بسم الله الرحمن الرحیم

جلسه نهم

درس تحقیق در عملیات

دوگانی

$$
\begin{array}{ll}
\text{maximize} & 2x_1 + 3x_2 \\
\text{subject to} & 4x_1 + 8x_2 \leq 12 \\
& 2x_1 + x_2 \leq 3 \\
& 3x_1 + 2x_2 \leq 4 \\
& x_1, x_2 \geq 0.
\end{array}
$$

$$\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & + & 3x_2 & & \\
\text{subject to} & 4x_1 & + & 8x_2 & \leq & 12 \\
 & 2x_1 & + & x_2 & \leq & 3 \\
 & 3x_1 & + & 2x_2 & \leq & 4 \\
 & \multicolumn{5}{l}{x_1, x_2 \geq 0.}
\end{array}$$

$$2x_1 + 3x_2 \leq 4x_1 + 8x_2 \leq 12.$$

$$\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & + & 3x_2 & & \\
\text{subject to} & 4x_1 & + & 8x_2 & \leq & 12 \\
& 2x_1 & + & x_2 & \leq & 3 \\
& 3x_1 & + & 2x_2 & \leq & 4 \\
& x_1, x_2 \geq 0. & & & &
\end{array}$$

$$2x_1 + 3x_2 \leq 2x_1 + 4x_2 \leq 6$$

$$
\begin{array}{llcrcr}
\text{maximize} & 2x_1 & + & 3x_2 & & \\
\text{subject to} & 4x_1 & + & 8x_2 & \leq & 12 \\
 & 2x_1 & + & x_2 & \leq & 3 \\
 & 3x_1 & + & 2x_2 & \leq & 4 \\
 & x_1, x_2 \geq 0. & & & &
\end{array}
$$

$$
2x_1 + 3x_2 = \frac{1}{3}(4x_1 + 8x_2 + 2x_1 + x_2) \leq \frac{1}{3}(12 + 3) = 5
$$

$$d_1x_1 + d_2x_2 \leq h$$

$$d_1 \geq 2,\ d_2 \geq 3$$

هدف

$$2x_1 + 3x_2 \leq d_1x_1 + d_2x_2 \leq h$$

برنامه ریزی [illegible]

$$d_1x_1 + d_2x_2 \le h$$

$$d_1 \ge 2,\ d_2 \ge 3$$

هدف

$$2x_1 + 3x_2 \le d_1x_1 + d_2x_2 \le h$$

$$
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 & 2x_1 & + & x_2 & \leq & 3 \\
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 & \multicolumn{5}{l}{x_1, x_2 \geq 0.}
\end{array}
$$

$$
(4y_1 + 2y_2 + 3y_3)x_1 + (8y_1 + y_2 + 2y_3)x_2 \leq 12y_1 + 3y_2 + 4y_3
$$

$$
\begin{array}{ll}
\text{maximize} & 2x_1 + 3x_2 \\
\text{subject to} & 4x_1 + 8x_2 \le 12 \qquad y_{۱} \\
 & 2x_1 + x_2 \le 3 \qquad y_{۲} \\
 & 3x_1 + 2x_2 \le 4 \qquad y_{۳} \\
 & x_1, x_2 \ge 0.
\end{array}
$$

۲۱

۸۱

$$\underbrace{\underbrace{(4y_1 + 2y_2 + 3y_3)}x_1 + (\overbrace{8y_1 + y_2 + 2y_3}^{}) x_2 \le 12y_1 + 3y_2 + 4y_3}_{۱۷}$$

تابع هدف = ۲x۱ + ۳x۲

$$
\begin{array}{lrcrcrcl}
\text{Minimize} & 12y_1 & + & 3y_2 & + & 4y_3 & & \\
\text{subject to} & 4y_1 & + & 2y_2 & + & 3y_3 & \geq & 2 \\
& 8y_1 & + & y_2 & + & 2y_3 & \geq & 3 \\
& \multicolumn{7}{l}{y_1, y_2, y_3 \geq 0.}
\end{array}
$$

$$\begin{array}{lrcrcrcl} \text{Minimize} & 12y_1 & + & 3y_2 & + & 4y_3 & & \\ \text{subject to} & 4y_1 & + & 2y_2 & + & 3y_3 & \geq & 2 \\ & 8y_1 & + & y_2 & + & 2y_3 & \geq & 3 \\ & \multicolumn{7}{l}{y_1, y_2, y_3 \geq 0.} \end{array}$$

