بسم الله الرحمن الرحیم

برنامه‌ریزی نیمه‌معین برای طراحی الگوریتم‌های تقریبی

جلسه هفتم: دوگانی

مرور

قضیه نهایی دوگانی (برای برنامه‌نویسی نیمه‌معین)

$$\begin{array}{ll} \text{maximize} & C \bullet X \\ \text{subject to} & A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m \\ & X \succeq 0. \end{array}$$

4.1.1 Theorem. *If the semidefinite program (4.1) is feasible and has a finite value γ, and if there is a positive definite matrix $\tilde{X}$ such that $A(\tilde{X}) = \mathbf{b}$, then the dual program*

$$\begin{array}{ll} \textit{minimize} & \mathbf{b}^T \mathbf{y} \\ \textit{subject to} & \sum_{i=1}^m y_i A_i - C \succeq 0 \end{array} \tag{4.2}$$

is feasible and has finite value $\beta = \gamma$.

کنج محدب بسته

4.2.1 Definition. *Let $K \subseteq V$ be a nonempty closed set.*[1] *K is called a closed convex cone if the following two conditions hold.*

(i) *For all $\mathbf{x} \in K$ and all nonnegative real numbers λ, we have $\lambda\mathbf{x} \in K$.*
(ii) *For all $\mathbf{x}, \mathbf{y} \in K$, we have $\mathbf{x} + \mathbf{y} \in K$.*

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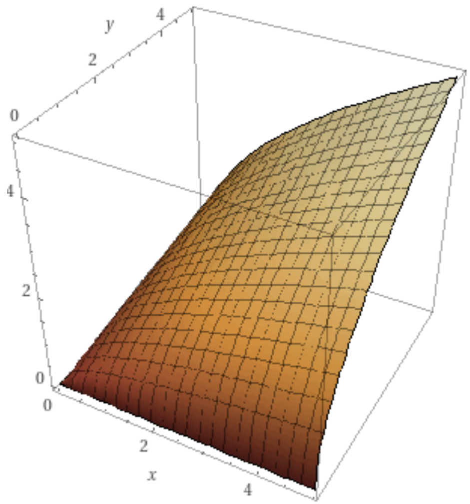

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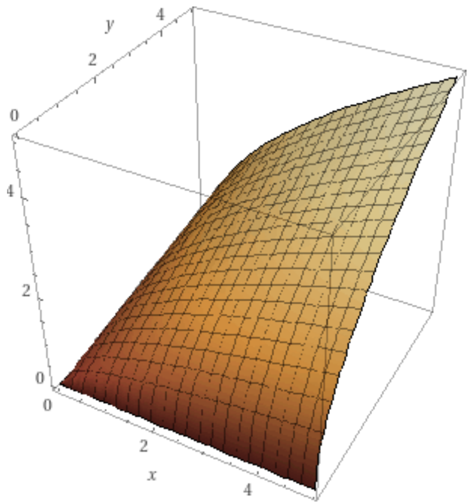

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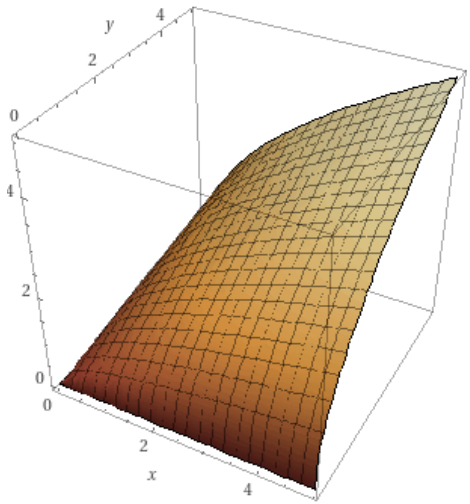

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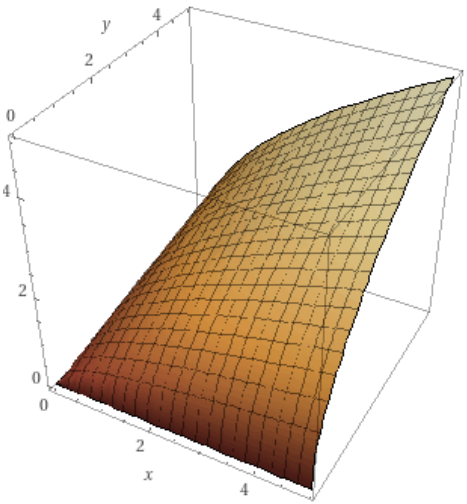

