بسم الله الرحمن الرحیم

جلسه هفتم

درس تحقیق در عملیات

روش سیمپلکس

سیمپلکس

یک مثال و چند مشکل

$$
\begin{array}{lrcrcl}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & -x_1 & + & x_2 & \leq & 1 \\
 & x_1 & & & \leq & 3 \\
 & & & x_2 & \leq & 2 \\
 & \multicolumn{3}{r}{x_1, x_2 \geq 0.} & &
\end{array}
$$

$$
\begin{array}{llcccl}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & -x_1 & + & x_2 & \leq & 1 \\
 & x_1 & & & \leq & 3 \\
 & & & x_2 & \leq & 2 \\
 & \multicolumn{5}{l}{x_1, x_2 \geq 0.}
\end{array}
$$

$$
\begin{array}{llcccccccl}
\text{maximize} & x_1 & + & x_2 & & & & & & \\
\text{subject to} & -x_1 & + & x_2 & + & x_3 & & & & = 1 \\
 & x_1 & & & & & + & x_4 & & = 3 \\
 & & & x_2 & & & & & + x_5 & = 2 \\
 & \multicolumn{9}{l}{x_1, x_2, \ldots, x_5 \geq 0,}
\end{array}
$$

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{lrcrcrcrcrcl} \text{maximize} & x_1 & + & x_2 & & & & & & & & \\ \text{subject to} & -x_1 & + & x_2 & + & x_3 & & & & & = & 1 \\ & x_1 & & & & & + & x_4 & & & = & 3 \\ & & & x_2 & & & & & + & x_5 & = & 2 \\ & \multicolumn{11}{l}{x_1, x_2, \ldots, x_5 \geq 0,} \end{array}$$

$$
\begin{array}{rcrcrcr}
x_3 & = & 1 & + & x_1 & - & x_2 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 2 & & & - & x_2 \\
\hline
z & = & & & x_1 & + & x_2
\end{array}
$$

$$B = \{3, 4, 5\}$$

$$
\begin{array}{rcrcrcr}
x_3 & = & 1 & + & x_1 & - & x_2 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 2 & & & - & x_2 \\
\hline
z & = & & & x_1 & + & x_2
\end{array}
$$

$$
\begin{array}{rcrcrcr}
x_3 & = & 1 & + & x_1 & - & x_2 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 2 & & & - & x_2 \\
\hline
z & = & & & x_1 & + & x_2
\end{array}
$$

1

$$
\begin{array}{rcl}
x_3 & = & 1 + x_1 - x_2 \quad \textbf{1} \\
x_4 & = & 3 - x_1 \\
x_5 & = & 2 - x_2 \\
\hline
z & = & x_1 + x_2
\end{array}
$$

$$
\begin{array}{rcrcrcr}
x_3 & = & 1 & + & x_1 & - & x_2 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 2 & & & - & x_2 \\
\hline
z & = & & & x_1 & + & x_2
\end{array}
$$

1

2

$$x_2 = 1 + x_1 - x_3$$

$$\begin{array}{lcrcrcr} x_3 & = & 1 & + & x_1 & - & x_2 \\ x_4 & = & 3 & - & x_1 & & \\ x_5 & = & 2 & & & - & x_2 \\ \hline z & = & & & x_1 & + & x_2 \end{array}$$

$$
\begin{array}{lcrcrcr}
x_2 & = & 1 & + & x_1 & - & x_3 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 1 & - & x_1 & + & x_3 \\
\hline
z & = & 1 & + & 2x_1 & - & x_3
\end{array}
$$

$$B = \{2, 4, 5\}$$

$$
\begin{array}{rcrcrcr}
x_2 & = & 1 & + & x_1 & - & x_3 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 1 & - & x_1 & + & x_3 \\
\hline
z & = & 1 & + & 2x_1 & - & x_3
\end{array}
$$

$$x_1 = \boxed{\qquad\qquad} \leftarrow$$

$$
\begin{array}{lcl}
x_2 &=& 1 + x_1 - x_3 \\
x_4 &=& 3 - x_1 \\
x_5 &=& 1 - x_1 + x_3 \\
\hline
z &=& 1 + 2x_1 - x_3
\end{array}
$$

