بسم الله الرحمن الرحیم

جلسه بیستم

درس تحقیق در عملیات

برنامەریزی محدب

یک مثال: کوچکترین گوی

برنامەریزی محدب

تعمیم از برنامه‌ریزی خطی

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برنامه‌ریزی خطی

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ s.t. \quad & Ax = b \end{aligned}$$

تعمیم از برنامه‌ریزی خطی

برنامه‌ریزی خطی

$$\begin{aligned} minimize \quad & c^\top x \\ s.t. \quad & Ax = b \end{aligned}$$

تعمیم

برنامه‌ریزی محدب

$$\begin{aligned} mininize \quad & f(x) \\ & x \in K \end{aligned}$$

تابع محدب

مجموعه محدب

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$$t \in [0,1]$$
$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

مجموعه محدب

$$t \in [0,1], x, y \in K$$
$$\Rightarrow tx + (1-t)y \in K$$

تعمیم از برنامه‌ریزی خطی

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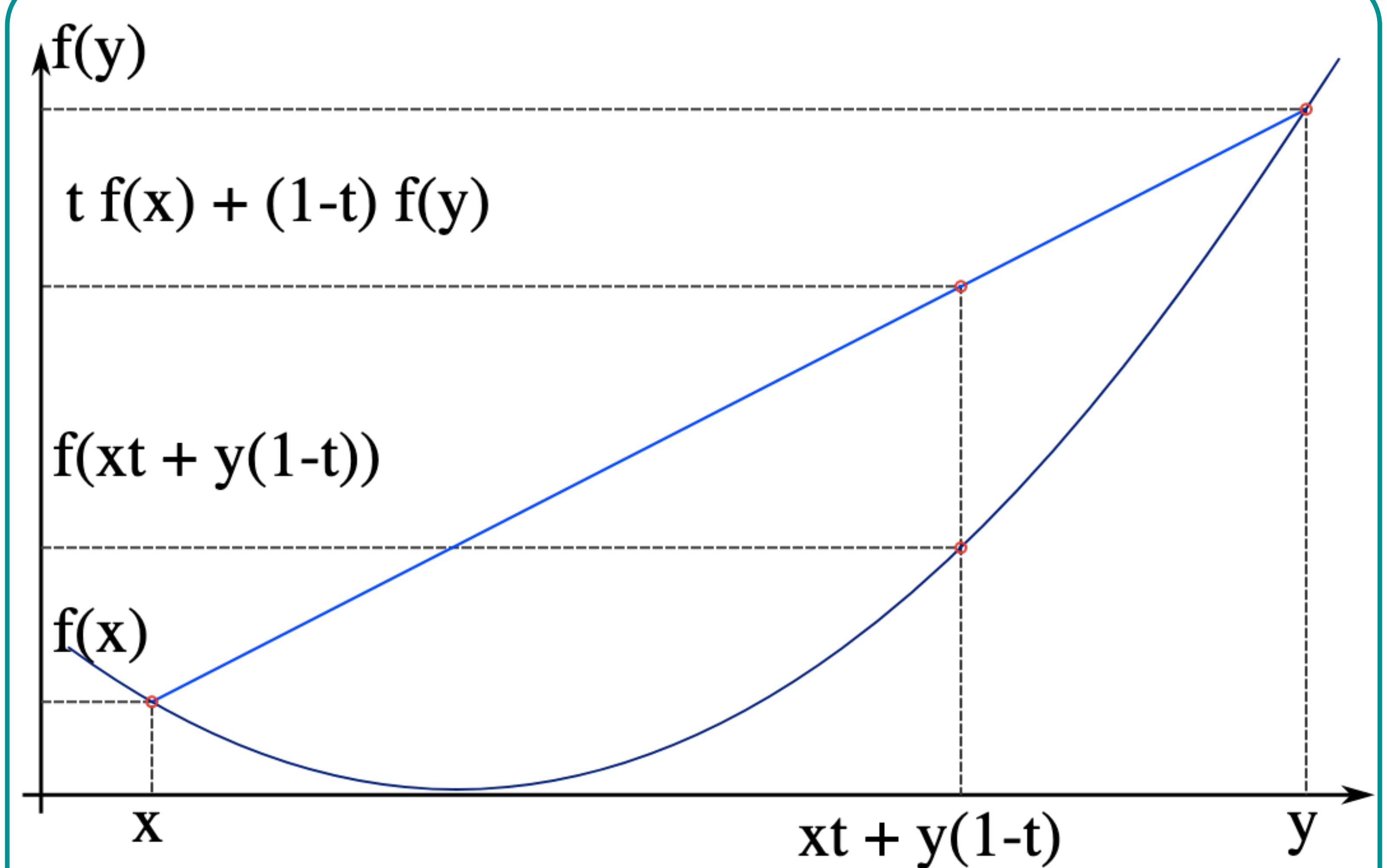
تعمیم



