بسم الله الرحمن الرحیم

جلسه هجدهم

خلاصه‌سازی برای مه‌داده

احساس فشردگی تنک

مرور

$C=(c+2)k$

(ϵ, Ck)-RIP

k_تنک

$$x \longrightarrow y = \Pi x$$

؟

الگوریتم بازیابی (ورودی: Π و y)

$$\min ||z||_1$$
$$\text{s.t.} \quad \Pi z = y$$

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$$\|\hat{x}-x\|_1 \leq c(k) \min_{y \text{ is } k-sparse} \|y-x\|_1$$

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مثال:

$$m = \Omega(\epsilon^{-2} k \log^4 n)$$

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فشردگی/زمان:

$n^2 k \log n \,/\, k \log n$

k-تنک

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مرور

فشردگی/زمان:
n²k log n / k log n

تنک‌ها:
فشردگی/زمان:
k² nnz(A) / k²

k_تنک

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Lemma 2 *Assume A satisfies the nullspace property of order $2k$ with constant $C < 2$. Then for x^* that minimizes $\|x^*\|_1$ subject to $Ax^* = Ax$ we have*

$$\|x - x^*\|_1 \leq \frac{2C}{2 - C} Err_1^k(x)$$

η که $A\eta=0$:

$||\eta||_1 \leq C||\eta_{-T}||_1$ برای هر |T|=۲k

Lemma 1 *Suppose that A satisfies RIP of order $(c+2)k$ with constant δ, $c > 1$. Then A satisfies the nullspace property of order $2k$ with constant $C = 1 + \sqrt{2/c}(1+\delta)(1-\delta)$.*

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جایگزین

RIP_1

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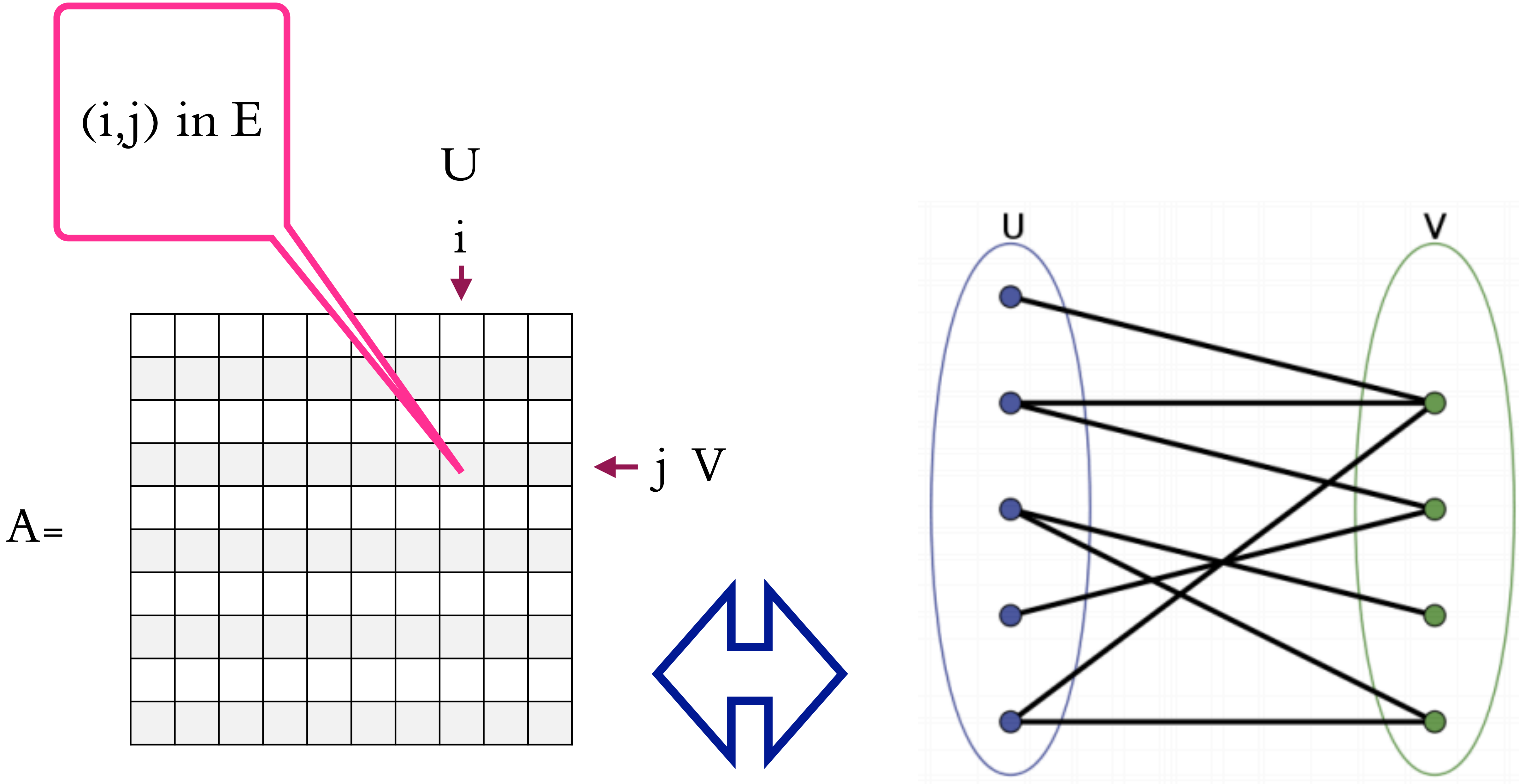
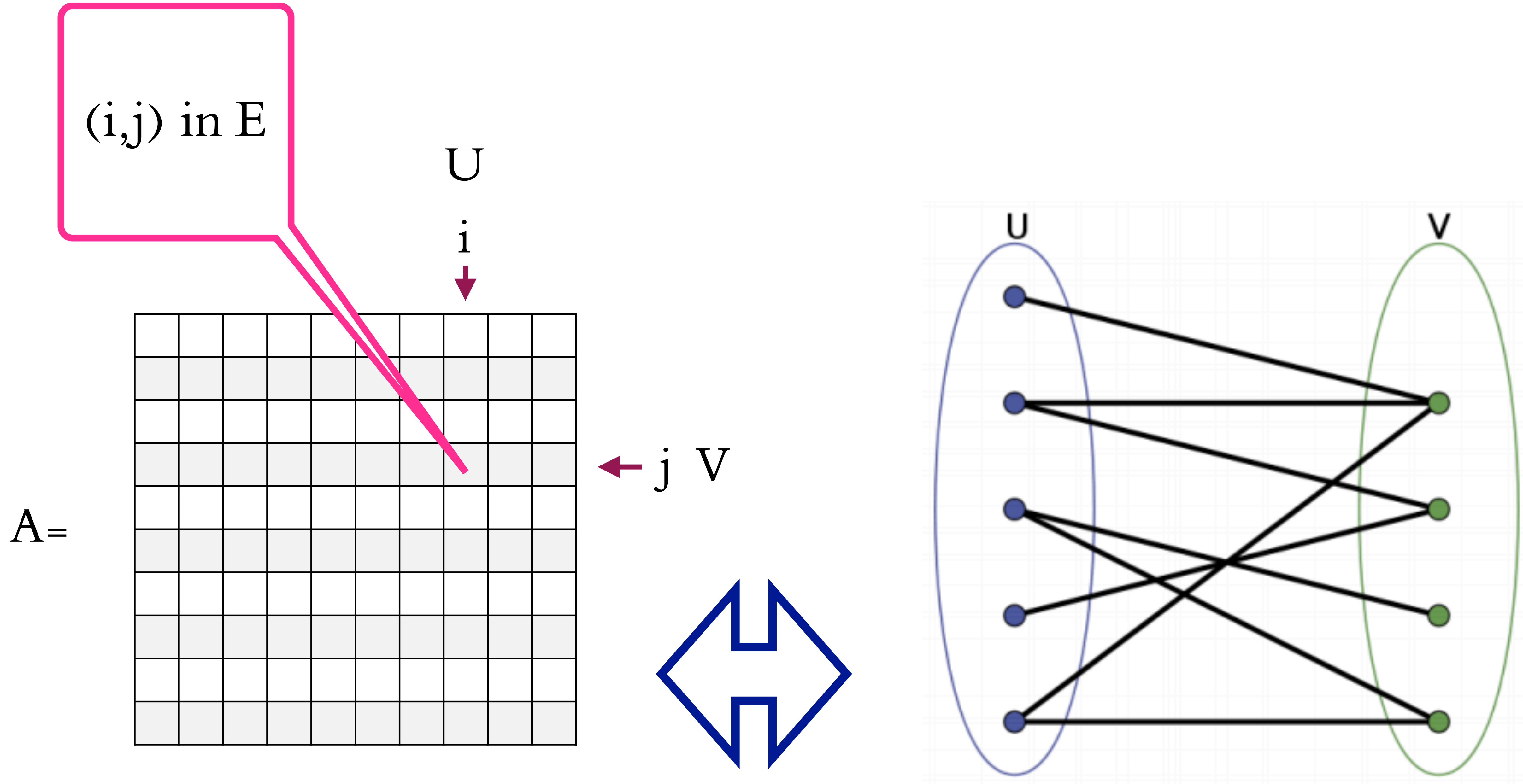
Definition 28. *We say a matrix* $\Pi \in \mathbb{R}^{m \times n}$ *satisfies the* (ε, k)-restricted isometry property *(or* RIP *for short) if for all* k*-sparse vectors* x *of unit Euclidean norm,*