4.75

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}$$

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$$\text{minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} \geq \mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0}$$

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0} \qquad \text{(P)}$$

$$\text{minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} \geq \mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0} \qquad \text{(D)}$$

for each feasible solution $\mathbf{x}$ *of* (P)

and each feasible solution $\mathbf{y}$ *of* (D)

$$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$$

قضیه دوگانی ضعیف

Duality theorem of linear programming

For the linear programs

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0} \tag{P}$$

and

$$\text{minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} \geq \mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0} \tag{D}$$

exactly one of the following possibilities occurs:

1. Neither (P) nor (D) has a feasible solution.
2. (P) is unbounded and (D) has no feasible solution.
3. (P) has no feasible solution and (D) is unbounded.
4. Both (P) and (D) have a feasible solution. Then both have an optimal solution, and if $\mathbf{x}^*$ is an optimal solution of (P) and $\mathbf{y}^*$ is an optimal solution of (D), then

   $$\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*.$$

   That is, the *maximum of* (P) *equals the minimum of* (D).

Duality theorem of linear programming

For the linear programs

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0} \tag{P}$$

and

$$\text{minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} \geq \mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0} \tag{D}$$

exactly one of the following possibilities occurs:

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   $$\boxed{\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*.}$$

   That is, the *maximum of* (P) *equals the minimum of* (D).

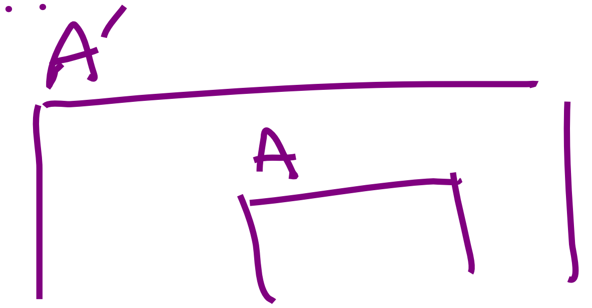
الگوریتم شدنی بودن سخت‌تر از الگوریتم بهینه‌سازی

$$\begin{aligned} \text{Maximize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \le \mathbf{b}, \\ & \mathbf{x} \ge \mathbf{0} \end{aligned}$$

الگوریتم شدنی بودن سخت‌تر از الگوریتم بهینه‌سازی

فرض: الگوریتم A هست که یک جواب شدنی پیدا می‌کند

حکم ⇐ الگوریتم A' هست که جواب بهینه را می‌دهد



$$\begin{array}{ll} \text{Maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} \le \mathbf{b}, \\ & \mathbf{x} \ge \mathbf{0} \end{array}$$

الگوریتم شدنی بودن سخت‌تر از الگوریتم بهینه‌سازی

$$
\begin{array}{ll}
\text{Maximize} & \mathbf{c}^T\mathbf{x} \\
\text{subject to} & A\mathbf{x} \leq \mathbf{b}, \\
 & A^T\mathbf{y} \geq \mathbf{c}, \\
 & \mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{y}, \\
 & \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}.
\end{array}
$$

دوگان برای همه

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \begin{Bmatrix} \leq \\ \geq \\ = \end{Bmatrix} b_i \qquad (C_i)$$

$$\left\{\begin{array}{l} y_i \geq 0 \\ y_i \leq 0 \\ y_i \in \mathbb{R} \end{array}\right\} \text{ if we have } \left\{\begin{array}{c} \leq \\ \geq \\ = \end{array}\right\} \text{ in } C_i$$

$$(Q_j)$$

$$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \left\{ \begin{array}{c} \geq \\ \leq \\ = \end{array} \right\} c_j$$

$$\text{if } x_j \text{ satisfies } \left\{ \begin{array}{l} x_j \geq 0 \\ x_j \leq 0 \\ x_j \in \mathbb{R} \end{array} \right\}$$