- الف) بسته
- ب) ضرب
- ج) جمع

- معادلا $\begin{pmatrix} x & z \\ z & y \end{pmatrix}$ مثبت نیمه‌معین

مسیر پیش رو

SEPARATION:
Separating a closed convex cone from a point by a hyperplane [argument: closest point]

CONE PROGRAMS:
(P)
$\max\{\langle \mathbf{c}, \mathbf{x}\rangle : \mathbf{b} - A(\mathbf{x}) \in L, \mathbf{x} \in K\}$
(D)
$\min\{\langle \mathbf{b}, \mathbf{y}\rangle : A^T(\mathbf{y}) - \mathbf{c} \in K^*, \mathbf{y} \in L^*\}$

FARKAS LEMMA:
the system $A\mathbf{x} = \mathbf{b}, \mathbf{x} \in K$
limit-feasible
XOR
$\exists \mathbf{y} : A^T(\mathbf{y}) \in K^*, \langle \mathbf{b}, \mathbf{y}\rangle < 0$

WEAK DUALITY for cone programs:
limit value of (P) $\leq$ value (D)
[easy]

REGULAR DUALITY for cone programs:
limit value of (P) $=$ value of (D)

$\exists$ interior point
$\Rightarrow$ limit value $=$ value

STRONG DUALITY for cone programs:
(P) feasible, finite value, interior point $\Rightarrow$
(D) feasible, same value

(also a version for equational form)

$(\mathrm{PSD}_n)^* = \mathrm{PSD}_n$

SDP DUALITY:
$\max\{C \bullet X : A_1 \bullet X = b_1, \ldots, A_m \bullet X = b_m, X \succeq 0\}$
feasible, finite value, interior point
$\Rightarrow$
$\min\{\mathbf{b}^T\mathbf{y} : y_1 A_1 + \cdots + y_m A_m - C \succeq 0\}$ feasible, same value

دوگان کنج

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$$K^* := \{\mathbf{y} \in V : \langle \mathbf{y}, \mathbf{x} \rangle \geq 0 \text{ for all } \mathbf{x} \in K\}$$

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  - y خارج از K^*. x هست که $y^T x < 0$.
  - $y+y'$ برای $||y'|| << 1$:

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- ج) جمع
  - $\langle y + y', x \rangle = \langle y, x \rangle + \langle y', x \rangle$

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4.2.3 Fact. *Let $K \subseteq V$, $L \subseteq W$ be closed convex cones. Then*

$$K \oplus L := \{(\mathbf{x}, \mathbf{y}) \in V \oplus W : \mathbf{x} \in K, \mathbf{y} \in L\}$$

is again a closed convex cone, the direct sum *of K and L.*

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- ب) برای $\tilde{\mathbf{y}} = (\tilde{x}, \tilde{y}, \tilde{z})$ که $\tilde{x}\tilde{y} < \tilde{z}^2/4$ یا $\tilde{x} < 0$ یا $\tilde{y} < 0$، یک بردار (x,y,z) هست که

  $\tilde{y}^\top x < 0$

- $\mathbf{x} = (\tilde{y}, \tilde{x}, -\sqrt{\tilde{x}\tilde{y}}) \in \triangleleft\!\!\!\bigcirc$ یا $\mathbf{x} = (\tilde{y}, \tilde{x}, \sqrt{\tilde{x}\tilde{y}}) \in \triangleleft\!\!\!\bigcirc$

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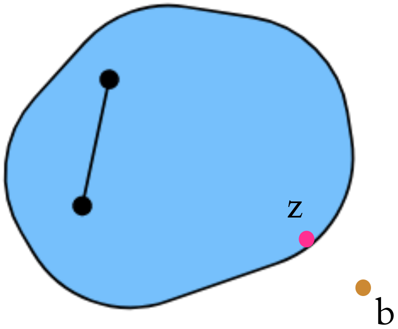
$$\tilde{\mathbf{y}}^T\mathbf{x} = 2\tilde{x}\tilde{y} - \tilde{z}\sqrt{\tilde{x}\tilde{y}} < 2\tilde{x}\tilde{y} - 2\tilde{x}\tilde{y} = 0.$$