$\rightarrow x_1 = 1$

$$
\begin{array}{lclclcr}
x_1 & = & 1 & + & x_3 & - & x_5 \\
x_2 & = & 2 & & & - & x_5 \\
x_4 & = & 2 & - & x_3 & + & x_5 \\
\hline
z & = & 3 & + & x_3 & - & 2x_5
\end{array}
$$

$$
\begin{array}{rcrcrcr}
x_1 & = & 1 & + & x_3 & - & x_5 \\
x_2 & = & 2 & & & - & x_5 \\
x_4 & = & 2 & - & x_3 & + & x_5 \\
\hline
z & = & 3 & + & x_3 & - & 2x_5
\end{array}
$$

$$
\begin{array}{rcrcrcr}
x_1 & = & 3 & - & x_4 & & \\
x_2 & = & 2 & & & - & x_5 \\
x_3 & = & 2 & - & x_4 & + & x_5 \\
\hline
z & = & 5 & - & x_4 & - & x_5
\end{array}
$$

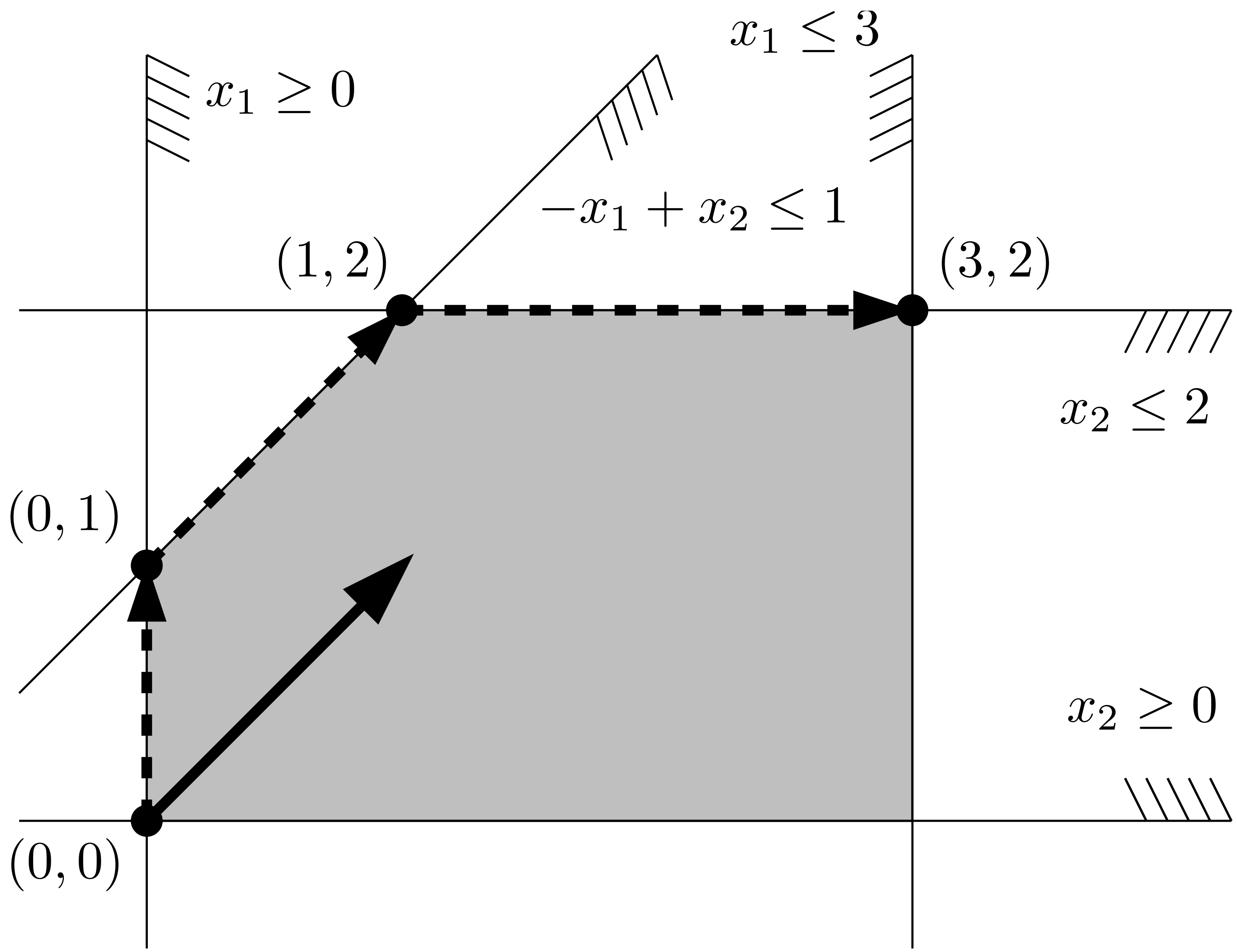
?

