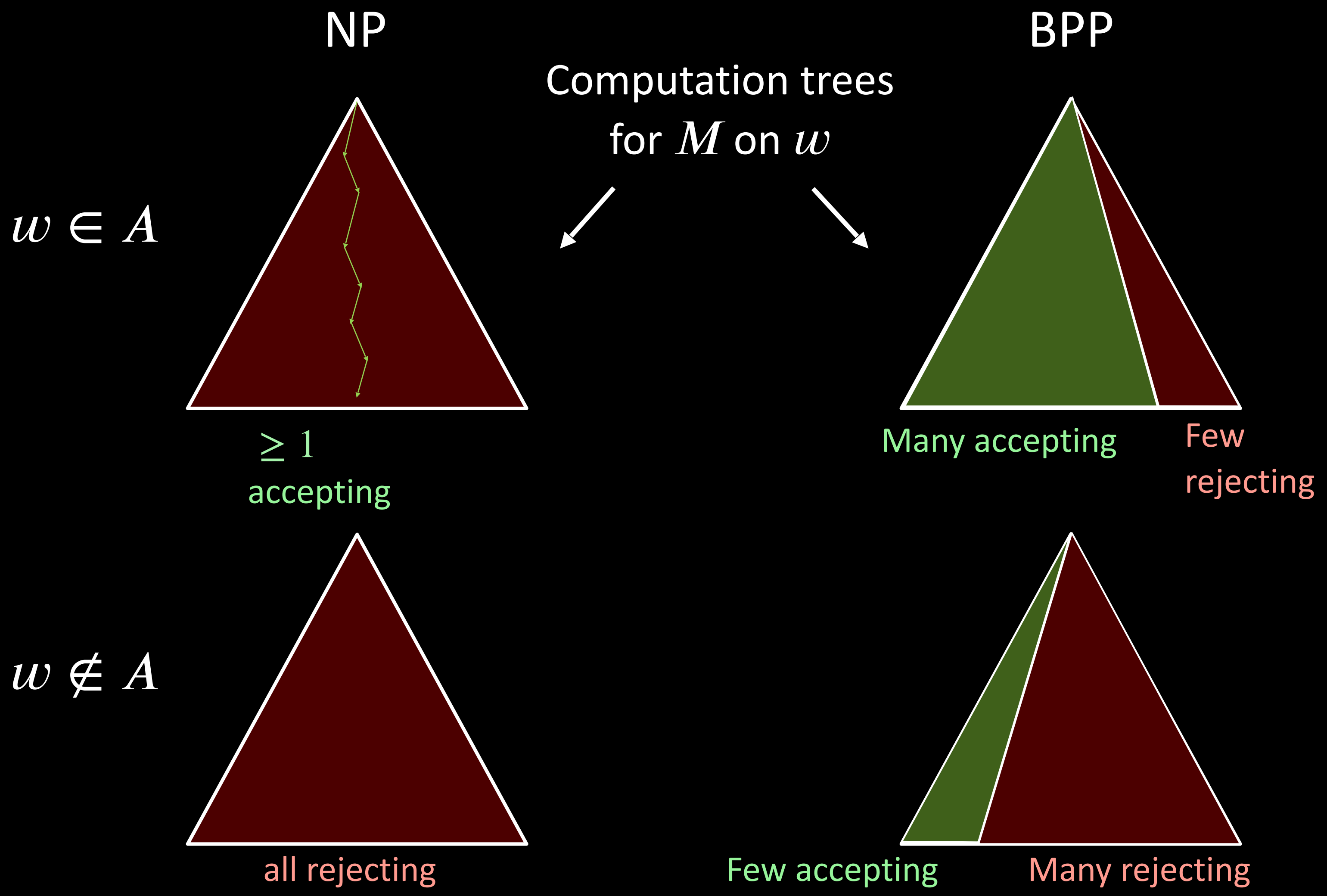

Check-in 23.1

Which of these are known to be true? Check all that apply.

(a) BPP is closed under union.

(b) BPP is closed under complement.

(c) P $\subseteq$ BPP

(d) BPP $\subseteq$ PSPACE

Check-in 23.1

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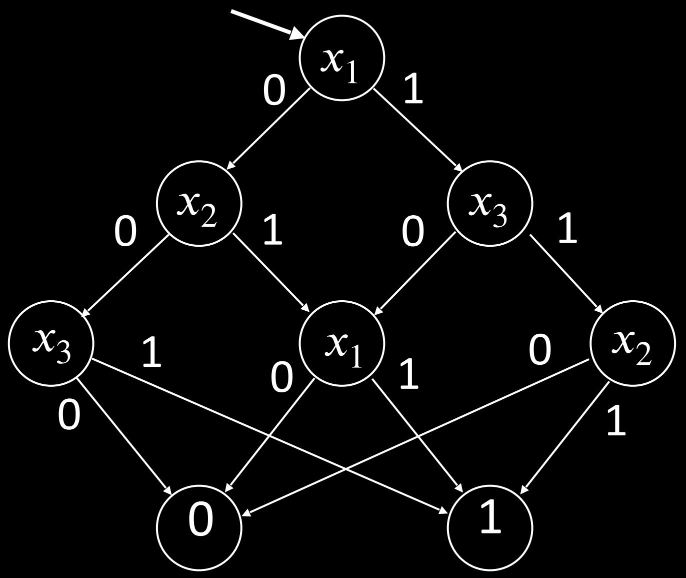

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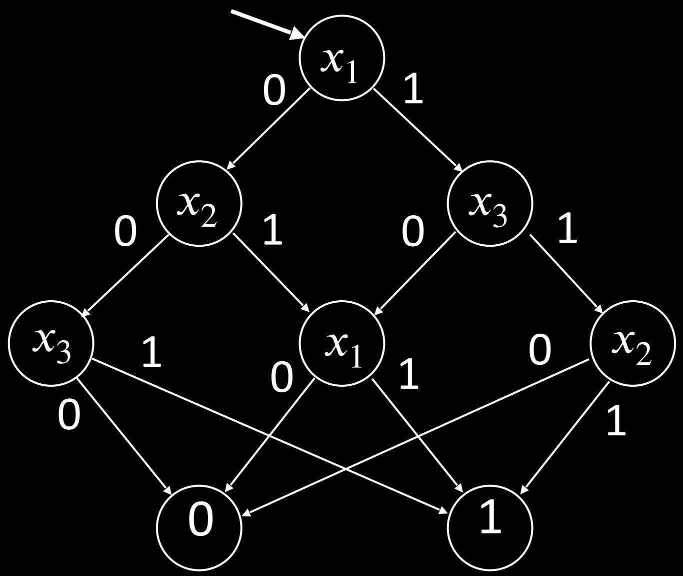

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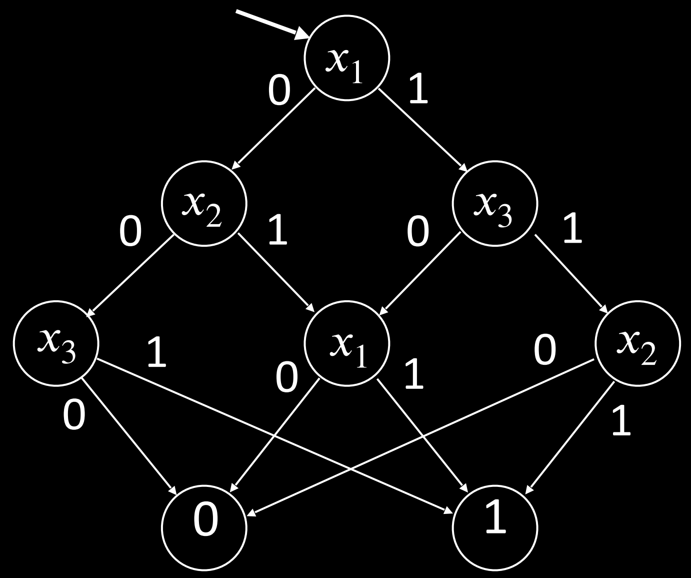

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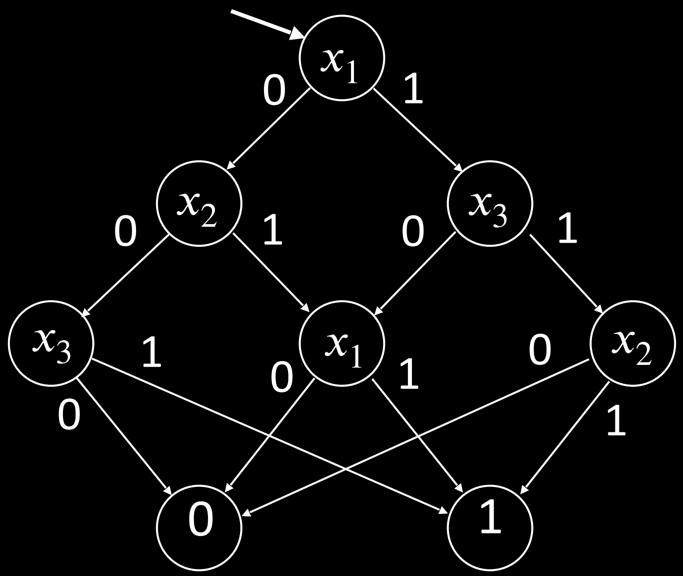

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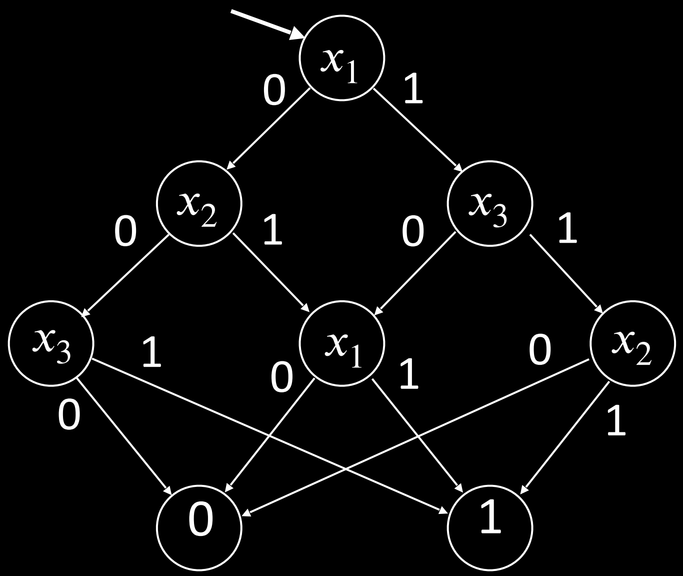

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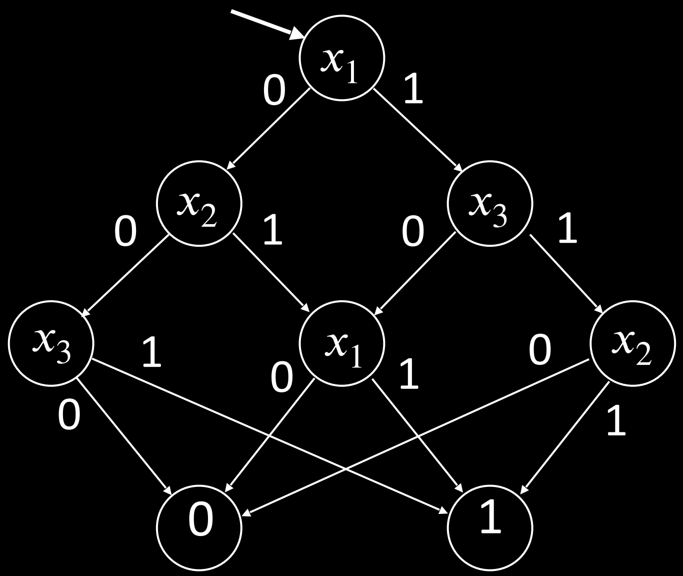

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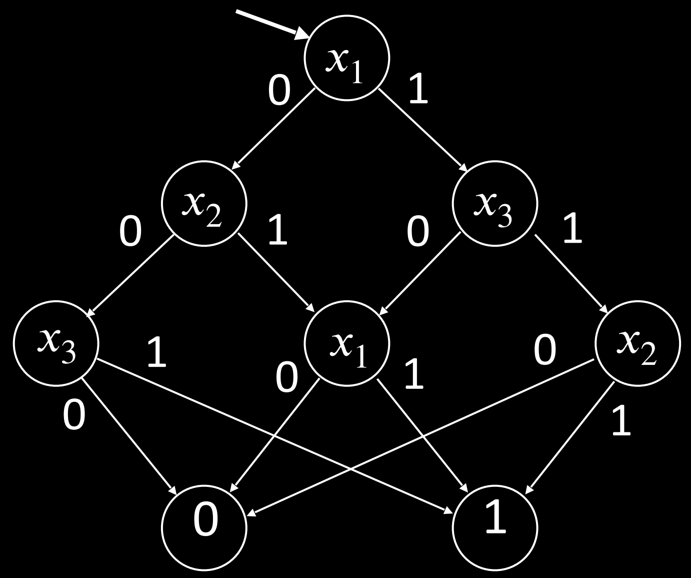

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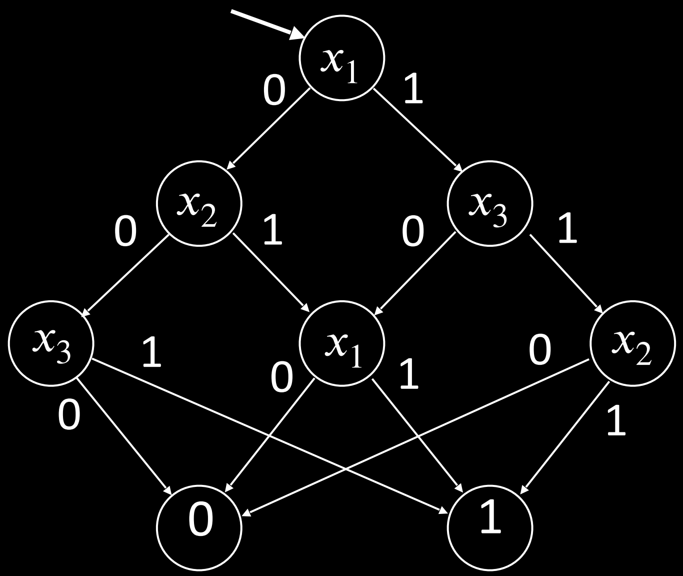

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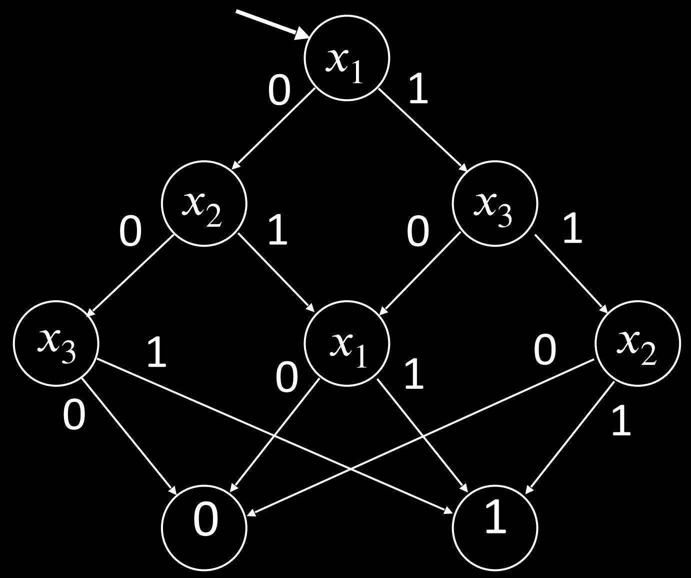

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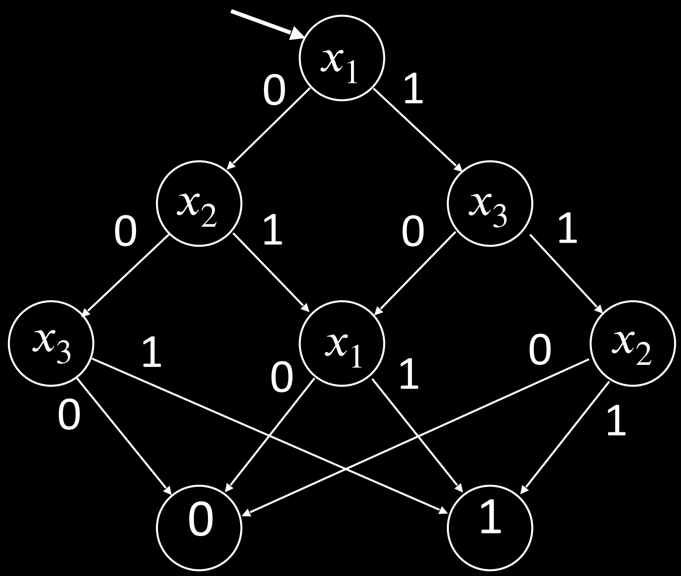

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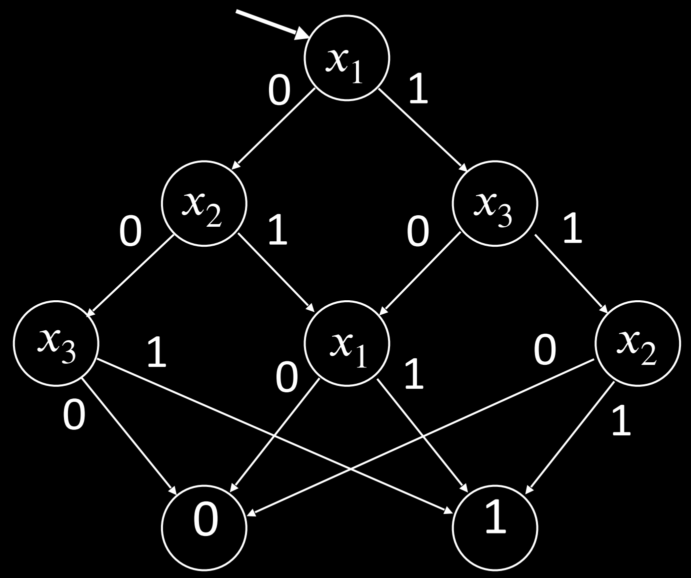

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Read-once Branching Programs

Defn: A BP is read-once if it never queries a variable more than once on any path from the start node to an output.

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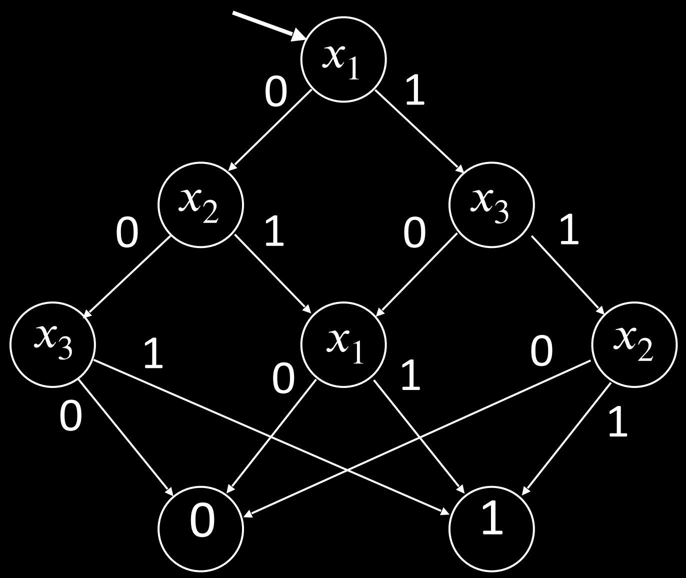

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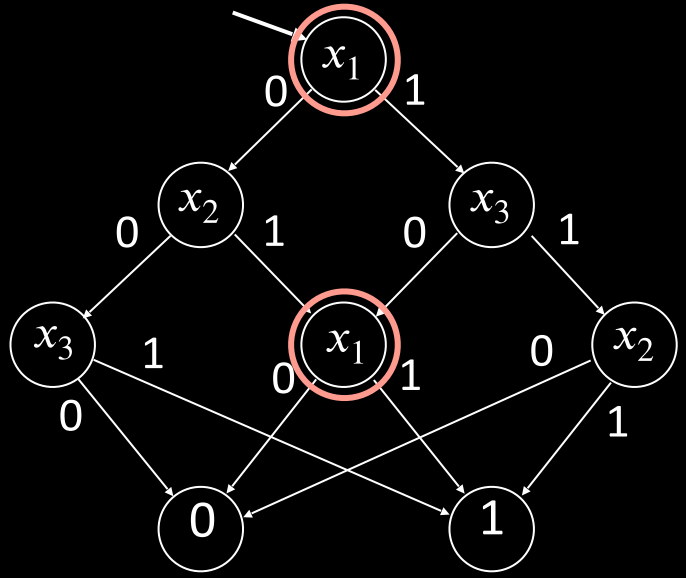

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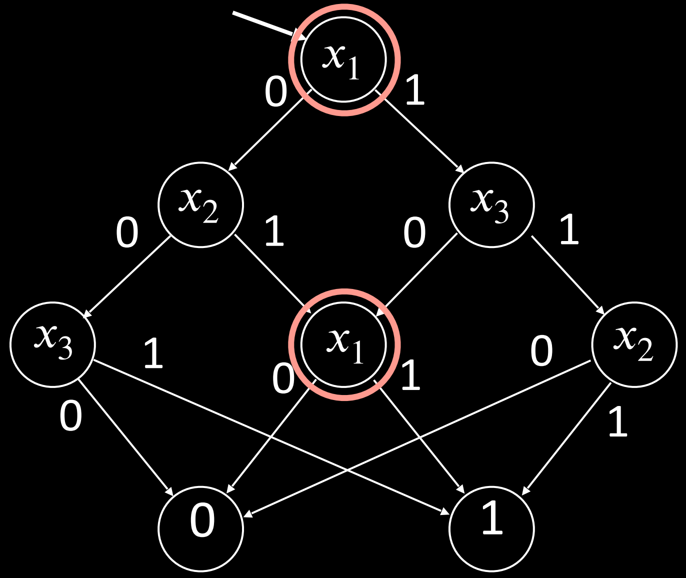

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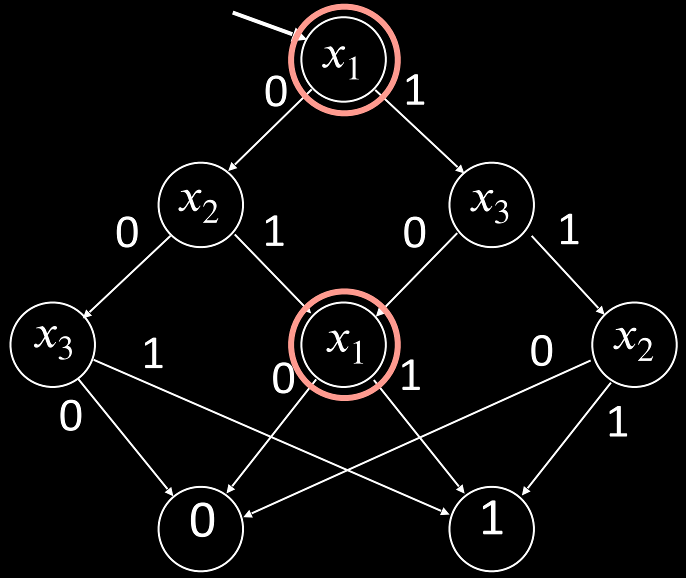

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Check-in 23.2

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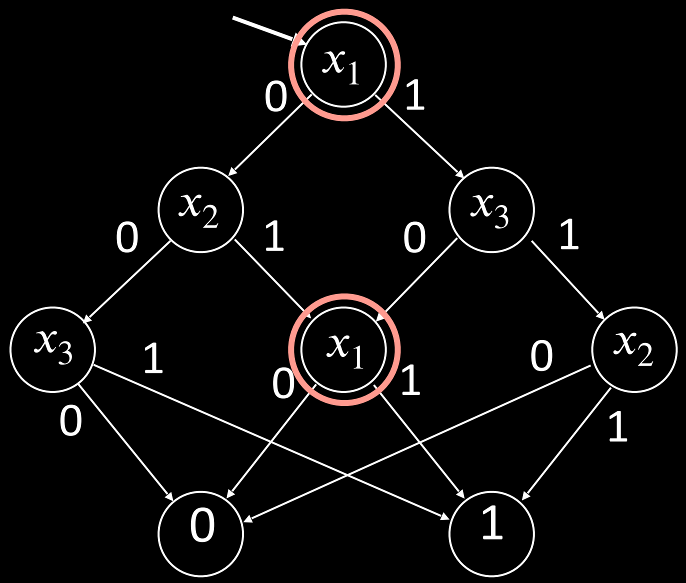
Assuming (as we will show) that EQROBP $\in$ BPP, can we use that to show EQBP $\in$ BPP by converting branching programs to read-once branching programs?

(a) Yes, there is no need to re-read inputs.

(b) No, we cannot do that conversion in general.

(c) No, the conversion is possible but not in polynomial-time.



Not read-once

Check-in 23.2

EQROBP $\in$ BPP

Theorem: EQ**ROBP** $\in$ BPP

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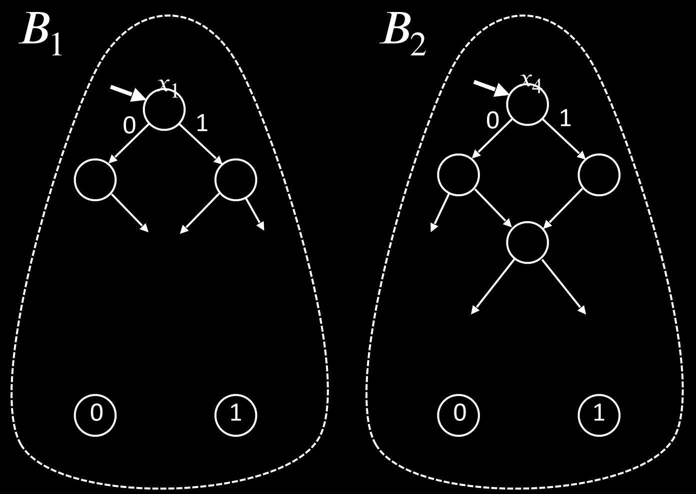

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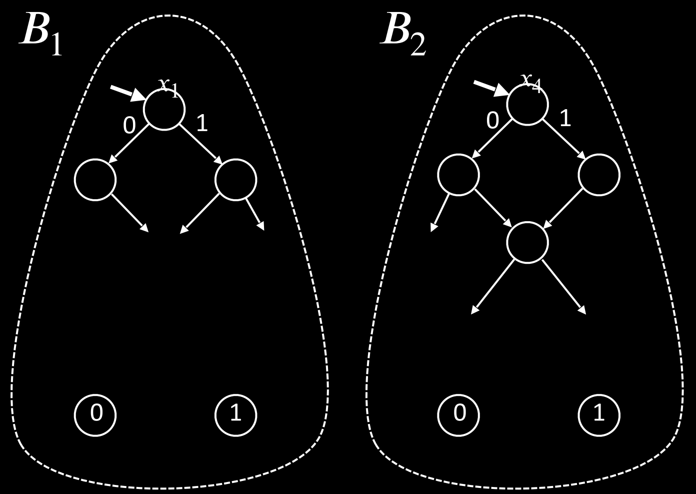

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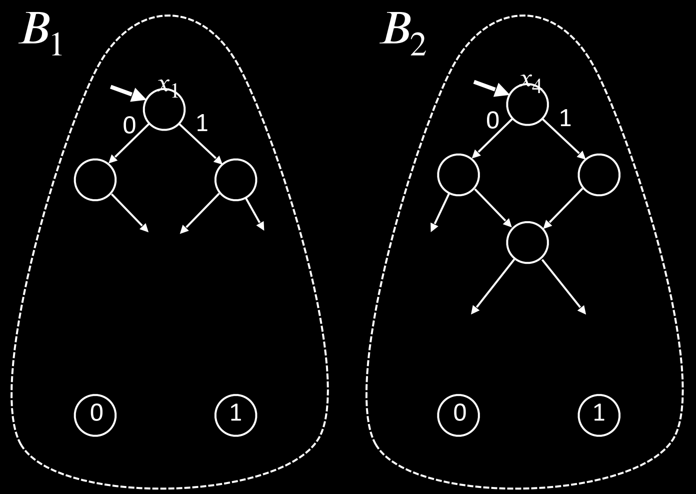

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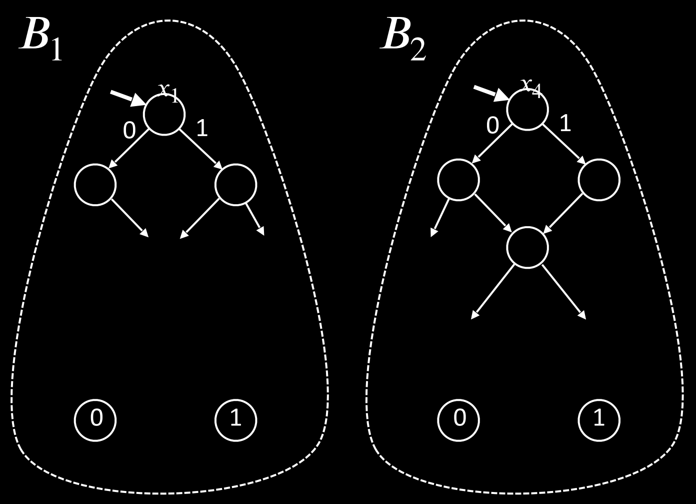

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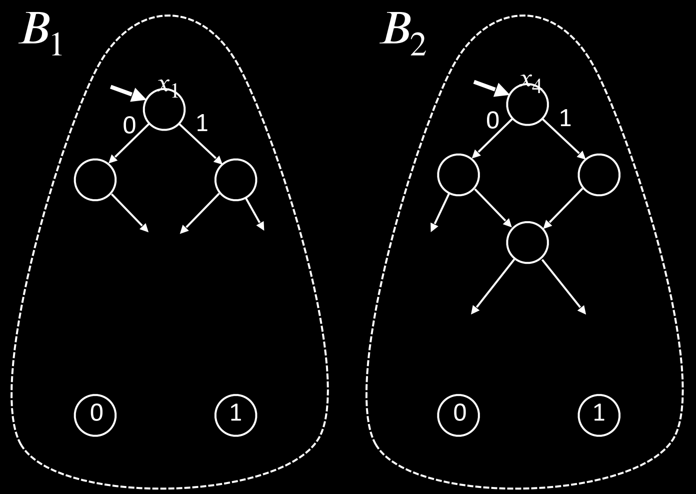

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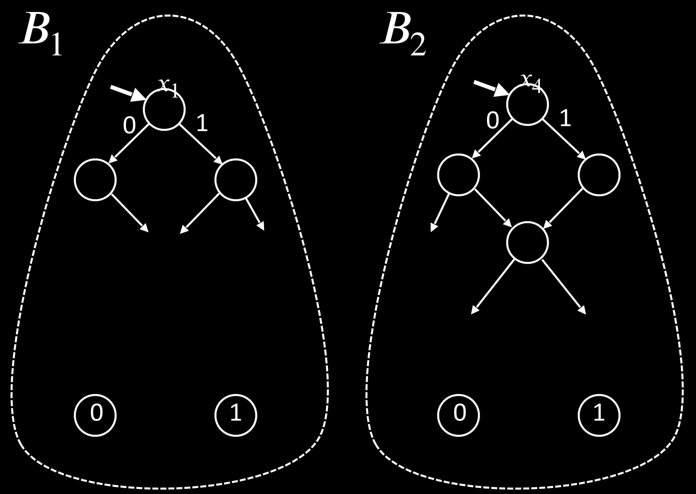

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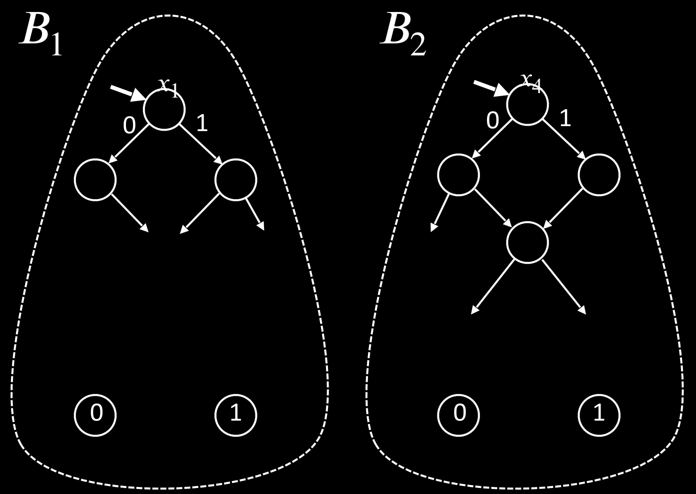

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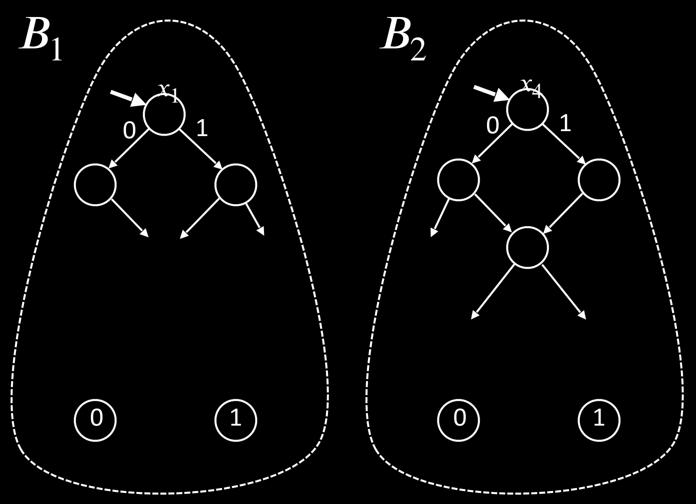

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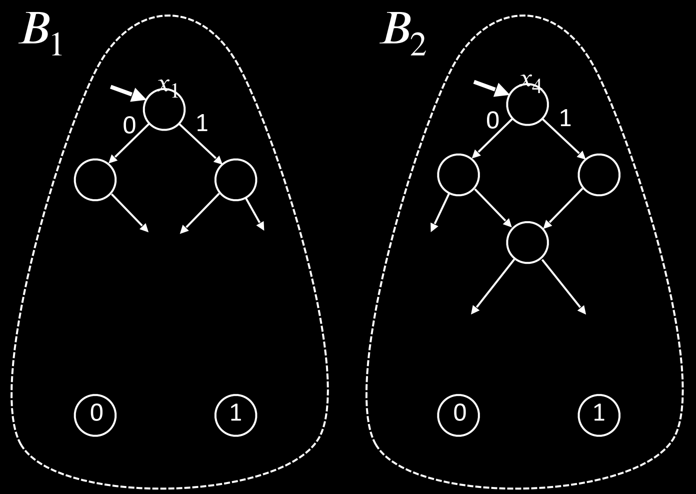

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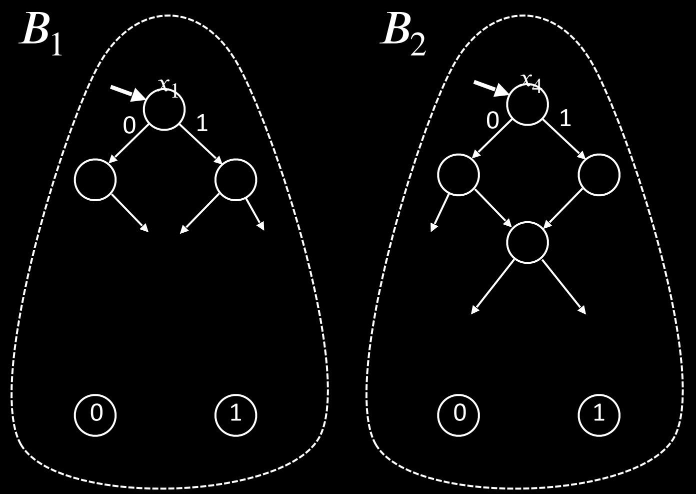

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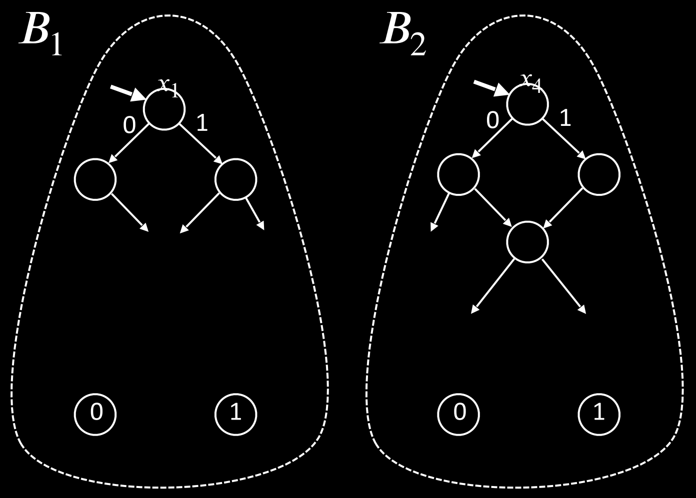

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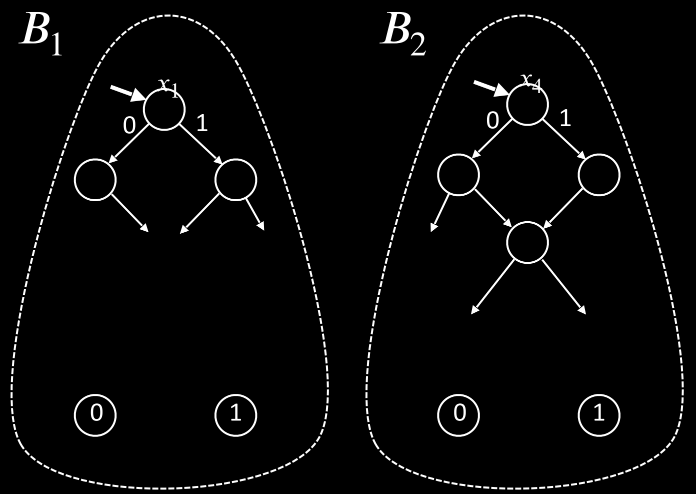

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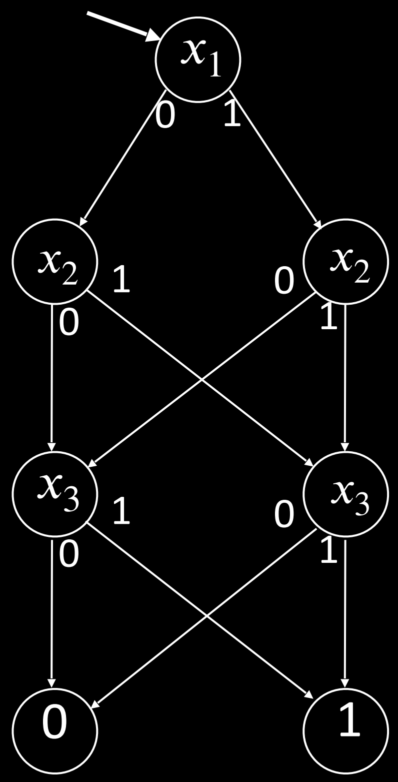
Boolean Labeling

Alternative way to view BP computation



Boolean Labeling

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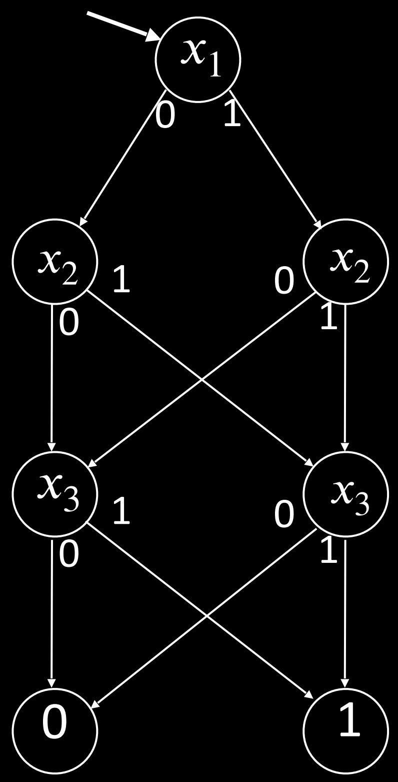

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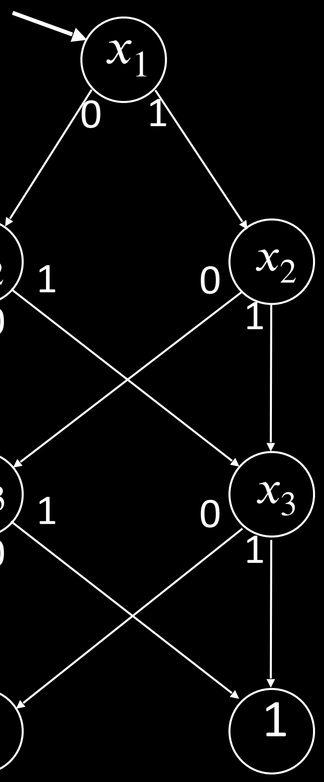

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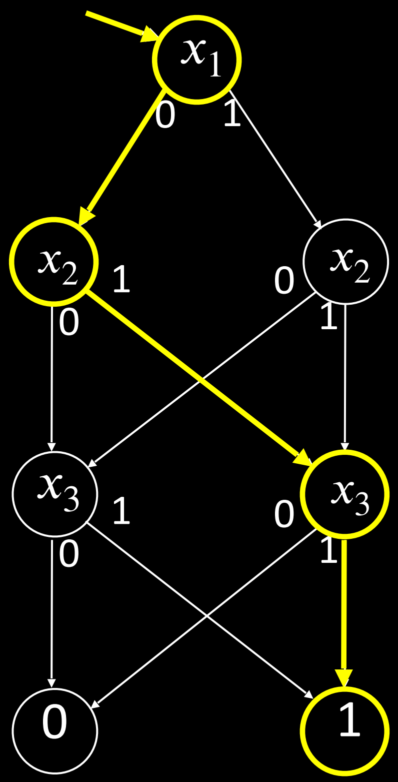

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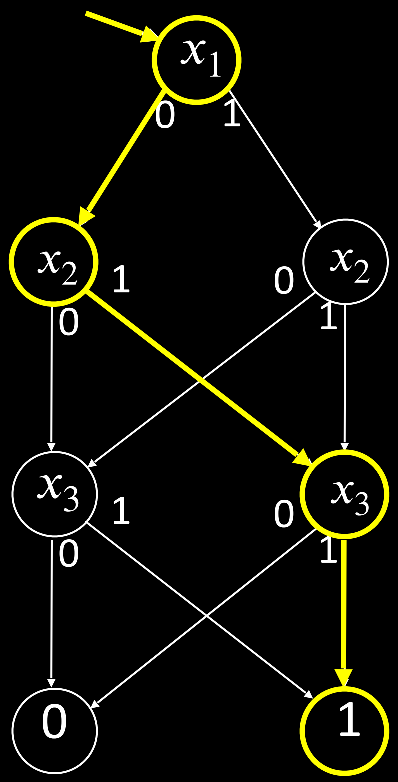

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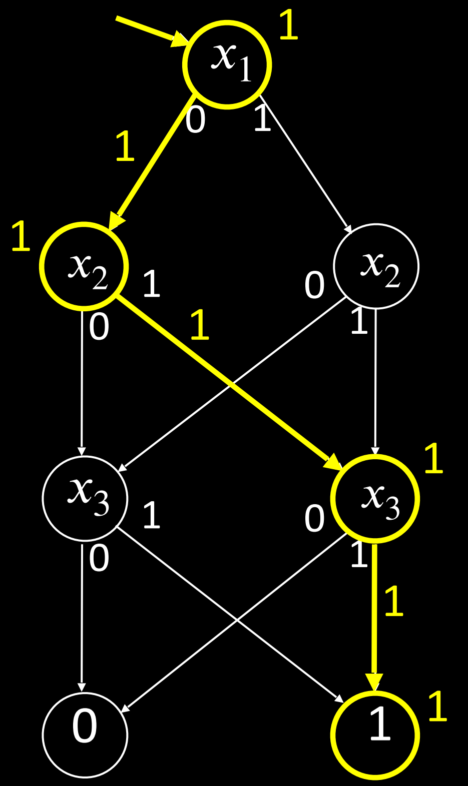

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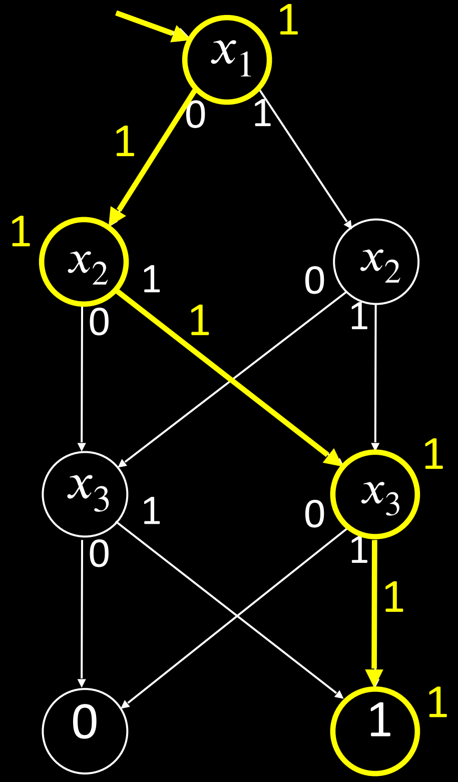

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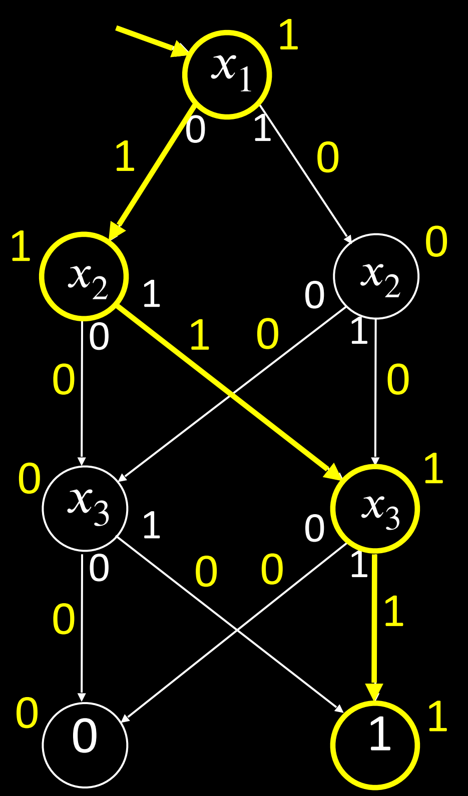

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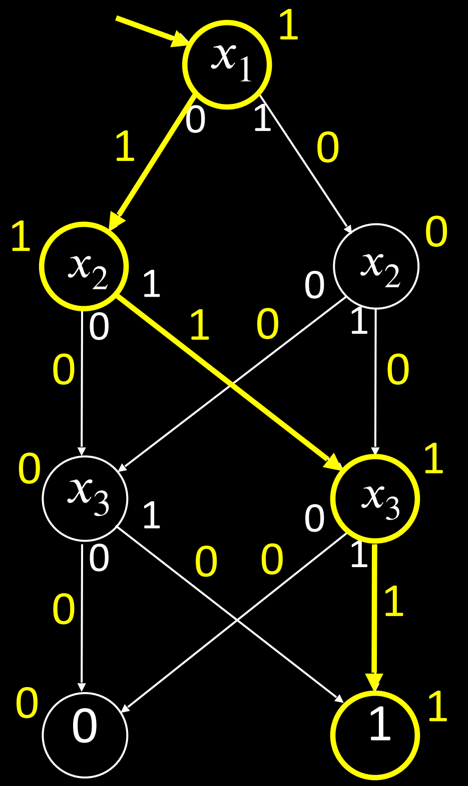

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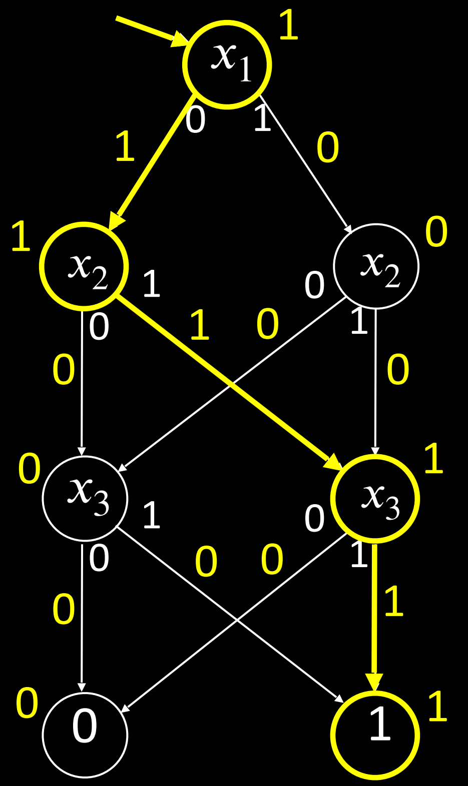

