بسم الله الرحمن الرحیم

برنامه‌ریزی نیمه‌معین برای طراحی الگوریتم‌های تقریبی

جلسه نهم: دوگانی (۴)

Project: People

مرور

کنج محدب بسته و دوگان کنج

4.2.1 Definition. *Let $K \subseteq V$ be a nonempty closed set.*[1] *K is called a closed convex cone if the following two conditions hold.*

(i) *For all $\mathbf{x} \in K$ and all nonnegative real numbers λ, we have $\lambda\mathbf{x} \in K$.*

(ii) *For all $\mathbf{x}, \mathbf{y} \in K$, we have $\mathbf{x} + \mathbf{y} \in K$.*

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(ii) *For all $\mathbf{x}, \mathbf{y} \in K$, we have $\mathbf{x} + \mathbf{y} \in K$.*

4.3.1 Definition. *Let $K \subseteq V$ be a closed convex cone. The set*

$$K^* := \{\mathbf{y} \in V : \langle \mathbf{y}, \mathbf{x} \rangle \geq 0 \text{ for all } \mathbf{x} \in K\}$$

is called the dual cone of K.

مثال: اگر K و L کنج محدب بسته باشند، آنگاه

$$K \oplus L := \{(\mathbf{x}, \mathbf{y}) \in V \oplus W : \mathbf{x} \in K, \mathbf{y} \in L\}$$

کنج محدب بسته است.

$$(K \oplus L)^* = K^* \oplus L^*.$$

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4.5.2 Definition. *Let* $A\colon V \to W$ *be a linear operator. A linear operator* $A^T\colon W \to V$ *is called an adjoint of* A *if*

$$\langle \mathbf{y}, A(\mathbf{x})\rangle = \langle A^T(\mathbf{y}), \mathbf{x}\rangle \text{ for all } \mathbf{x} \in V \text{ and } \mathbf{y} \in W.$$

4.5.2 Definition. *Let $A: V \to W$ be a linear operator. A linear operator $A^T: W \to V$ is called an adjoint of A if*

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4.5.3 Lemma. *Let $V = \mathrm{SYM}_n, W = \mathbb{R}^m$, and $A: V \to W$ defined by $A(X) = (A_1 \bullet X, A_2 \bullet X, \ldots, A_m \bullet X)$. Then*

$$A^T(\mathbf{y}) = \sum_{i=1}^{m} y_i A_i.$$

تعریف: نمایش ماتریسی عملگرهای خطی

فضای برداری:

$$W_1, W_2, \ldots, W_m$$

فضای برداری:

$$V_1, V_2, \ldots, V_n$$

عملگرهای خطی:

$$A_{ij} : V_j \to W_i$$

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$$\left(\begin{array}{c|c|c|c} A_{11} & A_{12} & \cdots & A_{1n} \\ \hline A_{21} & A_{22} & \cdots & A_{2n} \\ \hline \vdots & \vdots & & \vdots \\ \hline A_{m1} & A_{m2} & \cdots & A_{mn} \end{array}\right) : V_1 \oplus V_2 \oplus \cdots \oplus V_n \to W_1 \oplus W_2 \oplus \cdots \oplus W_m$$

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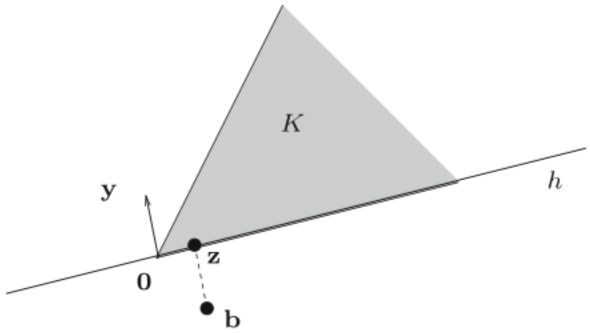
قضیه جداسازی