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مثالی از برنامه‌ریزی محدب

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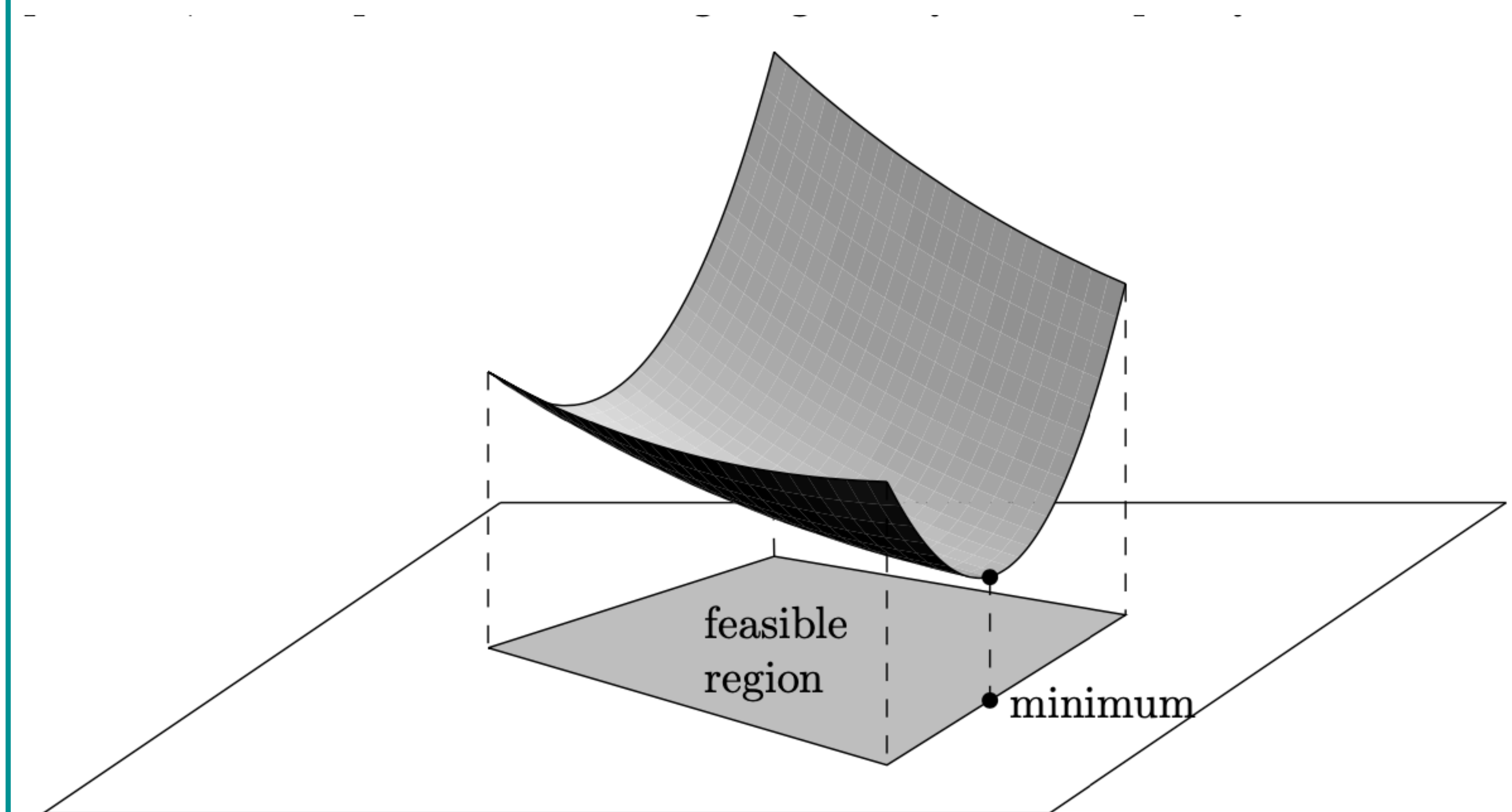

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تابع محدب

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مجموعه محدب

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & h_i(\mathbf{x}) = 0, \quad i = 1, \ldots, p, \end{aligned}$$

تابع محدب

تابع محدب

تابع خطی

تعمیم از برنامه‌ریزی خطی

برنامه‌ریزی خطی

$$
\begin{aligned}
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s.t. \quad & Ax = b
\end{aligned}
$$

تعمیم

میانه

تابع محدب

$$
\begin{aligned}
\text{Minimize} \quad & f(\mathbf{x}) \\
\text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\
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$$

برنامه‌ریزی محدب

تابع محدب

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\begin{aligned}
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تابع خطی

تابع محدب مشتق‌پذیر

قضیه: اگر f مشتق‌پذیر باشد:

$$\lambda \in [0,1] \quad f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

معادلند

$$f(\mathbf{x}) \geq f(\mathbf{z}) + \nabla f(\mathbf{z})(\mathbf{x}-\mathbf{z})$$