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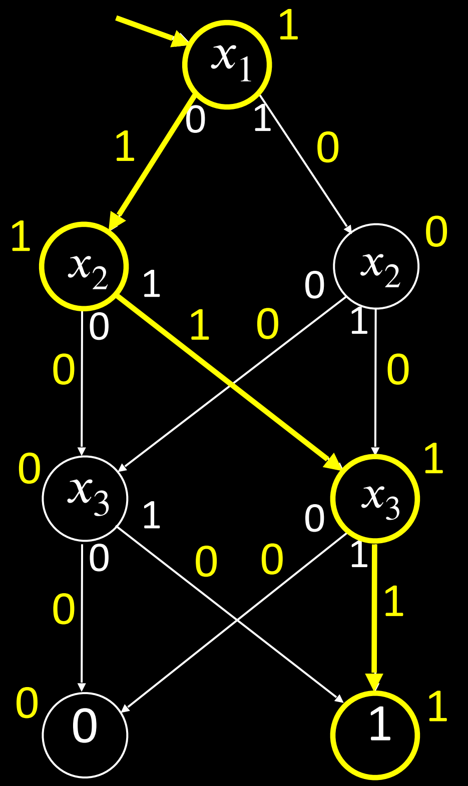

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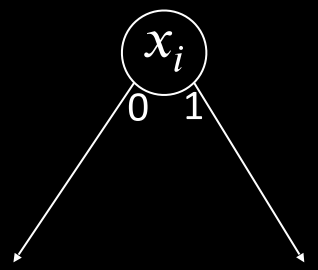

Label edges from nodes

Boolean Labeling

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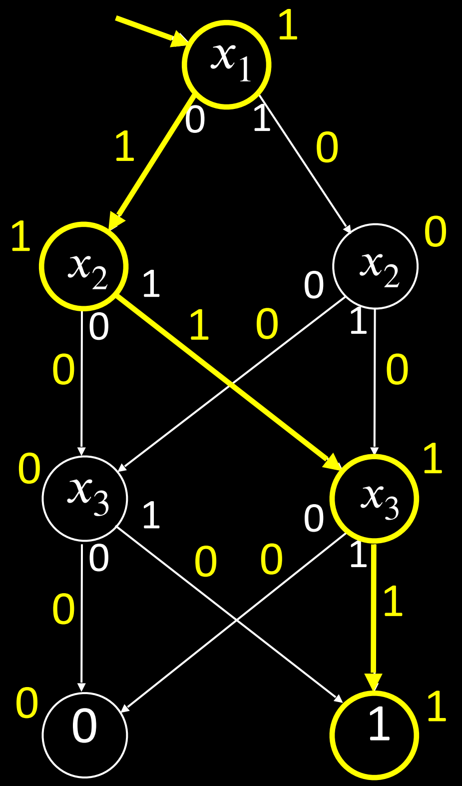

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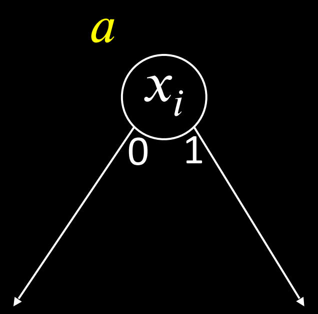

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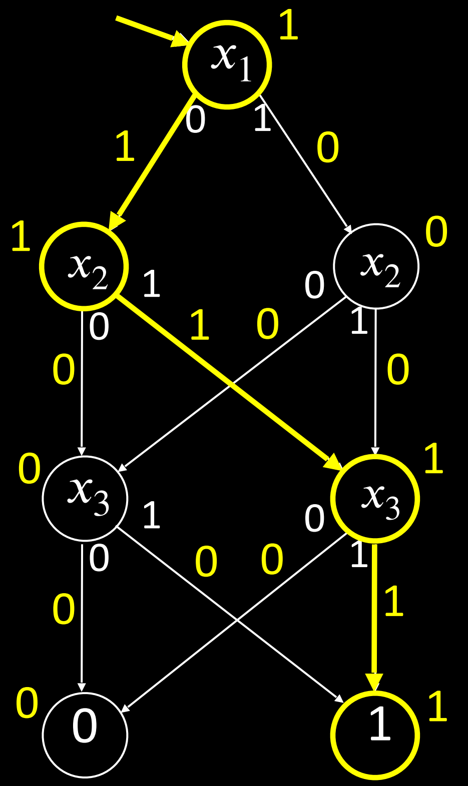

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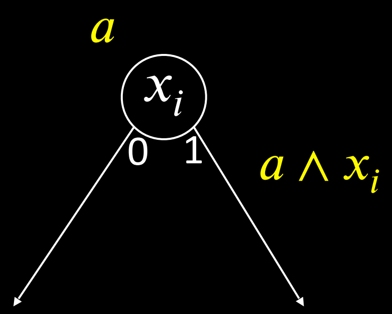

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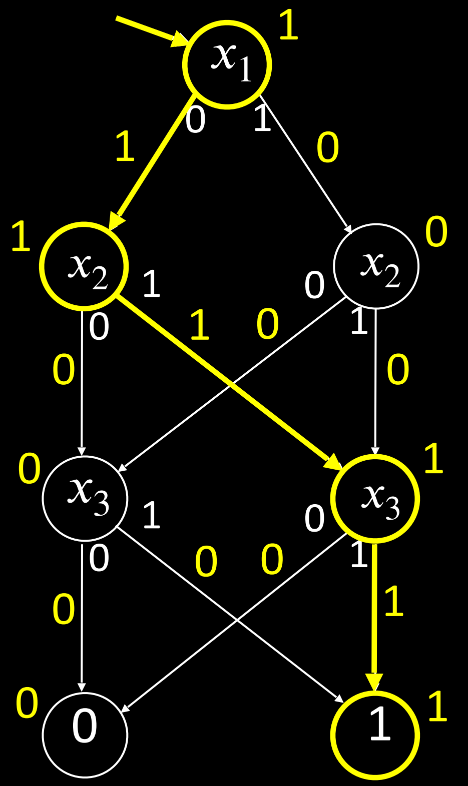

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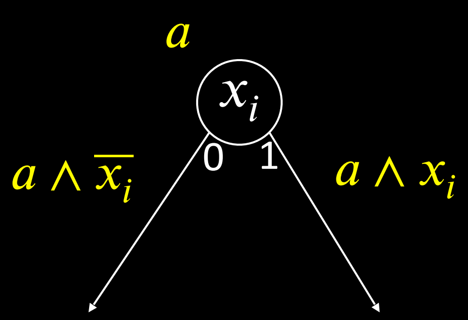

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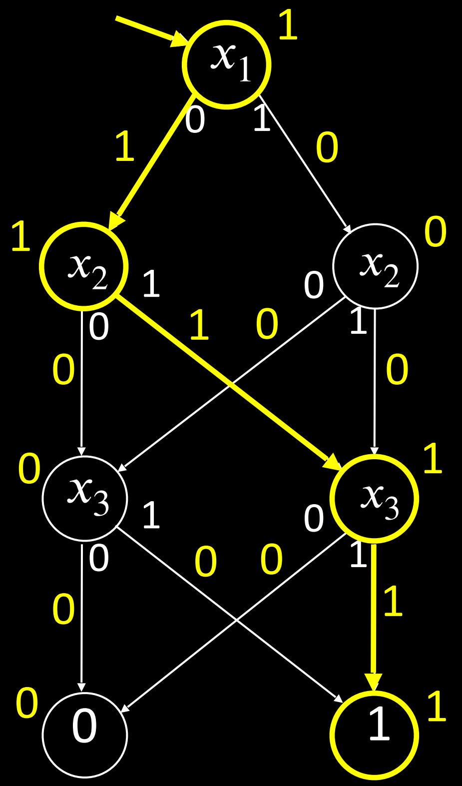

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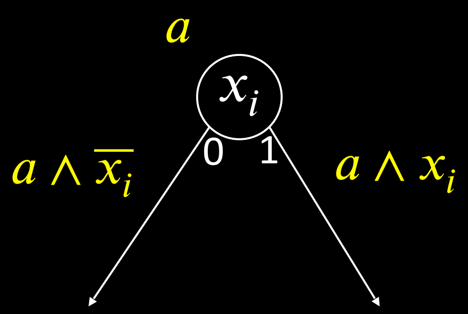

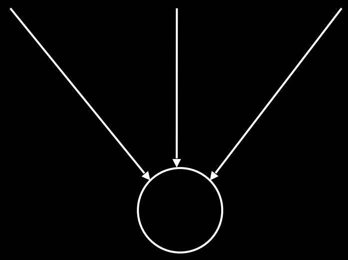

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Label nodes from incoming edges

Boolean Labeling

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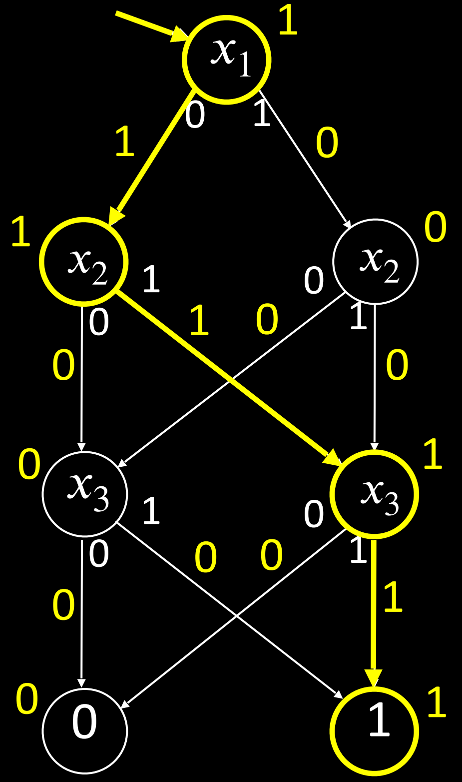

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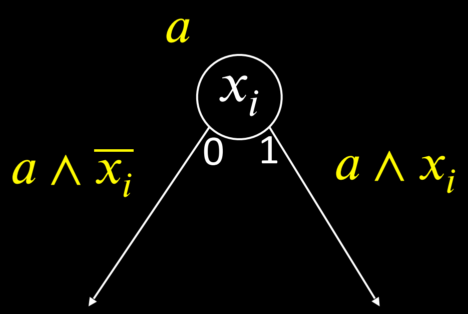

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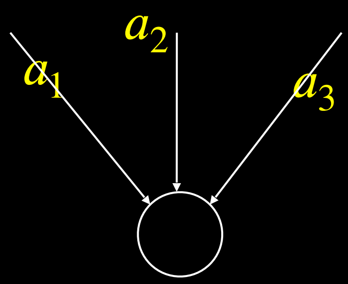

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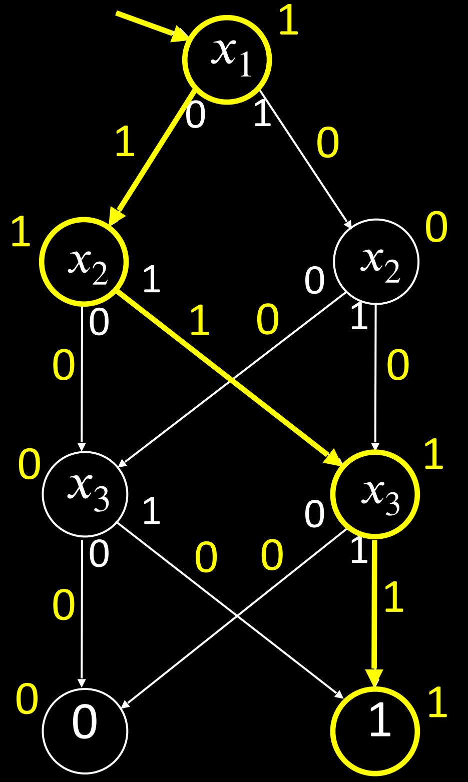

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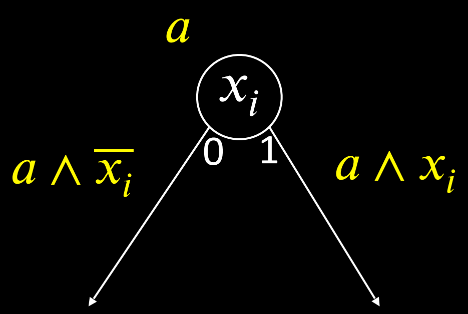

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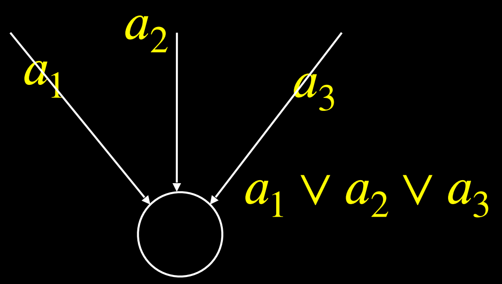

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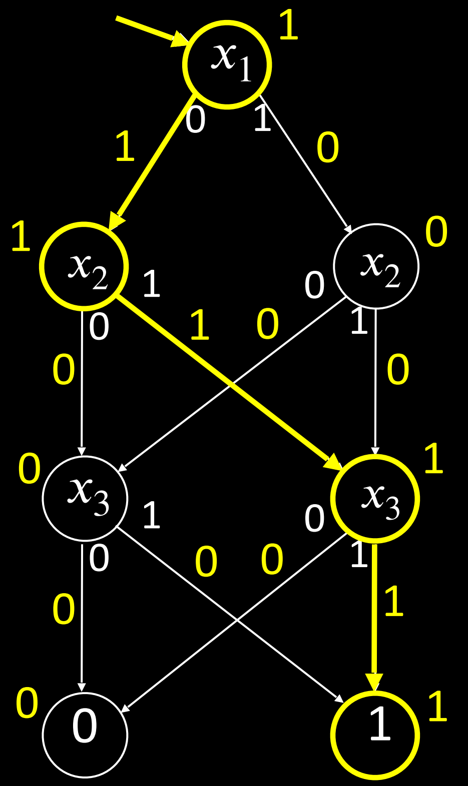
Boolean Labeling

Alternative way to view BP computation



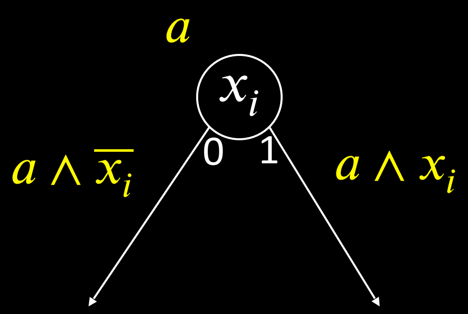
Show by example: Input is $x_1 = 0, \quad x_2 = 1, \quad x_3 = 1$

The BP follows its execution path.

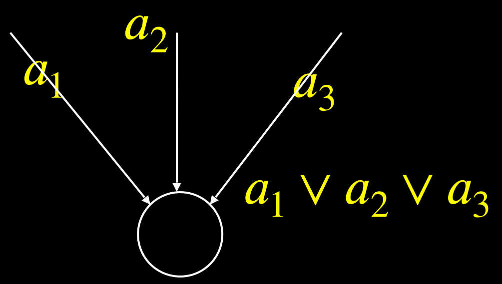
Label all nodes and edges on the execution path with 1 and off the execution path with 0.

Output the label of the output node 1.

Obtain the labeling inductively by using these rules:



Label edges from nodes



Label nodes from incoming edges

Arithmetization Method

Method: Simulate $\wedge$ and $\vee$ with $+$ and $\times$.

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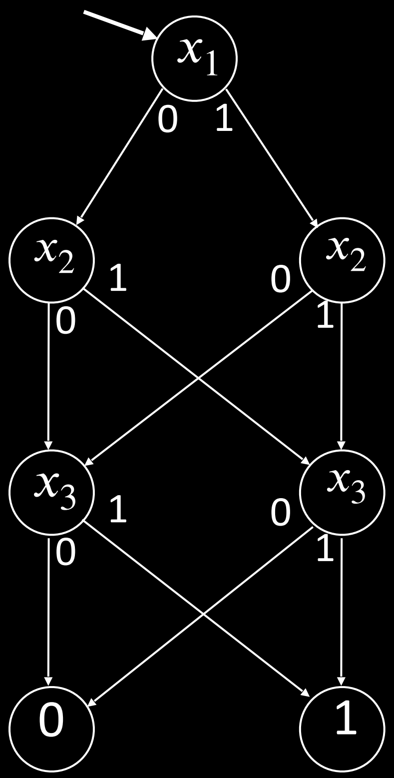

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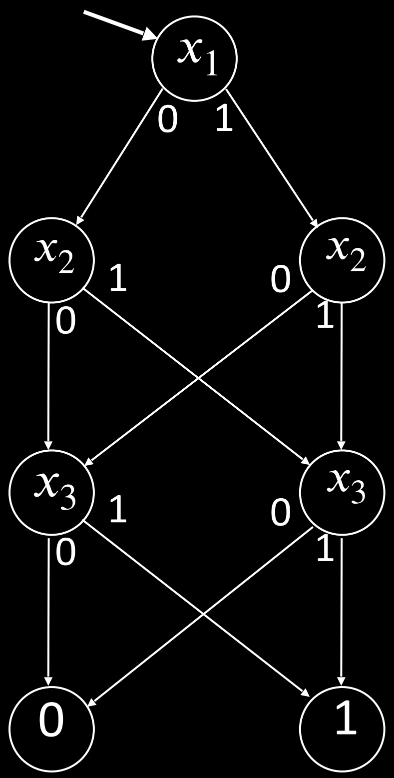

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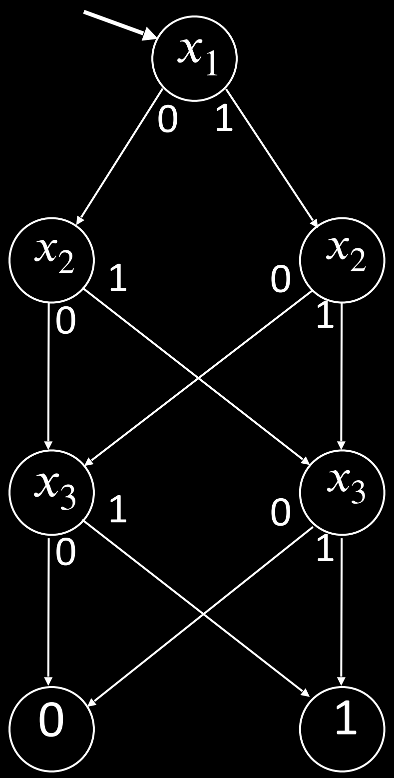

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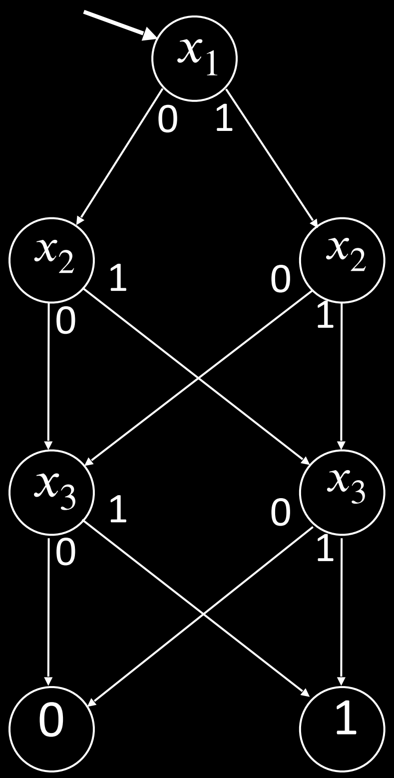

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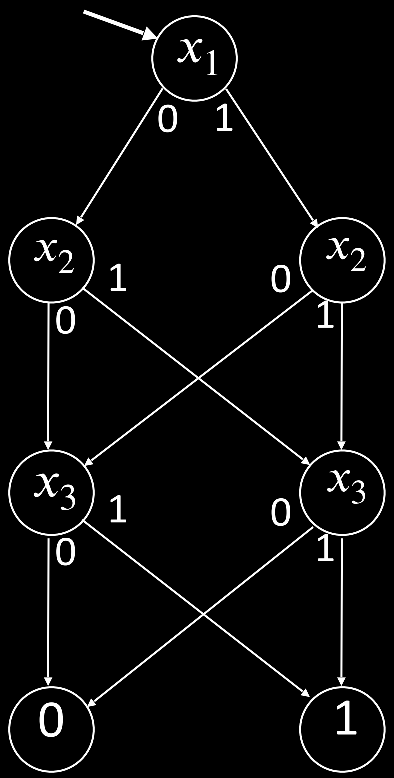

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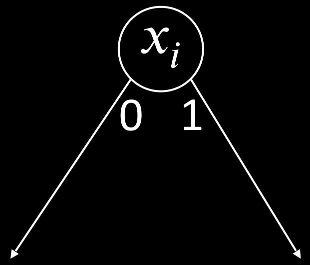

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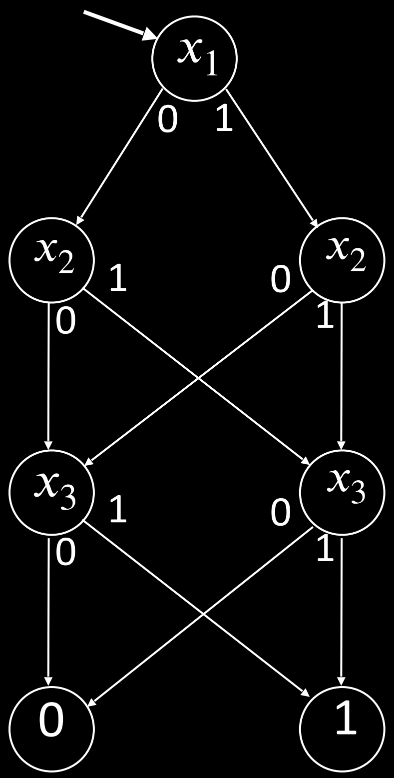

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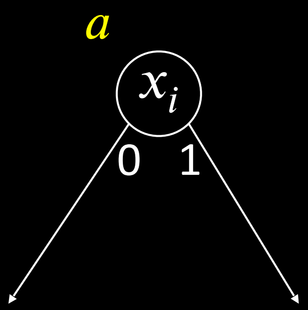

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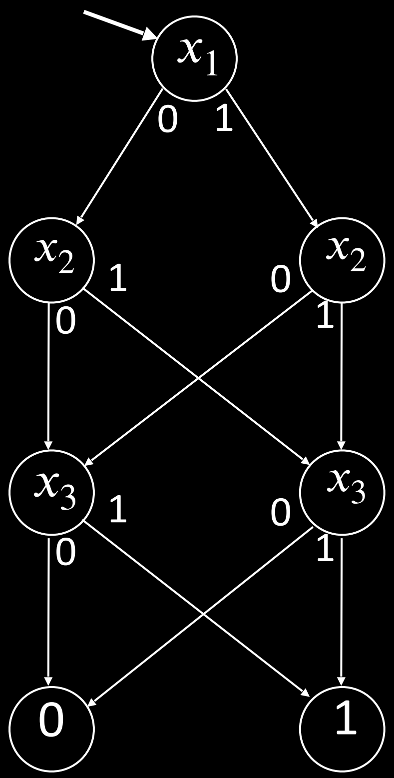

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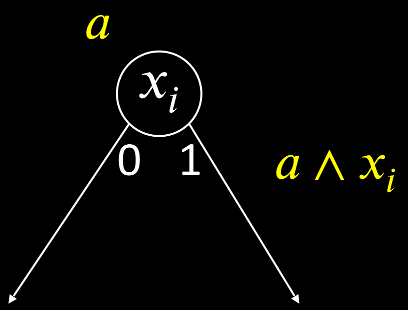

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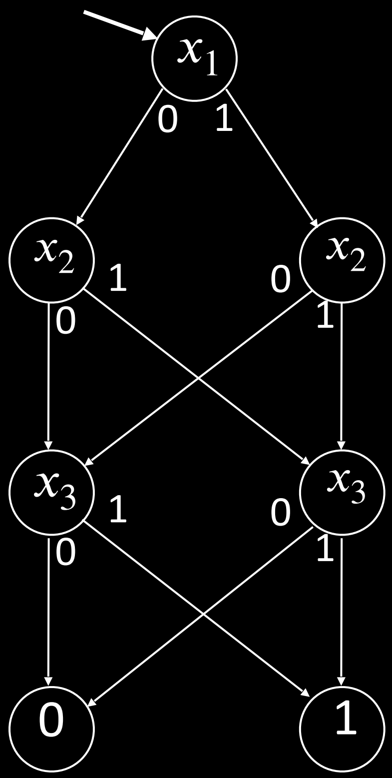

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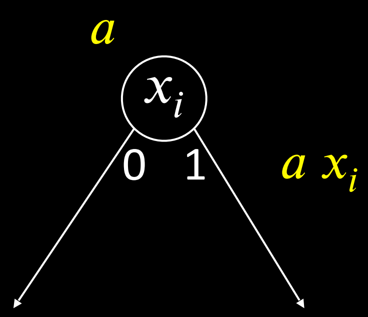

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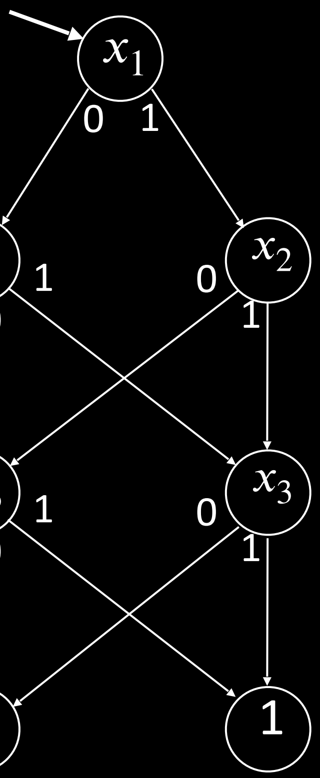

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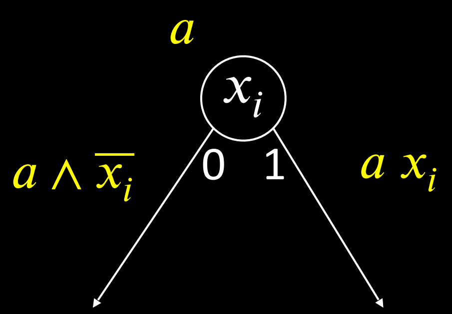

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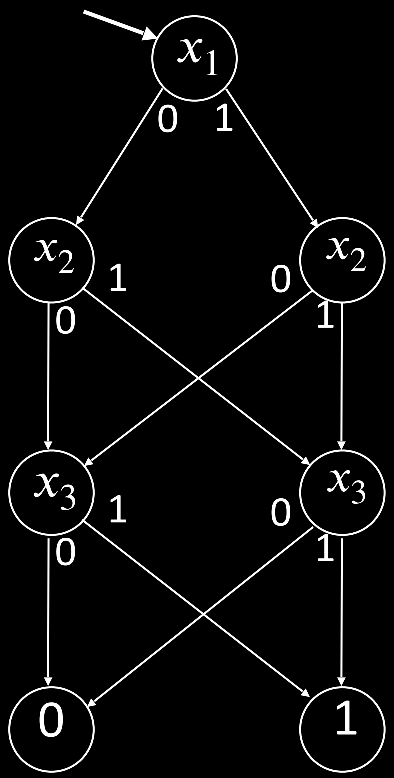

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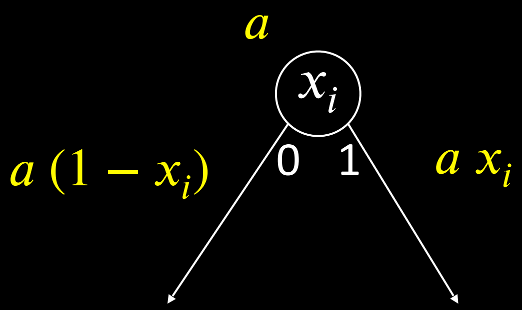

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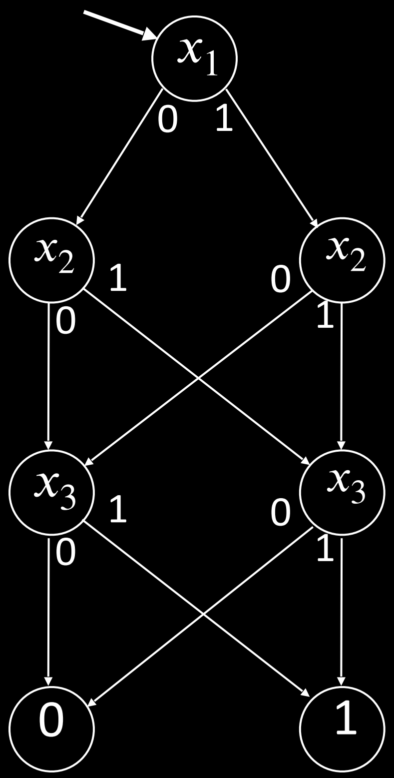

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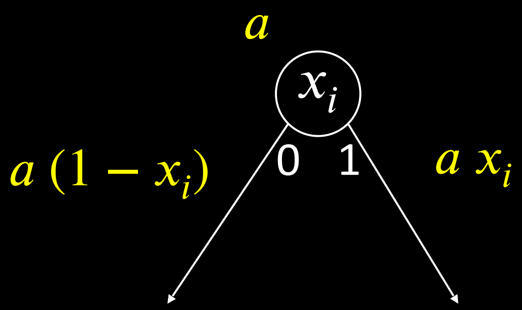

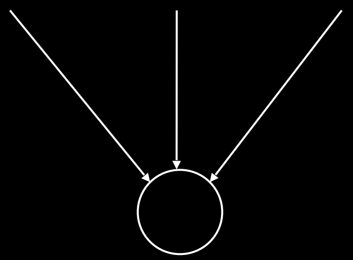

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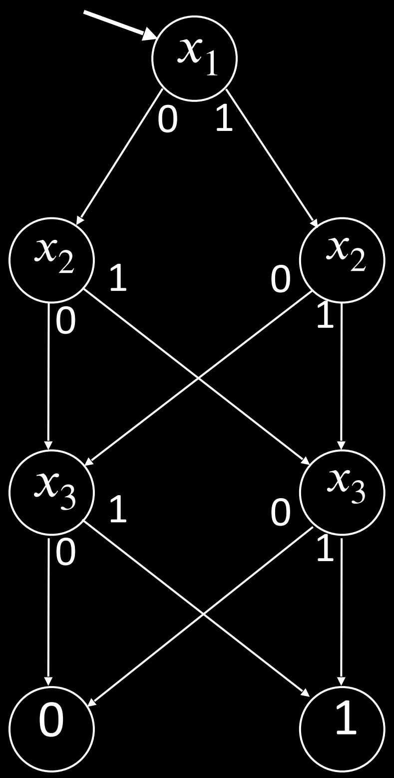

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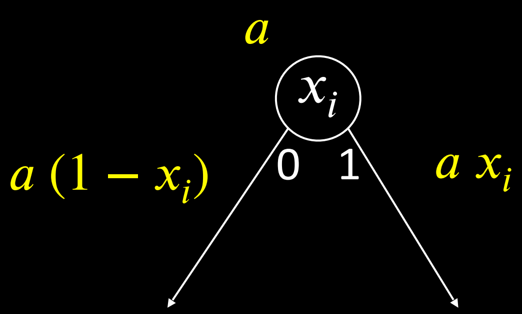

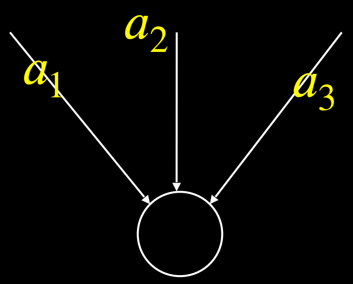

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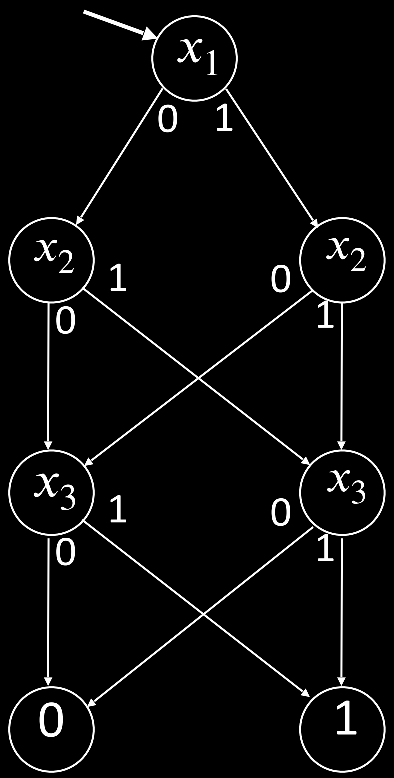

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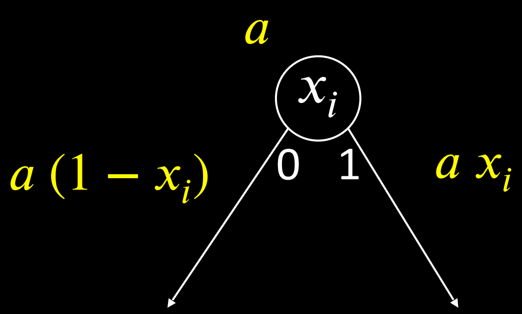

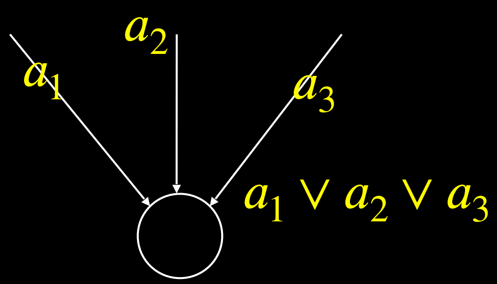

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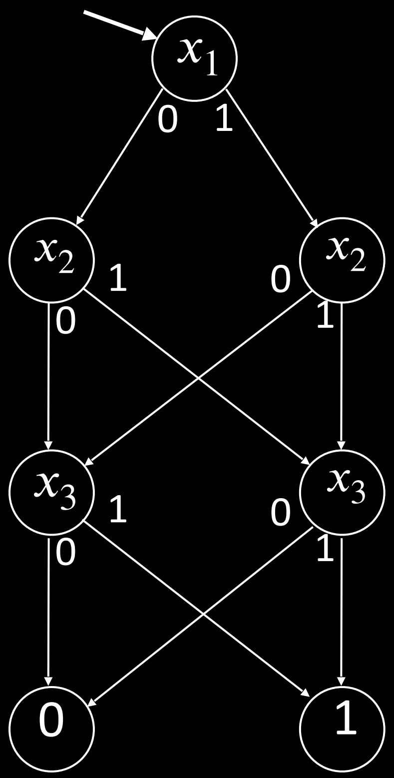

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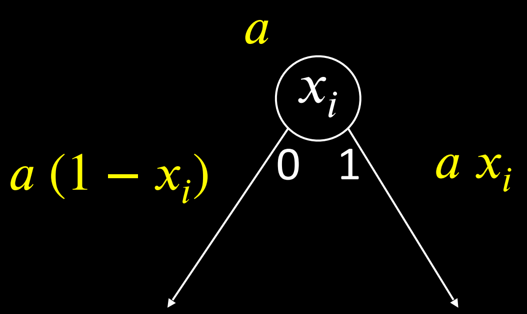

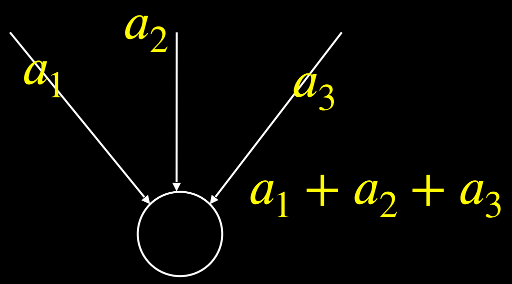

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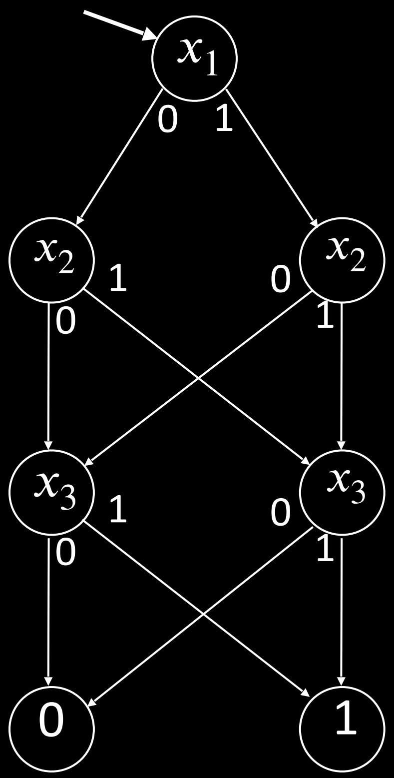

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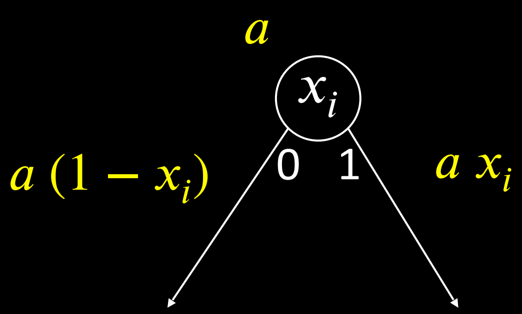

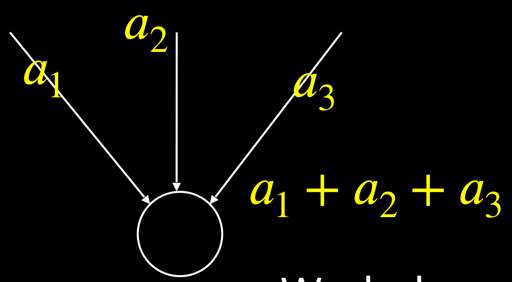

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The execution path can enter a node at most one time.

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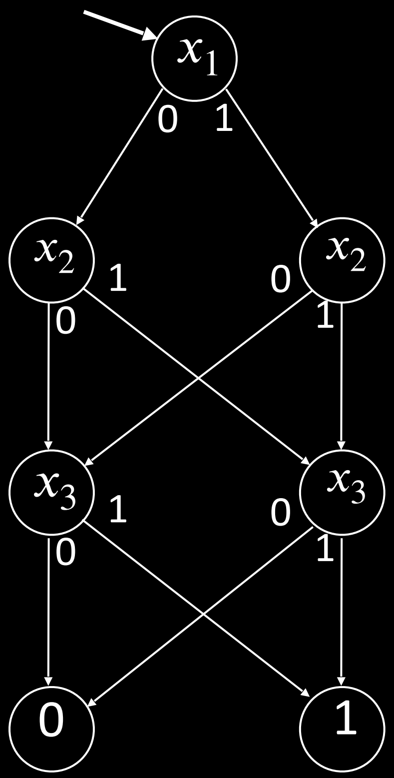

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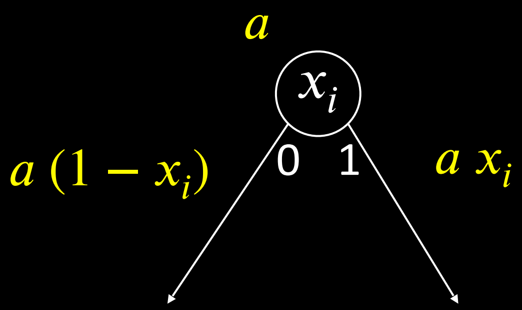

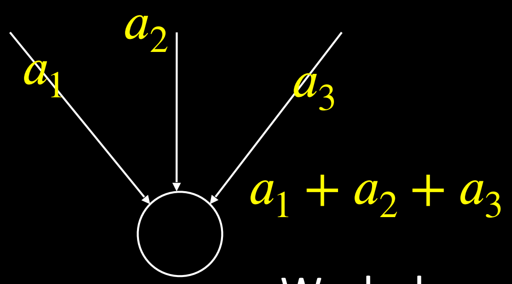

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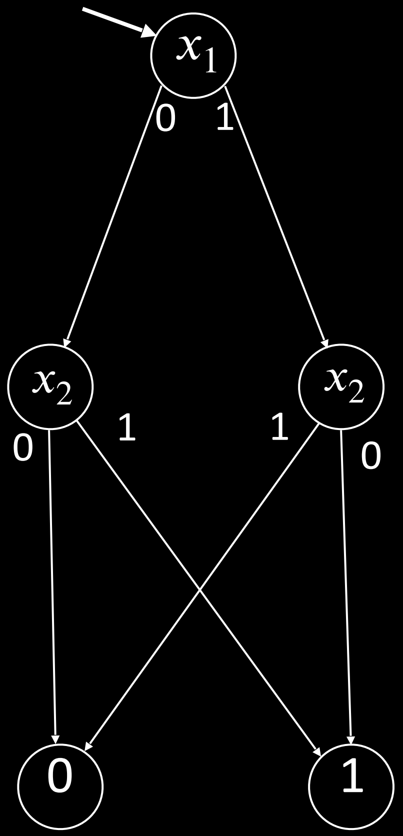

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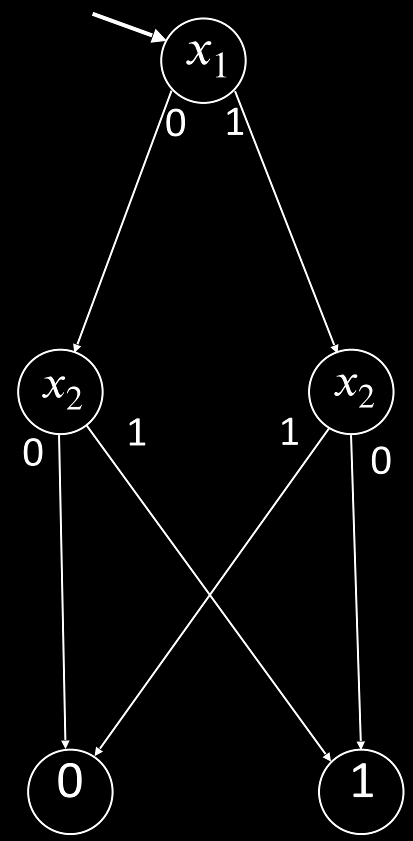

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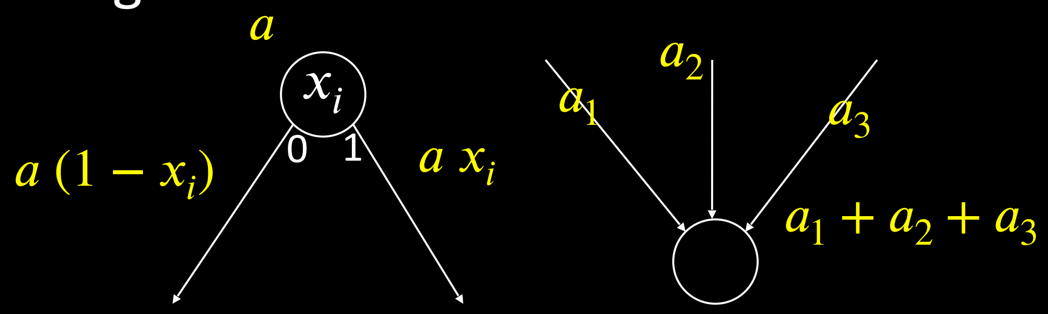

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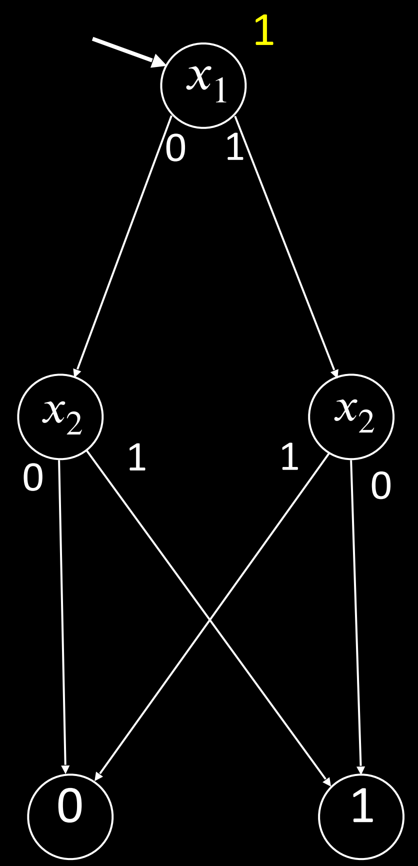

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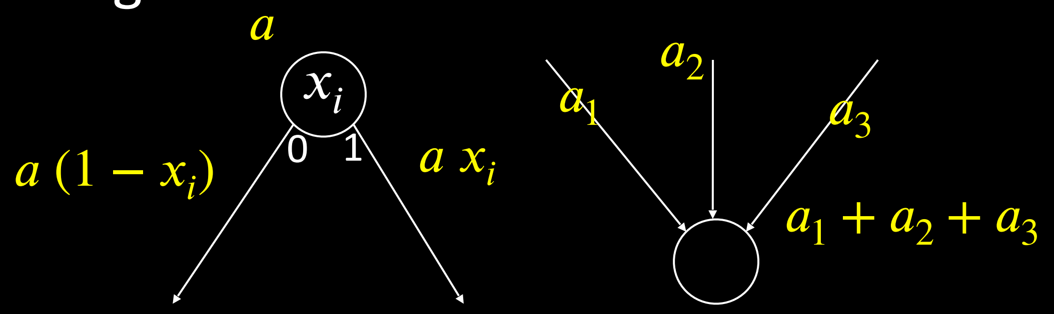

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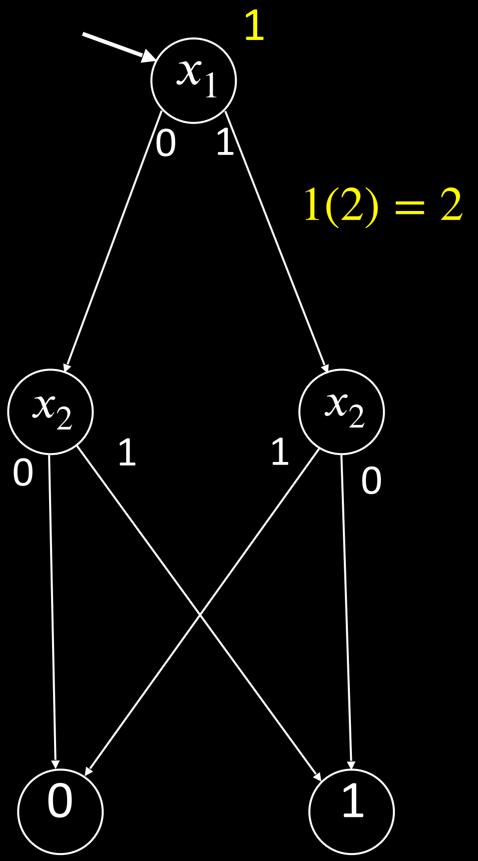

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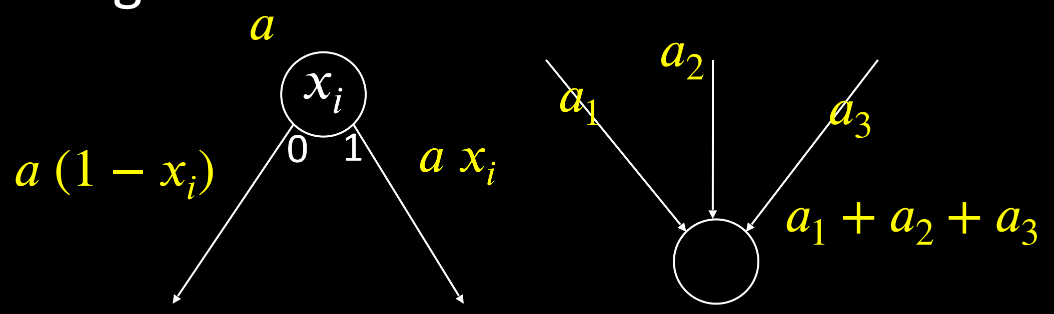

