بسم الله الرحمن الرحیم

نظریه علوم کامپیوتر

نظریه علوم کامپیوتر - بهار ۱۴۰۰-۱۴۰۱ - جلسه پنجم: محاسبه‌پذیری و محاسبه‌ناپذیری

Theory of computation - 002 - S05 - non-computability

TMs and Encodings – review

A TM has 3 possible outcomes for each input w:

1. *Accept* w (enter $q$acc)
2. *Reject* w by halting (enter $q$rej)
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Theorem: $A\text{DFA}$ is decidable

Proof: Give TM $D\text{A−DFA}$ that decides $A\text{DFA}$.

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Let $A_{\text{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA and } B \text{ accepts } w\}$

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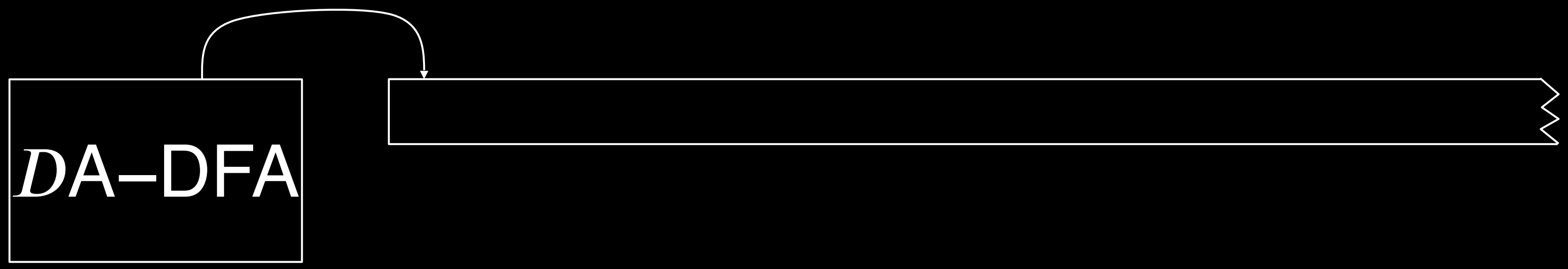

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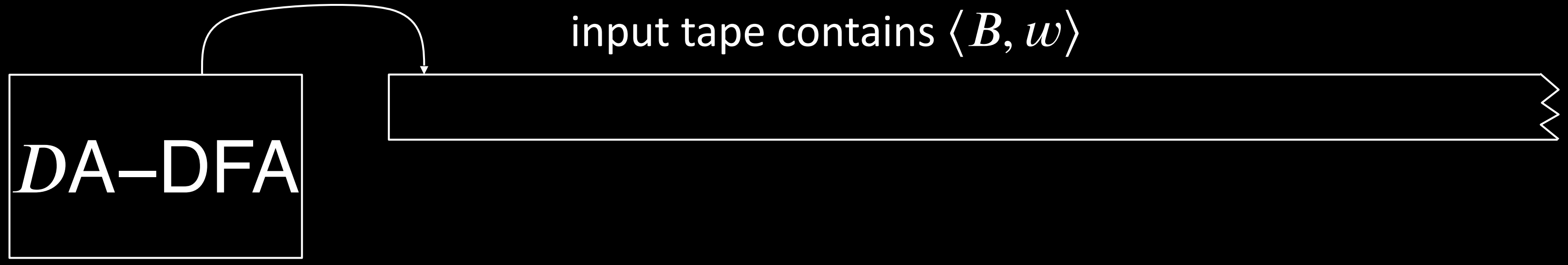

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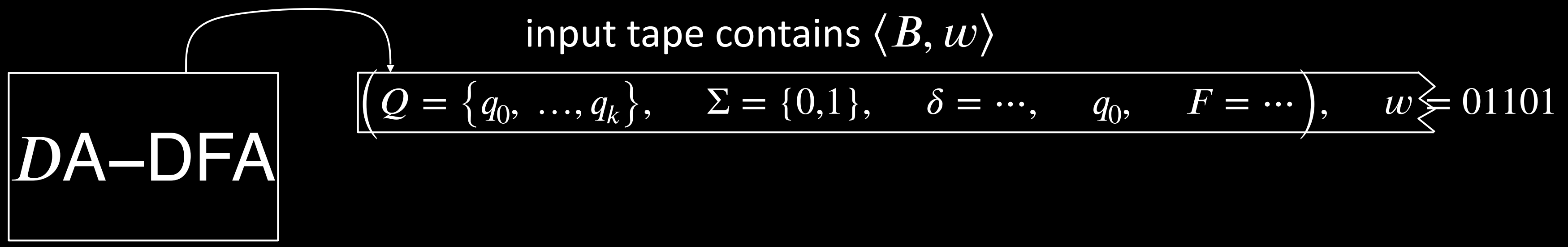

