برنامه‌ریزی نیمه‌معین برای طراحی الگوریتم‌های تقریبی

جلسه ششم: دوگانی

قضیه نهایی دوگانی (برای برنامه‌نویسی نیمه‌معین)

$$
\begin{aligned}
\text{maximize} \quad & C \bullet X \\
\text{subject to} \quad & A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m \\
& X \succeq 0.
\end{aligned}
$$

4.1.1 Theorem. *If the semidefinite program* (4.1) *is feasible and has a finite value* γ, *and if there is a positive definite matrix* $\tilde{X}$ *such that* $A(\tilde{X}) = \mathbf{b}$, *then the dual program*

$$
\begin{aligned}
\text{minimize} \quad & \mathbf{b}^T \mathbf{y} \\
\text{subject to} \quad & \sum_{i=1}^{m} y_i A_i - C \succeq 0
\end{aligned}
\qquad (4.2)
$$

is feasible and has finite value $\beta = \gamma$.

4.2.1 Definition. *Let $K \subseteq V$ be a nonempty closed set.*[1] *K is called a closed convex cone if the following two conditions hold.*

(i) *For all $\mathbf{x} \in K$ and all nonnegative real numbers λ, we have $\lambda\mathbf{x} \in K$.*
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مثال: مجموعه $K = \{0\}$ کنج محدب بسته است

4.2.2 Lemma. *The set* $\mathrm{PSD}_n \subseteq \mathrm{SYM}_n$ *of positive semidefinite matrices is a closed convex cone.*

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- معادلا: مکملش باز است
- $\tilde{x}$ هست که $\tilde{\mathbf{x}}^T M \tilde{\mathbf{x}} < 0$
- $M + \epsilon B$:
- $\max_B \tilde{x}^\top B \tilde{x}$

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- ج) $M + N$
- $\mathbf{x}^T (M + N)\mathbf{x} = \mathbf{x}^T M \mathbf{x} + \mathbf{x}^T N \mathbf{x} \geq 0$

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مثال: اگر K و L کنج محدب بسته باشند، آنگاه

$$K \oplus L := \{(\mathbf{x}, \mathbf{y}) \in V \oplus W : \mathbf{x} \in K, \mathbf{y} \in L\}$$

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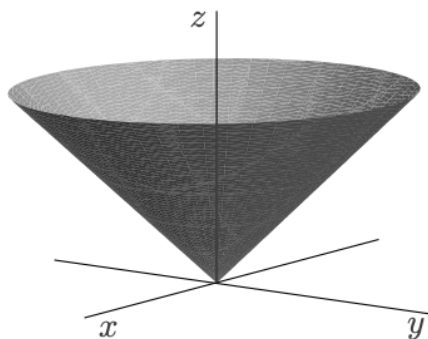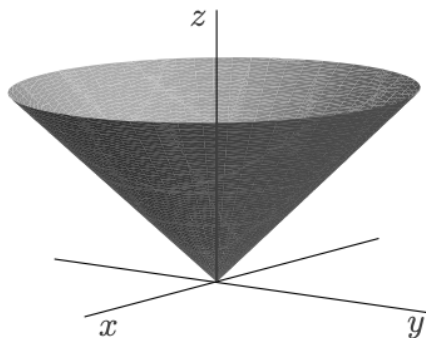
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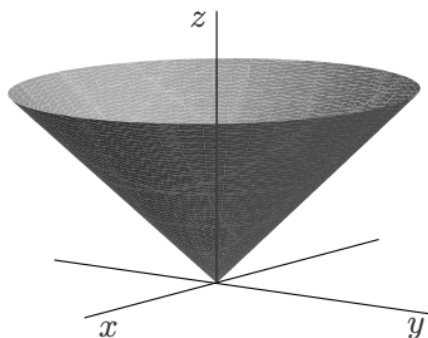
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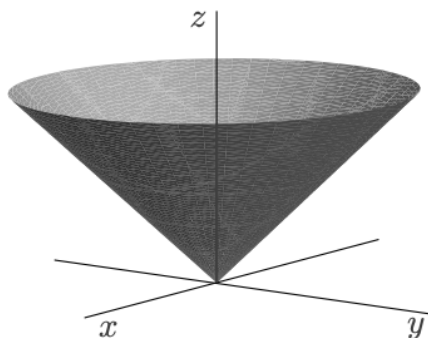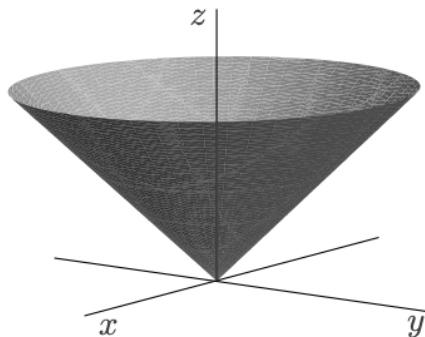
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- $(x, r) + (y, s) = (x + y, r + s)$