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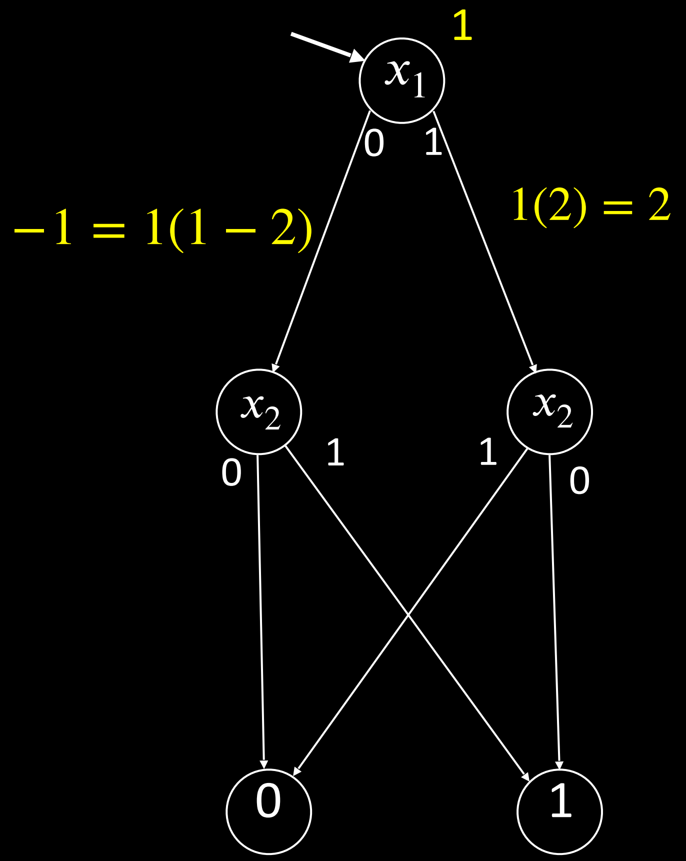

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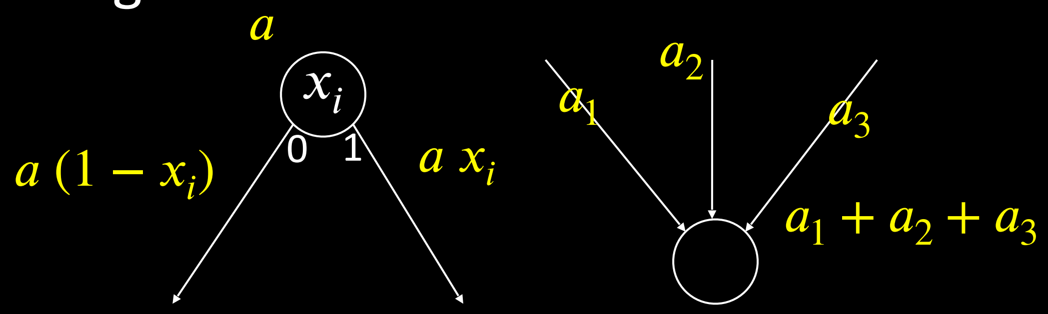

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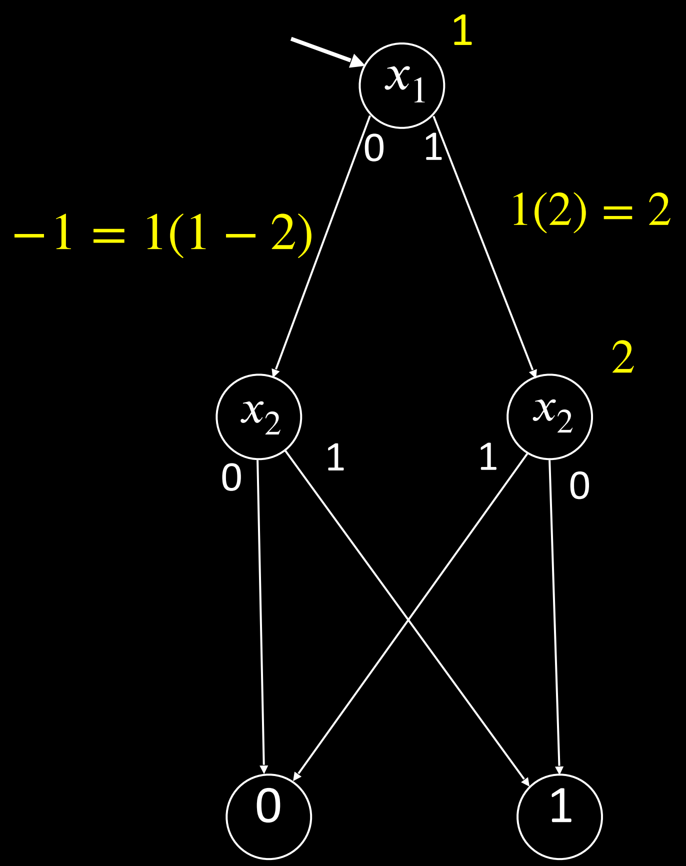

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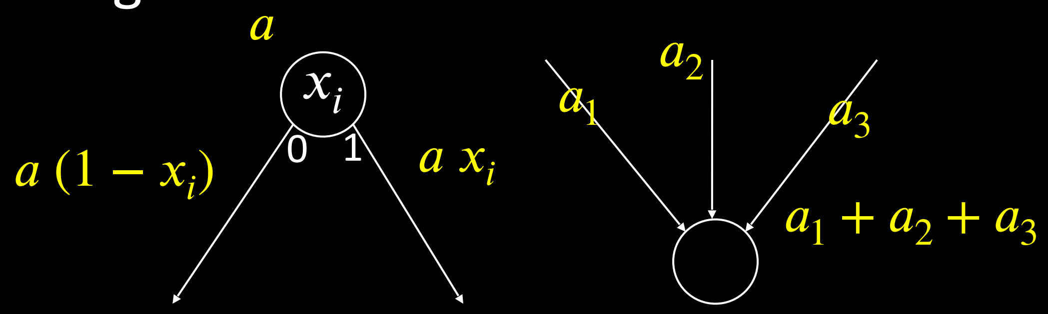

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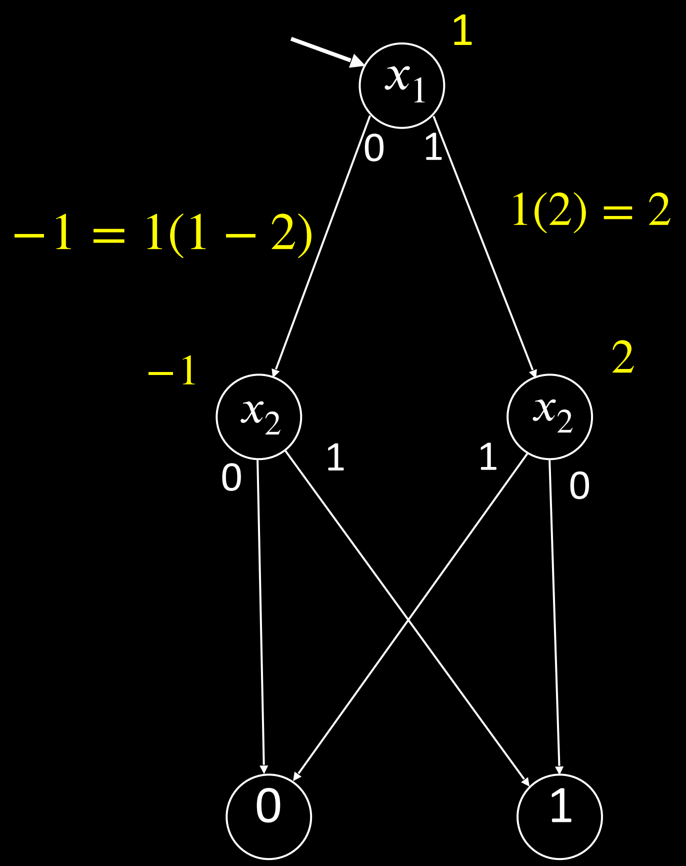

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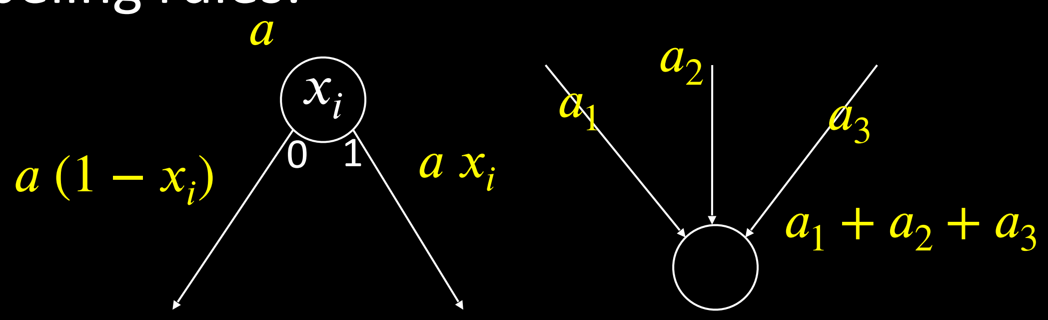

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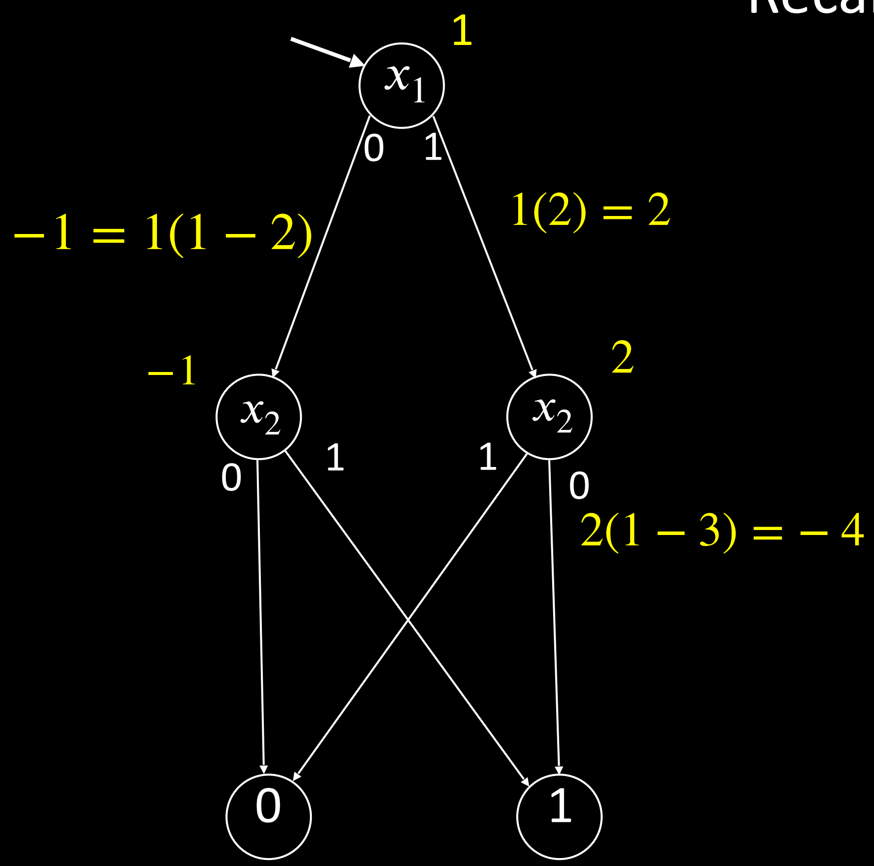

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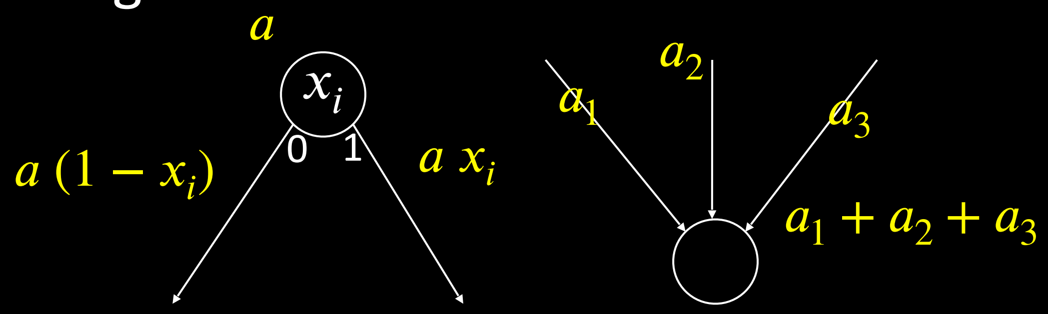

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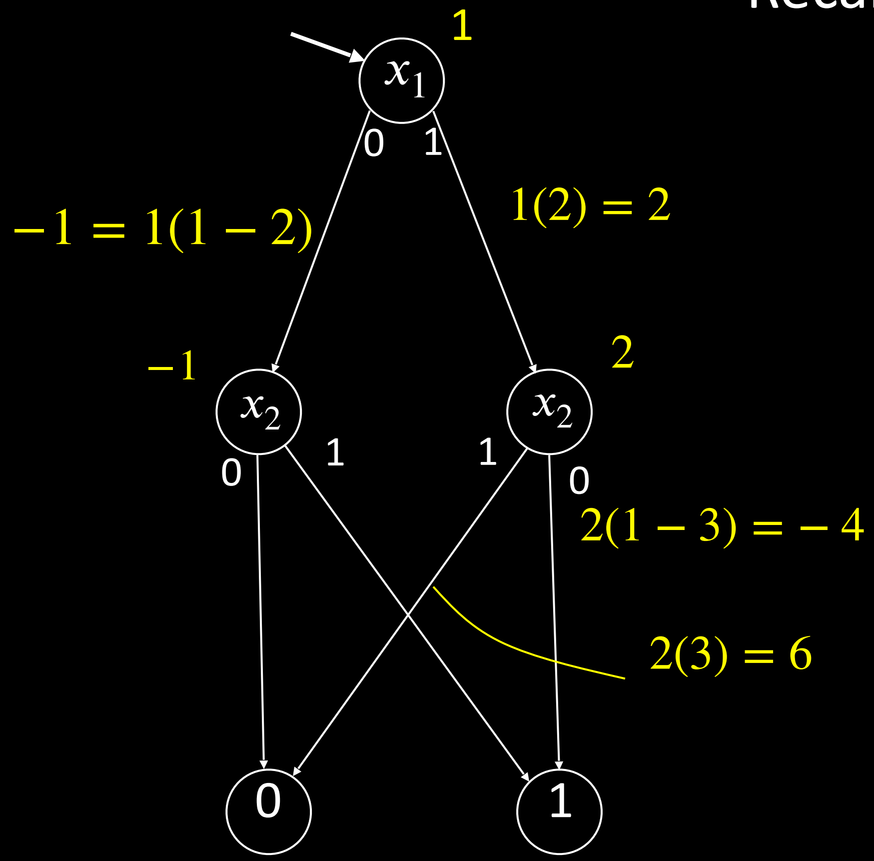

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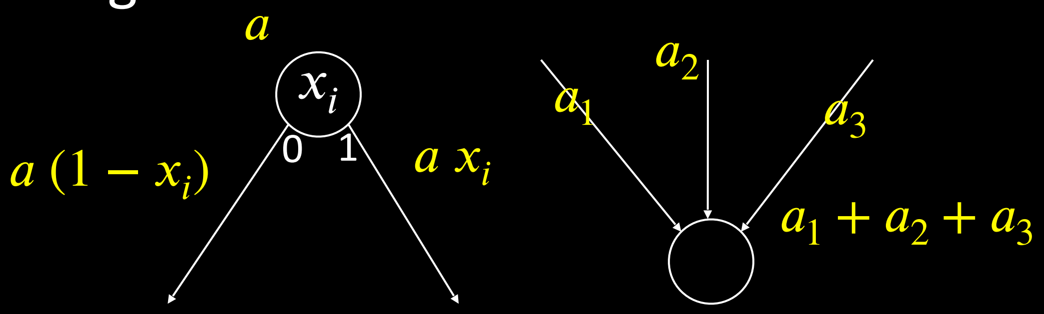

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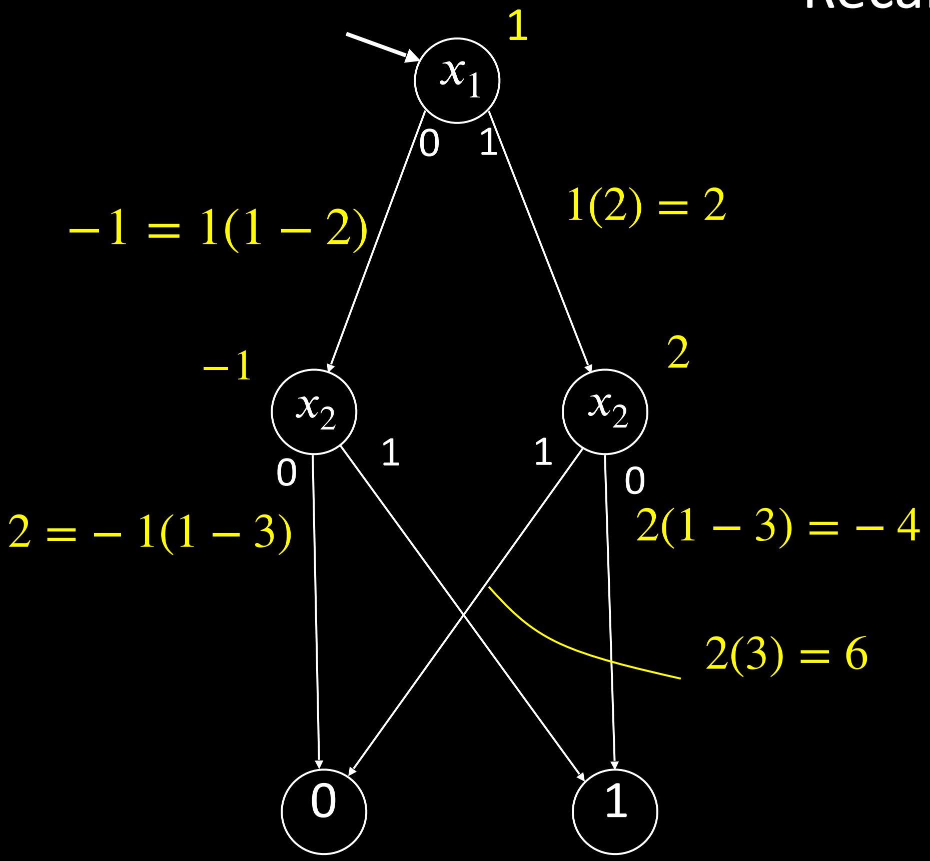

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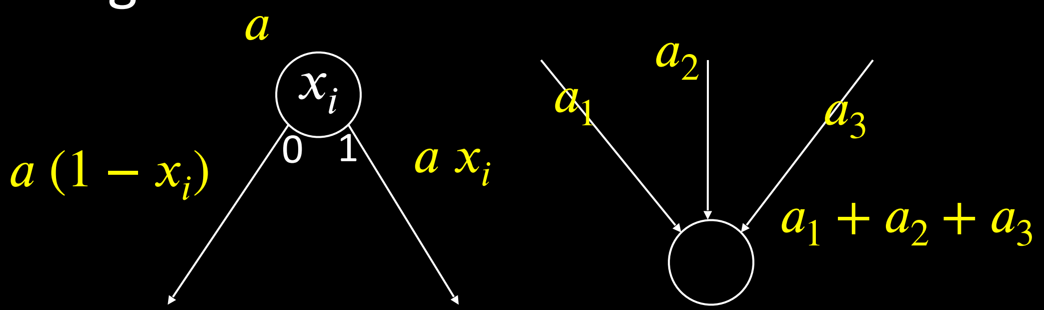

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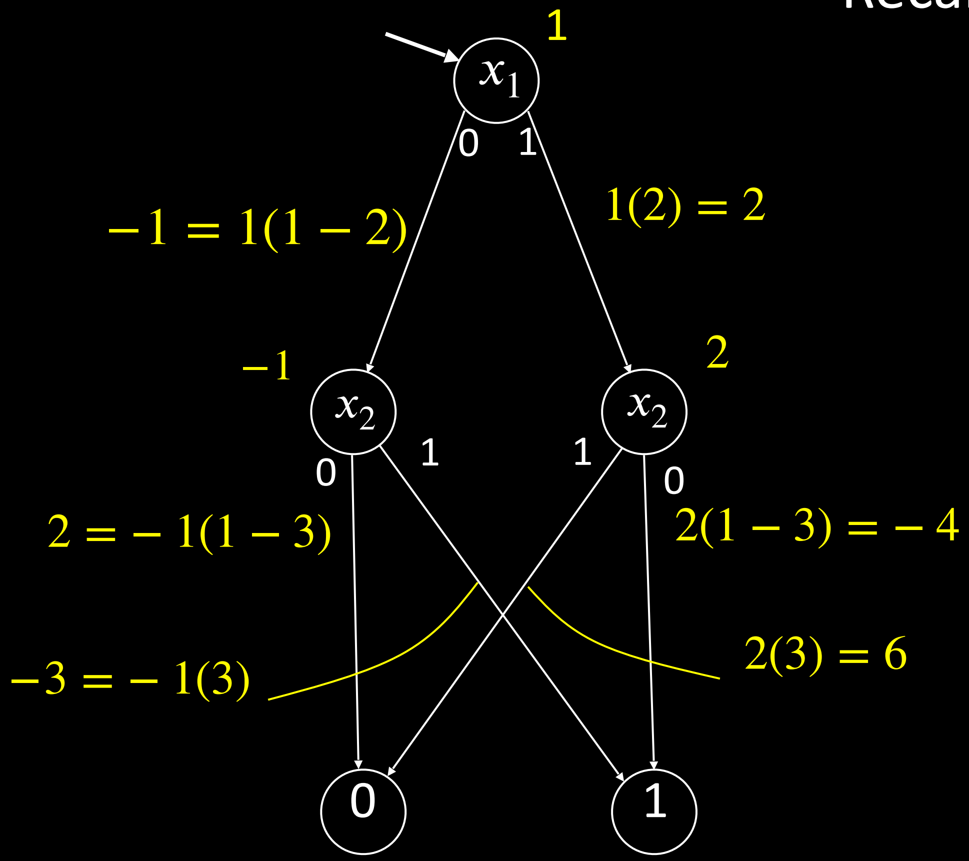

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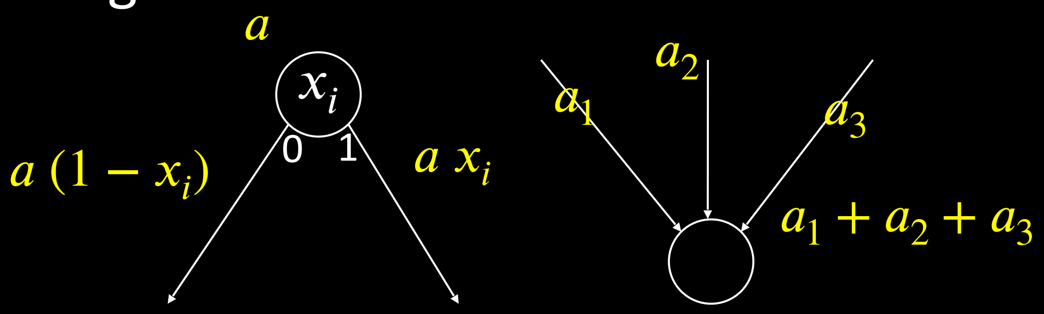

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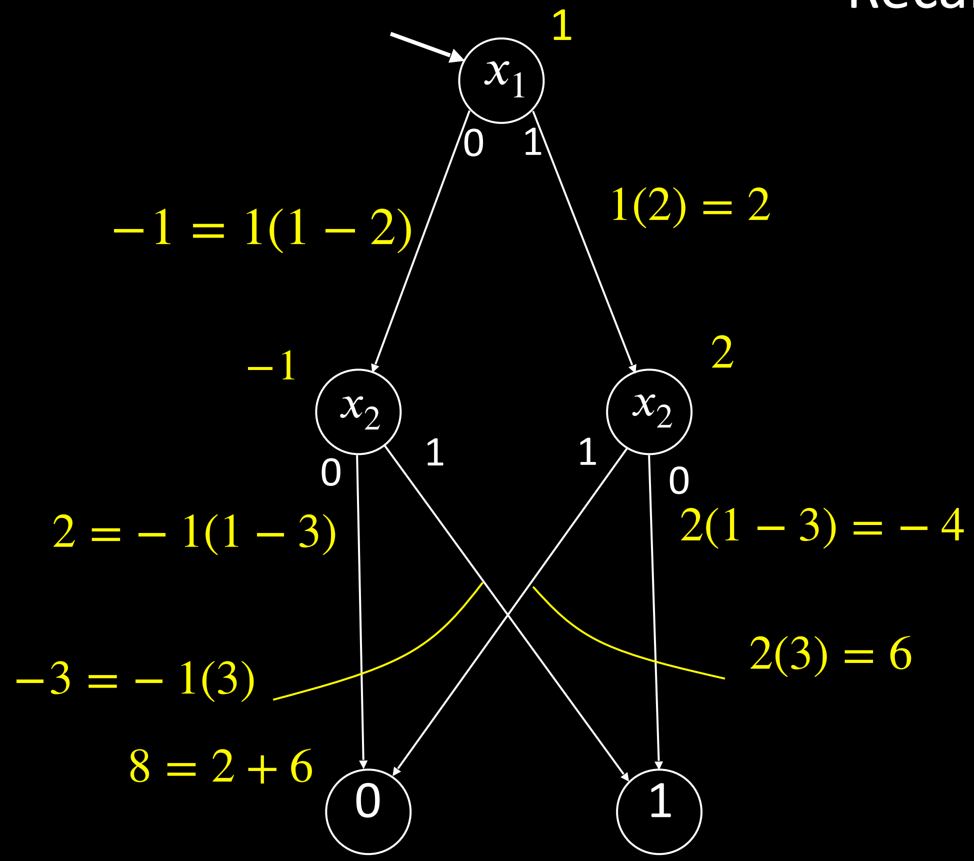

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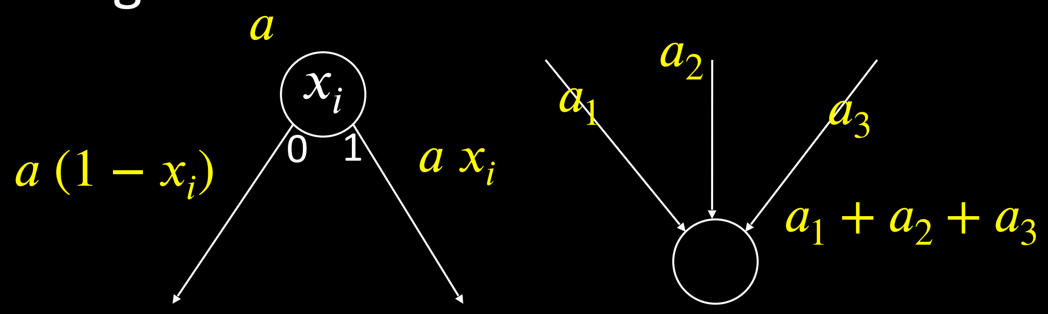

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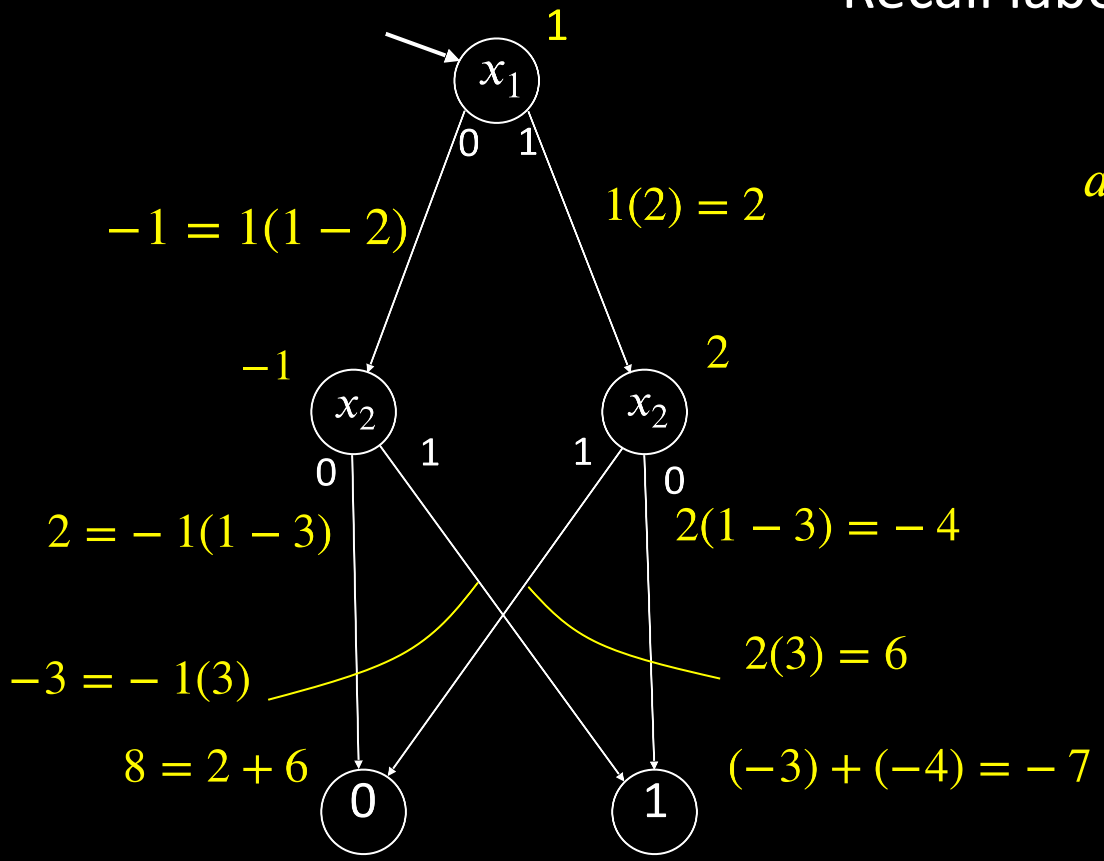

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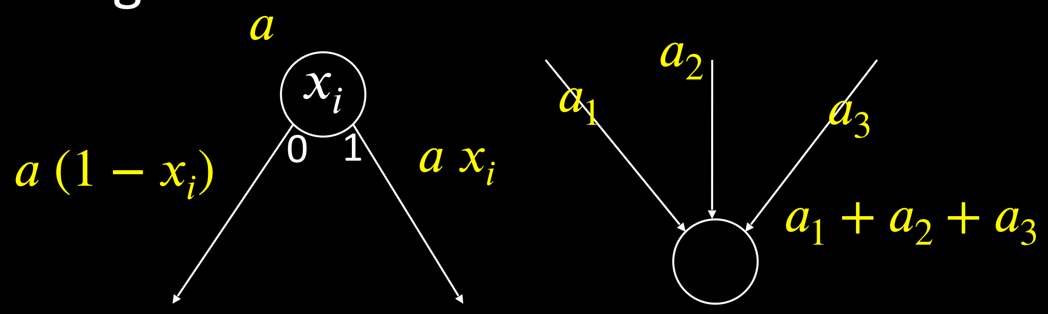

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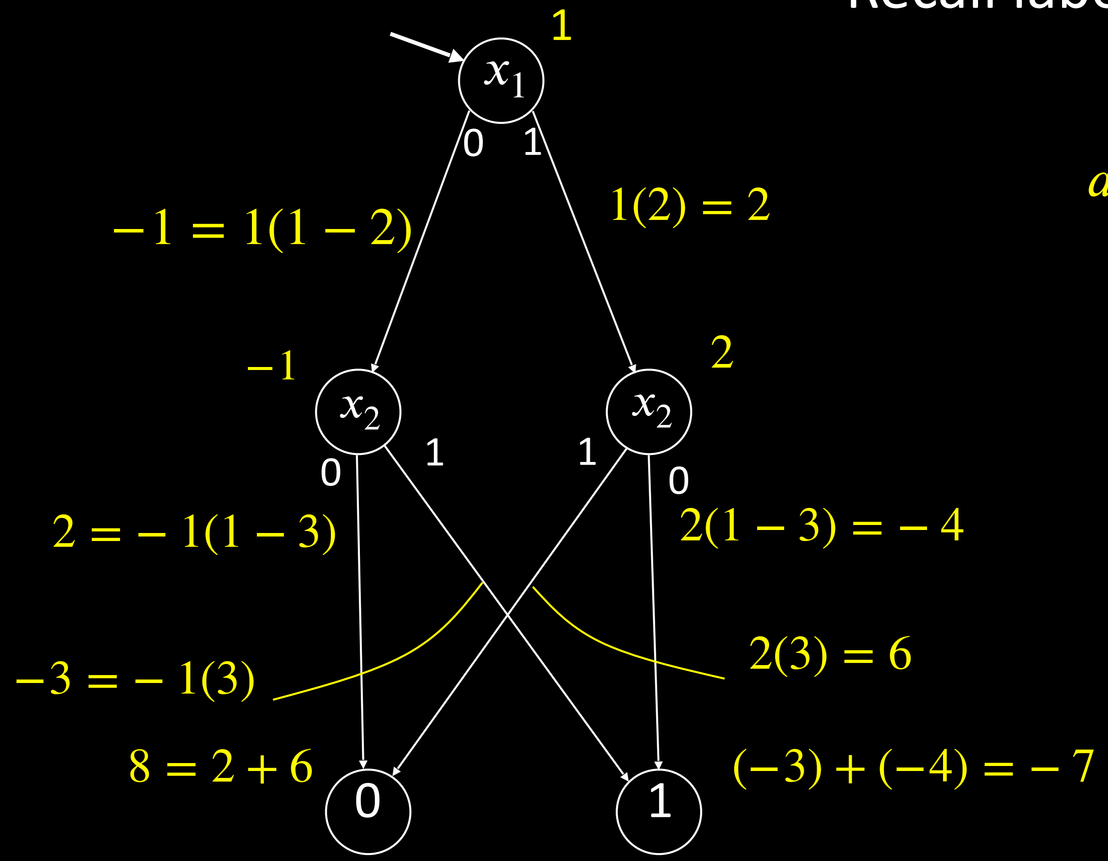

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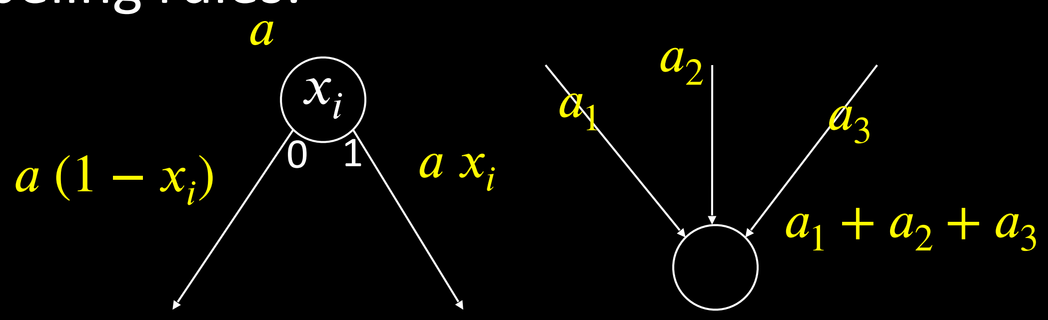

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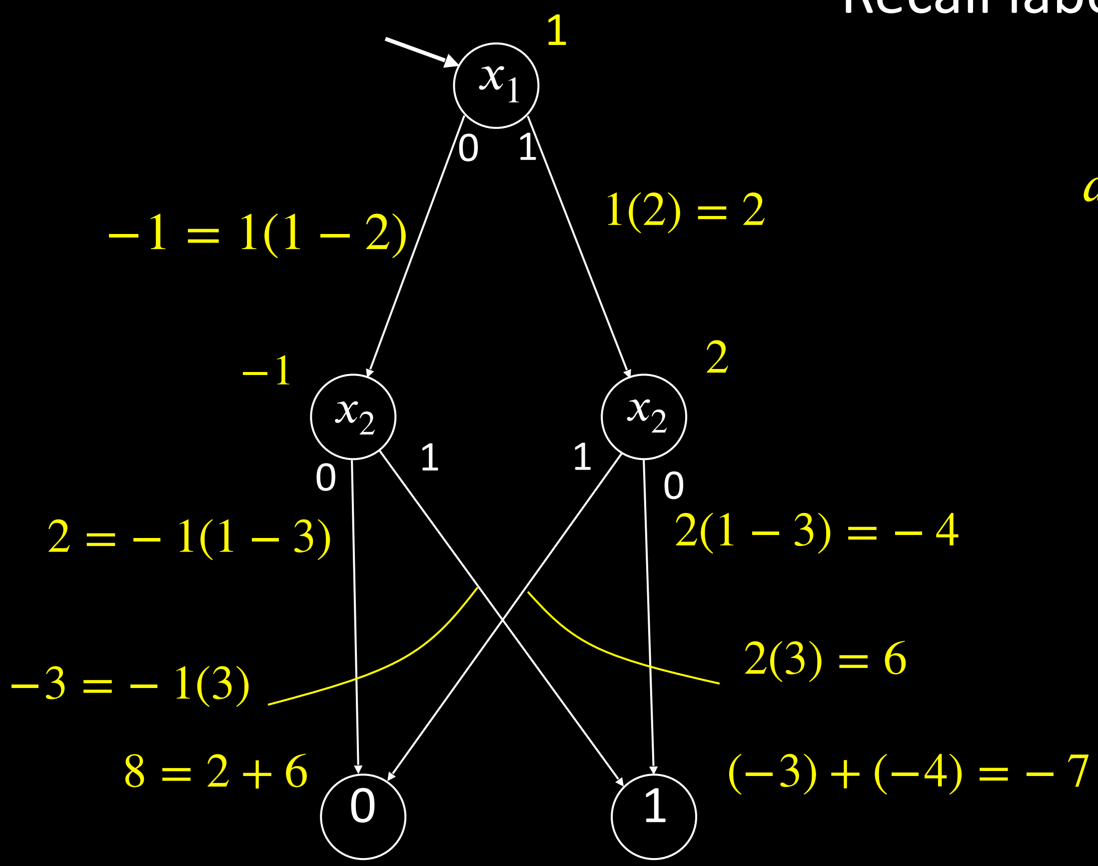

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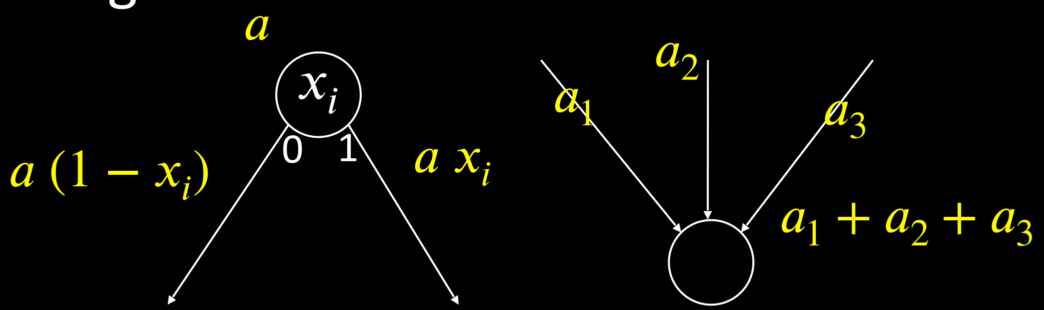

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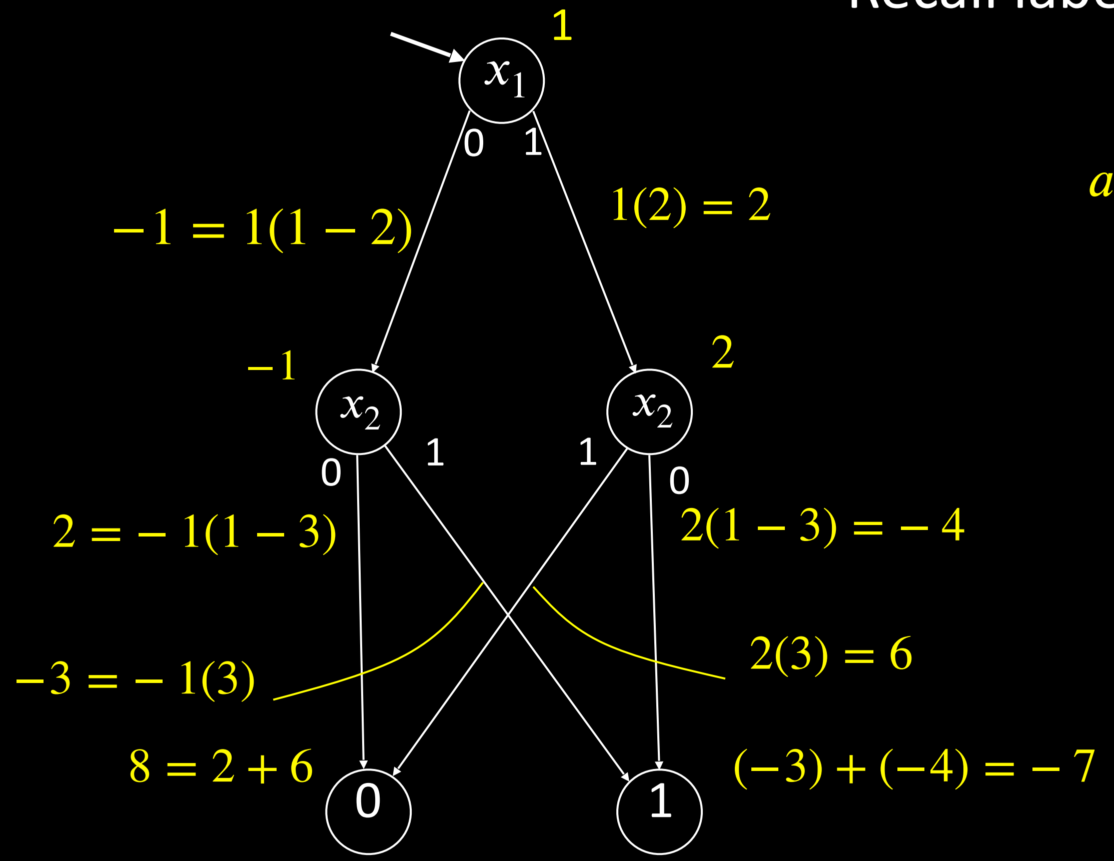

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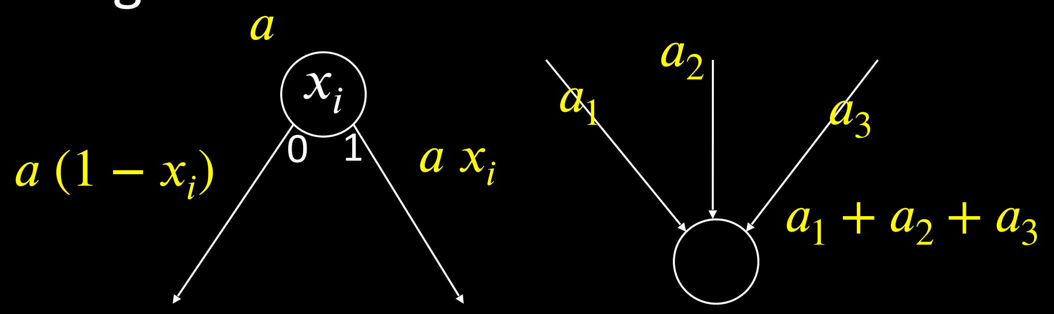

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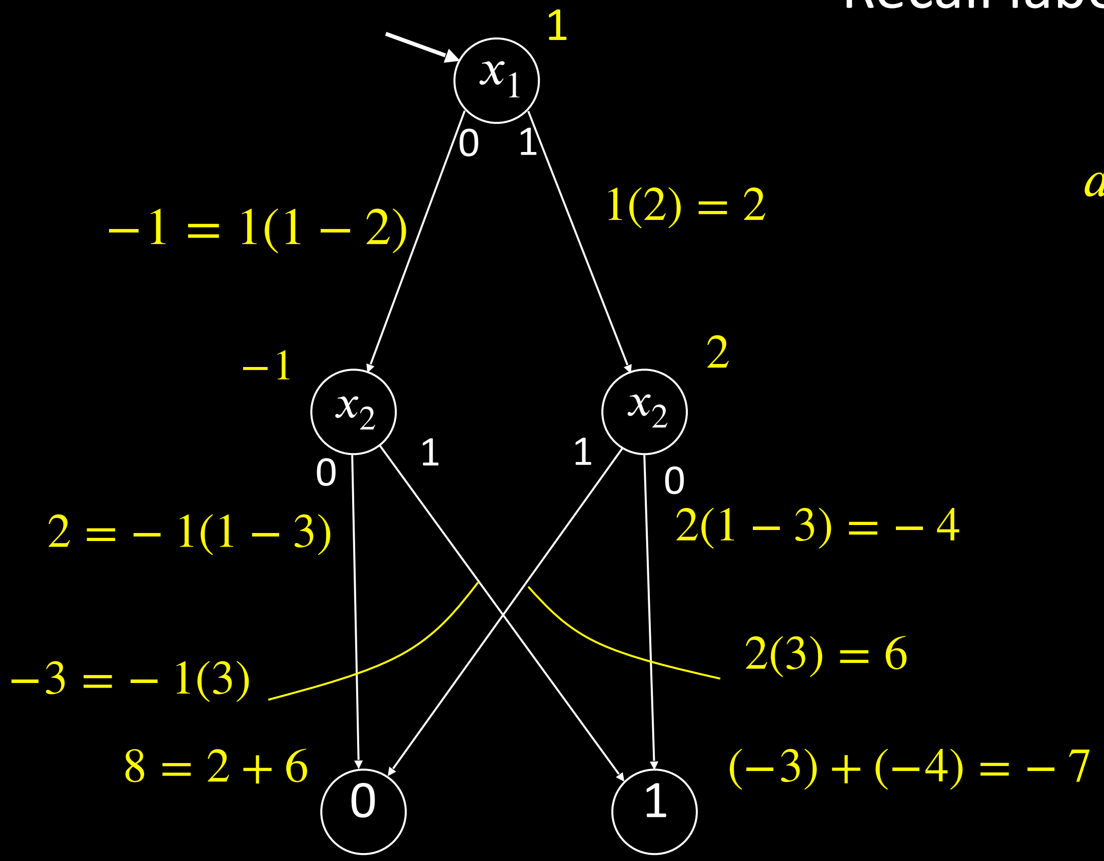

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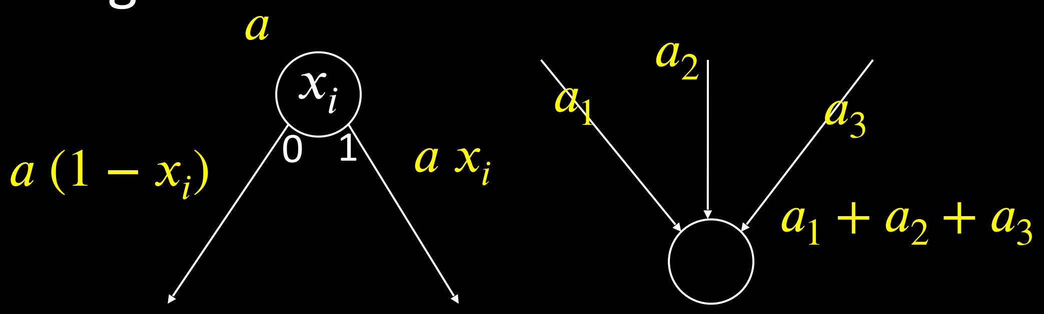