$$1 - \varepsilon \leq \|\Pi x\|_2^2 \leq 1 + \varepsilon.$$

Definition 63 (RIP_1)**.** *A matrix* A *is* (ϵ, k)*-*RIP_1 *if for all* k *sparse vector* v*,*

$$(1 - \epsilon) \|\mathrm{v}\|_1 \leq \|\mathrm{Av}\|_1 \leq (1 + \epsilon) \|\mathrm{v}\|_1$$


(i,j) in E
U
i
j V
A=
U
V



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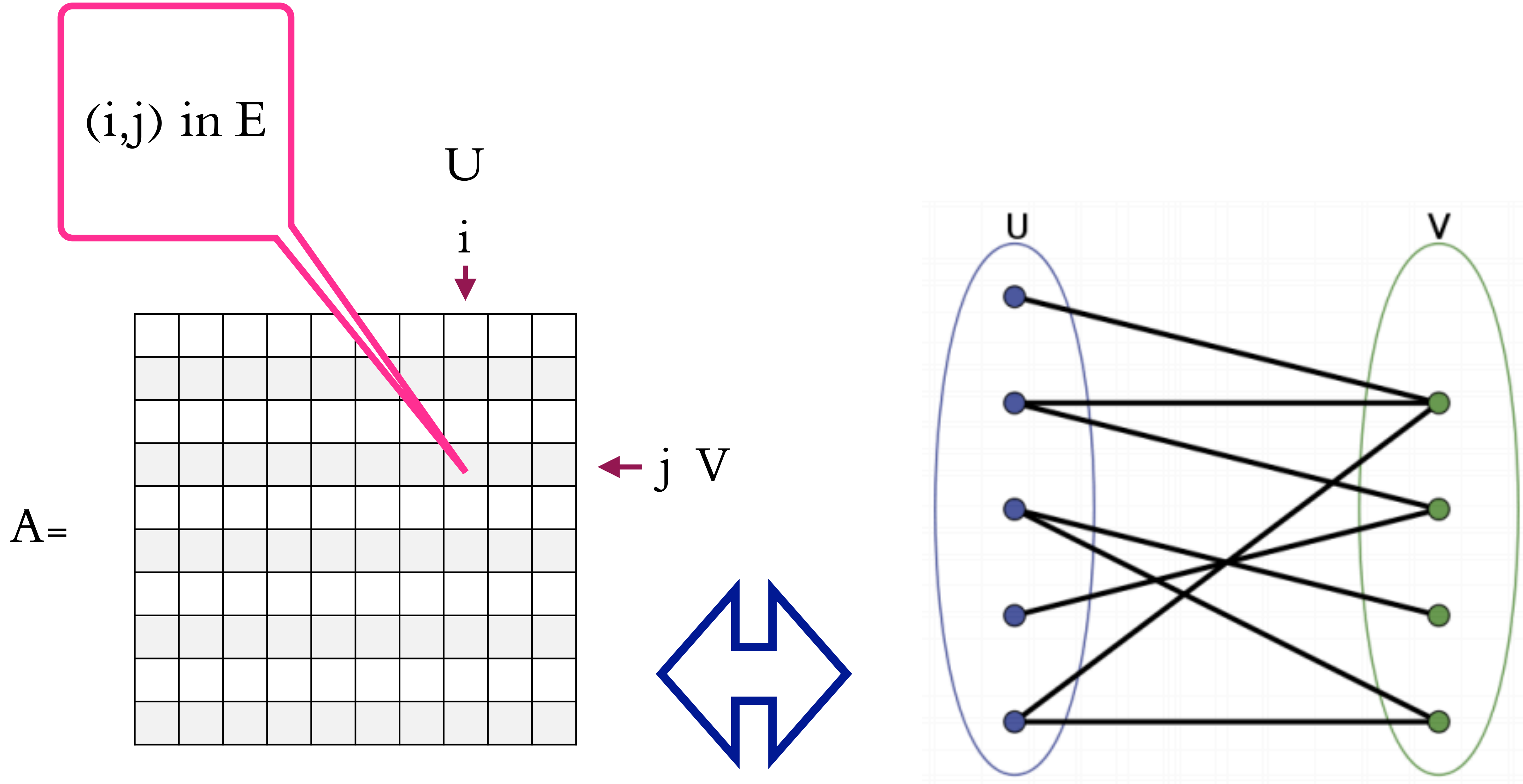

