بسم الله الرحمن الرحیم

یادگیری برخط

جلسه بیست و سوم:
بندیت ترکیبیاتی (۳)

الگوریتم کاهش آینه‌ای/پیروی از پیش‌روی منظم شده برای بندیت

Input Legendre potential F, action set $\mathcal{A}$ and learning rate $\eta > 0$

Choose $\bar{A}_1 = \operatorname{argmin}_{a \in \mathcal{A} \cap \operatorname{dom}(F)} F(a)$

for $t = 1, \ldots, n$ **do**

  Choose measure P_t on $\mathcal{A}$ with mean $\bar{A}_t$

  Sample action A_t from P_t and observe $\langle A_t, y_t \rangle$

  Compute estimate $\hat{Y}_t$ of the loss vector y_t

  Update:

$$\bar{A}_{t+1} = \operatorname{argmin}_{a \in \mathcal{A} \cap \operatorname{dom}(F)} \eta \langle a, \hat{Y}_t \rangle + D_F(a, \bar{A}_t) \qquad \text{(Mirror descent)}$$

$$\bar{A}_{t+1} = \operatorname{argmin}_{a \in \mathcal{A} \cap \operatorname{dom}(F)} \eta \sum_{s=1}^{t} \langle a, \hat{Y}_s \rangle + F(a) \qquad \text{(follow-the-regularised-leader)}$$

end for

Input Legendre potential F, action set $\mathcal{A}$ and learning rate $\eta > 0$

Choose $\bar{A}_1 = \operatorname{argmin}_{a \in \mathcal{A} \cap \operatorname{dom}(F)} F(a)$

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end for

Input Legendre potential F, action set $\mathcal{A}$ and learning rate $\eta > 0$
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    Choose measure P_t on $\mathcal{A}$ with mean $\bar{A}_t$
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$$\bar{A}_{t+1} = \operatorname{argmin}_{a \in \mathcal{A} \cap \operatorname{dom}(F)} \eta \sum_{s=1}^{t} \langle a, \hat{Y}_s \rangle + F(a) \qquad \text{(follow-the-regularised-leader)}$$

end for

THEOREM 28.10 (Regret of Mirror-Descent and FTRL with bandit feedback). *Suppose that Algorithm 16 is run with Legendre potential F, convex action set $\mathcal{A} \subset \mathbb{R}^d$ and learning rate $\eta > 0$ such that the loss estimators are unbiased: $\mathbb{E}[\hat{Y}_t \mid \bar{A}_t] = y_t$ for all $t \in [n]$. Then the regret for either variant of Algorithm 16, provided that they are well defined, is bounded by*

$$R_n(a) \leq \mathbb{E}\left[\frac{F(a) - F(\bar{A}_1)}{\eta} + \sum_{t=1}^{n} \langle \bar{A}_t - \bar{A}_{t+1}, \hat{Y}_t \rangle - \frac{1}{\eta} \sum_{t=1}^{n} D(\bar{A}_{t+1}, \bar{A}_t)\right].$$

بندیت معمولی	بندیت ترکیبیاتی	
	$\mathcal{A} \subseteq \{a \in \{0,1\}^d : \|a\|_1 \leq m\}$	مجموعه اعمال

بندیت ترکیبیاتی

بندیت معمولی

مجموعه اعمال

$$\mathcal{A} \subseteq \{a \in \{0,1\}^d : \|a\|_1 \leq m\}$$

$$y_t \in \{y : \sup_{a \in \mathcal{A}} |\langle a, y \rangle| \leq 1\}$$

مجموعه اعمال

بندیت ترکیبیاتی

$$\mathcal{A} \subseteq \{a \in \{0,1\}^d : \|a\|_1 \leq m\}$$

$$y_t \in [0,1]^d,$$

$$\Downarrow$$

$$|\langle A_t, y_t \rangle| \leq m$$

بندیت معمولی

$$y_t \in \{y : \sup_{a \in \mathcal{A}} |\langle a, y \rangle| \leq 1\}$$

	بندیت ترکیبیاتی	بندیت معمولی
مجموعه اعمال	$\mathcal{A} \subseteq \{a \in \{0,1\}^d : \|a\|_1 \leq m\}$ $y_t \in [0,1]^d$ $\Downarrow$ $\|\langle A_t, y_t \rangle\| \leq m$	$y_t \in \{y : \sup_{a \in \mathcal{A}} \|\langle a, y \rangle\| \leq 1\}$
بازخورد		$\langle A_t, y_t \rangle$

	بندیت ترکیباتی	بندیت معمولی
مجموعه اعمال	$\mathcal{A} \subseteq \{a \in \{0,1\}^d : \|a\|_1 \leq m\}$ $y_t \in [0,1]^d$ $\Downarrow$ $\lvert\langle A_t, y_t \rangle\rvert \leq m$	$y_t \in \{y : \sup_{a \in \mathcal{A}} \lvert\langle a, y \rangle\rvert \leq 1\}$
بازخورد	$(A_{t1}y_{t1}, \dots, A_{td}y_{td})$ نیمه‌ـ بندیت	$\langle A_t, y_t \rangle$