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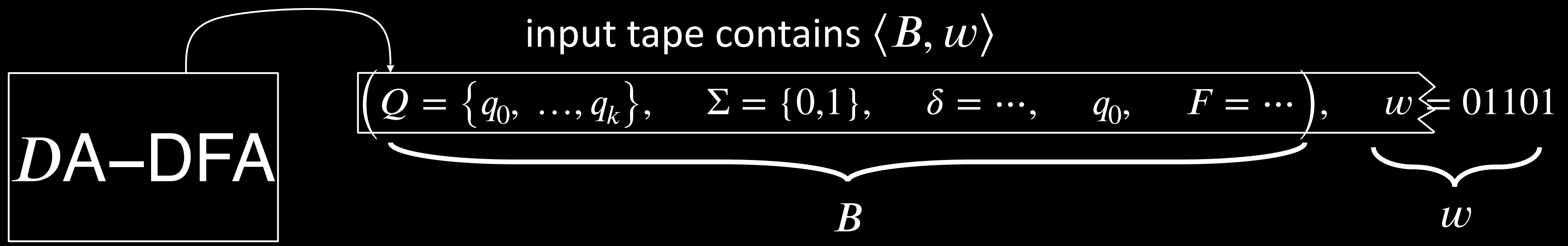

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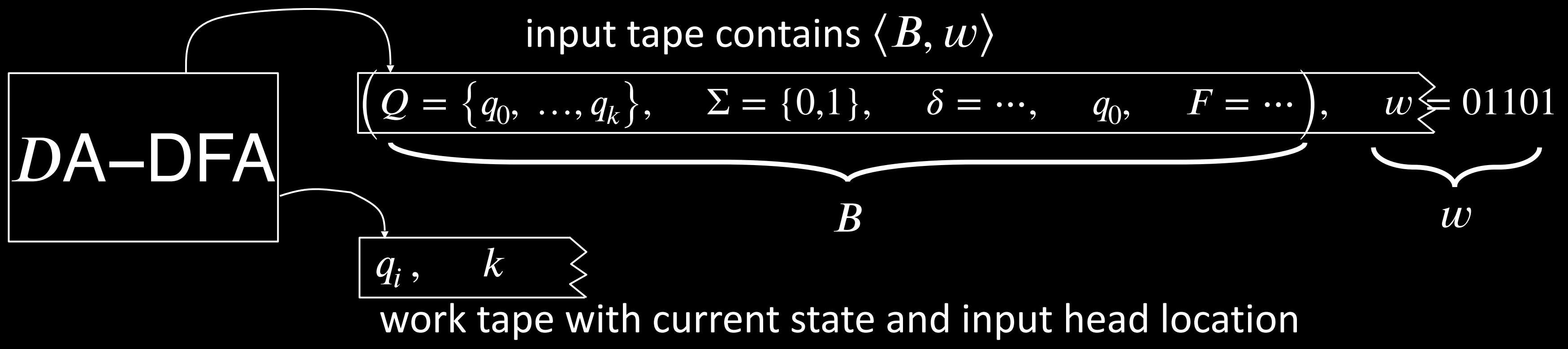

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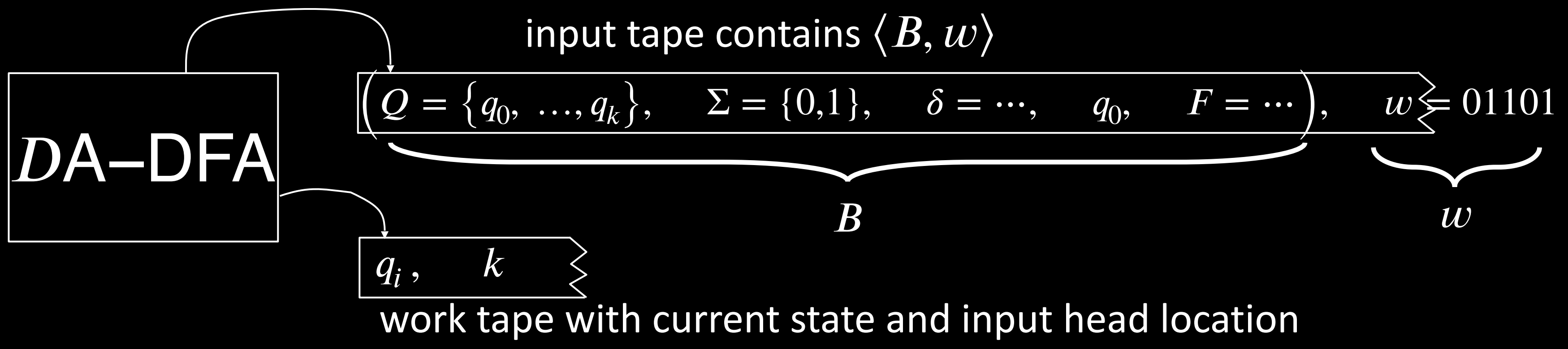