تابع محدب مشتق‌پذیر – تعبیر هندسی

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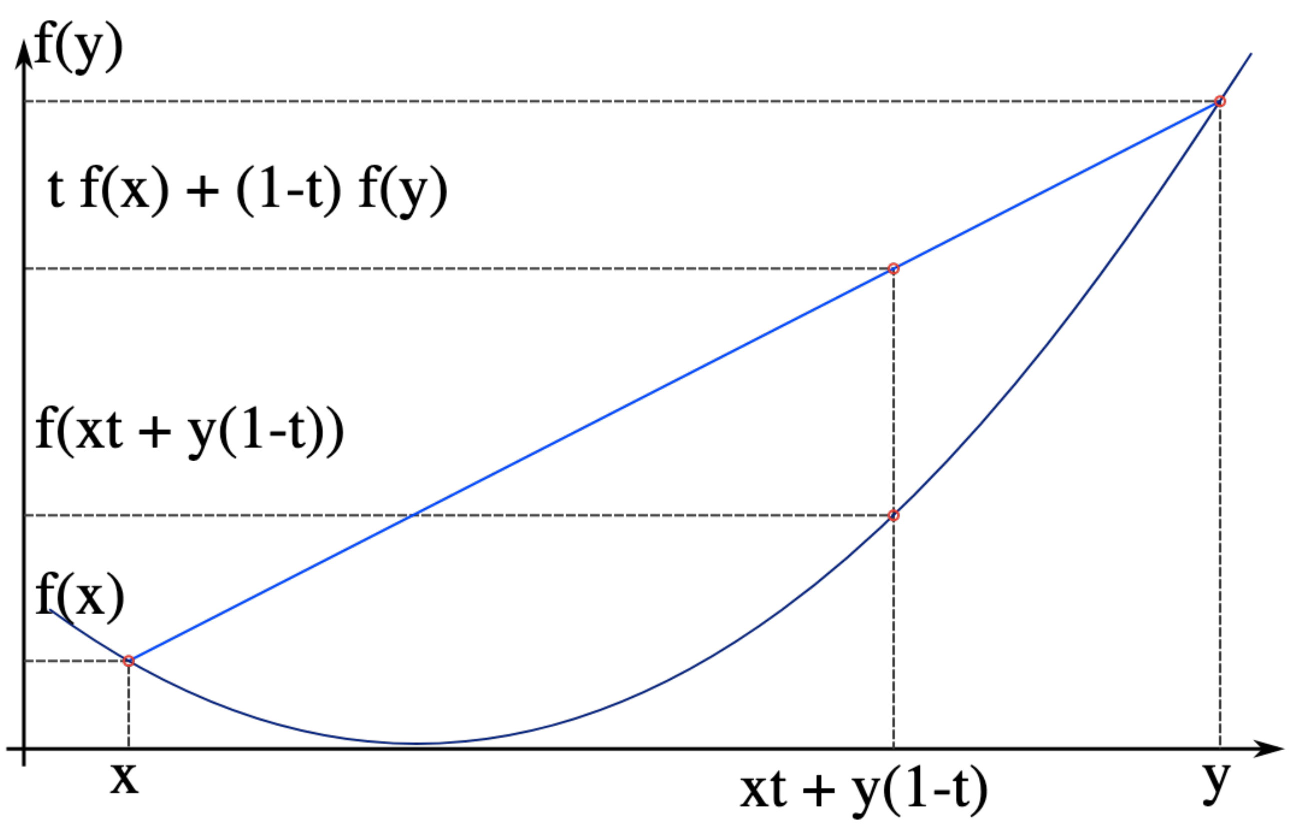

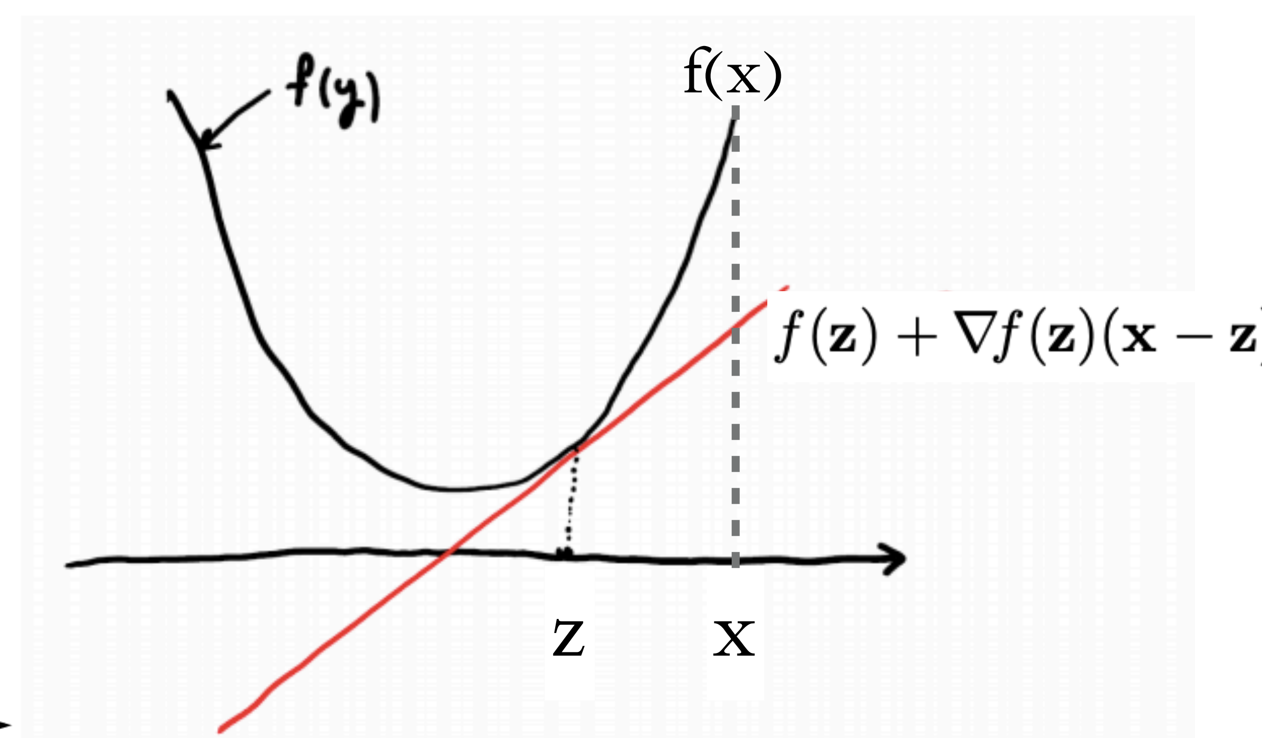

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اثبات:

$\Downarrow$:

$\Uparrow$:

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$$f(x + \lambda(y - x)) \leq f(x) + \lambda(f(y) - f(x))$$

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$$f(x + \lambda(y - x)) \le f(x) + \lambda(f(y) - f(x))$$

$$\Rightarrow f(y) - f(x) \ge \frac{f(x + \lambda(y-x)) - f(x)}{\lambda}, \forall \lambda \in (0,1]$$

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$$f(y)-f(x) \ge \nabla f^T(x)(y-x)$$

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$\Uparrow$:

$$f(x) \geq f(z) + \nabla f^T(z)(x - z)$$

$$z = \lambda x + (1 - \lambda)y$$

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$$\lambda f(x) + (1-\lambda) f(y) \ge f(z) + \nabla f^T(z)(\lambda x + (1-\lambda)y - z) = f(\lambda x + (1-\lambda) y)$$