بندیت ترکیباتی

مجموعه اعمال

$$\mathcal{A} \subseteq \{a \in \{0,1\}^d : \|a\|_1 \leq m\}$$

$$y_t \in [0,1]^d$$

$$\Downarrow$$

$$|\langle A_t, y_t \rangle| \leq m$$

بازخورد

$$(A_{t1}y_{t1}, \ldots, A_{td}y_{td})$$

نیمه‌ـ بندیت

بندیت معمولی

مجموعه اعمال

$$y_t \in \{y : \sup_{a \in \mathcal{A}} |\langle a, y \rangle| \leq 1\}$$

بازخورد

$$\langle A_t, y_t \rangle$$

$$R_n = \max_{a \in \mathcal{A}} \mathbb{E}\left[\sum_{t=1}^{n} \langle A_t - a, y_t \rangle\right]$$

	بندیت ترکیباتی	بندیت معمولی
مجموعه اعمال	$\mathcal{A} \subseteq \{a \in \{0,1\}^d : \|a\|_1 \leq m\}$ $y_t \in [0,1]^d$ $\Downarrow$ $\lvert\langle A_t, y_t\rangle\rvert \leq m$	$y_t \in \{y : \sup_{a\in\mathcal{A}} \lvert\langle a, y\rangle\rvert \leq 1\}$
بازخورد	$(A_{t1}y_{t1}, \ldots, A_{td}y_{td})$ نیمه‌ــ بندیت	$\langle A_t, y_t\rangle$
	$R_n = \max_{a\in\mathcal{A}} \mathbb{E}\left[\sum_{t=1}^{n}\langle A_t - a, y_t\rangle\right]$	$R_n = \max_{a\in\mathcal{A}} \mathbb{E}\left[\sum_{t=1}^{n}\langle A_t - a, y_t\rangle\right]$

	بندیت ترکیبیاتی	بندیت معمولی
مجموعه اعمال	$\mathcal{A} \subseteq \{a \in \{0,1\}^d : \|a\|_1 \leq m\}$ $y_t \in [0,1]^d$ $\Downarrow$ $\lvert\langle A_t, y_t \rangle\rvert \leq m$	$y_t \in \{y : \sup_{a \in \mathcal{A}} \lvert\langle a, y \rangle\rvert \leq 1\}$
بازخورد	$(A_{t1}y_{t1}, \ldots, A_{td}y_{td})$ نیمه‌ـ بندیت	$\langle A_t, y_t \rangle$
پشیمانی	$R_n = \max_{a \in \mathcal{A}} \mathbb{E}\left[\sum_{t=1}^{n} \langle A_t - a, y_t \rangle\right]$	$R_n = \max_{a \in \mathcal{A}} \mathbb{E}\left[\sum_{t=1}^{n} \langle A_t - a, y_t \rangle\right]$

صورت ۲: نیمه‌ـ بندیت + کاهش آینه‌ای

بازخورد: $(A_{t1}y_{t1}, \ldots, A_{td}y_{td})$

صورت ۲ : نیمه‌ـ بندیت + کاهش آینه‌ای

بازخورد: $(A_{t1}y_{t1}, \ldots, A_{td}y_{td})$

$$\hat{Y}_{ti} = \frac{A_{ti}y_{ti}}{\bar{A}_{ti}}$$

$$\bar{A}_{ti} = \mathbb{E}[A_{ti} \mid \mathcal{F}_{t-1}]$$

صورت ۲ : نیمه‌ـ‌بندیت + کاهش آینه‌ای

بازخورد: $(A_{t1}y_{t1}, \ldots, A_{td}y_{td})$

$$\Leftarrow \quad \hat{Y}_{ti} = \frac{A_{ti}y_{ti}}{\bar{A}_{ti}}$$

$$\bar{A}_{ti} = \mathbb{E}[A_{ti} \,|\, \mathcal{F}_{t-1}]$$

صورت ۲ : نیمه‌ـ بندیت + کاهش آینه‌ای

بازخورد: $(A_{t1}y_{t1}, \ldots, A_{td}y_{td})$

$$\mathbb{E}[\hat{Y}_{ti}] = \sum_{a \in A} P[a]\frac{a_i y_{ti}}{\bar{A}_{ti}} \Leftarrow \hat{Y}_{ti} = \frac{A_{ti}y_{ti}}{\bar{A}_{ti}}$$

$$\bar{A}_{ti} = \mathbb{E}[A_{ti} \mid \mathcal{F}_{t-1}]$$

صورت ۲: نیمه_بندیت + کاهش آینه‌ای

بازخورد: $(A_{t1}y_{t1}, \ldots, A_{td}y_{td})$

$$\hat{Y}_{ti} = \frac{A_{ti}y_{ti}}{\bar{A}_{ti}} \Rightarrow \mathbb{E}[\hat{Y}_{ti}] = \sum_{a \in A} P[a]\frac{a_i y_{ti}}{\bar{A}_{ti}}$$

