
بسم الله الرحمن الرحیم
جلسه سیزدهم
خلاصه‌سازی برای مه‌داده

مرور

- تبدیل JL: با داشتن n بردار در Rd، تولید n بردار در $\mathbb{R}^{O(\epsilon^{-2}\log n)}$ که تقریبا همه طول‌ها همانند.
  - نکته: تبدیلمان حساس به ورودی است.
  - زمان هر تبدیل: O(md) که $m = \epsilon^{-2}\log n$
- تبدیل سریع JL:
  - زمان تبدیل هر:
  - $O(d\log d + m)$
  - که $m = O(\epsilon^{-2}\log(\frac{1}{\delta})\log(\frac{d}{\delta}))$

O(m)

O(d)

هر سطر یک عدد ۱
یکنواخت در ستون‌ها

نمونه‌گیری

کانولوشن

ماتریس قطری با قطر تصادفی
+۱ یا −۱

$$\Pi = \frac{1}{\sqrt{m}} SHD$$

پخش

$H^\top H = d \cdot I$

$\max_{i,j} |H_{i,j}| \le 1$

O(d lg d)

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- در ادامه: ؟؟؟

سریع‌تر برای تنک‌ها

Fast JL Transforms

هدف

Theorem 5.0.1 (JL lemma [JL84]). *For any $\varepsilon \in (0,1)$ and any $X \subset \mathbb{R}^d$ for $|X| = n$ finite, there exists an embedding $f : X \to \mathbb{R}^m$ for $m = O(\varepsilon^{-2} \log n)$ such that*

$$\forall x, y \in X,\ (1-\varepsilon)\|x-y\|_2^2 \leq \|f(x) - f(y)\|_2^2 \leq (1+\varepsilon)\|x-y\|_2^2. \tag{5.1}$$