8.7.1 Fact. *Let $C \subseteq \mathbb{R}^n$ be a convex set and $f\colon \mathbb{R}^n \to \mathbb{R}$ a differentiable convex function. A vector $\mathbf{x}^*$ minimizes $f(\mathbf{x})$ over C if and only if*

$$\nabla f(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \geq 0 \quad \textit{for all } \mathbf{x} \in C.$$

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: $f(\mathbf{x}) \geq f(\mathbf{x}^*)$ برای هر x:

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$$\nabla f(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \geq 0$$

$\Rightarrow$:

$$\mathbf{x}(t) := \mathbf{x}^* + t(\mathbf{x} - \mathbf{x}^*) \in C, \quad t \in [0, 1]$$

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$\Leftarrow$:

برای هر x: $\quad f(\mathbf{x}) \geq f(\mathbf{x}^*)$

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x* کمینه کننده

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$\mathbf{x}(t) := \mathbf{x}^* + t(\mathbf{x} - \mathbf{x}^*) \in C, \quad t \in [0,1]$

x* کمینه کننده

$$\nabla f(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) = \frac{\partial}{\partial t} f(\mathbf{x}(t))|_{t=0} = \lim_{t \to 0} \frac{f(\mathbf{x}(t)) - f(\mathbf{x}^*)}{t} \geq 0$$

8.7.2 Proposition (Karush–Kuhn–Tucker conditions). *Let us consider the convex program*

$$\begin{array}{ll} \textit{minimize} & f(\mathbf{x}) \\ \textit{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

with f convex and differentiable, with continuous partial derivatives. A feasible solution $\mathbf{x}^* \in \mathbb{R}^n$ is optimal if and only if there is a vector $\tilde{\mathbf{y}} \in \mathbb{R}^m$ such that for all $j \in \{1, \ldots, n\}$,

$$\nabla f(\mathbf{x}^*)_j + \tilde{\mathbf{y}}^T \mathbf{a}_j \begin{cases} = 0 & \text{if } x_j^* > 0 \\ \geq 0 & \text{otherwise.} \end{cases}$$

Here $\mathbf{a}_j$ is the jth column of A.

8.7.2 Proposition (Karush–Kuhn–Tucker conditions). *Let us consider the convex program*

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:

$$\begin{aligned} \Big(\nabla f(\mathbf{x}^*) + \tilde{\mathbf{y}}^T A\Big)\mathbf{x}^* &= 0, \\ \Big(\nabla f(\mathbf{x}^*) + \tilde{\mathbf{y}}^T A\Big)\mathbf{x} &\geq 0. \end{aligned}$$

8.7.2 Proposition (Karush–Kuhn–Tucker conditions). *Let us consider the convex program*

$$
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with f convex and differentiable, with continuous partial derivatives. A feasible solution $\mathbf{x}^* \in \mathbb{R}^n$ is optimal if and only if there is a vector $\tilde{\mathbf{y}} \in \mathbb{R}^m$ such that for all $j \in \{1, \ldots, n\}$,

$$
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$$

Here $\mathbf{a}_j$ is the jth column of A.

$\Leftarrow$:

$$
\begin{array}{rcl}
\left(\nabla f(\mathbf{x}^*) + \tilde{\mathbf{y}}^T A\right)\mathbf{x}^* & = & 0, \\
\left(\nabla f(\mathbf{x}^*) + \tilde{\mathbf{y}}^T A\right)\mathbf{x} & \geq & 0.
\end{array}
\quad \Longrightarrow \quad \nabla f(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \geq 0
$$

:

$$\begin{array}{ll} \textit{minimize} & f(\mathbf{x}) \\ \textit{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

x^*: کمینه کننده

x*: کمینه کننده

$$\begin{array}{ll} minimize & f(\mathbf{x}) \\ subject\ to & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

x: شدنی $\mathbf{c}^T(\mathbf{x} - \mathbf{x}^*) \leq 0$

: $\boxed{\mathbf{c}^T = -\nabla f(\mathbf{x}^*)}$

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$\mathbf{x}$: شدنی $\quad \mathbf{c}^T(\mathbf{x} - \mathbf{x}^*) \leq 0$

$\mathbf{x}^*$: بهینه کننده

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

: $\boxed{\mathbf{c}^T = -\nabla f(\mathbf{x}^*)}$

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x: شدنی $\mathbf{c}^T(\mathbf{x} - \mathbf{x}^*) \leq 0$

دوگان

x*: بهینه کننده

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

$\tilde{\mathbf{y}}$: بهینه

$$\begin{array}{ll} \text{minimize} & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & A^T\mathbf{y} \geq \mathbf{c}, \end{array}$$

: $\boxed{\mathbf{c}^T = -\nabla f(\mathbf{x}^*)}$

x*: کمینه کننده

$$\begin{array}{ll} \textit{minimize} & f(\mathbf{x}) \\ \textit{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

x: شدنی $\mathbf{c}^T(\mathbf{x} - \mathbf{x}^*) \leq 0$

دوگان

x*: بهینه کننده

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

$\tilde{\mathbf{y}}$: بهینه

$$\begin{array}{ll} \text{minimize} & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & A^T\mathbf{y} \geq \mathbf{c}, \end{array}$$

$$(\tilde{\mathbf{y}}^T A - \mathbf{c}^T)\mathbf{x}^*$$

x*: کمینه کننده

$$\begin{array}{ll} \textit{minimize} & f(\mathbf{x}) \\ \textit{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

x: شدنی $\mathbf{c}^T(\mathbf{x} - \mathbf{x}^*) \leq 0$

$$\boxed{\mathbf{c}^T = -\nabla f(\mathbf{x}^*)}$$:

x*: بهینه کننده

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

دوگان

$\tilde{\mathbf{y}}$: بهینه

$$\begin{array}{ll} \text{minimize} & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & A^T\mathbf{y} \geq \mathbf{c}, \end{array}$$

$$(\tilde{\mathbf{y}}^T A - \mathbf{c}^T)\mathbf{x}^* = \mathbf{b}^T\tilde{\mathbf{y}} - \mathbf{c}^T\mathbf{x}^*$$

$$\boxed{\mathbf{c}^T = -\nabla f(\mathbf{x}^*)}$$

x*: کمینه کننده

$$\begin{array}{ll} \textit{minimize} & f(\mathbf{x}) \\ \textit{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

x: شدنی $\mathbf{c}^T(\mathbf{x} - \mathbf{x}^*) \leq 0$

x*: بهینه کننده

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

دوگان

$\tilde{\mathbf{y}}$: بهینه

$$\begin{array}{ll} \text{minimize} & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & A^T\mathbf{y} \geq \mathbf{c}, \end{array}$$

$$\boxed{\mathbf{b}^T\tilde{\mathbf{y}} = \mathbf{c}^T\mathbf{x}^*}$$

$$(\tilde{\mathbf{y}}^T A - \mathbf{c}^T)\mathbf{x}^* = \mathbf{b}^T\tilde{\mathbf{y}} - \mathbf{c}^T\mathbf{x}^* = 0$$

$$\mathbf{c}^T = -\nabla f(\mathbf{x}^*)$$

$$\begin{array}{ll} \textit{minimize} & f(\mathbf{x}) \\ \textit{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

x*: کمینه کننده

x: شدنی $\mathbf{c}^T(\mathbf{x} - \mathbf{x}^*) \leq 0$

دوگان

x*: بهینه کننده

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

$\tilde{\mathbf{y}}$: بهینه

$$\begin{array}{ll} \text{minimize} & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & A^T\mathbf{y} \geq \mathbf{c}, \end{array}$$

$$\mathbf{b}^T\tilde{\mathbf{y}} = \mathbf{c}^T\mathbf{x}^*$$

$$(\tilde{\mathbf{y}}^T A - \mathbf{c}^T)\mathbf{x}^* = \mathbf{b}^T\tilde{\mathbf{y}} - \mathbf{c}^T\mathbf{x}^* = 0$$

$$\nabla f(\mathbf{x}^*)_j + \tilde{\mathbf{y}}^T\mathbf{a}_j \geq 0$$

: $\boxed{\mathbf{c}^T = -\nabla f(\mathbf{x}^*)}$

x*: کمینه کننده

$$\begin{array}{ll} \textit{minimize} & f(\mathbf{x}) \\ \textit{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

x: شدنی $\mathbf{c}^T(\mathbf{x} - \mathbf{x}^*) \leq 0$

x*: بهینه کننده

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

دوگان

$\tilde{\mathbf{y}}$: بهینه

$$\begin{array}{ll} \text{minimize} & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & A^T\mathbf{y} \geq \mathbf{c}, \end{array}$$

$$\boxed{\mathbf{b}^T\tilde{\mathbf{y}} = \mathbf{c}^T\mathbf{x}^*}$$

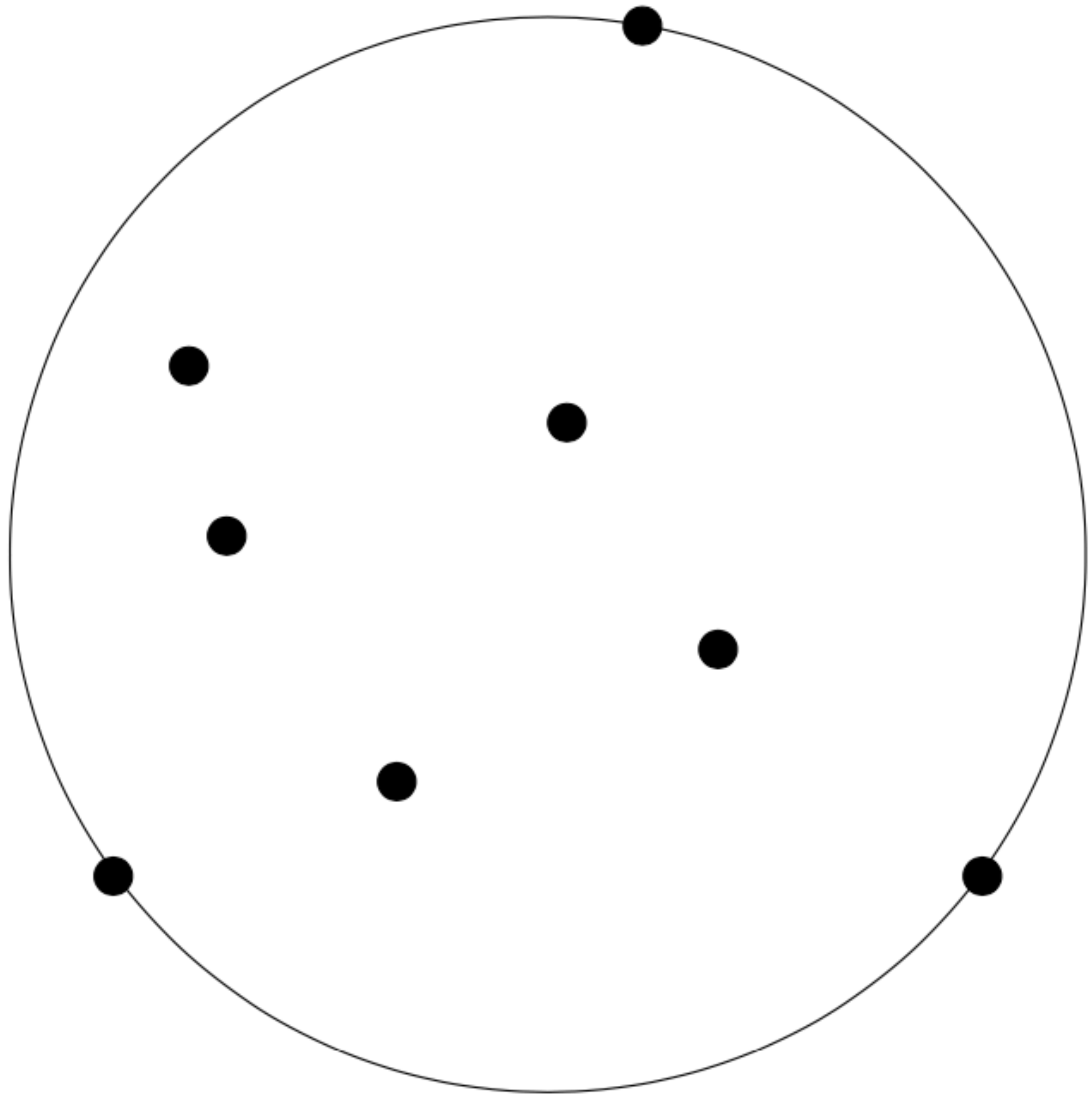
$$(\tilde{\mathbf{y}}^T A - \mathbf{c}^T)\mathbf{x}^* = \mathbf{b}^T\tilde{\mathbf{y}} - \mathbf{c}^T\mathbf{x}^* = 0$$

