In contrast to the earlier problems of the chapter, this model includes $\mathcal{NP}$-hard problems, such as the uncapacitated location problem. Hence we examine different integer programming formulations and heuristics.

2. ELEMENTARY PROPERTIES

There are many ways of defining and viewing both matroids and submodular functions. Here we introduce the definitions and the fundamental results that we will use later. First we study submodular functions (see Definition 1.4).

Proposition 2.1

i. *f is submodular if and only if*

$$\text{(a)} \qquad f(S \cup \{j\}) - f(S) \geqslant f(S \cup \{j, k\}) - f(S \cup \{k\}) \quad \text{for } j, k \in N, j \neq k, \text{ and } S \subseteq N \setminus \{j, k\}.$$

ii. *f is submodular and nondecreasing if and only if*

$$\text{(b)} \qquad f(T) \leqslant f(S) + \sum_{j \in T \setminus S} [f(S \cup \{j\}) - f(S)] \quad \text{for } S, T \subseteq N.$$

Proof. i. If f is submodular we obtain (a) by setting $S \leftarrow S \cup \{j\}$ and $T \leftarrow S \cup \{k\}$ in Definition 1.4.

If (a) holds, let $S = A \cap B$, $A \setminus B = \{j_1, \dots, j_r\}$, and $B \setminus A = \{k_1, \dots, k_s\}$. Then

$$\begin{aligned}
&f(B) - f(A \cap B) \\
&= \sum_{t=1}^{s} [f(S \cup \{k_1, \dots, k_t\}) - f(S \cup \{k_1, \dots, k_{t-1}\})] \\
&\geqslant \sum_{t=1}^{s} [f(S \cup \{k_1, \dots, k_t\} \cup \{j_1\}) - f(S \cup \{k_1, \dots, k_{t-1}\} \cup \{j_1\})] \\
&\qquad \vdots \\
&\geqslant \sum_{t=1}^{s} [f(S \cup \{k_1, \dots, k_t\} \cup \{j_1, \dots, j_r\}) - f(S \cup \{k_1, \dots, k_{t-1}\} \cup \{j_1, \dots, j_r\})] \\
&= \sum_{t=1}^{s} [f(A \cup \{k_1, \dots, k_t\}) - f(A \cup \{k_1, \dots, k_{t-1}\})] \\
&= f(A \cup B) - f(A).
\end{aligned}$$

ii. Let $T \setminus S = \{j_1, \dots, j_r\}$. Then

$$\begin{aligned}
f(T) \leqslant f(S \cup T) &= f(S) + \{f(S \cup T) - f(S)\} \\
&= f(S) + \sum_{t=1}^{r} \{f(S \cup \{j_1, \dots, j_t\}) - f(S \cup \{j_1, \dots, j_{t-1}\})\} \\
&\leqslant f(S) + \sum_{t=1}^{r} \{f(S \cup \{j_t\}) - f(S)\},
\end{aligned}$$

where the first inequality holds if f is nondecreasing, and the second one holds if f is submodular. Taking $T = S \cup \{j,k\}$ and $T = S \setminus \{k\}$ in (b) gives the converse. ∎