بسم الله الرحمن الرحیم

درس تحقیق در عملیات

ترم پاییز ۱۳۹۹-۱۴۰۰

بسم الله الرحمن الرحیم

جلسه هفدهم

درس تحقیق در عملیات

کاربرد جبرخطی در کدها

انگیزش

- ذخیره‌سازی داده روی DVD
- ممکن است خطا داشته باشیم
- مخابره ۴ بیت => ۱۶ حالت. چه کنیم؟
  - سه بار تکرار
  - 111001000111 ==> بازیابی
  - 111001000111
- یکی از N حالت را ارسال کنیم،
  - با n بیت
  - اگر حداکثر r خطا باشد ==> بازیابی‌پذیر

فاصله همینگ

تعریف: برای دو کلمه: $\mathbf{w}, \mathbf{w}' \in \{0,1\}^n$

$$d_H(\mathbf{w}, \mathbf{w}') := |\{j \in \{1, \ldots, n\} : w_j \neq w'_j\}|$$

$$|\mathbf{w}| := |\{j \in \{1, \ldots, n\} : w_j = 1\}|$$

$$\mathbf{w} \oplus \mathbf{w}' = ((w_1 + w'_1) \bmod 2, \ldots, (w_n + w'_n) \bmod 2) \in \{0,1\}^n$$

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$$d_H(\mathbf{w}, \mathbf{w}') = |\mathbf{w} \oplus \mathbf{w}'|$$

تعریف کد

8.4.1 Definition. *A code* $\mathcal{C} \subseteq \{0,1\}^n$ *has* **distance** d *if* $d_H(\mathbf{w}, \mathbf{w}') \geq d$ *for any two distinct words* $\mathbf{w}, \mathbf{w}'$ *in* $\mathcal{C}$. *For* $n, d \geq 0$, *let* $A(n,d)$ *denote the maximum cardinality of a code* $\mathcal{C} \subseteq \{0,1\}^n$ *with distance* d.

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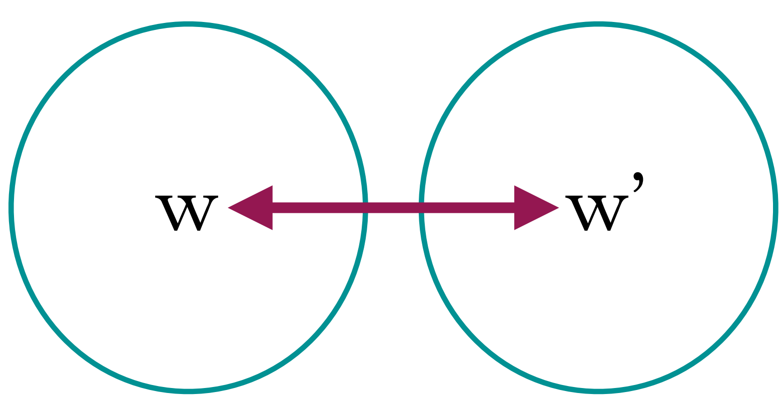
تصحیح خطا:

برای $r < d/2$، می‌توانیم r تا خطا را تصحیح کنیم.



حالت‌های خاص

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آن‌چه می‌دانیم

$$5312 \leq A(17,3) \leq 6552.$$

کران بالا برای A(n, d)

8.4.2 Lemma (Sphere-packing bound). *For all n and r,*

$$A(n, 2r+1) \leq \left\lfloor \frac{2^n}{\sum_{i=0}^{r} \binom{n}{i}} \right\rfloor .$$

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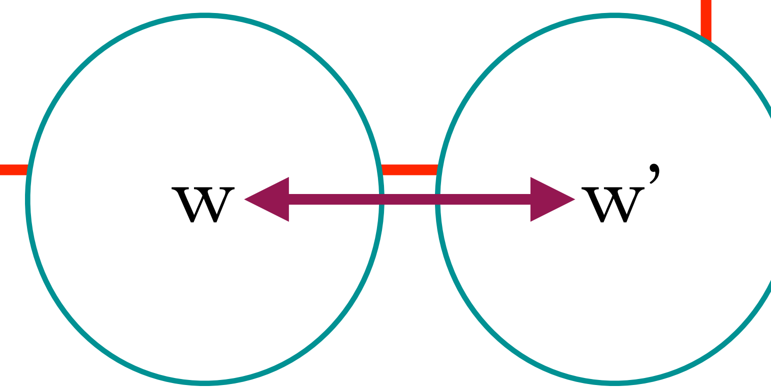

$$A(17,3) \leq \lfloor 131072/18 \rfloor = 7281$$

آنچه می‌دانیم

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کران بالای بهتر برای A(n,d)

8.4.3 Theorem (The Delsarte bound). *For integers n, i, t with $0 \le i$, $t \le n$, let us put*

$$K_t(n,i) = \sum_{j=0}^{\min(i,t)} (-1)^j \binom{i}{j}\binom{n-i}{t-j}$$

$$A(n,d) \le \begin{array}{lll} \textit{Maximize} & x_0 + x_1 + \cdots + x_n & \\ \textit{subject to} & x_0 = 1 & \\ & x_i = 0, & i = 1, 2, \ldots, d-1 \\ & \sum_{i=0}^{n} K_t(n,i) \cdot x_i \ge 0, & t = 1, 2, \ldots, n \\ & x_0, x_1, \ldots, x_n \ge 0. & \end{array}$$