$$\nabla f(\mathbf{x}^*)_j + \tilde{\mathbf{y}}^T\mathbf{a}_j \geq 0$$

$$\nabla f(\mathbf{x}^*)_j + \tilde{\mathbf{y}}^T\mathbf{a}_j = 0$$ اگر $x_j^* > 0$

یک مثال: کوچکترین گوی

سوال: کوچکترین گوی شامل نقاط

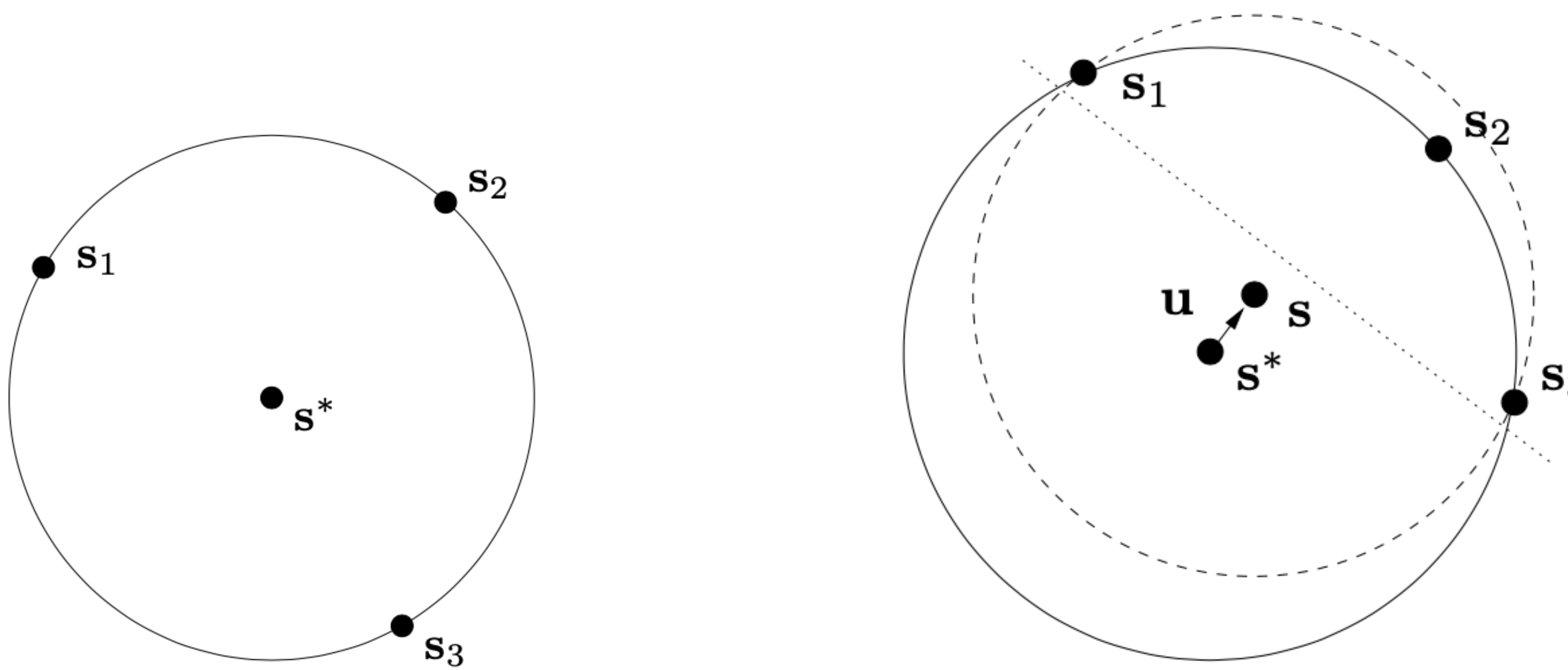


8.7.3 Lemma. *Let $S = \{\mathbf{s}_1, \ldots, \mathbf{s}_k\} \subseteq \mathbb{R}^d$ be a set of points on the boundary of a ball B with center $\mathbf{s}^* \in \mathbb{R}^d$. Then the following two statements are equivalent.*