$$= \frac{y_{ti}}{\bar{A}_{ti}} \sum_{a \in A} P[a] a_i$$

$$\bar{A}_{ti} = \mathbb{E}[A_{ti} \mid \mathcal{F}_{t-1}]$$

صورت ۲ : نیمه_بندیت + کاهش آینه‌ای

بازخورد: $(A_{t1}y_{t1}, \ldots, A_{td}y_{td})$

$$\hat{Y}_{ti} = \frac{A_{ti}y_{ti}}{\bar{A}_{ti}}$$

$$\bar{A}_{ti} = \mathbb{E}[A_{ti} \mid \mathcal{F}_{t-1}]$$

$$\begin{aligned}
\mathbb{E}[\hat{Y}_{ti}] &= \sum_{a\in A} P[a]\frac{a_i y_{ti}}{\bar{A}_{ti}} \Leftarrow \\
&= \frac{y_{ti}}{\bar{A}_{ti}}\sum_{a\in A} P[a]a_i \\
&= \frac{y_{ti}}{\bar{A}_{ti}}\mathbb{E}[A_{ti}]
\end{aligned}$$

صورت ۲ : نیمه_ بندیت + کاهش آینه‌ای

بازخورد: $(A_{t1}y_{t1}, \ldots, A_{td}y_{td})$

$$\hat{Y}_{ti} = \frac{A_{ti}y_{ti}}{\bar{A}_{ti}}$$

$$\bar{A}_{ti} = \mathbb{E}[A_{ti} \mid \mathcal{F}_{t-1}]$$

$$\mathbb{E}[\hat{Y}_{ti}] = \sum_{a \in A} P[a] \frac{a_i y_{ti}}{\bar{A}_{ti}} \Leftarrow$$

$$= \frac{y_{ti}}{\bar{A}_{ti}} \sum_{a \in A} P[a] a_i$$

$$= \frac{y_{ti}}{\bar{A}_{ti}} \mathbb{E}[A_{ti}]$$

$$\mathbb{E}[\hat{Y}_t \mid \mathcal{F}_{t-1}] = y_t$$

الگوریتم کاهش آینه‌ای برای بندیت، با تابع لژاندر منفی آنتروپی

Input $\mathcal{A}, \eta, F$

$\bar{A}_1 = \operatorname{argmin}_{a \in \operatorname{co}(\mathcal{A})} F(a)$

for $t = 1, \ldots, n$ **do**

Choose distribution P_t on $\mathcal{A}$ such that $\sum_{a \in \mathcal{A}} P_t(a)a = \bar{A}_t$

Sample $A_t \sim P_t$ and observe $A_{t1}y_{t1}, \ldots, A_{td}y_{td}$

Compute $\hat{Y}_{ti} = A_{ti}y_{ti}/\bar{A}_{ti}$ for all $i \in [d]$

Update $\bar{A}_{t+1} = \operatorname{argmin}_{a \in \operatorname{co}(\mathcal{A})} \eta\langle a, \hat{Y}_t\rangle + D_F(a, \bar{A}_t)$

end for

کران پشیمانی الگوریتم کاهش آینه‌ای برای بندیت، با تابع لژاندر منفی آنتروپی

قضیه:

