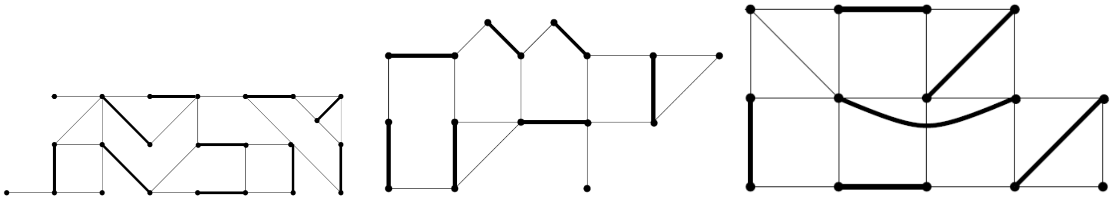
بهینه‌سازی ترکیبیاتی (پاییز ۹۵-۹۴) - تمرین سری سوم

سوال ۱:

Apply the matching augmenting algorithm to the matchings in the following graphs:



سوال ۲:

Consider the perfect matching polytope $P = \text{conv}\{\chi_M : M \text{ is a perfect matching in } G\}$. An edge is a line segment between two vertices $s = [\chi_M, \chi_N]$ such that $s = H \cap P$ for some hyperplane $H = \{x : w^T x = \lambda\}$ such that $w^T x \leq \lambda$ for all $x \in P$. Prove that $[\chi_M, \chi_N]$ is an edge if and only if $M \Delta N$ is a single cycle.

سوال ۳:

Prove that in a matrix, the maximum number of nonzero entries with no two in the same line (=row or column), is equal to the minimum number of lines that include all nonzero entries.

سوال ۴:

Let $G = (V, E)$ be a bipartite graph and let $b : V \to Z^+$. Show that G has a subgraph $G' = (V, E')$ such that $deg_{G'}(v) = b(v)$ for each $v \in V$ if and only if each $X \subseteq V$ contains at least $\frac{1}{2}(\sum_{v \in X} b(v) - \sum_{v \in V \setminus X} b(v))$ edges.

سوال ۵:

Let G = (V, E) be a graph. Show that the convex hull of the incidence vectors of matchings of size at least k and at most l is equal to the intersection of the matching polytope of G with the set $\{x | k \leq 1^T x \leq l\}$.

سوال ۶:

Let $A_1, \ldots, A_n$ be a collection of nonempty subsets of the finite set X so that each element in X is in exactly two sets among $A_1, \ldots, A_n$. Show that there exists a set Y intersecting all sets $A_1, \ldots, A_n$, and satisfying $|Y| \leq t$ if and only if for each subset I of $\{1, \ldots, n\}$ the number of components of $(A_i | i \in I)$ containing an odd number of sets in $(A_i | i \in I)$ is at most $2t - |I|$.

(Here a subset Y of X is called a component of $(A_i | i \in I)$ if it is a minimal nonempty subset of X with the property that for each $i \in I : A_i \cap Y = \emptyset or A_i \subseteq Y$.)

سوال ۷:

Let $G = (V, E)$ be a graph and let T be a subset of V. Then G has a matching covering T if and only if the number of odd components of $G - W$ contained in T is at most $|W|$, for each $W \subseteq V$.

سوال ۸:

Let $G = (V, E)$ be a graph and let $b : V \to Z^+$. Show that there exists a function $f : E \to Z^+$ so that for each $v \in V : X \sum_{e \in E, v \in e} f(e) = b(v)$ if and only if for each subset W of V the number (W) is at most $b(V\ W)$.

(Here for any subset W of V, $b(W) := \sum_{v \in W} b(v)$. Moreover, (W) denotes the following. Let U be the set of isolated vertices in the graph $G|W$ induced by W and let t denote the number of components C of the graph $G|W\backslash U$ with $b(C)$ odd. Then $(W) := b(U) + t$.)

سوال ۹:

Let $G = (V, E)$ be a graph and let $b : V \to Z^+$. Show that G has a subgraph $G' = (V, E')$ such that $deg_{G'}(v) = b(v)$ for each $v \in V$ if and only if for each two disjoint subsets U and W of V one has $\sum_{v \in U} b(v) \geq q(U, W) + \sum_{v \in W} (b(v) - d_{G-U}(v))$

Here $q(U, W)$ denotes the number of components K of $G - (U \cup W)$ for which $b(K)$ plus the number of edges connecting K and W, is odd. Moreover, $d_{G-U}(v)$ is the degree of v in the subgraph induced by $V\backslash U$.

سوال ۱۰:

A partially ordered set (or poset) is defined to be a set S together with a partial order on S, i.e. a relation $R \subseteq S \times S$ that is reflexive ($(x, x) \in R$ for all $x \in S$), anti-symmetric (if $(x, y) \in R$ and $(y, x) \in R$ then $x = y$), and transitive (if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$). Two elements $x, y \in S$ are called comparable if $(x, y) \in R$ or $(y, x) \in R$, otherwise they are incomparable. A chain (an antichain) is a subset of pairwise comparable (incomparable) elements of S. Use König's Theorem to prove the following theorem of Dilworth [1950]:

In a finite poset the maximum size of an antichain equals the minimum number of chains into which the poset can be partitioned.

Hint: Take two copies v' and v'' of each $v \in S$ and consider the graph with an edge $\{v', w''\}$ for each $(v, w) \in R$. (Fulkerson [1956])

سوال ۱۱:

Prove that every 3-regular graph with at most two bridges has a perfect matching. Is there a 3-regular graph without a perfect matching?

Hint: Use Tutte's Theorem.

سوال ۱۲:

Let $G = (V, E)$ be a bipartite graph and let $b : V \to Z^+$. Show that the maximum number of edges in a subset F of E so that each vertex v of G is incident with at most $b(v)$ of the edges in F, is equal to $\min_{X \subseteq V} \sum_{v \in X} b(v) + |E(V \backslash X)|$.

موفق باشید