جداسازی

4.4.2 Theorem. *Let $K \subseteq V$ be a closed convex cone, and let $\mathbf{b} \in V \setminus K$. Then there exists a vector $\mathbf{y} \in V$ such that*

$$\langle \mathbf{y}, \mathbf{x} \rangle \geq 0 \text{ for all } \mathbf{x} \in K, \text{ and } \langle \mathbf{y}, \mathbf{b} \rangle < 0.$$



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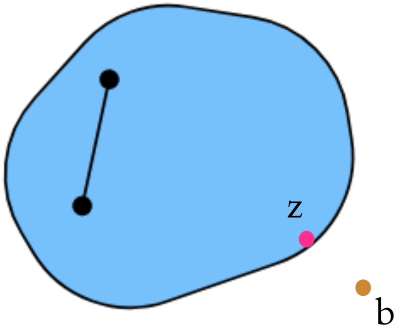

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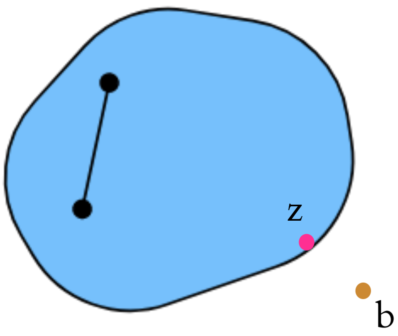
- وجود دارد:



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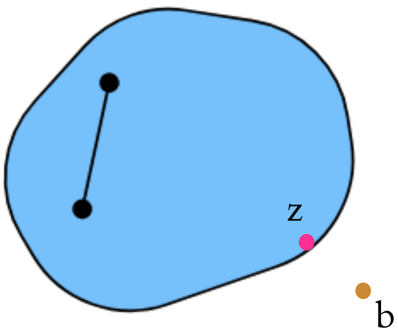

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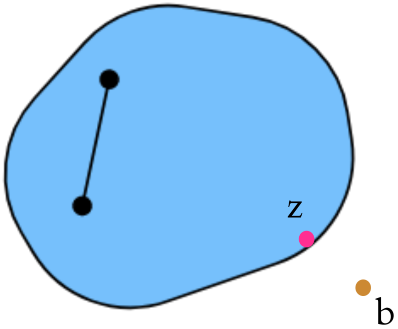

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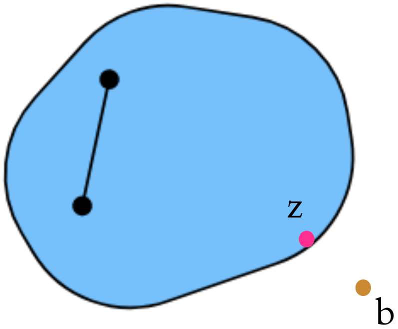

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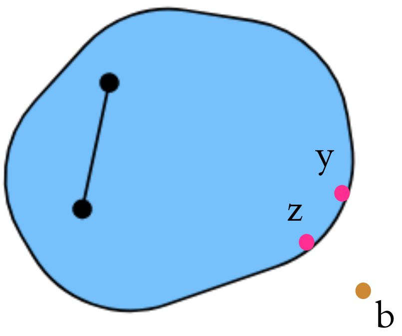

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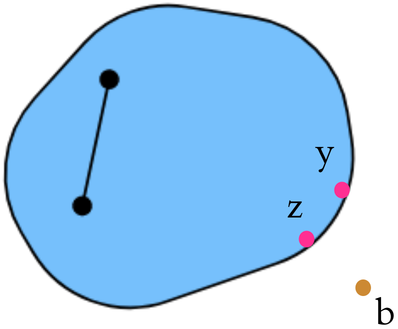