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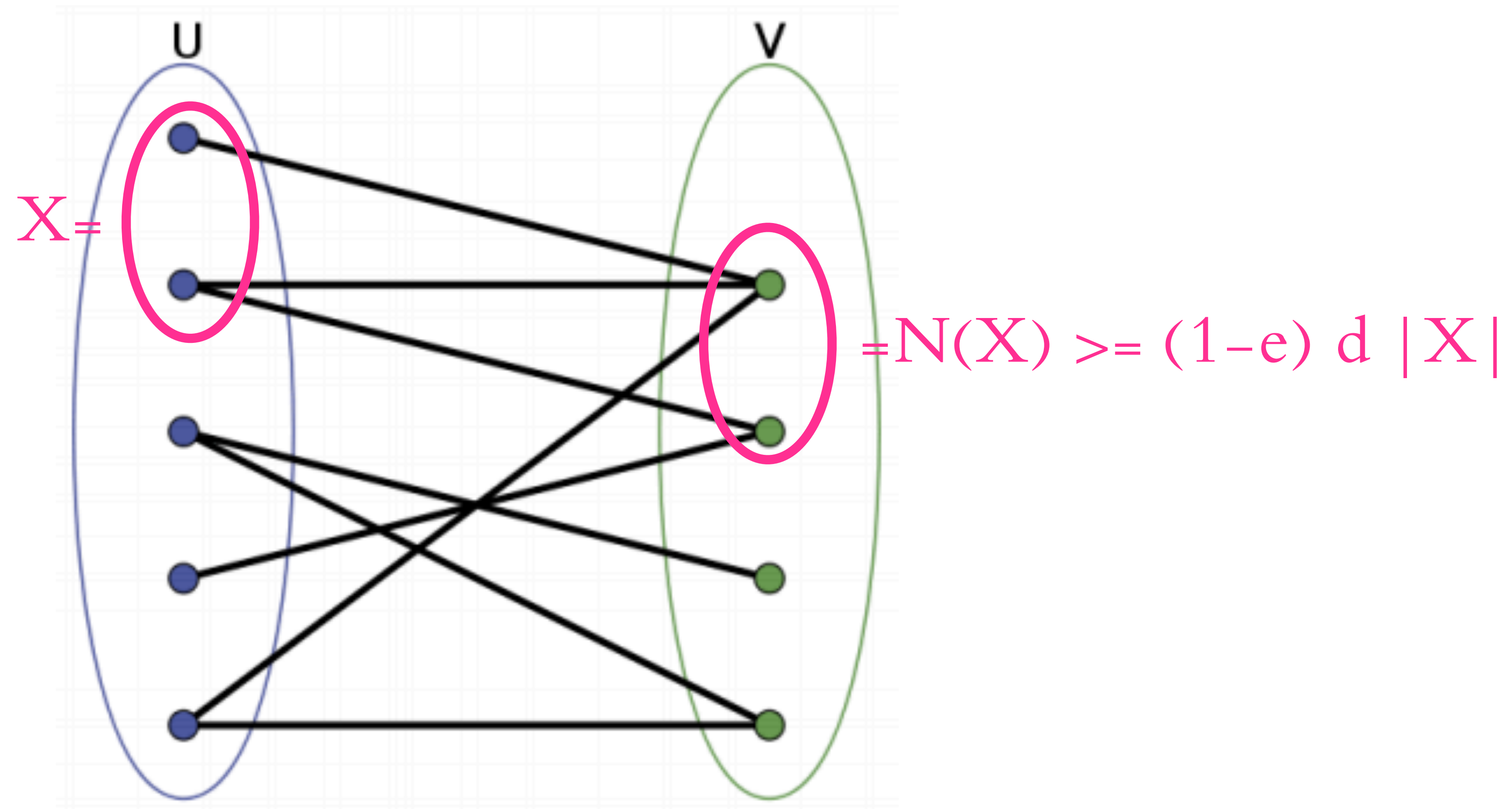
$$d(1-\epsilon)\left\|\mathrm{v}\right\|_1 \leq \left\|\mathrm{Av}\right\|_1 \leq d\left\|\mathrm{v}\right\|_1$$

Definition 64 (Expander)**.** A (l, ϵ)-*unbalanced expander* is a bipartite simple graph $G = (U, V, E)$, $|U| = n, |V| = m$, with left degree d such that for any $X \subset U$ with $|X| \le l$, the set of neighbors $N(X)$ of X has size $|N(X)| \ge (1 - \epsilon)d|X|$.

