The Small World Phenomenon

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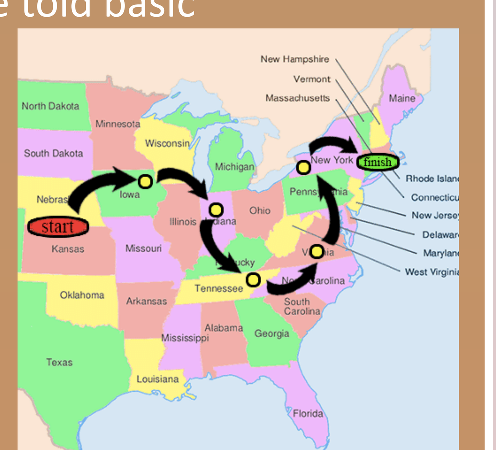
Introduction

The phenomenon of two strangers meeting in a strange place and discovering that they have a common acquaintance occurs surprisingly often. It is related to the equally counterintuitive fact that a chain of at most six intermediaries are sufficient to link most pairs of people in the world. This is the small world phenomenon. The name derives from the frequently heard comment of two freshly made acquaintance, on discovering that they have an acquaintance in common: "It is a small world, isn't it?"
The small-world effect could explain how rumors spread. It could explain how a blackout could propagate across an entire power grid, how year 2000 bugs could bring down vast computer systems, and perhaps even why neurons in the brain are connected the way they are.

Milgram's experiment

The popular conception of the small-world phenomenon may have arisen from a 1967 experiment by Harvard sociologist Stanley Milgram. Milgram's experiment developed out of a desire to learn more about the probability that two randomly selected people would know each other . The goal of the experiment was to find short chains of acquaintances linking pairs of people in the United States who did not know one another. In a typical run of the experiment, a source person in Nebraska would be given a letter to deliver to a target person in Massachusetts. The source would initially be told basic information about the target, including his address and Occupation;the source would then be instructed to send the letter to some-one she knew on a first-Name basis in an effort to transmit the letter to the target as efficaciously as possible.

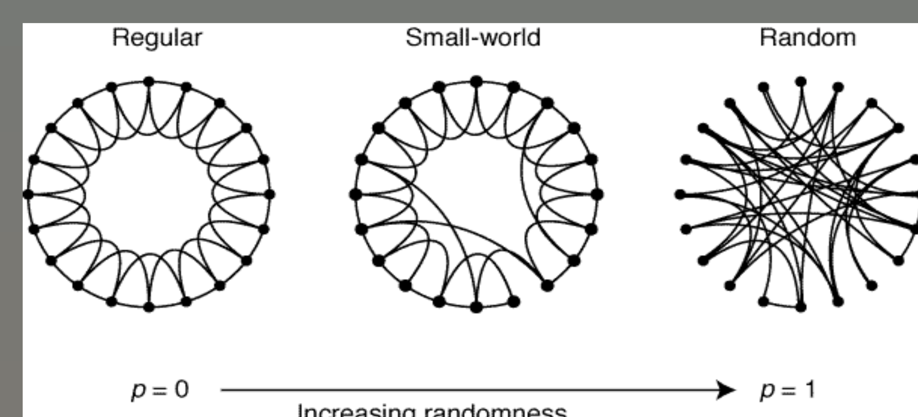


Anyone subsequently receiving the letter would be given the same instructions, and the chain of communication would continue until the target was reached. Over many trials, the average number of intermediate steps in a successful chain was found to lie between five and six, a quantity that has since entered popular culture as the "six degrees of separation" principle.