(i) *B is the unique smallest enclosing ball of S.*

(ii) *For every $\mathbf{u} \in \mathbb{R}^d$, there is an index $j \in \{1, 2, \ldots, k\}$ such that*

$$\mathbf{u}^T(\mathbf{s}_j - \mathbf{s}^*) \leq 0.$$



$$\mathbf{s} = \mathbf{s}^* + t\mathbf{u}$$

$$
\begin{aligned}
(\mathbf{s}_j - \mathbf{s})^T(\mathbf{s}_j - \mathbf{s}) &= (\mathbf{s}_j - \mathbf{s}^* - t\mathbf{u})^T(\mathbf{s}_j - \mathbf{s}^* - t\mathbf{u}) \\
&= (\mathbf{s}_j - \mathbf{s}^*)^T(\mathbf{s}_j - \mathbf{s}^*) + t^2\mathbf{u}^T\mathbf{u} - 2t\mathbf{u}^T(\mathbf{s}_j - \mathbf{s}^*) \\
&= r^2 + t^2 - 2t\mathbf{u}^T(\mathbf{s}_j - \mathbf{s}^*).
\end{aligned}
$$

: خط جداکننده برای هر u => کمینه بودن شعاع

برای هر u و t، فاصله یک s_j بیشتر شود

: خط جداکننده برای هر u <= کمینه بودن شعاع

هر نقطه s: یک نقطه s_j که ... منفی است <= فاصله بیشتر می‌شود.

8.7.4 Theorem. *Let $\mathbf{p}_1, \ldots, \mathbf{p}_n$ be points in $\mathbb{R}^d$, and let Q be the $d \times n$ matrix whose jth column is formed by the d coordinates of the point $\mathbf{p}_j$. Let us consider the optimization problem*

$$\begin{array}{ll} \text{minimize} & \mathbf{x}^T Q^T Q \mathbf{x} - \sum_{j=1}^n x_j \mathbf{p}_j^T \mathbf{p}_j \\ \text{subject to} & \sum_{j=1}^n x_j = 1 \\ & \mathbf{x} \geq \mathbf{0} \end{array} \tag{8.15}$$

in the variables $x_1, \ldots, x_n$. Then the objective function $f(\mathbf{x}) := \mathbf{x}^T Q^T Q \mathbf{x} - \sum_{j=1}^n x_j \mathbf{p}_j^T \mathbf{p}_j$ is convex, and the following statements hold.

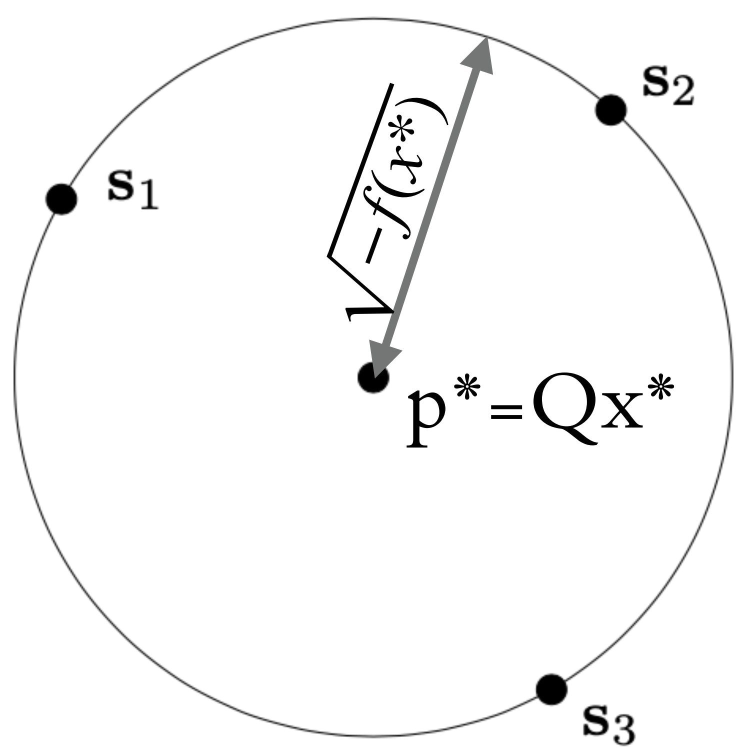
(i) *Problem (8.15) has an optimal solution $\mathbf{x}^*$.*

(ii) *There exists a point $\mathbf{p}^*$ such that $\mathbf{p}^* = Q\mathbf{x}^*$ holds for every optimal solution $\mathbf{x}^*$. Moreover, the ball with center $\mathbf{p}^*$ and squared radius $-f(\mathbf{x}^*)$ is the unique ball of smallest radius containing P.*

قضیه: ورودی: $\mathbf{p}_1, \ldots, \mathbf{p}_n$ *be points in* $\mathbb{R}^d$

نقاط $\mathbf{p_j}$ در ستون‌ها

$$\begin{array}{ll} \textit{minimize} & \mathbf{x}^T Q^T Q \mathbf{x} - \sum_{j=1}^{n} x_j \mathbf{p}_j^T \mathbf{p}_j \\ \textit{subject to} & \sum_{j=1}^{n} x_j = 1 \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$



اثبات:

$$\nabla f(\mathbf{x}^*)_j + \tilde{\mathbf{y}}^T \mathbf{a}_j \begin{cases} = 0 & if\ x_j^* > 0 \\ \geq 0 & otherwise. \end{cases} \qquad \text{(KKT)}$$

اثبات:

$$\nabla f(\mathbf{x}) = 2\mathbf{x}^T Q^T Q - (\mathbf{p}_1^T \mathbf{p}_1, \mathbf{p}_2^T \mathbf{p}_2, \ldots, \mathbf{p}_n^T \mathbf{p}_n)$$

$$\nabla f(\mathbf{x}^*)_j + \tilde{\mathbf{y}}^T \mathbf{a}_j \begin{cases} = 0 & \textit{if } x_j^* > 0 \\ \geq 0 & \textit{otherwise.} \end{cases} \qquad \text{(KKT)}$$