$$\text{minimize } \mathbf{b}^T\mathbf{y}$$

Dualization Recipe

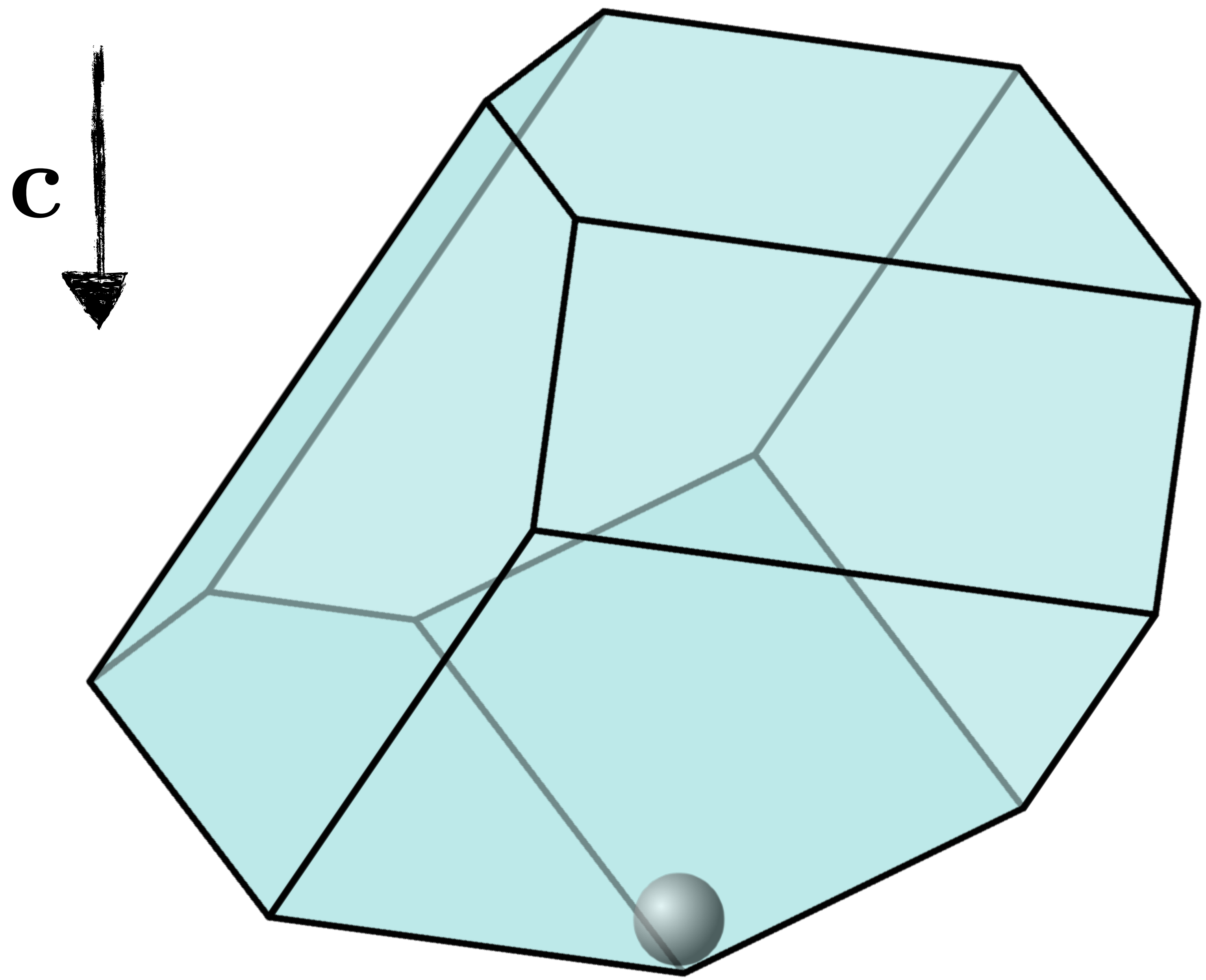
	Primal linear program	Dual linear program
Variables	$x_1, x_2, \ldots, x_n$	$y_1, y_2, \ldots, y_m$
Matrix	A	A^T
Right-hand side	$\mathbf{b}$	$\mathbf{c}$
Objective function	$\max \mathbf{c}^T\mathbf{x}$	$\min \mathbf{b}^T\mathbf{y}$
Constraints	ith constraint has $\leq$	$y_i \geq 0$
	$\geq$	$y_i \leq 0$
	$=$	$y_i \in \mathbb{R}$
	$x_j \geq 0$	jth constraint has $\geq$
	$x_j \leq 0$	$\leq$
	$x_j \in \mathbb{R}$	$=$

تفسیر فیزیکی!

