بسم الله الرحمن الرحیم

نظریه علوم کامپیوتر

نظریه علوم کامپیوتر - بهار ۱۴۰۰-۱۴۰۱ - جلسه دهم: پیچیدگی محاسبات

Theory of computation - 002 - S10 - computational complexity

Intro to Complexity Theory

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Defn: $f(n)$ is $O\big(g(n)\big)$ if $f(n) \leq cg(n)$ for some fixed c independent of n.

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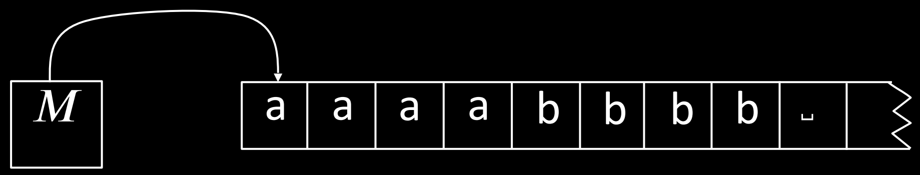

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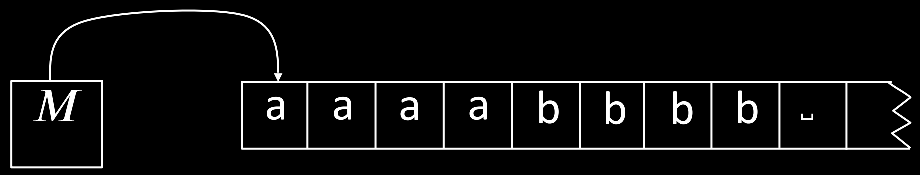

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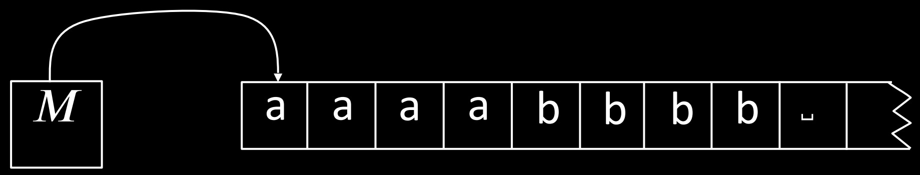

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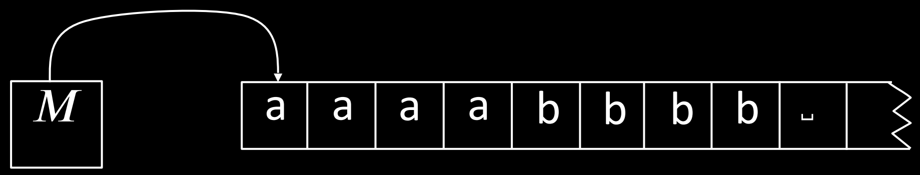

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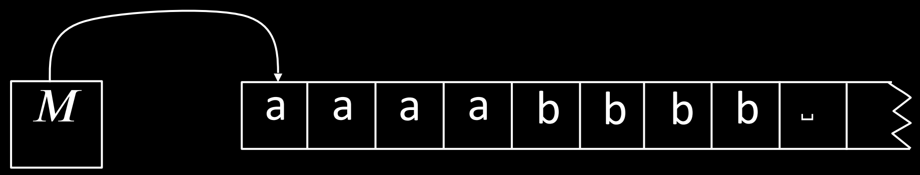

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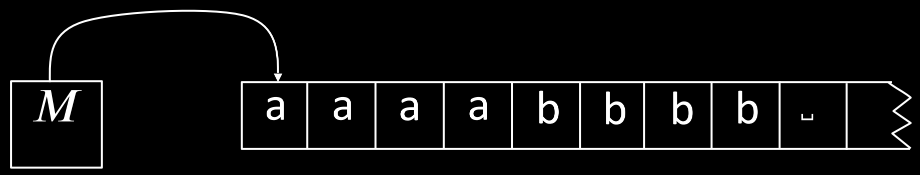

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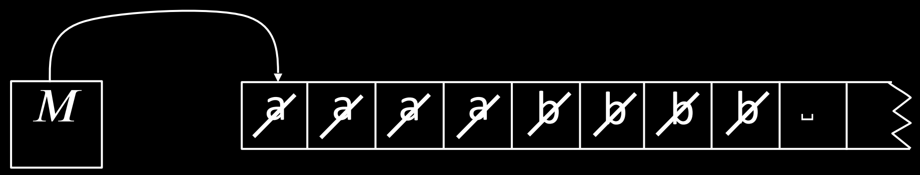

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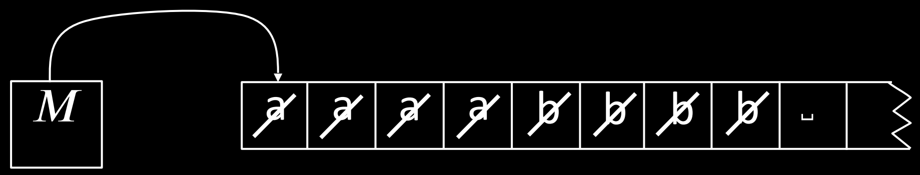

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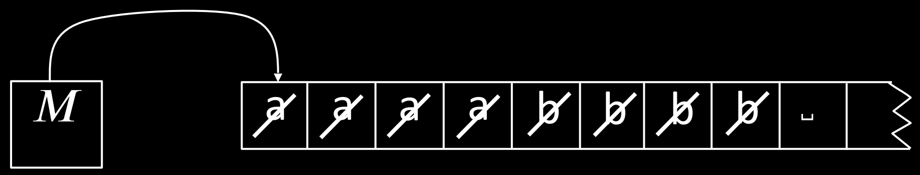

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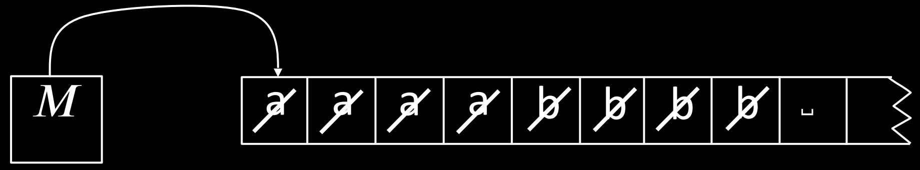

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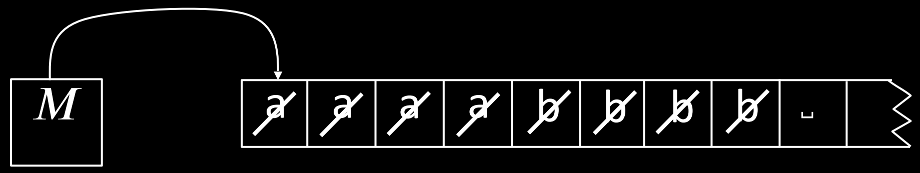

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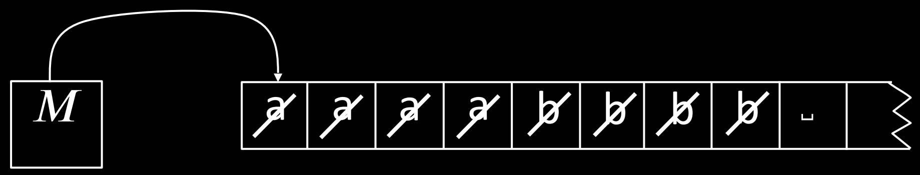

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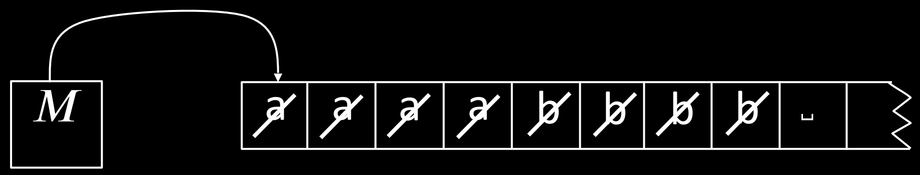

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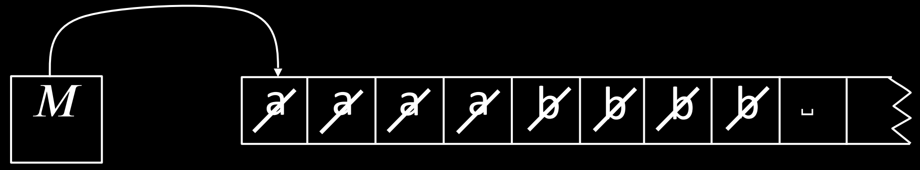

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Check-in 12.1

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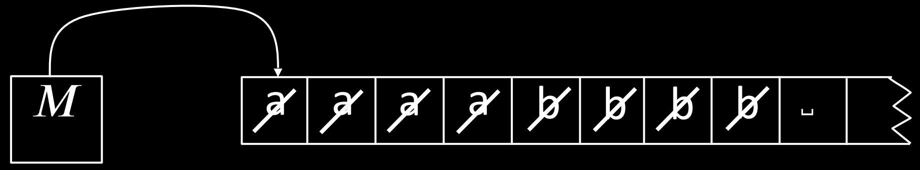

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How much improvement is possible in the bound for this theorem about 1-tape TMs deciding A?

(a) $O(n^2)$ is best possible.

(b) $O(n \log n)$ is possible.

(c) $O(n)$ is possible.

Check-in 12.1

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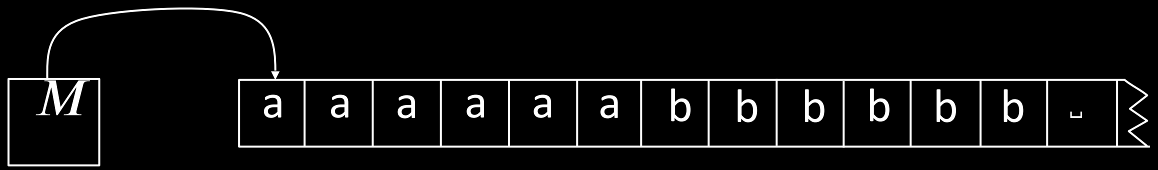

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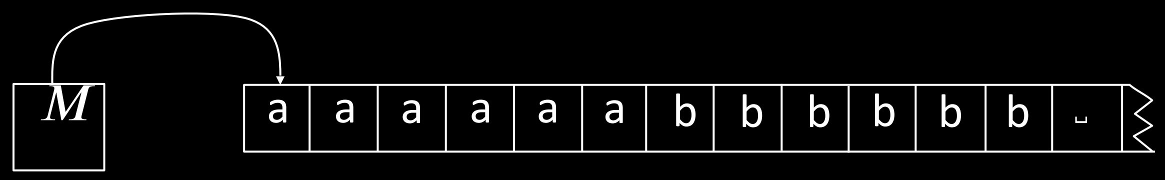

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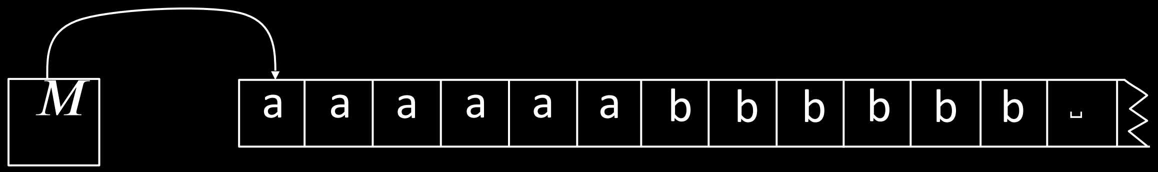

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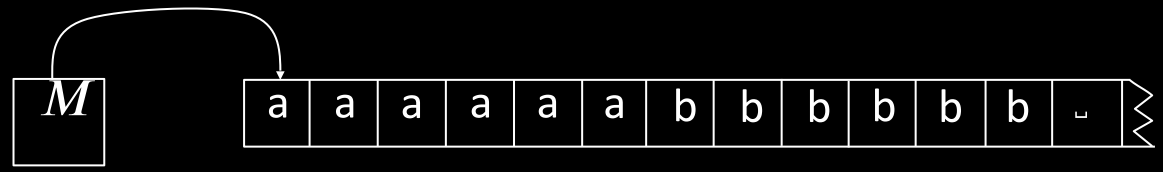
1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
2. Repeat until all crossed off.
   Scan tape, crossing off every other a and b.