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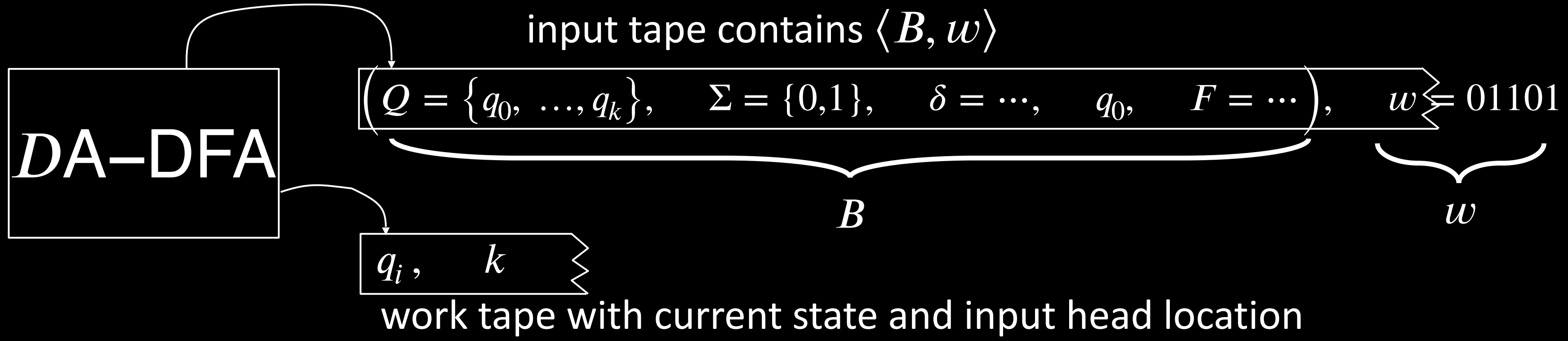

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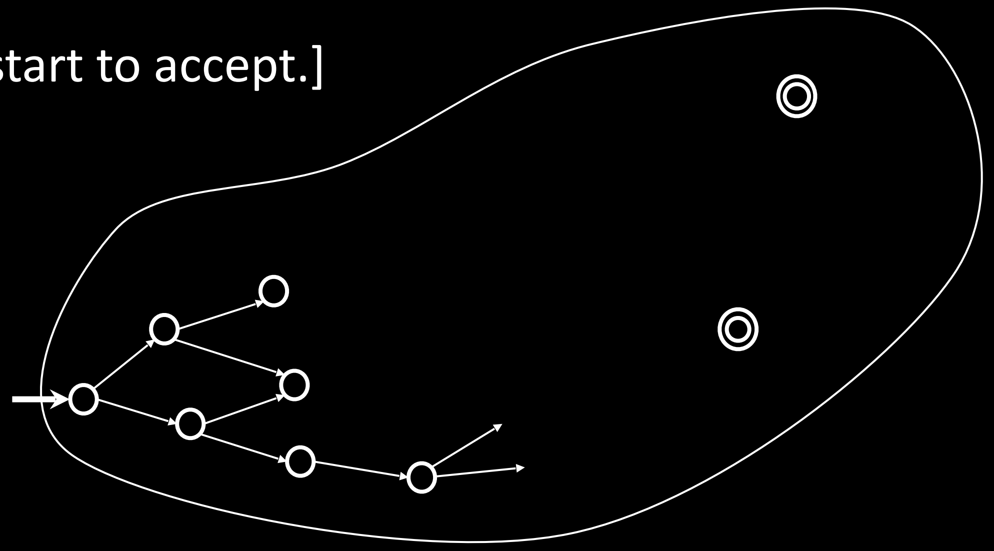

