Hardness of Approximation

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Full Outline

Outline

1. MaxCut Problem and PCP Theorem
2. Unique Games Conjecture
3. Analysis Notions
4. Proof of The Main Theorem

MaxCut Problem

Definition

Let $G = (V, E)$ be an unweighted graph. MaxCut problem is to find a set $S \subset V$ such that the number of edges between S and $\overline{S}$ is maximized.

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In the year 1995 Goemans and Williamson approximated the MaxCut problem with ratio

$$\min_{z \in [-1,1]} \left\{ \frac{2 \arccos z}{\pi(1-z)} \right\} = \alpha_{GW} \approx 0.878567$$

PCP Theorem

Theorem (PCP)

For a 3SAT formula ϕ, let $OPT(\phi)$ denote the maximum fraction of clauses that can be satisfied by any assignment. Thus $OPT(\phi) = 1$ if and only if ϕ is satisfiable. The PCP Theorem states that there is a universal constant $\alpha^* < 1$ and a polynomial time reduction that maps a 3SAT instance ϕ to another 3SAT instance ψ such that

- *(Yes Case): If $OPT(\phi) = 1$ then $OPT(\psi) = 1$.*
- *(No Case): If $OPT(\phi) < 1$ then $OPT(\psi) \leq \alpha^*$.*

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Håstad showed that in the above theorem we can set $\alpha^* = \frac{7}{8} + \epsilon$ for every $\epsilon > 0$. So we can't approximate 3SAT with a ratio better than $\frac{7}{8} + \epsilon$ unless $\mathrm{P} = \mathrm{NP}$.

Hardness of Approximation With PCP Theorem

Problem	Best Approx. Known	Best Inapprox. Known
Max-E3SAT	$\frac{7}{8}$	$\frac{7}{8} + \epsilon$
Independent Set	$\frac{1}{n}$	$\frac{1}{n^{1-\epsilon}}$
Metric TSP	$\frac{3}{2} - 10^{-36}$	$\frac{123}{122}$
MaxCut	α_{GW}	$\frac{16}{17} + \epsilon$
Vertex Cover	2	1.36
Max Acyclic Subgraph	$\frac{1}{2}$	$\frac{65}{66} + \epsilon$

Outline

Unique Games Problem

Definition

An instance $L = (G((U, V), E), [M], \{\pi_{vu}\})$ of the Unique Games Problem (UG) is a bipartite graph G with vertex set (U, V) and edges E. The set $[M]$ is a set of M labels. For each edge $(u, v) \in U \times V$, there is a permutation $\pi_{vu} : [M] \to [M]$. The problem is to find an assignment $\ell : U \cup V \to [M]$ of labels that maximizes the number of edges which are satisfied. (An edge (u, v) is satisfied if $\ell(u) = \pi_{vu}(\ell(v))$.)

Unique Games Conjecture

Definition

For some $\delta > 0$ and a given UG instance L, Gap-UG$_{1-\delta,\delta}$ is the decision problem of distinguishing between the following two cases:

- There is a labeling ℓ under which at least $(1-\delta)|E|$ edges are satisfied ($\text{OPT}(L) \geq (1-\delta)|E|$).
- For any labeling ℓ at most $\delta|E|$ edges are satisfied ($\text{OPT}(L) \leq \delta|E|$).

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Conjecture (Unique Games Conjecture)

For any $\delta > 0$, there is an M such that it is NP-hard to decide Gap-UG$_{1-\delta,\delta}$ on instances with label set of size M.

Unique Games Conjecture

Theorem

Assuming the UGC, if there is a polynomial-time reduction from Gap-$UG_{1-\delta,\delta}$ to Gap-$MaxCut_{c,s}$ then it is NP-hard to approximate MaxCut with ratio better than $\frac{s}{c}$

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Proof.

