بسم الله الرحمن الرحیم

تمرین‌های جایگشت و شمول و کمی گراف - درس ریاضیات گسسته نیمسال دوم ۹۲-۹۳ – دانشگاه شریف

8. In a set of n persons, any subset of four contains a person who knows the other three persons. Prove that there exists a person who knows all the others. (If A knows B then B knows A.)

6. We assign an arrow to each edge of a convex polyhedron, so that at least one arrow starts at each vertex, and at least one arrow arrives. Prove that there exist two faces of the polyhedron, so that you can trace their perimeters in the direction of the arrows (BWM).

10. In a set S of $2n$ persons there are two with an even number of common friends.

13. [*M23*] It is well known that half of the terms in the expansion of a determinant have a plus sign, and half have a minus sign. In other words, there are just as many permutations with an *even* number of inversions as with an *odd* number, when $n \geq 2$. Show that, in general, the number of permutations having a number of inversions congruent to t modulo m is $n!/m$, regardless of the integer t, whenever $n \geq m$.

1.1.27. (!) Let G be a graph with girth 5. Prove that if every vertex of G has degree at least k, then G has at least k^2+1 vertices. For $k=2$ and $k=3$, find one such graph with exactly k^2+1 vertices.

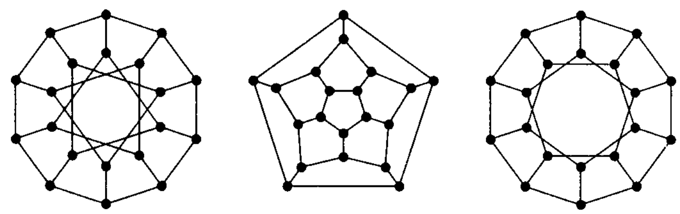
2. [*M20*] In the classical problem of Josephus (exercise 1.3.2–22), n men are initially arranged in a circle; the mth man is executed, the circle closes, and every mth man is repeatedly eliminated until all are dead. The resulting execution order is a permutation of $\{1, 2, \ldots, n\}$. For example, when $n = 8$ and $m = 4$ the order is 5 4 6 1 3 8 7 2; the inversion table corresponding to this permutation is 3 6 3 1 0 0 1 0.

Give a simple recurrence relation for the elements $b_1\, b_2 \ldots b_n$ of the inversion table in the general Josephus problem for n men, when every mth man is executed.

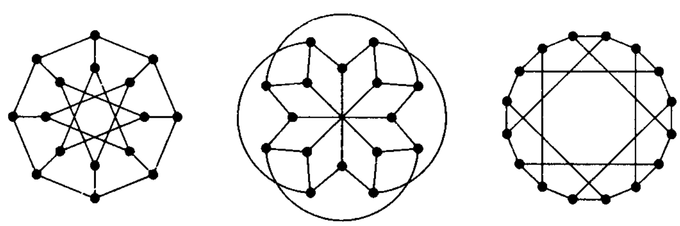
9. Two black knights stand on the lower corners of a 3×3 chessboard, and two white knights on the upper corners. White and black knights must be interchanged by legal moves onto free squares. Find the minimum number of moves needed (quoted by Lucas in 1894 from an earlier source in 1512).

1.1.14. **(!) Prove that removing opposite corner squares from an 8-by-8 checkerboard leaves a subboard that cannot be partitioned into 1-by-2 and 2-by-1 rectangles. Using the same argument, make a general statement about all bipartite graphs.**

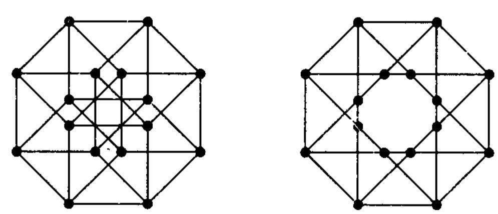
1.1.19. Determine which pairs of graphs below are isomorphic.



1.1.20. Determine which pairs of graphs below are isomorphic.



1.1.21. Determine whether the graphs below are bipartite and whether they are isomorphic. (The graph on the left appears on the cover of Wilson–Watkins [1990].)



1.1.26. (!) Let G be a graph with girth 4 in which every vertex has degree k. Prove that G has at least $2k$ vertices. Determine all such graphs with exactly $2k$ vertices.

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1.1.28. (+) *The Odd Graph* O_k. The vertices of the graph O_k are the k-element subsets of $\{1, 2, \ldots, 2k+1\}$. Two vertices are adjacent if and only if they are disjoint sets. Thus O_2 is the Petersen graph. Prove that the girth of O_k is 6 if $k \geq 3$.

1.1.30. Let G be a simple graph with adjacency matrix A and incidence matrix M. Prove that the degree of v_i is the ith diagonal entry in A^2 and in MM^T. What do the entries in position (i, j) of A^2 and MM^T say about G?

1.1.31. (!) Prove that a self-complementary graph with n vertices exists if and only if n or $n-1$ is divisible by 4. (Hint: When n is divisible by 4, generalize the structure of P_4 by splitting the vertices into four groups. For $n \equiv 1 \bmod 4$, add one vertex to the graph constructed for $n-1$.)

1.1.35. (!) Prove that K_n decomposes into three pairwise-isomorphic subgraphs if and only if $n+1$ is not divisible by 3. (Hint: For the case where n is divisible by 3, split the vertices into three sets of equal size.)

1.1.38. (!) Let G be a simple graph in which every vertex has degree 3. Prove that G decomposes into claws if and only if G is bipartite.

1.1.47. ($*$) *Edge-transitive versus vertex-transitive.*

a) Let G be obtained from K_n with $n \geq 4$ by replacing each edge of K_n with a path of two edges through a new vertex of degree 2. Prove that G is edge-transitive but not vertex-transitive.

b) Suppose that G is edge-transitive but not vertex-transitive and has no vertices of degree 0. Prove that G is bipartite.

c) Prove that the graph in Exercise 1.1.6 is vertex-transitive but not edge-transitive.