$$\|x + y\| \leq \|x\| + \|y\|$$

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$$\triangleleft\hspace{-0.4em}0 = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, xy \geq z^2\}$$

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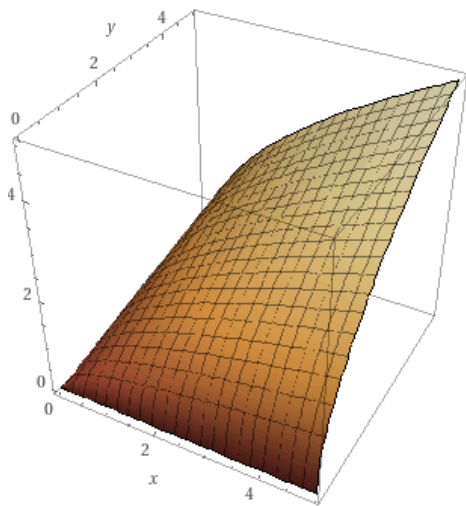$$\triangleleft\!\!) = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, xy \geq z^2\}$$
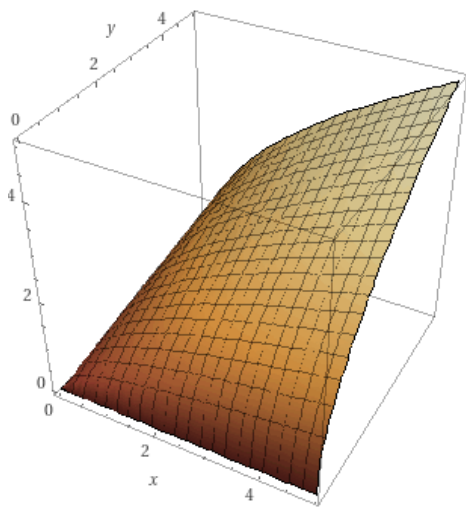
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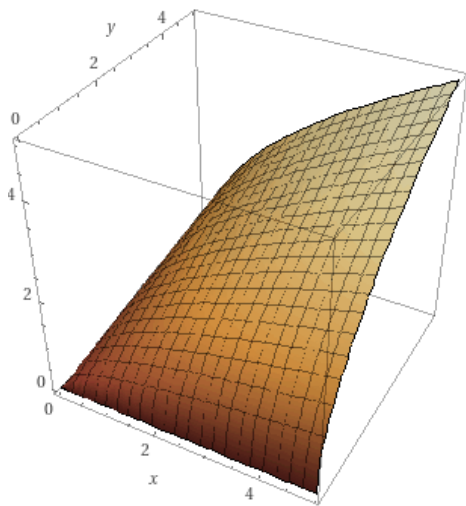<div dir="rtl">

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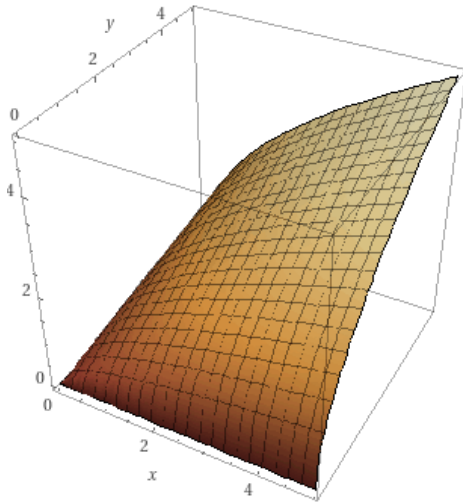(کنج بستنی واژگون) کنج محدب بسته است

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- معادلا $\begin{pmatrix} x & z \\ z & y \end{pmatrix}$ مثبت نیمه‌معین