بسم الله الرحمن الرحیم

برنامه‌ریزی نیمه‌معین برای طراحی الگوریتم‌های تقریبی

جلسه هفدهم: تابع درجه ۲ روی گراف

Project: People

مرور

نسخه SDP:

MaxQP[G]: maximizing a quadratic form on a graph

$$G = (V, E)$$

$$\max\left\{ \sum_{\{i,j\} \in E} a_{ij} x_i x_j : x_1, \ldots, x_n \in \{\pm 1\} \right\},$$

where a_{ij} are real weights on edges, generally both positive and negative.

SDP relaxation of MaxQP[G]

$$S_{\max} := \max\left\{ \sum_{\{i,j\} \in E} a_{ij} \mathbf{v}_i^T \mathbf{v}_j : \|\mathbf{v}_1\|, \ldots, \|\mathbf{v}_n\| \le 1 \right\}$$

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روش GW:؟

به ازای هر یال

ما => بهینه $\boldsymbol{\rho}$

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روش GW:؟

به ازای هر یال

ما >= بهینه ρ

؟

Let G be a (loopless) graph. The *Grothendieck constant* K_G of G is defined as

$$\sup \frac{S_{\max}}{\mathsf{Opt}},$$

where Opt is the optimum value of $\textsc{MaxQP}[G]$, $S_{\max}$ is the optimum of the SDP relaxation, and the supremum is over all choices of the edge weights a_{ij} (not all zeros).

Theorem (Alon et al. [AMMN06]). *For every graph G, we have*

$$K_G = O(\log \vartheta(\overline{G})),$$

where $\overline{G}$ is the complement of G and $\vartheta(.)$ is the Lovász theta function. Moreover, there is a randomized rounding algorithm which, for given G and weights a_{ij}, computes a solution of $\textsc{MaxQP}[G]$ with value at least $\Omega(S_{\max}/\log \vartheta(\overline{G}))$ in expected polynomial time.

$$\vartheta(\overline{G}) \leq \chi(G)$$

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$$\vartheta(\overline{G}) \leq \chi(G)$$

- => تقریب با ضریب ثابت
- گراف‌های ۲ـ بخشی
  - آیزینگ، نرم برشی
- گراف‌های با بزرگترین درجه ثابت

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- حل SDP
- گرد کردن

گردن کردن، تلاش ۱:

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$$E[Z_i Z_j] = E[(\gamma v_i)(\gamma v_j)]$$

$$Z_i := \gamma^T \mathbf{v}_i$$

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$$E[Z_i Z_j] = E[(\gamma v_i)(\gamma v_j)] = E[(\gamma \sum_t \alpha_t e_t)(\gamma \sum_t \beta_t e_t)]$$

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چی شد؟

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با احتمال خوبی $Z_i \in [-M, M]$

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==> ۱ و ـ۱ کردن Zi حداکثر M^2 ضرر می‌زند.

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خیلی ضرر می‌کنیم.

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$$R := S_{\max} - S_{\min}$$

اندازه بازه
تغییرات تابع
هدف

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$$M := 3\sqrt{1 + \ln \frac{R}{S_{\max}}}$$

الگوریتم

Randomized rounding for MAXQP[G]

1. Given $\mathbf{v}_1, \ldots, \mathbf{v}_n$ attaining $S_{\max}$, generate a random n-dimensional Gaussian $\boldsymbol{\gamma}$, and set $Z_i := \boldsymbol{\gamma}^T \mathbf{v}_i$, $i = 1, 2, \ldots, n$.
2. Compute R and M as above, and set

$$\tilde{Z}_i := \begin{cases} Z_i & \text{if } |Z_i| \leq M \\ 0 & \text{otherwise.} \end{cases}$$

3. Return $x_i := \tilde{Z}_i / M$, $i = 1, 2, \ldots, n$.

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$$? \geq E[\sum_{i,j \in E} a_{ij} \tilde{Z}_i \tilde{Z}_j]$$

مقدار جواب ما = $\sum_{i,j \in E} a_{ij} x_i x_j$

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$$\mathbf{E}\left[\sum_{\{i,j\}\in E} a_{ij}\tilde{Z}_i\tilde{Z}_j\right] = \sum_{\{i,j\}\in E} a_{ij}\Big(\mathbf{E}\left[Z_iZ_j\right] - \mathbf{E}\left[Z_iT_j\right] - \mathbf{E}\left[Z_jT_i\right] + \mathbf{E}\left[T_iT_j\right]\Big)$$

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$$= S_{\max} - \mathbf{E}\Big[\sum_{\{i,j\}\in E} a_{ij}(Z_iT_j + Z_jT_i)\Big] + \mathbf{E}\Big[\sum_{\{i,j\}\in E} a_{ij}T_iT_j\Big]$$