4.4.2 Theorem. *Let $K \subseteq V$ be a closed convex cone, and let $\mathbf{b} \in V \setminus K$. Then there exists a vector $\mathbf{y} \in V$ such that*

$$\langle \mathbf{y}, \mathbf{x} \rangle \geq 0 \text{ for all } \mathbf{x} \in K, \text{ and } \langle \mathbf{y}, \mathbf{b} \rangle < 0.$$



هدف نهایی:

$$
\begin{array}{lll}
\text{(D)} & \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\
 & \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\
 & & \mathbf{y} \in L^*.
\end{array}
$$

$$
\begin{array}{lll}
\text{(P)} & \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\
 & \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\
 & & \mathbf{x} \in K.
\end{array}
$$

مرحله ۱:

لم فارکاش:

پایینی شدنی حدی است XOR بالایی شدنی

$$A^T(\mathbf{y}) \in K^*, \langle \mathbf{b}, \mathbf{y} \rangle < 0$$

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$c = 0 \qquad L = \{0\}$

دوگانی قوی:

تحت شرایطی،

(P) شدنی حدی است با جواب کران‌دار

اگر و فقط اگر

(D) شدنی است با جواب کران‌دار

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4.7.3 Theorem. *The dual program (D) is feasible and has a finite value β if and only if the primal program (P) is limit-feasible and has a finite limit value γ. Moreover, $\beta = \gamma$.*

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$$\left(\begin{array}{c|c} A^T & -\mathbf{c} \\ \hline \mathrm{id} & \mathbf{0} \\ \hline 0 & 1 \end{array}\right)(\mathbf{y}, z) \in K^* \oplus L^* \oplus \mathbb{R}_+ \quad \Rightarrow \quad \langle(\mathbf{b}, -\beta), (\mathbf{y}, z)\rangle \geq 0$$

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\left(\begin{array}{c|c|c} A & \mathrm{id} & 0 \\ \hline -\mathbf{c}^T & \mathbf{0}^T & 1 \end{array} \right) (\mathbf{x}, \mathbf{x}', z) = (\mathbf{b}, -\beta),\ (\mathbf{x}, \mathbf{x}', z) \in (K^* \oplus L^* \oplus \mathbb{R}_+)^* = K \oplus L \oplus \mathbb{R}_+
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(P) Maximize $\langle \mathbf{c}, \mathbf{x} \rangle$
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پس (P) شدنی حدی است با مقداری بیشتر مساوی β

(P) Maximize $\langle \mathbf{c}, \mathbf{x} \rangle$
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نشدنی

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پس (P) شدنی حدی است با مقداری بیشتر مساوی β

(بنابر دوگانی ضعیف:) برنامه‌ریزی کنج (P) شدنی است با مقداری برابر با β

$$
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اگر P شدنی حدی باشد:

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این جواب + یک جواب حدی برای (P): جوابی بهتر برای (P)

دوگانی برنامەریزی نیمەمعین

$$\text{(P)} \quad \begin{array}{ll} \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & \mathbf{x} \in K. \end{array}$$

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4.7.1 Theorem. *If the primal program (P) is feasible, has a finite value* γ and *has an interior point* $\tilde{\mathbf{x}}$, *then the dual program (D) is also feasible and has the same value* γ.

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$$
\begin{array}{ll} \text{maximize} & C \bullet X \\ \text{subject to} & A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m \\ & X \succeq 0. \end{array}
$$