Assume the contrary. Let L be a UG instance. The reduction gives an instance MC of MaxCut such that if $\mathrm{OPT}(L) \geq (1-\delta)|E_L|$ then $\mathrm{OPT}(MC) \geq c\,|E_{MC}|$ and if $\mathrm{OPT}(L) \leq \delta\,|E_L|$ then $\mathrm{OPT}(MC) \leq s\,|E_{MC}|$. Run the supposed $\left(\frac{s}{c}\right)$-approximation algorithm. if $\mathrm{OPT}(L) \geq (1-\delta)|E_L|$ then this will produce a cut of size at least $\left(\frac{s}{c}\right) c\,|E_{MC}| = s\,|E_{MC}|$. If $\mathrm{OPT}(L) \leq \delta\,|E_L|$ the algorithm will be unable to produce a cut with size at least $s\,|E_{MC}|$. Thus we can distinguish the two cases and get a polynomial time algorithm for Gap-$\mathrm{UG}_{1-\delta,\delta}$. Contradiction. □

Main Theorem

Theorem (Main Theorem)

For every $\rho \in (-1, 0)$ and $\epsilon > 0$ there is some $\delta > 0$ and a polynomial-time reduction from Gap-UG$_{1-\delta,\delta}$ to Gap-MaxCut$_{\frac{1-\rho}{2}-\epsilon, \frac{1}{\pi}\arccos\rho+\epsilon}$.

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Assuming the UGC, the above theorem shows that there is no polynomial time approximation for the MaxCut problem with ratio better than

$$\min_{\rho\in(-1,0)} \left\{ \frac{\frac{1}{\pi}\text{arccos}\,\rho+\epsilon}{\frac{1-\rho}{2}-\epsilon} \right\} = \alpha_{GW} + \epsilon'$$

Unless $\text{P} = \text{NP}$.

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Fourier Expansion

Definition

Every real-valued function $f : \{-1, 1\}^n \to \mathbb{R}$ has a unique expansion such that

$$f(x) = \sum_{S \subseteq [n]} \hat{f}(S)\chi_S(x), \quad \chi_S(x) = \prod_{i \in S} x_i$$

The coefficients $\hat{f}(S)$ are known as Fourier coefficients and the entire sum is known as the Fourier expansion.

Inner Product

Definition

We can define an inner product for the space of all real-valued functions over $\{-1, 1\}^n$ such that

$$\langle f, g \rangle = 2^{-n} \sum_{x \in \{-1,1\}^n} f(x)g(x).$$

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With this definition it is clear that $||\chi_S|| = 1$ for all $S \subseteq [n]$ and $\langle \chi_{S_1}, \chi_{S_2}\rangle = 0$ for all $S_1 \neq S_2 \subseteq [n]$. So the functions χ_S, for all $S \subseteq [n]$, form an orthonormal basis for this space. This shows that

$$\hat{f}(S) = \langle f, \chi_S\rangle.$$

Influential Variables

Definition

for a function $f : \{-1,1\}^n \to \mathbb{R}$ the influence of a variable x_i is defined to be

$$\mathrm{Inf}_i(f) = \Pr_{x \in \{-1,1\}^n} \left(f(x) \neq f\left(x_1, x_2, \ldots, -x_i, \ldots, x_n\right)\right)$$

Example

- $\mathrm{Inf}_i\left(x_j\right) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ (Dictatorship function)
- $\mathrm{Inf}_i\left(\chi_S\right) = \begin{cases} 0 & i \notin S \\ 1 & i \in S \end{cases}$
- $\mathrm{Inf}_i(\text{majority}) = \Theta\left(\frac{1}{\sqrt{n}}\right), \quad \text{majority(x)} = \operatorname{sgn}\left(\sum_{i=1}^{n} x_i\right)$

Influential Variables

Theorem

for a function $f : \{-1, 1\}^n \to \mathbb{R}$ *and an integer number* $1 \leq i \leq n$,

$$Inf_i(f) = \sum_{S \ni i} \hat{f}(S)^2$$

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The Main Theorem

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For every $\rho \in (-1, 0)$ and $\epsilon > 0$ there is some $\delta > 0$ and a polynomial-time reduction from Gap-UG$_{1-\delta,\delta}$ to Gap-MaxCut$_{\frac{1-\rho}{2}-\epsilon, \frac{1}{\pi}\arccos\rho+\epsilon}$.