Lemma. *Let* $X_1, X_2, \ldots, X_n$ *and* $Y_1, Y_2, \ldots, Y_n$ *be real random variables with* $\mathbf{E}\left[X_i^2\right] \leq A$ *and* $\mathbf{E}\left[Y_i^2\right] \leq B$ *for all* i *(no independence assumed). Then*

$$\mathbf{E}\Bigg[\sum_{\{i,j\} \in E} a_{ij}(X_i Y_j + X_j Y_i)\Bigg] \leq 2R\sqrt{AB},$$

$$R := S_{\max} - S_{\min}$$

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$$U_i := \frac{1}{2}\left(\frac{X_i}{\sqrt{A}} + \frac{Y_i}{\sqrt{B}}\right), \quad V_i := \frac{1}{2}\left(\frac{X_i}{\sqrt{A}} - \frac{Y_i}{\sqrt{B}}\right)$$

$$(x+y)^2 \le (x+y)^2 + (x-y)^2 = 2(x^2+y^2)$$

$$\mathbf{E}\left[U_i^2\right] \le \frac{1}{2}E[(\frac{X_i}{\sqrt{A}})^2 + (\frac{Y_i}{\sqrt{B}})^2] = 1$$

Lemma. *Let* $X_1, X_2, \ldots, X_n$ *and* $Y_1, Y_2, \ldots, Y_n$ *be real random variables with* $\mathbf{E}\left[X_i^2\right] \leq A$ *and* $\mathbf{E}\left[Y_i^2\right] \leq B$ *for all* i *(no independence assumed). Then*

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$$(x-y)^2 \leq (x+y)^2 + (x-y)^2 = 2(x^2+y^2)$$

$$\mathbf{E}\left[V_i^2\right] \leq \frac{1}{2}E[(\frac{X_i}{\sqrt{A}})^2 + (\frac{Y_i}{\sqrt{B}})^2] = 1$$

$$U_i := \frac{1}{2}\left(\frac{X_i}{\sqrt{A}} + \frac{Y_i}{\sqrt{B}}\right), \quad V_i := \frac{1}{2}\left(\frac{X_i}{\sqrt{A}} - \frac{Y_i}{\sqrt{B}}\right)$$

$$\mathbf{E}\left[U_i^2\right] \leq 1 \quad \mathbf{E}\left[V_i^2\right] \leq 1$$

$$\mathbf{E}\Bigg[\sum_{\{i,j\}\in E} a_{ij}(X_iY_j + X_jY_i)\Bigg]$$

$$\frac{X_i}{\sqrt{A}}\frac{X_j}{\sqrt{A}} + \frac{X_i}{\sqrt{A}}\frac{Y_j}{\sqrt{B}} + \frac{Y_i}{\sqrt{B}}\frac{X_j}{\sqrt{A}} + \frac{Y_i}{\sqrt{B}}\frac{Y_j}{\sqrt{B}}$$

$$\frac{X_i}{\sqrt{A}}\frac{X_j}{\sqrt{A}} - \frac{X_i}{\sqrt{A}}\frac{Y_j}{\sqrt{B}} - \frac{Y_i}{\sqrt{B}}\frac{X_j}{\sqrt{A}} + \frac{Y_i}{\sqrt{B}}\frac{Y_j}{\sqrt{B}}$$

$$= \; 2\sqrt{AB}\left(\mathbf{E}\Bigg[\sum_{\{i,j\}\in E} a_{ij}U_iU_j\Bigg] - \mathbf{E}\Bigg[\sum_{\{i,j\}\in E} a_{ij}V_iV_j\Bigg]\right)$$

$$U_i := \frac{1}{2}\left(\frac{X_i}{\sqrt{A}} + \frac{Y_i}{\sqrt{B}}\right), \quad V_i := \frac{1}{2}\left(\frac{X_i}{\sqrt{A}} - \frac{Y_i}{\sqrt{B}}\right)$$

$$\mathbf{E}\left[U_i^2\right] \leq 1 \quad \mathbf{E}\left[V_i^2\right] \leq 1$$

قضیه: می‌توان بردارهایی ساخت با همین ضرب‌ها

$$\begin{aligned}\mathbf{E}\Big[\sum_{\{i,j\}\in E} a_{ij}(X_iY_j + X_jY_i)\Big] \\ = \ 2\sqrt{AB}\Big(\mathbf{E}\Big[\sum_{\{i,j\}\in E} a_{ij}U_iU_j\Big] - \mathbf{E}\Big[\sum_{\{i,j\}\in E} a_{ij}V_iV_j\Big]\Big) \\ \leq \ 2\sqrt{AB}(S_{\max} - S_{\min}) = 2R\sqrt{AB}.\end{aligned}$$

قضیه: می‌توان بردارهایی ساخت با همین ضرب‌ها

قضیه: اگر $\mathbf{E}\left[X_i^2\right] \leq 1$

آنگاه:

$$S_{\min} \leq \mathbf{E}\left[\sum_{\{i,j\}\in E} a_{ij} X_i X_j\right] \leq S_{\max}$$