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$$\Rightarrow A(17,3) \le 6553\tfrac{3}{5}$$

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ایده:

x_i: نسبت با فاصله i

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اثبات: ارائه یک جواب شدنی

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به ازای یک کد C:

$$\tilde{x}_i = \frac{1}{|\mathcal{C}|} \cdot \left| \{ (\mathbf{w}, \mathbf{w}') \in \mathcal{C}^2 : d_H(\mathbf{w}, \mathbf{w}') = i \} \right|,$$

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فاصله هیچ زوجی ۱ تا d-1 نیست: $x_i = 0,$

$$|\mathbf{w}|_I = |\{i \in I : w_i = 1\}|$$

$$d_H^I(\mathbf{w}, \mathbf{w}') = |\mathbf{w} \oplus \mathbf{w}'|_I$$

8.4.5 Lemma. *Let $I \subseteq \{1, 2, \ldots, n\}$ be a set of indices, and let $\mathcal{C} \subseteq \{0,1\}^n$. Then the number of pairs $(\mathbf{w}, \mathbf{w}') \in \mathcal{C}^2$ with $d_H^I(\mathbf{w}, \mathbf{w}')$ even is at least as large as the number of pairs $(\mathbf{w}, \mathbf{w}') \in \mathcal{C}^2$ with $d_H^I(\mathbf{w}, \mathbf{w}')$ odd. (In probabilistic*

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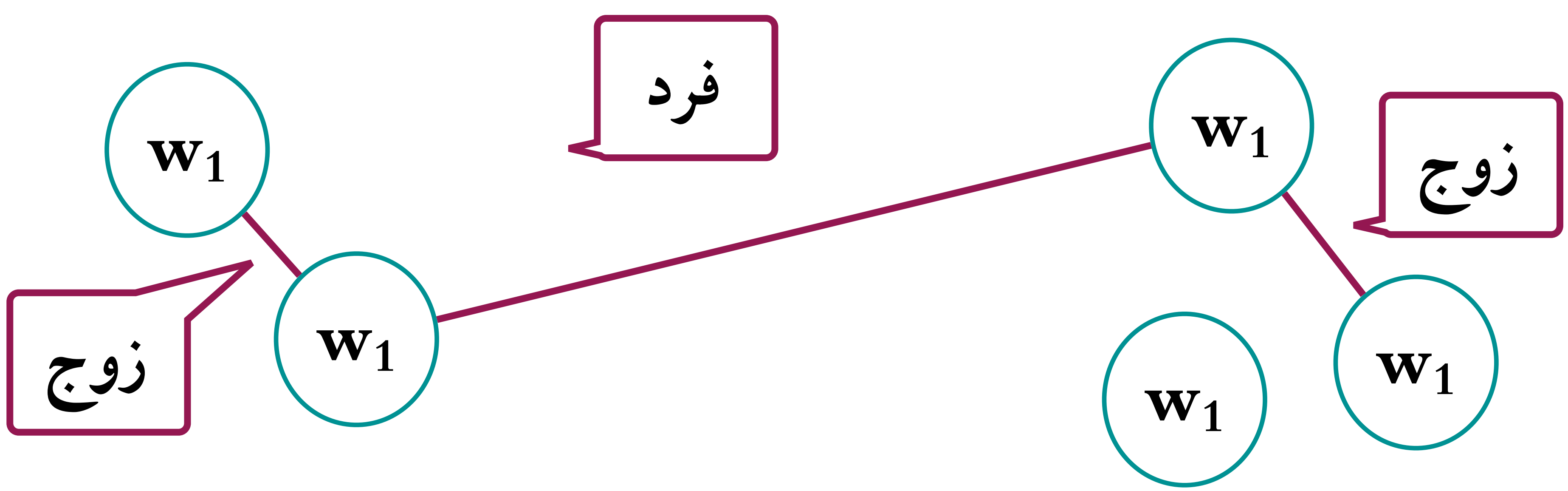