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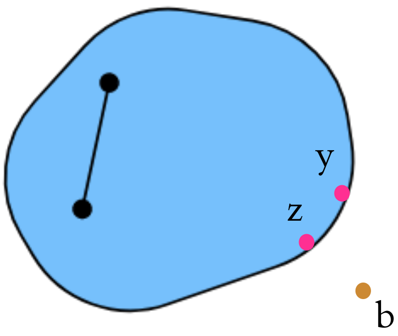

$$\|b - z - \alpha(y - z)\|^2$$

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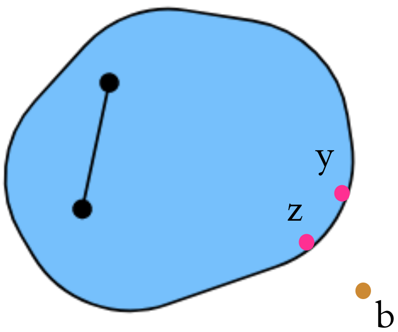

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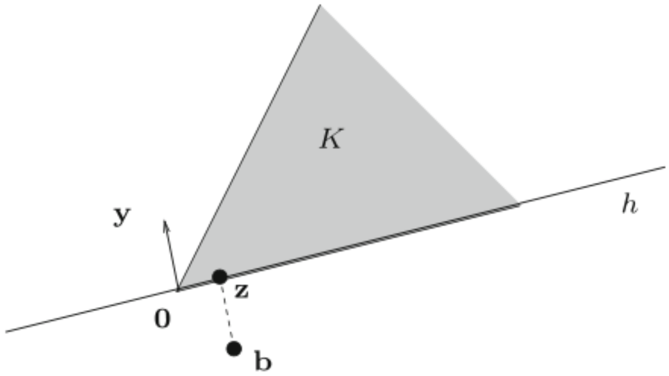

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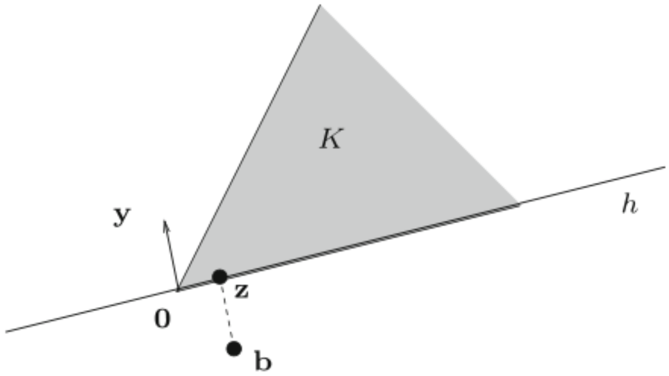

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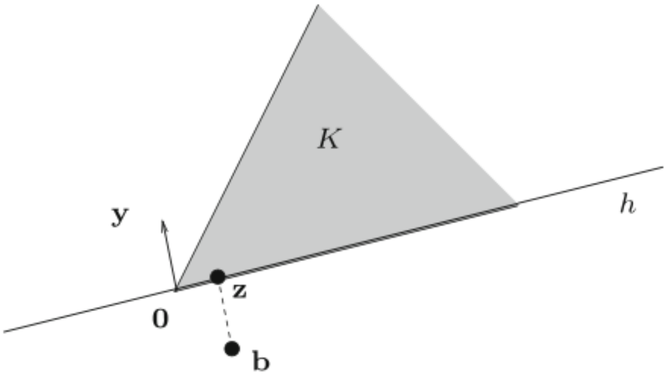

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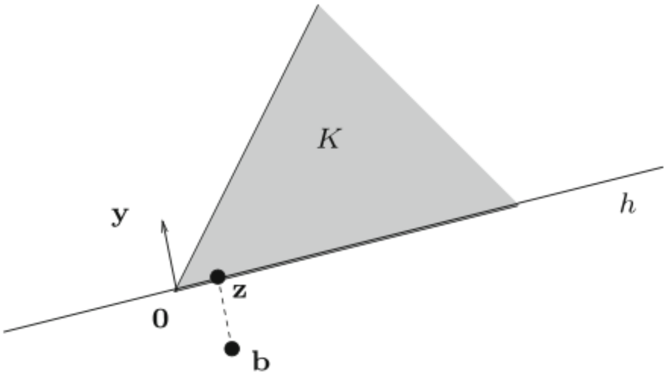