Deciding $A = \left\{ \mathrm{a}^k \mathrm{b}^k \,\middle|\, k \geq 0 \right\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

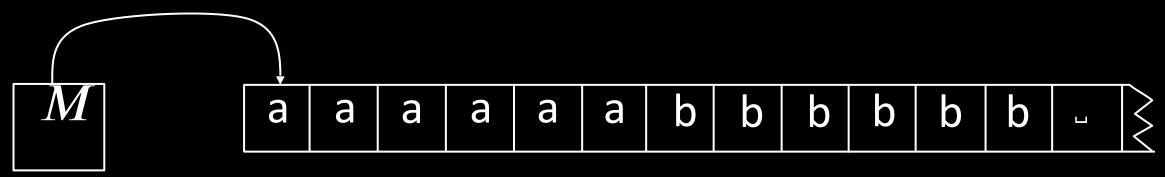
Proof:

$M =$ "On input w

1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
2. Repeat until all crossed off.

    Scan tape, crossing off every other a and b.

    Reject if even/odd parities disagree.



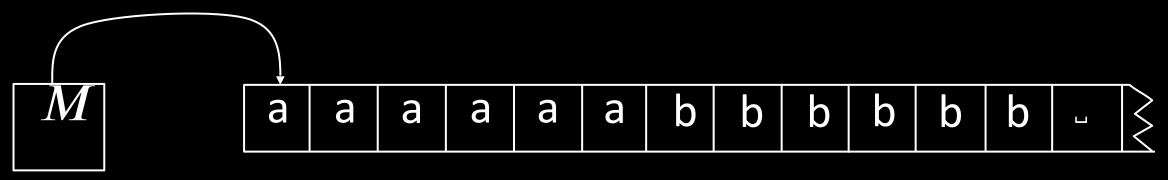
Deciding $A = \left\{ a^k b^k \,\middle|\, k \geq 0 \right\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in a^*b^*$. *Reject* if not.
2. Repeat until all crossed off.
   Scan tape, crossing off every other a and b.
   Reject if even/odd parities disagree.
3. Accept if all crossed off. "



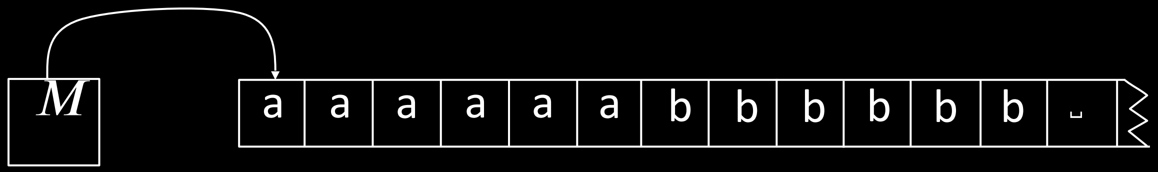
Deciding $A = \{a^k b^k \mid k \geq 0\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in a^*b^*$. *Reject* if not.
2. Repeat until all crossed off.
   Scan tape, crossing off every other a and b.
   Reject if even/odd parities disagree.
3. Accept if all crossed off. "



	Parities
a's	
b's	

Deciding $A = \{\mathrm{a}^k\mathrm{b}^k \mid k \geq 0\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

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$M =$ "On input w

1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
2. Repeat until all crossed off.
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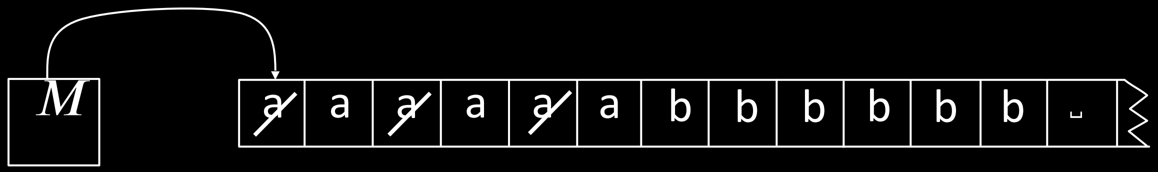
3. Accept if all crossed off. "



	Parities
a's	
b's	

Deciding $A = \left\{ \mathrm{a}^k \mathrm{b}^k \,\middle|\, k \geq 0 \right\}$ faster

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2. Repeat until all crossed off.
   Scan tape, crossing off every other a and b.
   Reject if even/odd parities disagree.
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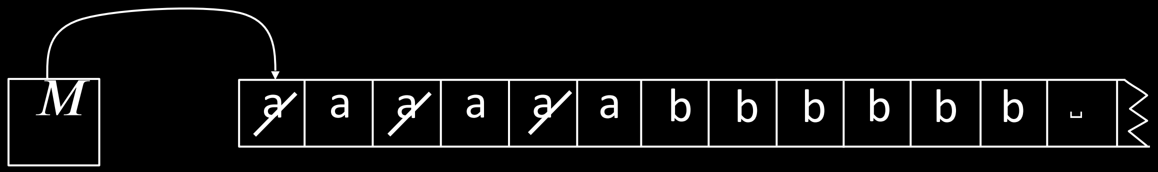


	Parities
a's	even (6)
b's	

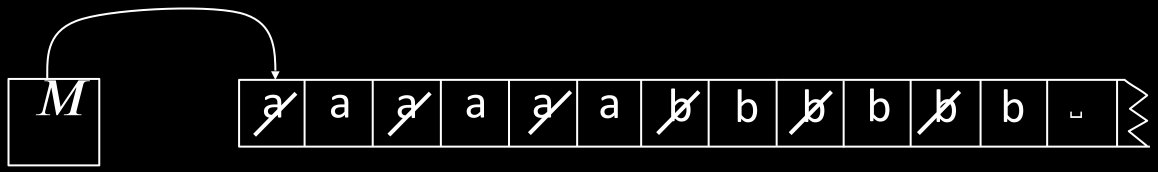
Deciding $A = \{a^k b^k \mid k \geq 0\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in a^*b^*$. *Reject* if not.
2. Repeat until all crossed off.
   Scan tape, crossing off every other a and b.
   Reject if even/odd parities disagree.
3. Accept if all crossed off. "



	Parities
a's	even (6)
b's	

Deciding $A = \left\{ \mathrm{a}^k \mathrm{b}^k \,\middle|\, k \geq 0 \right\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
2. Repeat until all crossed off.
   Scan tape, crossing off every other a and b.
   Reject if even/odd parities disagree.
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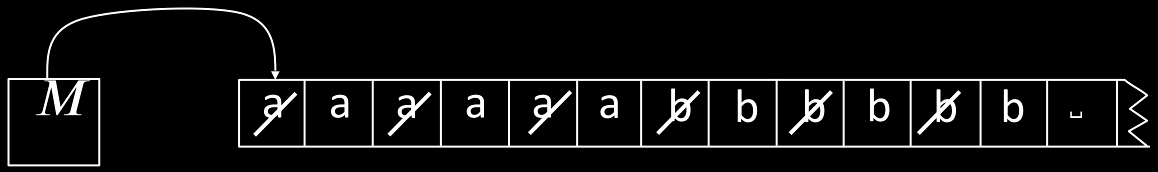

	Parities
a's	even (6)
b's	even (6)

Deciding $A = \left\{ \mathrm{a}^k \mathrm{b}^k \,\middle|\, k \geq 0 \right\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
2. Repeat until all crossed off.
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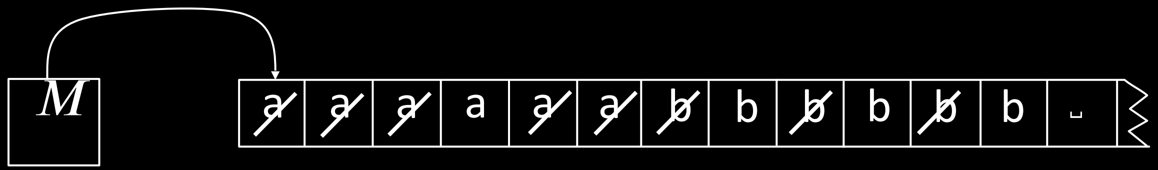


	Parities
a's	even (6)
b's	even (6)

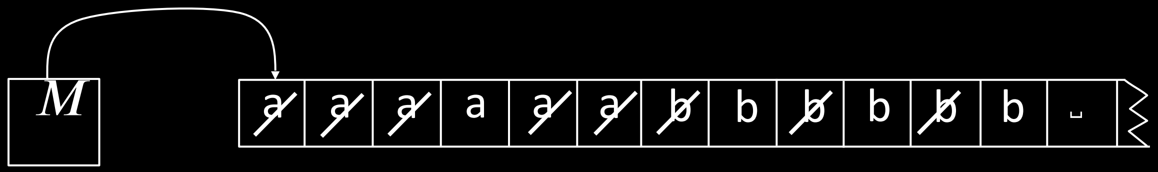
Deciding $A = \left\{ \mathrm{a}^k\mathrm{b}^k \,\middle|\, k \geq 0 \right\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
2. Repeat until all crossed off.
   Scan tape, crossing off every other a and b.
   Reject if even/odd parities disagree.
3. Accept if all crossed off. "



	Parities	
a's	even (6)	odd (3)
b's	even (6)	

Deciding $A = \left\{ \mathrm{a}^k \mathrm{b}^k \,\middle|\, k \geq 0 \right\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
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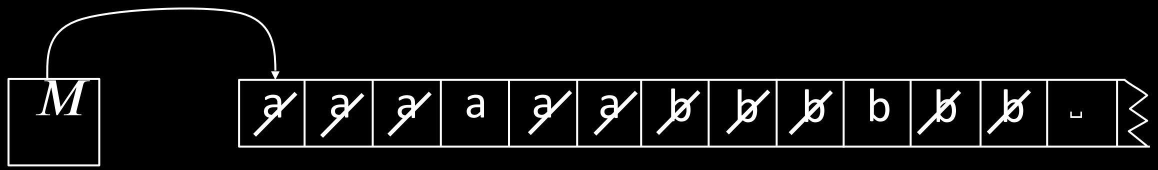

	Parities	
a's	even (6)	odd (3)
b's	even (6)	

Deciding $A = \left\{ \mathrm{a}^k \mathrm{b}^k \middle| \; k \geq 0 \right\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
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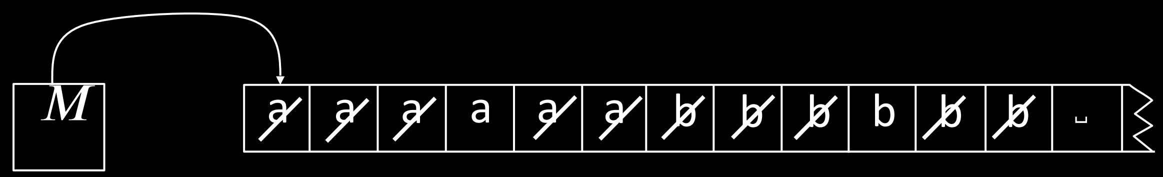

	Parities	
a's	even (6)	odd (3)
b's	even (6)	odd (3)

Deciding $A = \left\{ \mathrm{a}^k \mathrm{b}^k \,\middle|\, k \geq 0 \right\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
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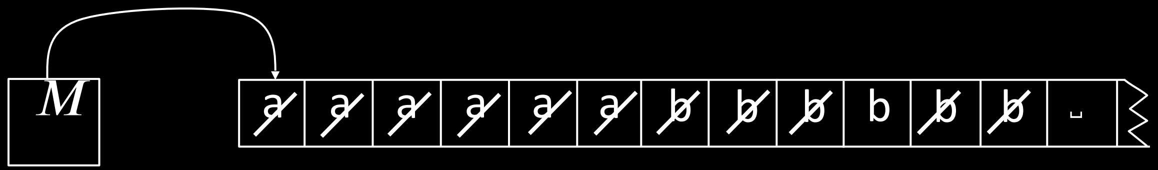

	Parities	
a's	even (6)	odd (3)
b's	even (6)	odd (3)

Deciding $A = \{a^k b^k \mid k \geq 0\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in a^*b^*$. *Reject* if not.
2. Repeat until all crossed off.

    Scan tape, crossing off every other a and b.