$$
\begin{array}{lcccccc}
x_1 & = & 3 & - & x_4 & & \\
x_2 & = & 2 & & & - & x_5 \\
x_3 & = & 2 & - & x_4 & + & x_5 \\
\hline
z & = & 5 & - & x_4 & - & x_5
\end{array}
$$

$$
\begin{array}{rcrcrcr}
x_1 & = & 3 & - & x_4 & & \\
x_2 & = & 2 & & & - & x_5 \\
x_3 & = & 2 & - & x_4 & + & x_5 \\
\hline
z & = & 5 & \underbrace{-} & \underbrace{x_4} & \underbrace{-} & \underbrace{x_5}_{۱۷\ ۰}
\end{array}
$$

$\forall$ جواب شدنی $z \le ۵$

تعبیر هندسی

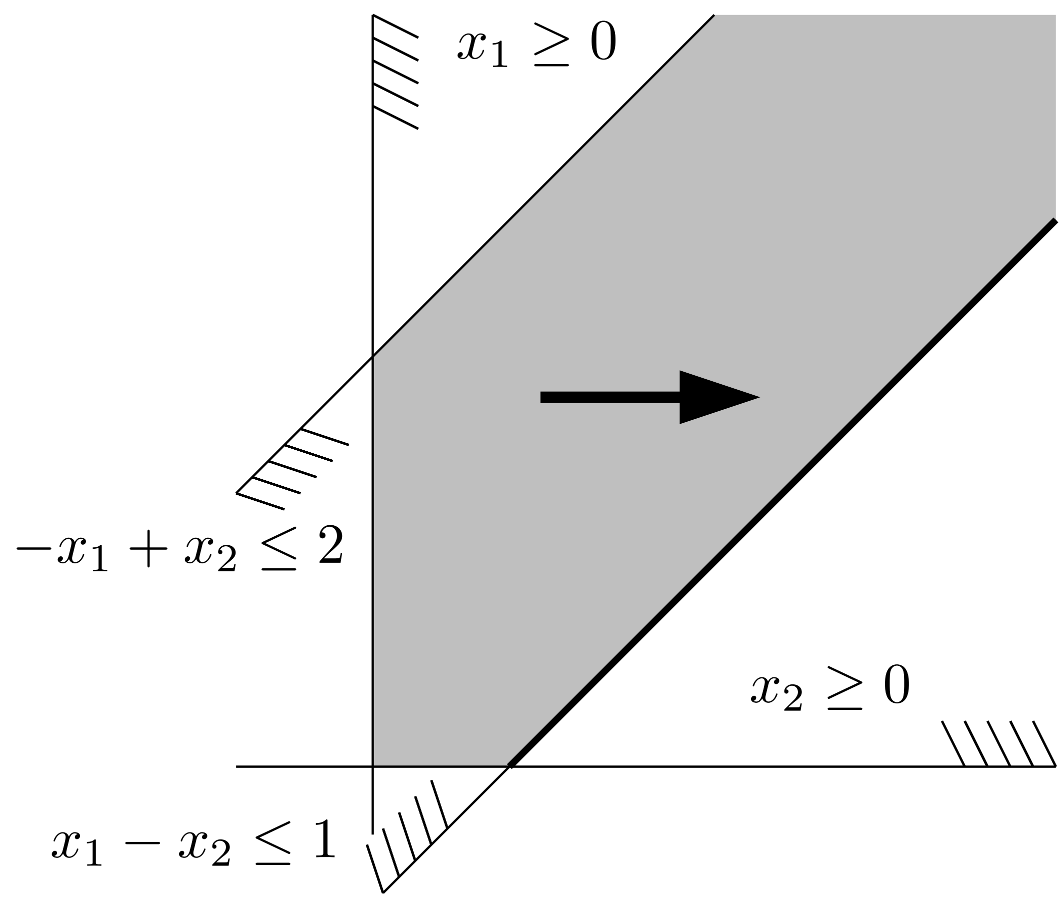


مشکل ۱:

$$
\begin{array}{ll}
\text{maximize} & x_1 \\
\text{subject to} & x_1 - x_2 \leq 1 \\
 & -x_1 + x_2 \leq 2 \\
 & x_1, x_2 \geq 0
\end{array}
$$

?

$$
\begin{array}{ll}
\text{maximize} & x_1 \\
\text{subject to} & x_1 - x_2 \le 1 \\
 & -x_1 + x_2 \le 2 \\
 & x_1, x_2 \ge 0
\end{array}
$$

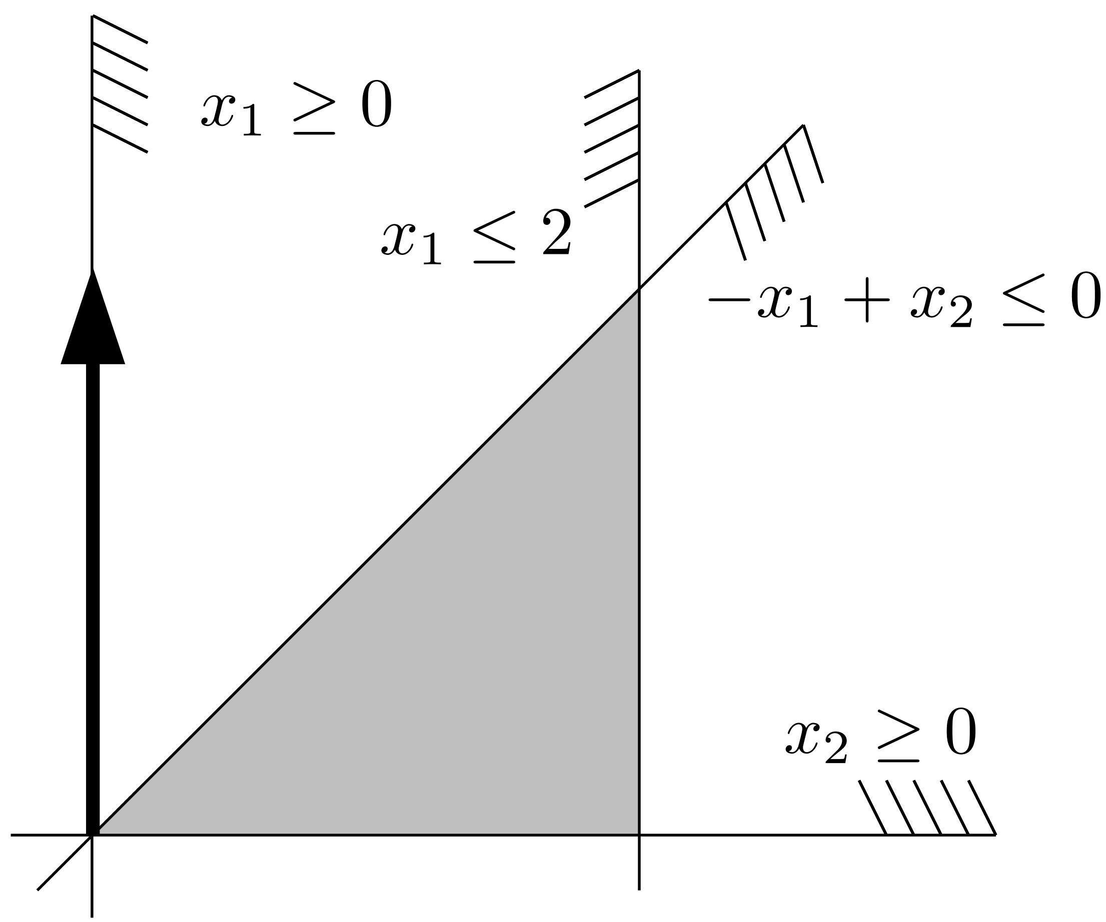


مشکل ۲: تبهگنی

$$
\begin{array}{ll}
\text{maximize} & x_2 \\
\text{subject to} & -x_1 + x_2 \leq 0 \\
 & x_1 \leq 2 \\
 & x_1, x_2 \geq 0.
\end{array}
$$

