بسم الله الرحمن الرحیم

نظریه علوم کامپیوتر

نظریه علوم کامپیوتر - بهار ۱۴۰۰-۱۴۰۱ - جلسه شانزدهم: سلسله‌مراتب زمان

Theory of computation - 002 - S16 - Time hierarchy

Contents

Last time:

- Log-space reducibility
- L = NL? question
- $PATH$ is NL-complete
- $\overline{2SAT}$ is NL-complete
- NL = coNL

Today: (Sipser §9.1)

- Time and Space Hierarchy Theorems

Review: Major Complexity Classes

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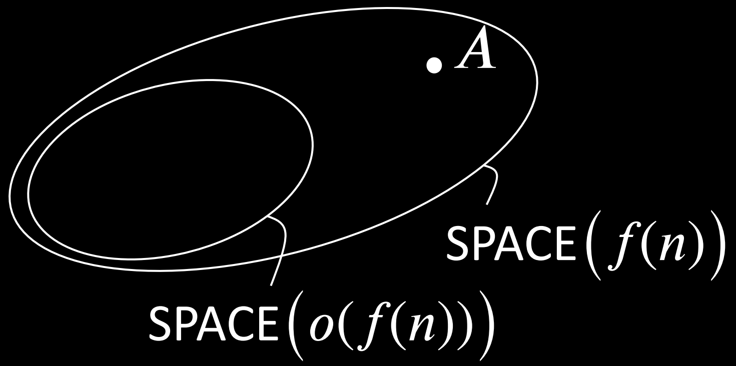

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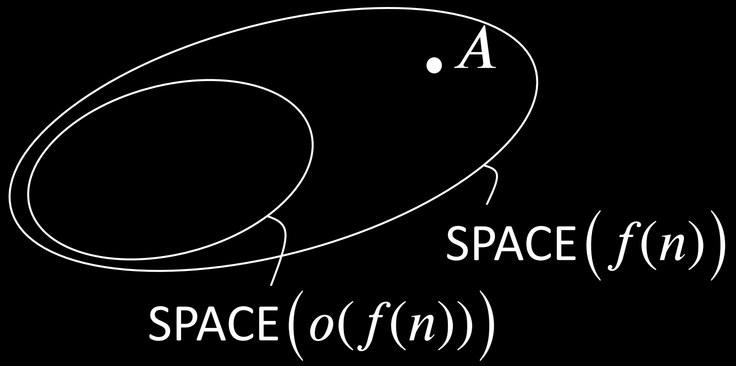

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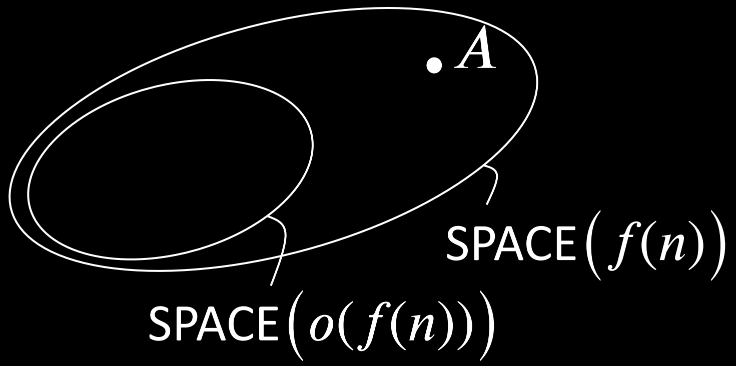

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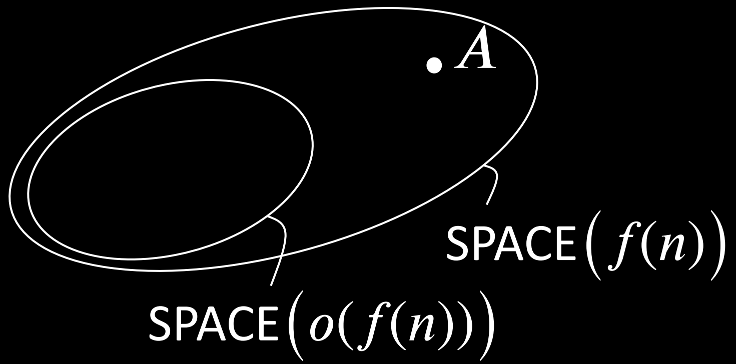

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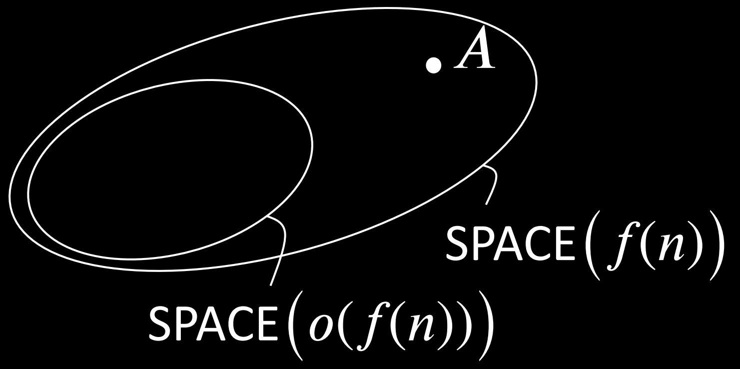