    Reject if even/odd parities disagree.
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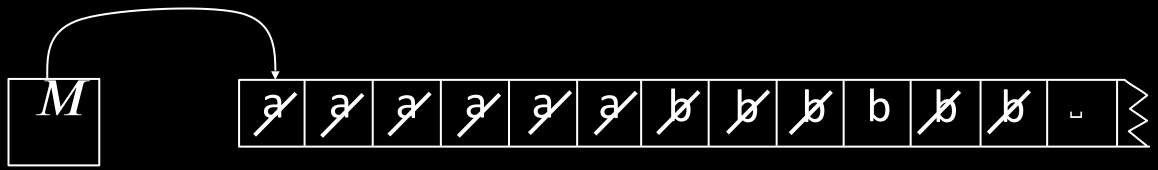

	Parities		
a's	even (6)	odd (3)	odd (1)
b's	even (6)	odd (3)	

Deciding $A = \{a^k b^k \mid k \geq 0\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in a^*b^*$. *Reject* if not.
2. Repeat until all crossed off.
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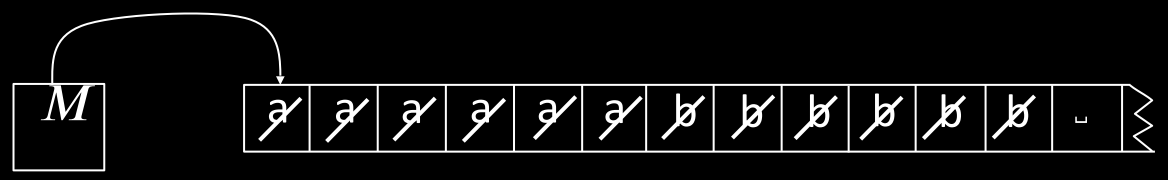

	Parities		
a's	even (6)	odd (3)	odd (1)
b's	even (6)	odd (3)	

Deciding $A = \left\{ \mathrm{a}^k \mathrm{b}^k \,\middle|\, k \geq 0 \right\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
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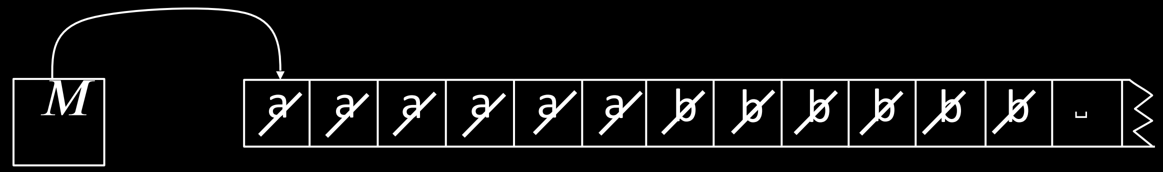


	Parities		
a's	even (6)	odd (3)	odd (1)
b's	even (6)	odd (3)	odd (1)

Deciding $A = \left\{ \mathrm{a}^k \mathrm{b}^k \,\middle|\, k \geq 0 \right\}$ faster

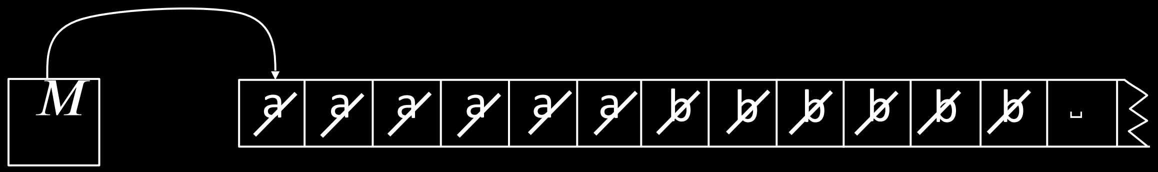
Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
2. Repeat until all crossed off.
   Scan tape, crossing off every other a and b.
   Reject if even/odd parities disagree.
3. Accept if all crossed off. "

Analysis:



	Parities		
a's	even (6)	odd (3)	odd (1)
b's	even (6)	odd (3)	odd (1)

Deciding $A = \left\{ \mathrm{a}^k\mathrm{b}^k \middle|\ k \geq 0 \right\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
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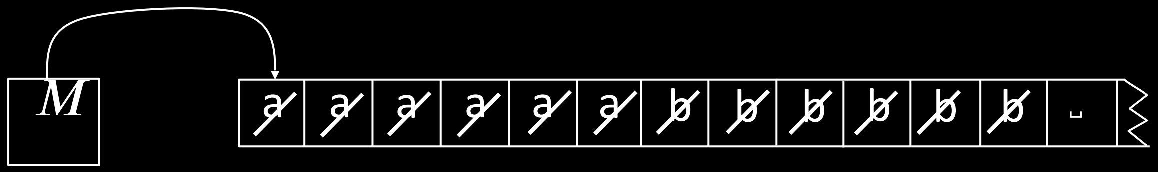
   Scan tape, crossing off every other a and b.
   Reject if even/odd parities disagree.
3. Accept if all crossed off. "

Analysis:

$O(n)$ steps



	Parities		
a's	even (6)	odd (3)	odd (1)
b's	even (6)	odd (3)	odd (1)

Deciding $A = \{ a^k b^k \mid k \geq 0 \}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

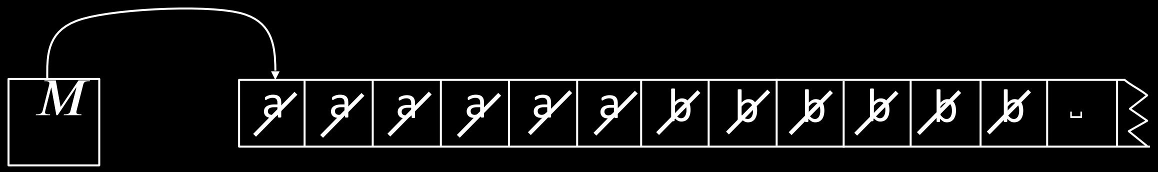
$M =$ "On input w

1. Scan tape to check if $w \in a^*b^*$. *Reject* if not.
2. Repeat until all crossed off.
   Scan tape, crossing off every other a and b.
   Reject if even/odd parities disagree.
3. Accept if all crossed off. "

Analysis:

$O(n)$ steps

$+O(\log n)$ iterations



	Parities		
a's	even (6)	odd (3)	odd (1)
b's	even (6)	odd (3)	odd (1)

Deciding $A = \{ \mathrm{a}^k \mathrm{b}^k \mid k \geq 0 \}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

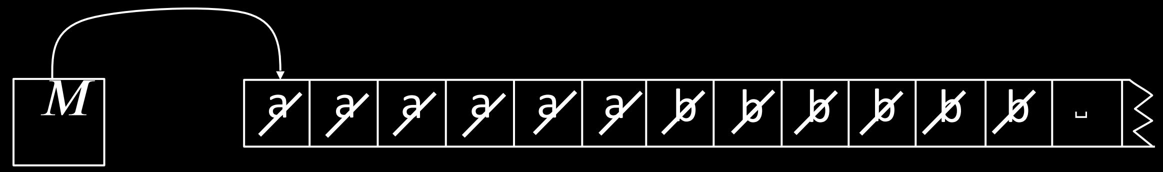
1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
2. Repeat until all crossed off.
   Scan tape, crossing off every other a and b.
   Reject if even/odd parities disagree.
3. Accept if all crossed off. "

Analysis:

$O(n)$ steps

$+O(\log n)$ iterations

$\times O(n)$ steps



	Parities		
a's	even (6)	odd (3)	odd (1)
b's	even (6)	odd (3)	odd (1)

Deciding $A = \{\mathrm{a}^k\mathrm{b}^k \mid k \geq 0\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
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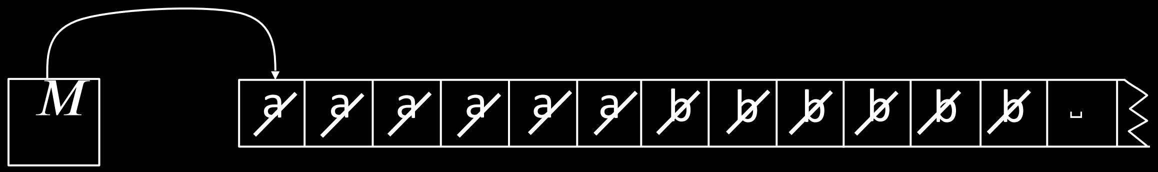
3. Accept if all crossed off. "

Analysis:

$O(n)$ steps

$+O(\log n)$ iterations

$\times O(n)$ steps



	Parities		
a's	even (6)	odd (3)	odd (1)
b's	even (6)	odd (3)	odd (1)

Deciding $A = \left\{ \text{a}^k\text{b}^k \,\middle|\, k \geq 0 \right\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in \text{a}^*\text{b}^*$. *Reject* if not.
2. Repeat until all crossed off.
   Scan tape, crossing off every other a and b.
   Reject if even/odd parities disagree.
3. Accept if all crossed off. "

Analysis:

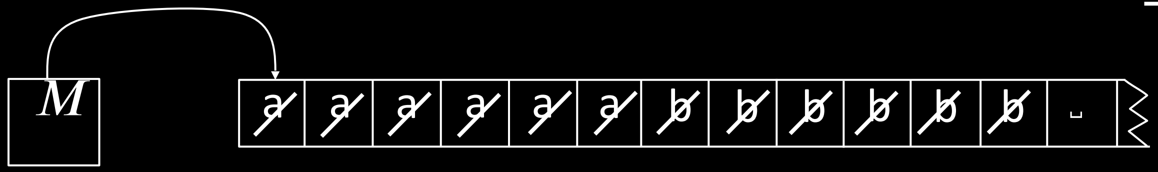
$O(n)$ steps

$+O(\log n)$ iterations

$\times O(n)$ steps

$O(n) + O(n\log n)$ steps

$= O(n\log n)$ steps



	Parities		
a's	even (6)	odd (3)	odd (1)
b's	even (6)	odd (3)	odd (1)

Deciding $A = \left\{ \mathrm{a}^k \mathrm{b}^k \,\middle|\, k \geq 0 \right\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
2. Repeat until all crossed off.
   Scan tape, crossing off every other a and b.
   Reject if even/odd parities disagree.
3. Accept if all crossed off. "

Analysis:

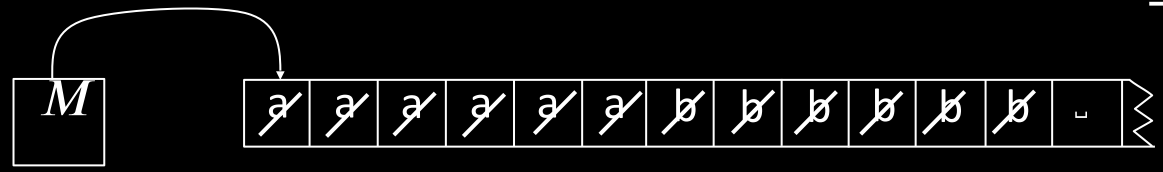
$O(n)$ steps

$+O(\log n)$ iterations

$\times\, O(n)$ steps

$O(n) + O(n\log n)$ steps

$= O(n\log n)$ steps



	Parities		
a's	even (6)	odd (3)	odd (1)
b's	even (6)	odd (3)	odd (1)

Further improvement? Not possible.

Deciding $A = \left\{ \mathrm{a}^k \mathrm{b}^k \,\middle|\, k \geq 0 \right\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
2. Repeat until all crossed off.
   Scan tape, crossing off every other a and b.
   Reject if even/odd parities disagree.
3. Accept if all crossed off. "

Analysis:

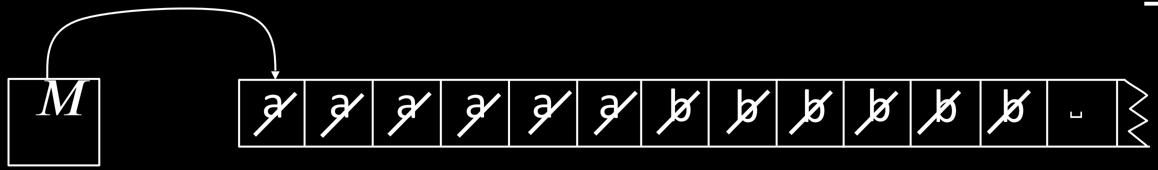
$O(n)$ steps

$+O(\log n)$ iterations

$\times\, O(n)$ steps

$O(n) + O(n\log n)$ steps

$= O(n\log n)$ steps



	Parities		
a's	even (6)	odd (3)	odd (1)
b's	even (6)	odd (3)	odd (1)

Further improvement? Not possible.

Theorem: A 1-tape TM M cannot decide A by using $o(n\log n)$ steps.