?

$$
\begin{array}{llcl}
\text{maximize} & x_2 & & \\
\text{subject to} & -x_1 + x_2 & \leq & 0 \\
& x_1 & \leq & 2 \\
& x_1, x_2 \geq 0. & &
\end{array}
$$



[illegible]

$x_0 = 1 \quad - x_4$

$x_0 = 2 \quad - 2x_2$

$z = \quad + x_2$

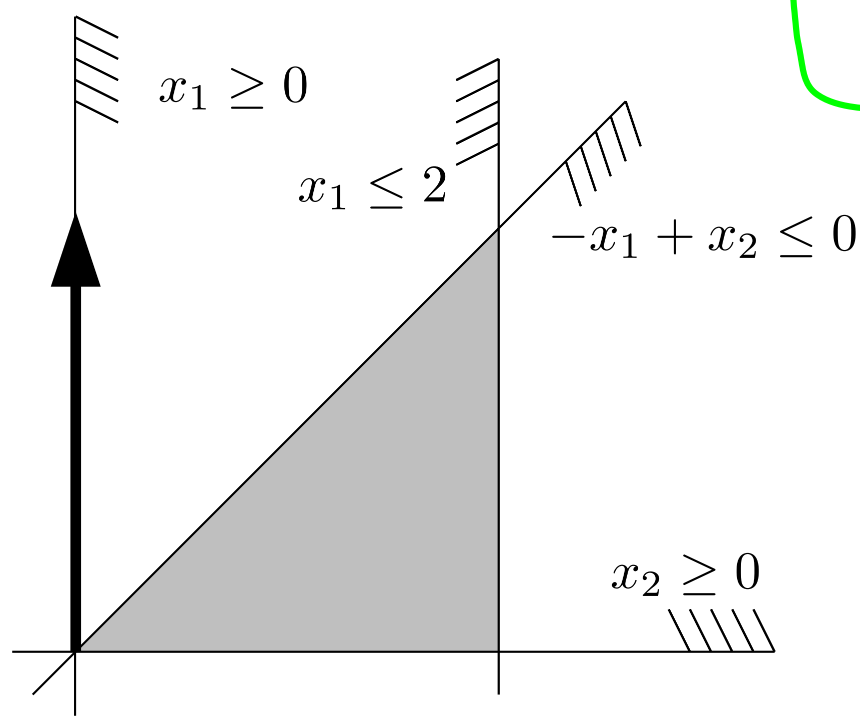
$$
\begin{array}{lrcrcl}
\text{maximize} & & & x_2 & & \\
\text{subject to} & -x_1 & + & x_2 & \leq & 0 \\
& x_1 & & & \leq & 2 \\
& x_1, x_2 \geq 0. & & & &
\end{array}
$$

$x_0 = 0 \quad - x_1$

$x_4 = 0 \quad - 2x_1$

$x_2 = 0 \quad - x_1$

$z = \quad + x_1 - x_2$



B

مشکل ۳:

$$\begin{array}{lrcrcrcl}
\text{maximize} & x_1 & + & 2x_2 & & & & \\
\text{subject to} & x_1 & + & 3x_2 & + & x_3 & = & 4 \\
 & & & 2x_2 & + & x_3 & = & 2 \\
 & \multicolumn{7}{l}{x_1, x_2, x_3 \geq 0.}
\end{array}$$

$$
\begin{array}{lrcrcrcl}
\text{maximize} & x_1 & + & 2x_2 & & & & \\
\text{subject to} & x_1 & + & 3x_2 & + & x_3 & = & 4 \\
& & & 2x_2 & + & x_3 & = & 2 \\
& \multicolumn{7}{l}{x_1, x_2, x_3 \geq 0.}
\end{array}
$$