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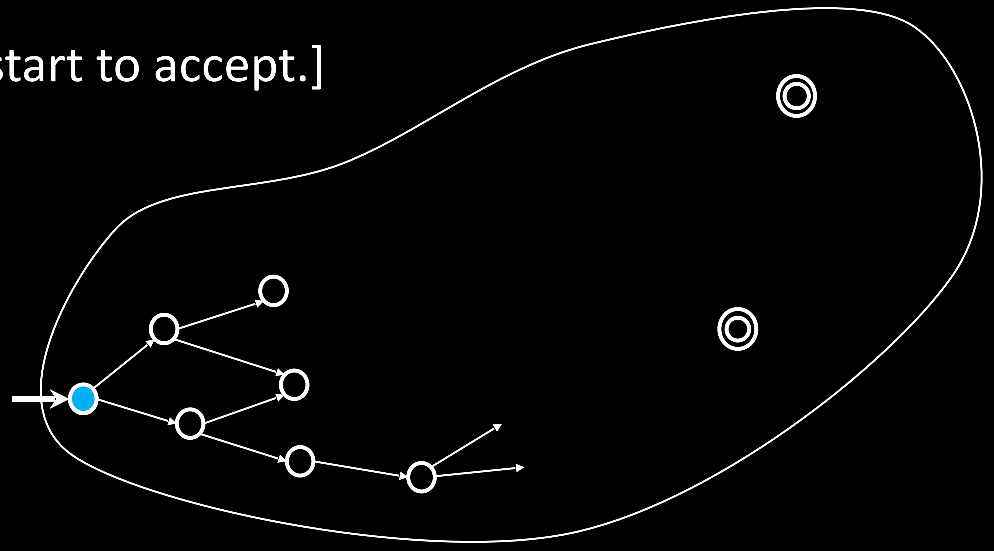

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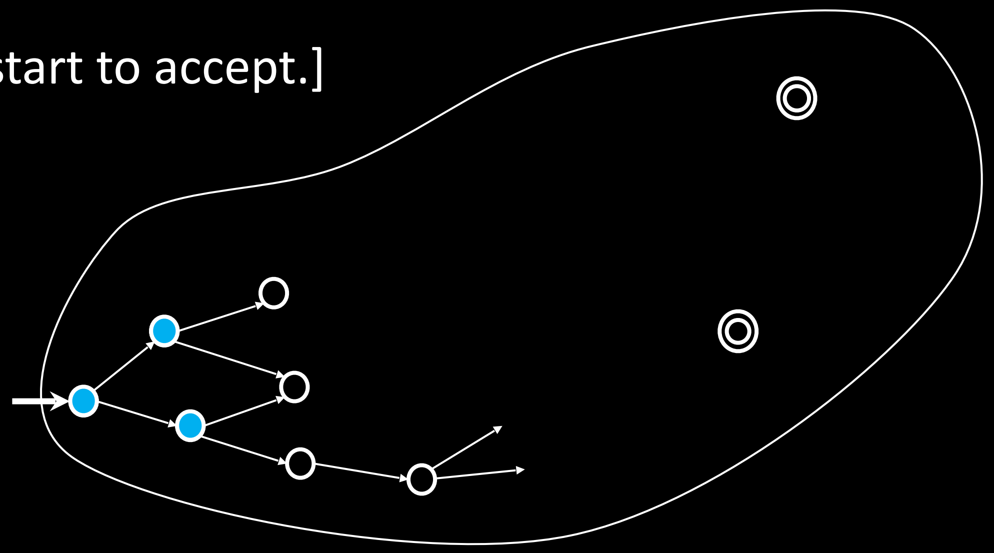

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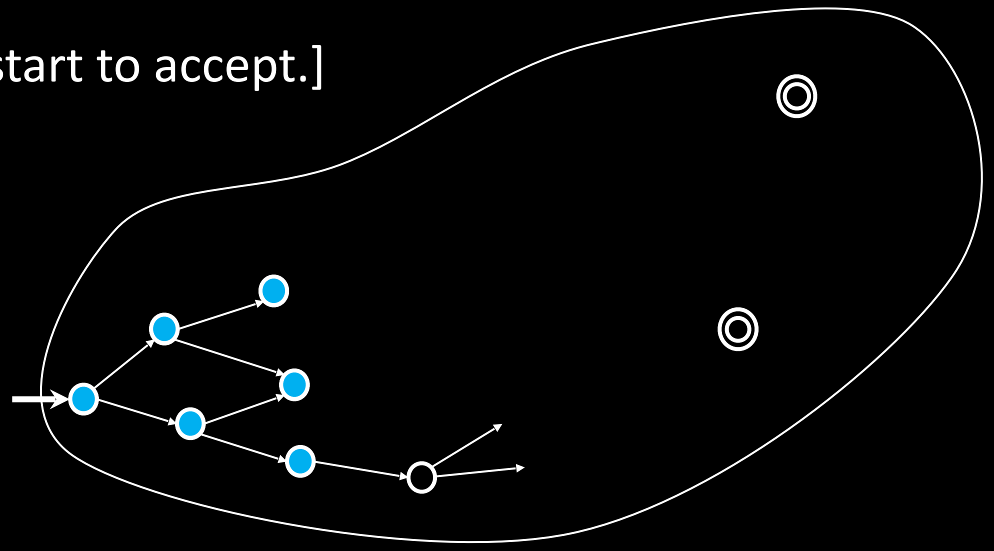