Πz

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$||z||_0 \ll d$

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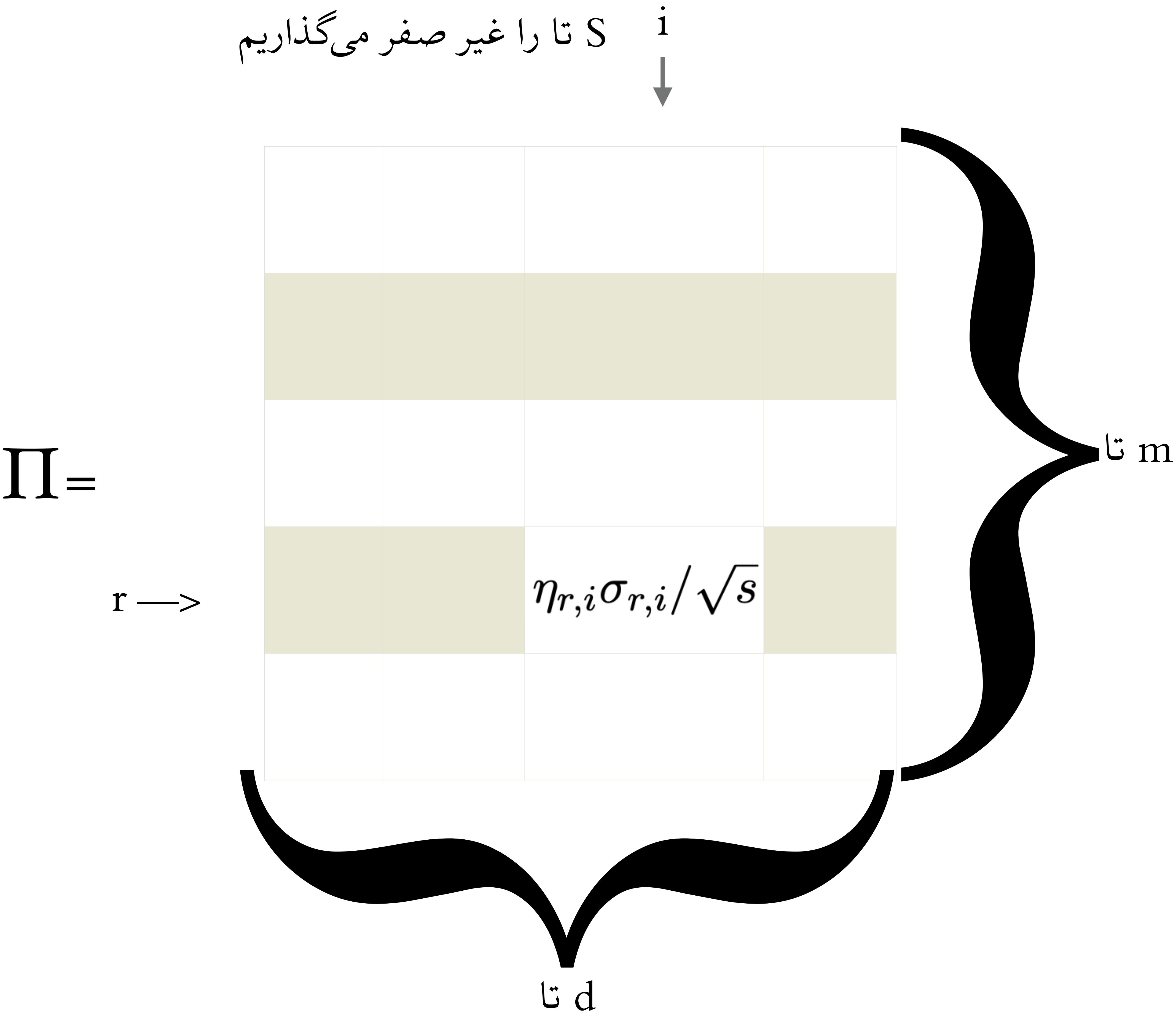
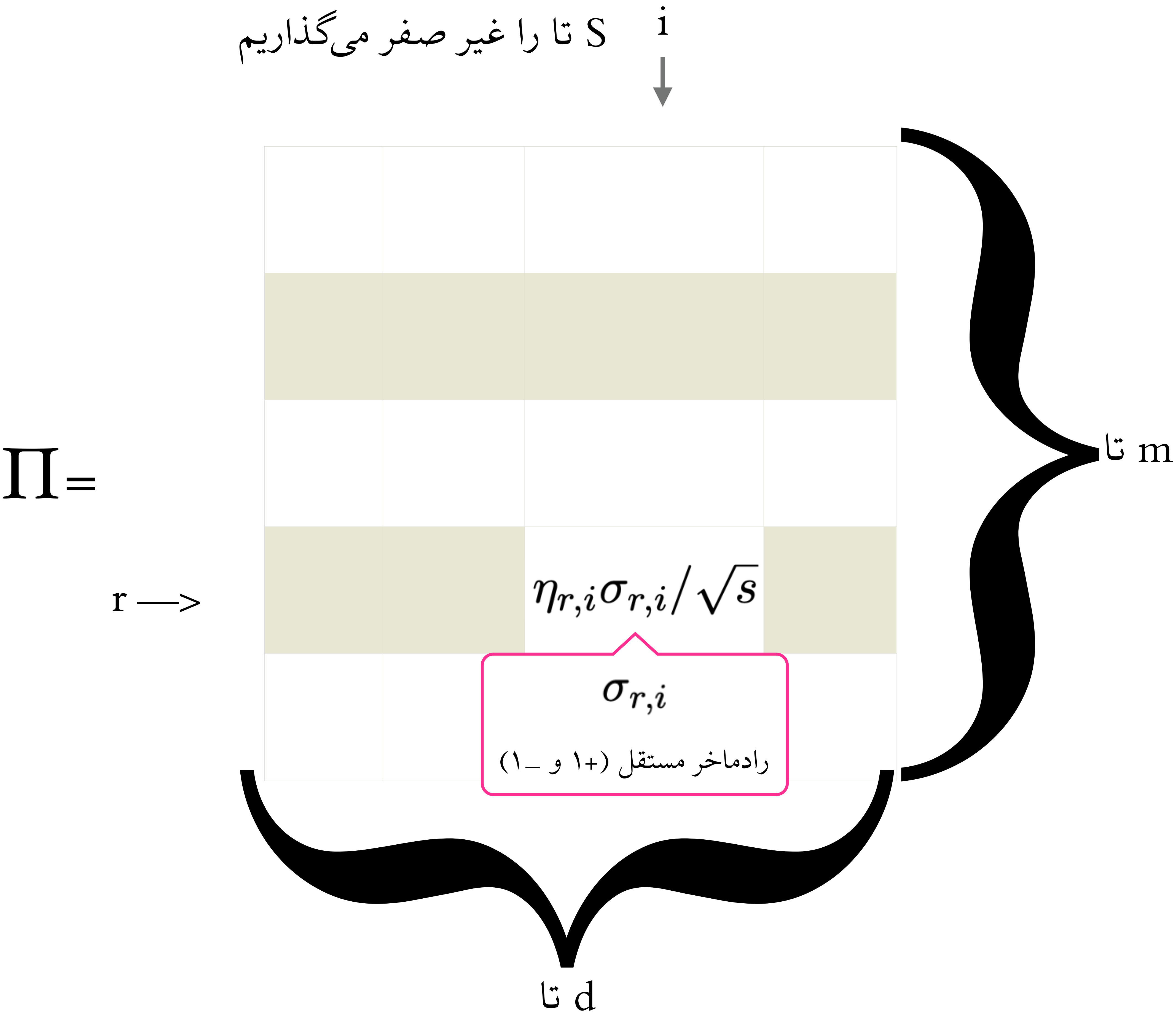
زمان اجرای ضرب:
$O(s||z||_0)$