deg() = d



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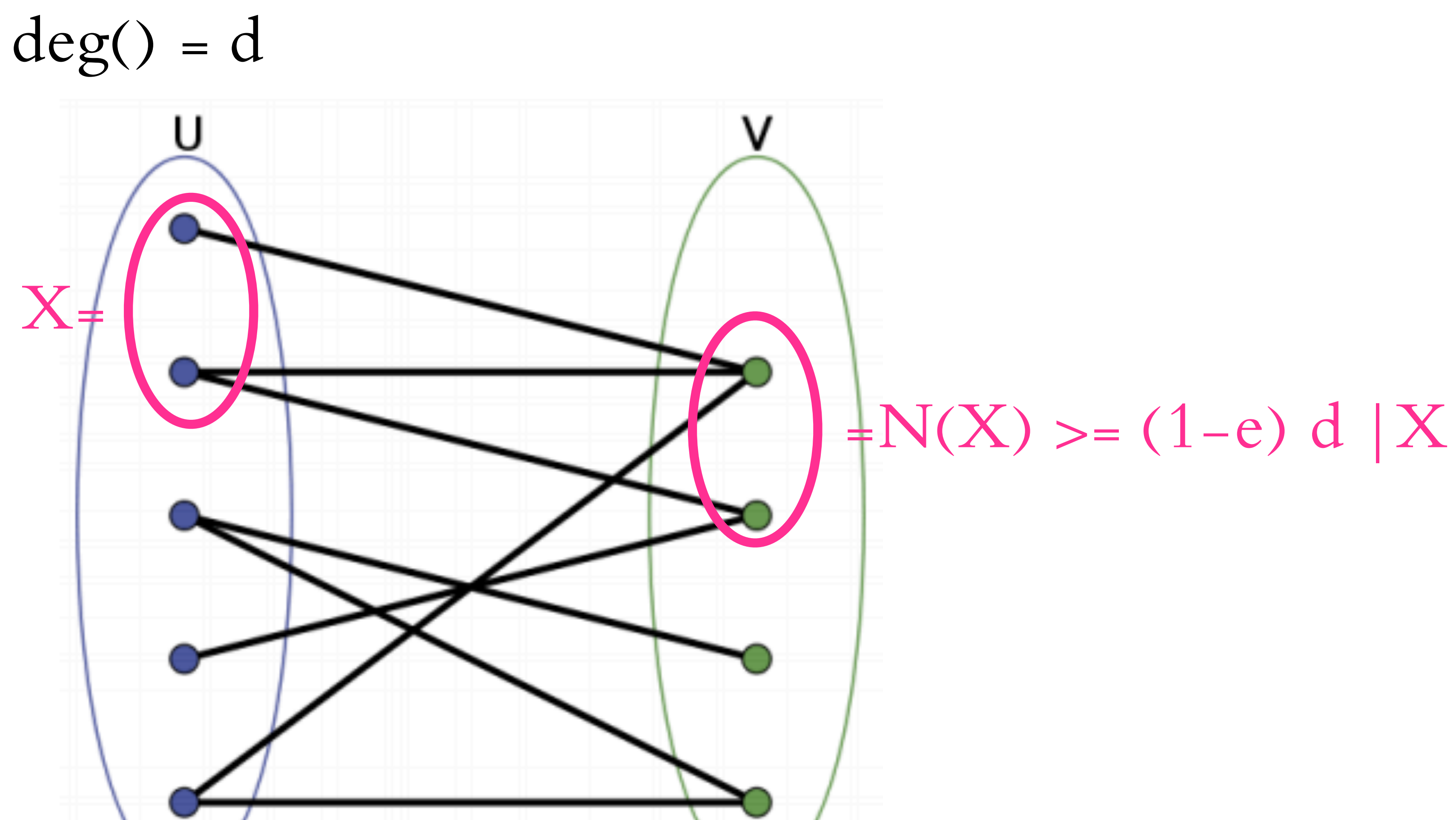

Claim 69. For any $n/2 \geq l \geq 1$, $\epsilon > 0$, there exists a (l, ϵ)-unbalanced expander with left degree $d = O(\log(n/l)/\epsilon)$ and right set size $O(ld/\epsilon) = O(l \log(n/l)/\epsilon^2)$.

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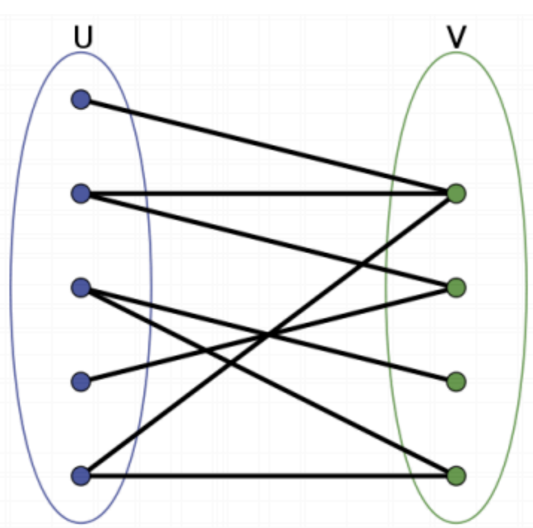
Theorem 65 ([BGIKS08]). *For $A \in \{0,1\}^{m \times n}$, if the underlying bipartite graph is a $(k, d(1-\frac{\epsilon}{2}))$ expander, then for all k sparse vector v*

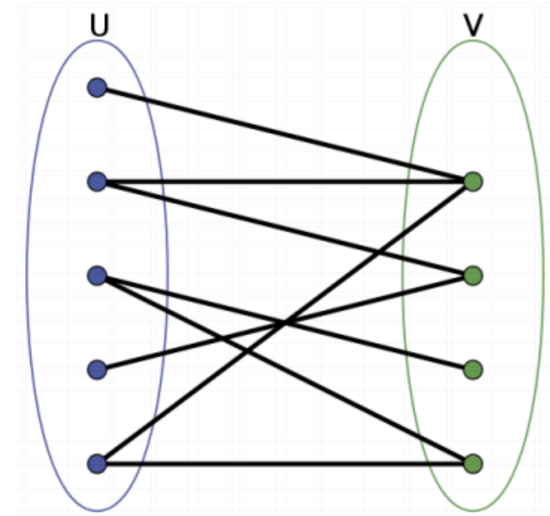
$$d(1-\epsilon)\left\|\mathrm{v}\right\|_1 \leq \left\|\mathrm{Av}\right\|_1 \leq d\left\|\mathrm{v}\right\|_1$$