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b}.$$

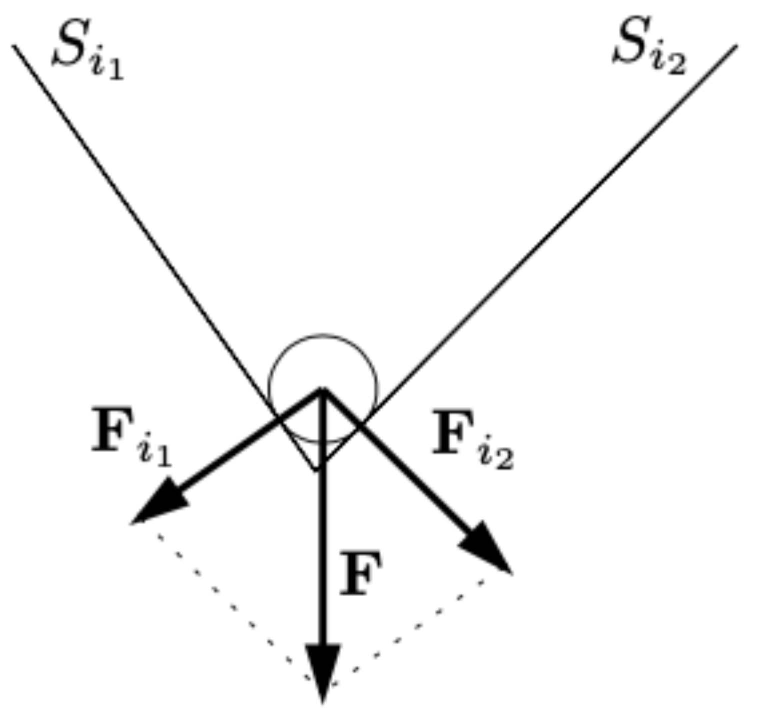
$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b}.$$

$$\text{minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} = \mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0}.$$


c

maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$.

minimize $\mathbf{b}^T\mathbf{y}$ subject to $A^T\mathbf{y} = \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.