i
S تا را غیر صفر می‌گذاریم
Π=
r —>
$\eta_{r,i}\sigma_{r,i}/\sqrt{s}$
m تا
d تا



S تا را غیر صفر می‌گذاریم
i
$\Pi =$
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رادماخر مستقل (+۱ و −۱)
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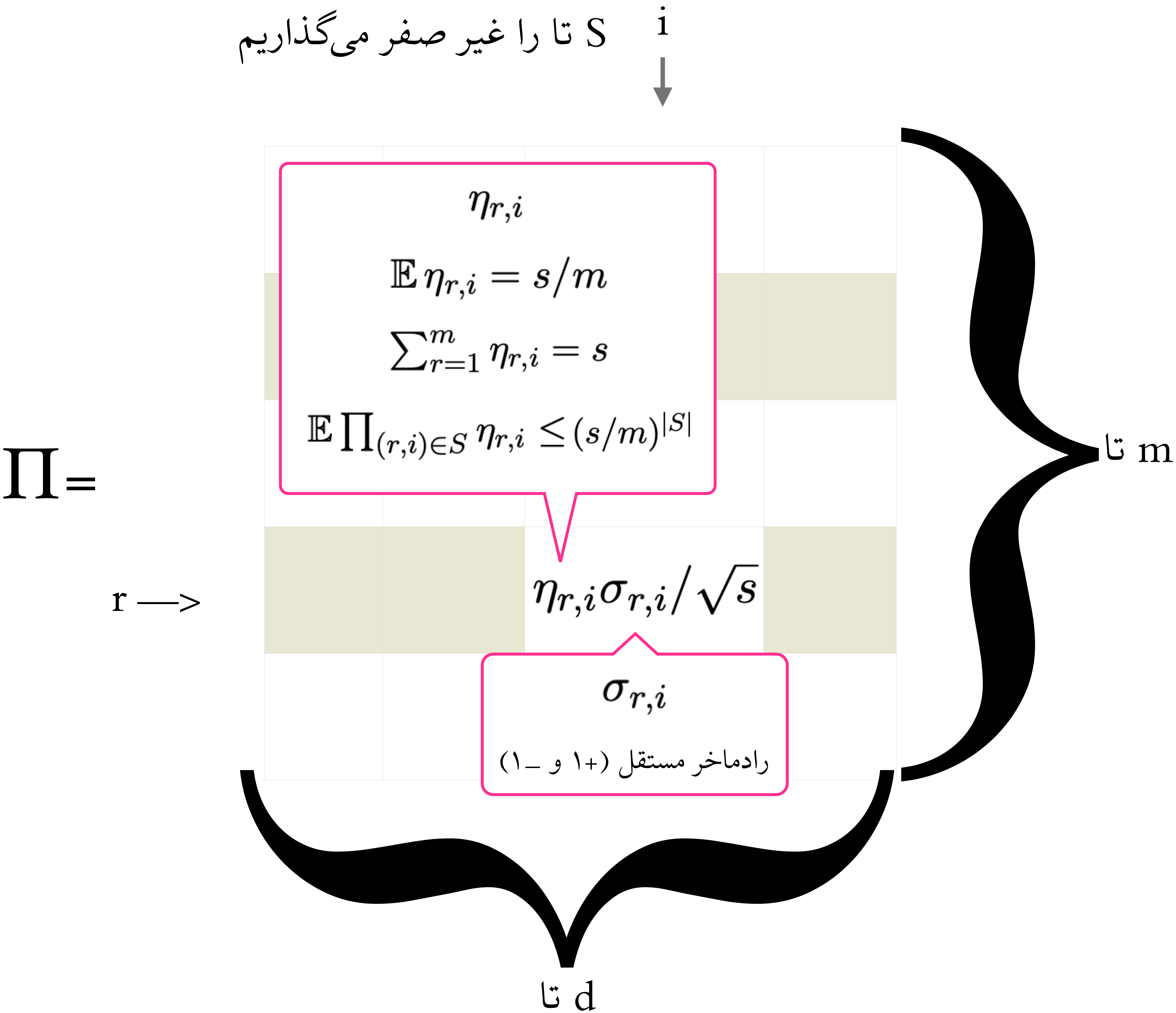