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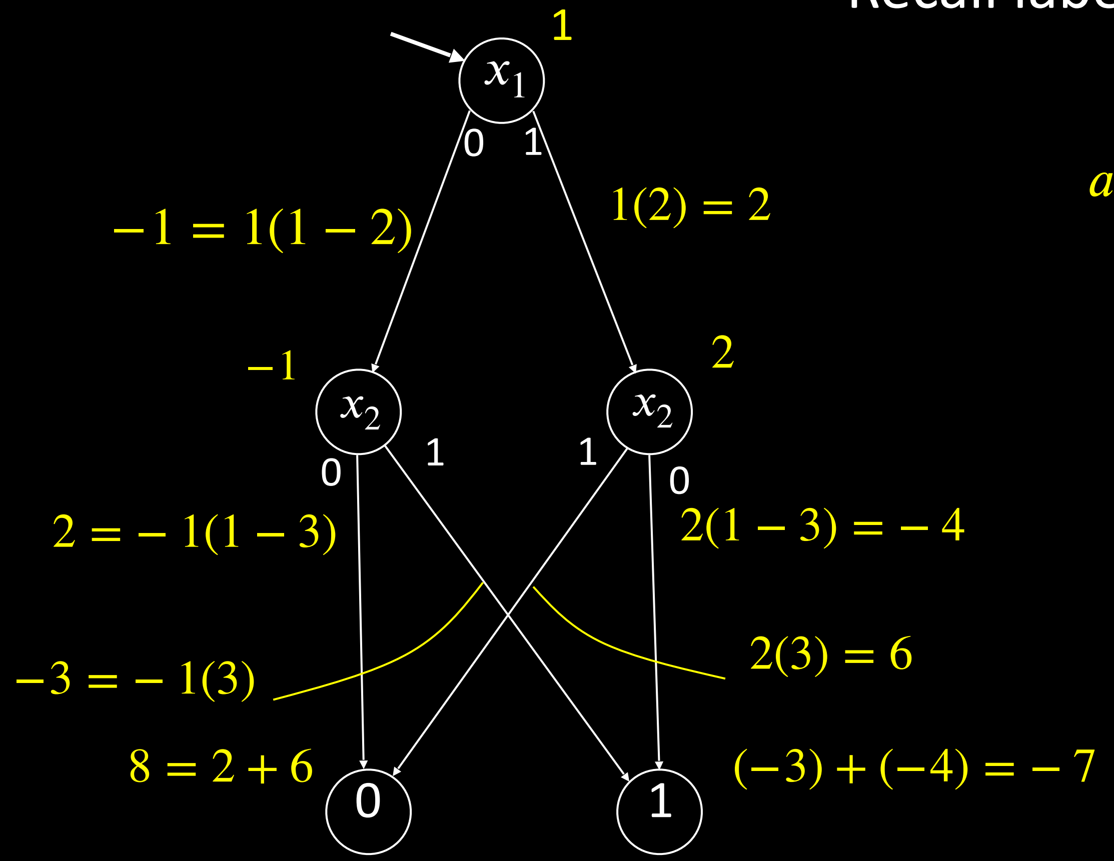

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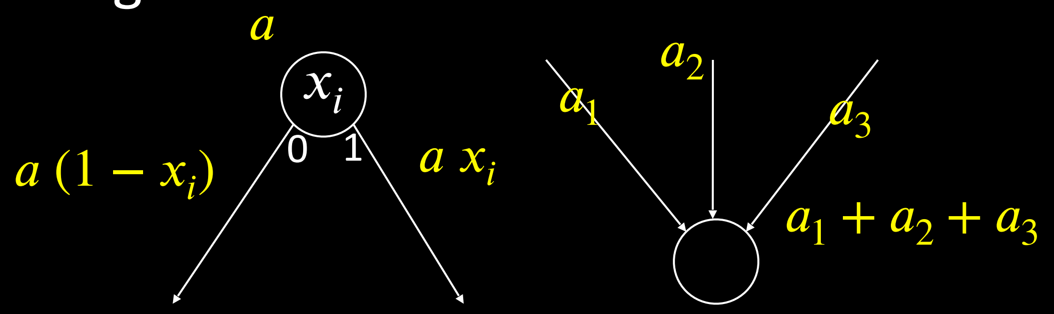

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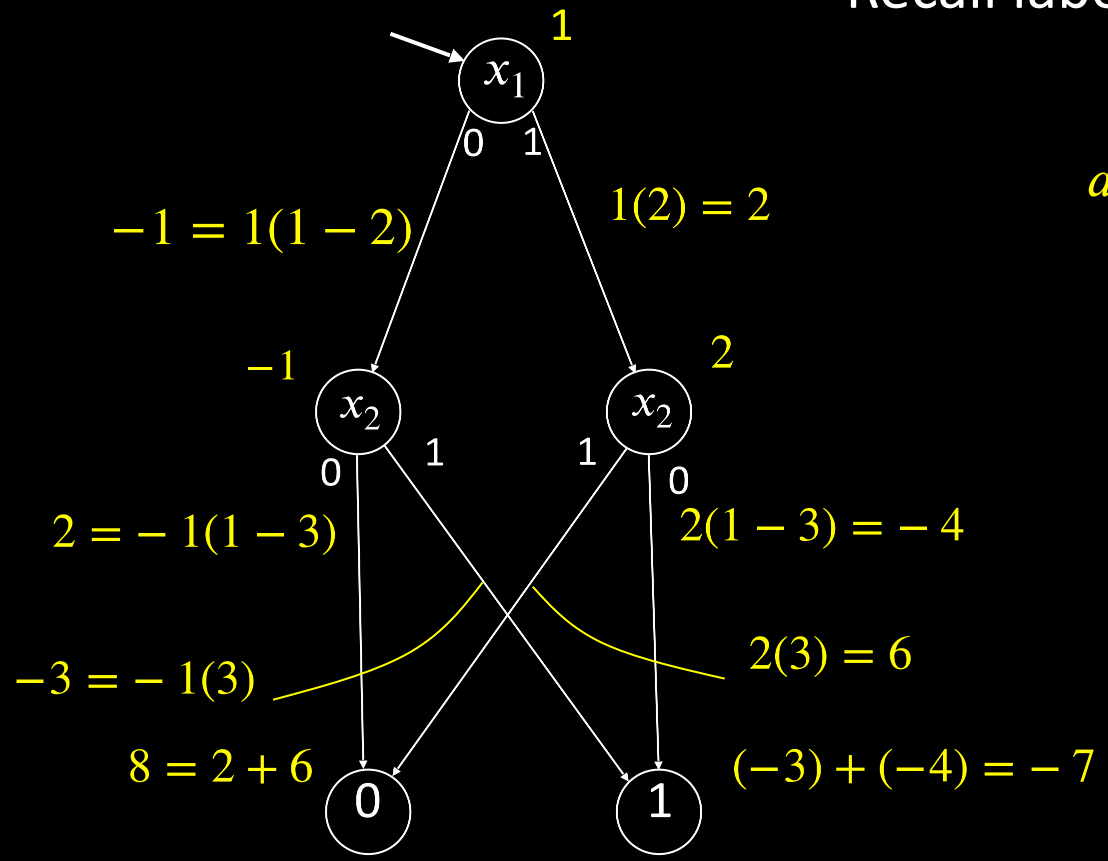

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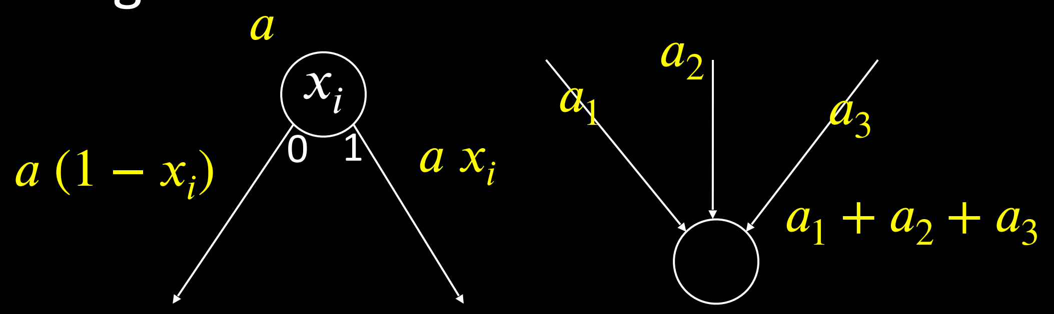

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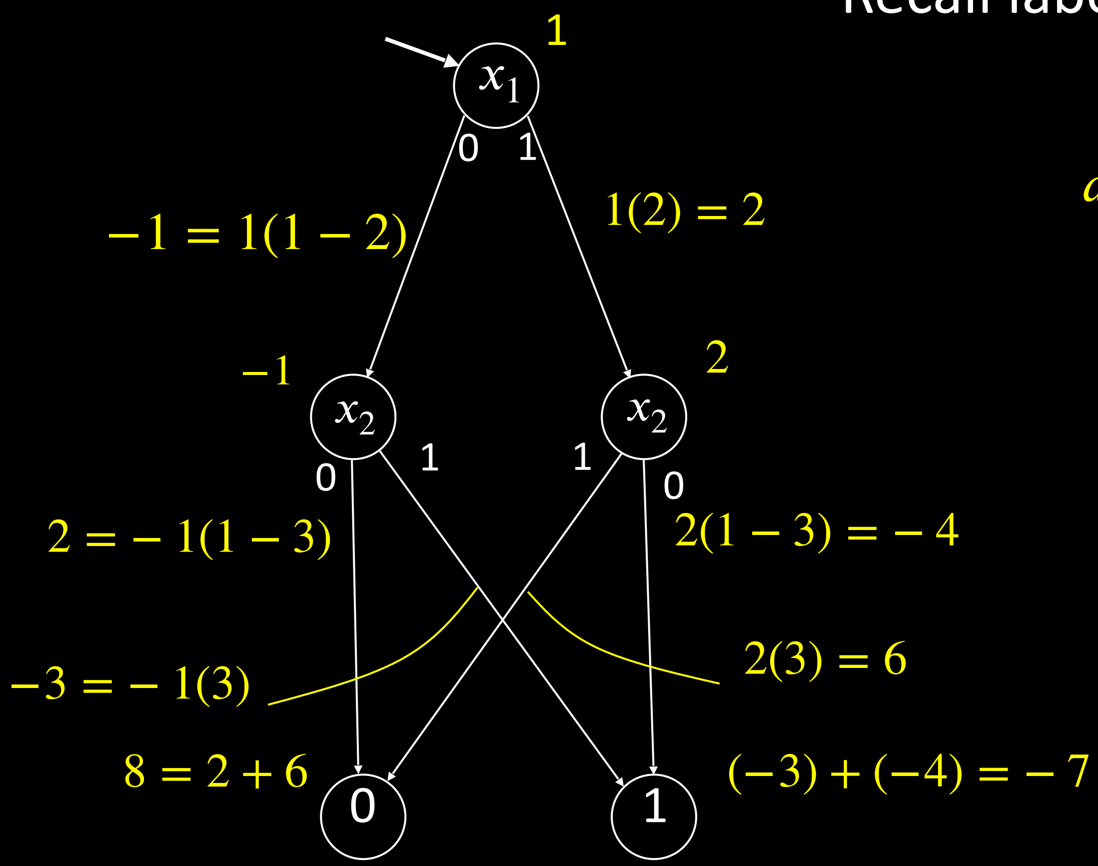

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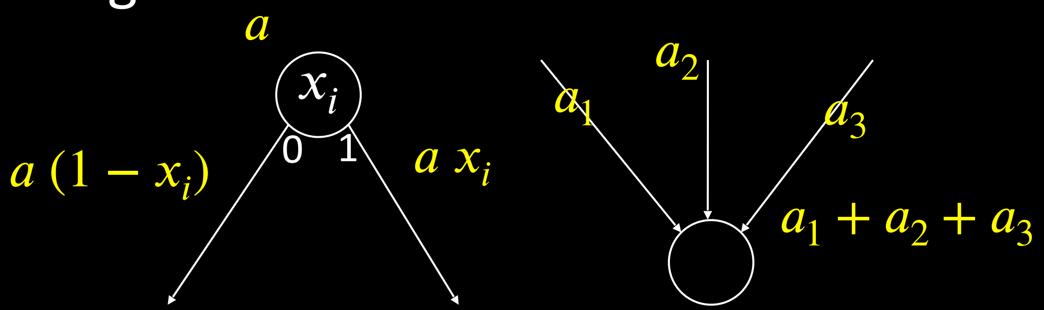

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Check-in 23.3

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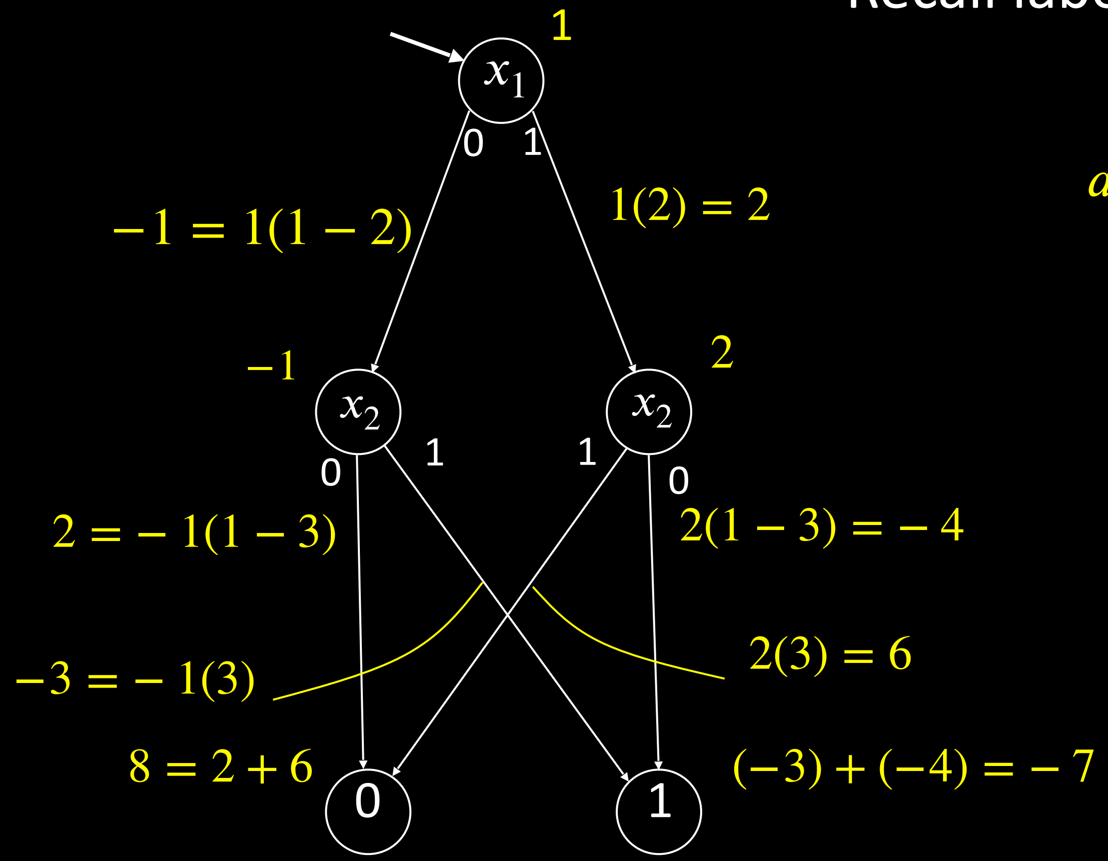

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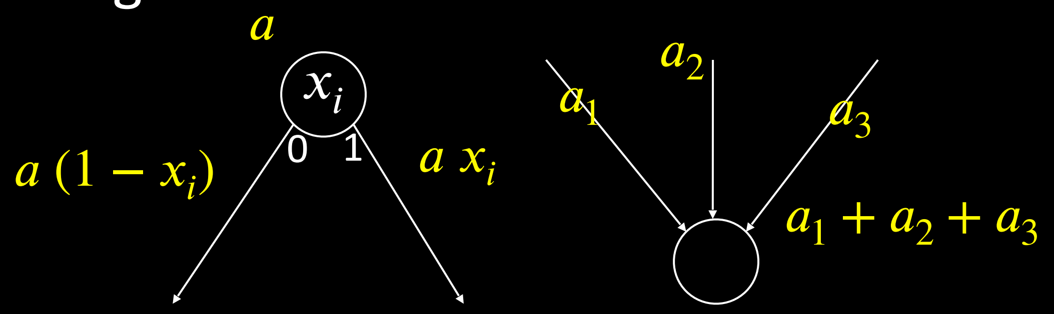

Check-in 23.3

What is the output for this branching program using the arithmetized interpretation if $x_1 = 1, \ x_2 = y$?

(a) $(1 - y)$

(b) $(y + 1)$

(c) y

Check-in 23.3

Quick review of today

1. Defined probabilistic Turing machines
2. Defined the class BPP
3. Sketched the amplification lemma
4. Introduced branching programs and read-once branching programs
5. Started the proof that EQROBP $\in$ BPP
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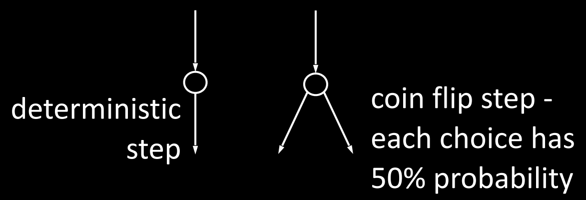

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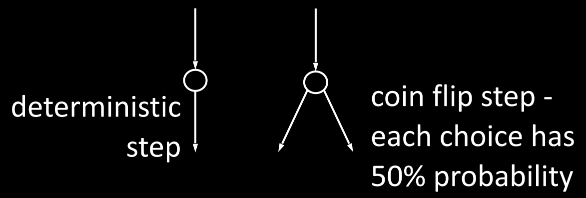

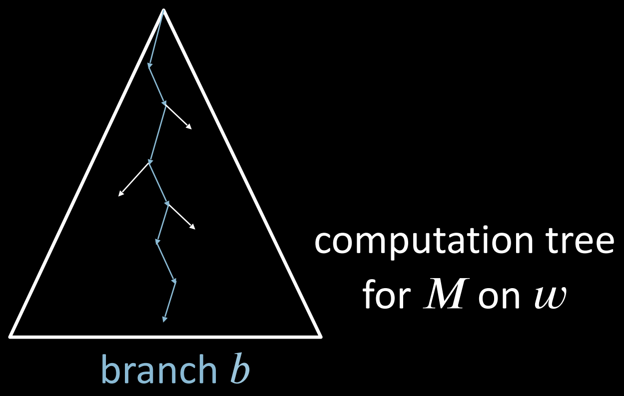

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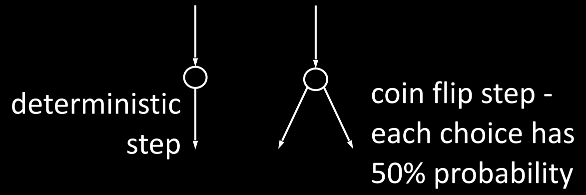

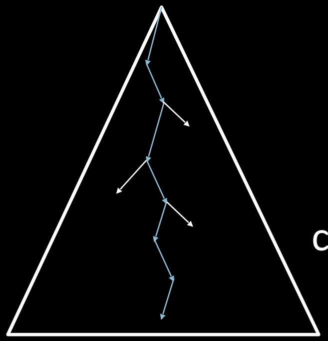

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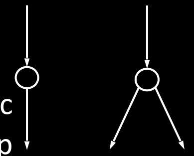

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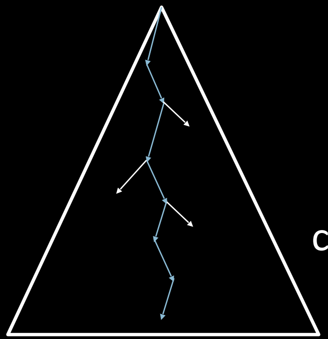

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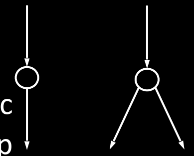

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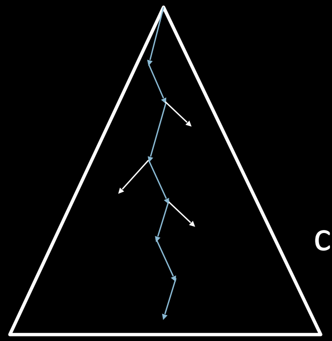

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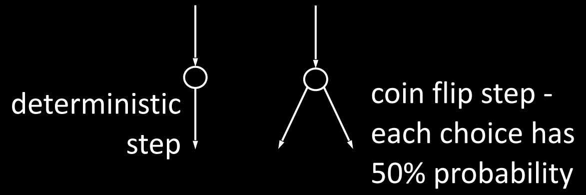

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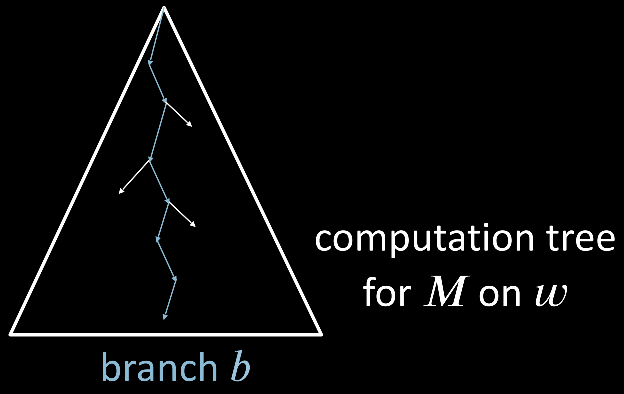

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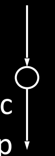

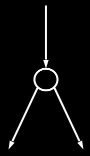

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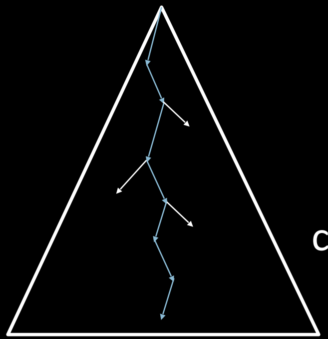

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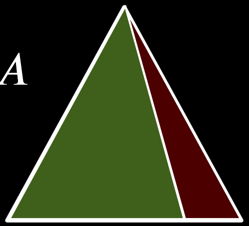

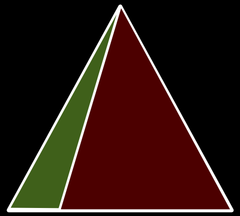

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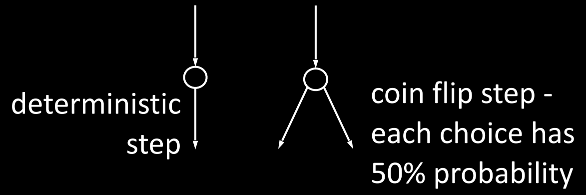

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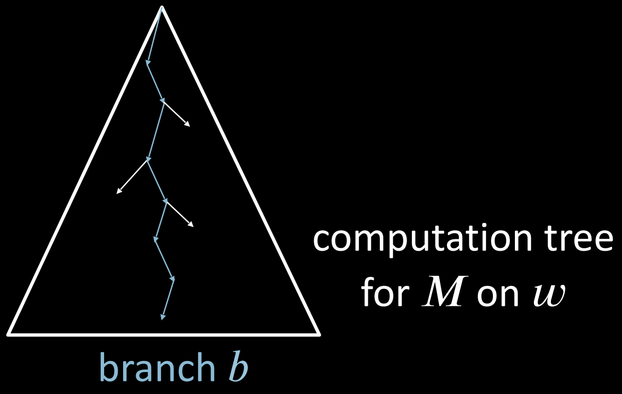

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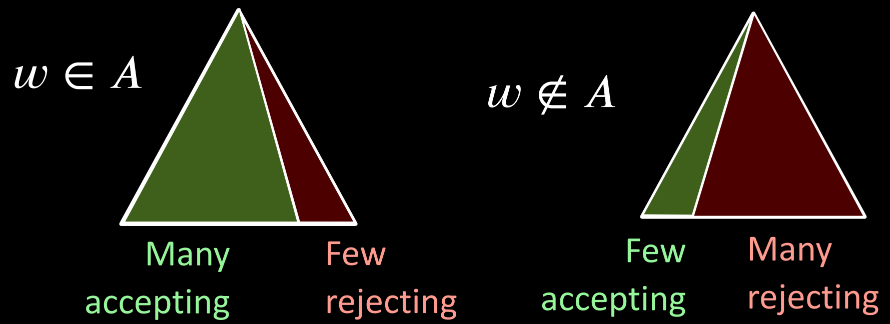

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Check-in 24.1

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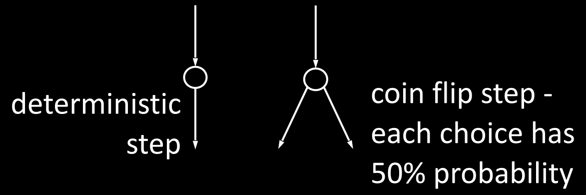

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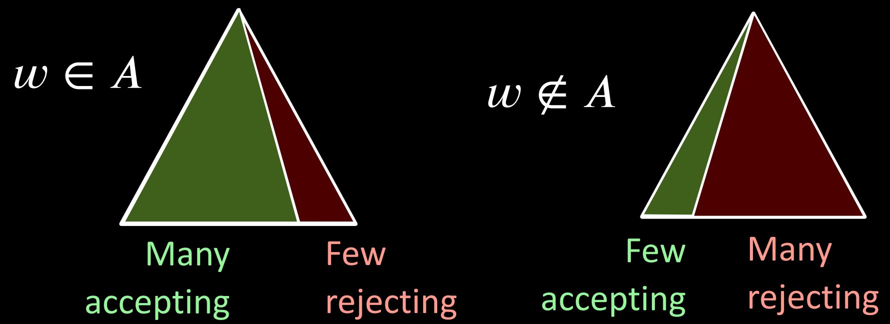
⇩

Amplification lemma: 2^{-p}



Check-in 24.1

Actually using a probabilistic algorithm presupposes a source of randomness. Can we use a standard pseudo-random number generator (PRG) as the source?

(a) Yes, but the result isn't guaranteed.

(b) Yes, but it will run in exponential time.

(c) No, a TM cannot implement a PRG.

(d) No, because that would show P = BPP.

Check-in 24.1

Review: Branching Programs

Defn: A branching program (BP) is a directed, acyclic (no cycles) graph that has

1. *Query nodes* labeled x_i and having two outgoing edges labeled 0 and 1.
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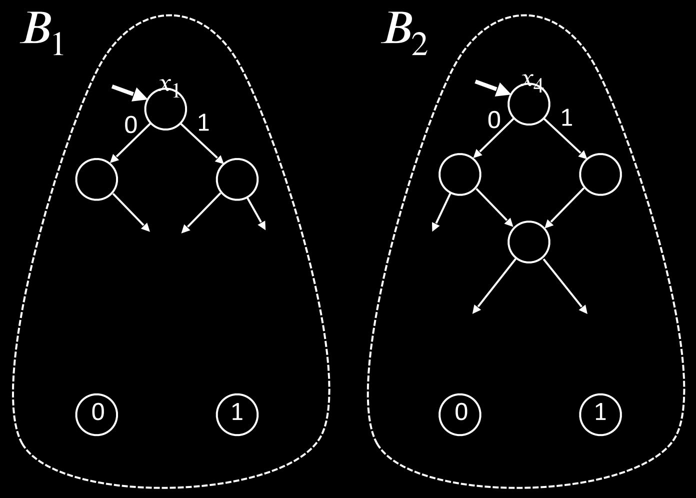
Theorem: EQ**BP** is coNP-complete (on pset 6)



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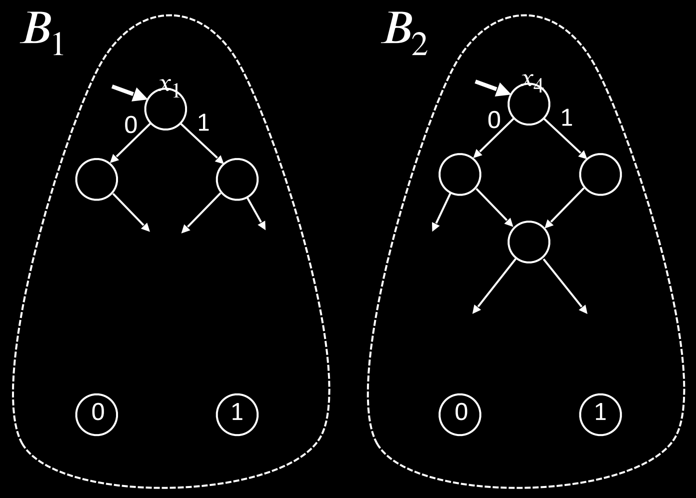

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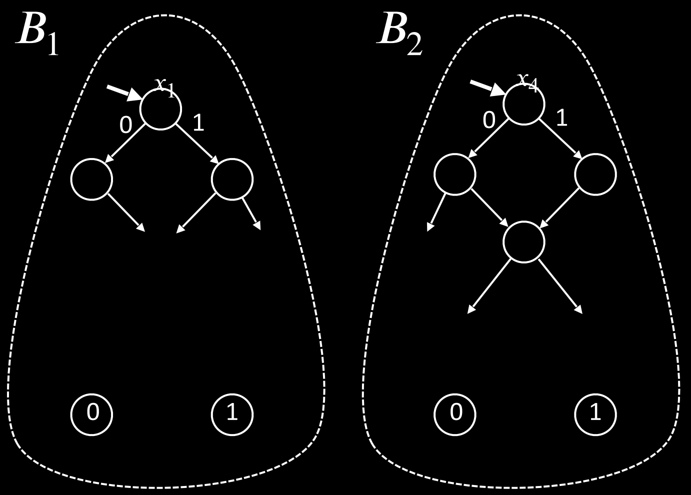

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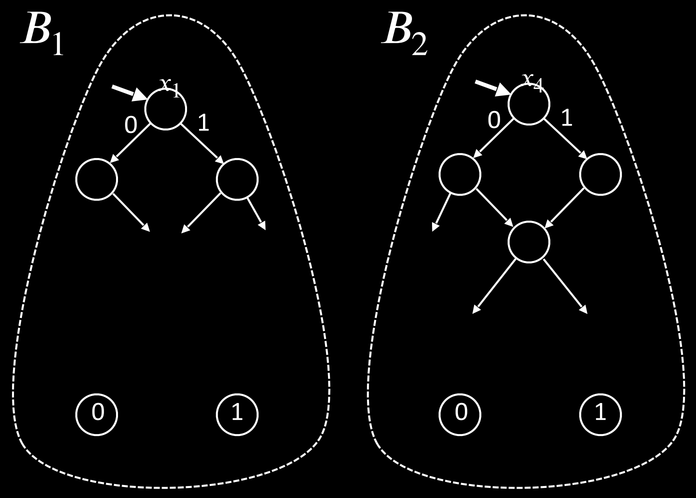

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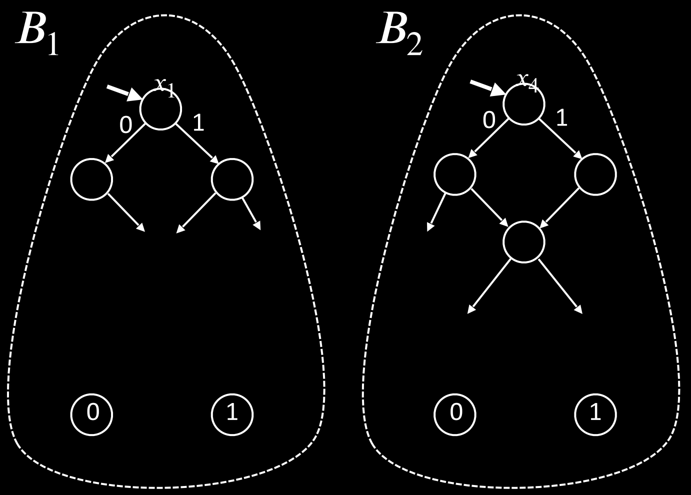

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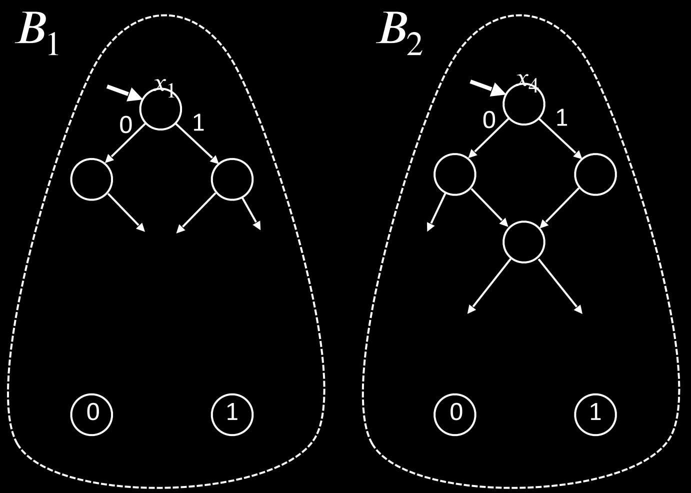

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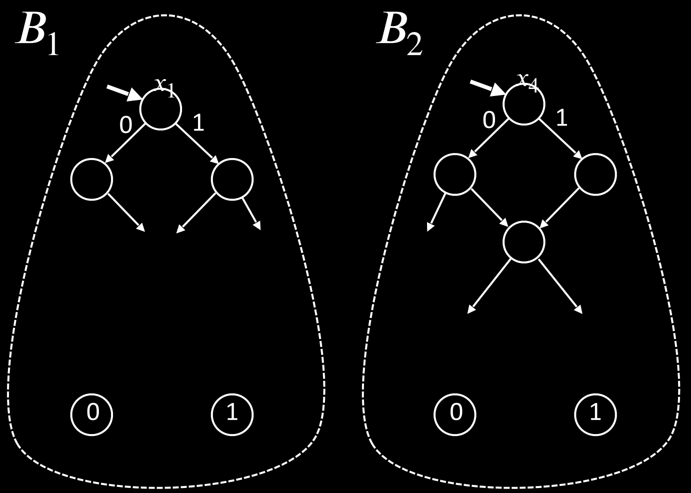

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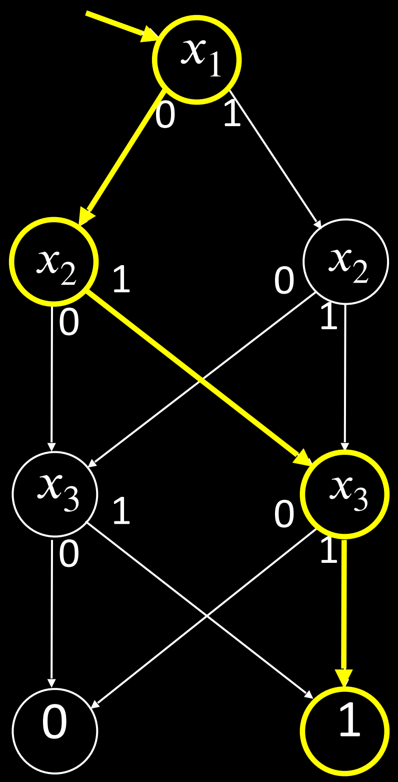

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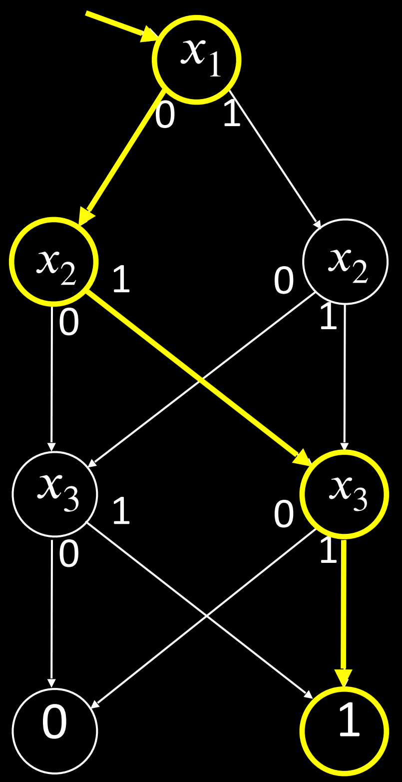

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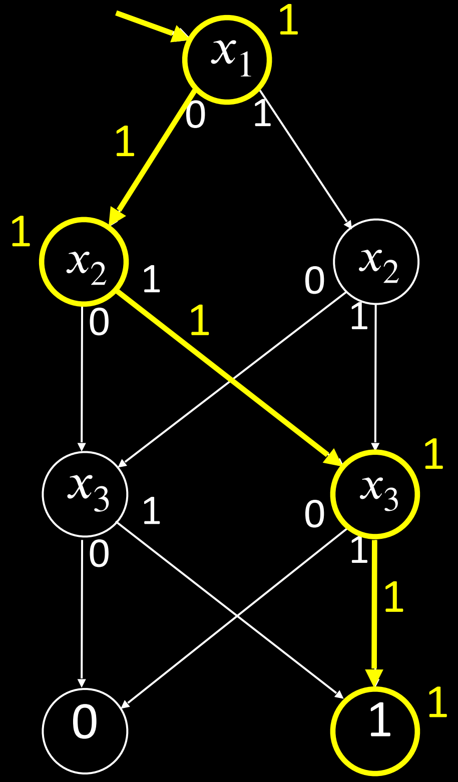

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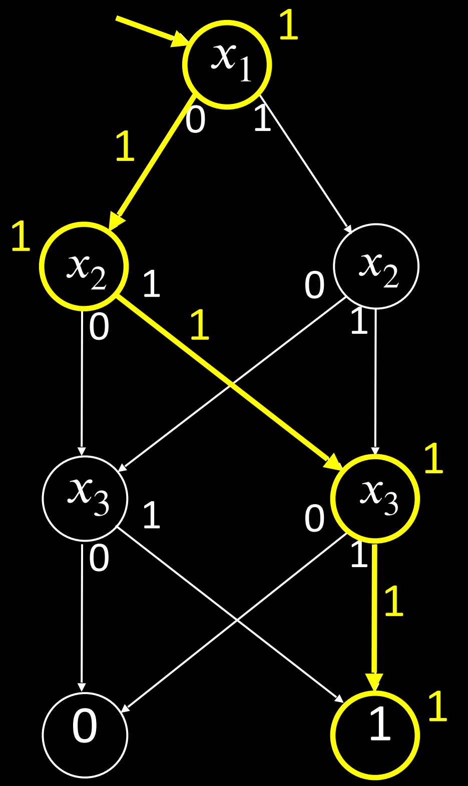

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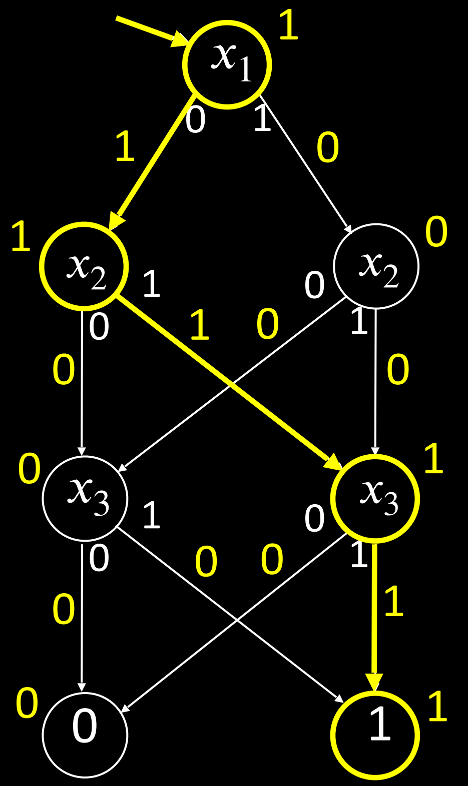

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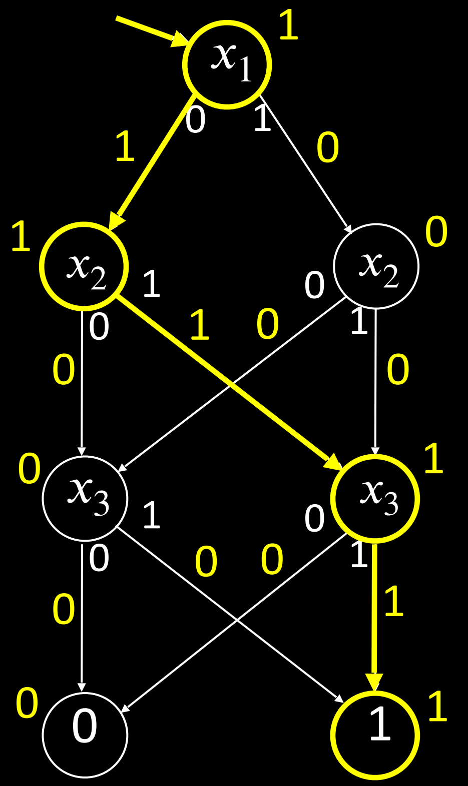

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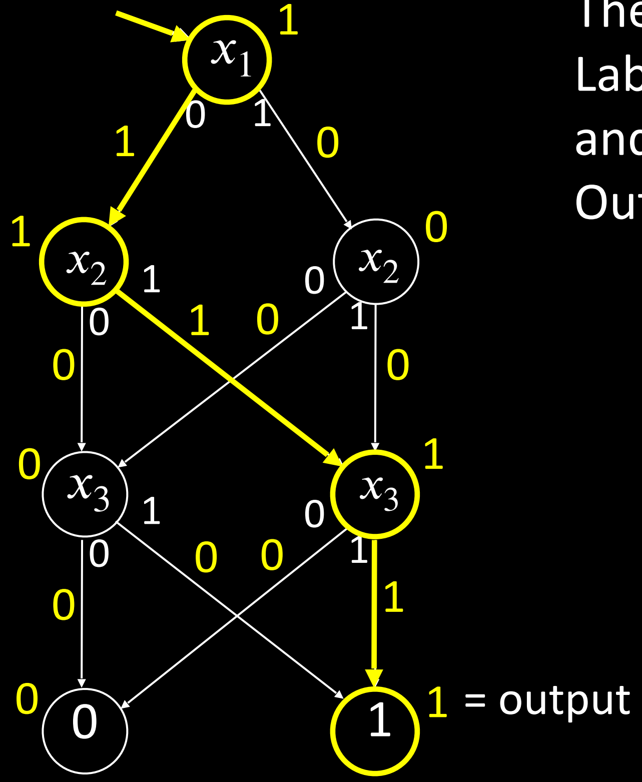

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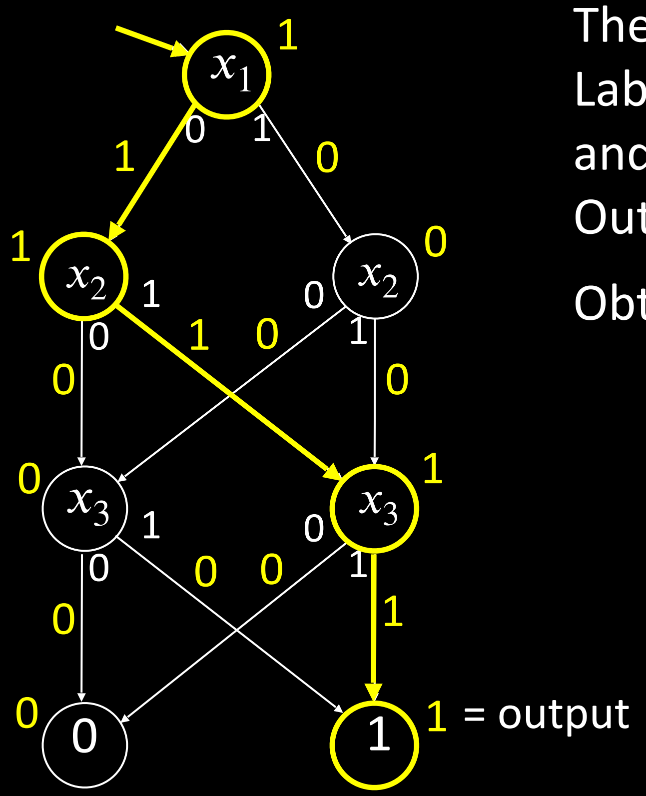

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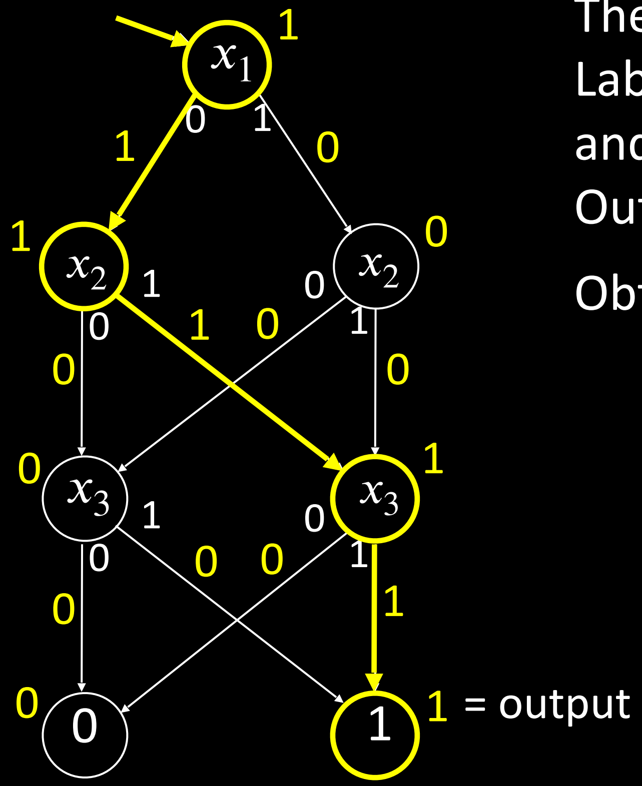

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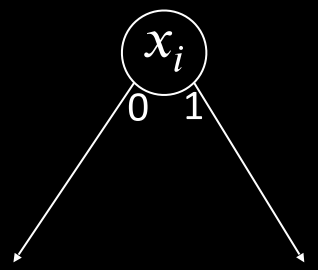

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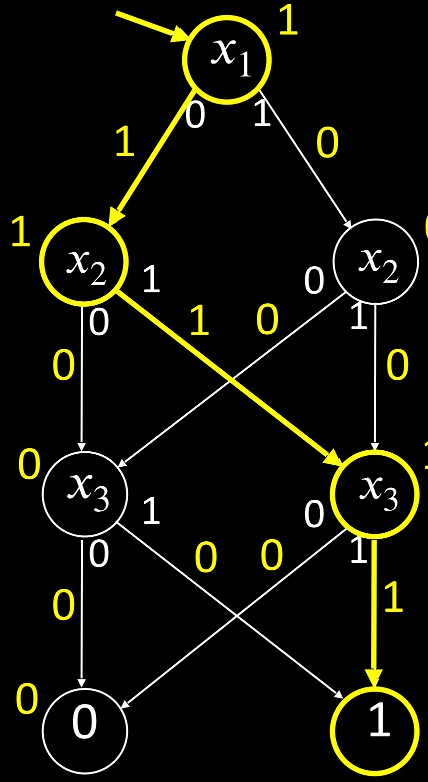

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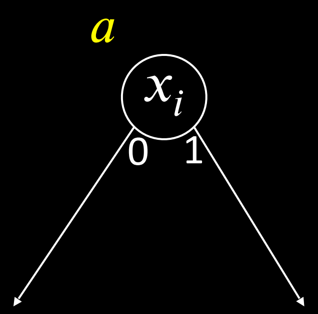

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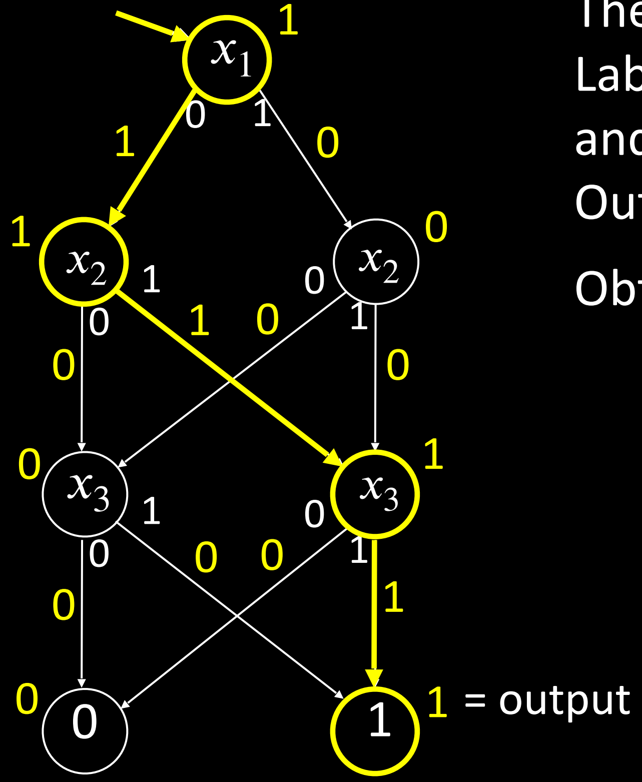

