بسم الله الرحمن الرحیم

جلسه سیزدهم

خلاصه‌سازی برای مه‌داده

مرور

- تبدیل JL: با داشتن n بردار در Rd، تولید n بردار در $\mathbb{R}^{O(\epsilon^{-2}\log n)}$ که تقریبا همه طول‌ها همانند.
  - نکته: تبدیلمان حساس به ورودی است.
  - زمان هر تبدیل: O(md) که $m = \epsilon^{-2}\log n$
- تبدیل سریع JL:
  - زمان تبدیل هر:
  - $O(d\log d + m)$
  - که $m = O(\epsilon^{-2}\log(\frac{1}{\delta})\log(\frac{d}{\delta}))$

$$\Pi = \frac{1}{\sqrt{m}} S H D$$

هر سطر یک عدد ۱ یکنواخت در ستون‌ها

O(m)

نمونه‌گیری

O(d)

کانولوشن

ماتریس قطری با قطر تصادفی ۱+ یا ۱-

پخش

$H^\top H = d \cdot I$

$\max_{i,j} |H_{i,j}| \leq 1$

O(d lg d)

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پخش

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O(d lg d)

- در ادامه: ؟؟؟

سریع‌تر برای تنک‌ها

Fast JL Transforms

هدف

Theorem 5.0.1 (JL lemma [JL84]). *For any $\varepsilon \in (0,1)$ and any $X \subset \mathbb{R}^d$ for $|X| = n$ finite, there exists an embedding $f : X \to \mathbb{R}^m$ for $m = O(\varepsilon^{-2} \log n)$ such that*

$$\forall x, y \in X, \ (1-\varepsilon)\|x-y\|_2^2 \leq \|f(x) - f(y)\|_2^2 \leq (1+\varepsilon)\|x-y\|_2^2. \tag{5.1}$$

Πz

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$||z||_0 \ll d$

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غیر صفر

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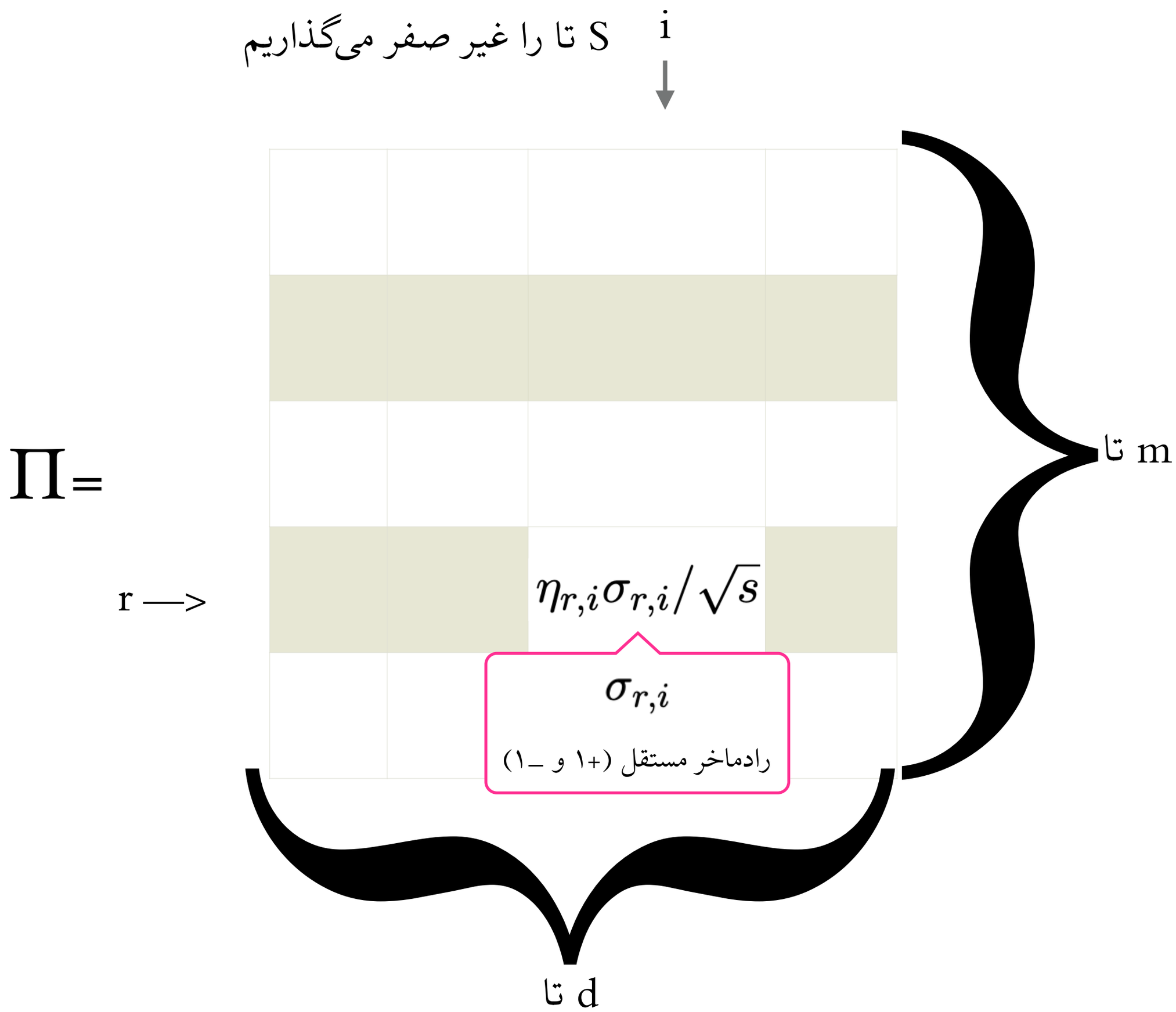
Πz