$$
\begin{array}{lrcrcrcrcrcl}
\text{Maximize} & & & & & & - & x_4 & - & x_5 & & \\
\text{subject to} & x_1 & + & 3x_2 & + & x_3 & + & x_4 & & & = & 4 \\
& & & 2x_2 & + & x_3 & & & + & x_5 & = & 2 \\
& \multicolumn{11}{l}{x_1, x_2, \ldots, x_5 \geq 0.}
\end{array}
$$

سیمپلکس

در حالت کلی

$$\boxed{\begin{array}{ll} \text{Maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{array}}$$

m-element set $B \subseteq \{1, 2, \ldots, n\}$

A_B is nonsingular

the (unique) solution of the system $A_B \mathbf{x}_B = \mathbf{b}$ is nonnegative.

simplex tableau $\mathcal{T}(B)$

$$
\begin{array}{lcccc}
\mathbf{x}_B & = & \mathbf{p} & + & Q\,\mathbf{x}_N \\
\hline
z & = & z_0 & + & \mathbf{r}^T\mathbf{x}_N
\end{array}
$$

simplex tableau $\mathcal{T}(B)$

x_u

$$\begin{array}{lcl} \mathbf{x}_B &=& \mathbf{p} + Q\,\mathbf{x}_N \\ \hline z &=& z_0 + \mathbf{r}^T\mathbf{x}_N \end{array}$$

$\boxed{x_v = \cdots - x_u}$

x_v

$$\mathbf{x}_N = \mathbf{0}$$

$$\mathbf{x}_B = \mathbf{p}$$

$$z = z_0$$

For each feasible basis B there exists exactly one simplex tableau

$$\mathbf{r} \leq \mathbf{0} \implies \text{optimal}\,!$$

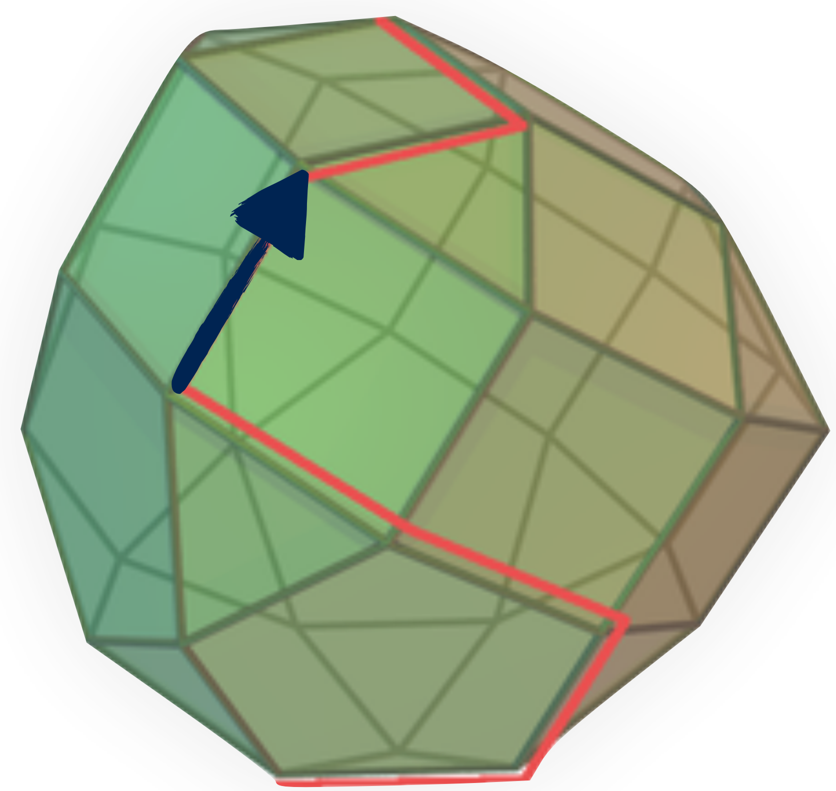
گام لولا! (pivot)