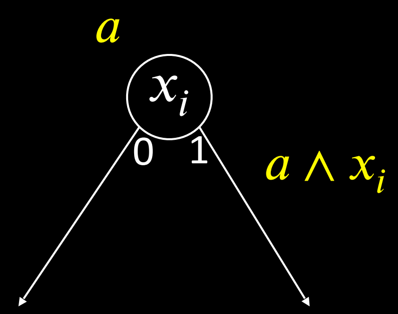

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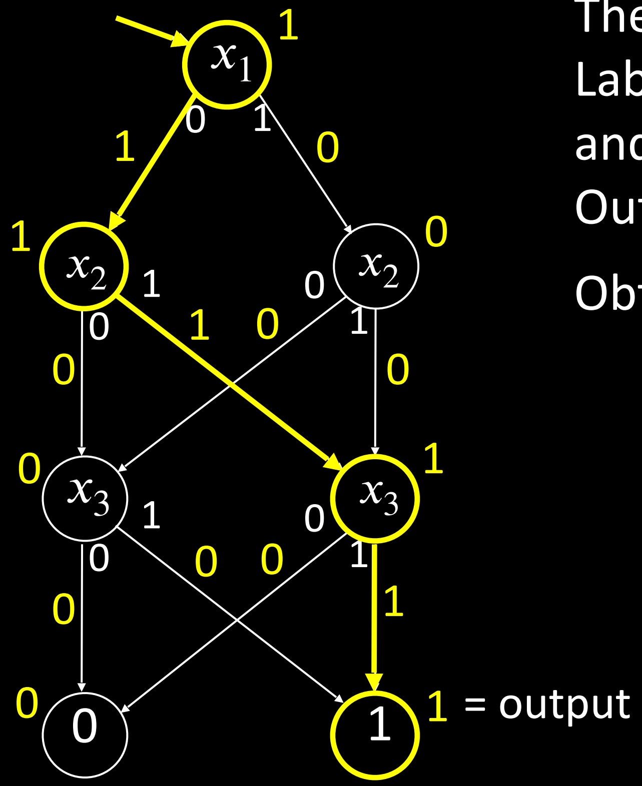

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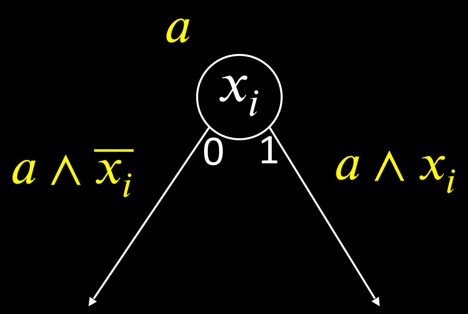

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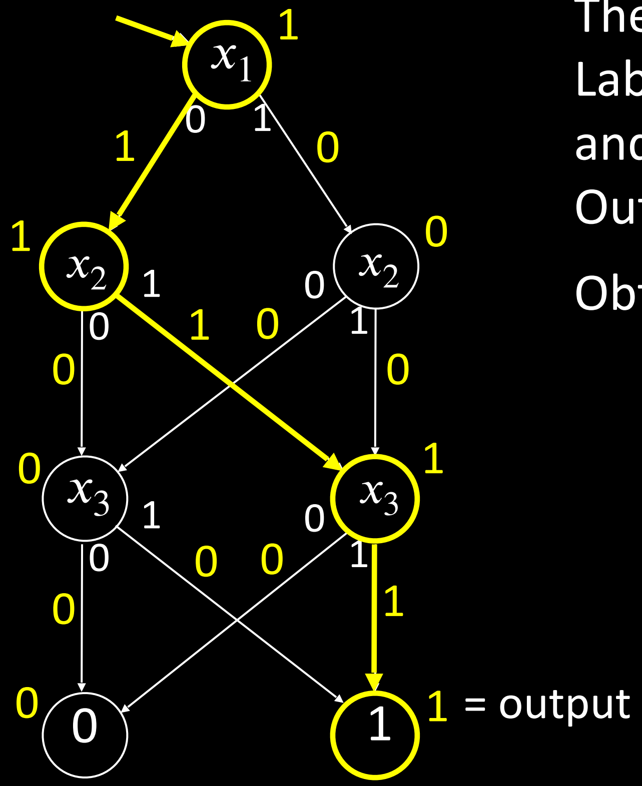

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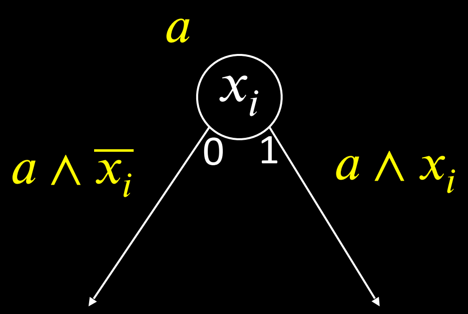

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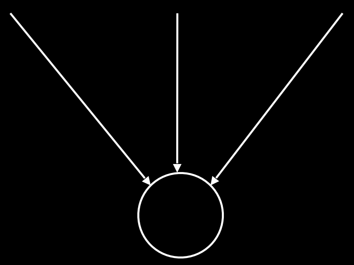

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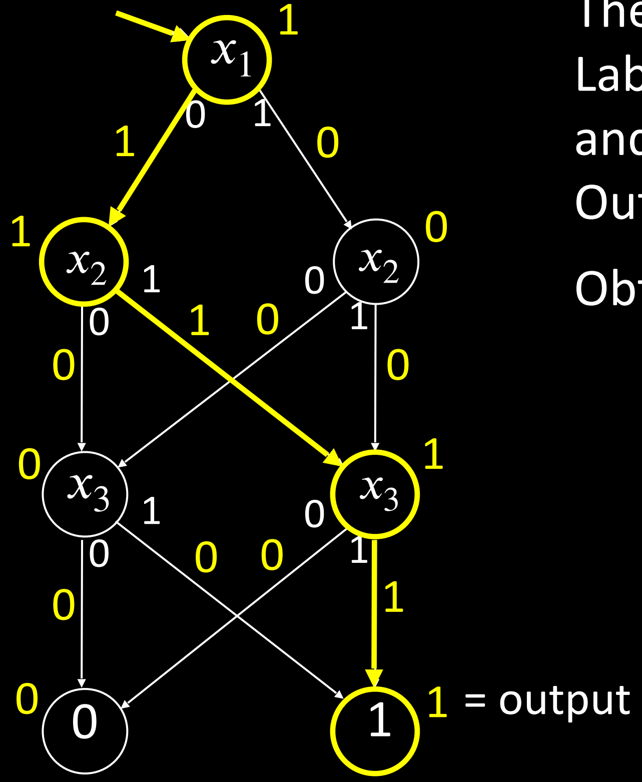

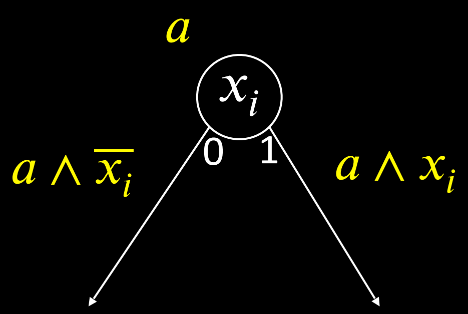

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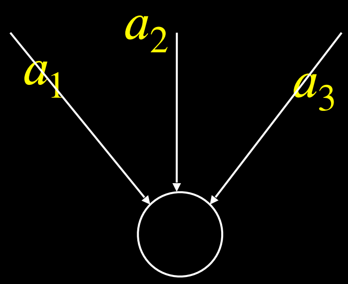

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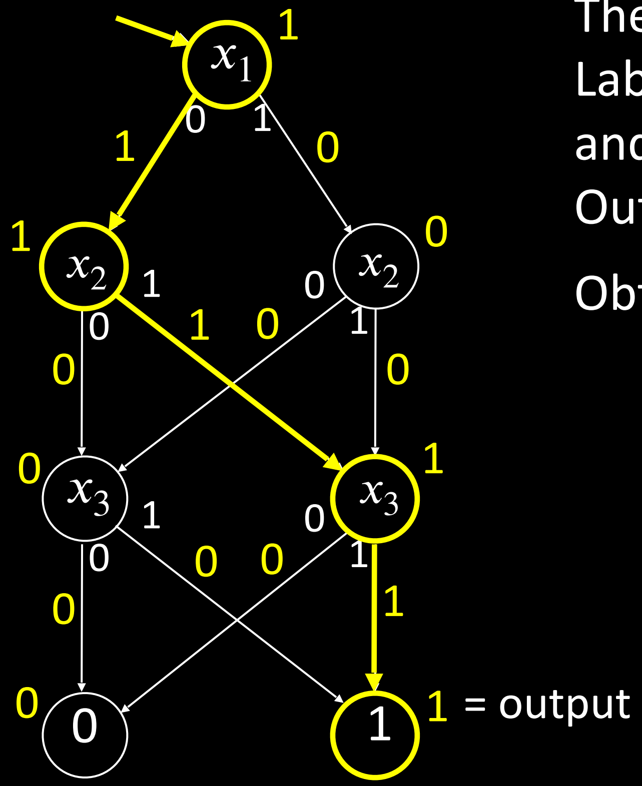

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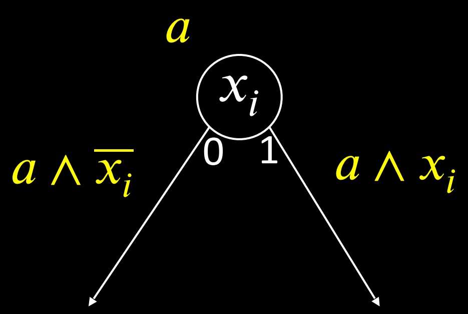

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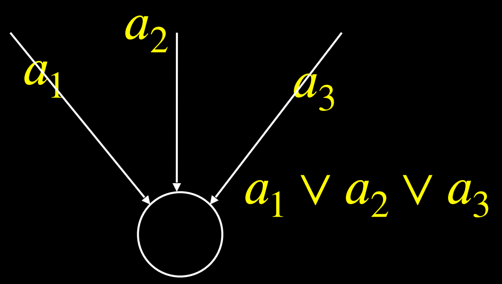

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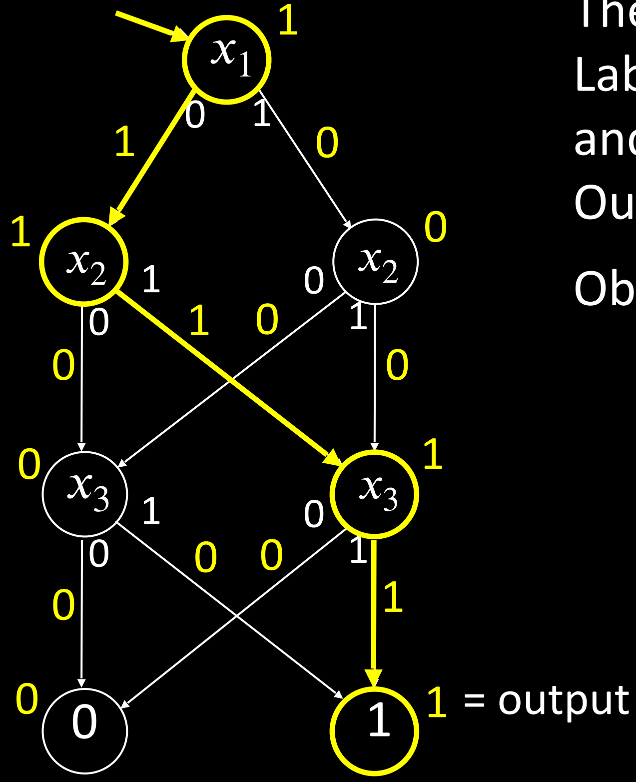

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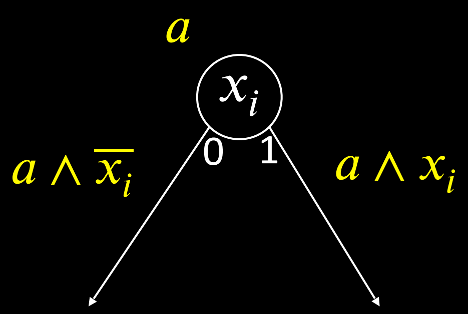

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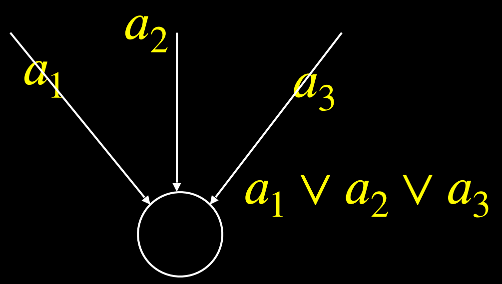

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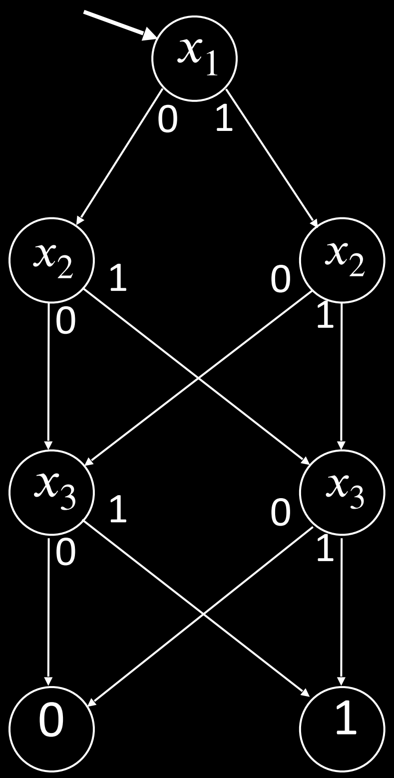

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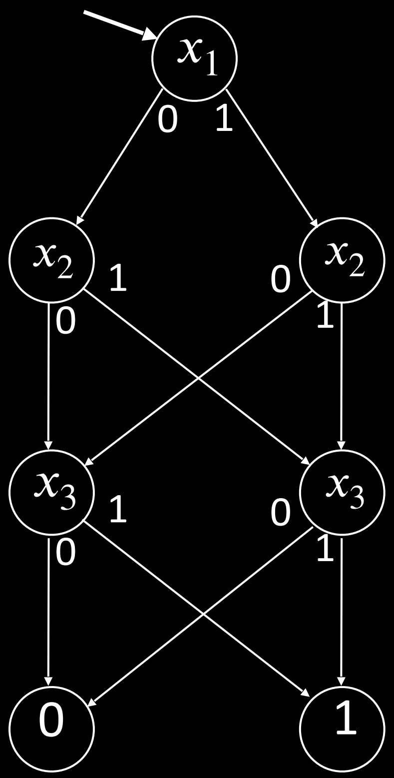

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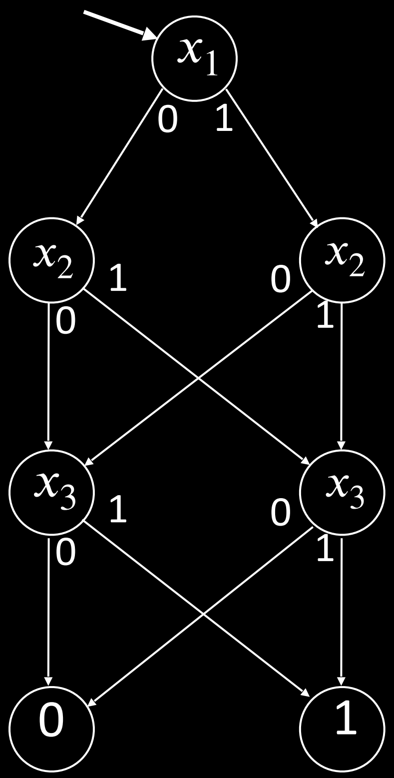

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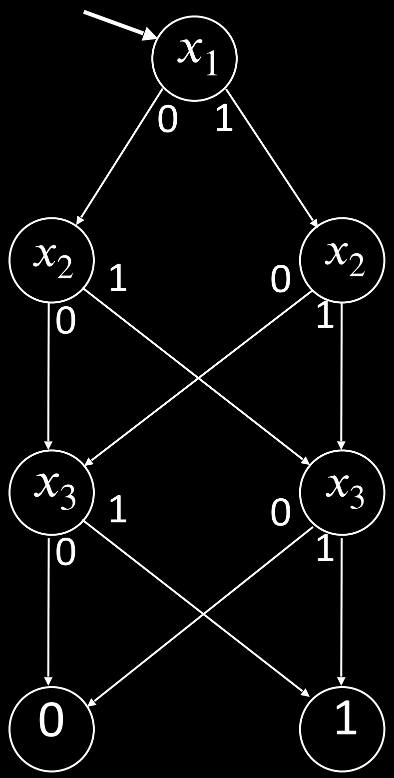

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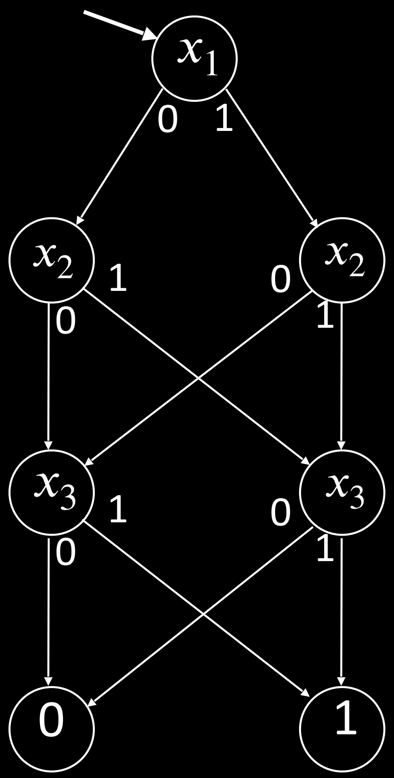

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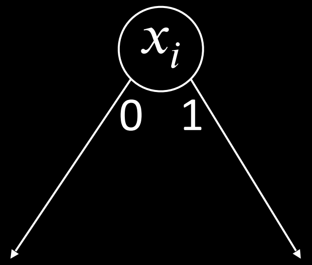

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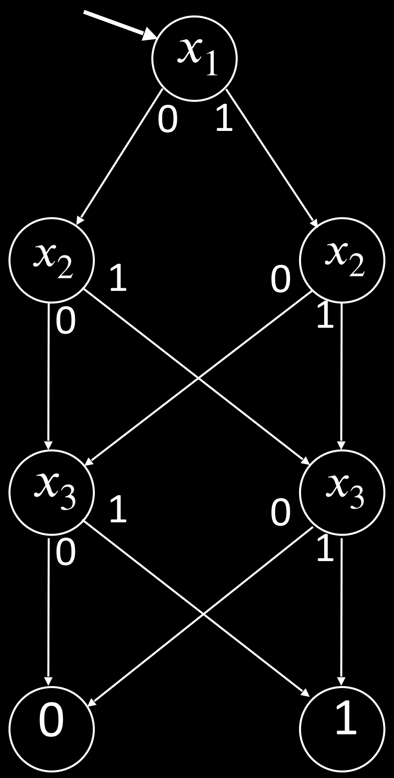

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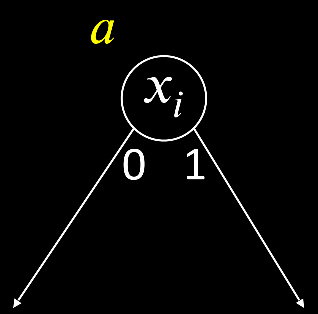

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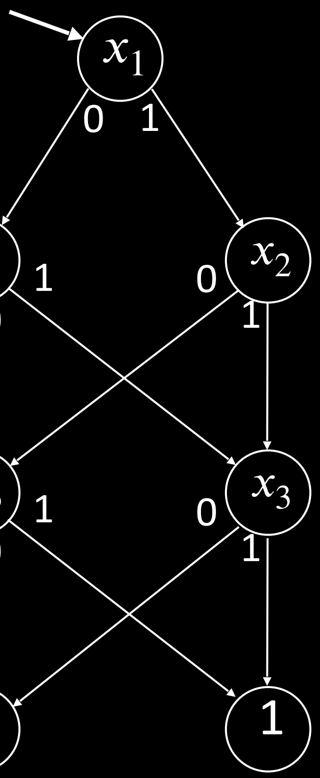

Replace Boolean labeling with arithmetical labeling

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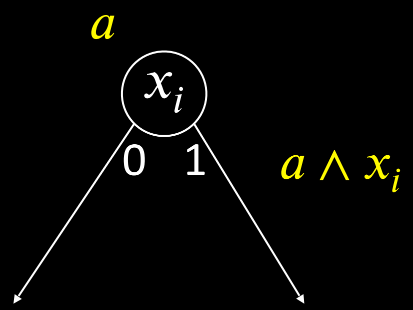

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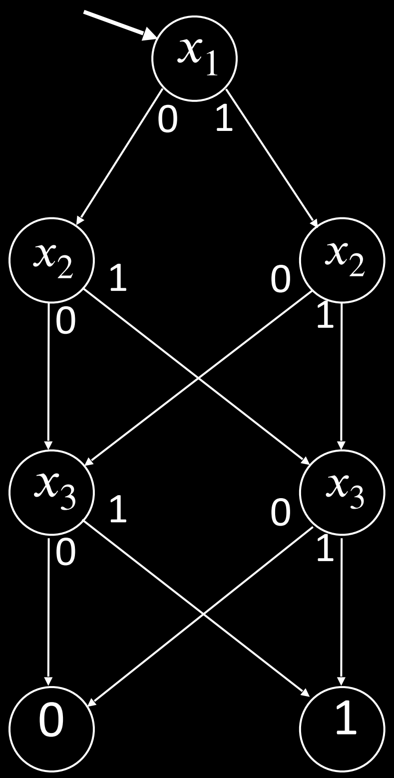

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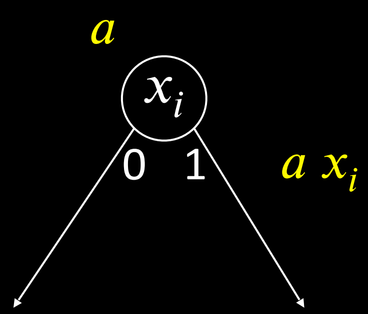

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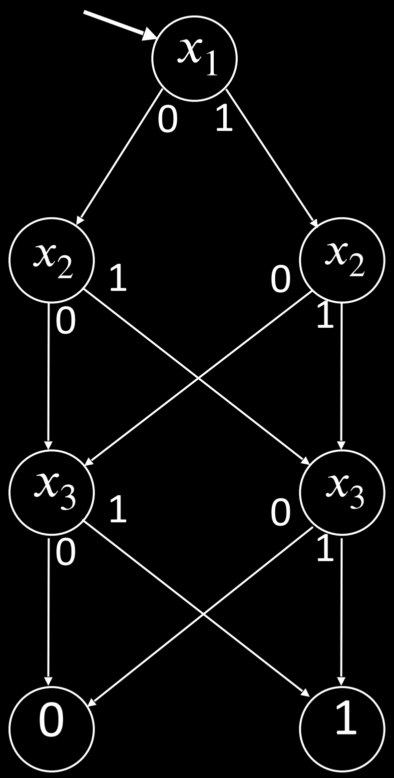

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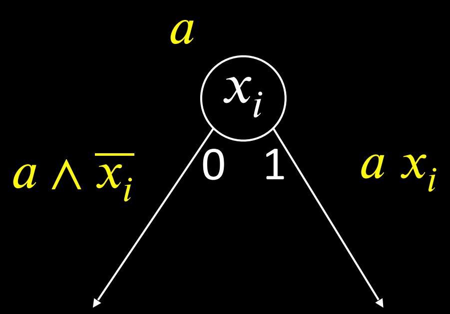

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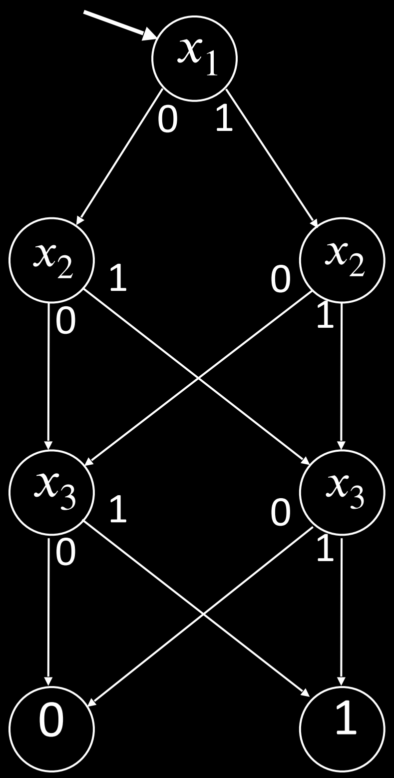

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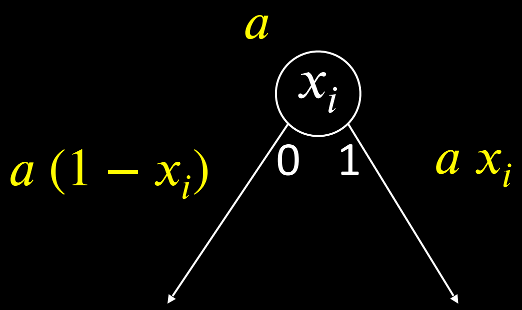

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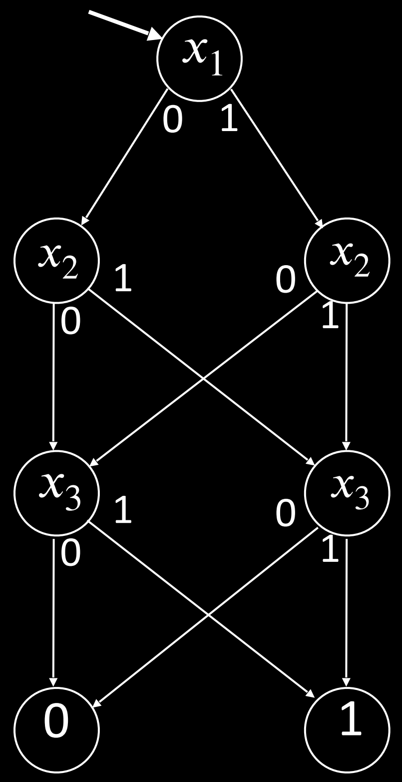

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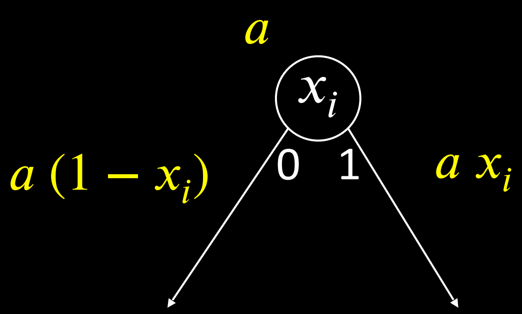

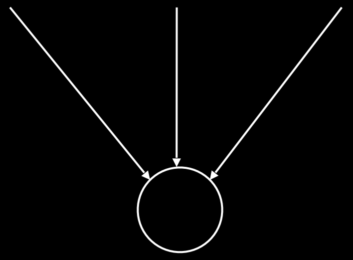

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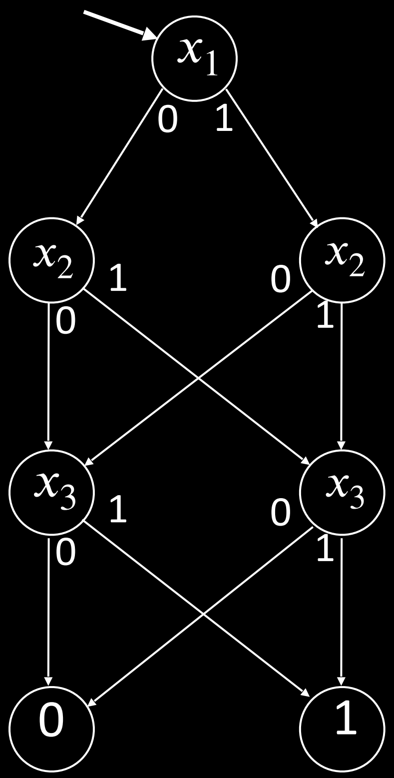

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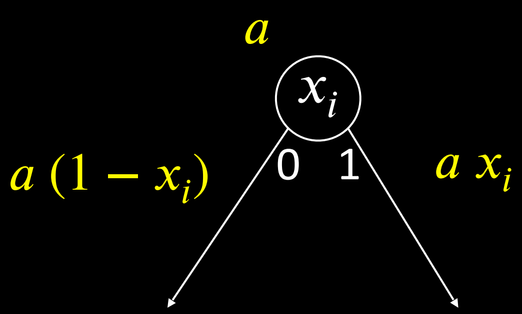

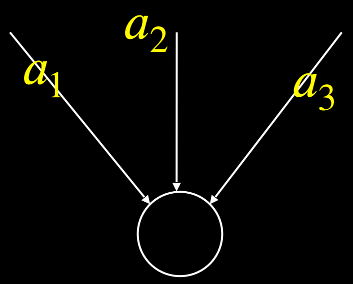

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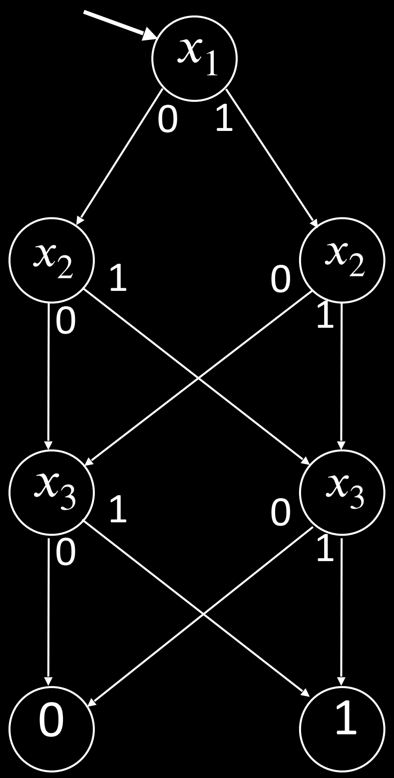

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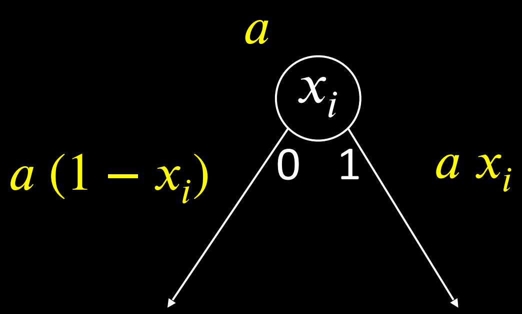

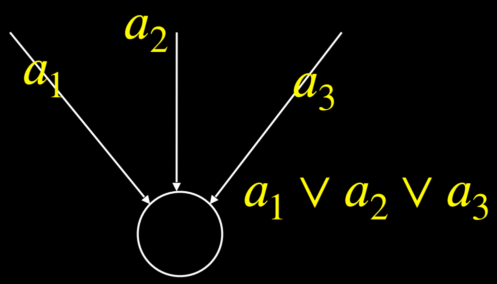

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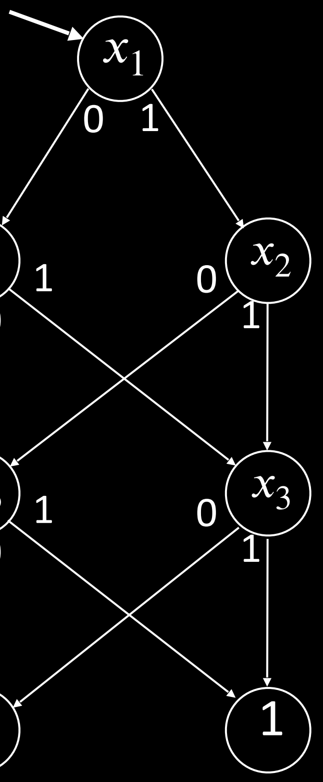

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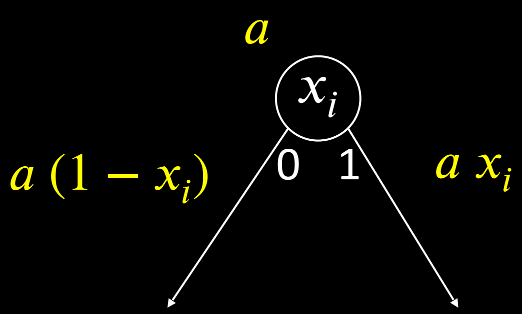

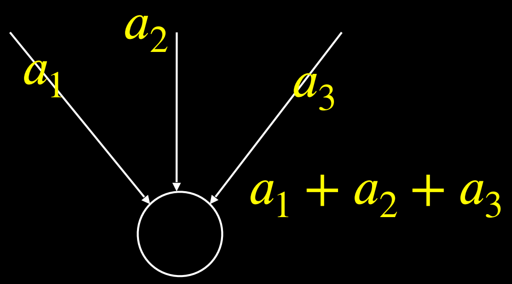

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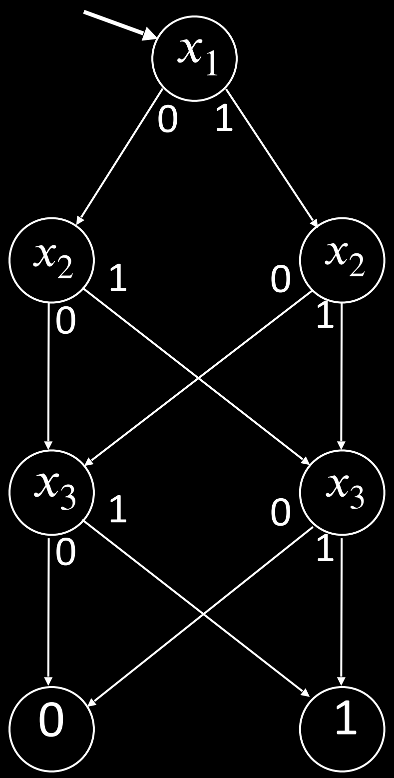

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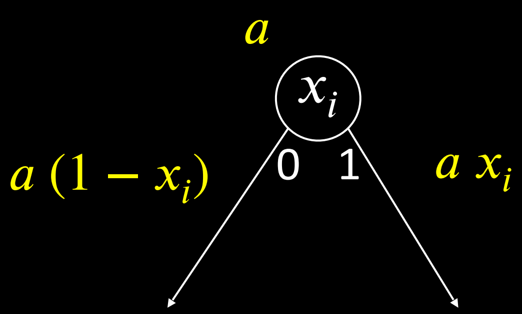

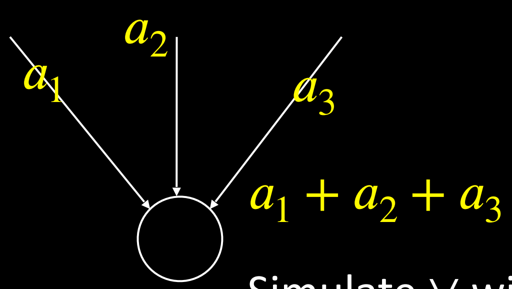

Simulate $\vee$ with $+$ because the BP is acyclic.
The execution path can enter a node
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$$
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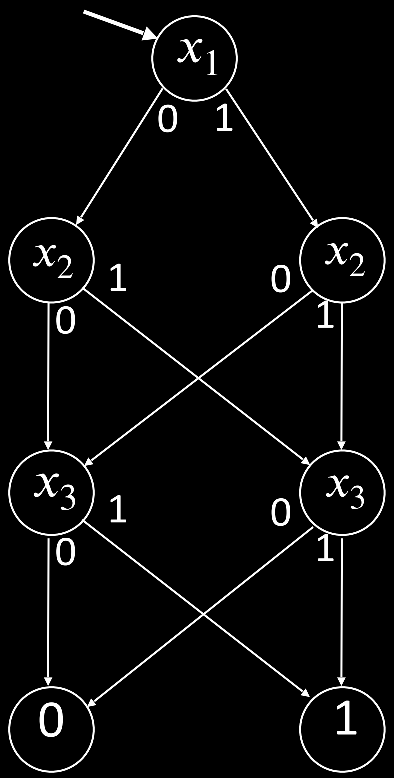

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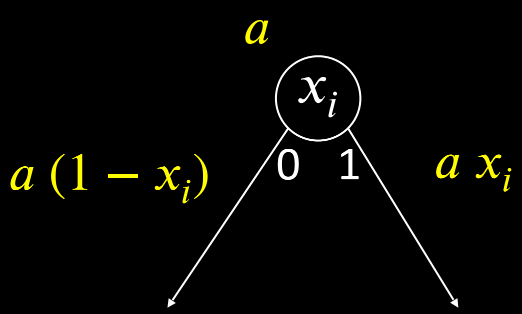

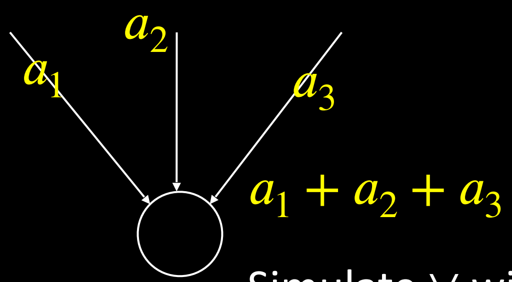

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Non-Boolean Labeling

Use the arithmetized interpretation of the BP’s computation to define its operation on non-Boolean inputs.

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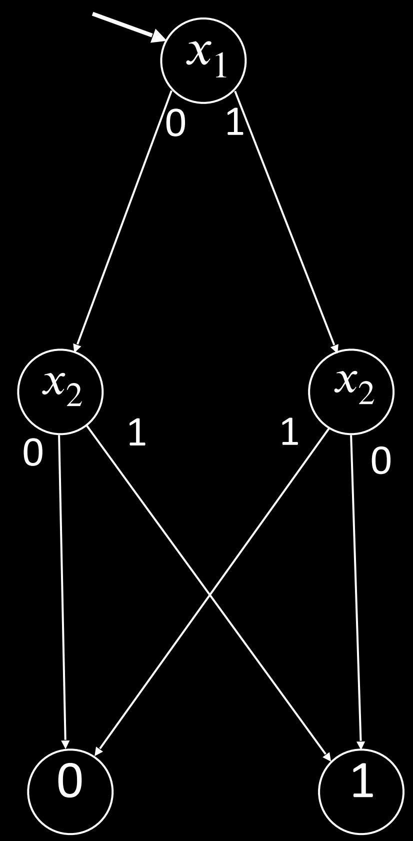

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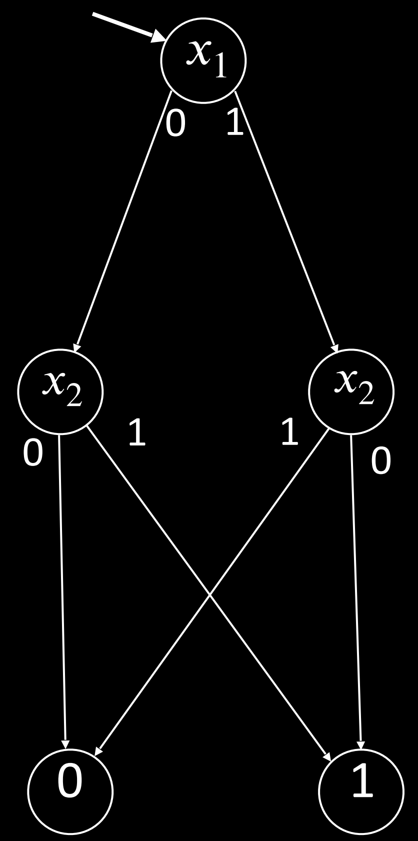

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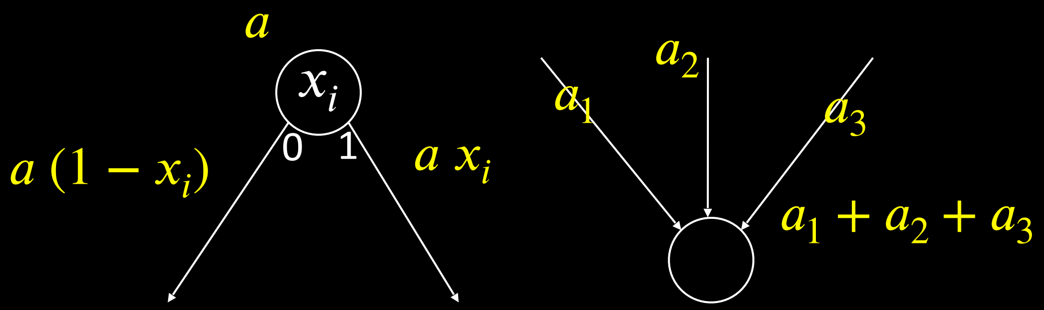

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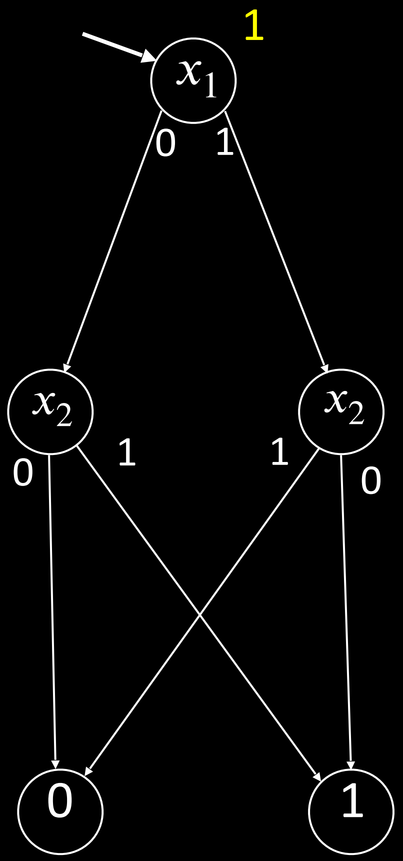

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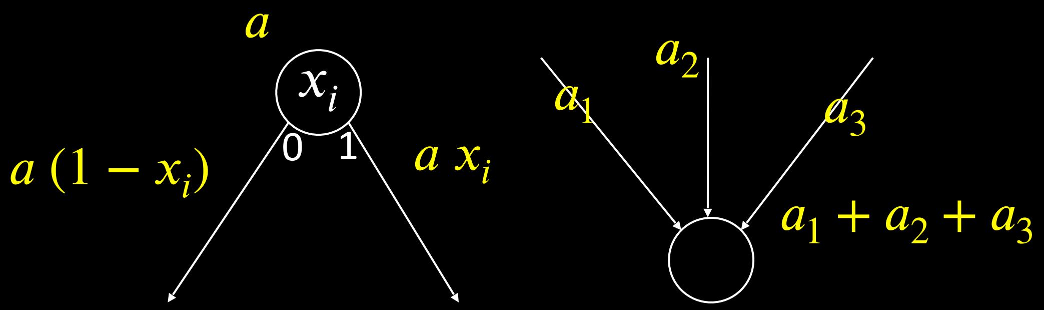

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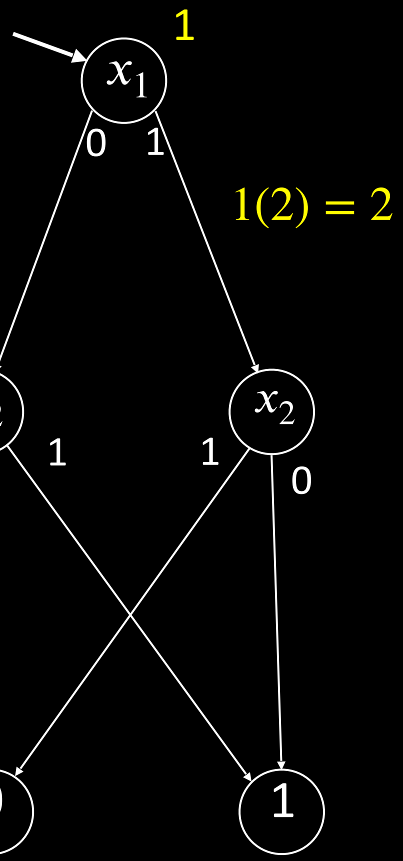

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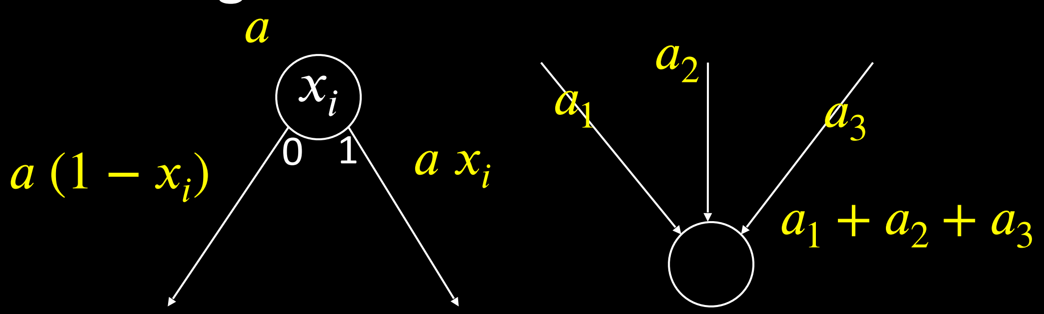

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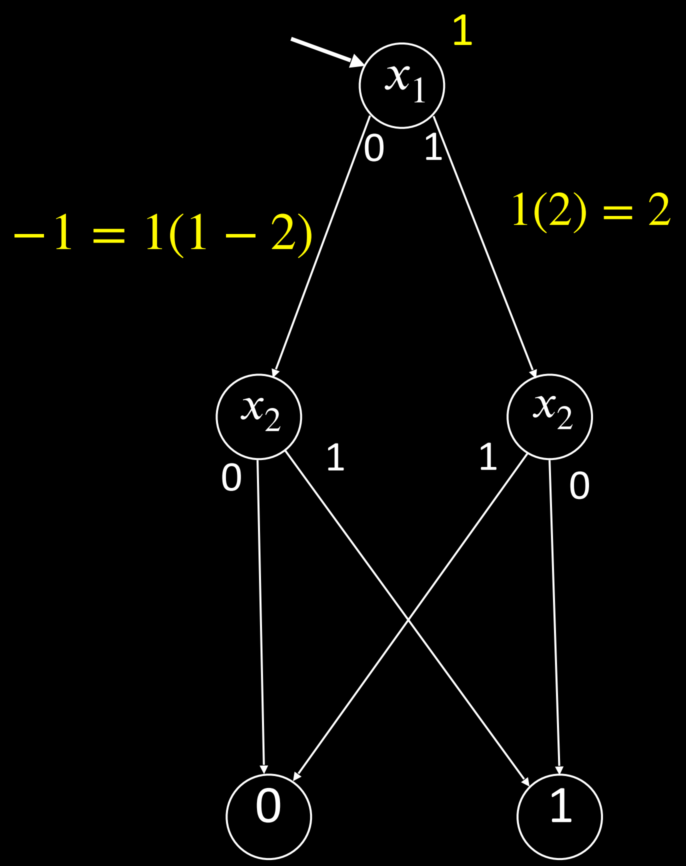

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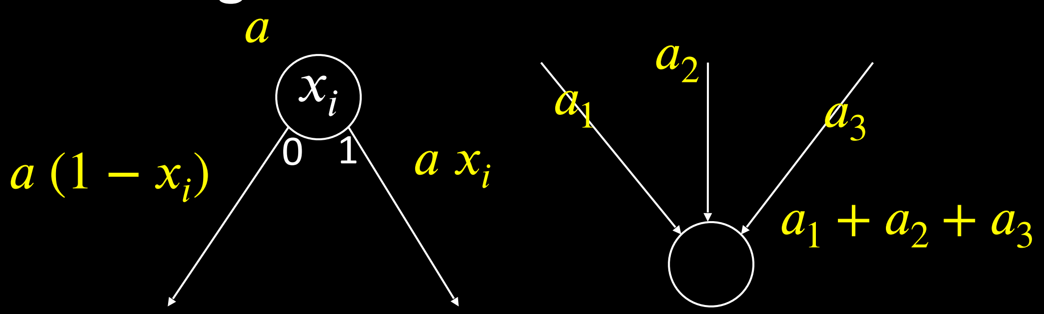