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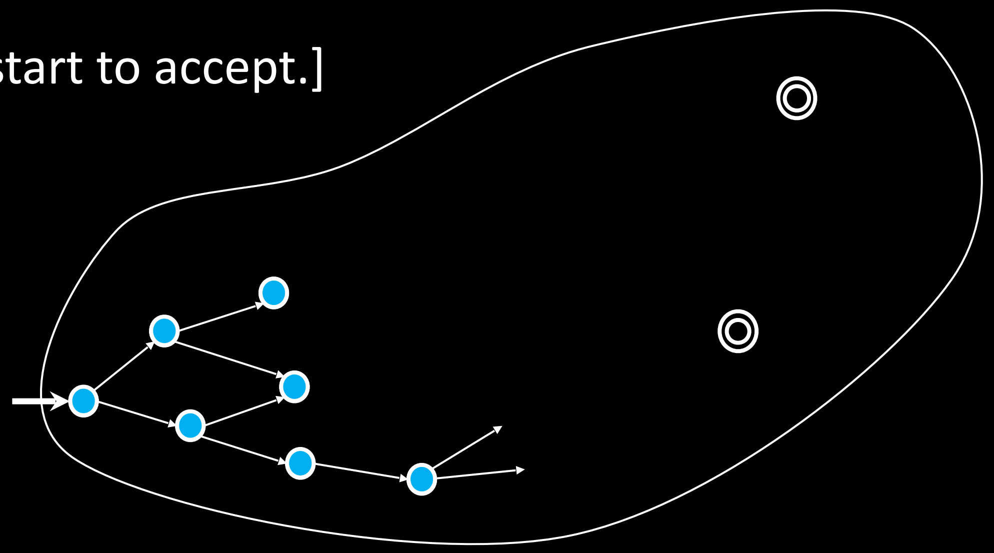

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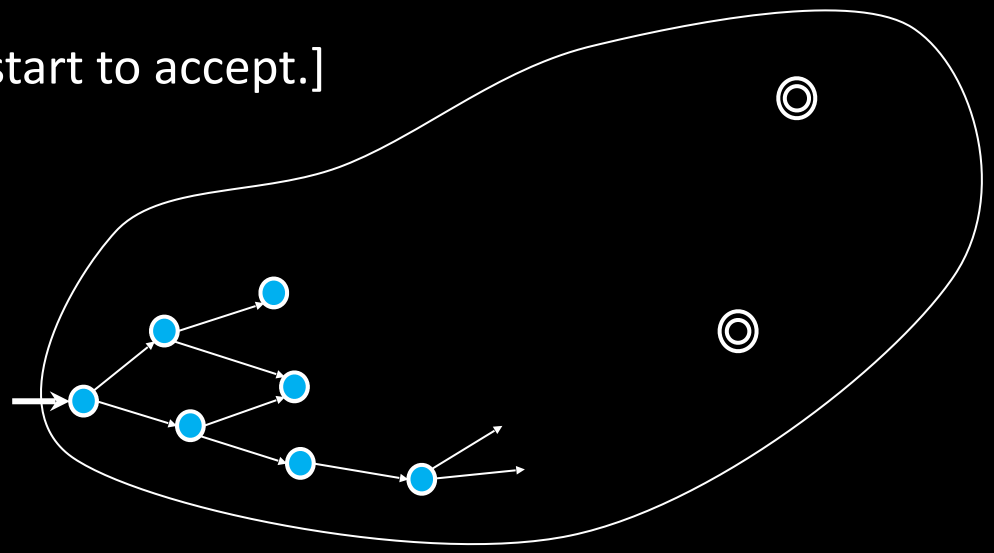