Deciding $A = \left\{ \mathrm{a}^k \mathrm{b}^k \,\middle|\, k \geq 0 \right\}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n\log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
2. Repeat until all crossed off.
   Scan tape, crossing off every other a and b.
   Reject if even/odd parities disagree.
3. Accept if all crossed off. "

Analysis:

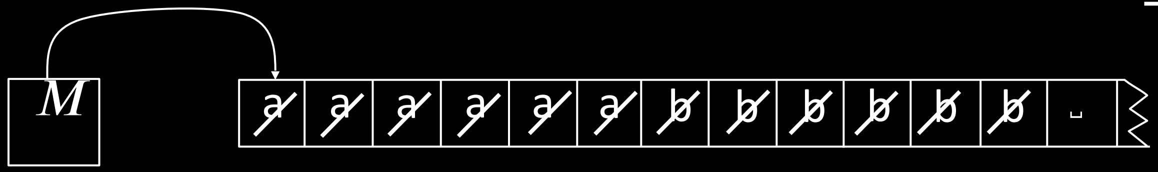
$O(n)$ steps

$+O(\log n)$ iterations

$\times\, O(n)$ steps

$O(n) + O(n\log n)$ steps

$= O(n\log n)$ steps



	Parities		
a's	even (6)	odd (3)	odd (1)
b's	even (6)	odd (3)	odd (1)

Further improvement? Not possible.

Theorem: A 1-tape TM M cannot decide A by using $o(n\log n)$ steps.

You are not responsible for knowing the proof.

Deciding $A = \{ \mathrm{a}^k \mathrm{b}^k \mid k \geq 0 \}$ faster

Theorem: A 1-tape TM M can decide A by using $O(n \log n)$ steps.

Proof:

$M =$ "On input w

1. Scan tape to check if $w \in \mathrm{a}^*\mathrm{b}^*$. *Reject* if not.
2. Repeat until all crossed off.
   Scan tape, crossing off every other a and b.
   Reject if even/odd parities disagree.
3. Accept if all crossed off. "

Analysis:

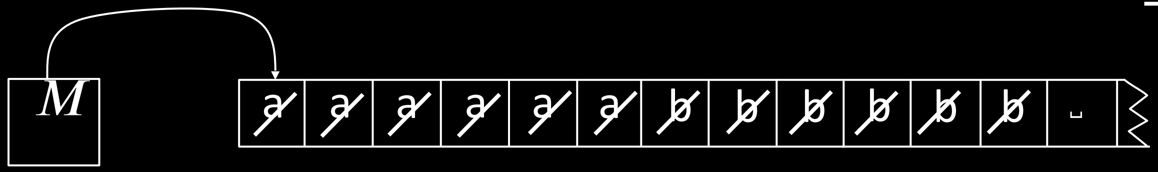
$O(n)$ steps

$+O(\log n)$ iterations

$\times O(n)$ steps

$O(n) + O(n \log n)$ steps

$= O(n \log n)$ steps



	Parities		
a's	even (6)	odd (3)	odd (1)
b's	even (6)	odd (3)	odd (1)

Further improvement? Not possible.

Theorem: A 1-tape TM M cannot decide A by using $o(n \log n)$ steps.

You are not responsible for knowing the proof.

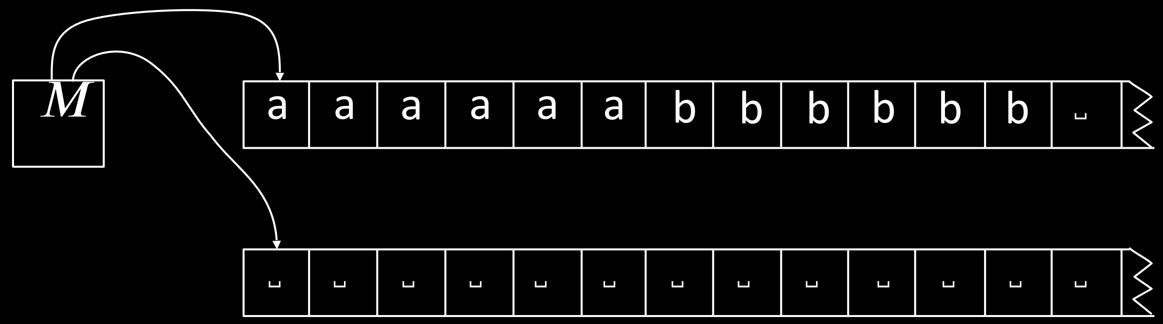
Deciding $A = \left\{ \mathrm{a}^k \mathrm{b}^k \,\middle|\, k \geq 0 \right\}$ even faster

Deciding $A = \{\mathrm{a}^k\mathrm{b}^k \mid k \geq 0\}$ even faster

Theorem: A multi-tape TM M can decide A using $O(n)$ steps.

Deciding $A = \{ \text{a}^k\text{b}^k \mid k \geq 0 \}$ even faster

Theorem: A multi-tape TM M can decide A using $O(n)$ steps.

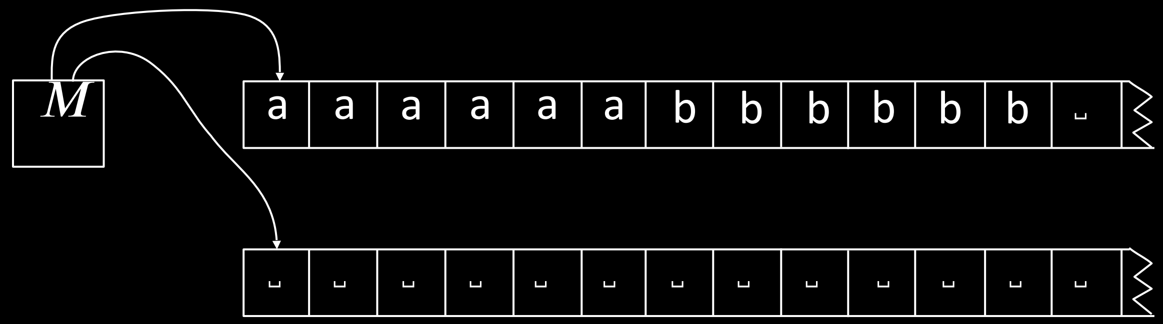


Deciding $A = \left\{ \mathrm{a}^k \mathrm{b}^k \,\middle|\, k \geq 0 \right\}$ even faster

Theorem: A multi-tape TM M can decide A using $O(n)$ steps.

$M =$ "On input w

1. Scan input to check if $w \in \mathrm{a}{*}\mathrm{b}^*$, *reject* if not.



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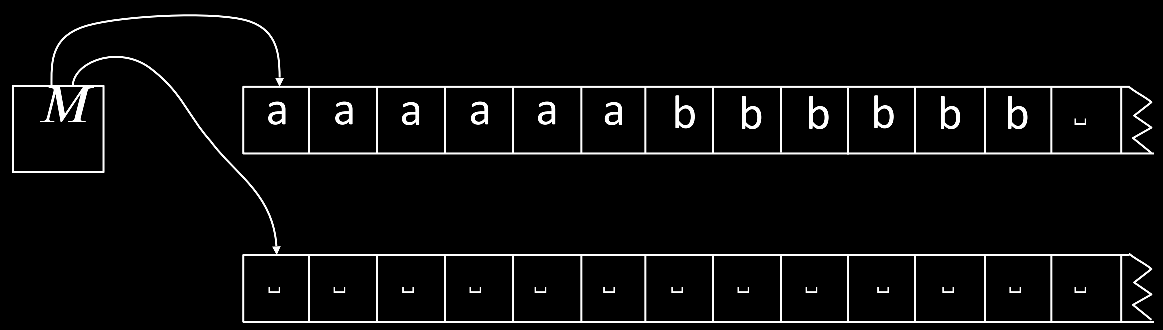

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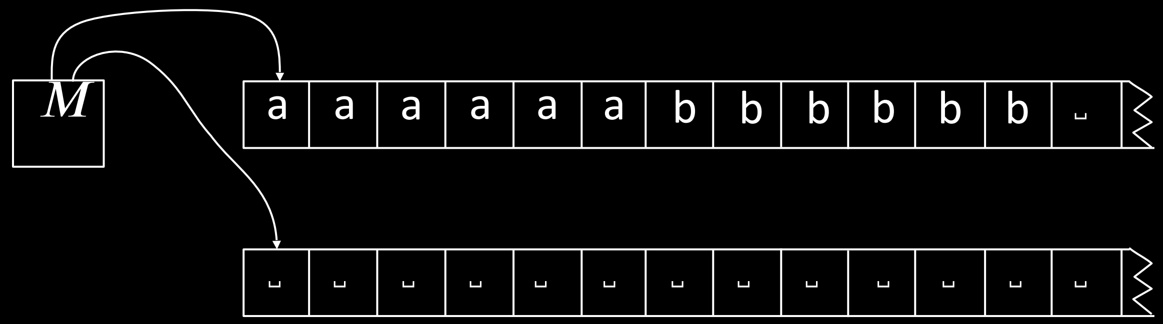

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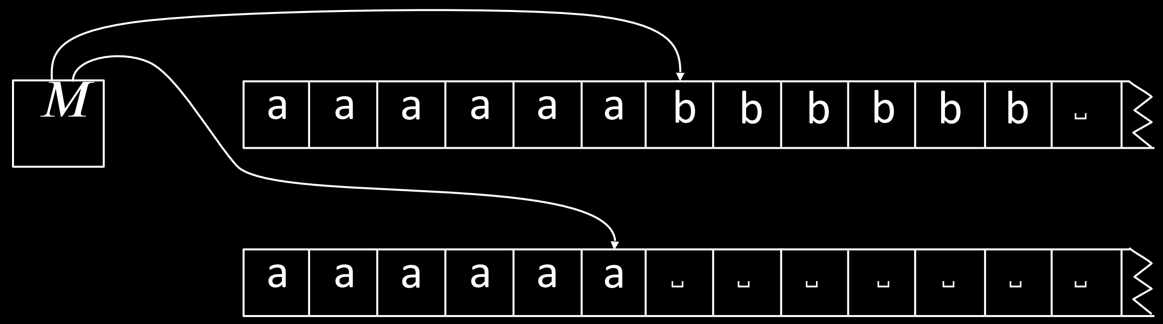

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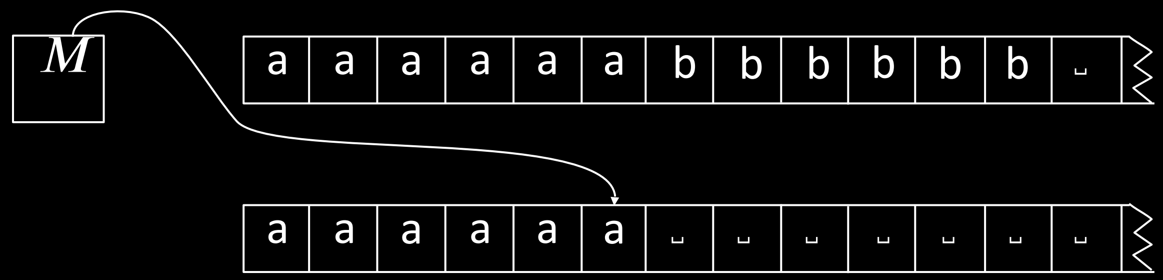

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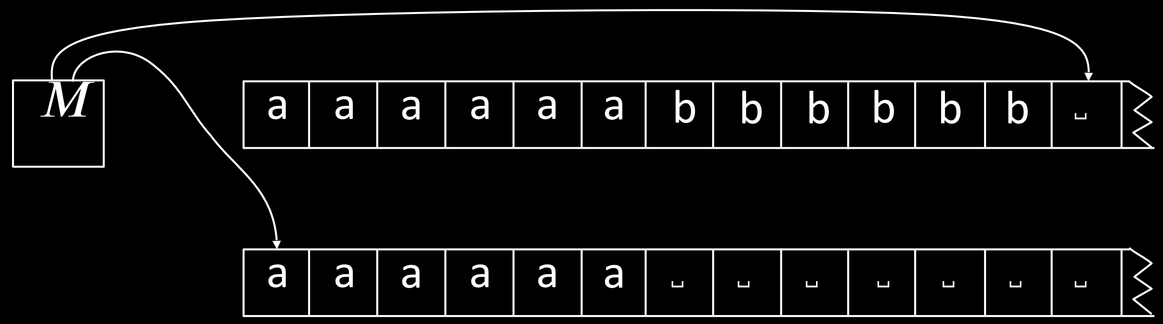

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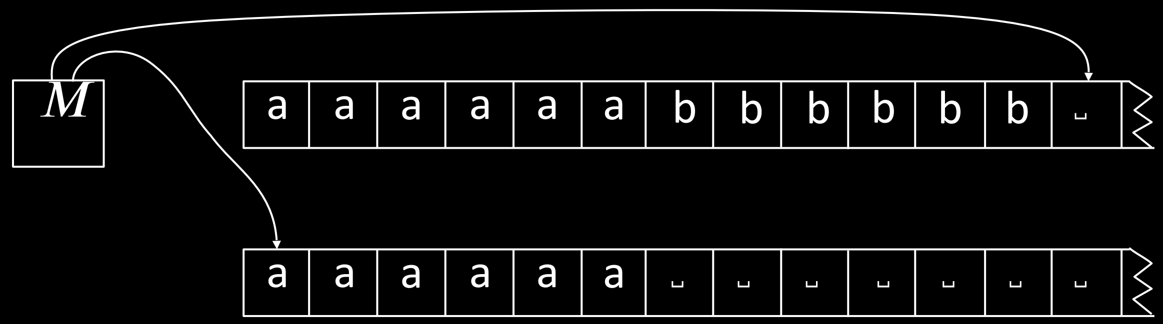