$$F(a) = \sum_{i=1}^{d} (a_i \log(a_i) - a_i) \qquad a \in [0, \infty)^d$$

$$F(a) = \infty \qquad \textit{otherwise.}$$

$$\eta = \sqrt{2m(1 + \log(d/m))/(nd)},$$

$$\Downarrow$$

$$R_n \leq \sqrt{2nmd(1 + \log(d/m))}$$

برای مسئله کوتاه‌ترین مسیر نیمه‌ـ بندیتی

Input $\mathcal{A}$, η, F

$\bar{A}_1 = \operatorname{argmin}_{a \in \operatorname{co}(\mathcal{A})} F(a)$

for $t = 1, \ldots, n$ **do**

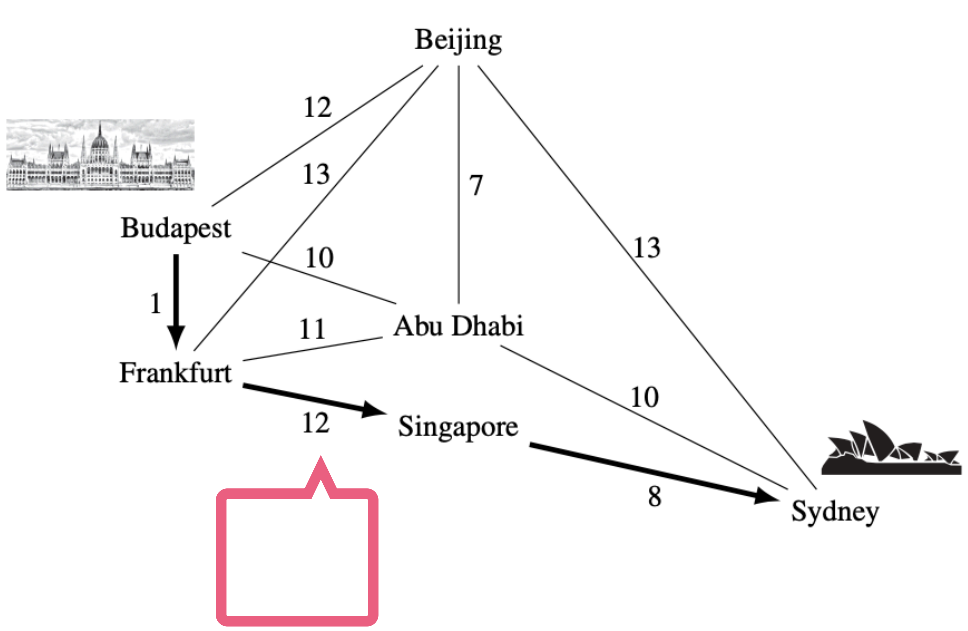
  Choose distribution P_t on $\mathcal{A}$ such that $\sum_{a \in \mathcal{A}} P_t(a)a = \bar{A}_t$

  Sample $A_t \sim P_t$ and observe $A_{t1}y_{t1}, \ldots, A_{td}y_{td}$

  Compute $\hat{Y}_{ti} = A_{ti}y_{ti}/\bar{A}_{ti}$ for all $i \in [d]$

  Update $\bar{A}_{t+1} = \operatorname{argmin}_{a \in \operatorname{co}(\mathcal{A})} \eta\langle a, \hat{Y}_t\rangle + D_F(a, \bar{A}_t)$

end for



کران پایین و الگوریتم کلی؟!

قضیه:

کران پشیمانی الگوریتم کاهش آینه‌ای برای بندیت، با تابع لژاندر منفی آنتروپی

$$\Downarrow$$

$$R_n \leq \sqrt{2nmd(1+\log(d/m))}$$

- اگر الگوریتم بندیتی با پشیمانی $Cn^{1-\epsilon}m^{\alpha}$ برای مسئله بزرگ‌ترین مسیر روی گراف داشته باشیم،
- زمان اجرای آن چند جمله‌ای نیست

صورت ۳: نیمه‌ـ بندیت + پیروی از پیش‌روی آشفته

صورت ۳: نیمه‌ـ بندیت + پیروی از پیش‌روی آشفته

$$a^* = \operatorname{argmin}_{a \in \mathcal{A}} \langle a, y \rangle$$

صورت ۳: نیمه‌ـ‌بندیت + پیروی از پیش‌روی آشفته

$$a^* = \operatorname{argmin}_{a \in \mathcal{A}} \langle a, y \rangle$$

پیروی از پیش‌روی آشفته:

Input $\mathcal{A}, n, \eta, \beta, Q$

$\hat{L}_0 = \mathbf{0} \in \mathbb{R}^d$

for $t = 1, \ldots, n$ **do**

محاسبه A_t بر حسب $\hat{L}_t$

Observe $A_{t1}y_{t1}, \ldots, A_{td}y_{td}$

محاسبه $\hat{Y}$

$\hat{L}_t = \hat{L}_{t-1} + \hat{Y}_t$

end for

$\hat{L}_{t-1} = \sum_{s=1}^{t-1} \hat{Y}_s$ $\quad \sim Q$

$$A_t = \operatorname{argmin}_{a \in \mathcal{A}} \langle a, \eta \hat{L}_{t-1} - Z_t \rangle,$$

$\hat{L}_{t-1} = \sum_{s=1}^{t-1} \hat{Y}_s$

$\sim Q$

$$A_t = \operatorname{argmin}_{a \in \mathcal{A}} \langle a, \eta \hat{L}_{t-1} - Z_t \rangle,$$

پیروی از پیش‌روی آشفته:

Input $\mathcal{A}, n, \eta, \beta, Q$

$\hat{L}_0 = \mathbf{0} \in \mathbb{R}^d$

for $t = 1, \ldots, n$ **do**

Sample $Z_t \sim Q$

Compute $A_t = \operatorname{argmax}_{a \in \mathcal{A}} \langle a, Z_t - \eta \hat{L}_{t-1} \rangle$

