بسم الله الرحمن الرحیم

تمرین‌های توابع مولد – درس ریاضیات گسسته نیمسال دوم ۹۲-۹۳ – دانشگاه شریف

8. (a) Prove that $F_{m+n} = F_m F_n + F_{m-1}F_{n-1}$ for $m, n \geq 0$ (with the convention that $F_{-1} = 0$).

(b) Use this to derive an algorithm for calculating F_n using only $c \log n$ arithmetic operations. [HINT: see Russian peasant multiplication (Exercise 12 of Chapter 2).]

(c) Given that multiplication is slower than addition, is this algorithm really better than one involving $n-1$ additions?

12 How many ways are there to put the numbers $\{1, 2, \ldots, 2n\}$ into a $2 \times n$ array so that rows and columns are in increasing order from left to right and from top to bottom? For example, one solution when $n = 5$ is

$$\begin{pmatrix} 1 & 2 & 4 & 5 & 8 \\ 3 & 6 & 7 & 9 & 10 \end{pmatrix}.$$

38 Find a closed form for the double generating function

$$M(w, z) = \sum_{m,n \geqslant 0} \min(m, n)\, w^m z^n .$$

42 A space probe has discovered that organic material on Mars has DNA composed of five symbols, denoted by (a, b, c, d, e), instead of the four components in earthling DNA. The four pairs cd, ce, ed, and ee never occur consecutively in a string of Martian DNA, but any string without forbidden pairs is possible. (Thus $bbcda$ is forbidden but $bbdca$ is OK.) How many Martian DNA strings of length n are possible? (When $n = 2$ the answer is 21, because the left and right ends of a string are distinguishable.)

49 This is a problem about powers and parity.

a Consider the sequence $\langle a_0, a_1, a_2, \ldots \rangle = \langle 2, 2, 6, \ldots \rangle$ defined by the formula

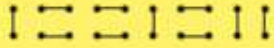
$$a_n = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n .$$

Find a simple recurrence relation that is satisfied by this sequence.

b Prove that $\lceil (1 + \sqrt{2})^n \rceil \equiv n \pmod 2$ for all integers $n > 0$.

c Find a number α of the form $(p + \sqrt{q})/2$, where p and q are positive integers, such that $\lfloor \alpha^n \rfloor \equiv n \pmod 2$ for all integers $n > 0$.

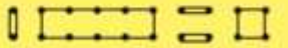
27 A $2 \times n$ domino tiling can also be regarded as a way to draw n disjoint lines in a $2 \times n$ array of points:



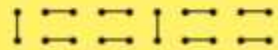
If we superimpose two such patterns, we get a set of cycles, since every point is touched by two lines. For example, if the lines above are combined with the lines

 ,

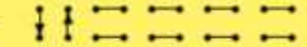
the result is

 .

The same set of cycles is also obtained by combining

 with  .

But we get a unique way to reconstruct the original patterns from the superimposed ones if we assign orientations to the vertical lines by using arrows that go alternately up/down/up/down/$\cdots$ in the first pattern and alternately down/up/down/up/$\cdots$ in the second. For example,

 +  =  .

The number of such oriented cycle patterns must therefore be $T_n^2 = F_{n+1}^2$, and we should be able to prove this via algebra. Let Q_n be the number of oriented $2 \times n$ cycle patterns. Find a recurrence for Q_n, solve it with generating functions, and deduce algebraically that $Q_n = F_{n+1}^2$.