بسم الله الرحمن الرحیم

تمرین‌های سوم گراف – درس ریاضیات گسسته نیمسال دوم ۹۲-۹۳ – دانشگاه شریف

**1.4.18.** (∗) Prove that a digraph having no cycle has a unique kernel.

**1.4.23.** Prove that every graph $G$ has an orientation $D$ that is "balanced" at each vertex, meaning that $\left|d_D^+(v) - d_D^-(v)\right| \le 1$ for every $v \in V(G)$.

**2.1.18.** (!) Prove that every tree with maximum degree $\Delta > 1$ has at least $\Delta$ vertices of degree 1. Show that this is best possible by constructing an $n$-vertex tree with exactly $\Delta$ leaves, for each choice of $n, \Delta$ with $n > \Delta \ge 2$.

**1.4.14.** (!) Let $G$ be an $n$-vertex digraph with no cycles. Prove that the vertices of $G$ can be ordered as $v_1, \ldots, v_n$ so that if $v_i v_j \in E(G)$, then $i < j$.

**1.4.10.** (!) Prove that a digraph is strongly connected if and only if for each partition of the vertex set into nonempty sets $S$ and $T$, there is an edge from $S$ to $T$.

**1.4.11.** (!) Prove that in every digraph, some strong component has no entering edges, and some strong component has no exiting edges.

**1.4.12.** Prove that in a digraph the connection relation is an equivalence relation, and its equivalence classes are the vertex sets of the strong components.

**1.4.17.** (∗) Prove that a (directed) odd cycle is a digraph with no kernel. Construct a digraph that has an odd cycle as an induced subgraph but does have a kernel.

**1.4.33.** (∗) Let $A$ and $B$ be two $m$ by $n$ matrices with entries in $\{0, 1\}$. An *exchange* operation substitutes a submatrix of the form $\binom{01}{10}$ for a submatrix of the form $\binom{10}{01}$ or vice versa. Prove that if $A$ and $B$ have the same list of row sums and have the same list of column sums, then $A$ can be transformed into $B$ by a sequence of exchange operations. Interpret this conclusion in the context of bipartite graphs. (Ryser [1957])