We present the reduction in two parts.

The Reduction, Part 1

Let $L = (G((U,V),E),[M],\{\pi_{vu}\})$ be an instance of Gap-UG$_{1-\delta,\delta}$. For each vertex in V, we introduce a cloud of 2^M vertices for MaxCut instance $MC = G'(V',E')$. So the set of vertices of MC is $V' = V \times \{-1,1\}^M$.

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- Pick $\mu \in \{-1,1\}^M$ such that $\Pr(\mu_i = 1) = \frac{1+\rho}{2}$ and $\Pr(\mu_i = -1) = \frac{1-\rho}{2}$. Then set $y = \mu x$, represents a vertex from the cloud of w.

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Finally the edge we output is $(v \times (x \circ \pi_{vu}), w \times (y \circ \pi_{wu}))$. This process defines a distribution over the edges of G', so we can think of G' as a weighted graph where the weights are equal to the relevant probabilities.

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We define a partition in G' by assigning each vertex $v \times x$ in the cloud of a vertex $v \in V$ according to $f_v : \{-1,1\}^M \to \{-1,1\}$ which is defined as follows

$$f_v(x) = x_{\ell(v)}.$$

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Therefore

$$\mathrm{val}(\mathrm{cut}(S,\overline{S})) = \sum_{e \in E'(S,\overline{S})} \mathrm{weight}(e) = \Pr_{e \sim D}(e \text{ is in the cut}).$$

The Reduction, Part 1

Suppose that $e = (v \times (x \circ \pi_{vu}), w \times (y \circ \pi_{wu}))$. Also assume that both edges relevant to e of the label cover are satisfied by ℓ. By the union bound we have

$$\begin{aligned} \Pr(vu \text{ or } wu \text{ not satisfied}) &\leq \Pr(vu \text{ not satisfied}) + \Pr(wu \text{ not satisfied}) \\ &\leq \delta + \delta = 2\delta. \end{aligned}$$

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Hence the probability that both edges relevant to e of the label cover are satisfied by ℓ is at least $1 - 2\delta$.

The Reduction, Part 1

Now we look at the assignment to the endpoints of e:

$$f_v\left(x \circ \pi_{vu}\right) = \left(x \circ \pi_{vu}\right)_{\ell(v)} = x_{\pi_{vu}(\ell(v))}$$

$$f_w\left(y \circ \pi_{wu}\right) = \left(y \circ \pi_{wu}\right)_{\ell(w)} = y_{\pi_{wu}(\ell(w))}$$

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Since u is a common neighbour of v and w, $\pi_{vu}(\ell(v)) = \ell(u) = \pi_{wu}(\ell(w))$. Thus according to D, the endpoints of e are on different sides with probability $\frac{1-\rho}{2}$.

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$\pi_{vu}(\ell(v)) = \ell(u) = \pi_{wu}(\ell(w))$. Thus according to D, the endpoints of e are on different sides with probability $\frac{1-\rho}{2}$.
In conclusion

$$\mathrm{val}(\mathrm{cut}(S, \overline{S})) \geq (1 - 2\delta)\left(\frac{1-\rho}{2}\right)$$

which is greater than $\frac{1-\rho}{2} - \epsilon$ if δ is chosen to be smaller than $\frac{\epsilon}{2}$.

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In this part we want to prove that if every labeling satisfies at most a $\delta|E|$ constraints, then G' has no cut with value more than $\frac{\arccos \rho}{\pi} + \epsilon$.

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In order to do this, we will prove the contrapositive: If G' has a cut with value at least $\frac{\arccos \rho}{\pi} + \epsilon$ then there is a labeling that satisfies at least $\delta|E|$ constraints.