S تا را غیر صفر می‌گذاریم
i
$\eta_{r,i}$
$\mathbb{E}\,\eta_{r,i} = s/m$
$\sum_{r=1}^{m} \eta_{r,i} = s$
$\mathbb{E}\prod_{(r,i)\in S} \eta_{r,i} \leq (s/m)^{|S|}$
m تا
$\prod =$
r —>
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Theorem 5.3.1. *As long as* $m \simeq \varepsilon^{-2}\log(1/\delta)$ *and* $s \simeq \varepsilon m$,

$$\forall z : \|z\|_2 = 1,\ \mathbb{P}_{\Pi}(|\|\Pi z\|_2^2 - 1| > \varepsilon) < \delta.$$

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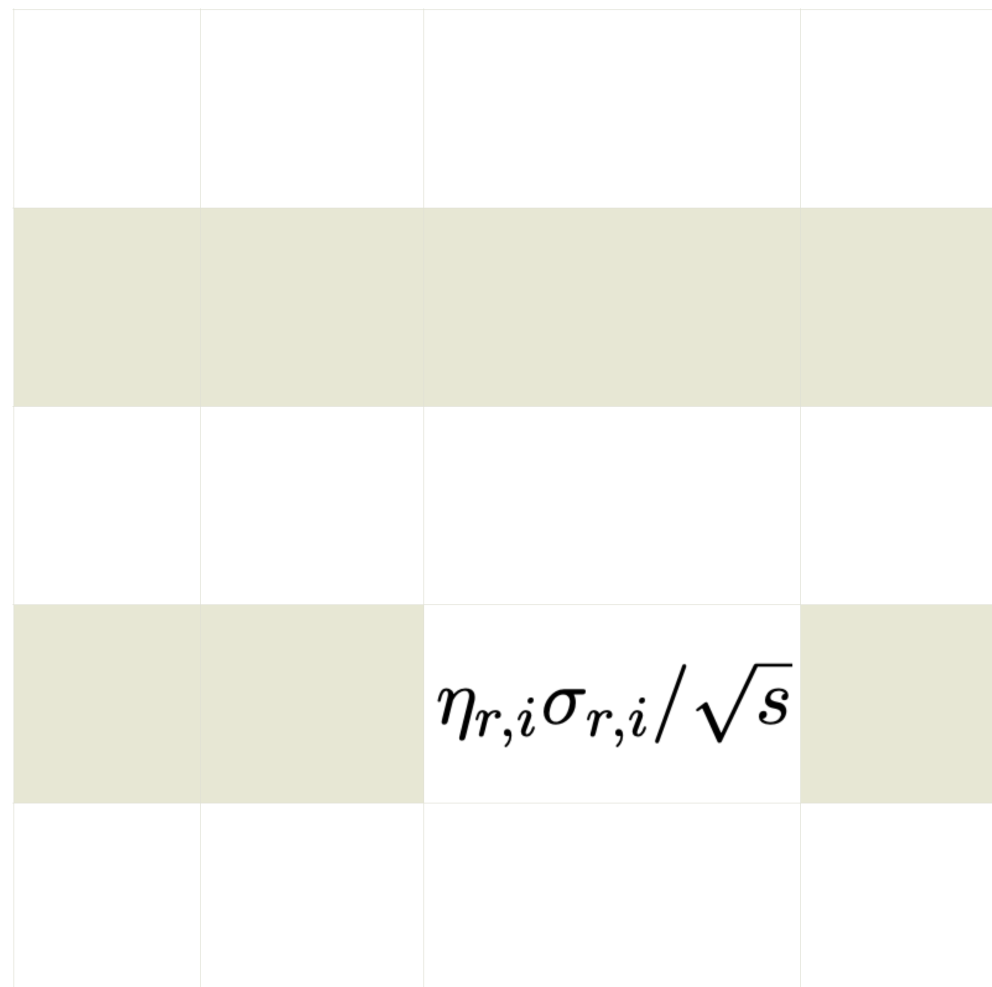