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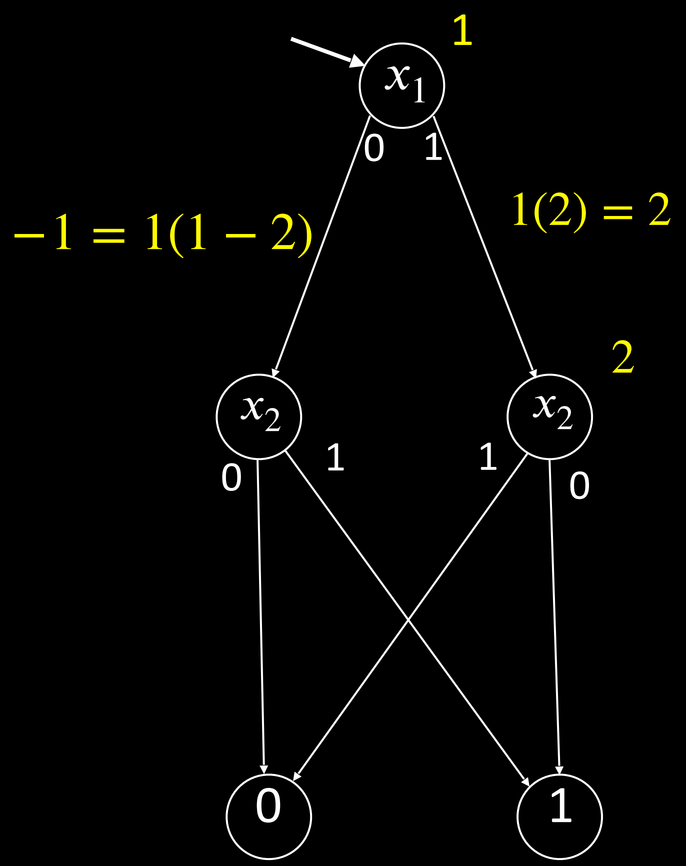

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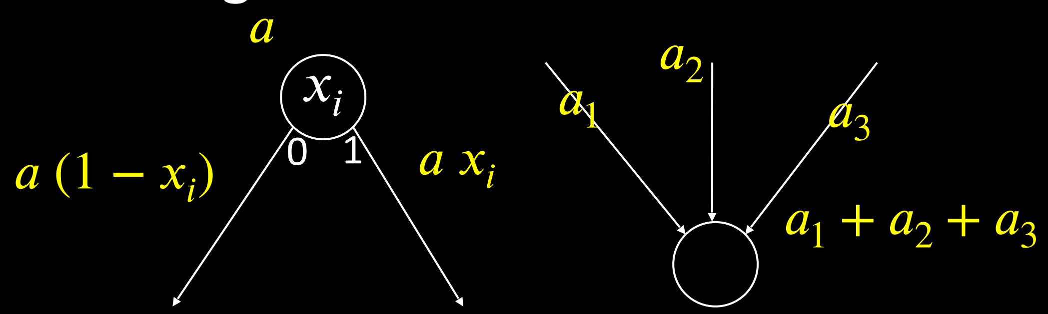

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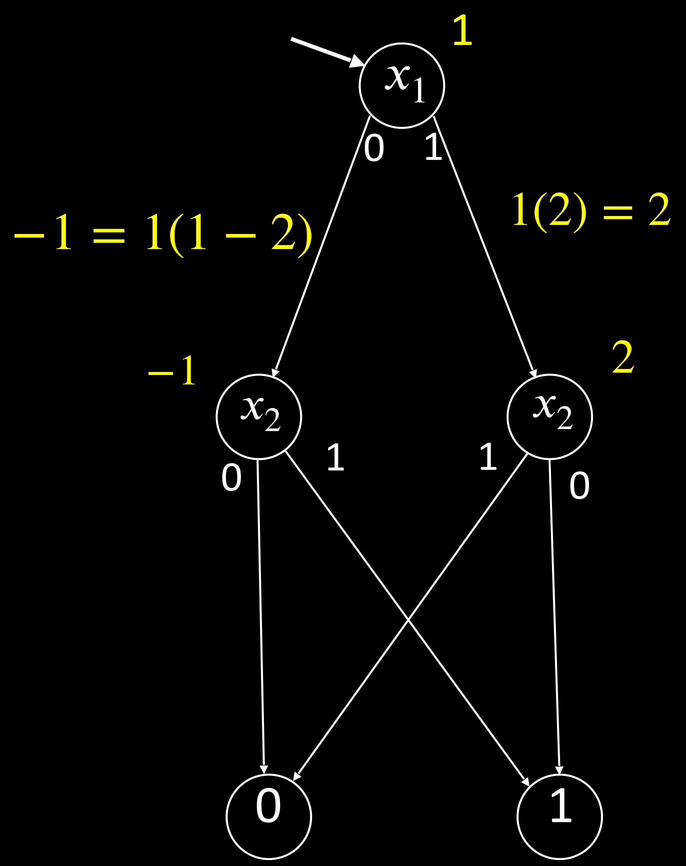

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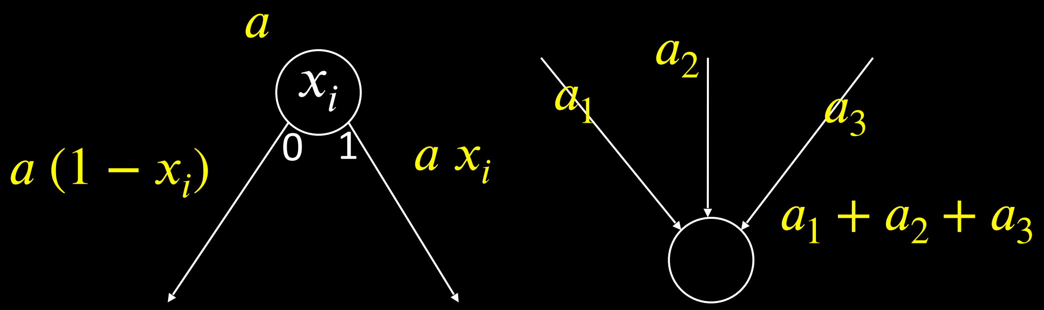

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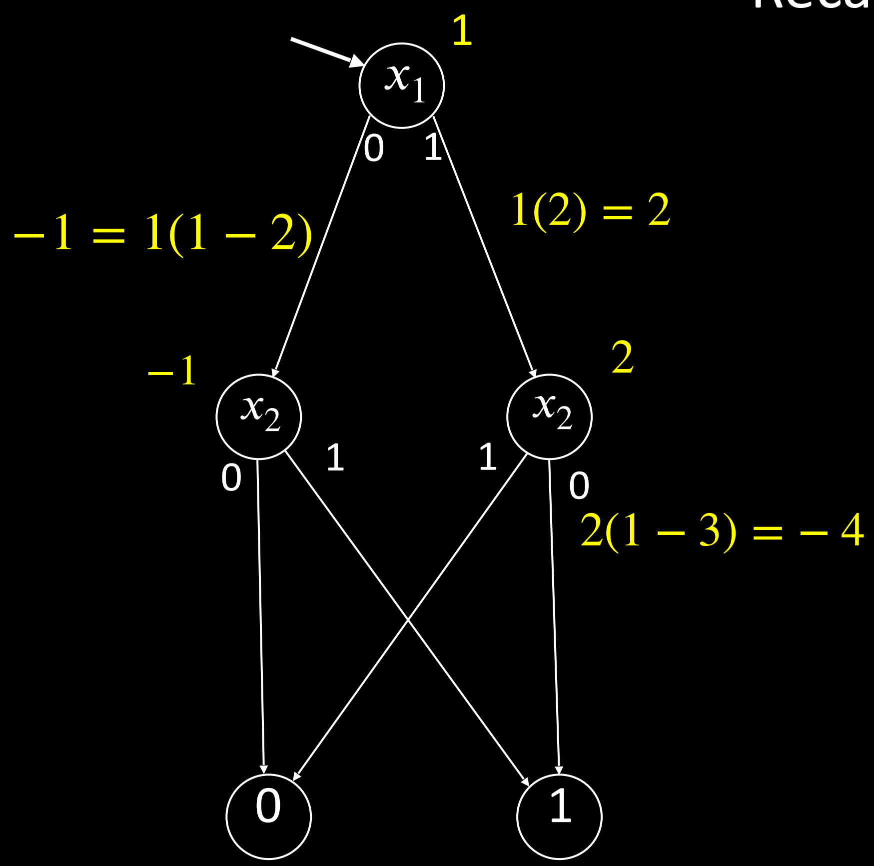

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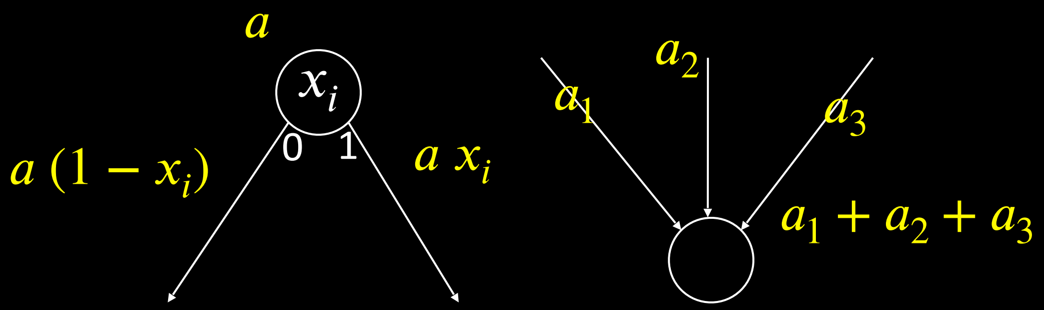

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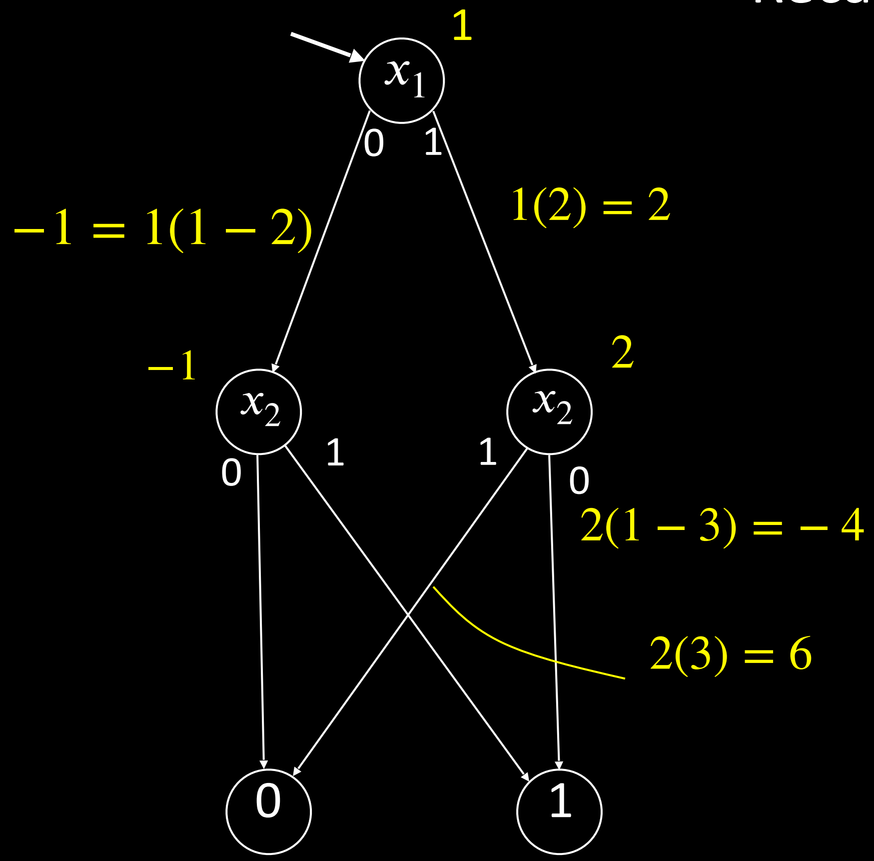

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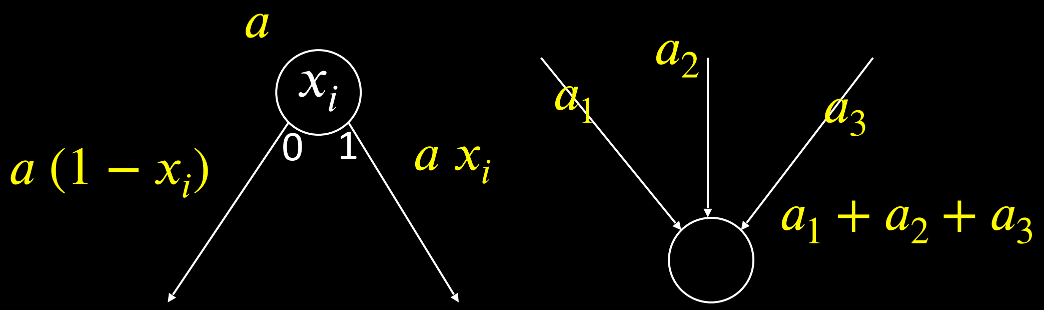

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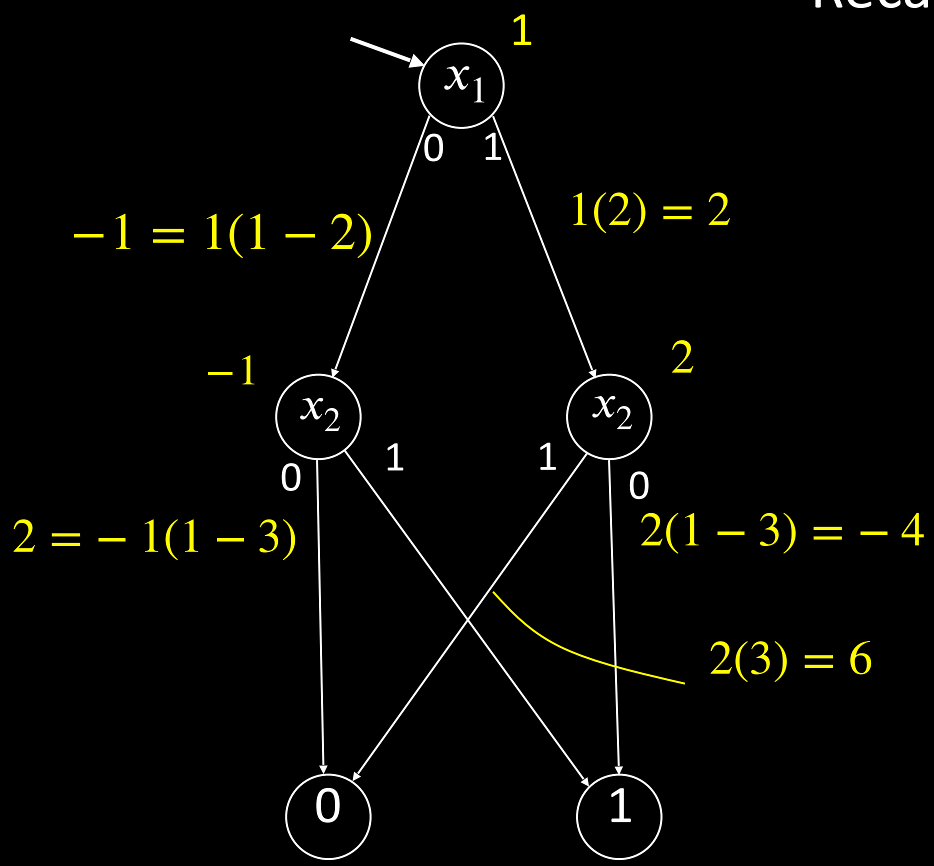

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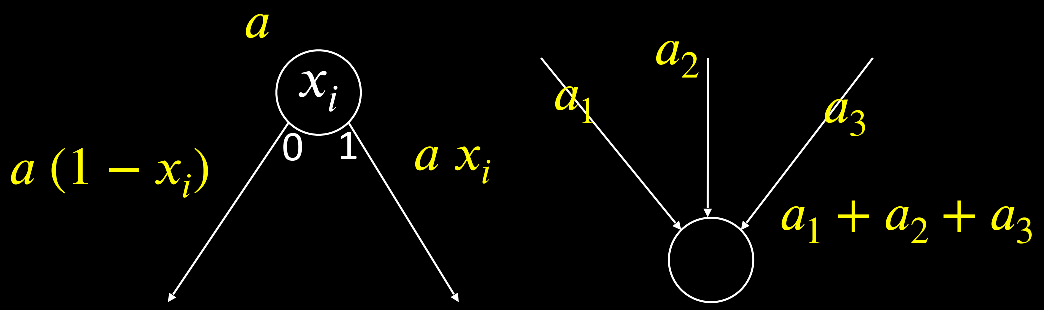

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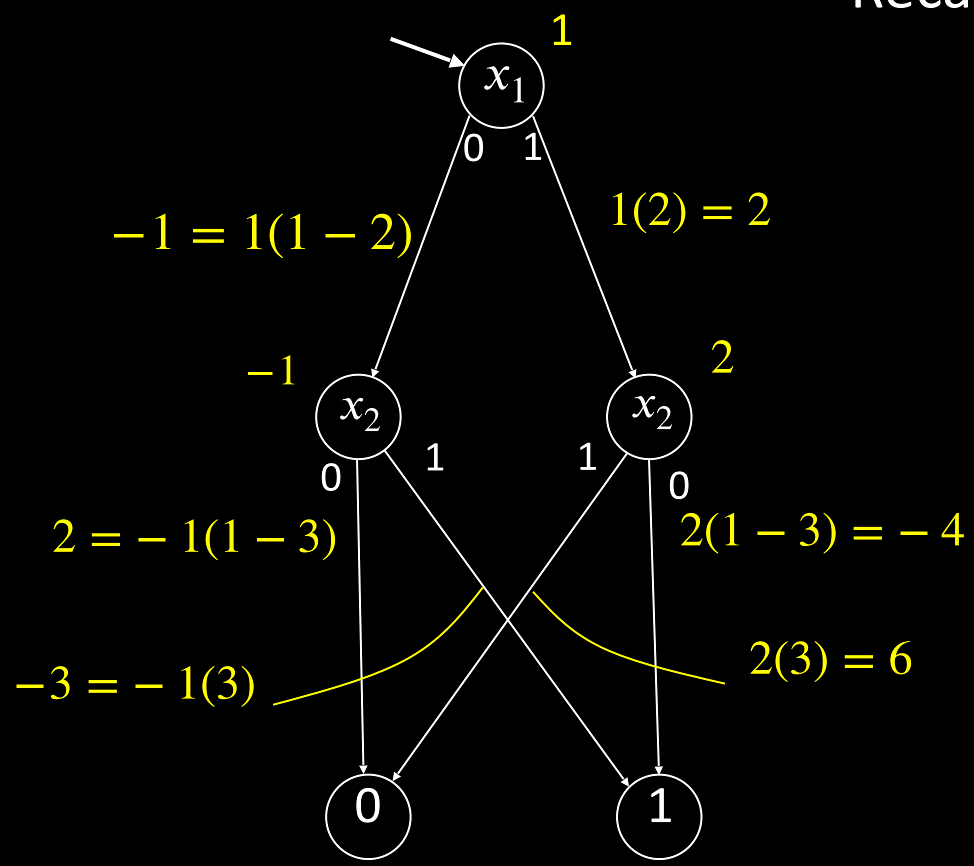

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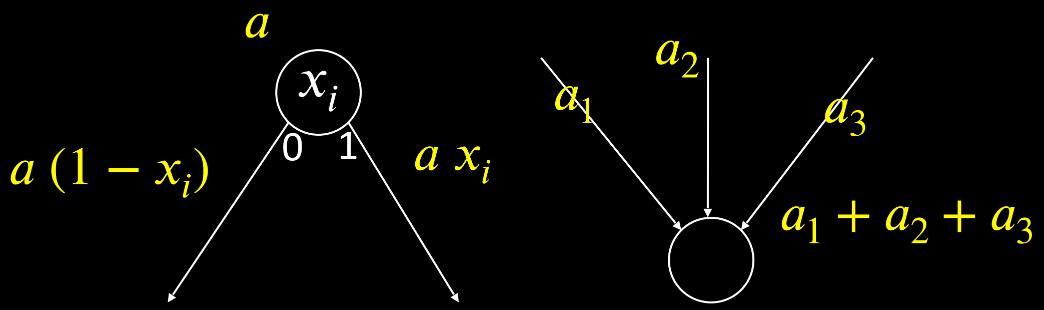

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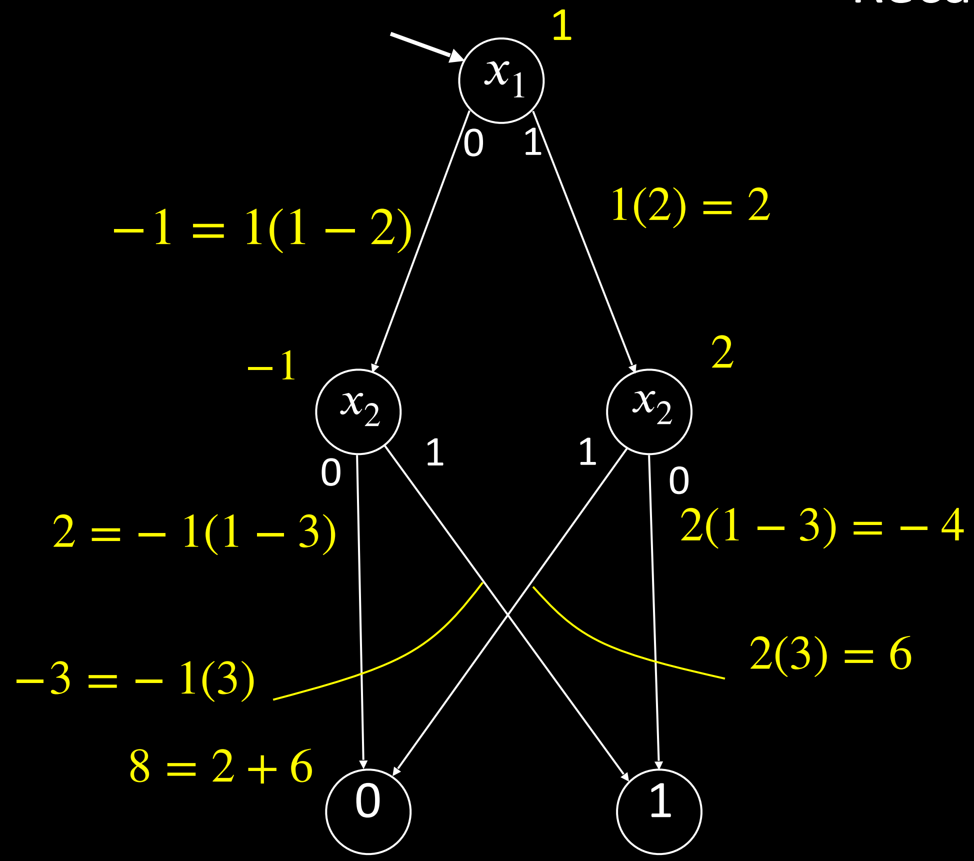

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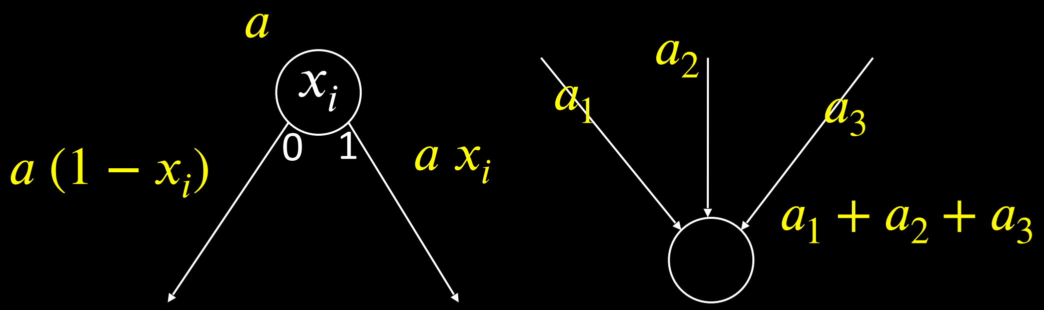

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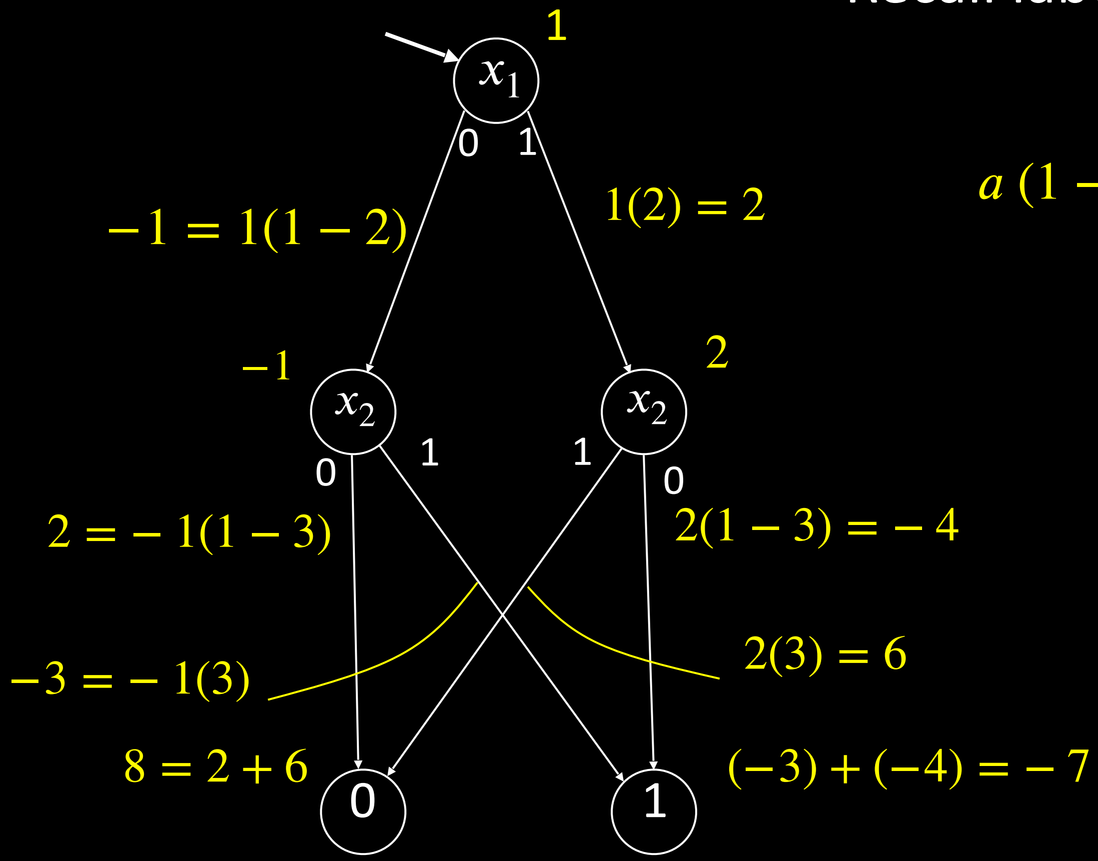

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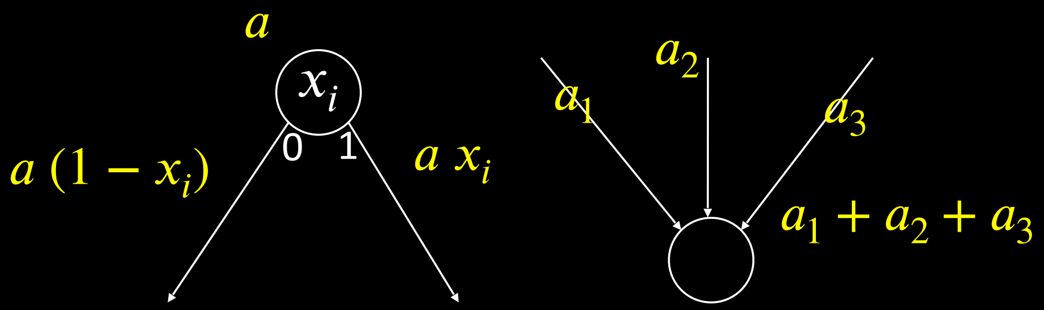

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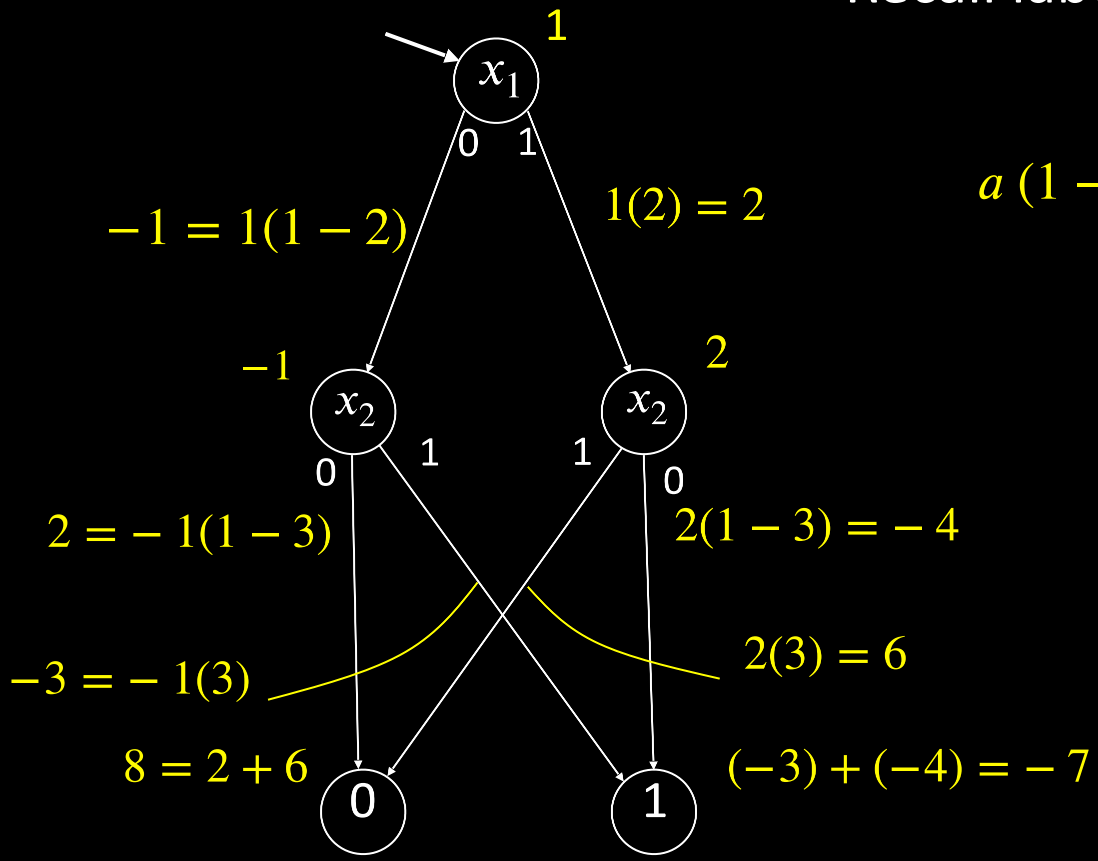

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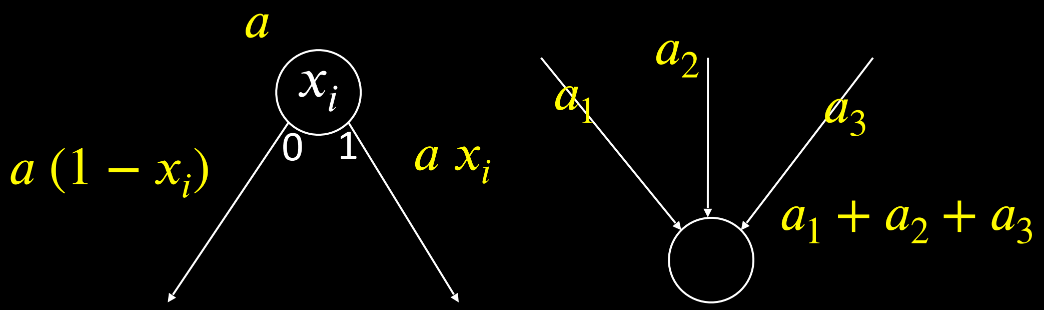

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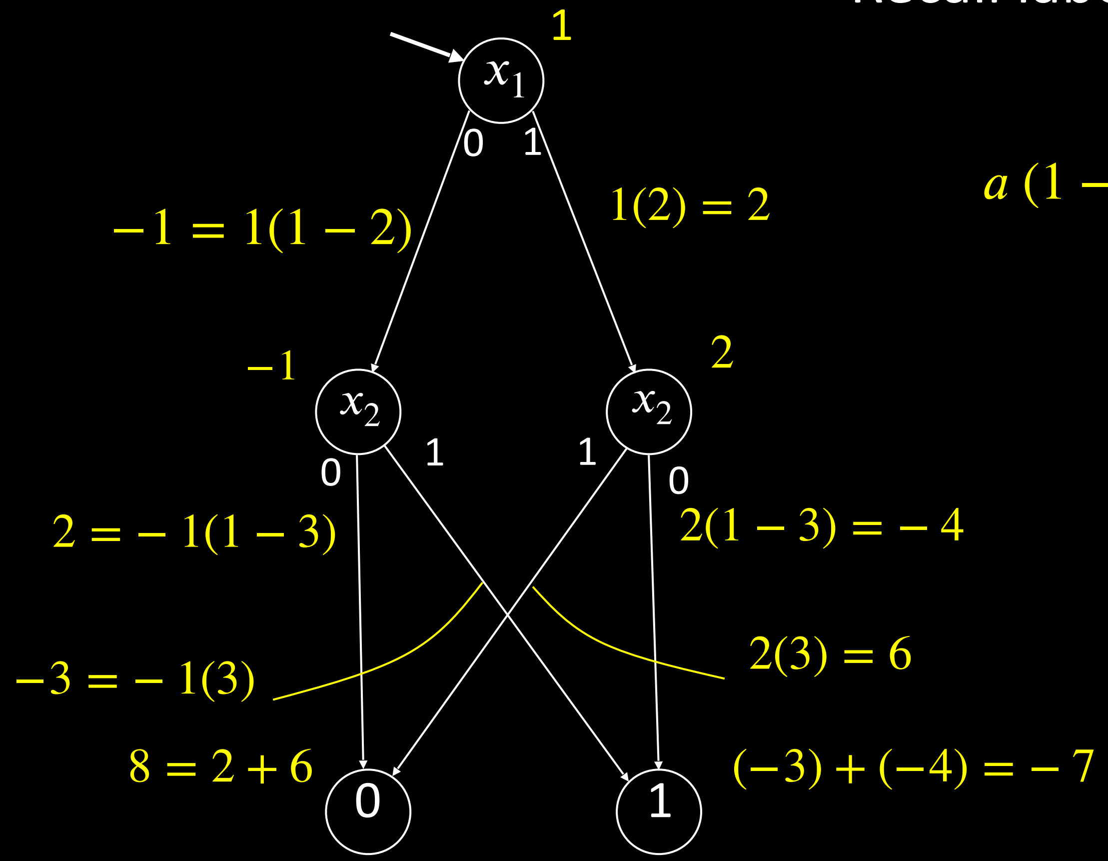

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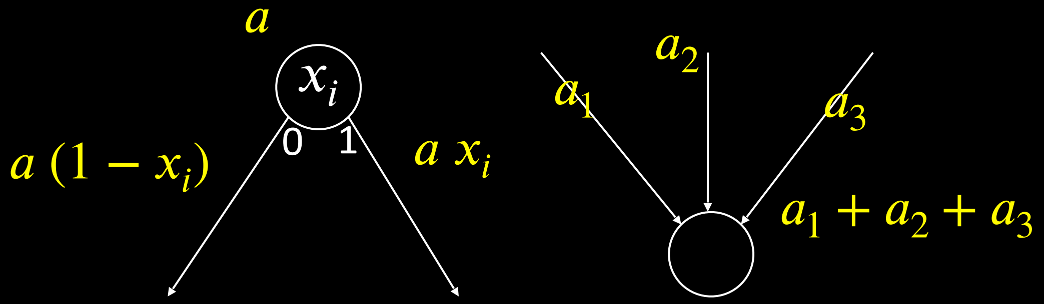

Algorithm sketch for EQROBP: “On input $\langle B_1, B_2 \rangle$

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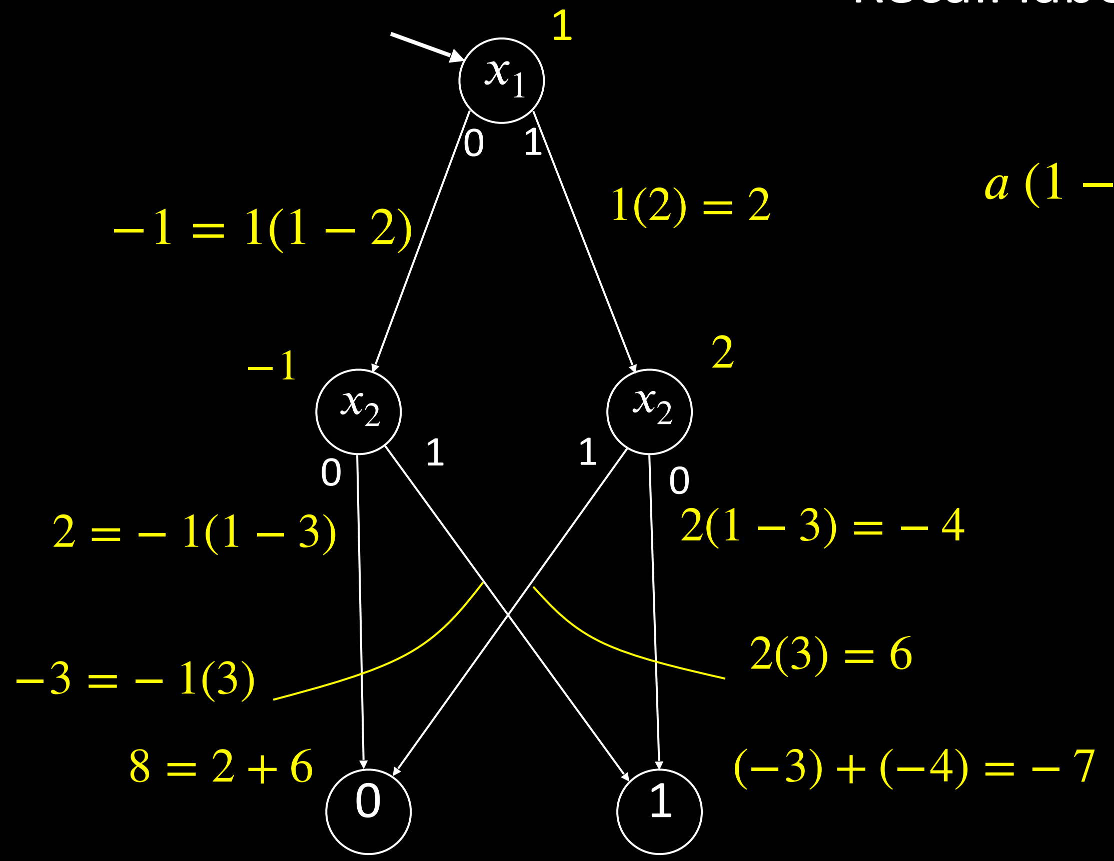

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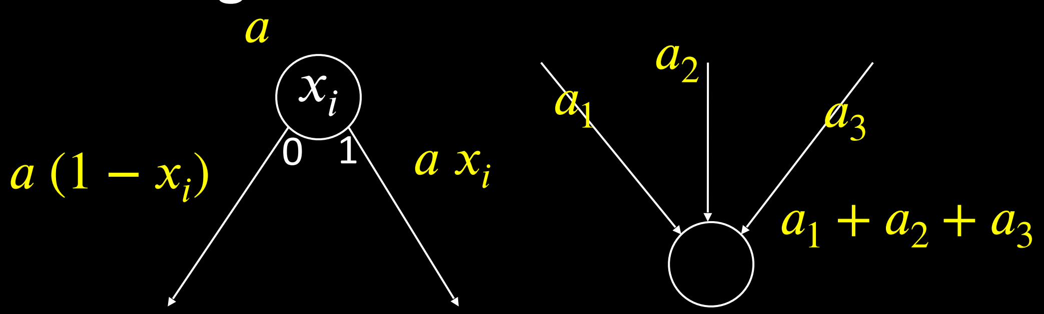


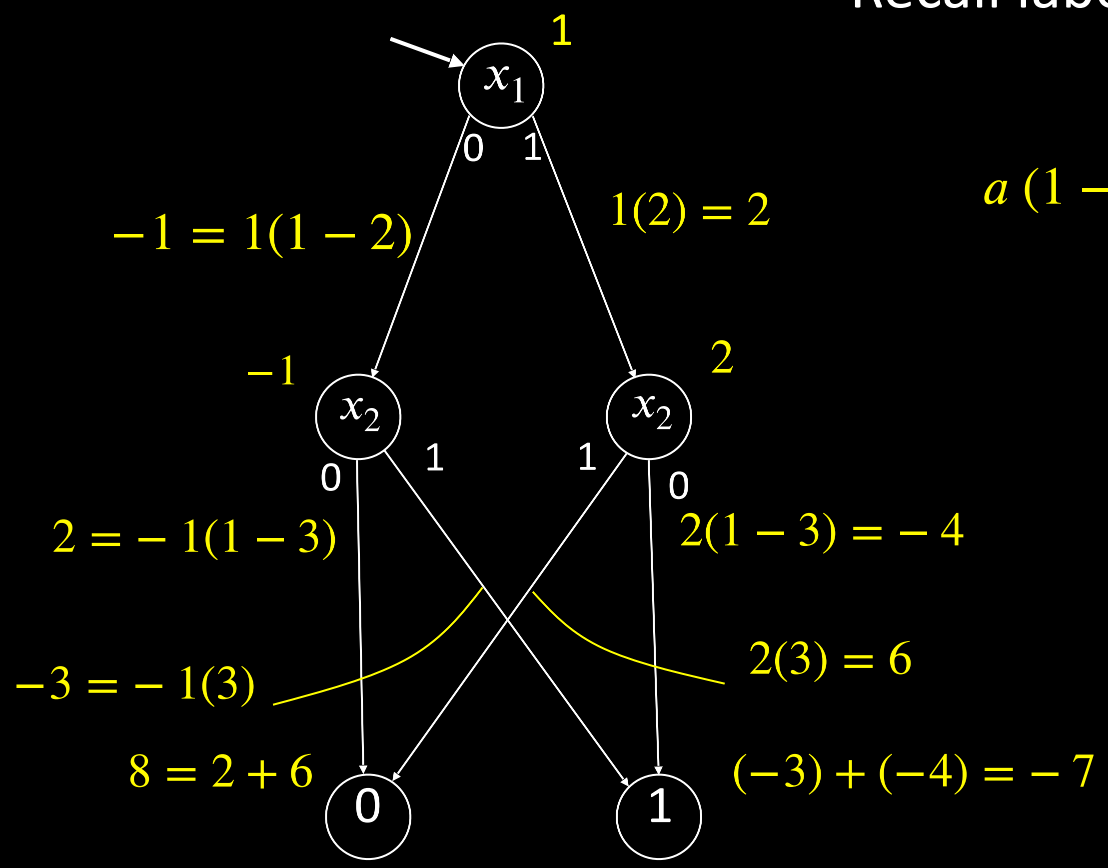
Algorithm sketch for EQROBP: "On input $\langle B_1, B_2 \rangle$

1. Pick a random *non-Boolean* input assignment.

Non-Boolean Labeling

Use the arithmetized interpretation of the BP's computation to define its operation on non-Boolean inputs.