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$K = \mathrm{PSD}_n$

$$
\text{(P)} \quad \begin{array}{ll} \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & \mathbf{x} \in K. \end{array}
$$

$$
\text{(D)} \quad \begin{array}{ll} \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\ \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\ & \mathbf{y} \in L^*. \end{array}
$$

$$
\text{(P)} \quad \begin{array}{lll} \text{maximize} & C \bullet X & \\ \text{subject to} & A_i \bullet X = b_i, & i = 1, 2, \ldots, m \\ & X \succeq 0. & \end{array}
$$

$$L = \{\mathbf{0}\}$$

$$K = \mathrm{PSD}_n$$

$$\text{(P)} \quad \begin{array}{ll} \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & \mathbf{x} \in K. \end{array}$$

$$\text{(D)} \quad \begin{array}{ll} \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\ \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\ & \mathbf{y} \in L^*. \end{array}$$

$$\text{(P)} \quad \begin{array}{lll} \text{maximize} & C \bullet X & \\ \text{subject to} & A_i \bullet X = b_i, & i = 1, 2, \ldots, m \\ & X \succeq 0. & \end{array}$$

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$$\text{(P)} \quad \text{Maximize} \quad \langle \mathbf{c}, \mathbf{x} \rangle \quad \text{subject to} \quad \mathbf{b} - A(\mathbf{x}) \in L, \quad \mathbf{x} \in K.$$

$$\text{(D)} \quad \text{Minimize} \quad \langle \mathbf{b}, \mathbf{y} \rangle \quad \text{subject to} \quad A^T(\mathbf{y}) - \mathbf{c} \in K^*, \quad \mathbf{y} \in L^*.$$

$$\begin{array}{lll} \text{(P)} & \text{maximize} & C \bullet X \\ & \text{subject to} & A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m \\ & & X \succeq 0. \end{array}$$

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$L = \{\mathbf{0}\}$

$K = \mathrm{PSD}_n$

$$\begin{array}{lll} \text{(D)} & \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\ & & y \in \mathbb{R}^m \end{array}$$

$L^* = \mathbb{R}^m$

$$\text{(P)} \quad \text{Maximize} \quad \langle \mathbf{c}, \mathbf{x} \rangle \quad \text{subject to} \quad \mathbf{b} - A(\mathbf{x}) \in L, \quad \mathbf{x} \in K.$$

$$\text{(D)} \quad \text{Minimize} \quad \langle \mathbf{b}, \mathbf{y} \rangle \quad \text{subject to} \quad A^T(\mathbf{y}) - \mathbf{c} \in K^*, \quad \mathbf{y} \in L^*.$$

$$
\begin{array}{lll}
\text{(P)} & \text{maximize} & C \bullet X \\
 & \text{subject to} & A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m \\
 & & X \succeq 0.
\end{array}
$$

$$L = \{\mathbf{0}\}$$

$$K = \mathrm{PSD}_n$$

$$
\begin{array}{lll}
\text{(D)} & \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\
 & & y \in \mathbb{R}^m
\end{array}
$$

$$K^* = ???$$

$$L^* = \mathbb{R}^m$$

(P) Maximize $\langle \mathbf{c}, \mathbf{x} \rangle$
subject to $\mathbf{b} - A(\mathbf{x}) \in L$
$\mathbf{x} \in K.$

(D) Minimize $\langle \mathbf{b}, \mathbf{y} \rangle$
subject to $A^T(\mathbf{y}) - \mathbf{c} \in K^*$
$\mathbf{y} \in L^*.$

(P) maximize $C \bullet X$
subject to $A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m$
$X \succeq 0.$

$L = \{\mathbf{0}\}$

$K = \mathrm{PSD}_n$

(D) Minimize $\langle \mathbf{b}, \mathbf{y} \rangle$

$$\sum_{i=1}^{m} y_i A_i - C \in K^*$$

$$y \in \mathbb{R}^m$$

$K^* = ???$

$L^* = \mathbb{R}^m$

4.7.5 Lemma. $(\mathrm{PSD}_n)^* = \mathrm{PSD}_n$.

$$
\text{(P)} \quad \begin{array}{ll} \text{Maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{subject to} & \mathbf{b} - A(\mathbf{x}) \in L \\ & \mathbf{x} \in K. \end{array}
$$

$$
\text{(D)} \quad \begin{array}{ll} \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\ \text{subject to} & A^T(\mathbf{y}) - \mathbf{c} \in K^* \\ & \mathbf{y} \in L^*. \end{array}
$$