The Reduction, Part 2

For every vertex $v \in V$, let $f_v : \{-1,1\}^M \to \{-1,1\}$ be assignments to G' vertices with cut value at least $\frac{\arccos \rho}{\pi} + \epsilon$. According to the definition of the reduction, the value of the cut is as follows:

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$$\begin{aligned}
\mathrm{val}(\mathrm{cut}(S, \overline{S})) &= \mathop{\mathbb{E}}_{u,v,w,x,y}\left[\frac{1 - f_v(x \circ \pi_{vu}) f_w(y \circ \pi_{wu})}{2}\right] \\
&= \frac{1}{2} - \frac{1}{2} \mathop{\mathbb{E}}_{u,x,y}\left[\mathop{\mathbb{E}}_{v,w}[f_v(x \circ \pi_{vu}) f_w(y \circ \pi_{wu})]\right] \\
&= \frac{1}{2} - \frac{1}{2} \mathop{\mathbb{E}}_{u,x,y}\left[\mathop{\mathbb{E}}_{v}[f_v(x \circ \pi_{vu})] \cdot \mathop{\mathbb{E}}_{w}[f_w(y \circ \pi_{wu})]\right] \\
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\end{aligned}$$

where we defined $g_u(z) = \mathop{\mathbb{E}}_{v \sim u}[f_v(z \circ \pi_{vu})]$.

Noise Stability

Definition

Let $f : \{-1,1\}^n \to \mathbb{R}$ and let $\rho \in [-1,1]$. The noise stability of f at ρ is defined as follows: Let x be a uniformly random string in $\{-1,1\}^n$ and let y be a 'ρ-correlated' copy; i.e. pick each bit y_i independently so that $\mathbb{E}[x_i y_i] = \rho$. Then the noise stability is defined to be

$$\mathbb{S}_\rho(f) = \mathop{\mathbb{E}}_{x,y}[f(x)f(y)].$$

The Reduction, Part 2

From the definition of noise stability we have

$$\frac{\text{arccos}\,\rho}{\pi} + \epsilon \le \text{val}(\text{cut}(S, \overline{S})) = \frac{1}{2} - \frac{1}{2}\mathop{\mathbb{E}}_{u}[\mathbb{S}_\rho(g_u)].$$

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From the definition of noise stability we have

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Thus $\mathop{\mathbb{E}}_{u}[\mathbb{S}_\rho(g_u)] \le 1 - \frac{2}{\pi}\arccos\rho - 2\epsilon$. Now Markov's inequality implies that

$$\begin{aligned}
\Pr_u\left(\mathbb{S}_\rho(g_u) \ge 1 - \frac{2}{\pi}\arccos\rho - \epsilon\right) &\le \frac{\mathop{\mathbb{E}}_{u}\left[\mathbb{S}_\rho(g_u)\right]}{1 - \frac{2}{\pi}\arccos\rho - \epsilon} \\
&\le \frac{1 - \frac{2}{\pi}\arccos\rho - 2\epsilon}{1 - \frac{2}{\pi}\arccos\rho - \epsilon} \le 1 - \frac{\epsilon}{2}.
\end{aligned}$$

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Therefore, at least $\frac{\epsilon}{2}$-fraction of vertices $u \in U$ satisfy

$$\mathbb{S}_\rho(g_u) \leq 1 - \frac{2}{\pi}\arccos\rho - \epsilon.$$

We call such vertices U-good.

Majority is Stablest Theorem

Theorem (Majority is Stablest)

Fix $\rho \in (-1, 0]$. Then for any $\epsilon > 0$ there is a small enough $\tau > 0$ such that if $f : \{-1, 1\}^n \to [-1, 1]$ is any function satisfying

$$Inf_i(f) \leq \tau \quad \textit{for all } i = 1, 2, \ldots, n,$$

then

$$\mathbb{S}_\rho(f) \geq 1 - \frac{2}{\pi} \arccos \rho - \epsilon.$$

The Reduction, Part 2

By the Majority is Stablest theorem, for any U-good vertex like u, there is a small enough $\tau > 0$ and a coordinate j that satisfies $\mathrm{Inf}_j(g_u) > \tau$. We label u as $\ell(u) = j$, and the other not good vertices arbitrarily. Then