$$||Y||_p = \sqrt[p]{\mathbb{E}[Y^p]}$$

Theorem 1.1.12 (Hanson-Wright inequality [HW71]). *For $\sigma_1, \ldots, \sigma_n$ independent Rademachers and $A \in \mathbb{R}^{n \times n}$, for all $p \geq 1$*

$$\|\sigma^\top A\sigma - \mathbb{E}\,\sigma^\top A\sigma\|_p \lesssim \sqrt{p} \cdot \|A\|_F + p \cdot \|A\|.$$

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نرم اپراتوری:

$$\max_{||x||_2=1} ||Ax||_2$$

برابر است با بزرگ‌ترین مقدار ویژه

Theorem 5.3.1. *As long as* $m \simeq \varepsilon^{-2} \log(1/\delta)$ *and* $s \simeq \varepsilon m$,

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جمع روی r = s

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$= ||z||_2^2 = 1$

جمع روی r = s

$$E = \|\Pi z\|_2^2 - 1 = \frac{1}{s} \sum_{r=1}^{m} \sum_{i \neq j} \eta_{r,i} \eta_{r,j} \sigma_{r,i} \sigma_{r,j} z_i z_j \stackrel{\text{def}}{=} \sigma^\top A_{z,\eta} \sigma$$

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$||E||_p$

امید روی σ و η

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$\mathbb{E}E = 0$

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امید روی σ

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$$\leq \|\sqrt{p}\cdot\|A_{z,\eta}\|_F + p\cdot\|A_{z,\eta}\|\|_p$$

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نامساوی مثلثی

$$||E||_p \leq \sqrt{p} \cdot \| \| A_{z,\eta} \|_F \|_p + p \cdot \| \boxed{\| A_{z,\eta} \|} \|_p$$

$z^{(r)}$: چیزی که اینجا می‌افتد

$A_{z,\eta}$ =?

$$\frac{1}{s} \sum_{r=1}^{m} \sum_{i \neq j} \eta_{r,i} \eta_{r,j} \sigma_{r,i} \sigma_{r,j} z_i z_j \stackrel{\text{def}}{=} \sigma^\top A_{z,\eta} \sigma$$

r,i r',j

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بلاک r

$$A_{z,\eta} =? \quad \begin{array}{cccc} & 0 & 0 & 0 \\ 0 & & 0 & 0 \\ 0 & 0 & (1/s)z^{(r)}(z^{(r)})^\top & 0 \\ 0 & 0 & 0 & \end{array}$$

بلاک r

حذف قطر

$z^{(r)}$: چیزی که اینجا می‌افتد

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$$||\frac{1}{s}(z^{(r)}z^{(r)\top} - D)||$$

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حذف قطر

بلاک r

$$||\frac{1}{s}(z^{(r)}z^{(r)\top} - D)||$$

$$\le \max\{\frac{1}{s}||zz^T||, \frac{1}{s}||D||\}$$

هر دو zz^T و D مثبت نیمه‌معین

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r,i   r’,j

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هر دو zz^T و D مثبت نیمه‌معین

$||z||_\infty^2 \le 1$

$$||E||_p \le \sqrt{p}\cdot \big\| \|A_{z,\eta}\|_F \big\|_p + p \cdot \big\| \boxed{\|A_{z,\eta}\|} \big\|_p$$

بلاک r

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حذف قطر

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هر دو $zz^\top$ و D مثبت نیمه‌معین

1

$||z||_\infty^2 \le 1$

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$A_{z,\eta}$ =?

بلاک r

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هر دو zz^T و D مثبت نیمه‌معین

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$||z||_\infty^2 \le 1$

ماتریس مثبت نیمه معین: $||A|| = \max_{||x||=1} x^T A x$

$$\le \frac{1}{s}$$

$$||E||_p \leq \sqrt{p} \cdot \boxed{\| \|A_{z,\eta}\|_F \|_p} + p \cdot \| \|A_{z,\eta}\| \|_p$$