Example: $x_1 = 2$, $x_2 = 3$ Output = -7



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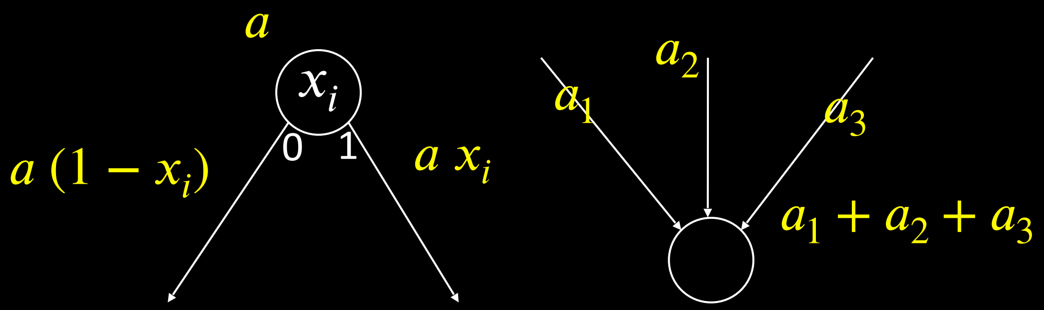

Algorithm sketch for EQROBP: "On input $\langle B_1, B_2\rangle$

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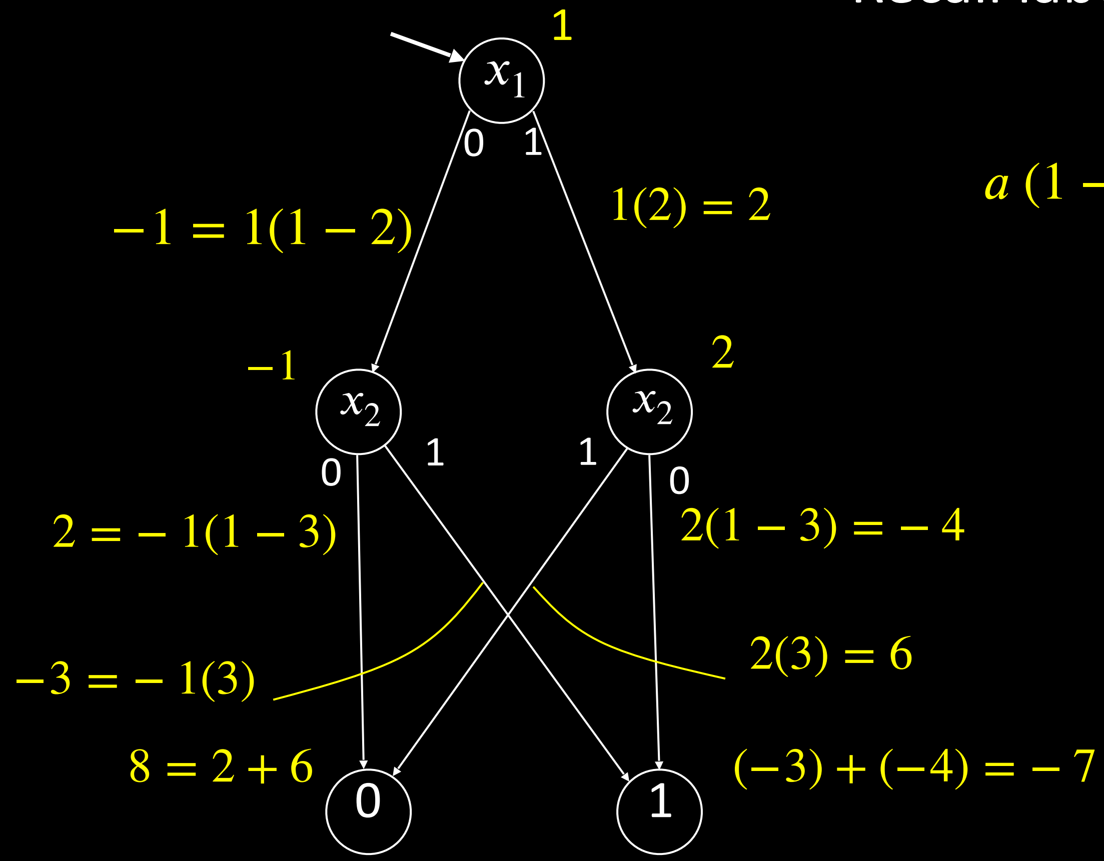

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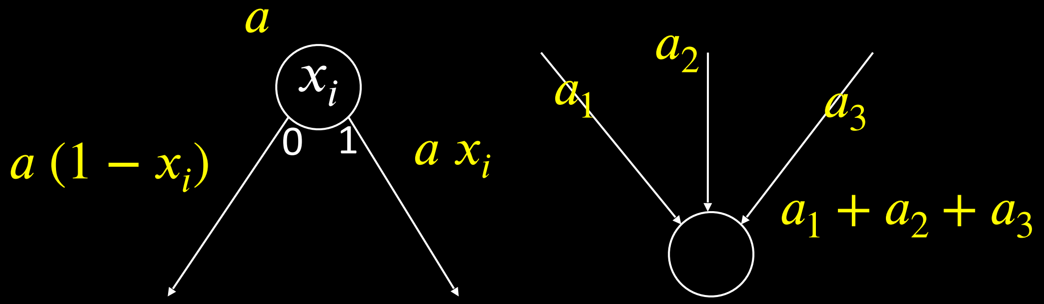

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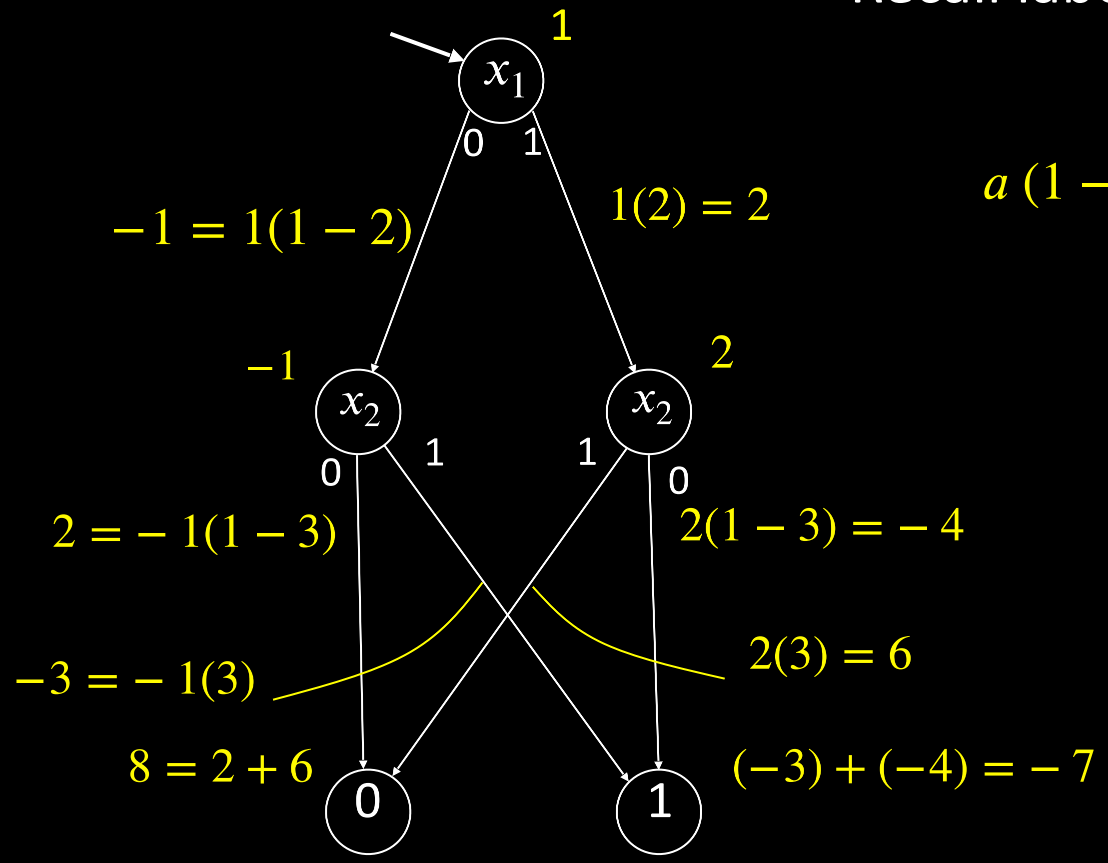

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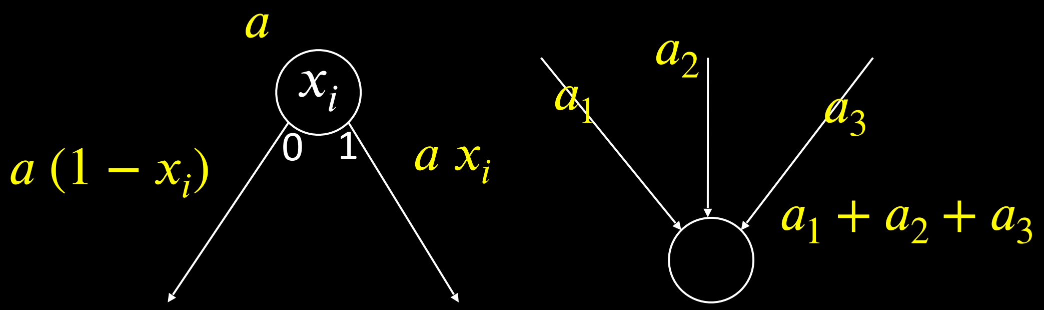

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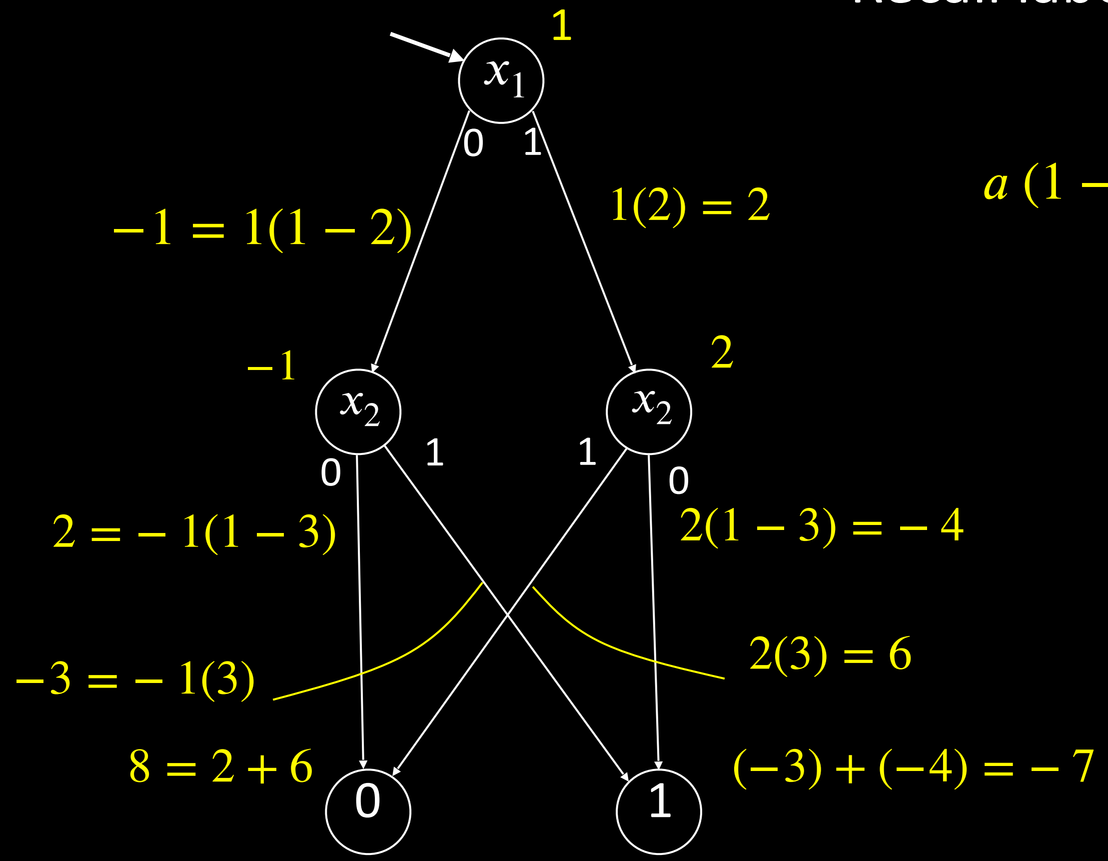

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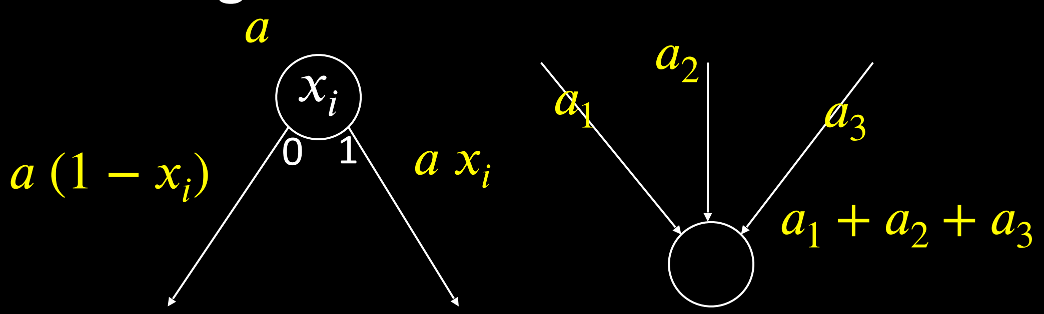

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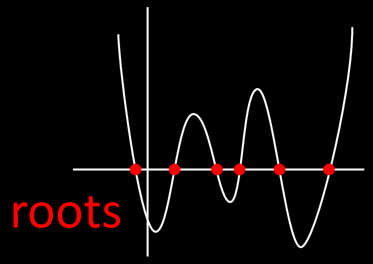
Roots of Polynomials



Let $p(x) = a_0 x^d + a_1 x^{d-1} + a_2 x^{d-2} + \cdots + a_d$ be a polynomial.

Corollary 2: If $p(x) \neq 0$ has degree $\leq d$ and we pick a random $r \in \mathbb{F}_q$, then $\Pr\left[\, p(r) = 0 \,\right] \leq \frac{d}{q}$.

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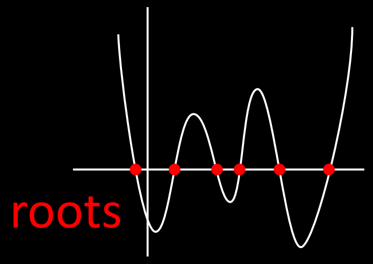

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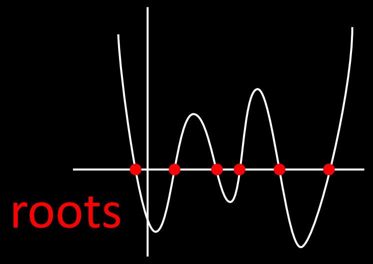

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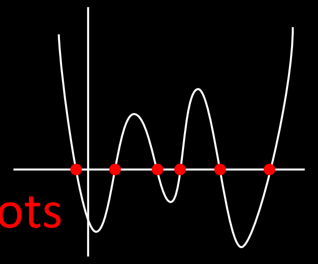

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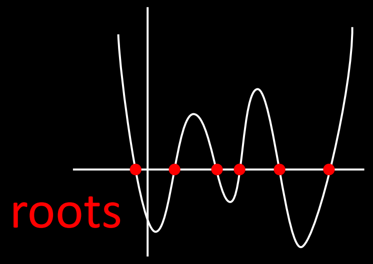

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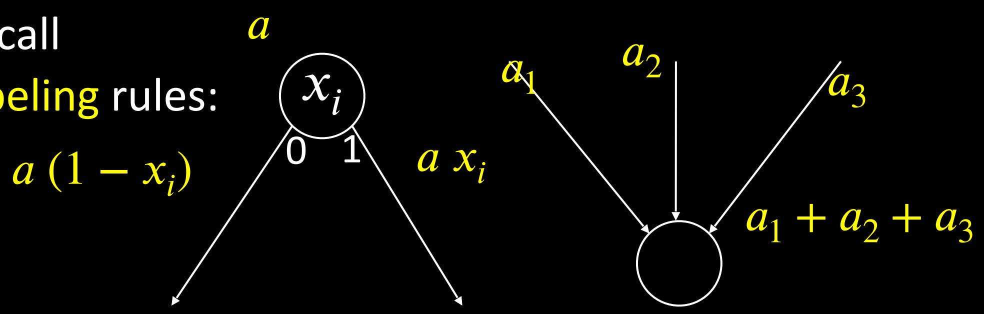

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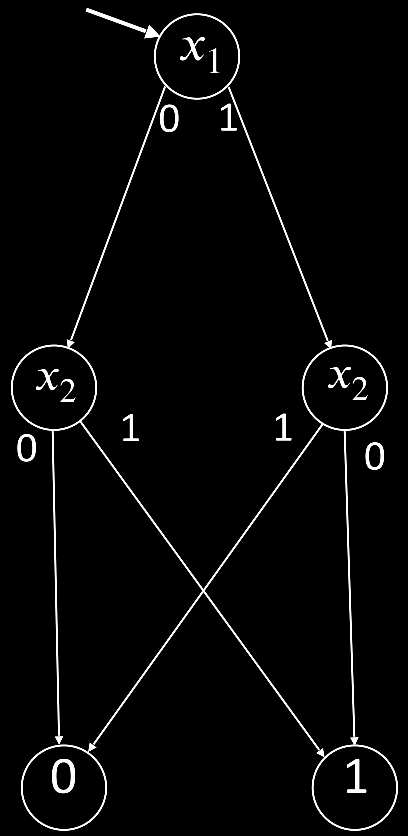

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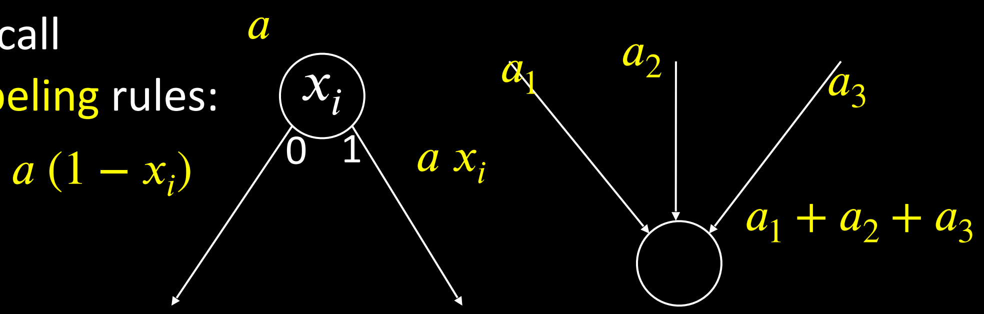

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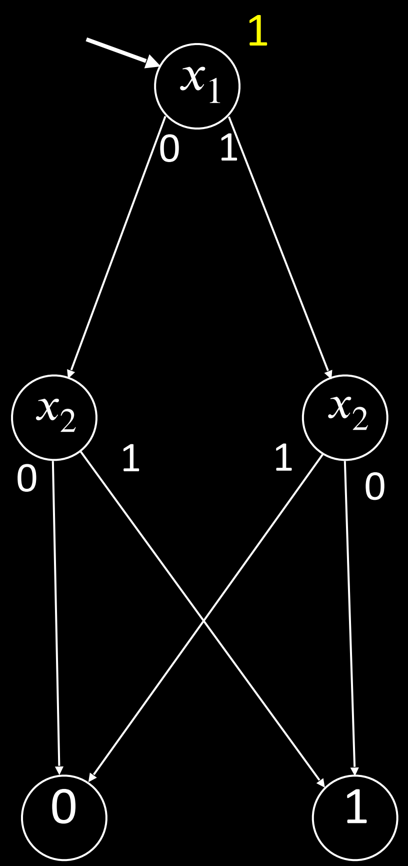

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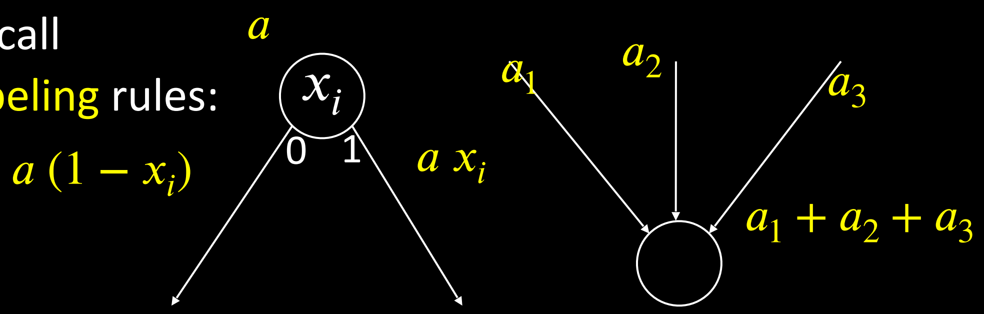

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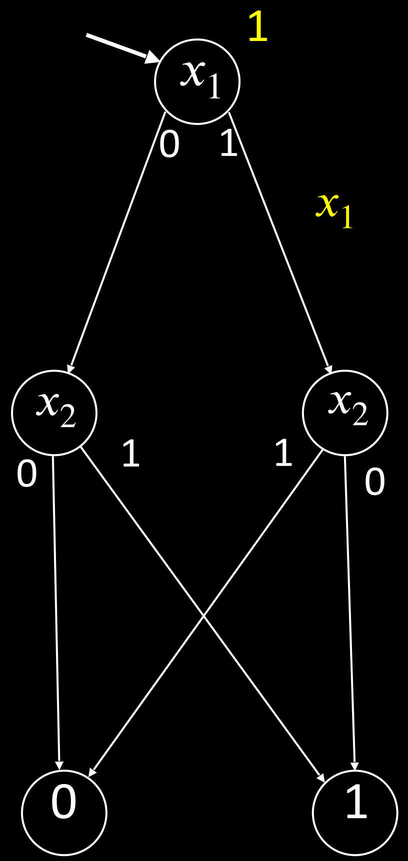

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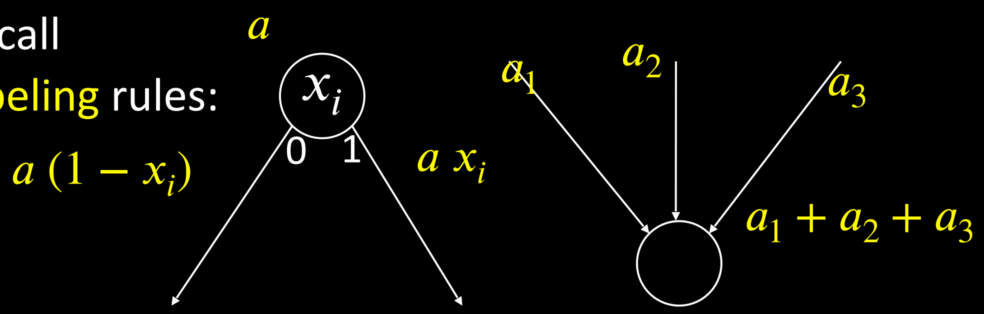

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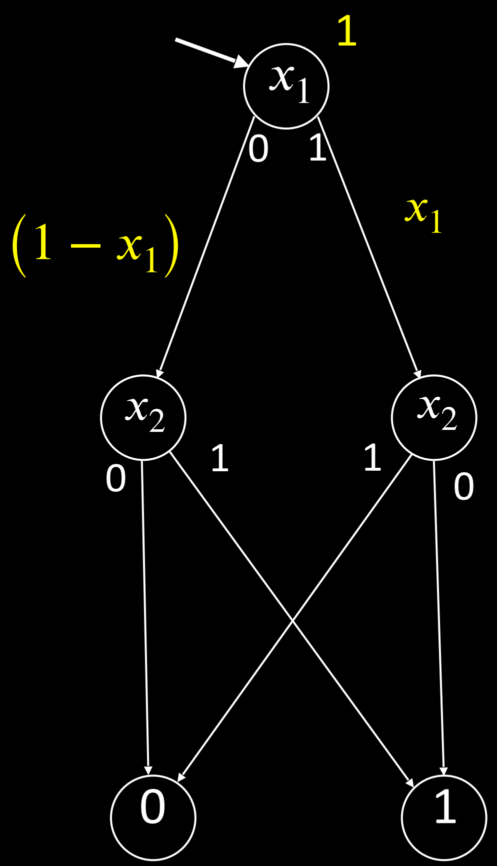

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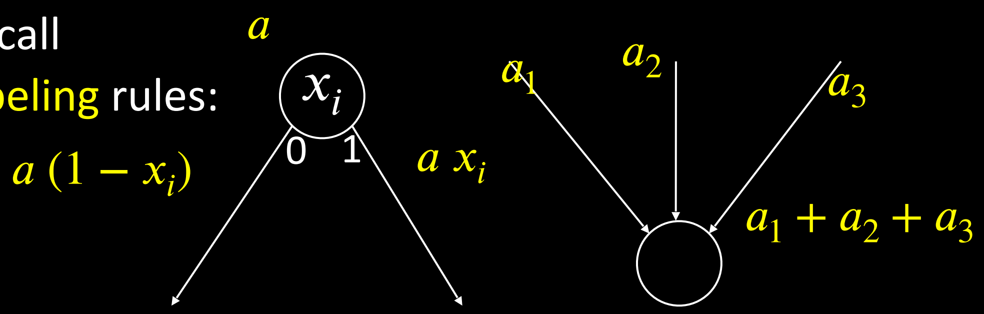

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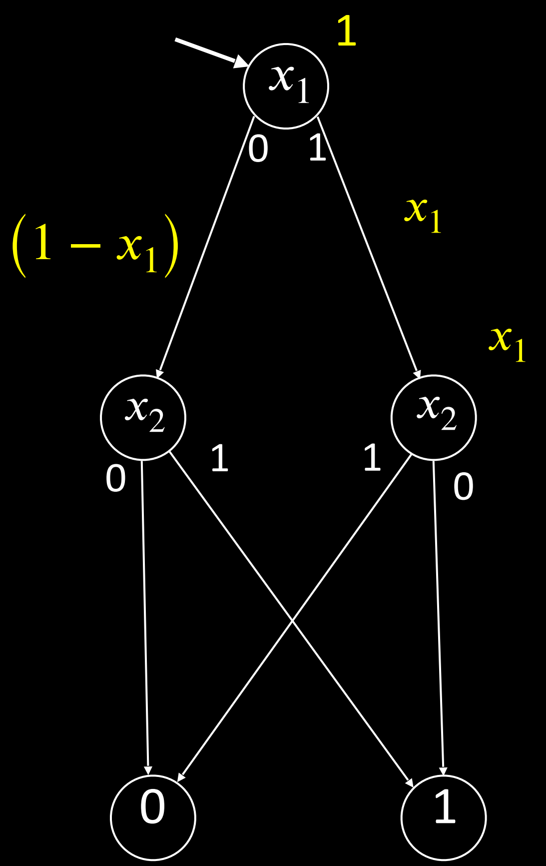

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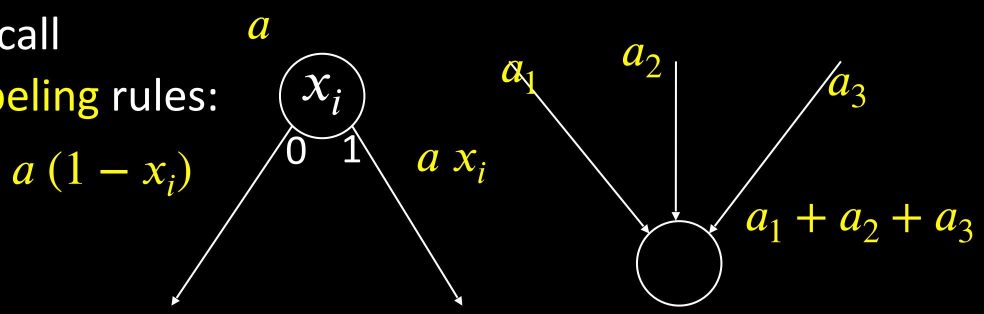

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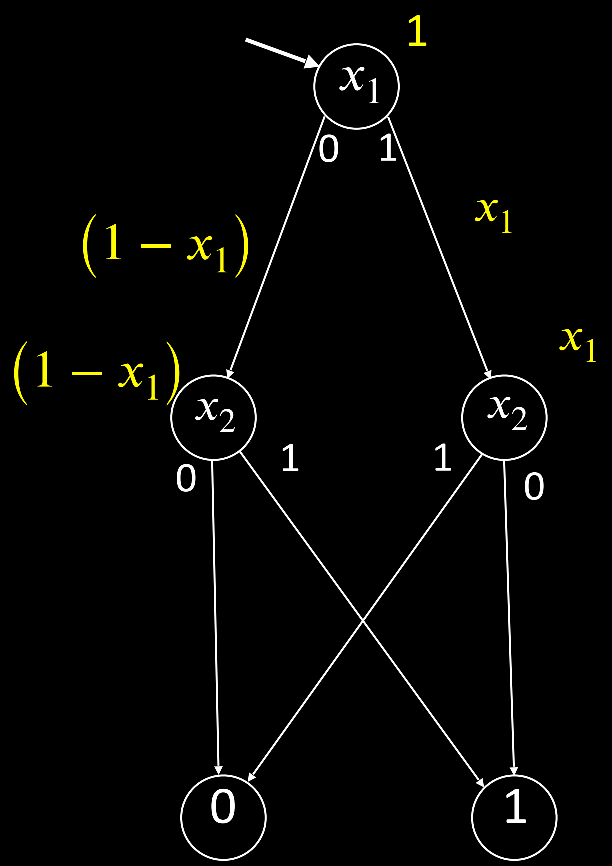

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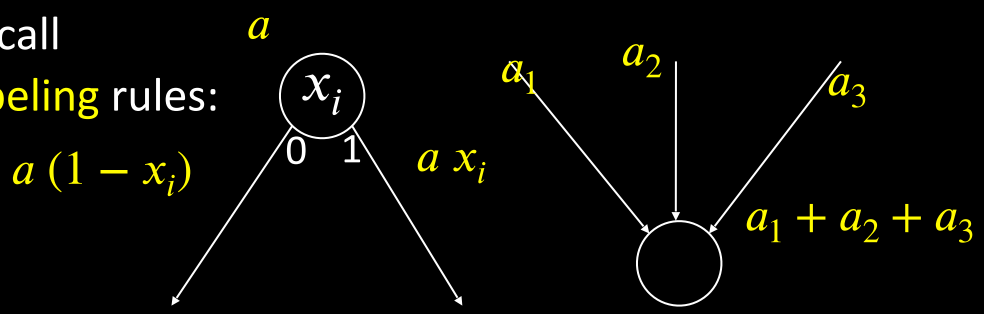

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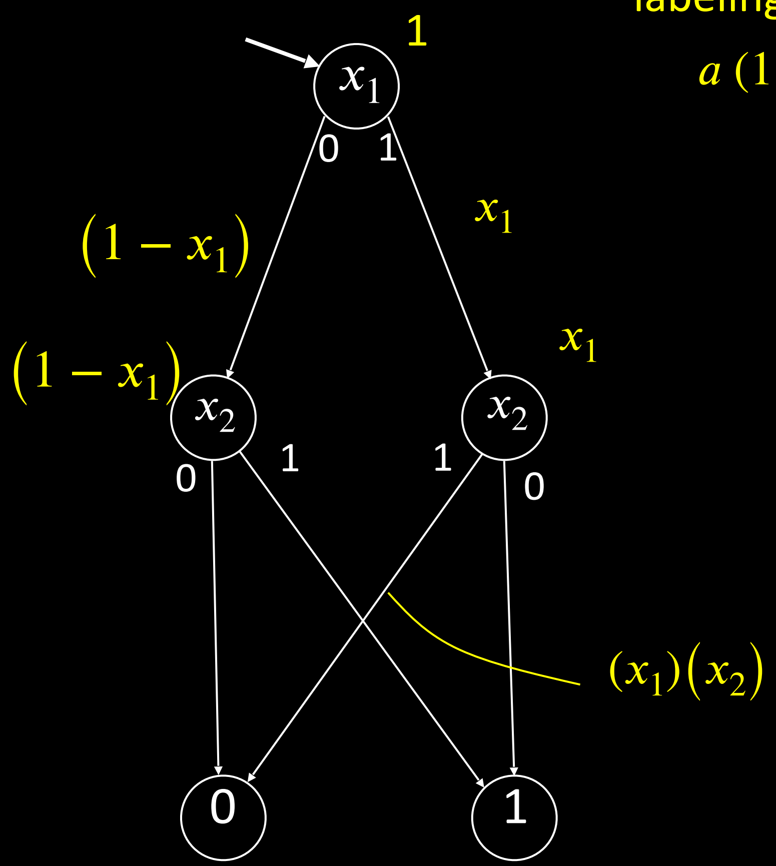

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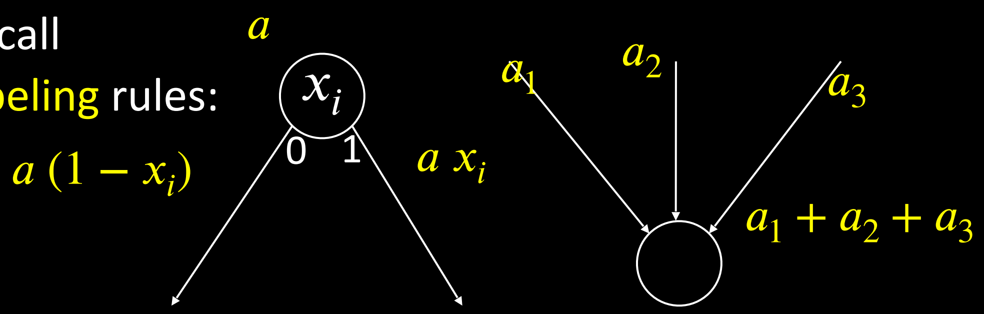

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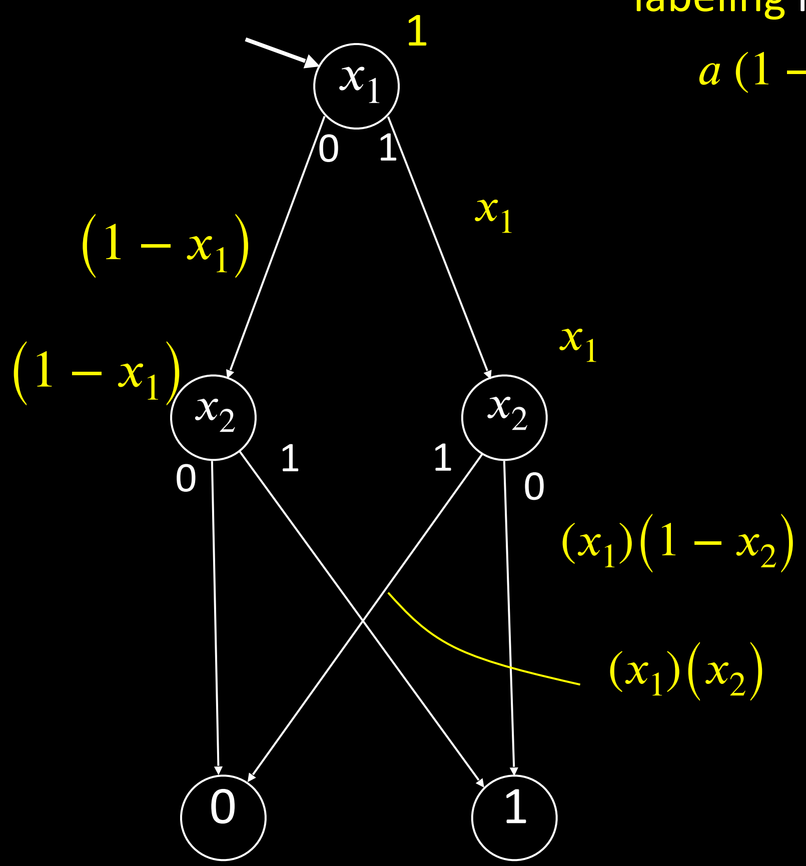

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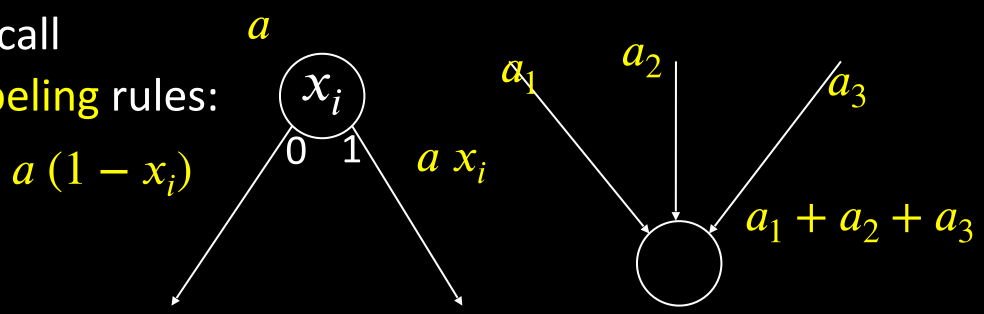

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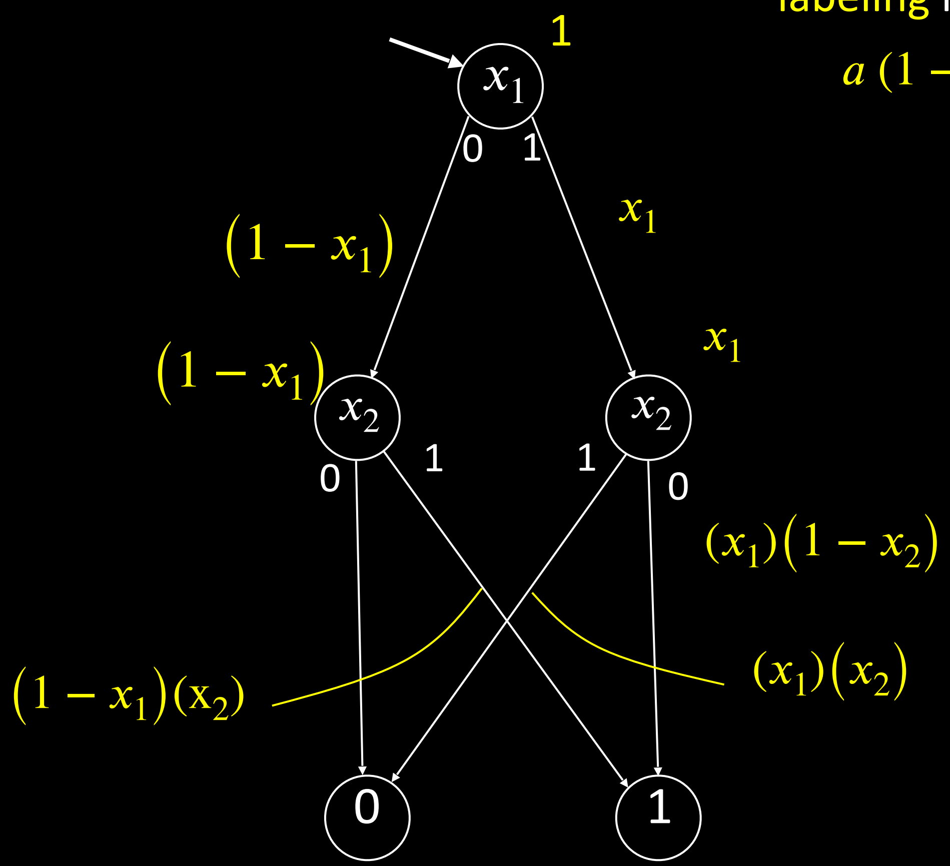

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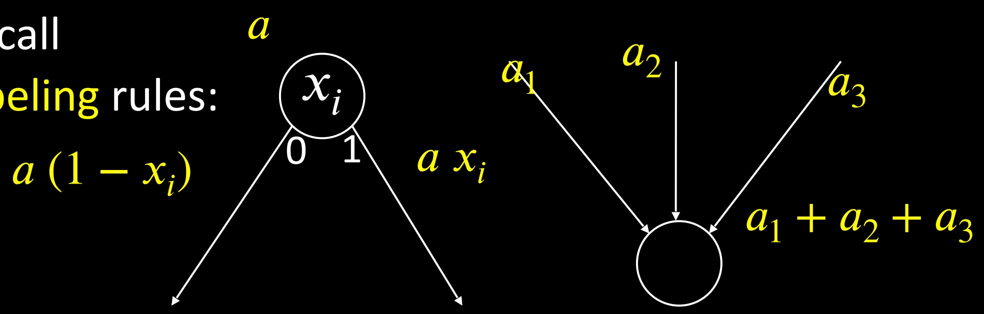

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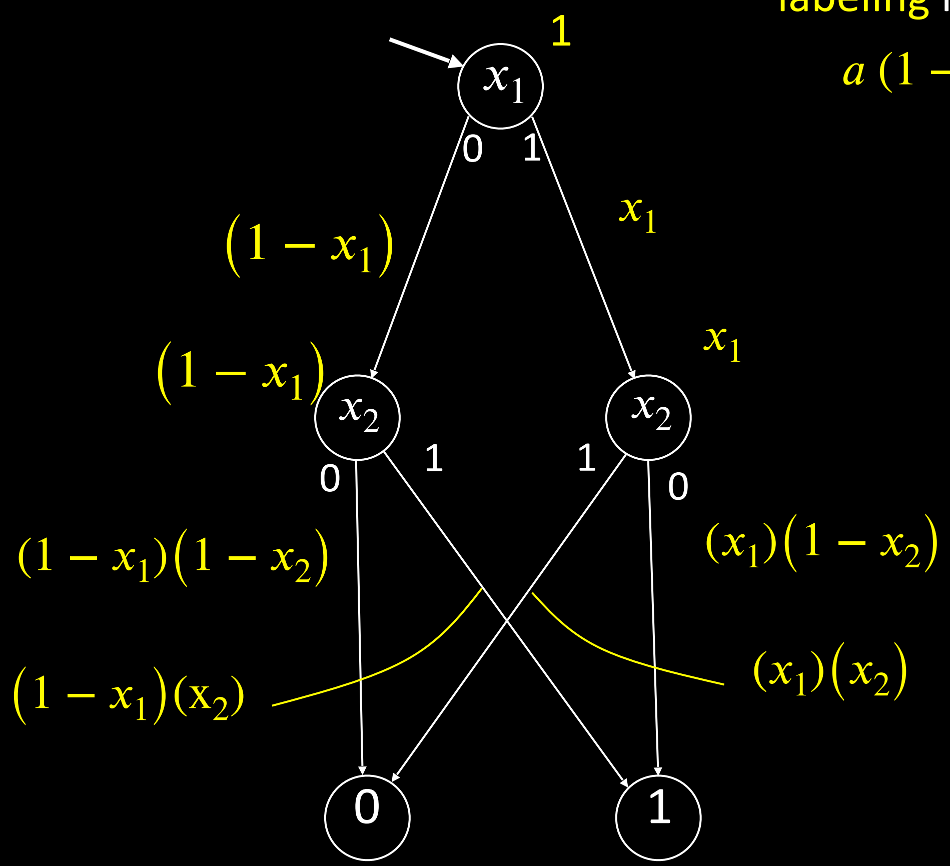

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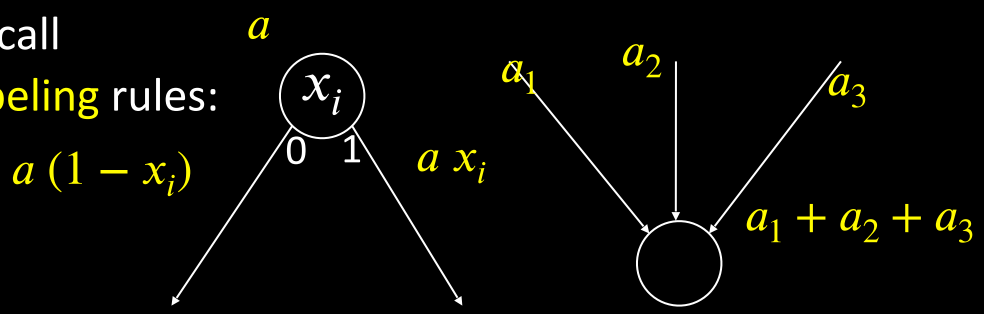

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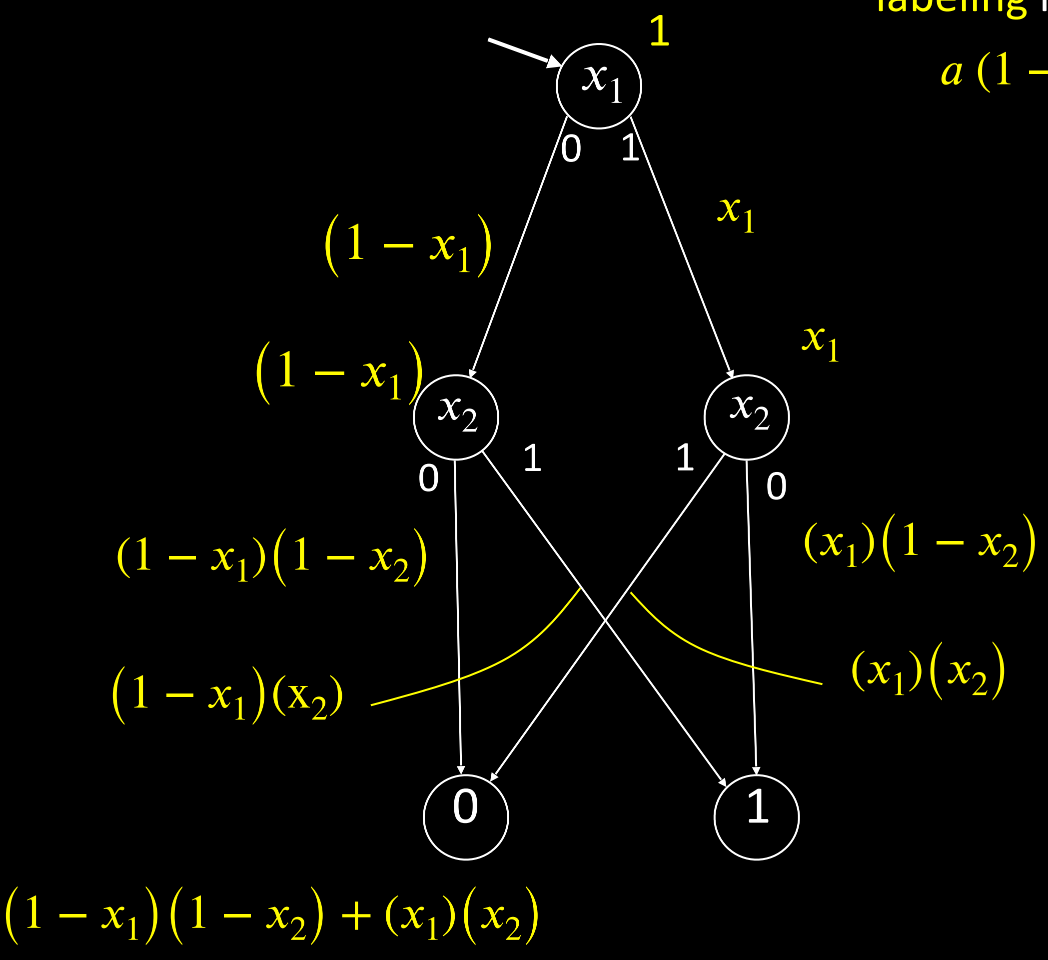

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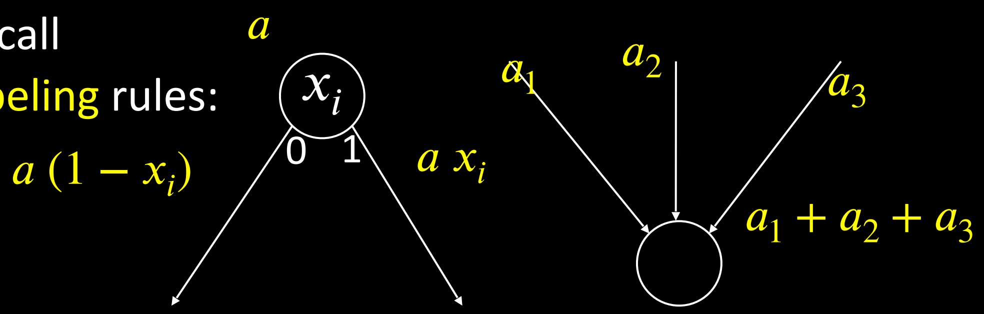

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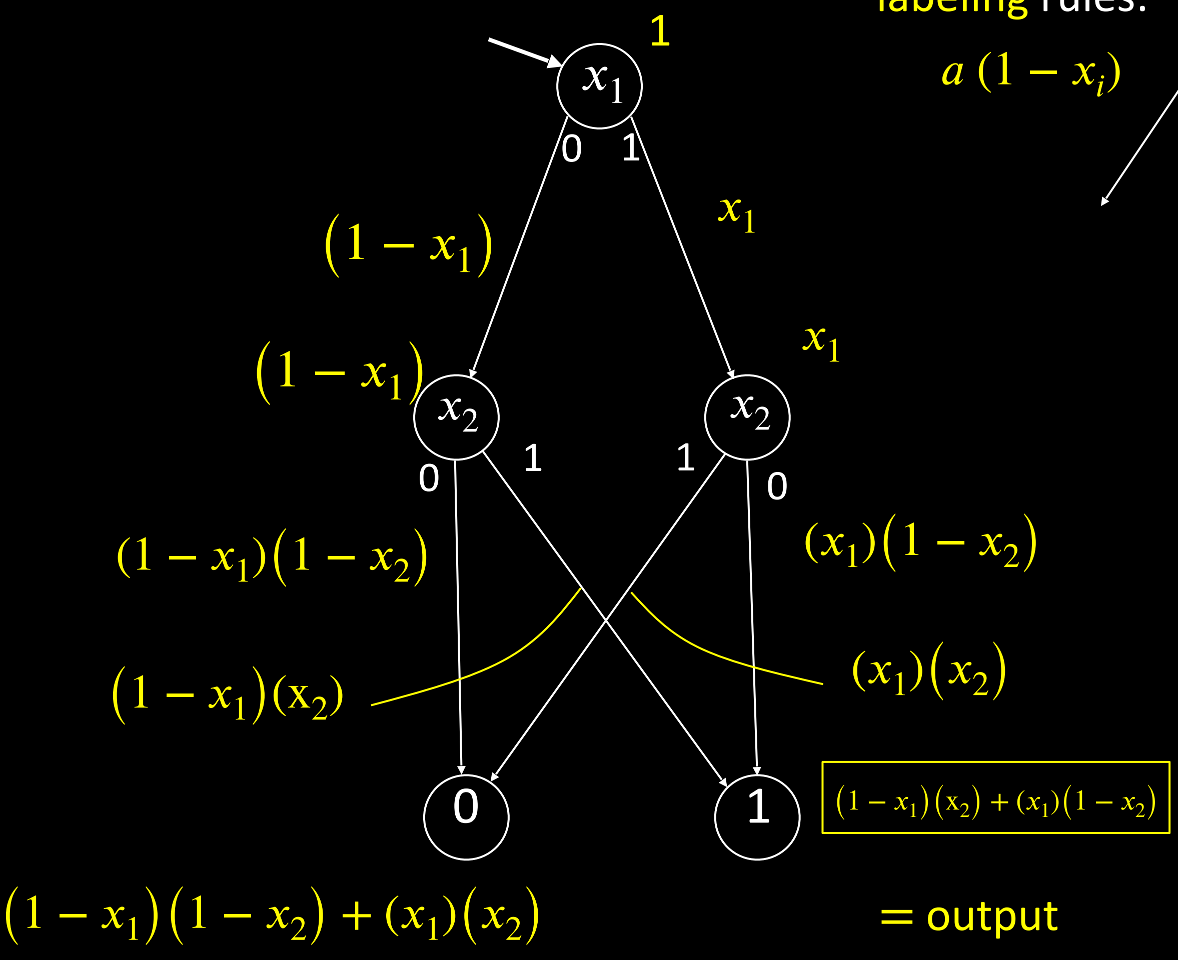

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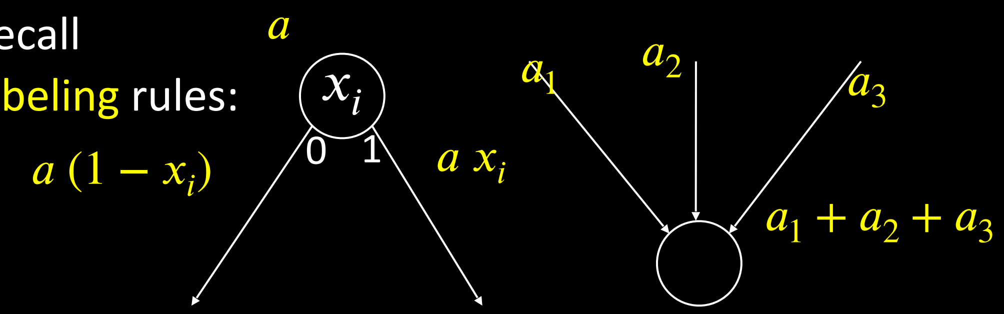