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$$\begin{aligned}
\tau < \sum_{S \ni j} \hat{g_u}(S)^2 &= \sum_{S \ni j} \mathop{\mathbb{E}}_{v} \left[\hat{f}_v(\pi_{vu}^{-1}(S)) \right]^2 \\
&\leq \sum_{S \ni j} \mathop{\mathbb{E}}_{v} \left[\hat{f}_v(\pi_{vu}^{-1}(S))^2 \right] \\
&= \mathop{\mathbb{E}}_{v} \left[\mathrm{Inf}_{\pi_{vu}^{-1}(j)}(f_v) \right] \\
&= \sum_{v \in N(u)} \frac{1}{|N(u)|} \mathrm{Inf}_{\pi_{vu}^{-1}(j)}(f_v) \qquad (1)
\end{aligned}$$

The Reduction, Part 2

We claim that for more than $\frac{\tau}{2}|N(u)|$ neighbours of u like v we have $\mathrm{Inf}_{\pi_{vu}^{-1}(j)}(f_v) \geq \frac{\tau}{2}$. Suppose the contrary. So for at least $\left(1 - \frac{\tau}{2}\right)|N(u)|$ neighbours of u like v we have $\mathrm{Inf}_{\pi_{vu}^{-1}(j)}(f_v) < \frac{\tau}{2}$. Then

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$$\begin{aligned}
\tau &\overset{(1)}{<} \sum_{v \in N(u)} \frac{1}{|N(u)|} \mathrm{Inf}_{\pi_{vu}^{-1}(j)}(f_v) \\
&< \frac{1}{|N(u)|}\left(\frac{\tau}{2}\left(1 - \frac{\tau}{2}\right)|N(u)| + 1 \cdot \frac{\tau}{2}|N(u)|\right) \\
&= \frac{\tau}{2}\left(2 - \frac{\tau}{2}\right) < \tau,
\end{aligned}$$

We claim that for more than $\frac{\tau}{2}|N(u)|$ neighbours of u like v we have $\mathrm{Inf}_{\pi_{vu}^{-1}(j)}(f_v) \geq \frac{\tau}{2}$. Suppose the contrary. So for at least $\left(1 - \frac{\tau}{2}\right)|N(u)|$ neighbours of u like v we have $\mathrm{Inf}_{\pi_{vu}^{-1}(j)}(f_v) < \frac{\tau}{2}$. Then

$$\begin{aligned}
\tau &\overset{(1)}{<} \sum_{v \in N(u)} \frac{1}{|N(u)|} \mathrm{Inf}_{\pi_{vu}^{-1}(j)}(f_v) \\
&< \frac{1}{|N(u)|} \left(\frac{\tau}{2} \left(1 - \frac{\tau}{2}\right) |N(u)| + 1 \cdot \frac{\tau}{2} |N(u)| \right) \\
&= \frac{\tau}{2} \left(2 - \frac{\tau}{2}\right) < \tau,
\end{aligned}$$

that clearly is a contradiction, hence our claim is proved. We call those vertices V-good.

In order to define labels for any $v \in V$ we define a set of candidates:

$$\mathrm{Cand}[v] = \left\{ i \in [M] : \ \mathrm{Inf}_i(f_v) \geq \frac{\tau}{2} \right\}$$

The Reduction, Part 2

In order to define labels for any $v \in V$ we define a set of candidates:

$$\mathsf{Cand}[v] = \left\{ i \in [M] : \ \mathsf{Inf}_i(f_v) \geq \frac{\tau}{2} \right\}$$

Thus for every V-good vertex like v, $\pi_{vu}^{-1}(j) \in \mathsf{Cand}[v]$. Moreover since

$$\sum_{i \in [M]} \mathsf{Inf}_i(f_v) \leq M,$$

we get that $|\mathsf{Cand}[v]| \leq \frac{2M}{\tau}$.