Observe $A_{t1} y_{t1}, \ldots, A_{td} y_{td}$

محاسبه $\hat{Y}$

$\hat{L}_t = \hat{L}_{t-1} + \hat{Y}_t$

end for

$$\boxed{A_t = \operatorname{argmin}_{a \in \mathcal{A}} \langle a, \eta \hat{L}_{t-1} - Z_t \rangle,}$$

$$\hat{Y}_{ti} \;=\; A_{ti} y_{ti} / P_{ti}$$

$$\boxed{A_t = \operatorname{argmin}_{a \in \mathcal{A}} \langle a, \eta \hat{L}_{t-1} - Z_t \rangle,}$$

$$\hat{Y}_{ti} = A_{ti} y_{ti} / P_{ti}$$

$$\boxed{P_{ti} = \mathbb{P}(A_{ti} = 1 \mid \mathcal{F}_{t-1}) = \bar{A}_{ti}}$$

$$\boxed{A_t = \operatorname{argmin}_{a\in\mathcal{A}} \langle a, \eta \hat{L}_{t-1} - Z_t \rangle,}$$

$$\hat{Y}_{ti} \;=\; A_{ti} y_{ti} / P_{ti}$$

$$P_{ti} \;=\; \mathbb{P}\left(A_{ti} = 1 \,|\, \mathcal{F}_{t-1}\right) \;=\; \bar{A}_{ti}$$

$$\bar{A}_t = \sum_{a\in\mathcal{A}} P_t(a) a$$

$$\boxed{A_t = \mathrm{argmin}_{a \in \mathcal{A}} \langle a, \eta \hat{L}_{t-1} - Z_t \rangle,}$$

$$\hat{Y}_{ti} \;=\; A_{ti} y_{ti} / P_{ti}$$

$$P_{ti} \;=\; \mathbb{P}\,(A_{ti} = 1 \,|\, \mathcal{F}_{t-1}) \;=\; \bar{A}_{ti}$$

$$\bar{A}_t = \sum_{a \in \mathcal{A}} P_t(a) a$$

$$P_t(a) = \mathbb{P}(a(Z_t - \eta \hat{L}_{t-1}) = a \,|\, \mathcal{F}_{t-1})$$

$$\boxed{A_t = \operatorname{argmin}_{a \in \mathcal{A}} \langle a, \eta \hat{L}_{t-1} - Z_t \rangle,}$$

$$\hat{Y}_{ti} = A_{ti} y_{ti} / P_{ti}$$

$$P_{ti} = \mathbb{P}(A_{ti} = 1 \mid \mathcal{F}_{t-1}) = \bar{A}_{ti}$$

$$\bar{A}_t = \sum_{a \in \mathcal{A}} P_t(a) a$$

$$P_t(a) = \mathbb{P}(a(Z_t - \eta \hat{L}_{t-1}) = a \mid \mathcal{F}_{t-1})$$

؟

محاسبه $P_{t,i}$؟

محاسبه $P_{t,i}$ برای

$$\hat{Y}_{ti} = A_{ti} y_{ti} / P_{ti}$$

محاسبه $P_{t,i}$ برای

$$\hat{Y}_{ti} = A_{ti} y_{ti} / P_{ti}$$

$$\boxed{A_t = \mathrm{argmin}_{a \in \mathcal{A}} \langle a, \eta \hat{L}_{t-1} - Z_t \rangle,}$$

$$P_{ti} = \mathbb{P}\left(A_{ti} = 1 \,|\, \mathcal{F}_{t-1}\right) = \bar{A}_{ti}$$

مطلوب: $1/P_{t,i}$

محاسبه $P_{t,i}$ برای

$$\hat{Y}_{ti} = A_{ti} y_{ti} / P_{ti}$$

$$\boxed{A_t = \operatorname{argmin}_{a \in \mathcal{A}} \langle a, \eta \hat{L}_{t-1} - Z_t \rangle,}$$

$$P_{ti} = \mathbb{P}\left(A_{ti} = 1 \,|\, \mathcal{F}_{t-1}\right) = \bar{A}_{ti}$$

مطلوب: $1/P_{t,i}$

LEMMA 30.3. *Let $U \in \{1, 2, \ldots\}$ be geometrically distributed with parameter $\theta \in [0, 1]$ so that $\mathbb{P}(U = j) = (1 - \theta)^{j-1}\theta$. Then $\mathbb{E}[U] = 1/\theta$.*

$$P_{ti} = \mathbb{P}\left(A_{ti} = 1 \,|\, \mathcal{F}_{t-1}\right) = \bar{A}_{ti}$$

$$\mathrm{Geometric}(P_{ti})$$

$$\hat{Y}_{ti} = \min(\beta, K_{ti}) A_{ti} y_{ti}$$

$$\mathbb{E}\left[K_{ti} A_{ti} y_{ti} \,|\, \mathcal{F}_{t-1}\right] = y_{ti}$$

پیروی از پیش‌روی آشفته:

Input $\mathcal{A}, n, \eta, \beta, Q$

$\hat{L}_0 = \mathbf{0} \in \mathbb{R}^d$

for $t = 1, \dots, n$ **do**

Sample $Z_t \sim Q$

Compute $A_t = \operatorname{argmax}_{a \in \mathcal{A}} \langle a, Z_t - \eta \hat{L}_{t-1} \rangle$