ماتریس $E[X_iX_j]$ مثبت نیمه‌معین است. بردارهای vi هست که همین ضرب‌ها را دارند

حکم: $\sum_{ij} x_i E[X_iX_j] x_j \geq 0$

$$\sum_{ij} x_i E[X_iX_j] x_j = E[\sum_{ij} x_i X_i X_j x_j] = E[(\sum_i x_i X_i)(\sum_i x_i X_i)] \geq 0$$

Lemma. *Let* $X_1, X_2, \ldots, X_n$ *and* $Y_1, Y_2, \ldots, Y_n$ *be real random variables with* $\mathbf{E}\left[X_i^2\right] \leq A$ *and* $\mathbf{E}\left[Y_i^2\right] \leq B$ *for all* i *(no independence assumed). Then*

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$$\mathbf{E}\left[Z_i^2\right] = \mathrm{Var}\left[Z_i\right] = 1$$

$$\mathbf{E}\left[T_i^2\right] \leq$$

$$A := \mathbf{E}\left[T_i^2\right] = \frac{2}{\sqrt{2\pi}} \int_M^\infty x^2 e^{-x^2/2}\,\mathrm{d}x.$$

$$\mathbf{E}\left[T_i^2\right] \le \sqrt{\frac{2}{\pi}}\left(M + \frac{1}{M}\right)e^{-M^2/2} \le Me^{-M^2/2}$$

$$\boxed{M + \tfrac{1}{M} \le \tfrac{10}{9}M}$$

$$\boxed{M = 3\sqrt{1 + \ln\underline{(R/S_{\max})}} \ge 3}$$

$$A \le Me^{-M^2/2}$$

$$A \leq M e^{-M^2/2} \qquad M = 3\sqrt{1 + \ln(R/S_{\max})}$$

$$A \leq 3\sqrt{R/S_{\max}} \cdot e^{-9/2} \left(S_{\max}/R\right)^{9/2}$$

$$M \leq 3\sqrt{R/S_{\max}}$$

$$e^{-\frac{9}{2}(1+\ln(R/S_{max}))}$$

$$\ln x \leq x - 1$$

$$< \frac{1}{10}\left(\frac{S_{\max}}{R}\right)^4 \leq \frac{1}{10}\left(\frac{S_{\max}}{R}\right)^2$$

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$$\geq S_{max} - 2R\sqrt{AB} \geq S_{max} - 2R\frac{1}{\sqrt{10}}\frac{S_{max}}{R}$$

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The expected value of the solution $x_1, \ldots, x_n$, i.e., of $\sum_{\{i,j\} \in E} a_{ij} x_i x_j$, is at least

$$\frac{1}{2} \frac{S_{\max}}{M^2} \geq S_{\max} \cdot \Omega\left(\frac{1}{1 + \log \frac{R}{S_{\max}}}\right).$$

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Lemma. *For every graph G and for every choice of edge weights (not all zeros), we have*

$$\frac{R}{S_{\max}} \leq \vartheta(\overline{G}).$$

Theorem (Alon et al. [AMMN06])**.** *For every graph G, we have*

$$K_G = O(\log \vartheta(\overline{G})),$$

where $\overline{G}$ is the complement of G and $\vartheta(.)$ is the Lovász theta function. Moreover, there is a randomized rounding algorithm which, for given G and weights a_{ij}, computes a solution of MaxQP$[G]$ *with value at least $\Omega(S_{\max}/\log\vartheta(\overline{G}))$ in expected polynomial time.*

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یک k_ رنگ‌آمیزی برداری : u_i

بردار تولید کننده $S_{\min}$: v_i

؟

جواب برای MaxQP با مقدار $\frac{-S_{\min}}{k-1}$

$$S_{\max} \geq \frac{-S_{\min}}{k-1}$$

$$k-1 \geq \frac{-S_{min}}{S_{max}}$$

$$k \geq \frac{S_{max}-S_{min}}{S_{max}}$$

یک k_ رنگ‌آمیزی برداری : u_i

بردار تولید کننده S_{min}: v_i

؟

جواب برای MaxQP با مقدار $\frac{-S_{\min}}{k-1}$

$$\mathbf{w}_i := \mathbf{u}_i \otimes \mathbf{v}_i$$

$$\sum_{\{i,j\}\in E} a_{ij}\mathbf{w}_i^T\mathbf{w}_j = \sum_{\{i,j\}\in E} a_{ij}(\mathbf{u}_i^T\mathbf{u}_j)(\mathbf{v}_i^T\mathbf{v}_j)$$

$$= -\frac{1}{k-1}\sum_{\{i,j\}\in E} a_{ij}\mathbf{v}_i^T\mathbf{v}_j = \frac{-S_{\min}}{k-1}$$

The expected value of the solution $x_1, \ldots, x_n$, i.e., of $\sum_{\{i,j\} \in E} a_{ij} x_i x_j$, is at least

$$\frac{1}{2} \frac{S_{\max}}{M^2} \geq S_{\max} \cdot \Omega\left(\frac{1}{1 + \log \frac{R}{S_{\max}}}\right).$$

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پایان