$||z||_0 \ll d$

زمان اجرای ضرب:
$O(s||z||_0)$

$$\Pi = \begin{array}{c} \\ \\ \\ r \longrightarrow \\ \\ \end{array} \overbrace{\begin{array}{c} \\ \\ \\ \\ \\ \end{array}}^{\text{i تا S را غیر صفر می‌گذاریم}} \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \\ \\ \end{array


S تا را غیر صفر می‌گذاریم
i
m تا
$\Pi =$
r —>
$\eta_{r,i}\sigma_{r,i}/\sqrt{s}$
$\sigma_{r,i}$
رادماخر مستقل (+۱ و ـ۱)
d تا

S تا را غیر صفر می‌گذاریم

i

$\Pi =$

r —>

$$\eta_{r,i}$$

$$\mathbb{E}\,\eta_{r,i} = s/m$$

$$\sum_{r=1}^{m} \eta_{r,i} = s$$

$$\mathbb{E} \prod_{(r,i)\in S} \eta_{r,i} \leq (s/m)^{|S|}$$

$$\eta_{r,i}\sigma_{r,i}/\sqrt{s}$$

$$\sigma_{r,i}$$

رادماخر مستقل (+۱ و ـ۱)

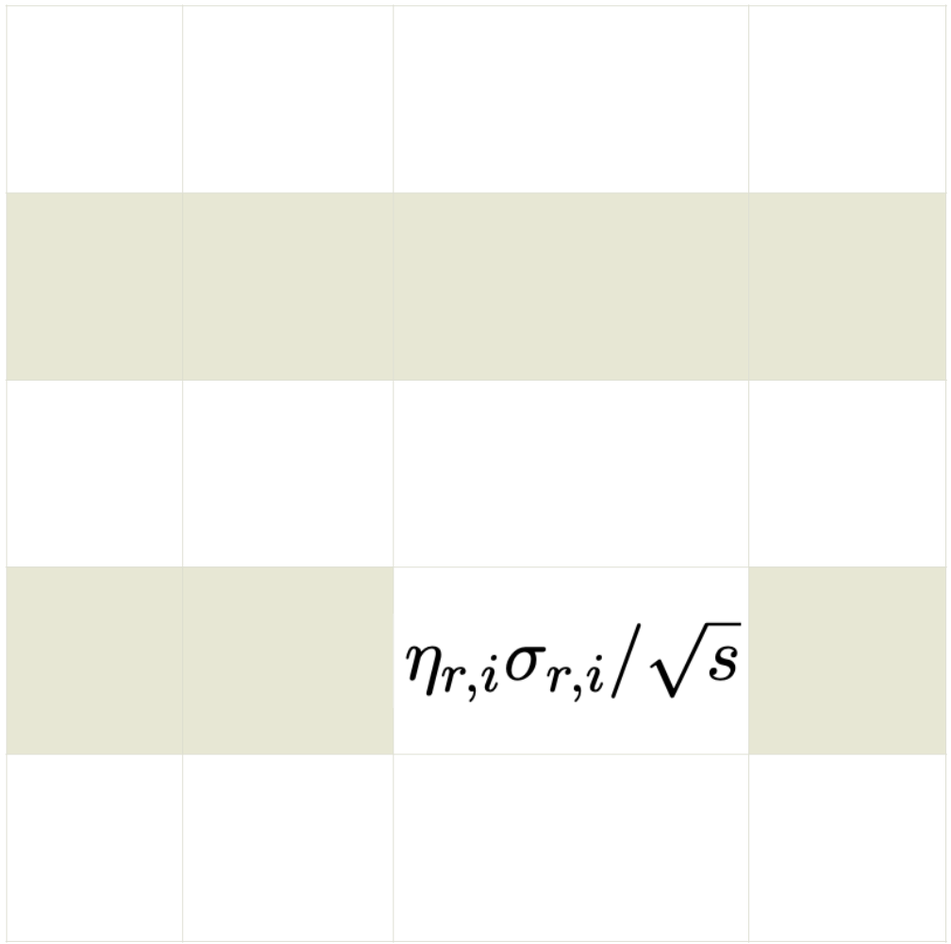
m تا

d تا

Theorem 5.3.1. *As long as* $m \simeq \varepsilon^{-2}\log(1/\delta)$ *and* $s \simeq \varepsilon m$,