$$\sum_{i \in D} y_i^* \mathbf{a}_i = \mathbf{c}.$$

پس: شدنی

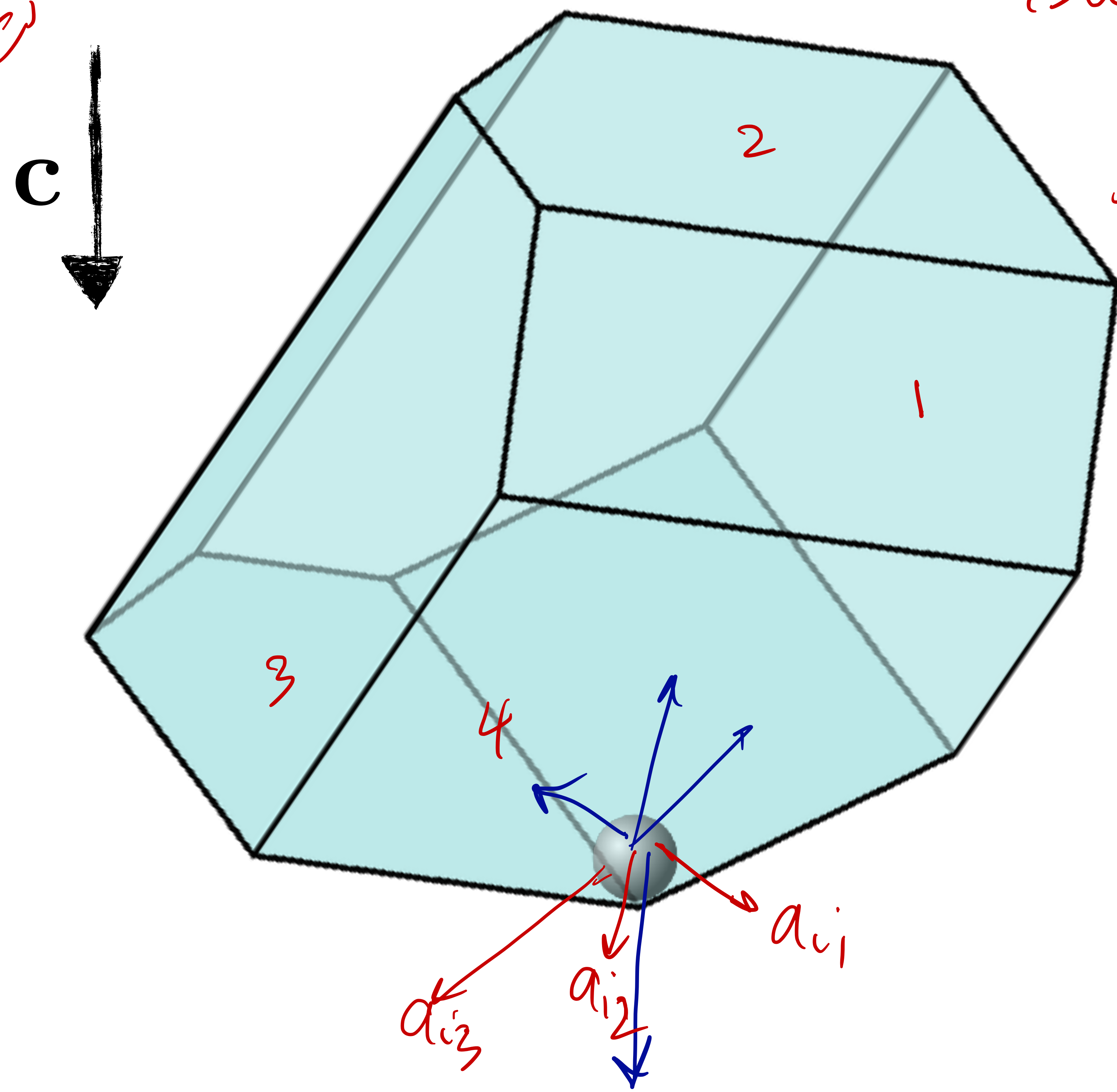
$$(\mathbf{y}^*)^T(A\mathbf{x}^* - \mathbf{b}) = 0$$

$$(\mathbf{y}^*)^T\mathbf{b} = (\mathbf{y}^*)^T A\mathbf{x}^* = \mathbf{c}^T\mathbf{x}^*$$

$i_1 \cdots i_k$
اندیس‌ها
قیدهای فعال

$$\sum_{j=1}^{k} y^*_{i_j} a_{i_j} = c$$

$i_1 \cdots i_k$
اندیس قیدهایی هستند که
در نقطه‌ی آن‌اند
وجود دارد که با کاهش c
گوشه دارد.



y^* را باید طوری
برای اعضا پیدا کنید

اثبات دوگانی به کمک روش سیمپلکس

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}$$

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}$$

$$\text{maximize } \overline{\mathbf{c}}^T\overline{\mathbf{x}} \text{ subject to } \bar{A}\overline{\mathbf{x}} = \mathbf{b} \text{ and } \overline{\mathbf{x}} \geq \mathbf{0}$$