$\leq 1/s$

$$\frac{1}{s} \sum_{r=1}^{m} \sum_{i \neq j} \eta_{r,i} \eta_{r,j} \sigma_{r,i} \sigma_{r,j} z_i z_j \stackrel{\text{def}}{=} \sigma^\top A_{z,\eta} \sigma$$

$$||E||_p \le \sqrt{p} \cdot \boxed{\| \| A_{z,\eta} \|_F \|_p} + p \cdot \| \| A_{z,\eta} \| \|_p$$

$\le 1/s$

$$\frac{1}{s} \sum_{r=1}^{m} \sum_{i \neq j} \eta_{r,i} \eta_{r,j} \sigma_{r,i} \sigma_{r,j} z_i z_j \stackrel{\text{def}}{=} \sigma^\top A_{z,\eta} \sigma$$

$$\| \| A_{z,\eta} \|_F \|_p = \| \| A_{z,\eta} \|_F^2 \|_{p/2}^{1/2}$$

$$||E||_p \leq \sqrt{p} \cdot \boxed{\| \|A_{z,\eta}\|_F \|_p} + p \cdot \| \|A_{z,\eta}\| \|_p$$

$\leq 1/s$

$$\frac{1}{s}\sum_{r=1}^{m}\sum_{i\neq j}\eta_{r,i}\eta_{r,j}\sigma_{r,i}\sigma_{r,j}z_i z_j \stackrel{\text{def}}{=} \sigma^\top A_{z,\eta}\sigma$$

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$$Q_{i,j} = \sum_{r=1}^{m} \eta_{r,i} \eta_{r,j}$$

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$$\| \|A_{z,\eta}\|_F \|_p = \| \|A_{z,\eta}\|_F^2 \|_{p/2}^{1/2}$$

$$Q_{i,j} = \sum_{r=1}^{m} \eta_{r,i}\eta_{r,j}$$

$$\|A_{z,\eta}\|_F^2 = \frac{1}{s^2} \sum_{i \neq j} z_i^2 z_j^2 \cdot Q_{i,j}$$

$$||E||_p \leq \sqrt{p} \cdot \boxed{\|\|A_{z,\eta}\|_F\|_p} + p \cdot \|\|A_{z,\eta}\|\|_p$$

$\leq 1/s$

$$\frac{1}{s}\sum_{r=1}^{m}\sum_{i\neq j}\eta_{r,i}\eta_{r,j}\sigma_{r,i}\sigma_{r,j}z_i z_j \stackrel{\text{def}}{=} \sigma^\top A_{z,\eta}\sigma$$

$$\|\|A_{z,\eta}\|_F\|_p = \|\|A_{z,\eta}\|_F^2\|_{p/2}^{1/2}$$

$$\leq \|\frac{1}{s^2}\sum_{i\neq j} z_i^2 z_j^2 \cdot Q_{i,j}\|_p^{1/2}$$

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$$\|A_{z,\eta}\|_F^2 = \frac{1}{s^2} \sum_{i \neq j} z_i^2 z_j^2 \cdot Q_{i,j}$$

$$\| \|A_{z,\eta}\|_F \|_p = \| \|A_{z,\eta}\|_F^2 \|_{p/2}^{1/2}$$

$$\leq \| \frac{1}{s^2} \sum_{i \neq j} z_i^2 z_j^2 \cdot Q_{i,j} \|_p^{1/2}$$

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$$\leq \frac{1}{s} \left(\sum_{i \neq j} z_i^2 z_j^2 \cdot \|Q_{i,j}\|_p \right)^{1/2}$$

$$\left\| \|A_{z,\eta}\|_F \right\|_p \leq \frac{1}{s} \left(\sum_{i \neq j} z_i^2 z_j^2 \cdot \|Q_{i,j}\|_p \right)^{1/2}$$

$$Q_{i,j} = \sum_{r=1}^{m} \eta_{r,i} \eta_{r,j}$$

برگه بعدی

$$\||\|A_{z,\eta}\|_F\|_p \leq \frac{1}{s}\left(\sum_{i\neq j} z_i^2 z_j^2 \cdot \|Q_{i,j}\|_p\right)^{1/2}$$

$$Q_{i,j} = \sum_{r=1}^{m} \eta_{r,i}\eta_{r,j}$$

$$p \simeq s^2/m$$

$$\|Q_{i,j}\|_p \lesssim p$$

برگه بعدی

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Theorem 35 (Bernstein's inequality). *Let $X_1, \ldots, X_n$ be independent random variables that are each at most K almost surely, and where*