$$\forall z : \|z\|_2 = 1,\ \mathbb{P}_{\Pi}(|\|\Pi z\|_2^2 - 1| > \varepsilon) < \delta.$$

$\Pi =$



$$||Y||_p = \sqrt[p]{\mathbb{E}[Y^p]}$$

Theorem 1.1.12 (Hanson-Wright inequality [HW71]). *For $\sigma_1, \ldots, \sigma_n$ independent Rademachers and $A \in \mathbb{R}^{n \times n}$, for all $p \geq 1$*

$$\|\sigma^\top A\sigma - \mathbb{E}\,\sigma^\top A\sigma\|_p \lesssim \sqrt{p} \cdot \|A\|_F + p \cdot \|A\|.$$

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$$\sqrt{\sum_{r,i} a_{r,i}^2}$$

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$$\sqrt{\sum_{r,i} a_{r,i}^2}$$

نرم اپراتوری:

$$\max_{||x||_2=1} x^T A x$$

برابر است با بزرگ‌ترین مقدار ویژه

Theorem 5.3.1. *As long as* $m \simeq \varepsilon^{-2}\log(1/\delta)$ *and* $s \simeq \varepsilon m$,

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$$= \frac{1}{s} \sum_{r=1}^{m} \Big[\sum_{i=1}^{d} z_i^2 \eta_{r,i} + \sum_{i \neq j} \eta_{r,i}\eta_{r,j}\sigma_{r,i}\sigma_{r,j} z_i z_j \Big]$$

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جمع روی s = r

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$= ||z||_2^2 = 1$

جمع روی r = s

$$E = \|\Pi z\|_2^2 - 1 = \frac{1}{s} \sum_{r=1}^{m} \sum_{i \neq j} \eta_{r,i} \eta_{r,j} \sigma_{r,i} \sigma_{r,j} z_i z_j \stackrel{\text{def}}{=} \sigma^\top A_{z,\eta} \sigma$$

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$||E||_p$

امید روی σ و η

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$$||E||_p = ||E - \mathbb{E}E||_p$$

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$\mathbb{E}E = 0$

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امید روی σ

امید روی η

$$\leq \|\sqrt{p}\cdot\|A_{z,\eta}\|_F + p\cdot\|A_{z,\eta}\|\|_p$$

امید روی η

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$$\leq \sqrt{p}\cdot\|\|A_{z,\eta}\|_F\|_p + p\cdot\|\|A_{z,\eta}\|\|_p$$

نامساوی مثلثی

$$||E||_p \leq \sqrt{p} \cdot \|\|A_{z,\eta}\|_F\|_p + p \cdot \|\boxed{\|A_{z,\eta}\|}\|_p$$

$A_{z,\eta}$ =?

$$\frac{1}{s}\sum_{r=1}^{m}\sum_{i\neq j}\eta_{r,i}\eta_{r,j}\sigma_{r,i}\sigma_{r,j}z_i z_j \stackrel{\text{def}}{=} \sigma^\top A_{z,\eta}\sigma$$

r,i r',j

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بلاک r

$A_{z,\eta}$ =?

	0	0	0
0		0	0
0	0	$(1/s)z^{(r)}(z^{(r)})^\top$	0
0	0	0	

حذف قطر

بلاک r

$$\frac{1}{s}\sum_{r=1}^{m}\sum_{i\neq j}\eta_{r,i}\eta_{r,j}\sigma_{r,i}\sigma_{r,j}z_i z_j \stackrel{\text{def}}{=} \sigma^\top A_{z,\eta}\sigma$$

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$$||\frac{1}{s}(zz^T - D)||$$

بلاک r

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حذف قطر

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$$||\frac{1}{s}(zz^T - D)||$$

$$\leq \max\{\frac{1}{s}||zz^T||, \frac{1}{s}||D||\}$$

هر سه
ماتریس مثبت
نیمه‌معین

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هر سه ماتریس مثبت نیمه‌معین

$||z||_\infty^2 \leq 1$

$$||E||_p \leq \sqrt{p} \cdot \| \|A_{z,\eta}\|_F \|_p + p \cdot \| \boxed{\|A_{z,\eta}\|} \|_p$$