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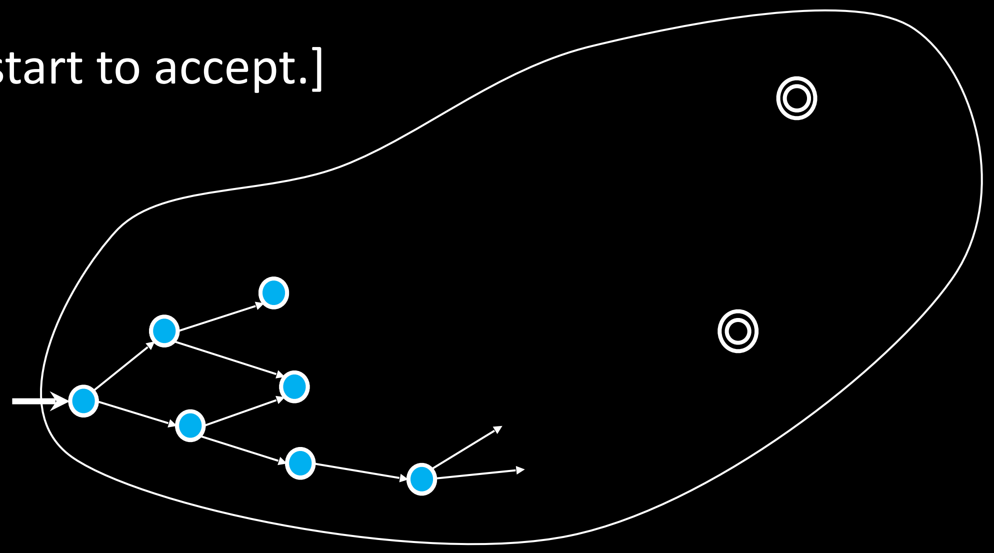

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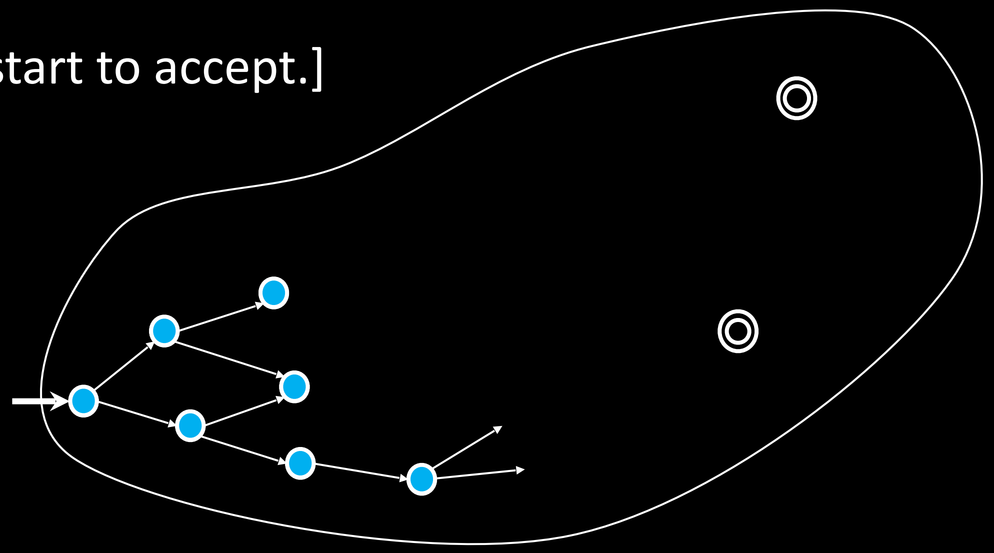

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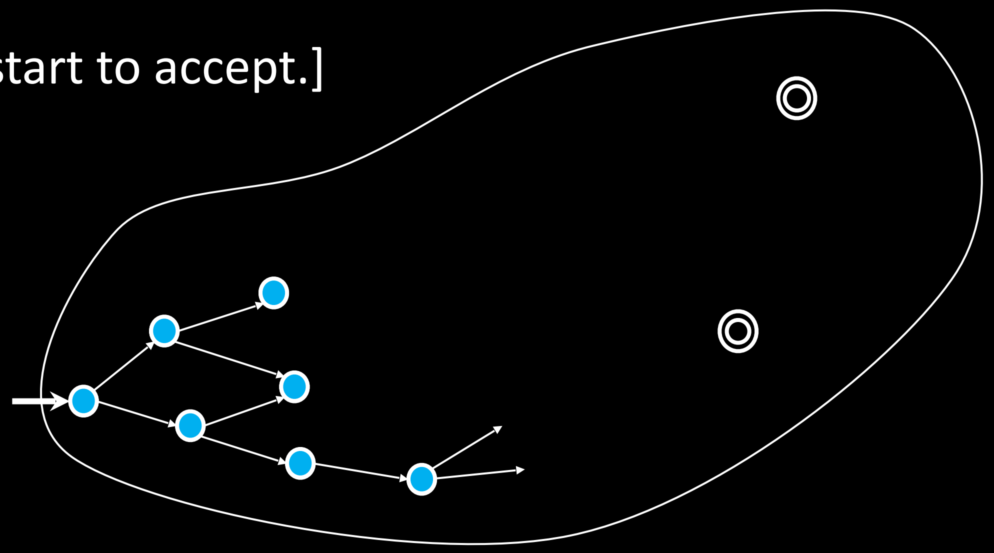