بردار v



بردار Av

Proof. $\|\mathrm{Av}\|_1 \leq d\,\|\mathrm{v}\|_1$



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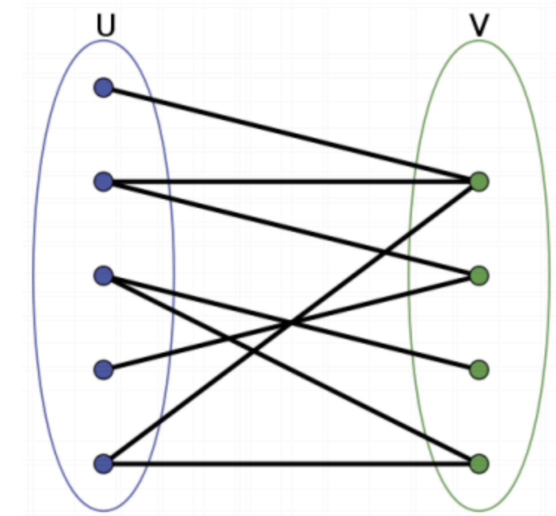

$d(1-\epsilon)\,\|\mathrm{v}\|_1 \leq \|\mathrm{Av}\|_1$:

Claim 66. $\|\mathrm{Av}\|_1 \geq \sum_{(i,j)=e\in E} r(e)|v_i|$

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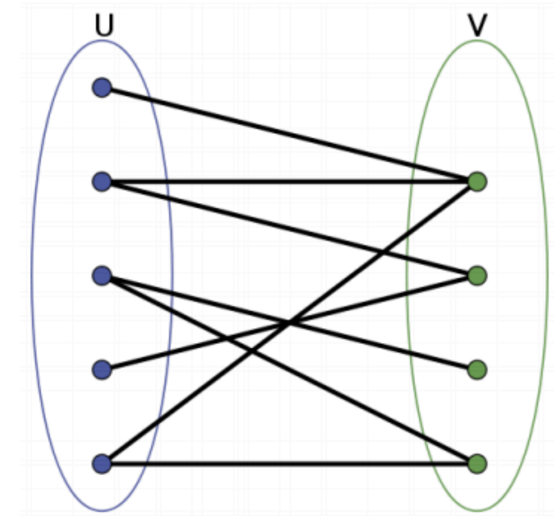

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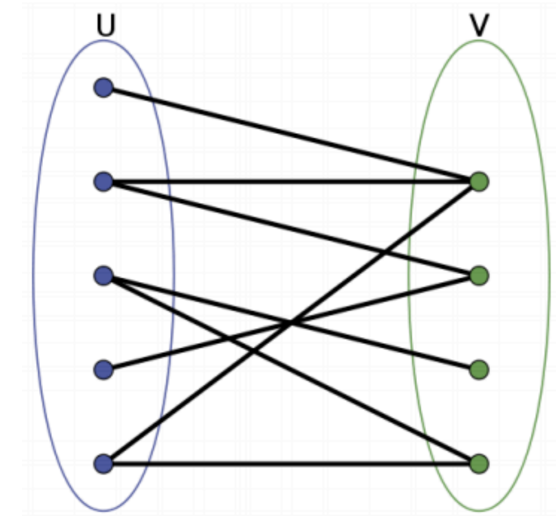

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$$\sum_{(i,j)=e\in E} r(e)|v_i| \geq \sum_{(i,j)=e\in E} r'(e)|v_i| \geq d\,\|\mathrm{v}\|_1 - \epsilon d\,\|\mathrm{v}\|_1$$

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$$\|\eta_S\|_1 \leq \alpha(\epsilon)\|\eta\|_1$$

