دانشکده علوم ریاضی

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$Question\, 1:$

G is a complete graph K_n، where n is even. What is the size of the maximum cut of G ?

$Question\, 2:$

We consider a communication channel، with input $X(t) \in \{1, ..., n\}$ ، and output $Y(t) \in \{1, ..., m\}$ ، for $t = 1, 2, ..$ (in seconds، say). The relation between the input and the output is given statistically:

$$p_{ij} = prob(Y(t) = i|X(t) = j), \quad i = 1, 2, ...m, \quad j = 1, 2, ..., n$$

The matrix $P \in R^{m \times n}$ is called the channel transition matrix، and the channel is called a discrete memoryless channel.

A famous result of Shannon states that information can be sent over the communication channel، with arbitrarily small probability of error، at any rate less than a number C، called the channel capacity، in bits per second. Shannon also showed that the capacity of a discrete memoryless channel can be found by solving an optimization problem. Assume that X has a probability distribution denoted $x \in R^n, i.e.,$

$$x_j = prob(X = j), \quad j = 1, ..., n$$

The mutual information between X and Y is given by

$$I(X; Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_j p_{ij} log_2 \frac{p_{ij}}{\sum_{k=1}^{n} x_k p_{ik}}$$

Then the channel capacity C is given by

$$C = \sup_x I(X;Y),$$

where the supremum is over all possible probability distributions for the input X، i.e.، over $x \succeq 0,\ 1^T x = 1.$

Show how the channel capacity can be computed with concave objective function.

Hint. Introduce the variable y = Px، which gives the probability distribution of the output Y ، and show that the mutual information can be expressed as

$$I(X;Y) = c^T x - \sum_{i=1}^{m} y_i log_2 y_i,$$

(The entropy of y is concave)، where $c_j = \sum_{i=1}^{m} p_{ij}, j = 1, 2, ..., n.$

$Question\, 3:$

Suppose that C and D are disjoint subsets of R^n.

Consider the set of $(a, b) \in R^{n+1}$ for which $a^T x \leq b$ for all $x \in C$ ، and $a^T x \geq b$ for all $x \in D$. Show that this set is a convex cone (which is the singleton $\{0\}$ if there is no hyperplane that separates C and D).

$Question\, 4:$

Suppose $K \subseteq R^2$ is a closed convex cone.

(a) Give a simple description of K in terms of the polar coordinates of its elements $(x = r(cos\theta, sin\theta)\ with\ r \geq 0)$.

(b) Give a simple description of K^*, and draw a plot illustrating the relation between K and K^*

$Question\, 5:$

Let $G = (V, E)$ be a graph. Show that $\vartheta(G)$ can be expressed as the value of the following optimization problem:

$$
\begin{aligned}
\vartheta(G) = min\ \ & \lambda_{max}(11^T + X) \\
s.t.\ \ & X_{ij} = 0,\ \ if\ \{i,j\} \in \bar{E},\ \ or\, i = j \\
& X \in SYM_n.
\end{aligned}
$$

$Question\, 6:$

Formulate the following optimization problems as semidefinite programs. The variable is $x \in R^n;\ \ F(x)$ is defined as

$$F(x) = F_0 + x_1F_1 + x_2F_2 + ... + x_nF_n$$

with $F_i \in S^m$. The domain of f in each subproblem is $dom\ f = \{x \in R^n | F(x) \succ 0\}$.

(a) Minimize $f(x) = max_{i=1,...k}\, c_i^T F(x)^{-1}c_i$ where $c_i \in R^m,\ \ i = 1, ..., k.$

(b) Minimize $f(x) = sup_{\|c\|_2 \leq 1}\, c^T F(x)^{-1}c$

Hint. Schur Complements

$Question\, 7:$

Suppose $A : R^n \rightarrow S^m$ is affine، i.e.،

$$A(x) = A_0 + x_1A_1 + x_2A_2 + ...x_nA_n$$

where $A_i \in S^m$. Let $\lambda_1(x) \geq \lambda_2(x) \geq ... \geq \lambda_m(x)$ denote the eigenvalues of $A(x)$. Show how to pose the following problems as SDPs.

(a) Minimize the maximum eigenvalue $\lambda_1(x)$

(b) Minimize the spread of the eigenvalues، $\lambda_1(x) - \lambda_m(x)$