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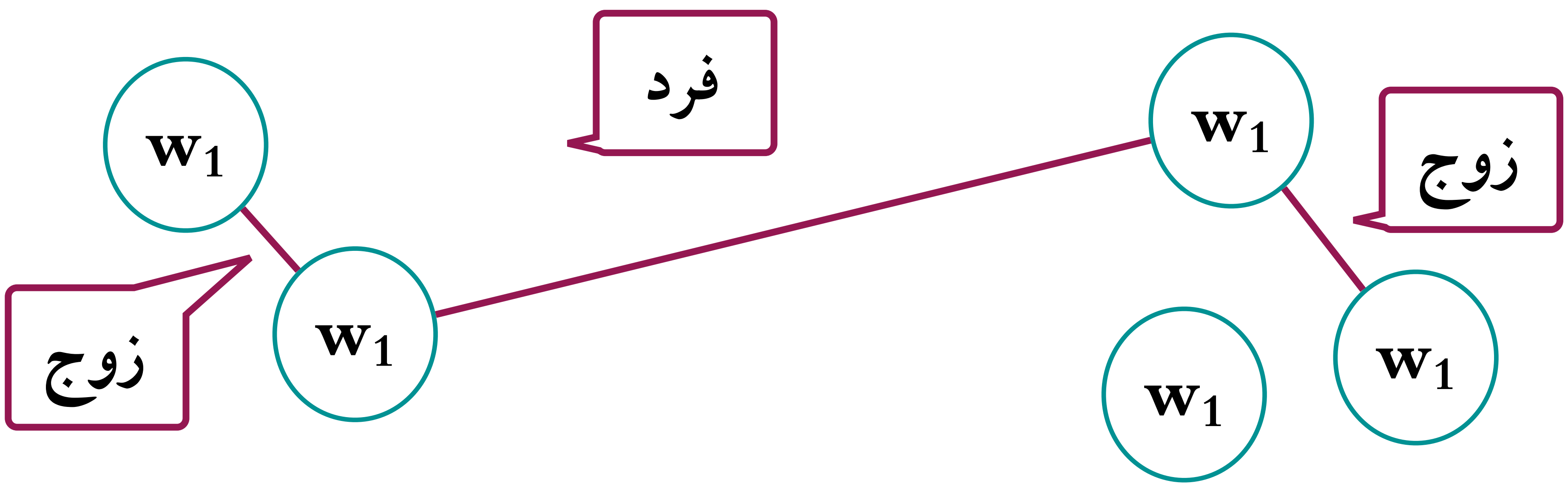

$$|\mathcal{E}|^2 + |\mathcal{O}|^2 \geq 2 \cdot |\mathcal{E}| \cdot |\mathcal{O}|$$

8.4.6 Corollary. *For every $\mathcal{C} \subseteq \{0,1\}^n$ and every $\mathbf{v} \in \{0,1\}^n$ we have*

$$\sum_{(\mathbf{w},\mathbf{w}') \in \mathcal{C}^2} (-1)^{(\mathbf{w} \oplus \mathbf{w}')^T \mathbf{v}} \geq 0.$$

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&= \left(\sum_{\mathbf{w}\in\mathcal{C}} (-1)^{\mathbf{w}^T\mathbf{v}}\right)^2 \geq 0.
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i := $\mathbf{d_H(w,w')}$

اثبات:

$$0 \leq \sum_{(\mathbf{w},\mathbf{w}')\in\mathcal{C}^2} \sum_{\mathbf{v}\in\{0,1\}^n:\,|\mathbf{v}|=t} (-1)^{\overbrace{(\mathbf{w}\oplus\mathbf{w}')^T\mathbf{v}}^{\mathbf{j}}}$$

$$\downarrow$$

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$$0 \leq \sum_i K_t(n,i)\ \tilde{x}_i$$

$$= \frac{1}{|C|}\,|\{(w,w') : \ldots\}|$$

8.4.3 Theorem (The Delsarte bound). *For integers n, i, t with $0 \leq i$, $t \leq n$, let us put*

$$A(n,d) \leq \begin{array}{lll} \textit{Maximize} & x_0 + x_1 + \cdots + x_n & \\ \textit{subject to} & x_0 = 1 & \\ & x_i = 0, & i = 1, 2, \ldots, d-1 \\ & \sum_{i=0}^{n} K_t(n,i) \cdot x_i \geq 0, & t = 1, 2, \ldots, n \\ & x_0, x_1, \ldots, x_n \geq 0. & \end{array}$$

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$$K_t(n,i) = \sum_{j=0}^{\min(i,t)} (-1)^j \binom{i}{j} \binom{n-i}{t-j}$$

$$\Rightarrow A(17,3) \leq 6553\tfrac{3}{5}$$

آنچه می‌دانیم $5312 \leq A(17,3) \leq 6552.$

??

ایده:

۱) فرض خلف: $|C| = 6553$

۲) بررسی LP

$A(17,3) \leq 6553\frac{3}{5}$

??

آنچه می‌دانیم

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۱) فرض خلف: $|C| = 6553$

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قسمتی از اثبات قبلی: ...

$$0 \leq \sum_{(\mathbf{w},\mathbf{w}')\in\mathcal{C}^2} \sum_{\mathbf{v}\in\{0,1\}^n:\, |\mathbf{v}|=t} (-1)^{(\mathbf{w}\oplus\mathbf{w}')^T\mathbf{v}}$$

$$\sum_{v:|v|=t} \sum_{w,w'} (-1)^{(w_1\oplus w_1')^\top v} \geq 0$$

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چون |C| فرد

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MWU

- https://www.cs.cmu.edu/afs/cs.cmu.edu/academic/class/15859-f11/www/
- 15-859(E): Linear and Semidefinite Programming (Advanced Algorithms) Fall 2011
- Lecturers: Anupam Gupta and Ryan O'Donnell
- Time: TR 12:00-1:20, GHC 4303
- Course Blog: http://lpsdp.wordpress.com/
- Office Hours: by appointment
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