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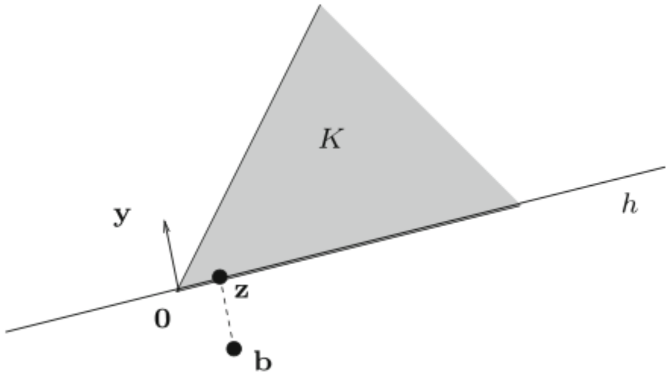

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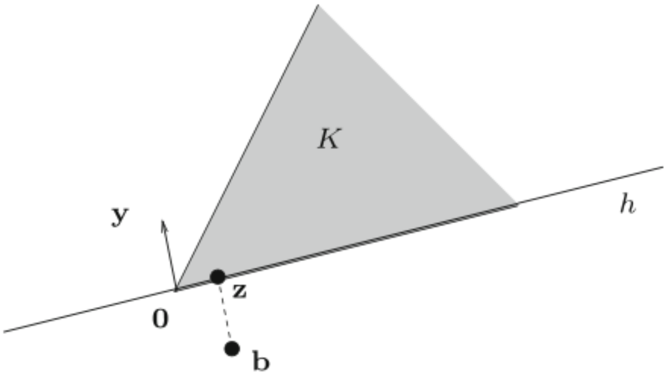

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$$\|b - z - \alpha z\|^2 \qquad = \|b - z\|^2 + \alpha^2 \|z\|^2 - 2\alpha (b - z)^\top z$$

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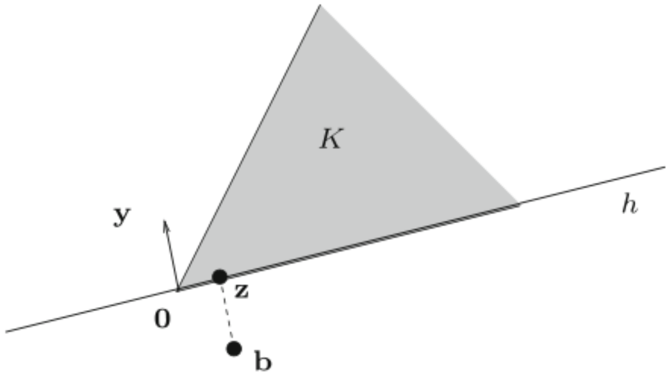

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  - برای α کمی کوچکتر از صفر یا کمی بزرگ‌تر از صفر

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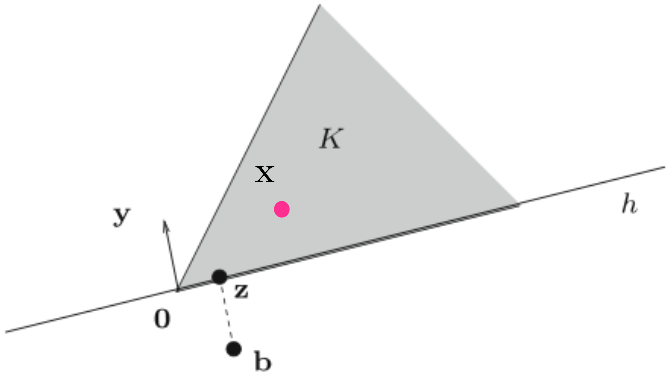

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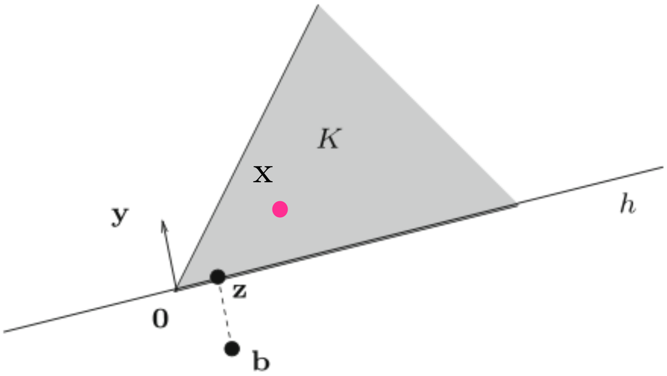