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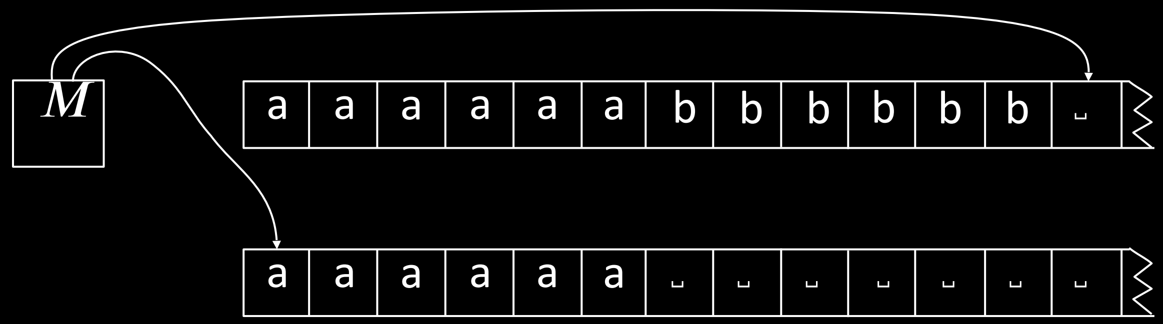

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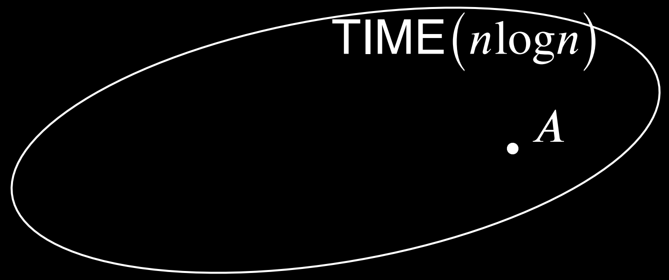

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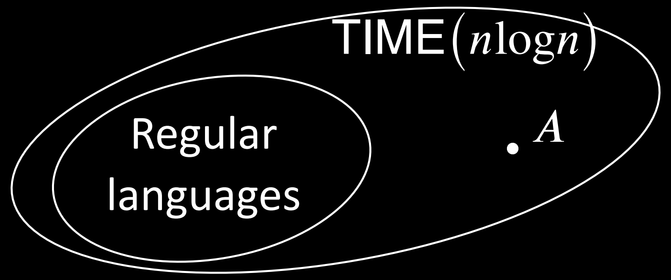

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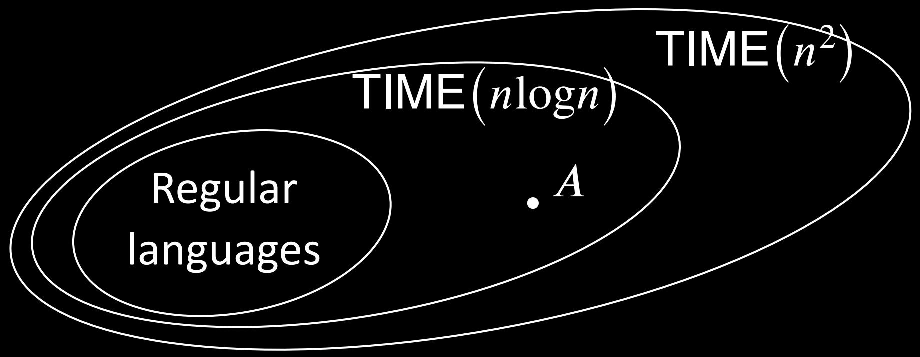

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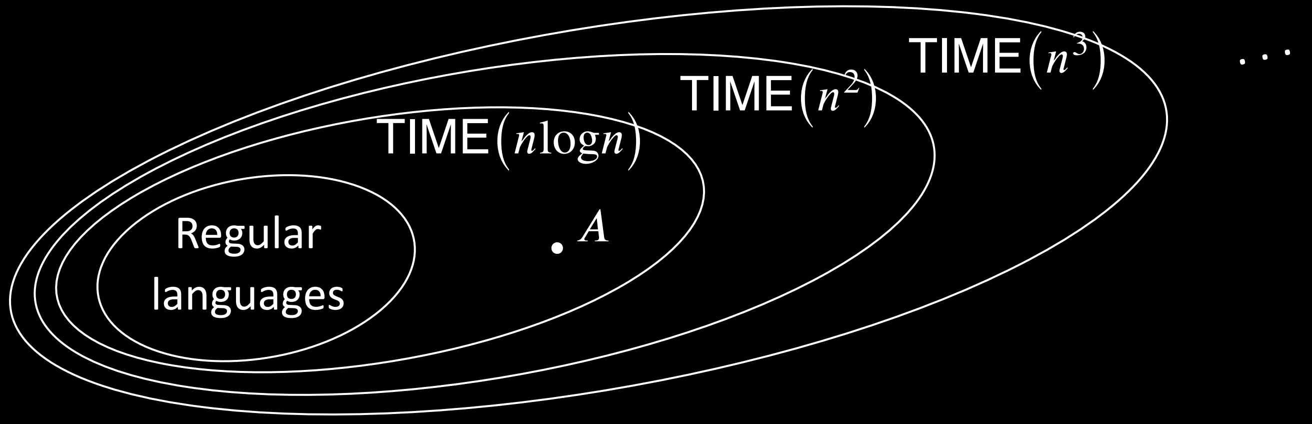

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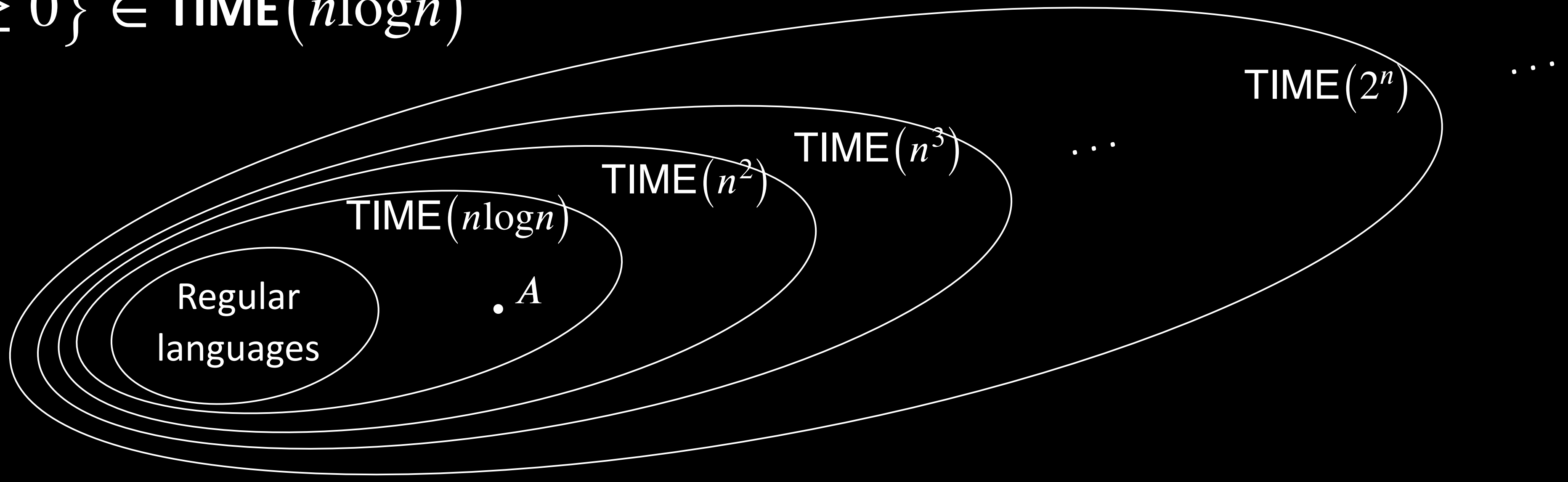

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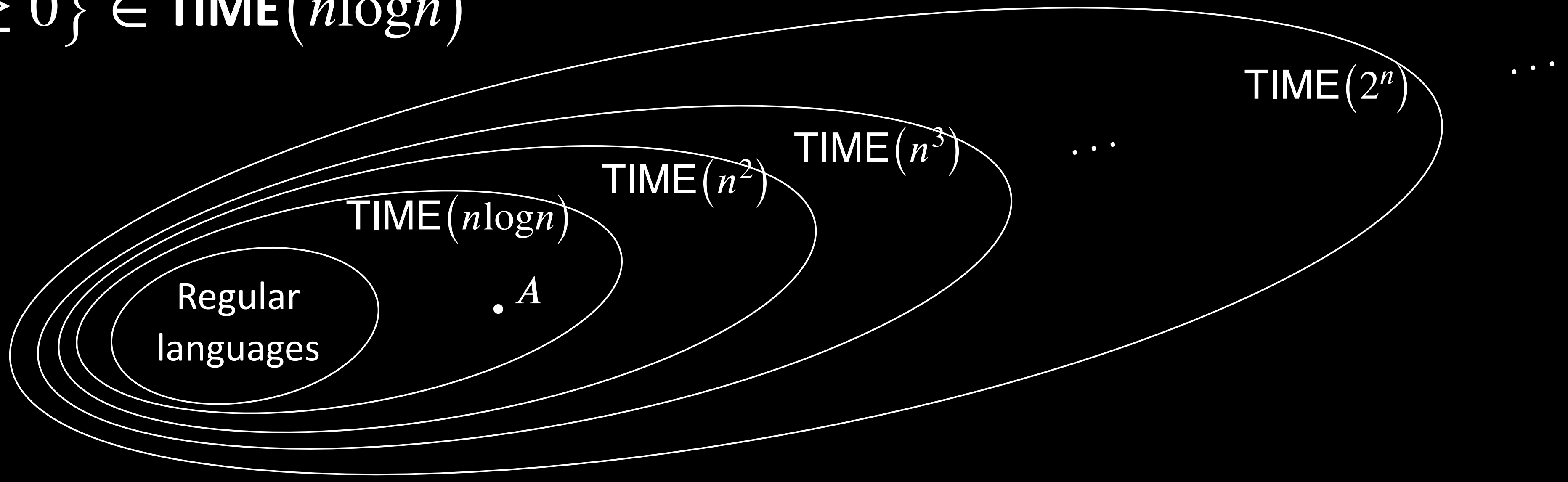

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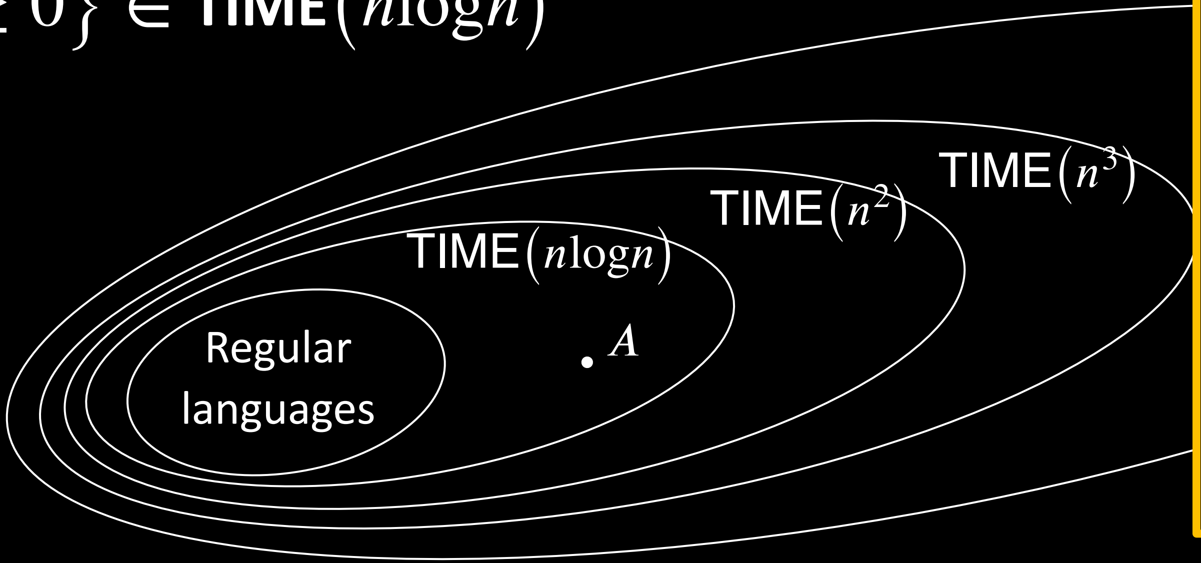

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Let

$B = \{\, ww^{\mathcal{R}} \mid w \in \{\mathrm{a}, \mathrm{b}\}^* \,\}$.