بلاک r

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	0	0	0
0		0	0
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هر سه ماتریس مثبت نیمه‌معین

1

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حذف قطر

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هر سه ماتریس مثبت نیمه‌معین

1

$||z||_\infty^2 \leq 1$

$$\leq \frac{1}{s}$$

$$||E||_p \leq \sqrt{p} \cdot \boxed{\|\|A_{z,\eta}\|_F\|_p} + p \cdot \|\|A_{z,\eta}\|\|_p$$

$\leq 1/s$

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$$\|\|A_{z,\eta}\|_F\|_p = \|\|A_{z,\eta}\|_F^2\|_{p/2}^{1/2}$$

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$$\|\|A_{z,\eta}\|_F\|_p = \|\|A_{z,\eta}\|_F^2\|_{p/2}^{1/2}$$

$$Q_{i,j} = \sum_{r=1}^{m}\eta_{r,i}\eta_{r,j}$$

$$||E||_p \leq \sqrt{p} \cdot \boxed{\|\|A_{z,\eta}\|_F\|_p} + p \cdot \|\|A_{z,\eta}\|\|_p$$

$\leq 1/s$

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$$\leq \frac{1}{s}\left(\sum_{i\neq j} z_i^2 z_j^2 \cdot \|Q_{i,j}\|_p\right)^{1/2}$$

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Theorem 35 (Bernstein's inequality). *Let $X_1, \ldots, X_n$ be independent random variables that are each at most K almost surely, and where*

Then for all $p \ge 1$

σ^2: واریانس X

$$\|\sum_{i=1}^{n} X_i - \mathbb{E}\sum_{i} X_i\|_p \lesssim \sigma\sqrt{p} + Kp.$$

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$$\sigma\sqrt{p} + Kp \le p + p \lesssim p$$

$$||Q_{i,j} - EQ_{i,j}||_p \le \frac{s}{\sqrt{m}} \cdot \frac{s}{\sqrt{m}} + \frac{s^2}{m} = 2\frac{s^2}{m}$$

$$p \simeq s^2/m$$

$$\|Q_{i,j}\|_p \lesssim p$$

$$\le \frac{1}{\sqrt{m}}$$

$$\|\|A_{z,\eta}\|_F\|_p \le \frac{1}{s}\left(\sum_{i\neq j} z_i^2 z_j^2 \cdot \|Q_{i,j}\|_p\right)^{1/2}$$

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$$||Q_{i,j} - EQ_{i,j}||_p \le \frac{s}{\sqrt{m}} \cdot \frac{s}{\sqrt{m}} + \frac{s^2}{m} = 2\frac{s^2}{m}$$

$$||Q_{i,j}||_p \le 2\frac{s^2}{m} + ||EQ_{i,j}||_p \approx p$$

$$p \simeq s^2/m$$

$$\|Q_{i,j}\|_p \lesssim p$$

$$\le \frac{1}{\sqrt{m}}$$

$$||E||_p \leq \sqrt{p} \cdot \underbrace{\|\|A_{z,\eta}\|_F\|_p}_{\leq 1/\sqrt{m}} + p \cdot \underbrace{\|\|A_{z,\eta}\|\|_p}_{\leq 1/s}$$

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$$\boxed{|E|}$$

$$\mathbb{P}(|\|\Pi x\|_2^2 - 1| > \varepsilon)$$

$$||E||_p \leq \sqrt{p} \cdot \underbrace{\|\|A_{z,\eta}\|_F\|_p}_{\leq 1/\sqrt{m}} + p \cdot \underbrace{\|\|A_{z,\eta}\|\|_p}_{\leq 1/s}$$

$$\boxed{p \simeq s^2/m} \quad \leq s/m + s/m \lesssim s/m$$

$$\mathbb{P}(|\|\Pi x\|_2^2 - 1| > \varepsilon) \quad = P[\,|\overbrace{E}^{|\mathrm{E}|}|^p > \epsilon^p]$$

$$||E||_p \leq \sqrt{p} \cdot \underbrace{\|\|A_{z,\eta}\|_F\|_p}_{\leq 1/\sqrt{m}} + p \cdot \underbrace{\|\|A_{z,\eta}\|\|_p}_{\leq 1/s}$$

$$\boxed{p \simeq s^2/m} \leq s/m + s/m \lesssim s/m$$

$$\mathbb{P}(|\|\Pi x\|_2^2 - 1| > \varepsilon) \overset{|E|}{=} P[\,|E|^p > \epsilon^p] \overset{\text{ماركوف}}{<} \epsilon^{-p}\, \mathbb{E}[\,|E|^p]$$