$$\mathcal{T}(B) \longrightarrow \mathcal{T}(B')$$

$$B' = (B \setminus \{u\}) \cup \{v\}$$

x_v enters the basis

x_u leaves the basis



$$B = \{k_1, k_2, \ldots, k_m\},\ k_1 < k_2 < \cdots < k_m$$

$$N = \{\ell_1, \ell_2, \ldots, \ell_{n-m}\},\ \ell_1 < \ell_2 < \cdots < \ell_{n-m}$$

$$x_{k_i} = p_i + \sum_{j=1}^{n-m} q_{ij} x_{\ell_j}$$

$$v = \ell_\beta$$

$$u = k_\alpha$$

$$q_{\alpha\beta} < 0 \quad \text{and} \quad -\frac{p_\alpha}{q_{\alpha\beta}} = \min\left\{-\frac{p_i}{q_{i\beta}} : q_{i\beta} < 0,\ i = 1, 2, \ldots, m\right\}$$

$B' = (B \setminus \{u\}) \cup \{v\}$ *is again a feasible basis.*

اثبات:

$$\begin{array}{lcccc} \mathbf{x}_B & = & \mathbf{p} & + & Q\,\mathbf{x}_N \\ \hline z & = & z_0 & + & \mathbf{r}^T\mathbf{x}_N \end{array}$$

u: حذف

v: جدید

اثبات:

$$
\begin{array}{llll}
\mathbf{x}_B & = \mathbf{p} & + & Q\,\mathbf{x}_N \\
\hline
z & = z_0 & + & \mathbf{r}^T\mathbf{x}_N
\end{array}
$$

u: حذف

v: جدید

۱

(تابلو جدید) <—— xu = ۰

اثبات:

$$\begin{array}{lcccc} \mathbf{x}_B & = & \mathbf{p} & + & Q\,\mathbf{x}_N \\ \hline z & = & z_0 & + & \mathbf{r}^T\mathbf{x}_N \end{array}$$

u: حذف

v: جدید

۱ (تابلو جدید) <—— xu = ۰

۲ حکم: ستون‌های 'B مستقلند

اثبات:

$$\begin{array}{lclcl} \mathbf{x}_B & = & \mathbf{p} & + & Q\,\mathbf{x}_N \\ \hline z & = & z_0 & + & \mathbf{r}^T\mathbf{x}_N \end{array}$$

u: حذف

v: جدید

۱

(تابلو جدید) —-> xu = ०

۲

حکم: ستون‌های B’ مستقلند

= (I ?)

اثبات:

$$
\begin{array}{lcccc}
\mathbf{x}_B & = & \mathbf{p} & + & Q\,\mathbf{x}_N \\
\hline
z & = & z_0 & + & \mathbf{r}^T\mathbf{x}_N
\end{array}
$$

u: حذف

v: جدید

۱

(تابلو جدید) <— xu = ۰

۲

حکم: ستون‌های B’ مستقلند

$A_B^{-1}A_{B'}$ = (I ?)

اثبات:

$$\begin{array}{lcl} \mathbf{x}_B & = \mathbf{p} & + \ Q\,\mathbf{x}_N \\ \hline z & = z_0 & + \ \mathbf{r}^T \mathbf{x}_N \end{array}$$

u: حذف

v: جدید

۱

(تابلو جدید) —> xu = ۰

۲

حکم: ستون‌های B’ مستقلند

$A_B^{-1} A_{B'}$ = (I ?)

غیرصفر u,v: ضریب —> x <- b <- x’

اثبات:

$$\frac{\mathbf{x}_B = \mathbf{p} + Q\,\mathbf{x}_N}{z = z_0 + \mathbf{r}^T\mathbf{x}_N}$$

u: حذف

v: جدید

۱ (تابلو جدید) <—— xu = ۰

۲ حکم: ستون‌های B’ مستقلند

$A_B x_B = b$

$A_{B'} x_{B'} = b$

$A_B^{-1} b = x_B$

$x_B \qquad x_{B'}$

$$A_B^{-1}A_{B'}$$

= (I ?)

$$x_B = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & u & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{B'} \end{bmatrix}$$

$x_u = u\, x_v$