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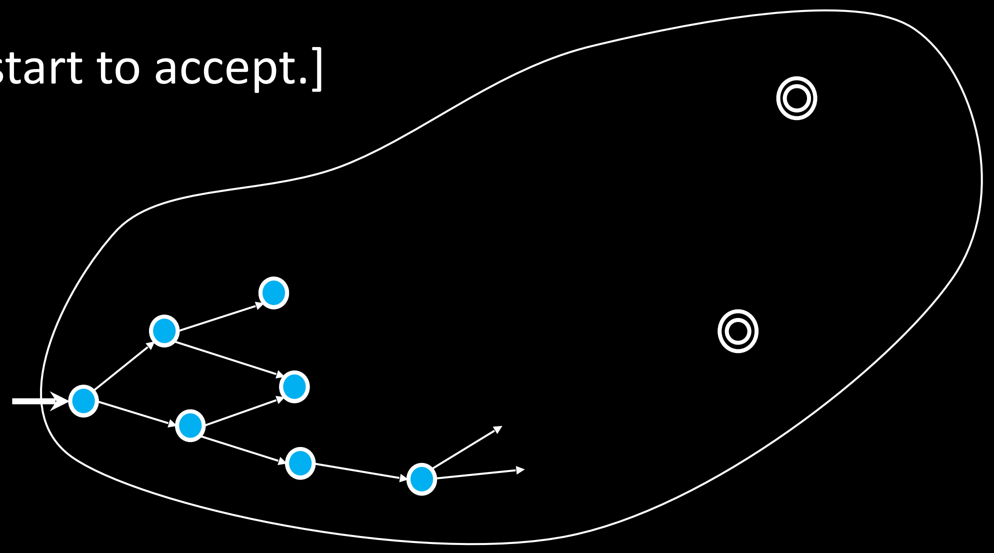

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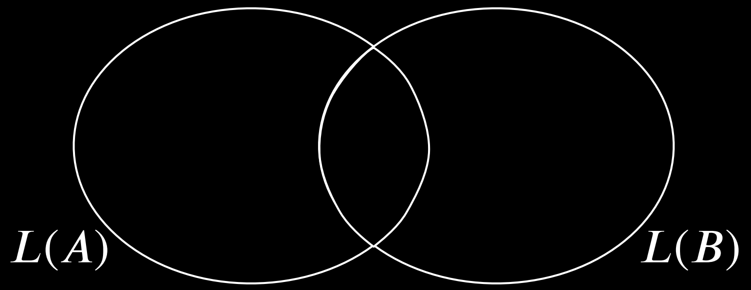

Symmetric difference

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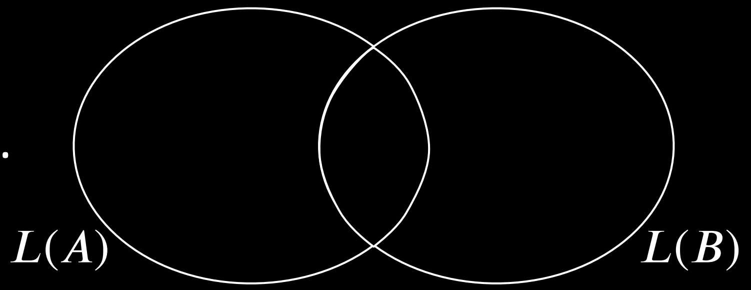
1. Construct DFA C where $L(C) = \left(L(A) \cap \overline{L(B)}\right) \cup \left(\overline{L(A)} \cap L(B)\right)$.



Symmetric difference

Equivalence problem for DFAs

Let $EQ_{\text{DFA}} = \{\langle A, B\rangle \mid A$ and B are DFAs and $L(A) = L(B)\}$

Theorem: EQ_{DFA} is decidable

Proof: Give TM $D_{\text{EQ-DFA}}$ that decides EQ_{DFA} .

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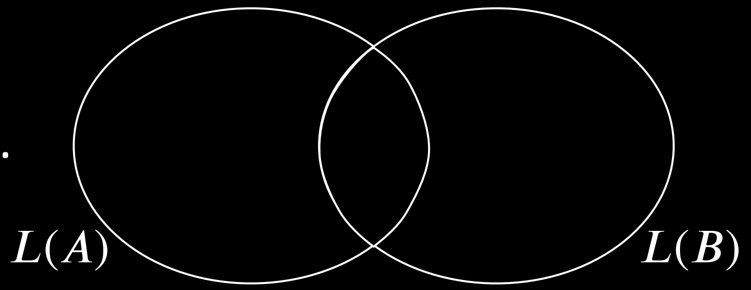