Then for all $p \ge 1$

σ^2: واریانس X

$$\Big\| \sum_{i=1}^{n} X_i - \mathbb{E} \sum_i X_i \Big\|_p \lesssim \sigma\sqrt{p} + Kp.$$

$$p \simeq s^2/m$$

$$\|Q_{i,j}\|_p \lesssim p$$

$$\le \frac{1}{\sqrt{m}}$$

برگه بعدی

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$$E[Q_{i,j}] = m \times \frac{s^2}{m^2} = \frac{s^2}{m}$$

$$p \simeq s^2 / m$$

$$\| Q_{i,j} \|_p \lesssim p$$

$$\leq \frac{1}{\sqrt{m}}$$

برگه بعدی

$$\|\|A_{z,\eta}\|_F\|_p \le \frac{1}{s}\left(\sum_{i\neq j} z_i^2 z_j^2 \cdot \|Q_{i,j}\|_p\right)^{1/2}$$

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$$E[Q_{i,j}^2] = E[\sum_{r,r'} \eta_{r,i}\eta_{r,j}\eta_{r',i}\eta_{r',j}] \le \sum_r (\frac{s}{m})^2 + \sum_{r,r'} (\frac{s}{m})^4 \lesssim s^2/m$$

$$p \simeq s^2/m$$

$$\|Q_{i,j}\|_p \lesssim p$$

$$\le \frac{1}{\sqrt{m}}$$

برگه بعدی

$$\| \| A_{z,\eta} \|_F \|_p \leq \frac{1}{s} \left(\sum_{i \neq j} z_i^2 z_j^2 \cdot \| Q_{i,j} \|_p \right)^{1/2}$$

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$$Var[Q_{i,j}] = \frac{s^2}{m} + \left(\frac{s^2}{m}\right)^2 - \left(\frac{s^2}{m}\right)^2 = s^2/m$$

$$p \simeq s^2/m$$

$$\| Q_{i,j} \|_p \lesssim p$$

$$\leq \frac{1}{\sqrt{m}}$$

برگه بعدی

$$\| \| A_{z,\eta} \|_F \|_p \leq \frac{1}{s} \left(\sum_{i \neq j} z_i^2 z_j^2 \cdot \| Q_{i,j} \|_p \right)^{1/2}$$

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$$Var[Q_{i,j}] = \frac{s^2}{m} + \left(\frac{s^2}{m}\right)^2 - \left(\frac{s^2}{m}\right)^2 = s^2/m$$

$$\sigma \sqrt{p} + Kp \leq p + p \lesssim p$$

$$p \simeq s^2/m$$

$$\| Q_{i,j} \|_p \lesssim p$$

$$\leq \frac{1}{\sqrt{m}}$$

برگه بعدی

$$\| \| A_{z,\eta} \|_F \|_p \le \frac{1}{s} \left(\sum_{i \neq j} z_i^2 z_j^2 \cdot \|Q_{i,j}\|_p \right)^{1/2}$$

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$$E[Q_{i,j}^2] = E[\sum_{r,r'} \eta_{r,i} \eta_{r,j} \eta_{r',i} \eta_{r',j}] \le \sum_r (\frac{s}{m})^2 + \sum_{r,r'} (\frac{s}{m})^4 \lesssim s^2/m$$

$$Var[Q_{i,j}] = \frac{s^2}{m} + \left(\frac{s^2}{m}\right)^2 - \left(\frac{s^2}{m}\right)^2 = s^2/m$$

$$\sigma \sqrt{p} + Kp \le p + p \lesssim p$$

$$||Q_{i,j} - EQ_{i,j}||_p \le \frac{s}{\sqrt{m}} \cdot \frac{s}{\sqrt{m}} + \frac{s^2}{m} = 2\frac{s^2}{m}$$

برگه بعدی

$$p \simeq s^2/m$$

$$\|Q_{i,j}\|_p \lesssim p$$

$$\le \frac{1}{\sqrt{m}}$$

$$\|\|\|A_{z,\eta}\|_F\|_p \leq \frac{1}{s}\left(\sum_{i\neq j} z_i^2 z_j^2 \cdot \|Q_{i,j}\|_p\right)^{1/2}$$