What is the smallest function t such that $B \in \text{TIME}\big(t(n)\big)$?

(a) $O(n)$

(b) $O\big(n\log n\big)$

(c) $O(n^2)$

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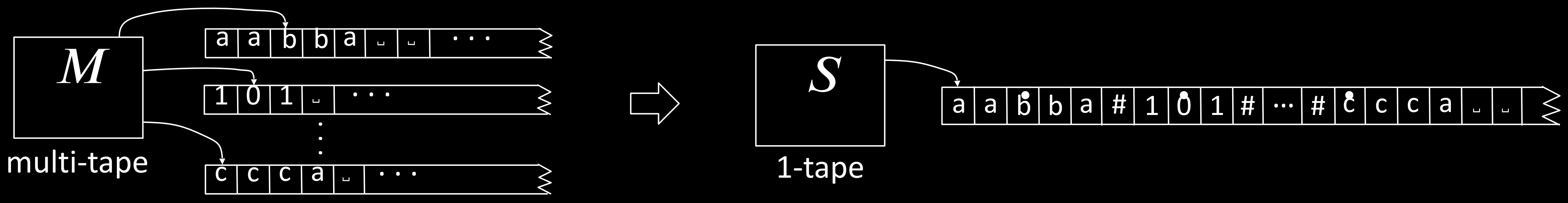

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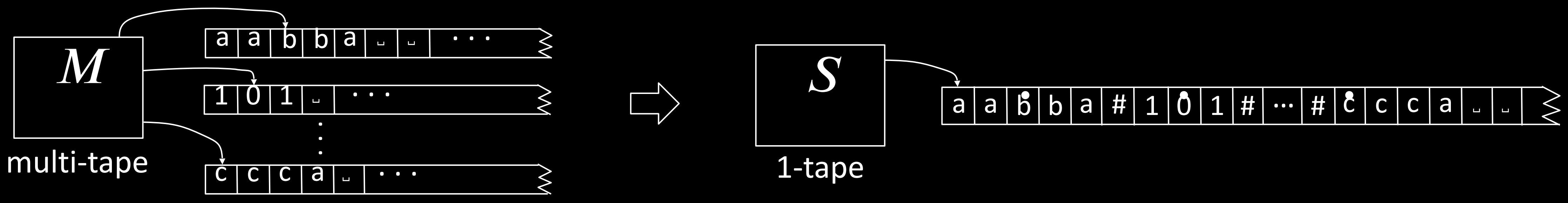

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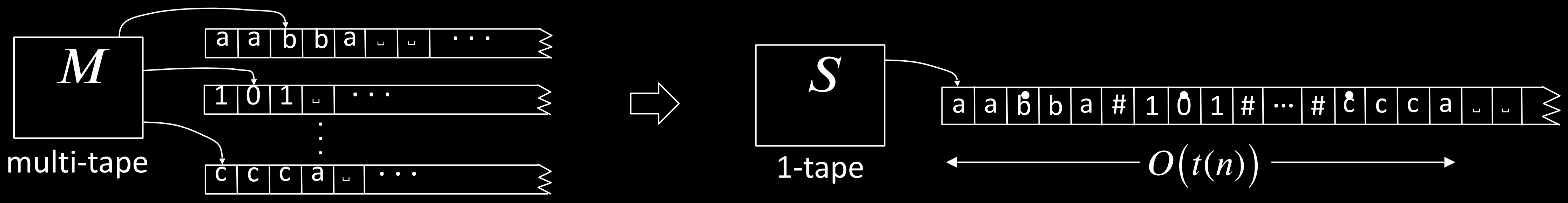

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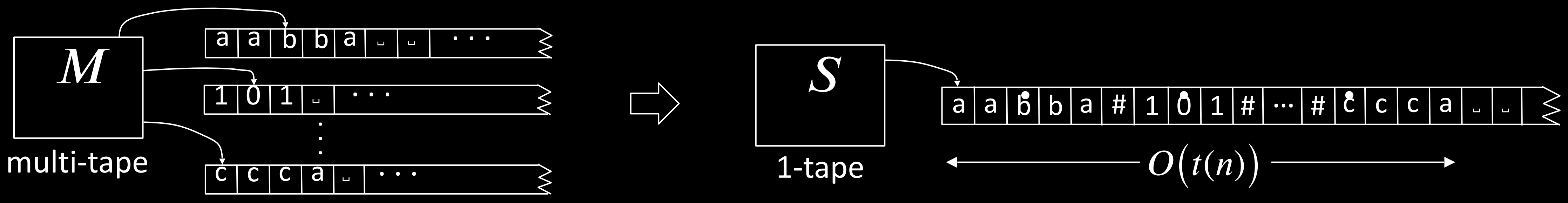

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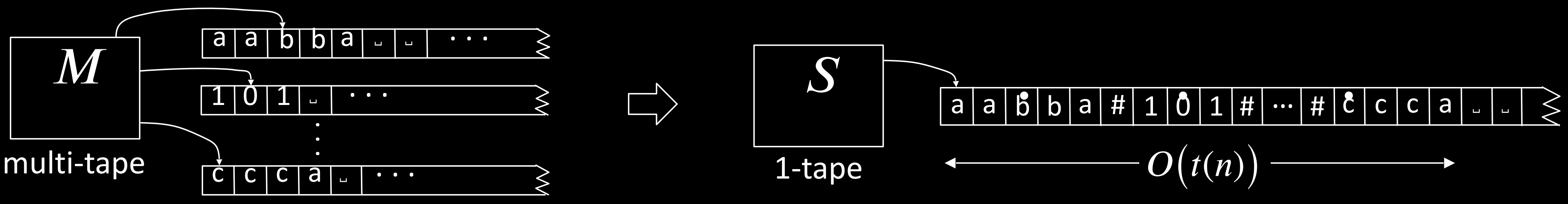

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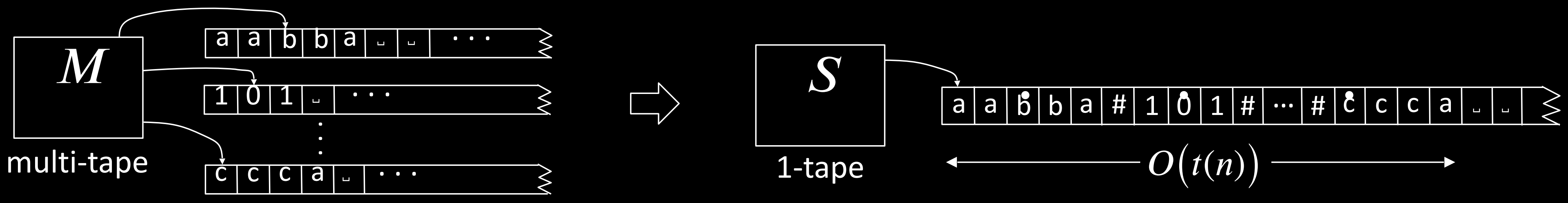

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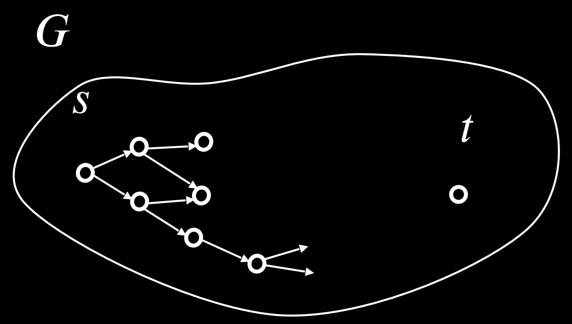

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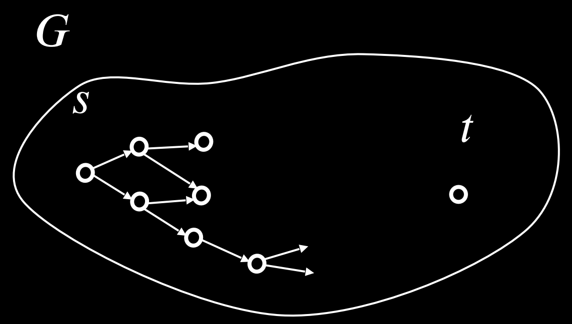

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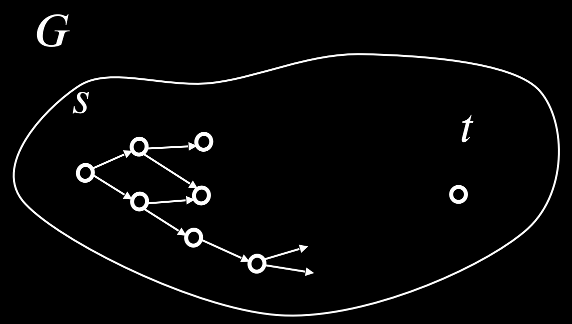

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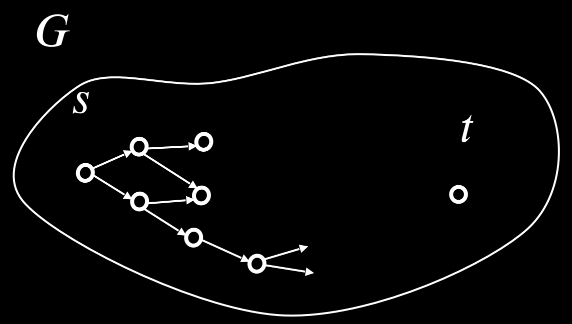
Defn: P $= \bigcup_k \text{TIME}(n^k)$

$=$ polynomial time decidable languages

- Invariant for all reasonable deterministic models
- Corresponds roughly to realistically solvable problems

Example: $PATH = \{\langle G, s, t \rangle \mid G$ is a directed graph with a path from s to t $\}$



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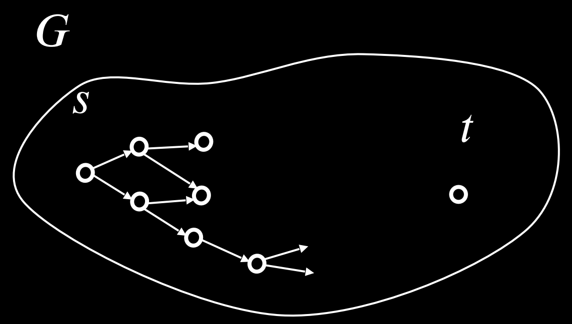

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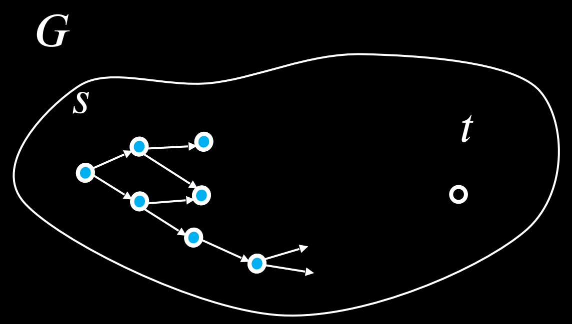

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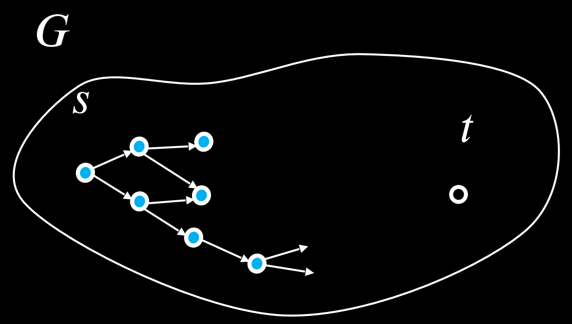

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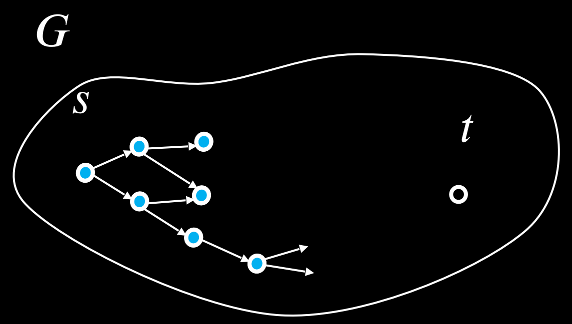

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Each stage should be clearly polynomial and the total number of steps polynomial.