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we get that $|\mathrm{Cand}[v]| \leq \frac{2M}{\tau}$.
Now we assign each $v \in V$ uniformly at random a label in $\mathrm{Cand}[v]$. Finally for any U-good vertex like u and V-good vertex like v, if the candidate that was chosen for v is $\pi_{vu}^{-1}(\ell(u))$, then the edge (u, v) is satisfied. Therefore at least $\left(\frac{\epsilon}{2} \cdot \frac{\tau}{2} \cdot \frac{\tau}{2M}\right)$-fraction of the edges are satisfied. So we just need to set

$$\delta = \min \left\{ \frac{\epsilon}{2}, \ \frac{\epsilon}{2} \cdot \frac{\tau}{2} \cdot \frac{\tau}{2M} \right\}$$

and this completes the proof. $\square$

Raghavendra's Result

Theorem

For every CSP $\mathcal{C}$, the integrality gap $\alpha_{\mathcal{C}}$ for the canonical semidefinite relaxation is same as the inapproximability threshold for the CSP, modulo the Unique Games Conjecture.

Hardness of Approximation With The UGC

Problem	Best Approx. Known	Best Inapprox. Known	Inapprox. Known Under UGC
MaxCut	α_{GW}	$\frac{16}{17} + \epsilon$	$\alpha_{GW} + \epsilon$
Vertex Cover	2	1.36	$2 - \epsilon$
Max Acyclic Subgraph	$\frac{1}{2}$	$\frac{65}{66} + \epsilon$	$\frac{1}{2} + \epsilon$
Feedback Arc Set	$O(\log n)$	1.36	$\omega(1)$
Coloring 3-colorable Graphs	$n^{0.199}$	5	$\omega(1)$[1]

[1] Under a stronger conjecture.

Algorithms

Algorithm	Value of Solution Found on $1-\delta$ Satisfiable Instance With M Labels and n Vertices
Khot	$1-O\left(n^2\delta^{\frac{1}{5}}\sqrt{\log\left(\frac{1}{\delta}\right)}\right)$
Trevisan	$1-O\left(\sqrt[3]{\delta\log n}\right)$
Gupta, Talwar	$1-O\left(\delta\log n\right)$
Charikar, Makarychev, Makarychev	$1-O\left(\sqrt{\delta\log M}\right)$ $M^{-\frac{\delta}{2}}$
Chlamtac, Makarychev, Makarychev	$1-O\left(\delta\sqrt{\log n\log M}\right)$

Algorithms

Algorithm	Value of Solution Found on $1-\delta$ Satisfiable Instance With M Labels and n Vertices
Arora, Khot, Kolla, Steurer, Tulsiani, Vishnoi	$1 - O\left(\delta \frac{1}{\lambda} \log\left(\frac{\lambda}{\delta}\right)\right)$ on graphs with eigenvalue gap λ.
Arora, Barak, Steurer	$1-\delta^\alpha$ for some $0 < \alpha < 1$ in time $\exp\left(n^{\delta^\alpha}\right)$

I also recommend this talk in MIT about Lasserre hierarchy. As far as we know, it is possible (though not likely) that a constant number of rounds of the Lasserre hierarchy already gives an efficient algorithm for the Unique Games problem, disproving the UGC.

First Steps Towards Proving The UGC

Theorem

For any $\delta > 0$, there is an M such that it is NP-hard to decide Gap-UG$_{\frac{1}{2},\delta}$ on instances with label set of size M.

Subhash Khot: 'As far as the author knows (and we have consulted the algorithmic experts), the known algorithmic attacks on the Unique Games problem work equally well whether the completeness is ≈ 1 or whether it is $\frac{1}{2}$. Thus, the implication that Gap-UG$_{\frac{1}{2},\delta}$ is NP-hard is a compelling evidence, in our opinion, that the known algorithmic attacks are (far) short of disproving the Unique Games Conjecture.'