Observe $A_{t1} y_{t1}, \dots, A_{td} y_{td}$

For each $i \in [d]$ sample $K_{ti} \sim \text{Geometric}(P_{ti})$

For each $i \in [d]$ compute $\hat{Y}_{ti} = \min(\beta, K_{ti}) A_{ti} y_{ti}$

$\hat{L}_t = \hat{L}_{t-1} + \hat{Y}_t$

end for

کاهش آینه‌ای = پیش‌روی آشفته

پیروی از پیش‌روی آشفته:

$\hat{L}_{t-1} = \sum_{s=1}^{t-1} \hat{Y}_s$

$\sim Q$

$$A_t = \operatorname{argmin}_{a \in \mathcal{A}} \langle a, \eta \hat{L}_{t-1} - Z_t \rangle,$$

کاهش آینه‌ای:

؟

$$\bar{A}_t = \operatorname{argmin}_{a \in \operatorname{co}(\mathcal{A})} \langle a, \eta \hat{Y}_{t-1} \rangle + D_F(a, \bar{A}_{t-1}).$$

$$\bar{A}_t = \mathrm{argmin}_{a \in \mathrm{co}(\mathcal{A})} \langle a, \eta \hat{Y}_{t-1} \rangle + D_F(a, \bar{A}_{t-1}).$$

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y) \rangle$$

$$\bar{A}_t = \mathrm{argmin}_{a \in \mathrm{co}(\mathcal{A})} \langle a, \eta \hat{Y}_{t-1} \rangle + D_F(a, \bar{A}_{t-1}).$$

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y) \rangle$$

$$\nabla(\ldots) = 0$$

$$\bar{A}_t = \operatorname{argmin}_{a \in \mathrm{co}(\mathcal{A})} \langle a, \eta \hat{Y}_{t-1} \rangle + D_F(a, \bar{A}_{t-1}).$$

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y) \rangle$$

$$\nabla(\ldots) = 0 \Rightarrow \quad \eta \hat{Y}_{t-1} + \nabla F(\bar{A}_t) - \nabla F(\bar{A}_{t-1}) = 0$$

$$\bar{A}_t = \mathrm{argmin}_{a \in \mathrm{co}(\mathcal{A})} \langle a, \eta \hat{Y}_{t-1} \rangle + D_F(a, \bar{A}_{t-1}).$$

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y) \rangle$$

$$\nabla(\ldots) = 0 \Rightarrow \quad \eta \hat{Y}_{t-1} + \nabla F(\bar{A}_t) - \nabla F(\bar{A}_{t-1}) = 0$$

$$\Rightarrow \quad \nabla F(\bar{A}_t) = \nabla F(\bar{A}_{t-1}) - \eta \hat{Y}_{t-1} = -\eta \hat{L}_{t-1}$$

$$\bar{A}_t = \mathrm{argmin}_{a \in \mathrm{co}(\mathcal{A})} \langle a, \eta \hat{Y}_{t-1} \rangle + D_F(a, \bar{A}_{t-1}).$$

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y) \rangle$$

$$\nabla(\ldots) = 0 \Rightarrow \quad \eta \hat{Y}_{t-1} + \nabla F(\bar{A}_t) - \nabla F(\bar{A}_{t-1}) = 0$$

$$\Rightarrow \quad \nabla F(\bar{A}_t) = \nabla F(\bar{A}_{t-1}) - \eta \hat{Y}_{t-1} = -\eta \hat{L}_{t-1}$$

$$\bar{A}_t = \nabla F^*(-\eta \hat{L}_{t-1})$$

$$\bar{A}_t = \text{argmin}_{a \in \text{co}(\mathcal{A})} \langle a, \eta \hat{Y}_{t-1} \rangle + D_F(a, \bar{A}_{t-1}).$$

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y) \rangle$$

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$$A_t = \text{argmin}_{a \in \mathcal{A}} \langle a, \eta \hat{L}_{t-1} - Z_t \rangle,$$

$$\bar{A}_t = \mathrm{argmin}_{a \in \mathrm{co}(\mathcal{A})} \langle a, \eta \hat{Y}_{t-1} \rangle + D_F(a, \bar{A}_{t-1}).$$

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$$\bar{A}_t = \mathbb{E}[A_t \,|\, \mathcal{F}_{t-1}] = \mathbb{E}\left[\mathrm{argmin}_{a \in \mathcal{A}} \langle a, \eta \hat{L}_{t-1} - Z_t \rangle \,\middle|\, \mathcal{F}_{t-1}\right]$$

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$$\begin{aligned} \bar{A}_t = \mathbb{E}[A_t \mid \mathcal{F}_{t-1}] &= \mathbb{E}\left[\mathrm{argmin}_{a \in \mathcal{A}} \langle a, \eta \hat{L}_{t-1} - Z_t \rangle \,\middle|\, \mathcal{F}_{t-1}\right] \\ &= \mathbb{E}\left[\mathrm{argmax}_{a \in \mathcal{A}} \langle a, Z_t - \eta \hat{L}_{t-1} \rangle \,\middle|\, \mathcal{F}_{t-1}\right] \end{aligned}$$

$$\bar{A}_t = \mathrm{argmin}_{a \in \mathrm{co}(\mathcal{A})} \langle a, \eta \hat{Y}_{t-1} \rangle + D_F(a, \bar{A}_{t-1}).$$