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$$

4.1.1 Theorem. *If the semidefinite program (4.1) is feasible and has a finite value γ, and if there is a positive definite matrix $\tilde{X}$ such that $A(\tilde{X}) = \mathbf{b}$, then the dual program*

$$
\begin{array}{ll} \text{minimize} & \mathbf{b}^T \mathbf{y} \\ \text{subject to} & \sum_{i=1}^m y_i A_i - C \succeq 0 \end{array} \tag{4.2}
$$

is feasible and has finite value $\beta = \gamma$.

$$
\text{(D)} \qquad \begin{array}{ll} \text{Minimize} & \langle \mathbf{b}, \mathbf{y} \rangle \\ & \sum_{i=1}^{m} y_i A_i - C \in \mathrm{PSD}_n \\ & y \in \mathbb{R}^m \end{array}
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4.7.5 Lemma. $(\mathrm{PSD}_n)^* = \mathrm{PSD}_n$.

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قضیه:

هر ماتریس متقارن M حقیقی را می‌توان به صورت زیر نوشت:

$$M = \sum_{i=1}^{n} \lambda_i \beta_i \beta_i^T.$$

که β_i بردار ویژه‌های M هستند و λ_i مقدار ویژه مربوط به بردار ویژه β_i است.

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اگر ضرب BA معنی‌دار باشد:

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- اگر X مثبت نیمه‌معین => $X \in \mathrm{PSD}^*$
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- اگر $M \in \mathrm{PSD}^*$ => M مثبت نیمه‌معین

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- اگر X مثبت نیمه‌معین <= $X \in \mathrm{PSD}^*$
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$$X \bullet Y = \mathrm{Tr}\left(\sum_{i=1}^{n} \lambda_i ß_i ß_i^T Y\right)$$

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$$X \bullet Y = \mathrm{Tr}\left(\sum_{i=1}^{n} \lambda_i ß_i ß_i^T Y\right) = \sum_{i=1}^{n} \lambda_i \,\mathrm{Tr}(ß_i ß_i^T Y) = \sum_{i=1}^{n} \lambda_i \,\mathrm{Tr}(ß_i^T Y ß_i)$$

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$$
\begin{aligned}
X \bullet Y &= \mathrm{Tr}\left(\sum_{i=1}^{n} \lambda_i \mathrm{ß}_i \mathrm{ß}_i^T Y\right) = \sum_{i=1}^{n} \lambda_i \,\mathrm{Tr}(\mathrm{ß}_i \mathrm{ß}_i^T Y) \\
&= \sum_{i=1}^{n} \lambda_i \,\mathrm{Tr}(\mathrm{ß}_i^T Y \mathrm{ß}_i) = \sum_{i=1}^{n} \lambda_i \mathrm{ß}_i^T Y \mathrm{ß}_i \geq 0
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\end{aligned}
$$

- اگر $M \in \mathrm{PSD}^*$ <= M مثبت نیمه‌معین

$$0 \leq M \bullet \mathbf{x}\mathbf{x}^T$$

تعریف PSD^*

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$$\begin{aligned} X \bullet Y &= \mathrm{Tr}\left(\sum_{i=1}^{n} \lambda_i \text{ß}_i \text{ß}_i^T Y\right) = \sum_{i=1}^{n} \lambda_i \,\mathrm{Tr}(\text{ß}_i \text{ß}_i^T Y) \\ &= \sum_{i=1}^{n} \lambda_i \,\mathrm{Tr}(\text{ß}_i^T Y \text{ß}_i) = \sum_{i=1}^{n} \lambda_i \text{ß}_i^T Y \text{ß}_i \geq 0 \end{aligned}$$

- اگر $M \in \mathrm{PSD}^*$ <= M مثبت نیمه‌معین

$$0 \leq M \bullet \mathbf{x}\mathbf{x}^T = \mathrm{Tr}((M\mathbf{x})\mathbf{x}^T)$$