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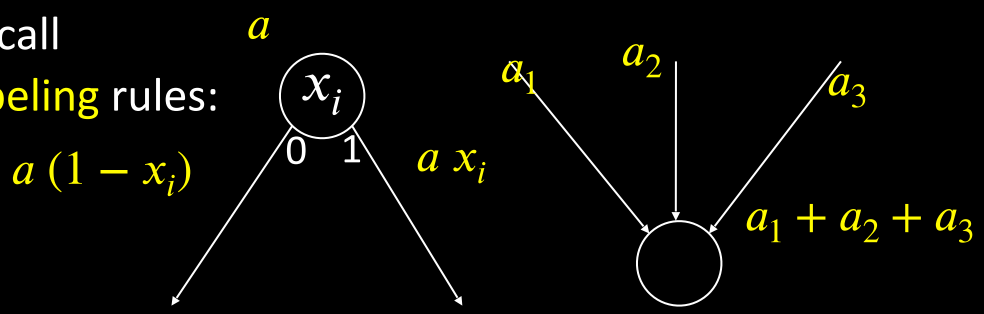

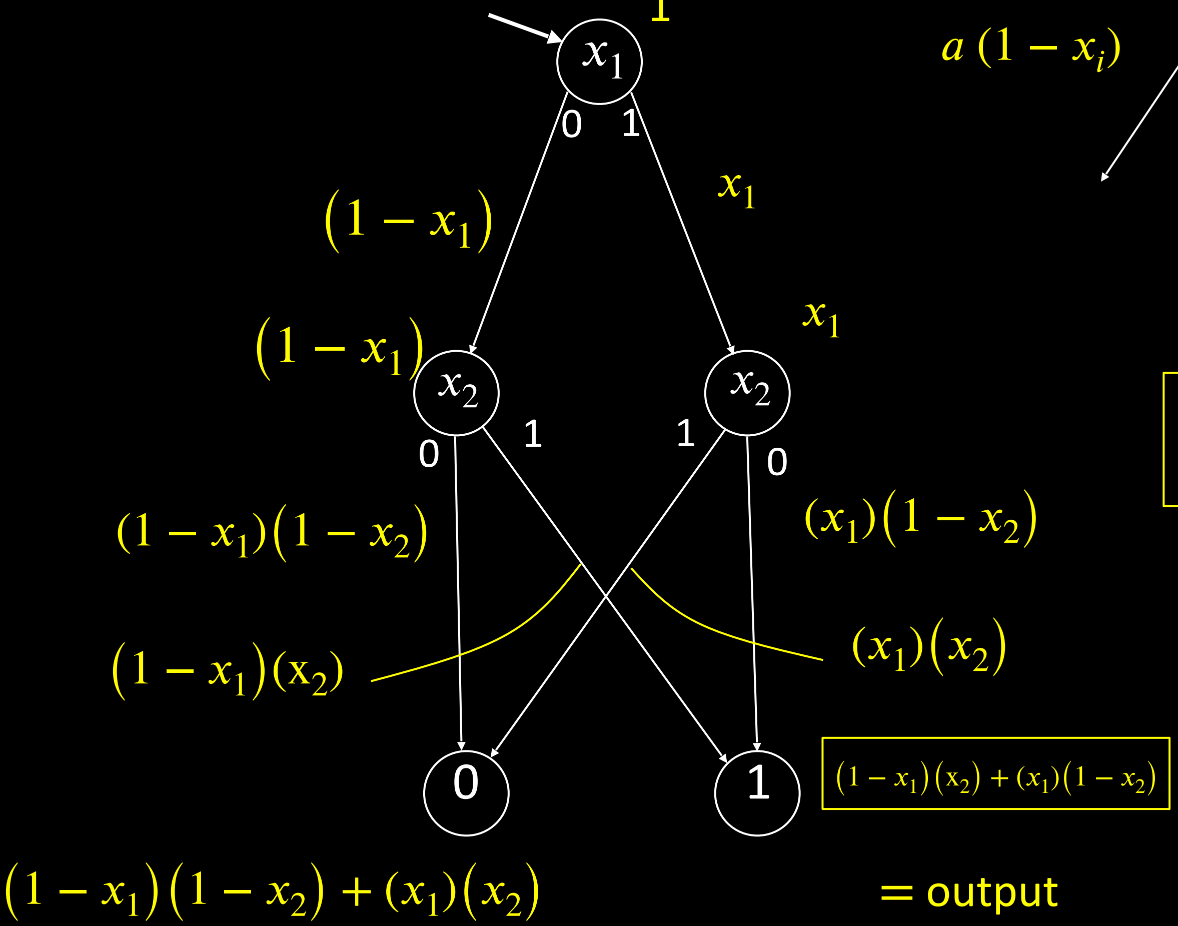

form of output

$$
\begin{aligned}
= \; & (1 - x_1)\;(x_2)\;(1 - x_3)\;(x_4)\;\cdots\;(1 - x_m) \\
+ \; & (x_1)\;(x_2)\;(x_3)\;(1 - x_4)\;\cdots\;(x_m) \\
+ \; & (x_1)\;(1 - x_2)(1 - x_3)\;(x_4)\;\cdots\;(x_m) \\
& \vdots \\
+ \; & (x_1)\;(x_2)\;(1 - x_3)\;(x_4)\;\cdots\;(x_m)
\end{aligned}
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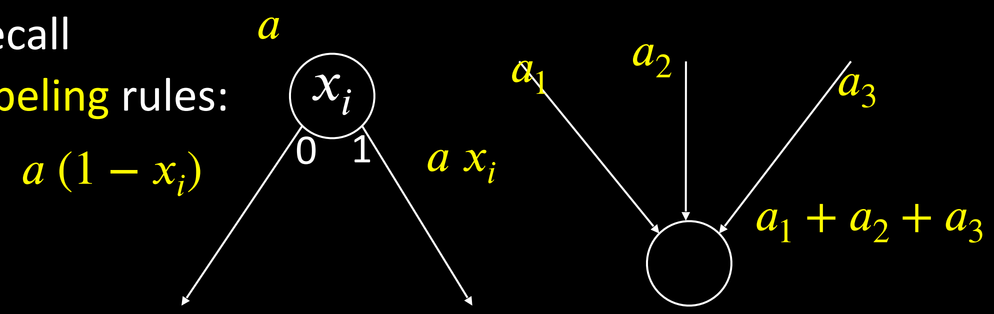

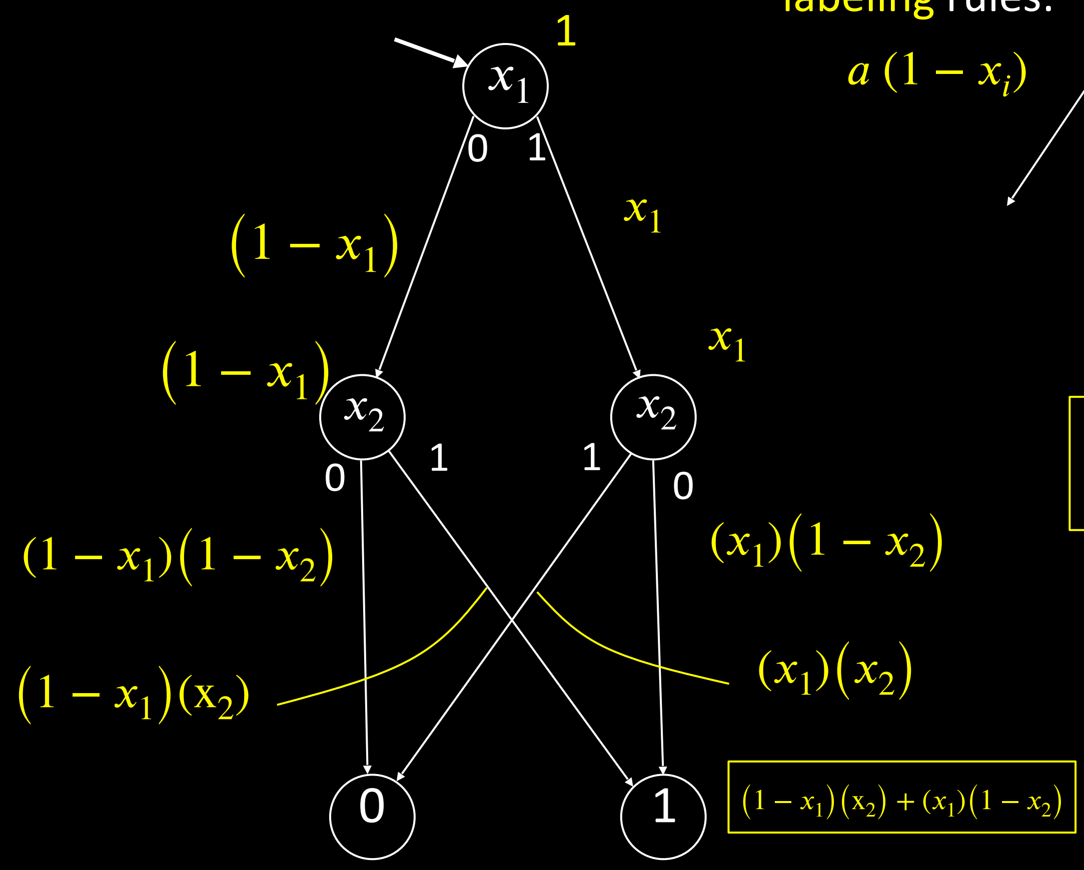

$(1-x_1)(1-x_2)+(x_1)(x_2)$ $=$ output

form of output

$$
\begin{aligned}
= \; & (1-x_1)\;(x_2)^3\;(1-x_3)\;(x_4)\;\cdots\;(1-x_m)\\
+ \; & (x_1)\;(x_2)\;(x_3)\;(1-x_4)\;\cdots\;(x_m)\\
+ \; & (x_1)\;(1-x_2)(1-x_3)\;(x_4)\;\cdots\;(x_m)\\
& \vdots\\
+ \; & (x_1)\;(x_2)\;(1-x_3)\;(x_4)\;\cdots\;(x_m)
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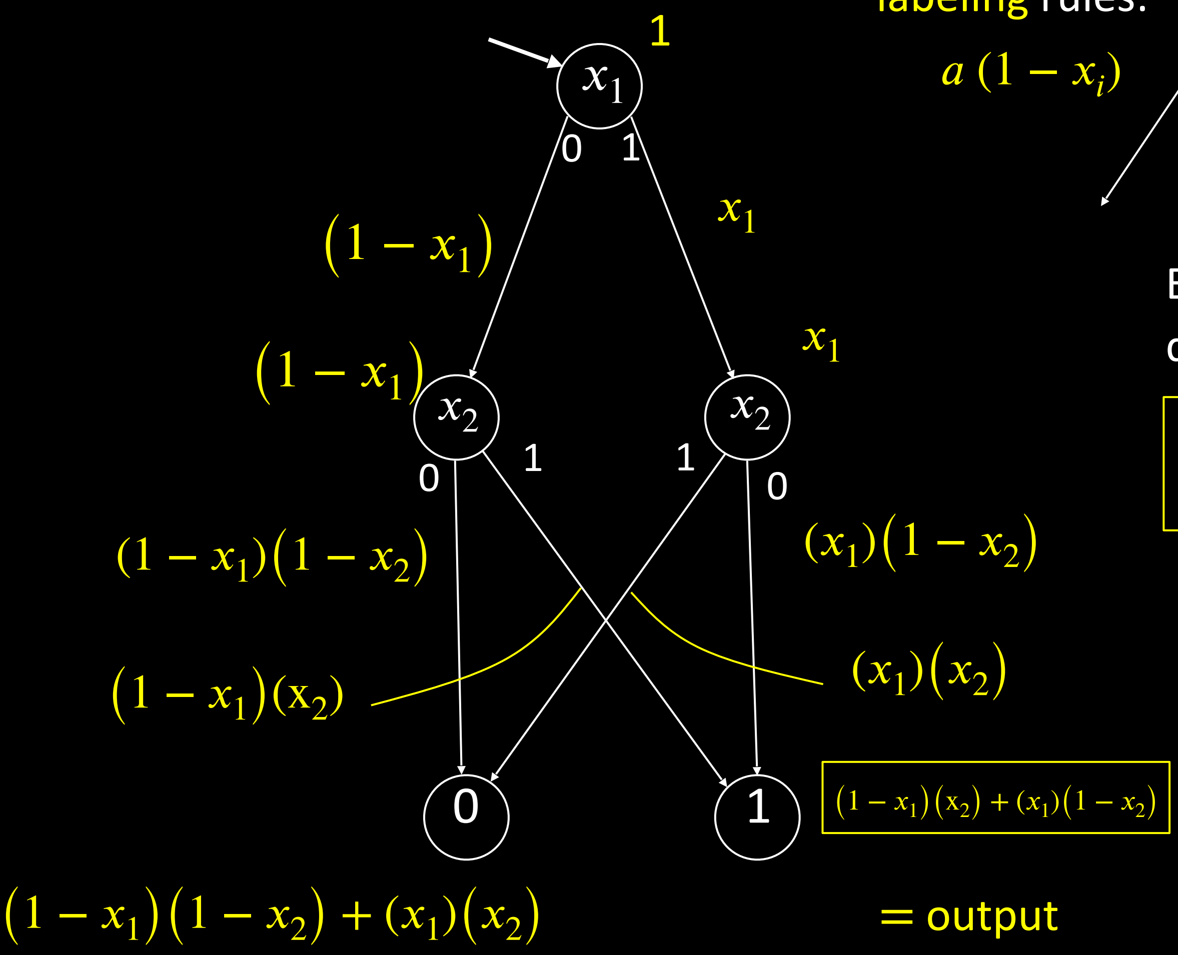

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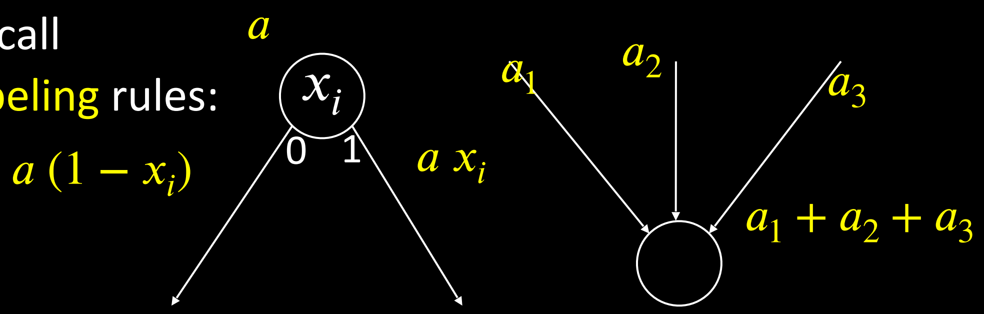

Exponents ≤ 1 due to "read-once"

$\boxed{\text{form of output}}$

$$
\begin{aligned}
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+ \ & (x_1)\ (1 - x_2)(1 - x_3)\ (x_4)\ \cdots\ (x_m) \\
& \vdots \\
+ \ & (x_1)\ (x_2)\ (1 - x_3)\ (x_4)\ \cdots\ (x_m)
\end{aligned}
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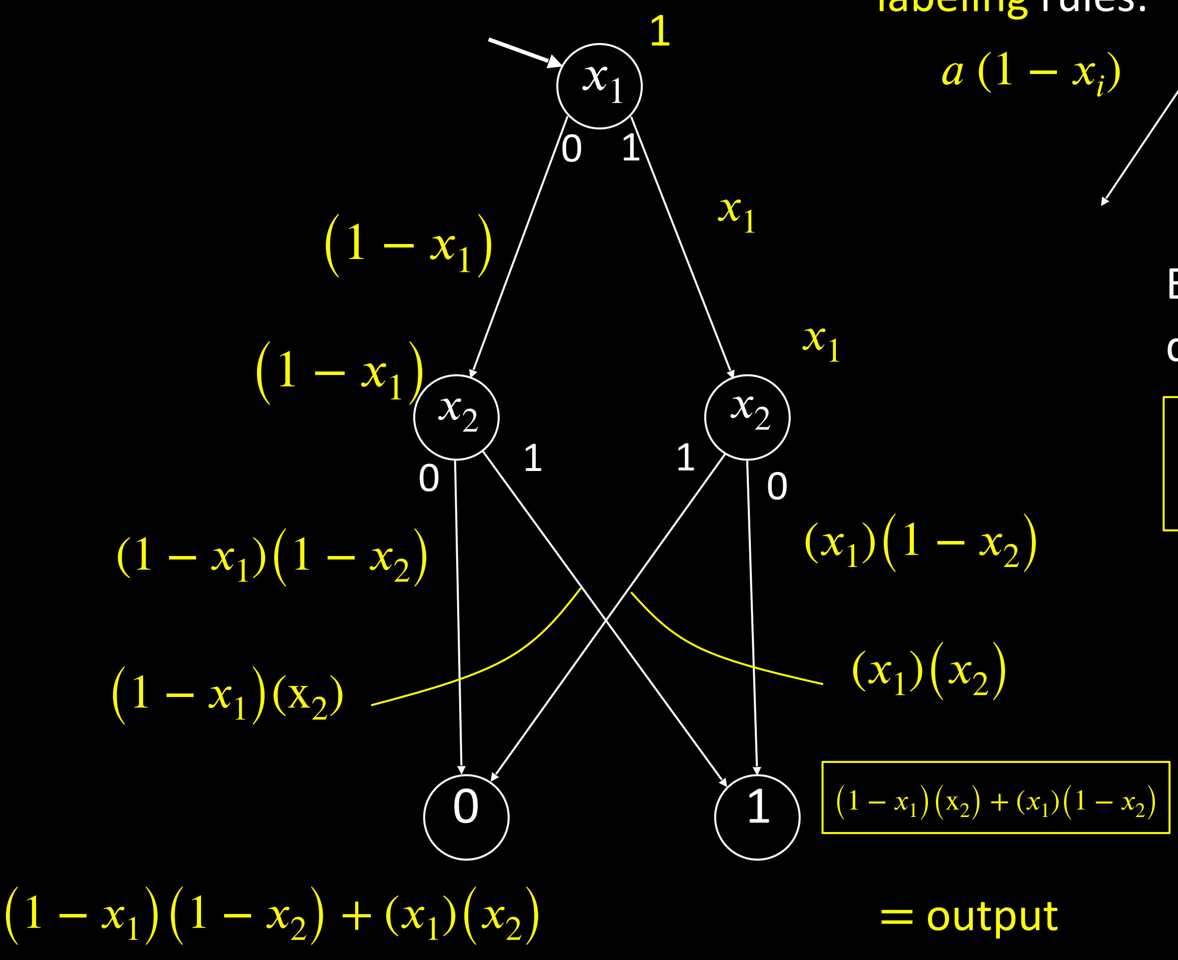

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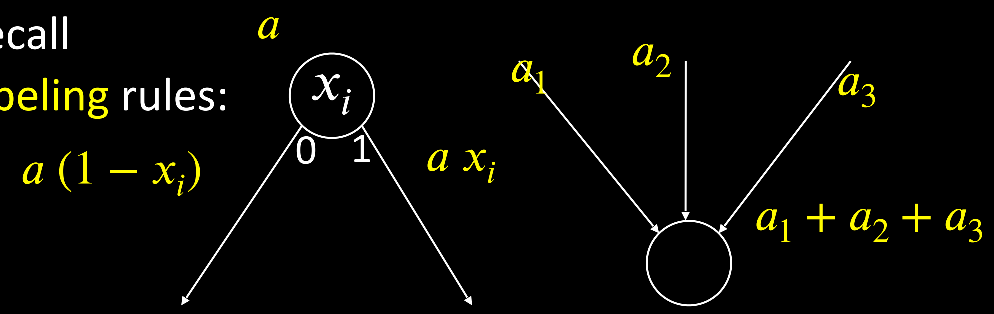

Exponents ≤ 1
due to "read-once"

$$
\begin{aligned}
\boxed{\text{form of output}} = \; & (1-x_1)\,(x_2)^{\cancel{2}}\,(1-x_3)\,(x_4)\cdots(1-x_m) \\
+ \; & (x_1)\,(x_2)\,(x_3)\,(1-x_4)\cdots(x_m) \\
+ \; & (x_1)\,(1-x_2)(1-x_3)\,(x_4)\cdots(x_m) \\
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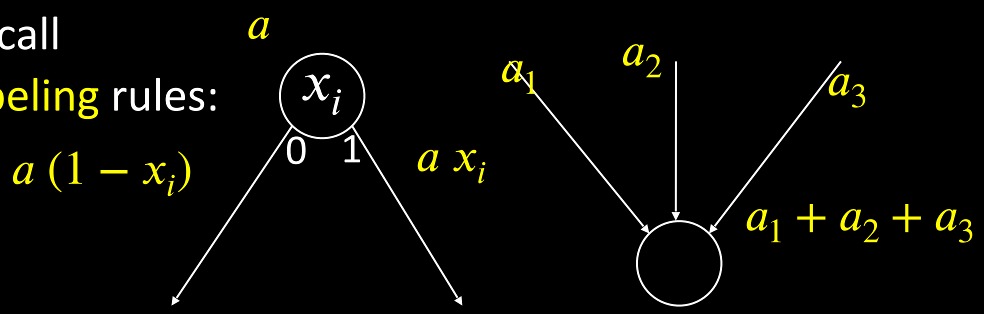
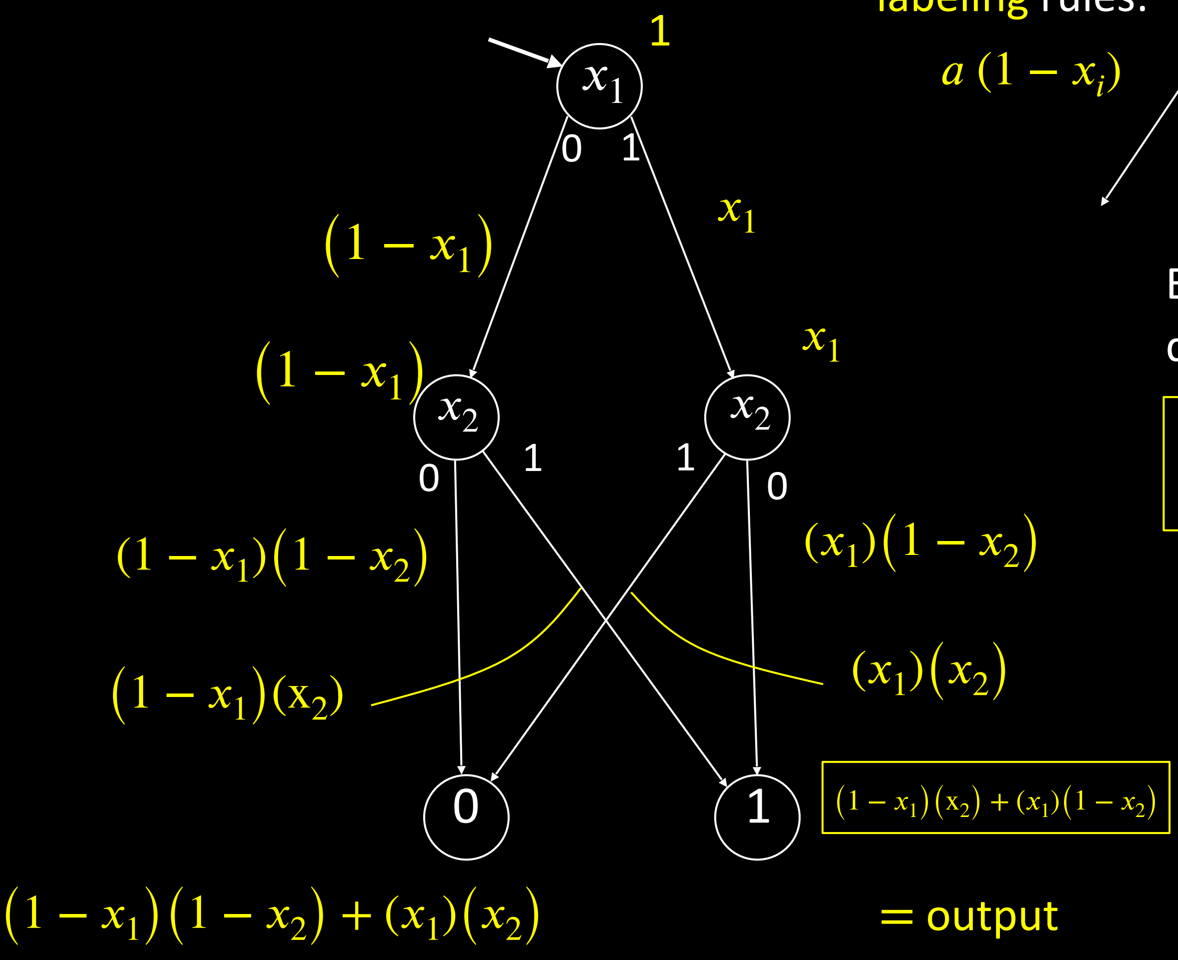




$=$ output

Exponents ≤ 1 due to "read-once"

Assume read <u>exactly</u> once so that for each i (x_i) or $(1-x_i)$ appears in every row

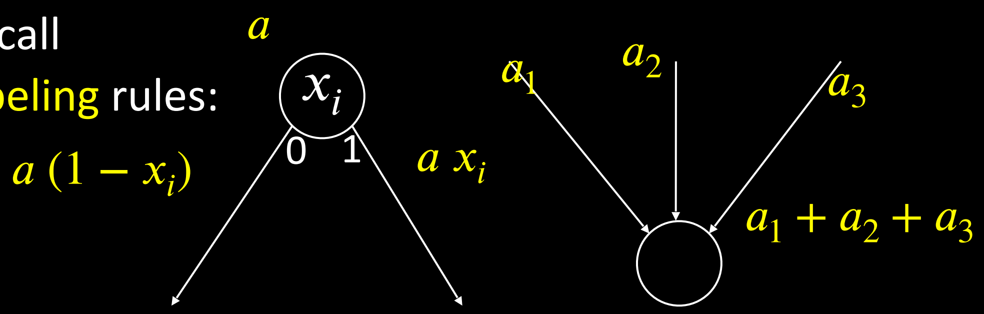
$\boxed{\text{form of output}}$

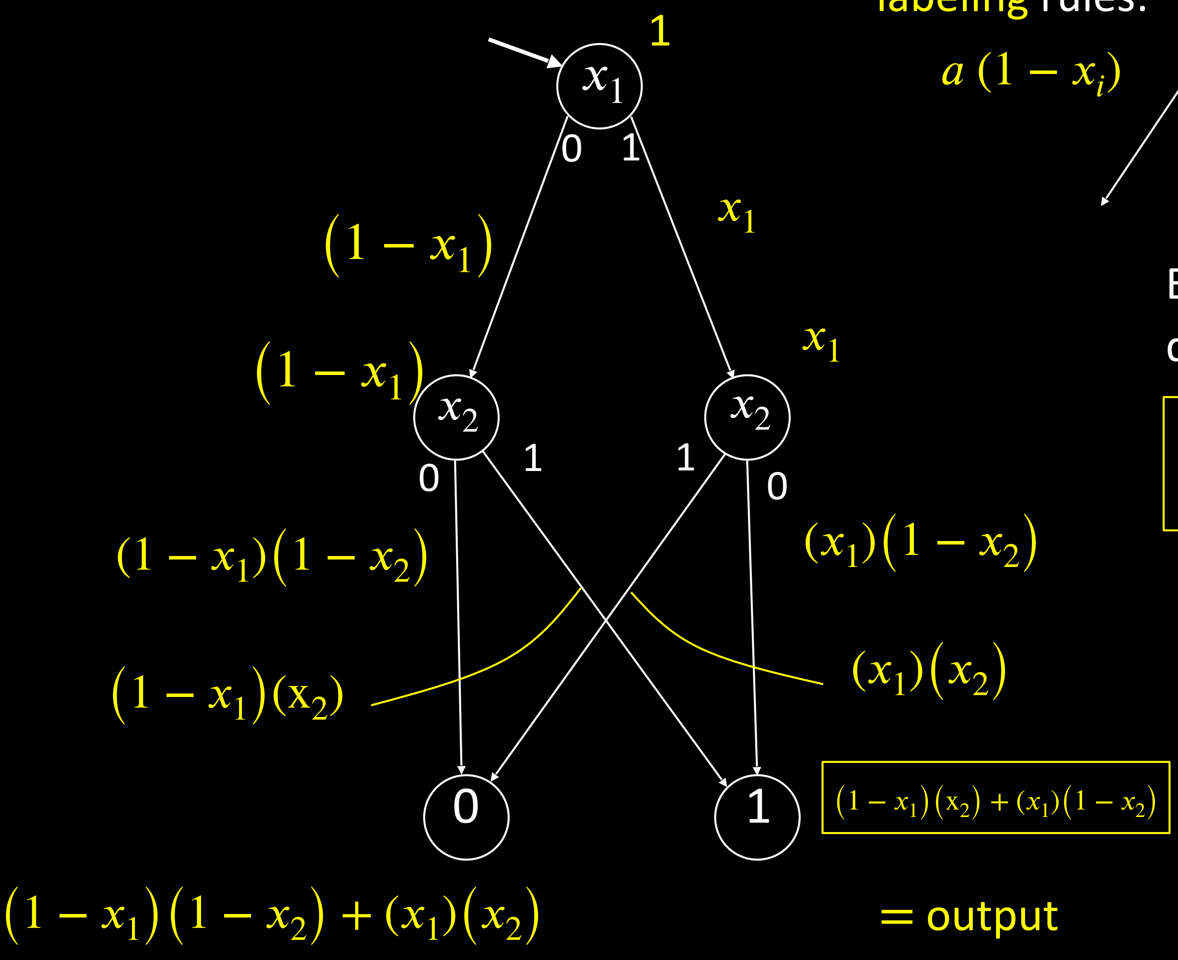
$$
\begin{aligned}
&= (1-x_1)\ (x_2)^{\cancel{2}}\ (1-x_3)\ (x_4)\ \cdots\ (1-x_m)\\
&+ (x_1)\ (x_2)\ (x_3)\ (1-x_4)\ \cdots\ (x_m)\\
&+ (x_1)\ (1-x_2)(1-x_3)\ (x_4)\ \cdots\ (x_m)\\
&\quad\vdots\\
&+ (x_1)\ (x_2)\ (1-x_3)\ (x_4)\ \cdots\ (x_m)
\end{aligned}
$$

Symbolic Execution

Leave the x_i as variables and obtain an expression in the x_i for the output of the BP.

Recall **labeling rules:**





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Corresponds to the TRUE rows in the truth table of the Boolean function

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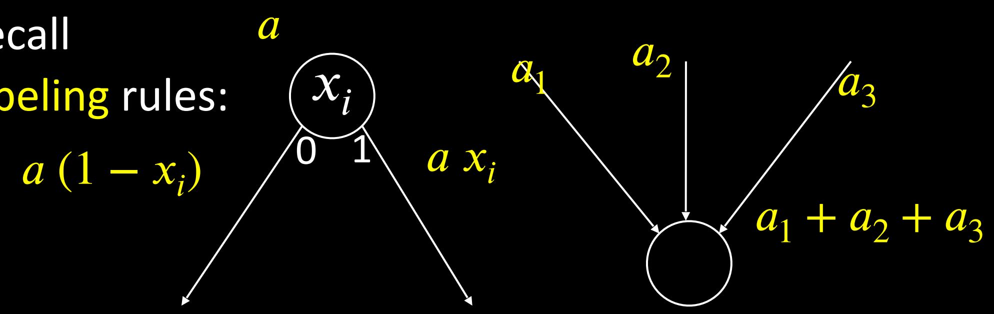

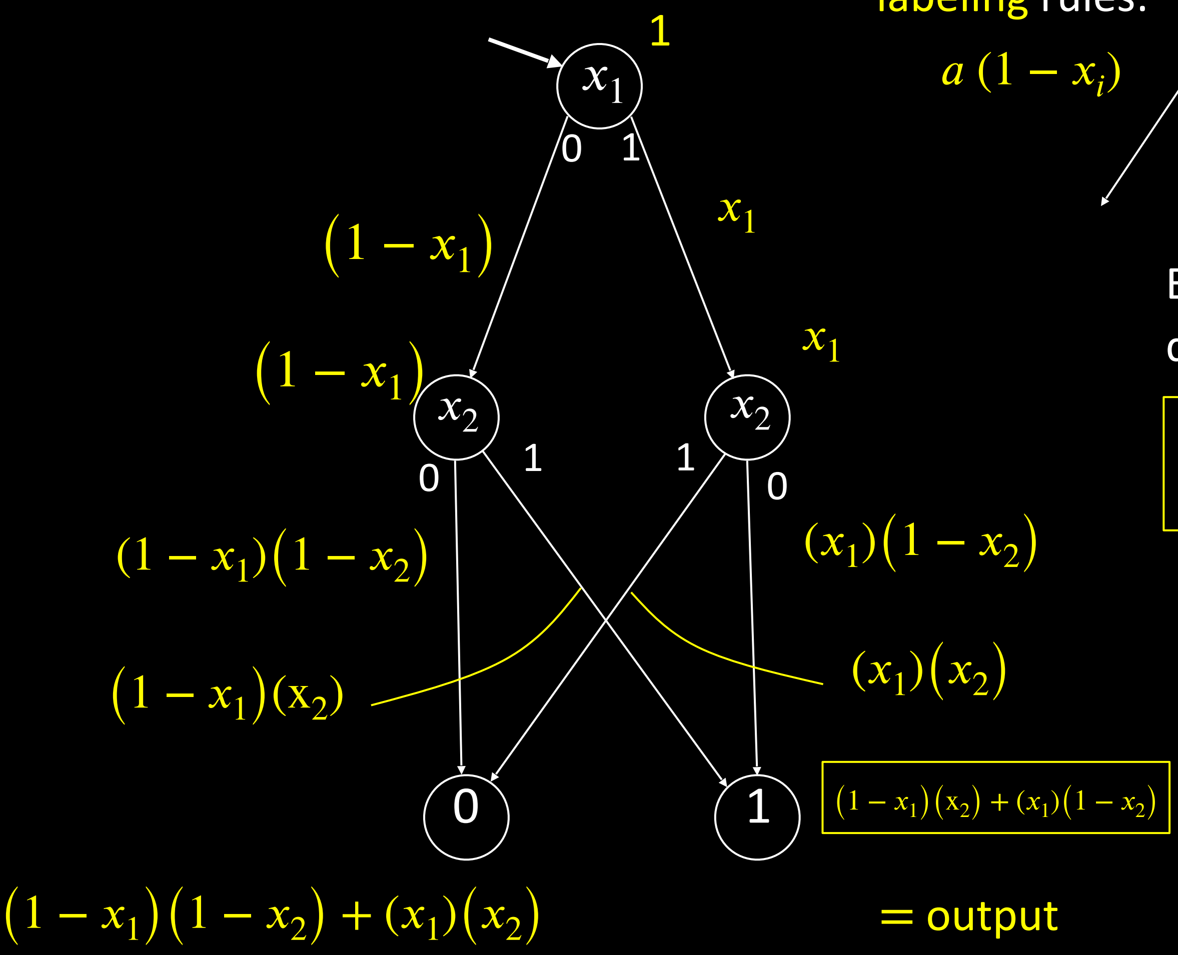

$= \text{output}$

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EQROBP $\in$ BPP

Algorithm for EQROBP $=$ "On input $\langle B_1, B_2 \rangle$ [on variables $x_1, \ldots, x_m$]

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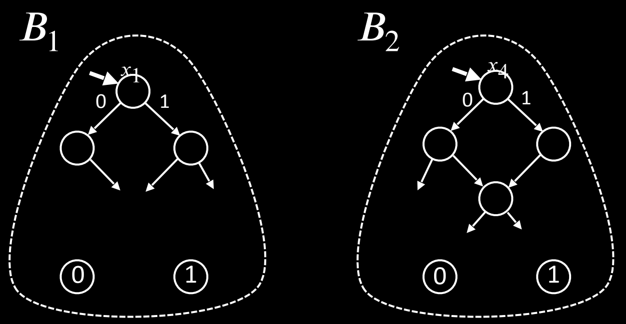

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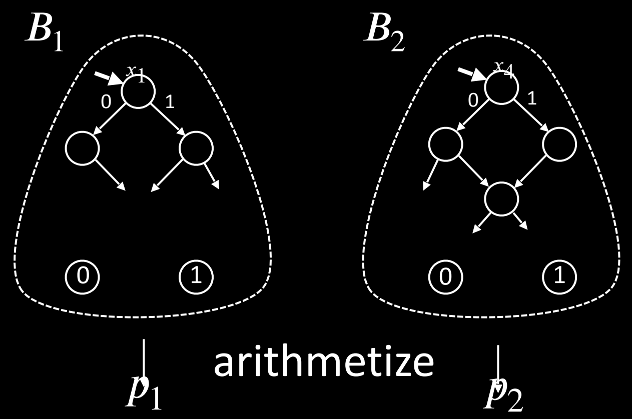

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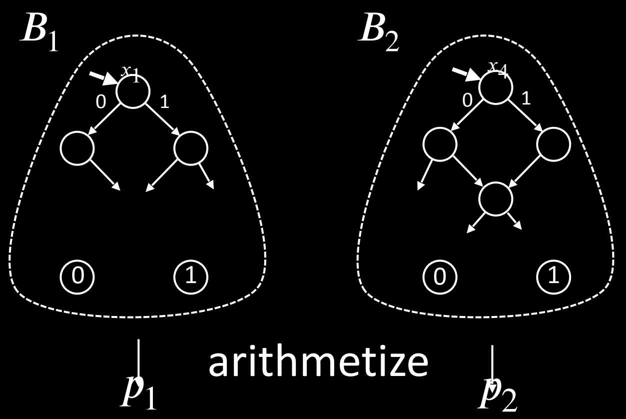

p_1 and p_2 each have the form:

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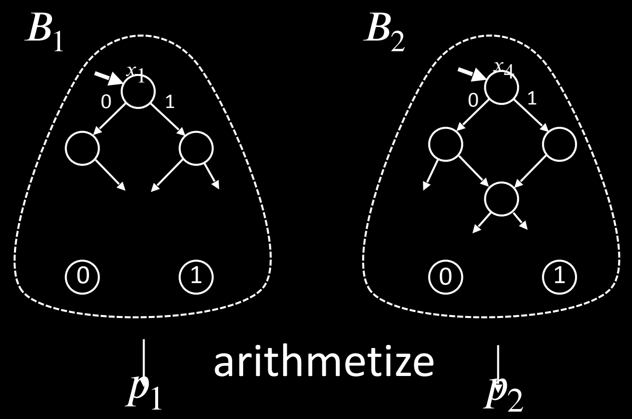

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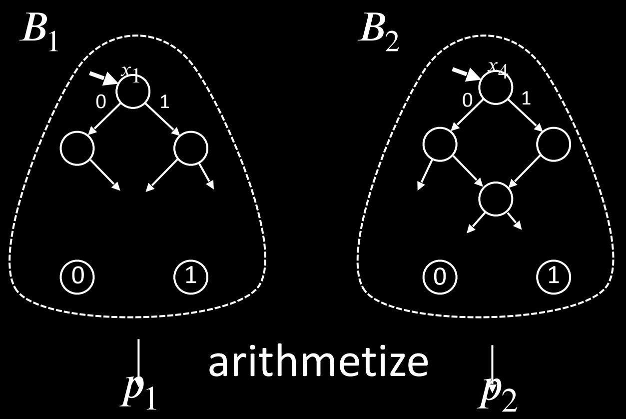

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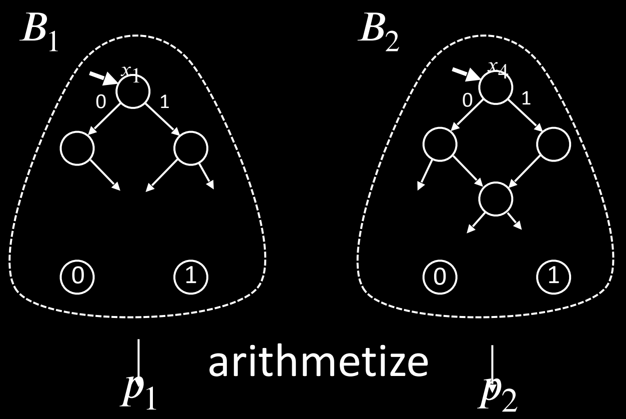

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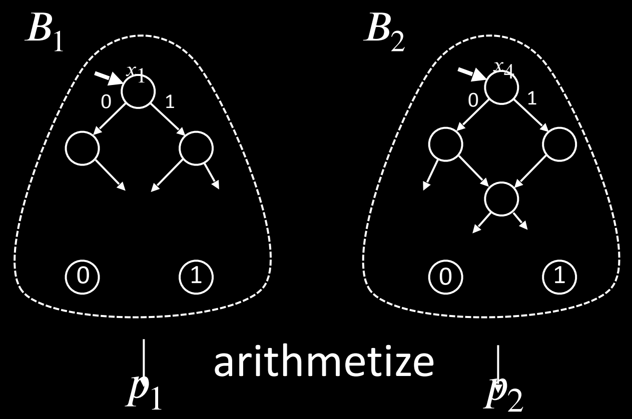

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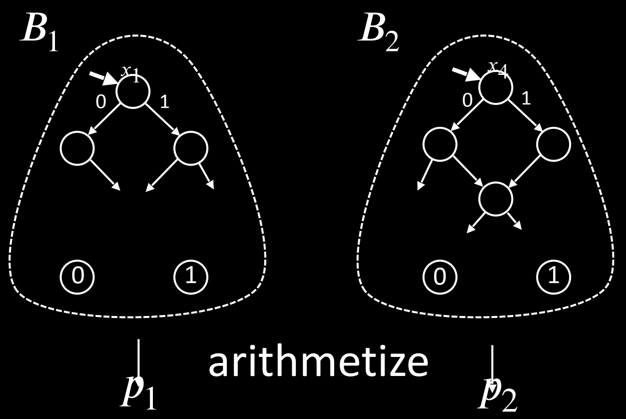

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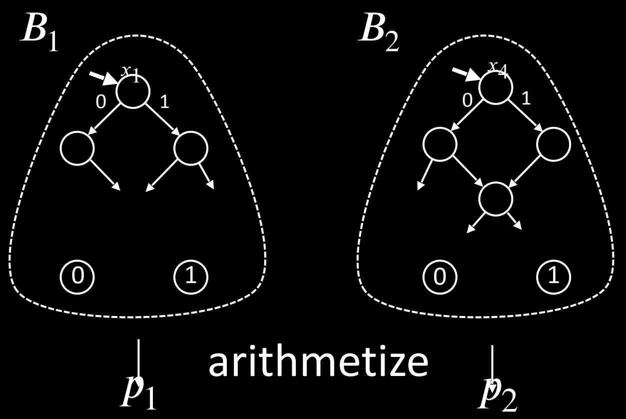

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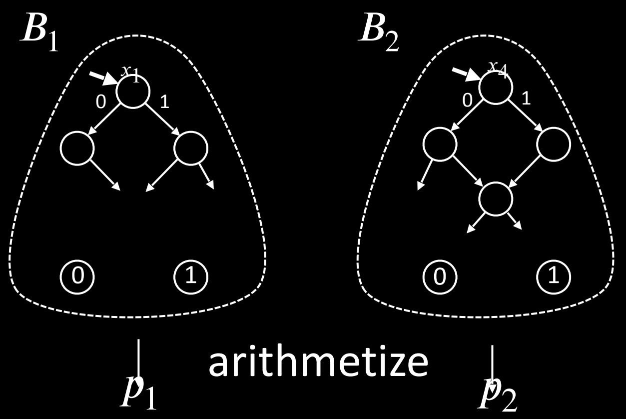

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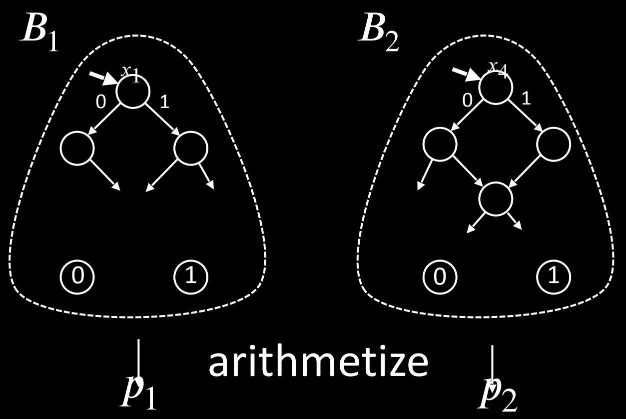

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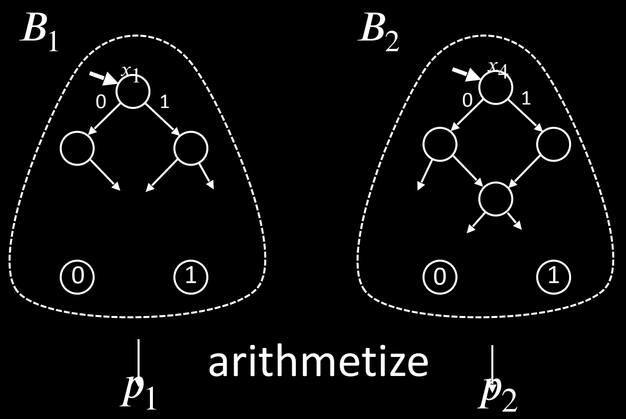

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Check-in 24.2

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Check-in 24.2

If the BPs were not read-once, the polynomials might have exponents ≥ 1. Where would the proof fail?

(a) $B_1 \equiv B_2$ implies they agree on all Boolean inputs

(b) Agreeing on all Boolean inputs implies $p_1 = p_2$

(c) Having $p_1 = p_2$ implies p_1 and p_2 always agree

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Thus p_1 and p_2 <u>always</u> agree (even on non-Boolean inputs).

Proof (2): If $B_1 \not\equiv B_2$ then $p_1 \neq p_2$ so $p = p_1 - p_2 \neq 0$.

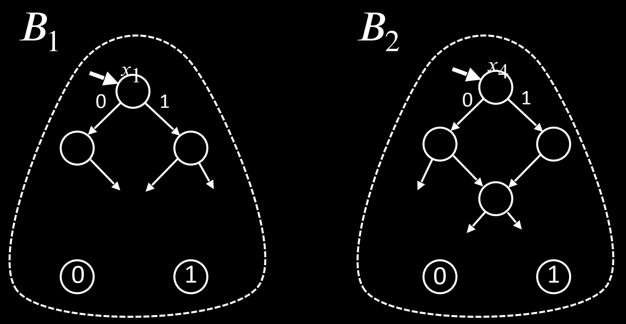
From Schwartz-Zippel, $\Pr\left[\, p_1(r) = p_2(r) \,\right] \leq \frac{dm}{q} \leq \frac{m}{3m} = \frac{1}{3}$.

p_1 and p_2 each have the form:

$$
\begin{aligned}
& (1-x_1)\ (x_2)\ (1-x_3)\ (x_4)\ \cdots\ (1-x_m) \\
+\ & (x_1)\ (x_2)\ (x_3)\ (1-x_4)\ \cdots\ (x_m) \\
+\ & (x_1)\ (1-x_2)(1-x_3)\ (x_4)\ \cdots\ (x_m) \\
& \vdots \\
+\ & (x_1)\ (x_2)\ (1-x_3)\ (x_4)\ \cdots\ (x_m)
\end{aligned}
$$

Check-in 24.2

$EQ_{\text{ROBP}} \in$ BPP



p_1 arithmetize p_2

Check-in 24.3

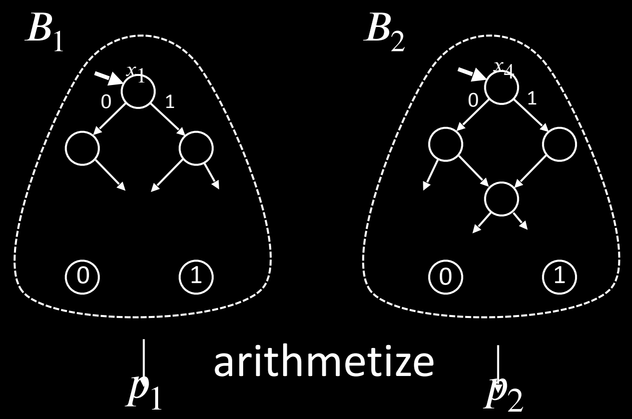
$EQ_{\text{ROBP}} \in$ BPP

Check-in 24.3

If p_1 and p_2 were exponentially large expressions, would that be a problem for the time complexity?

(a) Yes, but luckily they are polynomial in size.

(b) No, because we can evaluate them without writing them down.



Quick review of today

1. Simulated Read-once Branching Programs by polynomials
2. Gave probabilistic polynomial equality testing method
3. Showed EQROBP $\in$ BPP

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