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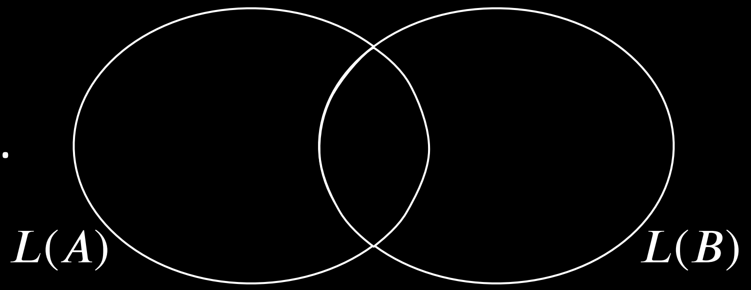

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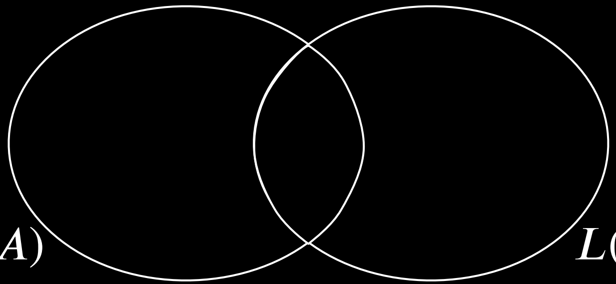

$L(A)$ $L(B)$

Symmetric difference

Check-in 7.1

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Check-in 7.1

Let $EQ_{\text{REX}} = \{\langle R_1, R_2\rangle \mid R_1$ and R_2 are regular expressions and $L(R_1) = L(R_2)\}$

Can we now conclude that EQ_{REX} is decidable?

a) Yes, it follows immediately from things we've already shown.

b) Yes, but it would take significant additional work.

c) No, intersection is not a regular operation.

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Recall Chomsky Normal Form (CNF) only allows rules:

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B → b

Lemma 1: Can convert every CFG into CNF.
Proof and construction in book.

Lemma 2: If H is in CNF and $w \in L(H)$ then every derivation of w has $2|w| - 1$ steps.
Proof: exercise.

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Corollary: Every CFL is decidable.

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Check-in 7.2

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Check-in 7.2

Can we conclude that APDA is decidable?

a) Yes.

b) No, PDAs may be nondeterministic.

c) No, PDAs may not halt.

2. *Accept* if DA–CFG accepts
   Reject if it rejects."

Recall Chomsky Normal Form (CNF) only allows rules:

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Lemma 1: Can convert every CFG into CNF. Proof and construction in book.

Lemma 2: If H is in CNF and $w \in L(H)$ then every derivation of w has $2|w| - 1$ steps. Proof: exercise.

Check-in 7.2

Emptiness Problem for CFGs

Let ECFG $= \{\langle G\rangle \mid G$ is a CFG and $L(G) = \varnothing\ \}$

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S $\rightarrow$ R T a

R $\rightarrow$ T b

T $\rightarrow$ a

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S → RTa

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1. Mark all occurrences of terminals in G.

$S \rightarrow RTa$

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Emptiness Problem for CFGs

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Theorem: E_{CFG} is decidable

Proof:

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1. Mark all occurrences of terminals in G.
2. Repeat until no new variables are marked

$S \to RTa$

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Check-in 7.3

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Check-in 7.3

Why can’t we use the same technique we used to show EQDFA is decidable to show that EQCFG is decidable?

a) Because CFGs are generators and DFAs are recognizers.

b) Because CFLs are closed under union.

c) Because CFLs are not closed under complementation and intersection.

Check-in 7.3

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Let ATM $= \{\langle M, w\rangle \mid M$ is a TM and M accepts $w\}$

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4. ~~*Reject* if M never halts.~~" Not a legal TM action.

Acceptance Problem for TMs

Let $A_{TM} = \{\langle M, w\rangle \mid M$ is a TM and M accepts $w\}$

Theorem: A_{TM} is not decidable

Proof: Later ….

Theorem: A_{TM} is T-recognizable

Proof: The following TM U recognizes A_{TM}

$U =$ "On input $\langle M, w\rangle$

1. Simulate M on input w.
2. *Accept* if M halts and accepts.
3. *Reject* if M halts and rejects.
4. ~~*Reject* if M never halts.~~" Not a legal TM action.

Turing's original "Universal Computing Machine"

Acceptance Problem for TMs

Let ATM $= \{\langle M, w\rangle \mid M$ is a TM and M accepts $w\}$

Theorem: ATM is not decidable

Proof: Later ….

Theorem: ATM is T-recognizable

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