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- اگر X مثبت نیمه‌معین <= $X \in \mathrm{PSD}^*$
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$$
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X \bullet Y &= \mathrm{Tr}\left(\sum_{i=1}^{n} \lambda_i \text{ß}_i \text{ß}_i^T Y\right) = \sum_{i=1}^{n} \lambda_i \,\mathrm{Tr}(\text{ß}_i \text{ß}_i^T Y) \\
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\end{aligned}
$$

- اگر $M \in \mathrm{PSD}^*$ <= M مثبت نیمه‌معین

$$0 \leq M \bullet \mathbf{x}\mathbf{x}^T = \mathrm{Tr}((M\mathbf{x})\mathbf{x}^T) = \mathrm{Tr}(\mathbf{x}^T M \mathbf{x})$$

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$$\begin{aligned} X \bullet Y &= \mathrm{Tr}\left(\sum_{i=1}^{n} \lambda_i \ss_i \ss_i^T Y\right) = \sum_{i=1}^{n} \lambda_i \, \mathrm{Tr}(\ss_i \ss_i^T Y) \\ &= \sum_{i=1}^{n} \lambda_i \, \mathrm{Tr}(\ss_i^T Y \ss_i) = \sum_{i=1}^{n} \lambda_i \ss_i^T Y \ss_i \geq 0 \end{aligned}$$

- اگر $M \in \mathrm{PSD}^*$ <= M مثبت نیمه‌معین

$$0 \leq M \bullet \mathbf{x}\mathbf{x}^T = \mathrm{Tr}((M\mathbf{x})\mathbf{x}^T) = \mathrm{Tr}(\mathbf{x}^T M \mathbf{x}) = \mathbf{x}^T M \mathbf{x}$$

تعریف PSD^*

الگوریتم هازان

برنامه‌ریزی خطی روی اسپکتراهدرون

$$
\begin{array}{ll}
\text{maximize} & C \bullet X \\
\text{subject to} & A_i \bullet X \leq b_i, \quad i = 1, 2, \ldots, m \\
& \mathrm{Tr}(X) = 1 \\
& X \succeq 0.
\end{array}
$$

$$
\begin{array}{ll}
\text{maximize} & C \bullet X \\
\text{subject to} & A_i \bullet X \leq b_i, \quad i = 1, 2, \ldots, m \\
& \operatorname{Tr}(X) = 1 \\
& X \succeq 0.
\end{array}
$$

5.1.1 Theorem. *Let γ_{opt} be the value of the semidefinite program (5.1), and let $\varepsilon \in (0, 1]$ be given. Hazan's algorithm either finds a matrix $\tilde{X} \in \operatorname{Spect}_n$ such that*

$$
\frac{\gamma_{\text{opt}} - C \bullet \tilde{X}}{\|C\|_F} \leq 2\varepsilon, \quad \frac{A_i \bullet \tilde{X} - b_i}{\|A_i\|_F} \leq \varepsilon, \quad i = 1, 2, \ldots, m,
$$

or correctly reports that (5.1) has no feasible solution. The number of arithmetic operations is bounded by

$$
O\left(\frac{N(\log n)(\log m)\log(1/\varepsilon)}{\varepsilon^3}\right)
$$

with high probability, assuming that $1/\varepsilon$ is bounded by some fixed polynomial function of n.

الگوریتم نقطه درونی

برنامه‌ریزی نیمه‌معین

$$
\begin{array}{ll}
\text{Maximize} & C \bullet X \\
\text{subject to} & A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m, \\
 & X \succeq 0.
\end{array}
$$

ایده:

$$
\begin{array}{ll}
\text{Maximize} & f_\mu(X) := C \bullet X + \mu \ln \det X \\
\text{subject to} & A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m, \\
& X \succ 0,
\end{array}
\tag{6.2}
$$

پایان