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$$\boxed{\nabla F^*(-\eta \hat{L}_{t-1}) \ = \mathbb{E}\left[\operatorname{argmax}_{a\in\mathcal{A}}\langle a, Z_t - \eta \hat{L}_{t-1}\rangle \,\middle|\, \mathcal{F}_{t-1}\right]}$$

$$\nabla F^*(x) = \int_{\mathbb{R}^d} \operatorname{argmax}_{a\in\operatorname{co}(\mathcal{A})}\langle a, x+z\rangle \, dQ(z)$$

$$\nabla \int_{\mathbb{R}^d} \phi(x+z)\, dQ(z) = \int_{\mathbb{R}^d} a(x+z)\, dQ(z)$$

$$\boxed{\phi(x) = \max_{a\in\mathcal{A}}\langle a, x\rangle} \qquad \boxed{a(x) = \operatorname{argmax}_{a\in\mathcal{A}}\langle a, x\rangle}$$

$$F^*(x) = \int_{\mathbb{R}^d} \phi(x+z) dQ(z)$$

قضیه: (پشمانی پیروی از پیش‌روی آشفته)

$$\text{Q:} \quad q(z) = 2^{-d} \exp(-\|z\|_1)$$

$$\eta = \sqrt{\frac{2(1+\log(d))}{(1+e^2)dnm}} \qquad \beta = \left\lceil \frac{1}{\eta m} \right\rceil$$

$$\Downarrow$$

$$R_n \leq m\sqrt{2(1+e^2)nd(1+\log(d))}.$$

اثبات

$$R_n(a) = \mathbb{E}\left[\sum_{t=1}^{n} \langle A_t - a, y_t \rangle\right]$$

اثبات

$$\begin{aligned} R_n(a) &= \mathbb{E}\left[\sum_{t=1}^{n} \langle A_t - a, y_t \rangle\right] \\ &= \mathbb{E}\left[\sum_{t=1}^{n} \langle \bar{A}_t - a, y_t \rangle\right] \end{aligned}$$

اثبات

$$
\begin{aligned}
R_n(a) &= \mathbb{E}\left[\sum_{t=1}^{n}\langle A_t - a, y_t\rangle\right] \\
&= \mathbb{E}\left[\sum_{t=1}^{n}\langle \bar{A}_t - a, y_t\rangle\right] \\
&= \mathbb{E}\left[\sum_{t=1}^{n}\langle \bar{A}_t - a, \hat{Y}_t\rangle\right] + \mathbb{E}\left[\sum_{t=1}^{n}\langle \bar{A}_t - a, y_t - \hat{Y}_t\rangle\right]
\end{aligned}
$$

اثبات

$$
\begin{aligned}
R_n(a) &= \mathbb{E}\left[\sum_{t=1}^{n} \langle A_t - a, y_t \rangle\right] \\
&= \mathbb{E}\left[\sum_{t=1}^{n} \langle \bar{A}_t - a, y_t \rangle\right] \\
&= \mathbb{E}\left[\sum_{t=1}^{n} \langle \bar{A}_t - a, \hat{Y}_t \rangle\right] + \mathbb{E}\left[\sum_{t=1}^{n} \langle \bar{A}_t - a, y_t - \hat{Y}_t \rangle\right] \\
&\leq \frac{\operatorname{diam}_F(\mathcal{A})}{\eta} + \mathbb{E}\left[\frac{1}{\eta}\sum_{t=1}^{n} D_F(\bar{A}_t, \bar{A}_{t+1})\right] + \mathbb{E}\left[\sum_{t=1}^{n} \langle \bar{A}_t - a, y_t - \hat{Y}_t \rangle\right]
\end{aligned}
$$

$$R_n(a) \leq \frac{\operatorname{diam}_F(\mathcal{A})}{\eta} + \mathbb{E}\left[\frac{1}{\eta}\sum_{t=1}^{n} D_F(\bar{A}_t, \bar{A}_{t+1})\right] + \mathbb{E}\left[\sum_{t=1}^{n}\langle \bar{A}_t - a, y_t - \hat{Y}_t\rangle\right]$$

$$R_n(a) \leq \frac{\mathrm{diam}_F(\mathcal{A})}{\eta} + \mathbb{E}\left[\frac{1}{\eta}\sum_{t=1}^{n} D_F(\bar{A}_t, \bar{A}_{t+1})\right] + \mathbb{E}\left[\sum_{t=1}^{n}\langle \bar{A}_t - a, y_t - \hat{Y}_t\rangle\right]$$

$$\leq m(1 + \log(d))$$

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$$\leq \frac{e^2 m d \eta^2}{2}$$

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$$R_n \le \frac{m(1+\log(d))}{\eta} + \frac{e^2 dnm\eta}{2} + \frac{dnm\eta}{2}$$