$$\overline{\mathbf{x}} = (x_1, \ldots, x_{n+m})$$

$$\overline{\mathbf{c}} = (c_1, \ldots, c_n, 0, \ldots, 0)$$

$$\bar{A} = (A \,|\, I_m)$$

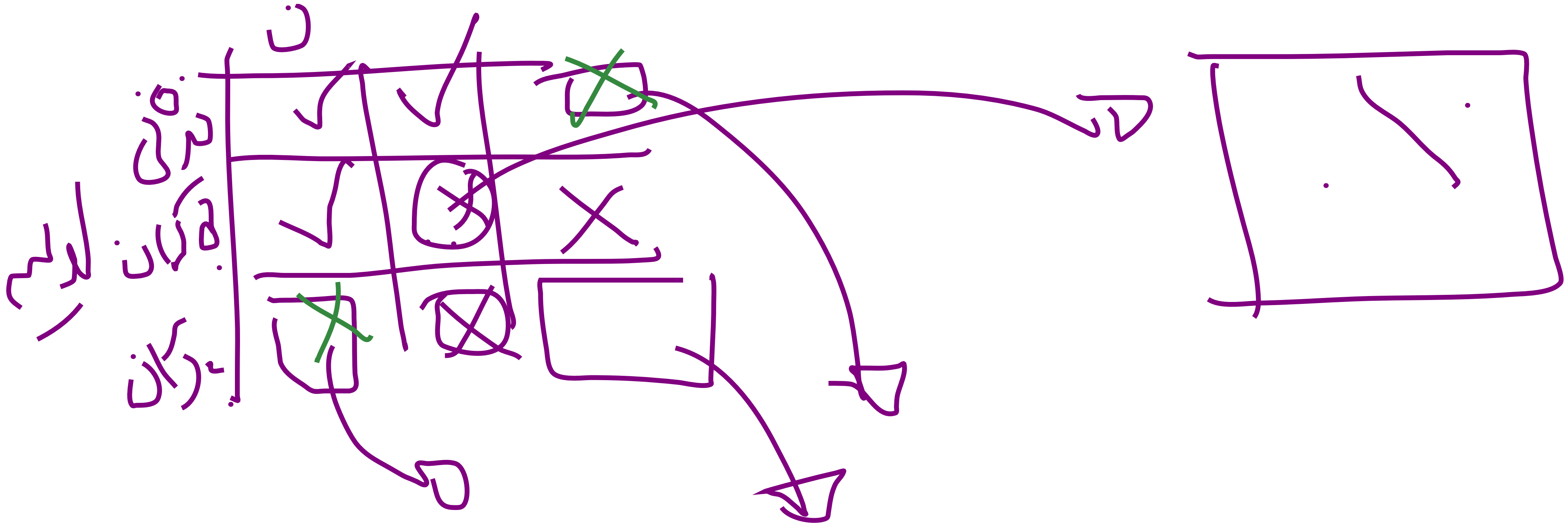
$$\begin{array}{lcl} \mathbf{x}_B & = & \mathbf{p} + Q\,\mathbf{x}_N \\ \hline z & = & z_0 + \mathbf{r}^T \mathbf{x}_N \end{array}$$

$\mathbf{y}^* = (\bar{\mathbf{c}}_B^T \bar{A}_B^{-1})^T$ یک جواب شدنی از مساله‌ی دوگان است و

$$\mathbf{c}^T \bar{\mathbf{x}}^* = \mathbf{b}^T \mathbf{y}^*$$

اگر این را اثبات کنیم:

تمام!



$\mathbf{y}^* = (\bar{\mathbf{c}}_B^T \bar{A}_B^{-1})^T$ یک جواب شدنی از مساله‌ی دوگان است و

$$\mathbf{c}^T\bar{\mathbf{x}}^* = \mathbf{b}^T\mathbf{y}^*$$

اگر این را اثبات کنیم:
تمام!

قسمت اول اثبات: $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$

$$\overline{\mathbf{x}}_B^* = \bar{A}_B^{-1}\mathbf{b}$$

$$\overline{\mathbf{x}}_N^* = \mathbf{0},$$

$$\mathbf{c}^T\mathbf{x}^* = \overline{\mathbf{c}}^T\overline{\mathbf{x}}^*$$

قسمت اول اثبات: $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$

$$\overline{\mathbf{x}}_B^* = \bar{A}_B^{-1}\mathbf{b}$$

$$\overline{\mathbf{x}}_N^* = \mathbf{0},$$

$$\mathbf{c}^T\mathbf{x}^* = \overline{\mathbf{c}}^T\overline{\mathbf{x}}^* = \overline{\mathbf{c}}_B^T\overline{\mathbf{x}}_B^*$$

قسمت اول اثبات: $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$

$$\overline{\mathbf{x}}_B^* = \bar{A}_B^{-1}\mathbf{b}$$

$$\overline{\mathbf{x}}_N^* = \mathbf{0},$$

$$\mathbf{c}^T\mathbf{x}^* = \overline{\mathbf{c}}^T\overline{\mathbf{x}}^* = \overline{\mathbf{c}}_B^T\overline{\mathbf{x}}_B^* = \overline{\mathbf{c}}_B^T(\bar{A}_B^{-1}\mathbf{b})$$