اثبات:

$$\nabla f(\mathbf{x}) = 2\mathbf{x}^T Q^T Q - (\mathbf{p}_1^T \mathbf{p}_1, \mathbf{p}_2^T \mathbf{p}_2, \ldots, \mathbf{p}_n^T \mathbf{p}_n)$$

$$\nabla f(\mathbf{x}^*)_j + \tilde{\mathbf{y}}^T \mathbf{a}_j \begin{cases} = 0 & \textit{if } x_j^* > 0 \\ \geq 0 & \textit{otherwise.} \end{cases} \quad \text{(KKT)}$$

$$2\mathbf{p}_j^T \mathbf{p}^* - \mathbf{p}_j^T \mathbf{p}_j + \mu \begin{cases} = 0 & \text{if } x_j^* > 0 \\ \geq 0 & \text{otherwise.} \end{cases} \quad \text{(KKT)}$$

$$\mathbf{p}^* = Q\mathbf{x}^* = \sum_{j=1}^{n} x_j^* \mathbf{p}_j$$

اثبات:

$$\nabla f(\mathbf{x}) = 2\mathbf{x}^T Q^T Q - (\mathbf{p}_1^T \mathbf{p}_1, \mathbf{p}_2^T \mathbf{p}_2, \ldots, \mathbf{p}_n^T \mathbf{p}_n)$$

$$\nabla f(\mathbf{x}^*)_j + \tilde{\mathbf{y}}^T \mathbf{a}_j \begin{cases} = 0 & \textit{if } x_j^* > 0 \\ \geq 0 & \textit{otherwise.} \end{cases} \qquad \text{(KKT)}$$

$$2\mathbf{p}_j^T \mathbf{p}^* - \mathbf{p}_j^T \mathbf{p}_j + \mu \begin{cases} = 0 & \text{if } x_j^* > 0 \\ \geq 0 & \text{otherwise.} \end{cases} \qquad \text{(KKT)}$$

$$\mathbf{p}^* = Q\mathbf{x}^* = \sum_{j=1}^{n} x_j^* \mathbf{p}_j$$

$$\|\mathbf{p}_j - \mathbf{p}^*\|^2 \begin{cases} = \mu + \mathbf{p}^{*T} \mathbf{p}^* & \text{if } x_j^* > 0 \\ \leq \mu + \mathbf{p}^{*T} \mathbf{p}^* & \text{otherwise.} \end{cases}$$

اثبات:

$$\nabla f(\mathbf{x}) = 2\mathbf{x}^T Q^T Q - (\mathbf{p}_1^T \mathbf{p}_1, \mathbf{p}_2^T \mathbf{p}_2, \ldots, \mathbf{p}_n^T \mathbf{p}_n)$$

$$\nabla f(\mathbf{x}^*)_j + \tilde{\mathbf{y}}^T \mathbf{a}_j \begin{cases} = 0 & \text{if } x_j^* > 0 \\ \geq 0 & \text{otherwise.} \end{cases} \quad \text{(KKT)}$$

$$2\mathbf{p}_j^T \mathbf{p}^* - \mathbf{p}_j^T \mathbf{p}_j + \mu \begin{cases} = 0 & \text{if } x_j^* > 0 \\ \geq 0 & \text{otherwise.} \end{cases} \quad \text{(KKT)}$$

$$\mathbf{p}^* = Q\mathbf{x}^* = \sum_{j=1}^n x_j^* \mathbf{p}_j$$

$$\|\mathbf{p}_j - \mathbf{p}^*\|^2 \begin{cases} = \mu + \mathbf{p}^{*T} \mathbf{p}^* & \text{if } x_j^* > 0 \\ \leq \underbrace{\mu + \mathbf{p}^{*T} \mathbf{p}^*}_{\text{مربع شعاع}} & \text{otherwise.} \end{cases}$$

اثبات:

$$\nabla f(\mathbf{x}) = 2\mathbf{x}^T Q^T Q - (\mathbf{p}_1^T\mathbf{p}_1, \mathbf{p}_2^T\mathbf{p}_2, \ldots, \mathbf{p}_n^T\mathbf{p}_n)$$

$$\nabla f(\mathbf{x}^*)_j + \tilde{\mathbf{y}}^T\mathbf{a}_j \begin{cases} = 0 & \textit{if } x_j^* > 0 \\ \geq 0 & \textit{otherwise.} \end{cases} \qquad \text{(KKT)}$$

$$2\mathbf{p}_j^T\mathbf{p}^* - \mathbf{p}_j^T\mathbf{p}_j + \mu \begin{cases} = 0 & \text{if } x_j^* > 0 \\ \geq 0 & \text{otherwise.} \end{cases} \qquad \text{(KKT)}$$

$$\mathbf{p}^* = Q\mathbf{x}^* = \textstyle\sum_{j=1}^n x_j^*\mathbf{p}_j$$

$$\|\mathbf{p}_j - \mathbf{p}^*\|^2 \begin{cases} = \mu + \mathbf{p}^{*T}\mathbf{p}^* & \text{if } x_j^* > 0 \\ \leq \underbrace{\mu + \mathbf{p}^{*T}\mathbf{p}^*}_{\text{مربع شعاع}} & \text{otherwise.} \end{cases}$$

اثبات:

به ازای هر u:

$$\sum_{j\in F} x_j^* \mathbf{u}^T(\mathbf{p}_j - \mathbf{p}^*) = \mathbf{u}^T\left(\sum_{j\in F} x_j^* \mathbf{p}_j - \sum_{j\in F} x_j^* \mathbf{p}^*\right) = \mathbf{u}^T(\mathbf{p}^* - \mathbf{p}^*) = 0.$$

<= هست j که: $\mathbf{u}^T(\mathbf{p}_j - \mathbf{p}^*) \leq 0$

<= یکتایی گوی