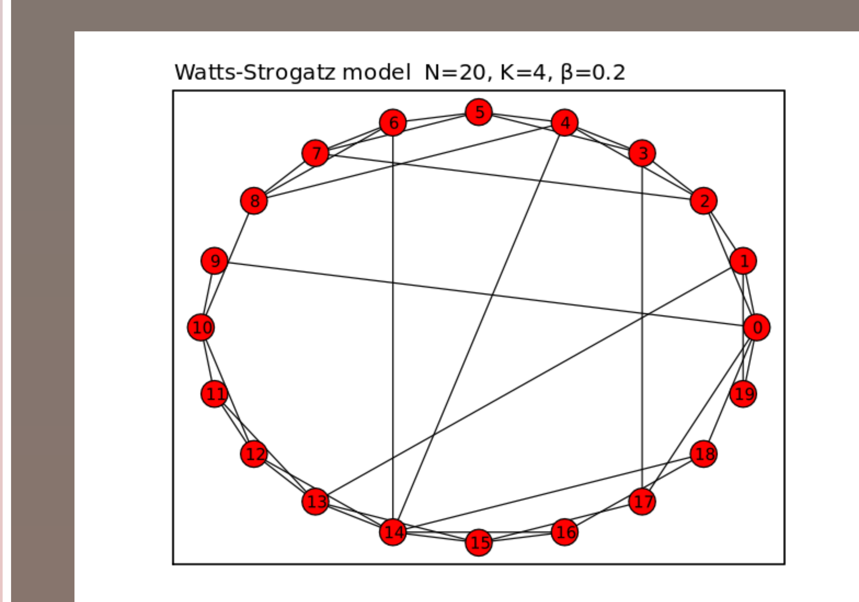
Watts and Strogatz model

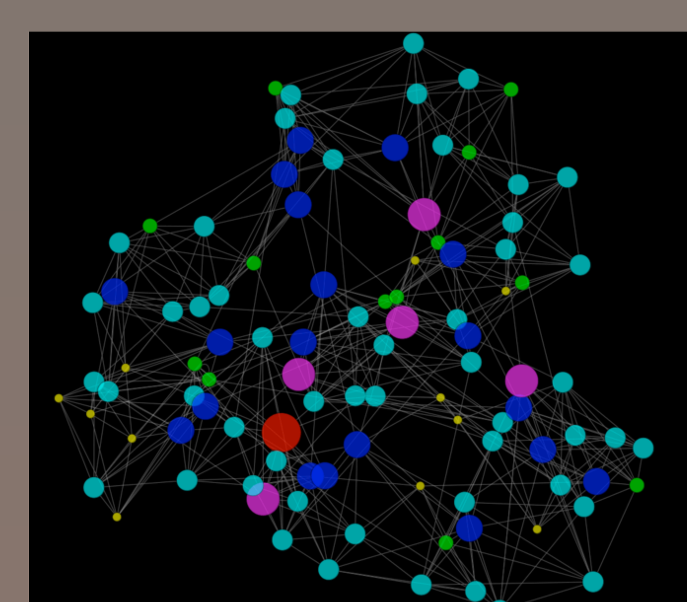
"Why is the small-world phenomenon surprising?" asks Strogatz. "Why shouldn't it be obvious that we're only 6 degrees of separation apart, or some other small number?" Mathematically minded people, he explains, often approach the question with a simple calculation: suppose I have 100 friends, each of whom also has 100 friends. A hundred times 100 makes 10,000 friends of my friends. If each of those 10,000 people has 100 friends, there will be 1 million people 3 degrees away from me. Five steps away, there are 10 billion. "So a lot of people would say it's not surprising that the degree of separation is small," concludes Strogatz, "because within 5 steps you've done the whole planet."

But there's a big assumption in that calculation. It presumes that each 100 friends are 100 new people. If everyone chose their friends at random from the entire world, the assumption would be valid, but we clearly don't. "The world that we live in is not at all random," as Watts points out. "We are very much constrained by our socioeconomic status, our geographical location, our background, our education and our profession, our interests and hobbies. All these things make our circle of acquaintances highly nonrandom."



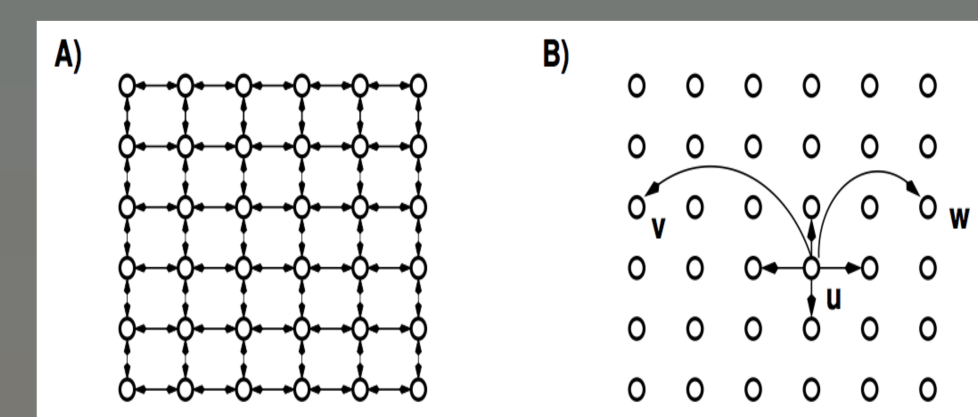
The model:
Given the desired number of nodes N , the mean degree K (assumed to be an even integer), and a special parameter β, satisfying $0 \le \beta \le 1$ and $N \gg K \gg \ln(N) \gg 1$,the model constructs an undirected graph with N nodes and NK/2 edges in the following way:
1.Construct a regular ring lattice, a graph with N nodes each connected to K neighbors, K/2 on each side. That is, if the nodes are labeled $n_0 \dots n_{N-1}$, there is an edge (n_i , n_j) if and only if $0 < | i – j | \bmod (n - K/2) \le K/2$.
2.For every node $n_i = n_0 , \dots , n_{N-1}$ take every edge (n_i , n_j) with $i < j$, and rewire it with probability β . Rewiring is done by replacing (n_i , n_j) with (n_i , n_k) where k is chosen with uniform probability from all possible values that avoid self-loops ($k \neq i$) and link duplication (there is no edge $(n_i , n_{k'})$ with k' = k at this point in the algorithm).