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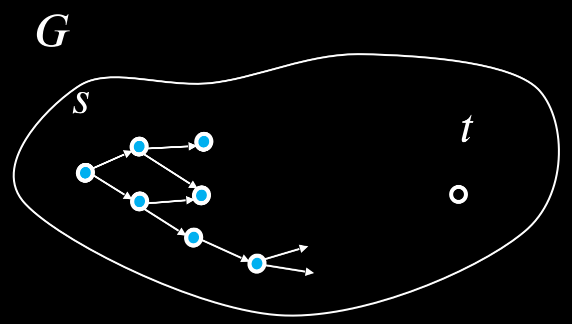

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PATH and *HAMPATH*

$PATH$ and $HAMPATH$

Example: $HAMPATH = \Big\{ \langle G, s, t \rangle \,\Big|\,$ G is a directed graph with a path from s to t and the path goes through every node of G $\Big\}$

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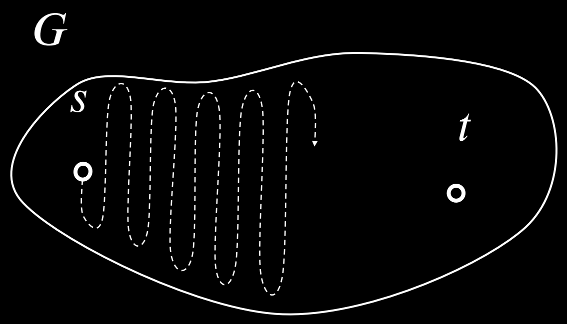

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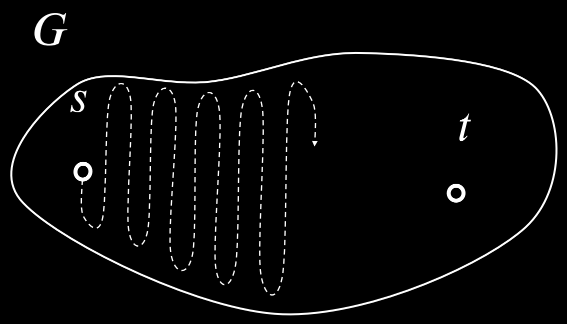

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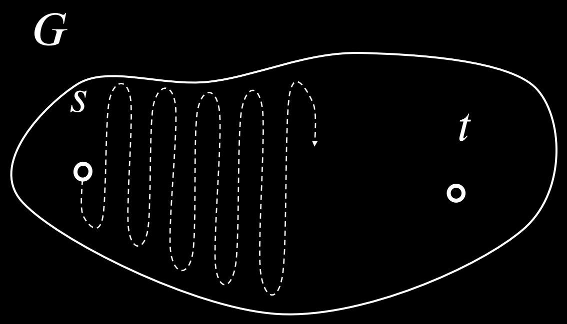

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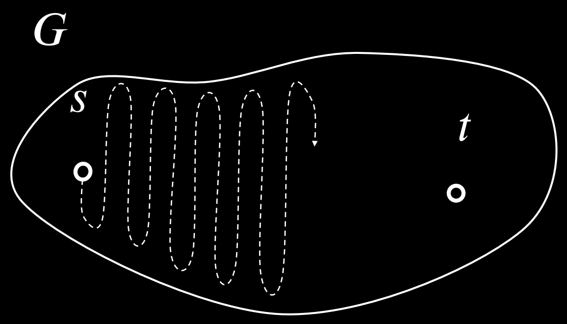

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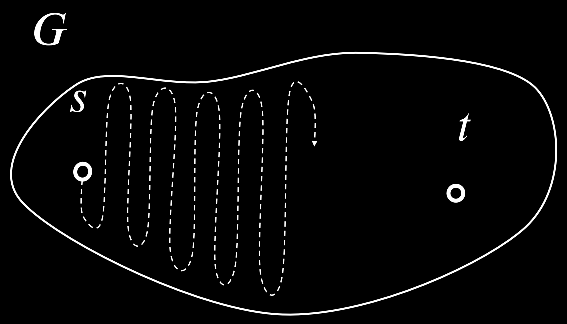

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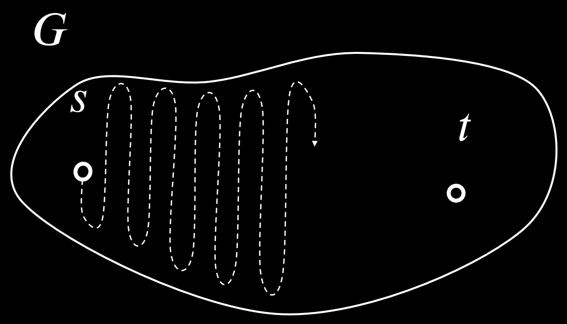

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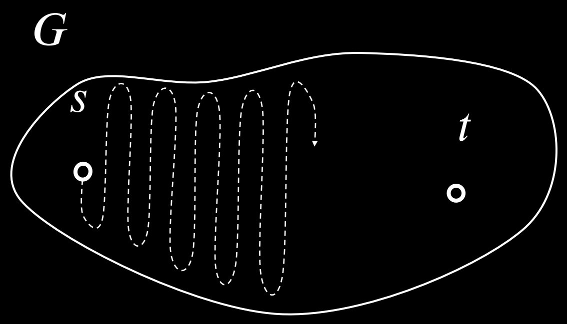

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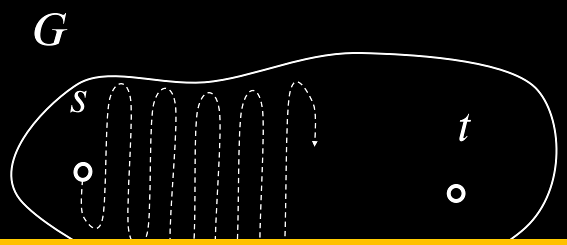

Check-in 12.3

Is $HAMPATH \in$ P ?

(a) Definitely Yes. You have a polynomial-time algorithm.

(b) Probably Yes. It should be similar to showing $PATH \in$ P.

(c) Toss up.

(d) Probably No. Hard to beat the exponential algorithm.

(e) Definitely No. You can prove it!

Quick Review

Defn: $\text{TIME}\big(t(n)\big) = \{B \mid$ some deterministic 1-tape TM M decides B and M runs in time $O\big(t(n)\big)\}$

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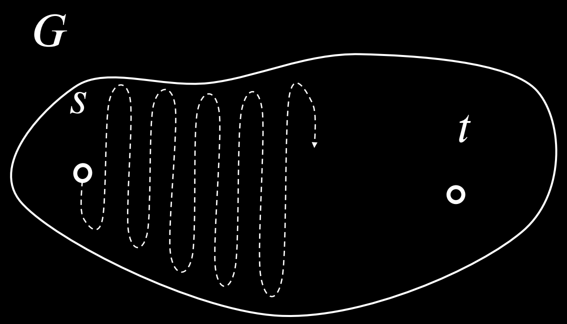

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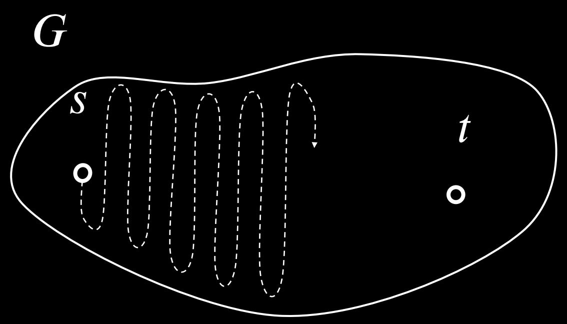

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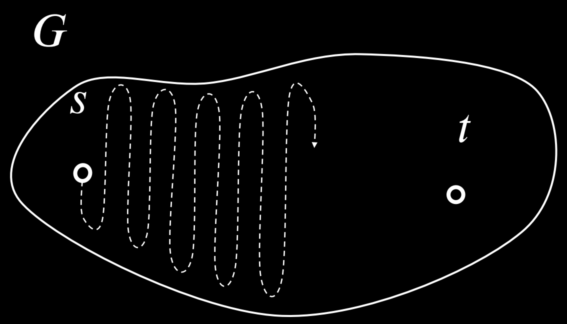

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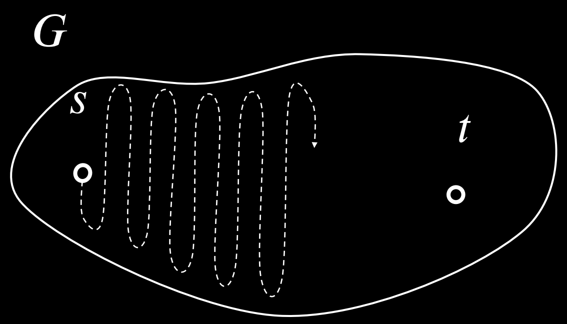

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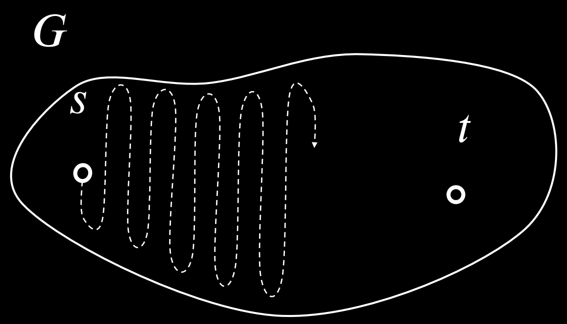

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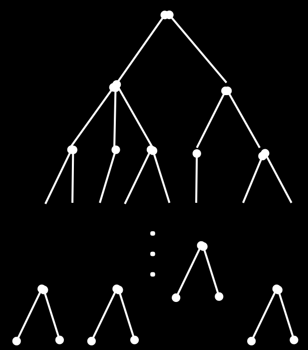

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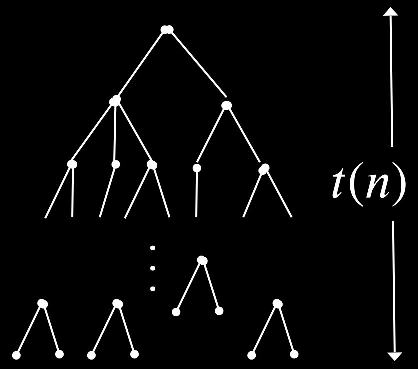

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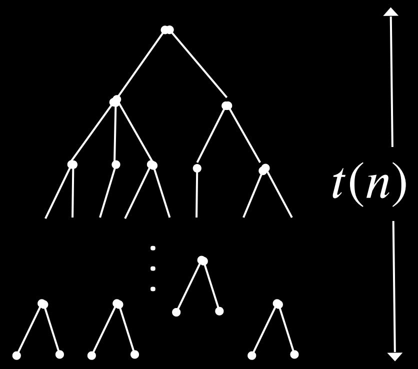

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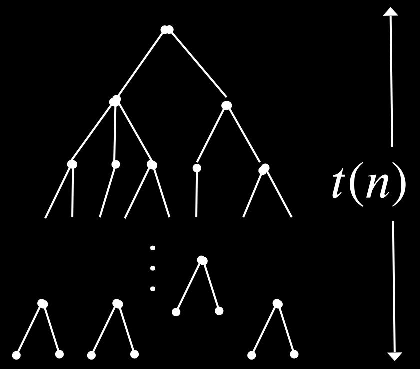

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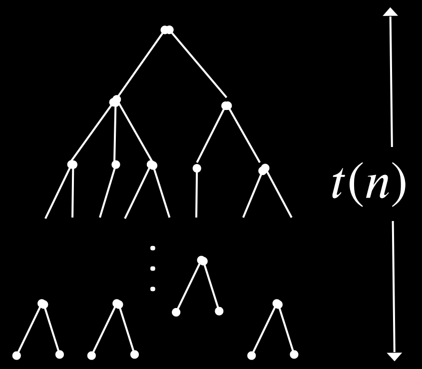

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2. *Accept* if $v_1 = s$
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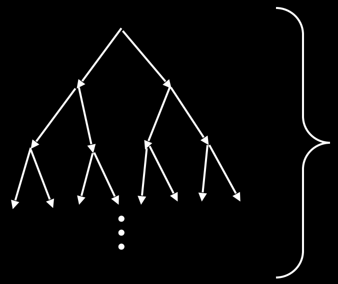

Guess bits of v_1

$HAMPATH \in$ NP

Theorem: $HAMPATH \in$ NP

Proof:

"On input $\langle G, s, t\rangle$ (Say G has m nodes.)