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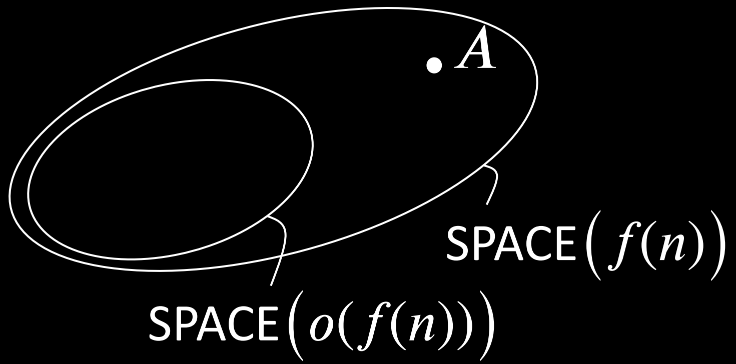

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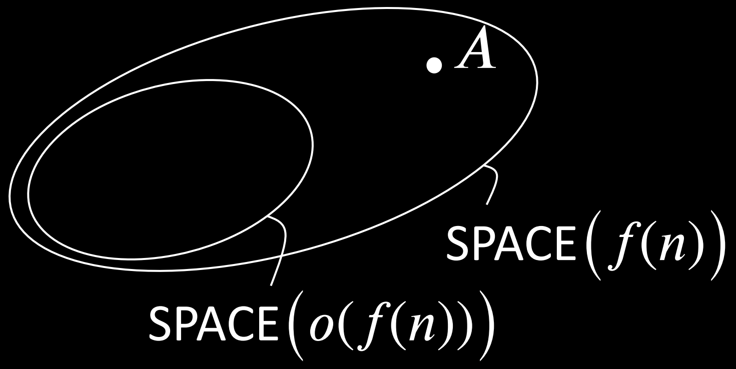

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Cost of moving it adds a $O\Big(\log\big(f(n)\big)\Big)$ overhead factor. So to halt within $O\big(f(n)\big)$ time, D stops when the counter reaches $f(n)/\log\big(f(n)\big)$.

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*Note: D can simulate M with a log factor time overhead due to the step counter.

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D(<M>1) rejects iff M(<M>10^k) accepts

Why do we lose a factor of $\log\big(f(n)\big)$?

D must halt within $O\big(f(n)\big)$ time.

To do so, D counts the number of steps it uses and stops if the limit is exceeded. The counter has size $\log\big(f(n)\big)$ and is stored on the tape.
It must be kept near the current head location.
Cost of moving it adds a $O\Big(\log\big(f(n)\big)\Big)$ overhead factor. So to halt within $O\big(f(n)\big)$ time, D stops when the counter reaches $f(n)/\log\big(f(n)\big)$.

Time Hierarchy Theorem (2/2)

Goal: Exhibit $A \in \text{TIME}\big(f(n)\big)$ but $A \notin \text{TIME}\Big(o\big(f(n)/\log\big(f(n)\big)\big)\Big)$

$A = L(D)$ where

1) D runs in $O\big(f(n)\big)$ time
2) D ensures that $L(D) \neq L(M)$ for every TM M that runs in $o\big(f(n)/\log\big(f(n)\big)\big)$ time.

$D =$ "On input w
1. Compute $f(n)$.
2. If $w \neq \langle M \rangle 10^*$ for some TM M, *reject*.
3. Simulate* M on w for $f(n)/\log\big(f(n)\big)$ steps.

   Accept if M rejects,

   Reject if M accepts or hasn't halted."

*Note: D can simulate M with a log factor time overhead due to the step counter.

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Recap: Separating Complexity Classes

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE$$

$$\underbrace{NL \neq PSPACE}_{\text{Space Hierarchy Theorem}}$$

$NL \subseteq SPACE\left(\log^2 n\right) \subsetneq SPACE(n) \subseteq$

PSPACE

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Check-in 21.3

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Check-in 21.3

Consider these two famous unsolved questions:

1. Does L = P?
2. Does P = PSPACE?

What do the hierarchy theorems tell us about these questions?

a) Nothing
b) At least one of these has answer "NO"
c) At least one of these has answer "YES"

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Defn: EXPTIME = $\bigcup_k \text{TIME}\left(2^{(n^k)}\right)$

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Time Hierarchy Theorem

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$\text{NL} \neq \text{PSPACE}$, $\text{PSPACE} \neq \text{EXPSPACE}$

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$\text{NL} \neq \text{PSPACE}$ (Space Hierarchy Theorem), $\text{P} \neq \text{EXPTIME}$ (Time Hierarchy Theorem), $\text{PSPACE} \neq \text{EXPSPACE}$ (Space Hierarchy Theorem)

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Let A be any language.

Defn: A TM M with oracle for A, written M^A, is a TM equipped with a "black box" that can answer queries "is $x \in A$?" for free.

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Theorem: There is an oracle A where $\mathrm{P}^A = \mathrm{NP}^A$

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If both TMs were oracle TMs D^A and M^A with the same oracle A,
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Therefore, if we could prove P $\neq$ NP by a diagonalization,
we would also prove that $\mathrm{P}^A \neq \mathrm{NP}^A$ for every oracle A.

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Therefore, if we could prove P $\neq$ NP by a diagonalization,
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But that is false!

Oracles and P versus NP

Theorem: There is an oracle A where $\text{P}^A = \text{NP}^A$

Proof: Let $A = TQBF$

$\text{NP}^{TQBF} \subseteq$ NPSPACE $=$ PSPACE $\subseteq \text{P}^{TQBF}$

Relevance to the P versus NP question

Recall: We showed $EQ_{\text{REX}\uparrow} \notin$ PSPACE.

Could we show $SAT \notin$ P using a similar method? NO.

Reason: Suppose YES.
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Which of these are known to be true?
Check all that apply.

(a) $\text{P}^{SAT} = \text{P}^{\overline{SAT}}$

(b) $\text{NP}^{SAT} = \text{coNP}^{SAT}$

(c) MIN-FORMULA $\in \text{P}^{TQBF}$

(d) $\text{NP}^{TQBF} = \text{coNP}^{TQBF}$

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