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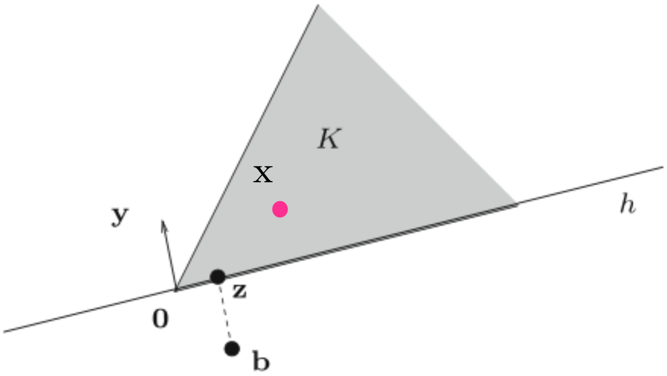

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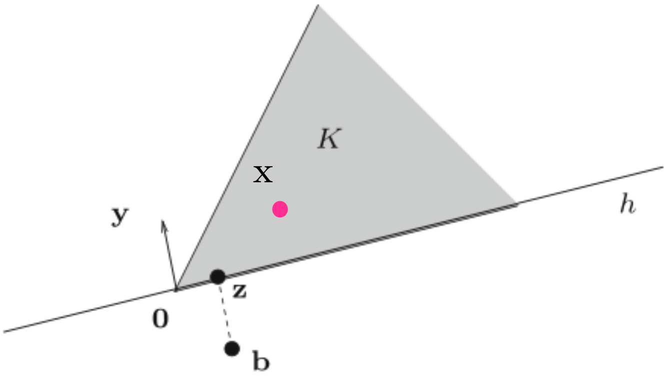

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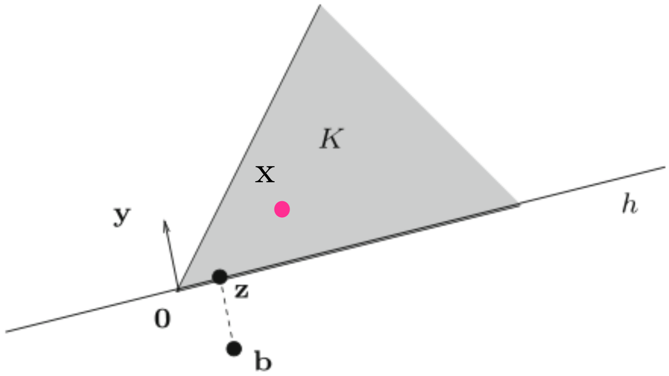

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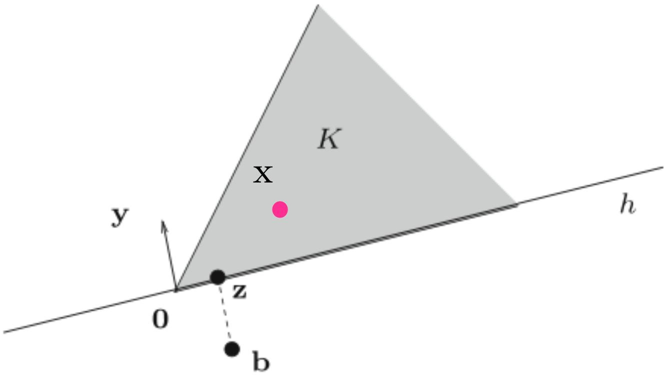