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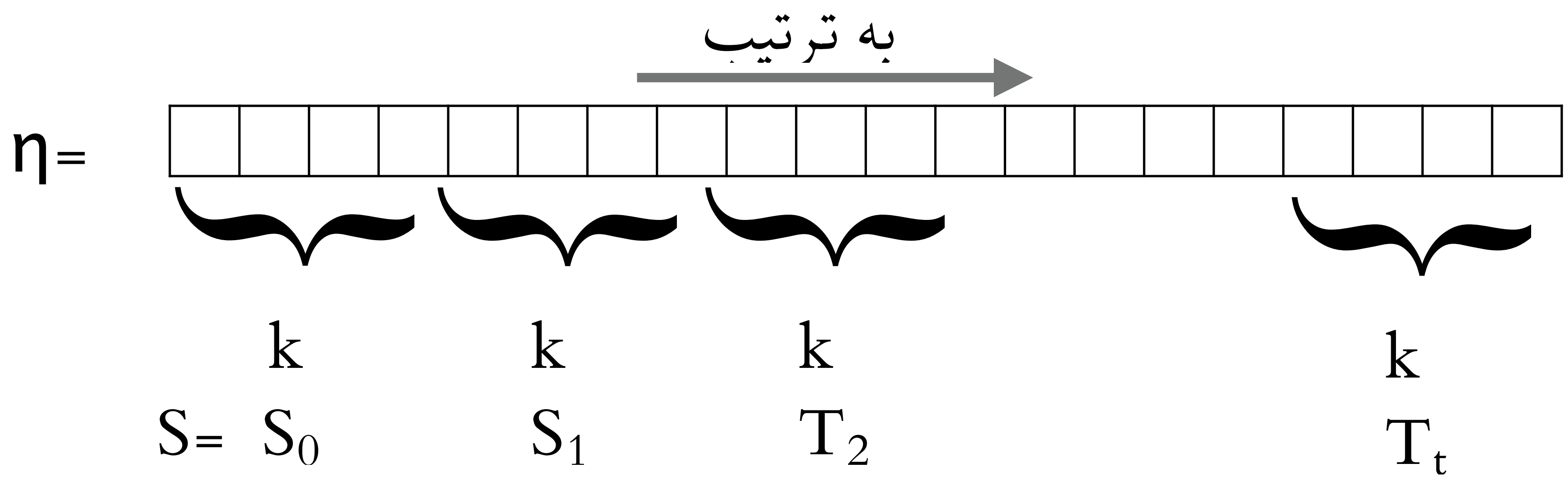