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$$R_n \le \frac{m(1+\log(d))}{\eta} + \frac{e^2 dnm\eta}{2} + \frac{dnm\eta}{2}$$

$$\le m\sqrt{2(1+e^2)nd(1+\log(d))}$$

$$
\begin{aligned}
F(a) &= \sup_{x \in \mathbb{R}^d} \left(\langle a, x \rangle - F^*(x) \right) \\
&= \sup_{x \in \mathbb{R}^d} \left(\langle a, x \rangle - \mathbb{E}[\max_{b \in \mathcal{A}} \langle b, x + Z \rangle] \right) \\
&\geq -\mathbb{E}[\max_{b \in \mathcal{A}} \langle b, Z \rangle] \\
&\geq -m\mathbb{E}[\|Z\|_\infty] \\
&= -m \sum_{i=1}^{d} \frac{1}{d} \geq -m(1 + \log(d))
\end{aligned}
$$

x=0

به جای همه درایه‌ها بیشترین

تمرین

$$
\mathrm{diam}_F(\mathcal{A}) = \max_{a,b \in \mathcal{A}} F(a) - F(b) \leq m(1 + \log(d))
$$

نامثبت

$$
\begin{aligned}
D_F(\bar{A}_t, \bar{A}_{t+1}) &= D_{F^*}(\nabla F(\bar{A}_{t+1}), \nabla F(\bar{A}_t)) \\
&= D_{F^*}(-\eta \hat{L}_{t-1} - \eta \hat{Y}_t, -\eta \hat{L}_{t-1}) \\
&= \frac{\eta^2}{2} \|\hat{Y}_t\|^2_{\nabla^2 F^*(\xi)}
\end{aligned}
$$

$$\xi = -\eta \hat{L}_{t-1} - \alpha \eta \hat{Y}_t$$

$$
\leq \frac{e^2 \eta^2}{2} \sum_{i=1}^{d} P_{ti} \hat{Y}_{ti} \sum_{j=1}^{d} \hat{Y}_{tj}
$$

$$\nabla^2 F^*(\xi)_{ij} \leq e^2 P_{ti}$$

$$
\leq \frac{e^2 \eta^2}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} P_{ti} K_{ti} A_{ti} K_{tj} A_{tj}
$$

$$
\begin{aligned}
\mathbb{E}[D_F(\bar{A}_t, \bar{A}_{t+1})] &\leq \frac{e^2 \eta}{2} \mathbb{E}\left[\sum_{i=1}^{d} \sum_{j=1}^{d} P_{ti} K_{ti} A_{ti} K_{tj} A_{tj}\right] \\
&= \frac{e^2 \eta^2}{2} \mathbb{E}\left[\sum_{i=1}^{d} \sum_{j=1}^{d} \frac{A_{ti} A_{tj}}{P_{tj}}\right] \leq \frac{e^2 m d \eta^2}{2}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[\hat{Y}_{ti} \mid \mathcal{F}_t] = \mathbb{E}[\min(\beta, K_{ti}) A_{ti} y_{ti} \mid \mathcal{F}_t] &= A_{ti} y_{ti} \mathbb{E}[\min(\beta, K_{ti}) \mid \mathcal{F}_t] \\
&= y_{ti} \, \frac{A_{ti}}{P_{ti}} (1 - (1 - P_{ti})^\beta),
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[\sum_{t=1}^n \langle \bar{A}_t - a, y_t - \hat{Y}_t \rangle\right] &\le \mathbb{E}\left[\sum_{t=1}^n \langle \bar{A}_t, y_t - \hat{Y}_t \rangle\right] \\
&= \mathbb{E}\left[\sum_{t=1}^n \sum_{i=1}^d y_{ti} P_{ti} (1 - P_{ti})^\beta\right] \le \frac{dn}{2\beta} = \frac{dnm\eta}{2}
\end{aligned}
$$

پیروی از پیش‌روی آشفته:

Input $\mathcal{A}, n, \eta, \beta, Q$

$\hat{L}_0 = \mathbf{0} \in \mathbb{R}^d$

for $t = 1, \ldots, n$ **do**

$$q(z) = 2^{-d}\exp(-\|z\|_1)$$

  Sample $Z_t \sim Q$

  Compute $A_t = \operatorname{argmax}_{a \in \mathcal{A}} \langle a, Z_t - \eta \hat{L}_{t-1} \rangle$

  Observe $A_{t1}y_{t1}, \ldots, A_{td}y_{td}$

  For each $i \in [d]$ sample $K_{ti} \sim \text{Geometric}(P_{ti})$

  For each $i \in [d]$ compute $\hat{Y}_{ti} = \min(\beta, K_{ti}) A_{ti} y_{ti}$

  $\hat{L}_t = \hat{L}_{t-1} + \hat{Y}_t$

end for

$$\beta = \left\lceil \frac{1}{\eta m} \right\rceil$$