قسمت اول اثبات: $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$

$$\overline{\mathbf{x}}_B^* = \bar{A}_B^{-1}\mathbf{b}$$
$$\overline{\mathbf{x}}_N^* = \mathbf{0},$$

$$\mathbf{c}^T\mathbf{x}^* = \overline{\mathbf{c}}^T\overline{\mathbf{x}}^* = \overline{\mathbf{c}}_B^T\overline{\mathbf{x}}_B^* = \overline{\mathbf{c}}_B^T(\bar{A}_B^{-1}\mathbf{b}) = (\overline{\mathbf{c}}_B^T\bar{A}_B^{-1})\mathbf{b}$$

قسمت اول اثبات: $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$

$$\overline{\mathbf{x}}_B^* = \bar{A}_B^{-1}\mathbf{b}$$
$$\overline{\mathbf{x}}_N^* = \mathbf{0},$$

$$\mathbf{c}^T\mathbf{x}^* = \overline{\mathbf{c}}^T\overline{\mathbf{x}}^* = \overline{\mathbf{c}}_B^T\overline{\mathbf{x}}_B^* = \overline{\mathbf{c}}_B^T(\bar{A}_B^{-1}\mathbf{b}) = (\overline{\mathbf{c}}_B^T\bar{A}_B^{-1})\mathbf{b} = (\mathbf{y}^*)^T\mathbf{b} = \mathbf{b}^T\mathbf{y}^*.$$

قسمت دوم اثبات: $A^T\mathbf{y}^* \geq \mathbf{c}$ and $\mathbf{y}^* \geq \mathbf{0}$

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$$\bar{A}^T\mathbf{y}^* \geq \bar{\mathbf{c}}$$

قسمت دوم اثبات: $A^T\mathbf{y}^* \geq \mathbf{c}$ and $\mathbf{y}^* \geq \mathbf{0}$

$$\bar{A}^T\mathbf{y}^* \geq \bar{\mathbf{c}}$$

$$\bar{A}^T(\bar{\mathbf{c}}_B^T\bar{A}_B^{-1})^T = (\bar{\mathbf{c}}_B^T\bar{A}_B^{-1}\bar{A})^T$$

قسمت دوم اثبات: $A^T\mathbf{y}^* \geq \mathbf{c}$ and $\mathbf{y}^* \geq \mathbf{0}$

$$\bar{A}^T\mathbf{y}^* \geq \bar{\mathbf{c}}$$

$$\bar{A}^T(\bar{\mathbf{c}}_B^T\bar{A}_B^{-1})^T = \underbrace{(\bar{\mathbf{c}}_B^T\bar{A}_B^{-1}\bar{A})^T}_{\mathbf{w}}$$

قسمت دوم اثبات: $A^T\mathbf{y}^* \geq \mathbf{c}$ and $\mathbf{y}^* \geq \mathbf{0}$

$$\bar{A}^T\mathbf{y}^* \geq \overline{\mathbf{c}}$$

$$\mathbf{w}_B = (\overline{\mathbf{c}}_B^T \bar{A}_B^{-1} \bar{A}_B)^T = (\overline{\mathbf{c}}_B^T I_m)^T = \overline{\mathbf{c}}_B$$

قسمت دوم اثبات: $A^T\mathbf{y}^* \geq \mathbf{c}$ and $\mathbf{y}^* \geq \mathbf{0}$

$$\bar{A}^T\mathbf{y}^* \geq \overline{\mathbf{c}}$$

$$\mathbf{w}_B = (\overline{\mathbf{c}}_B^T \bar{A}_B^{-1} \bar{A}_B)^T = (\overline{\mathbf{c}}_B^T I_m)^T = \overline{\mathbf{c}}_B$$

$$\mathbf{w}_N = (\overline{\mathbf{c}}_B^T \bar{A}_B^{-1} \bar{A}_N)^T = \overline{\mathbf{c}}_N - \mathbf{r} \geq \overline{\mathbf{c}}_N$$

$$\begin{array}{lcccc} \mathbf{x}_B & = & \mathbf{p} & + & Q\,\mathbf{x}_N \\ \hline z & = & z_0 & + & \mathbf{r}^T\mathbf{x}_N \end{array}$$