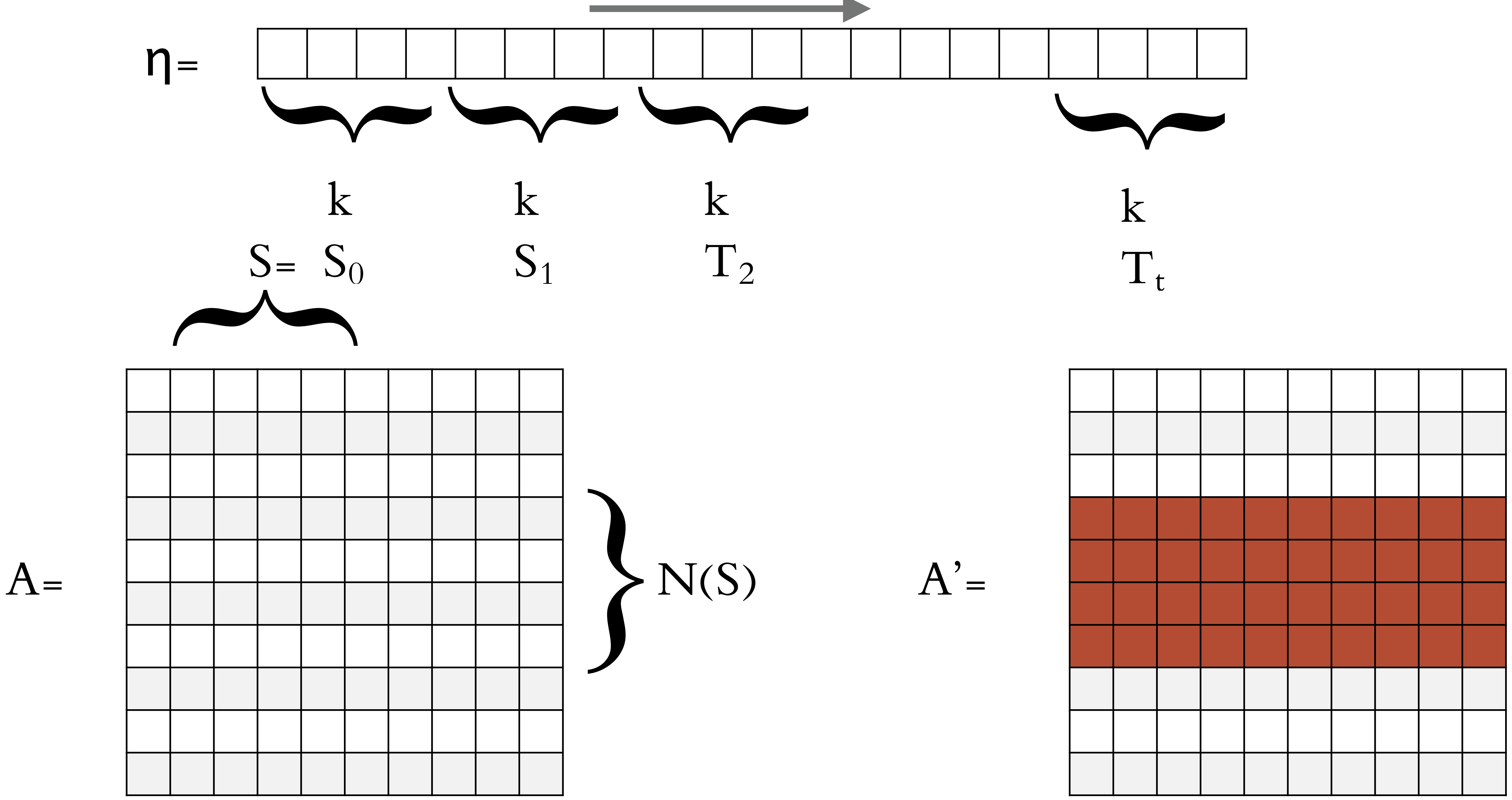
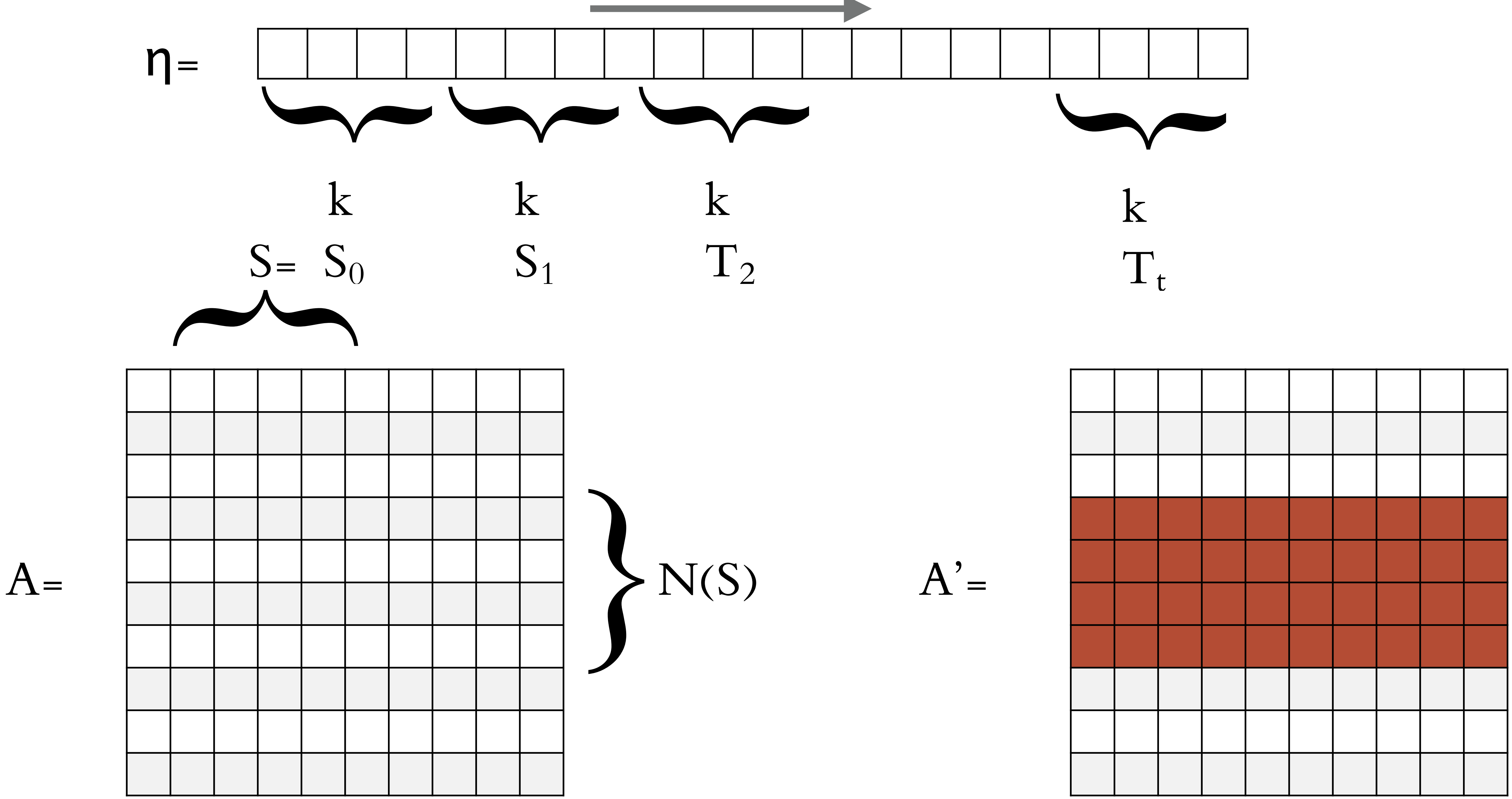
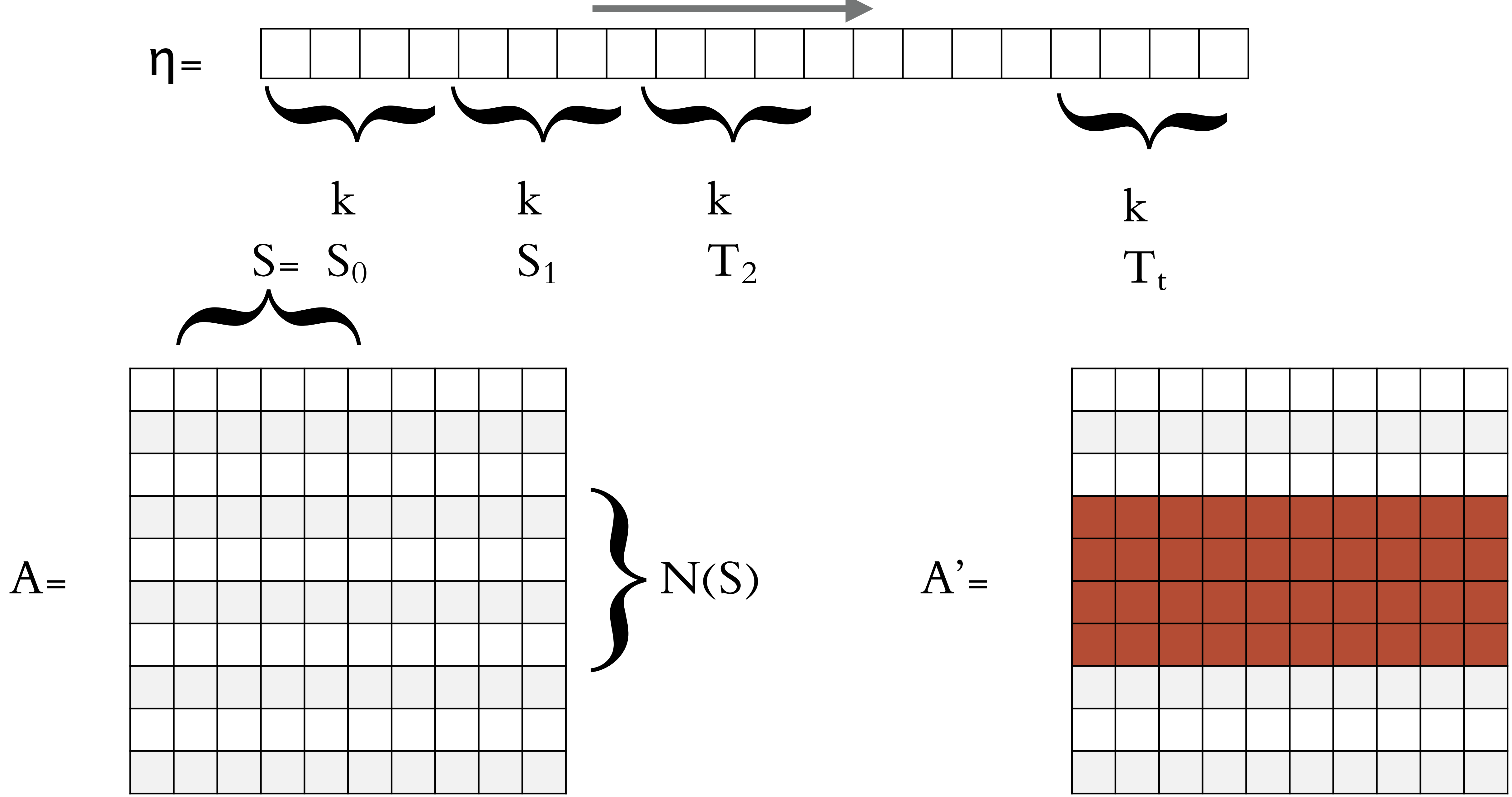