Kleinberg's model

In a paper published in ACM 2000 Article by Jon Kleinberg called "The Small-World Phenomenon: An Algorithmic Perspective"
He claims that the other existing works about " the small world " specially Watts-Strogatz model are just answering the question of :
(*) Why should there exist short chains of acquaintances linking together arbitrary pairs of strangers in a small-world network?
But in this paper he wants to answer the second question :
(**) Why should arbitrary pairs of strangers be able to ***find*** short chains of acquaintances that link them together?
So he starts to study the phenomenon in an algorithmic point of view and the algorithm called "decentralized algorithms"
In fact he constructs a new model of this phenomenon by making some changes in Watts-Strogatz model .
In this new model he begins from a two-dimensional grid and allow for edges to be directed.



(A) A two-dimensional grid network with n = 6, p = 1, and q = 0.
(B) The contacts of a node u with p = 1 and q = 2. v and w are the two long-range contacts.

the nodes in the grid (representing individuals in the social network) are identified with the set of *lattice points* in an n × n square, $\{(i,j) : i \in \{1,2,\dots,n\}, j \in \{1,2,\dots,n\}\}$, and he defines the lattice distance between two nodes (i, j) and (k, l) to be the number of "*lattice steps*" separating them:
$D((i, j), (k, l)) = |k - i| + |l - j|$.

For a universal constant $p \ge 1$, the node u has <u>a directed edge to every other node within lattice distance p</u> . These are its *local contacts* . For universal constants $q \ge 0$ and $r \ge 0$, <u>we also construct directed edges from u to q other nodes</u> (the long-range contacts) using independent random trials; the i^{th} directed edge from u has endpoint v with probability distribution proportional to : $[D(u, v)]^{-r} / \Sigma_v [D(u, v)]^{-r}$.
When r = 0, we have the uniform distribution over long-range contacts, the distribution used in the basic network model of Watts and Strogatz . One's long-range contacts are chosen independently of their position on the grid. As r increases, the long-range contacts of a node become more and more *clustered* in its vicinity on the grid. Thus, r serves as a basic structural parameter measuring how widely "networked" the underlying society of nodes is.

The decentralized algorithm

The whole knowledge that message-holder u has in these model can be summarized in the three following :
(i) the set of local contacts among all nodes.
(ii) the location, on the lattice, of the target t.
(iii) the locations and long-range contacts of all nodes that have come in contact with the message.
So that the decentralized algorithm A can has the following simple rule: in each step, the current message-holder u chooses a contact that is as close to the target t as possible, in the sense of lattice distance. (Note that algorithm A makes use of even less information than is allowed: the current message holder does not need to know anything about the set of previous message holders.) To analyze an execution of algorithm A, we say that it is in phase j if the lattice distance from the current message holder to the target is between 2j and 2j+1. He shows that in phase j, the expected time before the current message holder has a long-range contact within lattice distance 2j of t is bounded proportionally to logn; at this point, phase j will come to an end. As there are at most 1 + logn phases, a bound proportional to (log n)2 follows. Interestingly, the analysis matches our intuition, and Milgram's description, of how a short chain is found in real life: "The geographic movement of the message from Nebraska to Massachusetts is striking. There is a progressive closing in on the target area as each new person is added to the chain".

References

- Jon Kleinberg , " The Small-World Phenomenon: An Algorithmic Perspective",(2000).
-M. Kochen, Ed., The Small World (Ablex, Norwood, 1989).
-Shulman, Polly (1 December 1998)."From Muhammad Ali to Grandma Rose ".DISCOVER magazine. Retrieved 13 August 2010
- Watts,D.J ; Strogatz,S.H. (1998). "Collective dynamics of 'small-world' network "
-Stanley Milgram, "The Small World Problem", Psychology Today, 1967
-Travers, Jeffrey, and Stanley Milgram, "An Experimental Study of the Small World Problem", Sociometry 32(4, Dec. 1969)