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σ^2: واریانس X

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$$E[Q_{i,j}] = m \times \frac{s^2}{m^2} = \frac{s^2}{m}$$

$$E[Q_{i,j}^2] = E[\sum_{r,r'} \eta_{r,i}\eta_{r,j}\eta_{r',i}\eta_{r',j}] \leq \sum_r (\frac{s}{m})^2 + \sum_{r,r'} (\frac{s}{m})^4 \lesssim s^2/m$$

$$Var[Q_{i,j}] = \frac{s^2}{m} + \left(\frac{s^2}{m}\right)^2 - \left(\frac{s^2}{m}\right)^2 = s^2/m$$

$$\sigma\sqrt{p} + Kp \leq p + p \lesssim p$$

$$||Q_{i,j} - EQ_{i,j}||_p \leq \frac{s}{\sqrt{m}} \cdot \frac{s}{\sqrt{m}} + \frac{s^2}{m} = 2\frac{s^2}{m}$$

$$||Q_{i,j}||_p \leq 2\frac{s^2}{m} + ||EQ_{i,j}||_p \approx p$$

برگه بعدی

$$p \simeq s^2/m$$

$$\|Q_{i,j}\|_p \lesssim p$$

$$\leq \frac{1}{\sqrt{m}}$$

$$Q_{i,j} = \sum_s Y_s$$

$Y_s :=$ اگر s_امین ۱ در ستون i،
در سطر r باشد، $\eta_{r,j} = 1$

$$Q_{i,j} = \sum_{r=1}^{m} \eta_{r,i}\eta_{r,j}$$

Theorem 35 (Bernstein's inequality). *Let $X_1, \ldots, X_n$ be independent random variables that are each at most K almost surely, and where*

Then for all $p \geq 1$

σ^2: واریانس X

$$\|\sum_{i=1}^{n} X_i - \mathbb{E}\sum_i X_i\|_p \lesssim \sigma\sqrt{p} + Kp.$$

$$||\sum_t Y_t||_p \leq ||\sum_t Z_t||_p \lesssim \sqrt{s^2/m}\cdot\sqrt{p} + p \simeq p.$$

$$(\sum_t Y_t)^p = \sum_{t_1,\ldots,t_p} Y_{t_1}\ldots Y_{t_p}$$
$$\leq \sum_{t_1,\ldots,t_p} Z_{t_1}\ldots Z_{t_p}$$

متغیر برنولی مستقل

$$Q_{i,j} = \sum_s Y_s$$

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Theorem 35 (Bernstein's inequality). *Let $X_1, \ldots, X_n$ be independent random variables that are each at most K almost surely, and where*

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متغیر برنولی مستقل

$$||E||_p \leq \sqrt{p} \cdot \| \|A_{z,\eta}\|_F \|_p + p \cdot \| \|A_{z,\eta}\| \|_p$$

$\leq 1/\sqrt{m}$

$\leq 1/s$

$$||E||_p \leq \sqrt{p} \cdot \| \|A_{z,\eta}\|_F \|_p + p \cdot \| \|A_{z,\eta}\| \|_p$$

$\leq 1/\sqrt{m}$

$\leq 1/s$

$p \simeq s^2/m$

$\leq s/m + s/m \lesssim s/m$

$$||E||_p \leq \sqrt{p} \cdot \| \| A_{z,\eta} \|_F \|_p + p \cdot \| \| A_{z,\eta} \| \|_p$$

$\leq 1/\sqrt{m}$

$\leq 1/s$

$\boxed{p \simeq s^2/m}$ $\leq s/m + s/m \lesssim s/m$

$\boxed{|\mathrm{E}|}$

$$\mathbb{P}(|\|\Pi x\|_2^2 - 1| > \varepsilon)$$

$$||E||_p \leq \sqrt{p} \cdot \| \| A_{z,\eta} \|_F \|_p + p \cdot \| \| A_{z,\eta} \| \|_p$$

$\leq 1/\sqrt{m}$

$\leq 1/s$

$p \simeq s^2/m$

$$\leq s/m + s/m \lesssim s/m$$

|E|

$$\mathbb{P}(| \| \Pi x \|_2^2 - 1 | > \varepsilon) = P[|E|^p > \epsilon^p]$$

$$||E||_p \leq \sqrt{p} \cdot \| \| A_{z,\eta} \|_F \|_p + p \cdot \| \| A_{z,\eta} \| \|_p$$

$\leq 1/\sqrt{m}$

$\leq 1/s$

$p \simeq s^2/m$

$$\leq s/m + s/m \lesssim s/m$$

|E|

ماركوف

$$\mathbb{P}(| \|\Pi x\|_2^2 - 1| > \varepsilon) = P[|E|^p > \epsilon^p] < \epsilon^{-p} \, \mathbb{E}[|E|^p]$$