1. Nondeterministically write a sequence $v_1, v_2, \ldots, v_m$ of m nodes.
2. *Accept* if $v_1 = s$
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   each (v_i, v_{i+1}) is an edge
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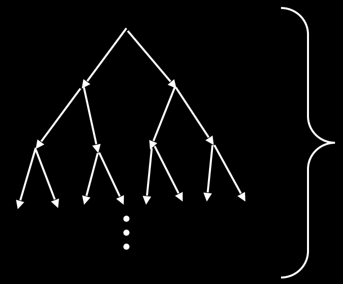

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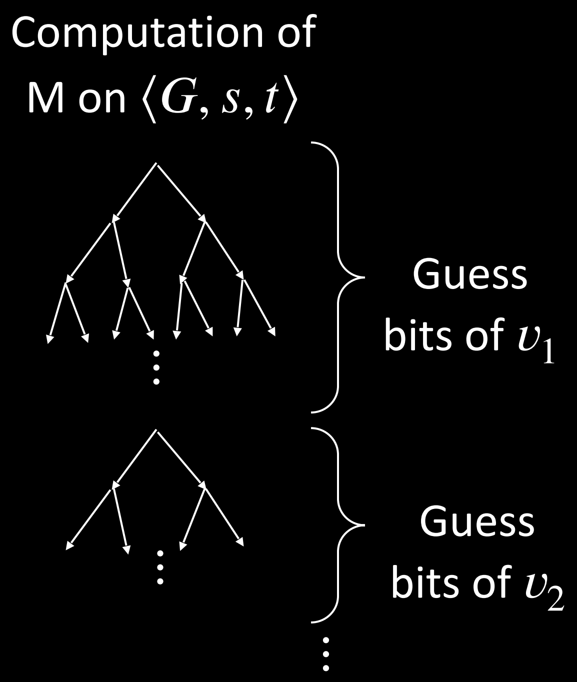

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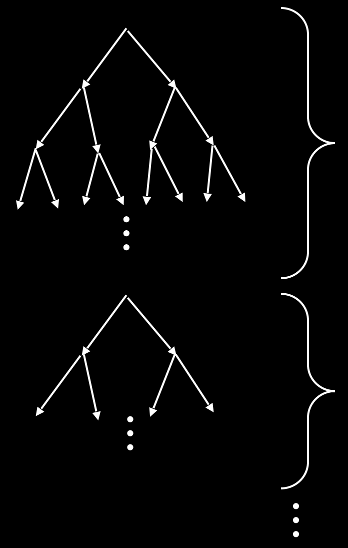

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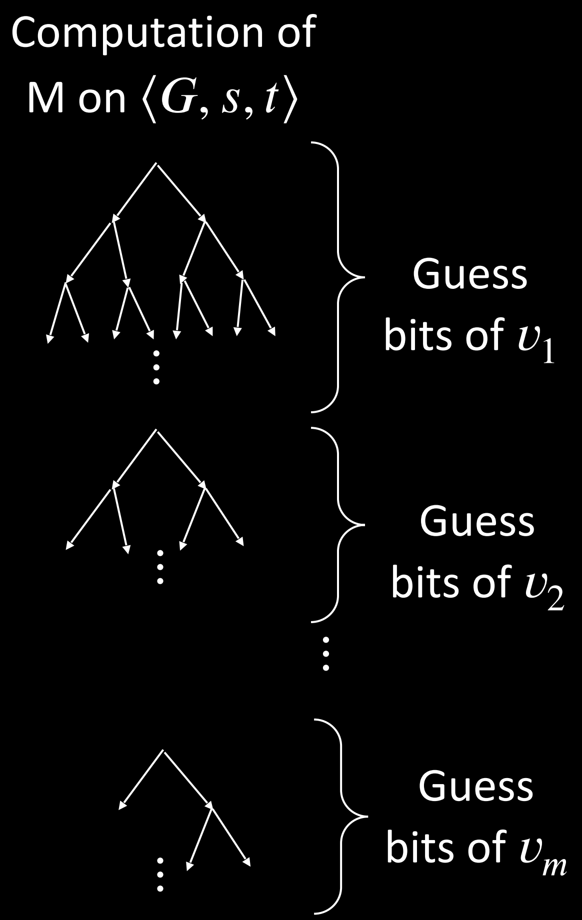

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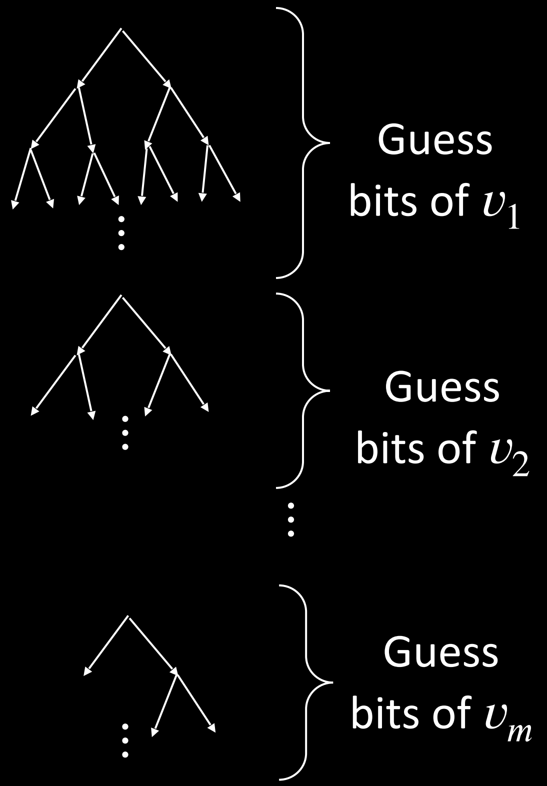

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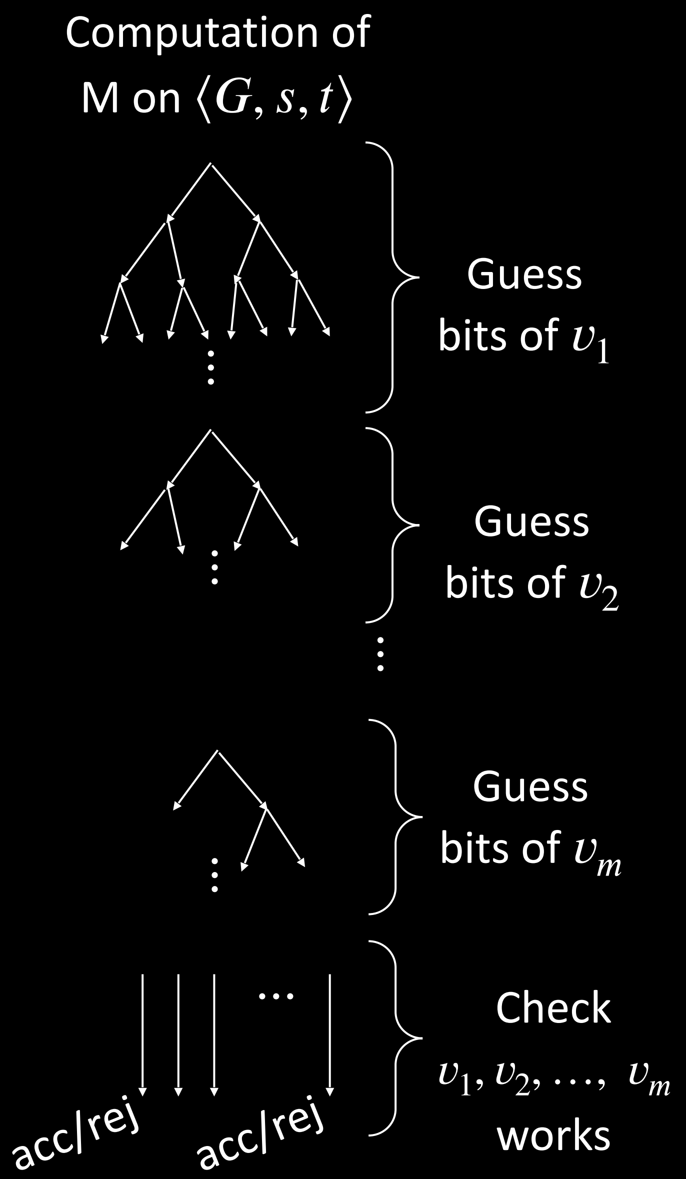

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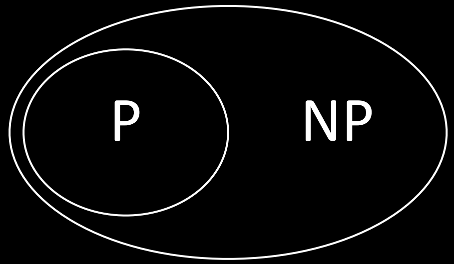

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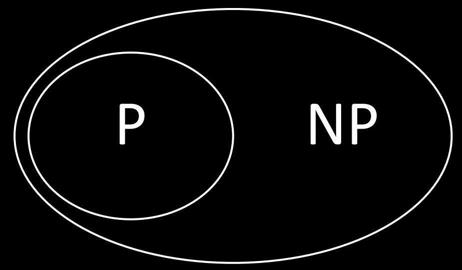

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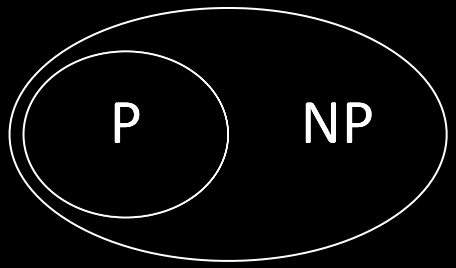

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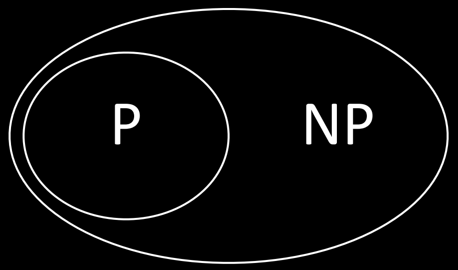

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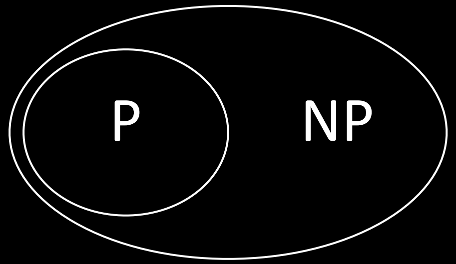

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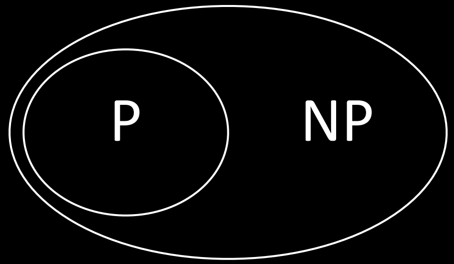

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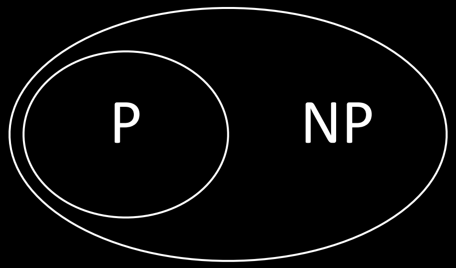

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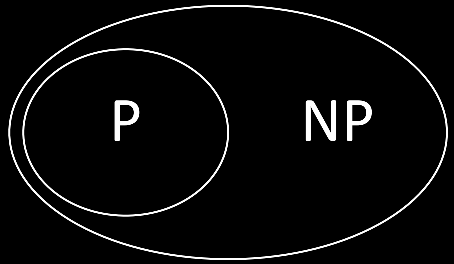

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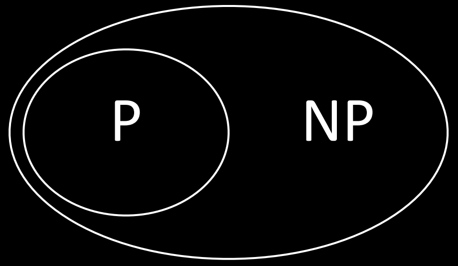

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Let $\overline{HAMPATH}$ be the complement of $HAMPATH$.

So $\langle G, s, t\rangle \in \overline{HAMPATH}$ if G does <u>not</u> have a Hamiltonian path from s to t.

Is $\overline{HAMPATH} \in$ NP?

(a) Yes, we can invert the accept/reject output of the NTM for $HAMPATH$.

(b) No, we cannot give a short certificate for a graph not to have a Hamiltonian path.

(c) I don't know.