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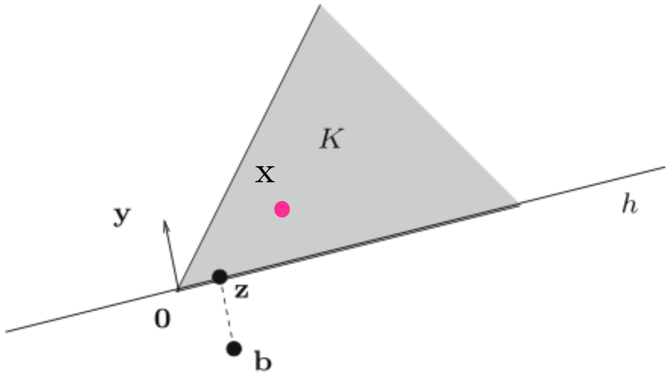

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4.3.1 Definition. *Let $K \subseteq V$ be a closed convex cone. The set*

$$K^* := \{\mathbf{y} \in V : \langle \mathbf{y}, \mathbf{x} \rangle \geq 0 \text{ for all } \mathbf{x} \in K\}$$

is called the dual cone of K.

4.4.1 Lemma. *Let $K \subseteq V$ be a closed convex cone. Then $(K^*)^* = K$.*

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4.4.1 Lemma. *Let $K \subseteq V$ be a closed convex cone. Then $(K^*)^* = K$.*

- الف) $b \in K \Rightarrow b \in K^{**}$:
  - (معادلا) به ازای هر $y \in K^*$: $\langle y, b \rangle \geq 0$
- ب) $b \notin K \Rightarrow b \notin K^{**}$:
  - (قضیه جداسازی): y هست که $\langle y, b \rangle < 0$ و $\langle y, x \rangle \geq 0$ برای هر $x \in K$

لم فارکاش

نسخه خطی

4.5.1 Lemma (Farkas). *Let $A \in \mathbb{R}^{m \times n}$ be an $m \times n$ matrix, and let $\mathbf{b} \in \mathbb{R}^m$. Then*

- *Either the system $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ has a solution $\mathbf{x} \in \mathbb{R}^n$.*
- *Or the system $A^T\mathbf{y} \geq \mathbf{0}$, $\mathbf{b}^T\mathbf{y} < 0$ has a solution $\mathbf{y} \in \mathbb{R}^m$.*

but not both.

نسخه خطی

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نسخه کنجی:

$$A(\mathbf{x}) = \mathbf{b}, \mathbf{x} \in K,$$

؟؟؟

$$
\begin{array}{ll}
\text{maximize} & C \bullet X \\
\text{subject to} & A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m \\
 & X \succeq 0.
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$$A : \mathrm{SYM}_n \to \mathbb{R}^m$$

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4.5.2 Definition. *Let $A: V \to W$ be a linear operator. A linear operator $A^T: W \to V$ is called an adjoint of A if*

$$\langle \mathbf{y}, A(\mathbf{x}) \rangle = \langle A^T(\mathbf{y}), \mathbf{x} \rangle \text{ for all } \mathbf{x} \in V \text{ and } \mathbf{y} \in W.$$

4.5.3 Lemma. *Let* $V = \mathrm{SYM}_n, W = \mathbb{R}^m$, *and* $A\colon V \to W$ *defined by* $A(X) = (A_1 \bullet X, A_2 \bullet X, \ldots, A_m \bullet X)$. *Then*

$$A^T(\mathbf{y}) = \sum_{i=1}^{m} y_i A_i.$$

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4.5.4 Lemma. *Let* $K \subseteq V$ *be a closed convex cone, and* $C = \{A(\mathbf{x}) : \mathbf{x} \in K\}$. *Then* $\overline{C}$, *the closure of* C, *is a closed convex cone.*

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4.5.5 Definition. *Let $K \subseteq V$ be a closed convex cone. The system*

$$A(\mathbf{x}) = \mathbf{b}, \;\; \mathbf{x} \in K$$

is called limit-feasible if there exists a sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ such that $\mathbf{x}_k \in K$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} A(\mathbf{x}_k) = \mathbf{b}.$$

4.5.6 Lemma (Farkas lemma for cones). *Let $K \subseteq V$ be a closed convex cone, and $\mathbf{b} \in W$.* Either *the system*

$$A(\mathbf{x}) = \mathbf{b}, \mathbf{x} \in K$$

is limit-feasible, or *the system*

$$A^T(\mathbf{y}) \in K^*, \langle \mathbf{b}, \mathbf{y} \rangle < 0$$

has a solution, but not both.