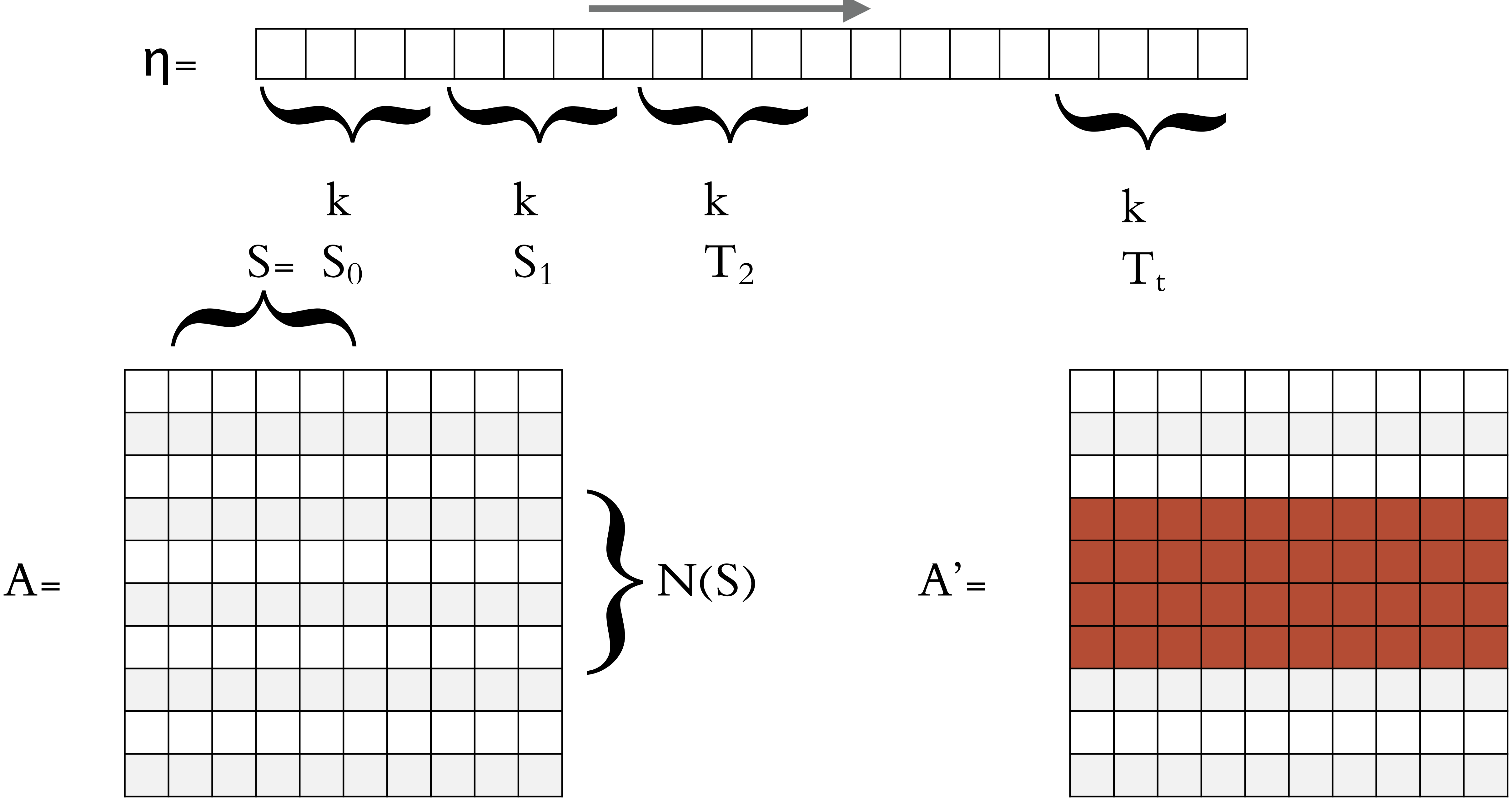
η=
k
k
k
k
S=
S0
S1
T2
Tt
A=
N(S)
A'=



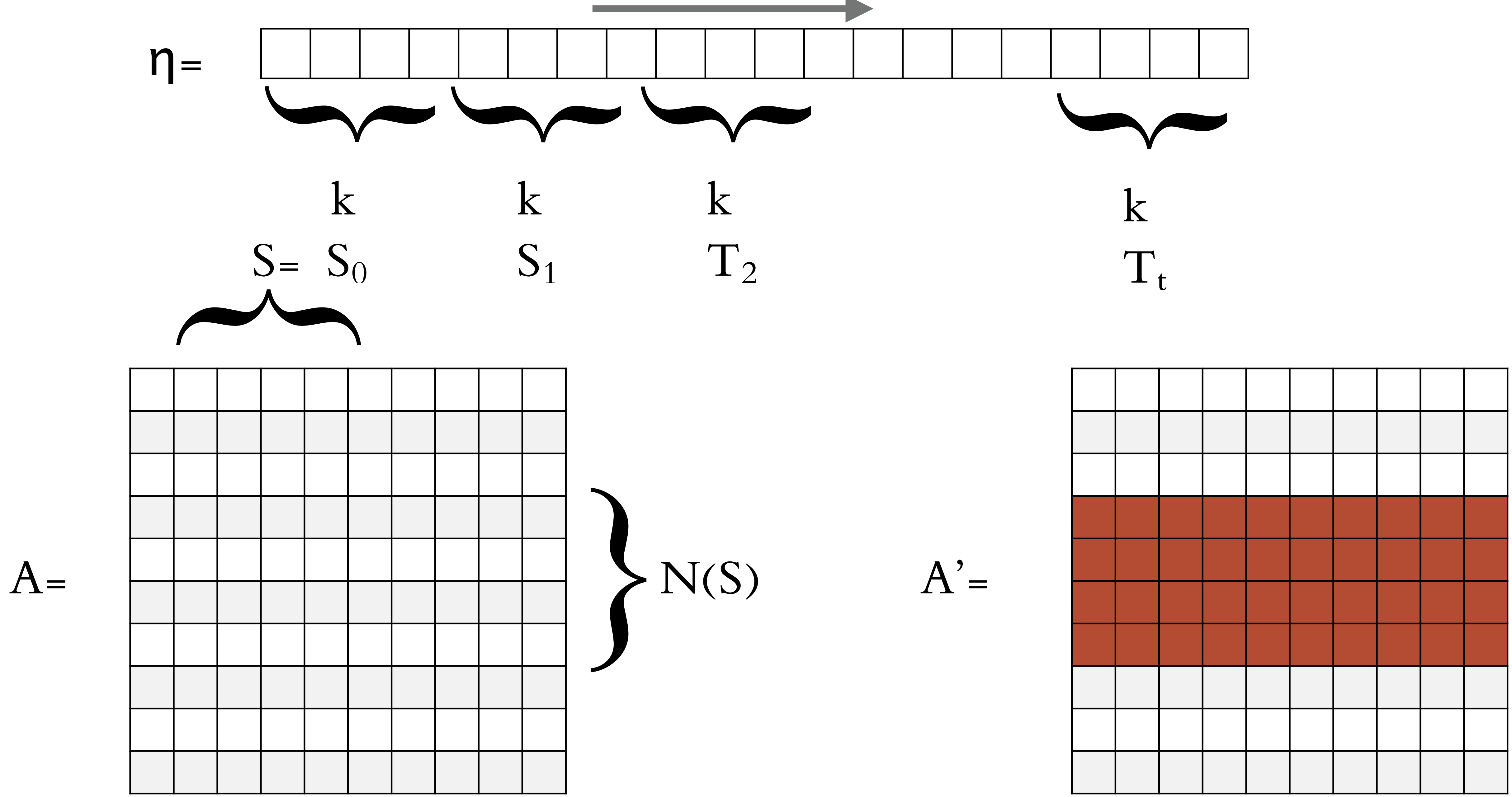
$$\|A'\eta_S\|_1$$



$$\|A'\eta_S\|_1 \;=\; \|A\eta_S\|_1$$

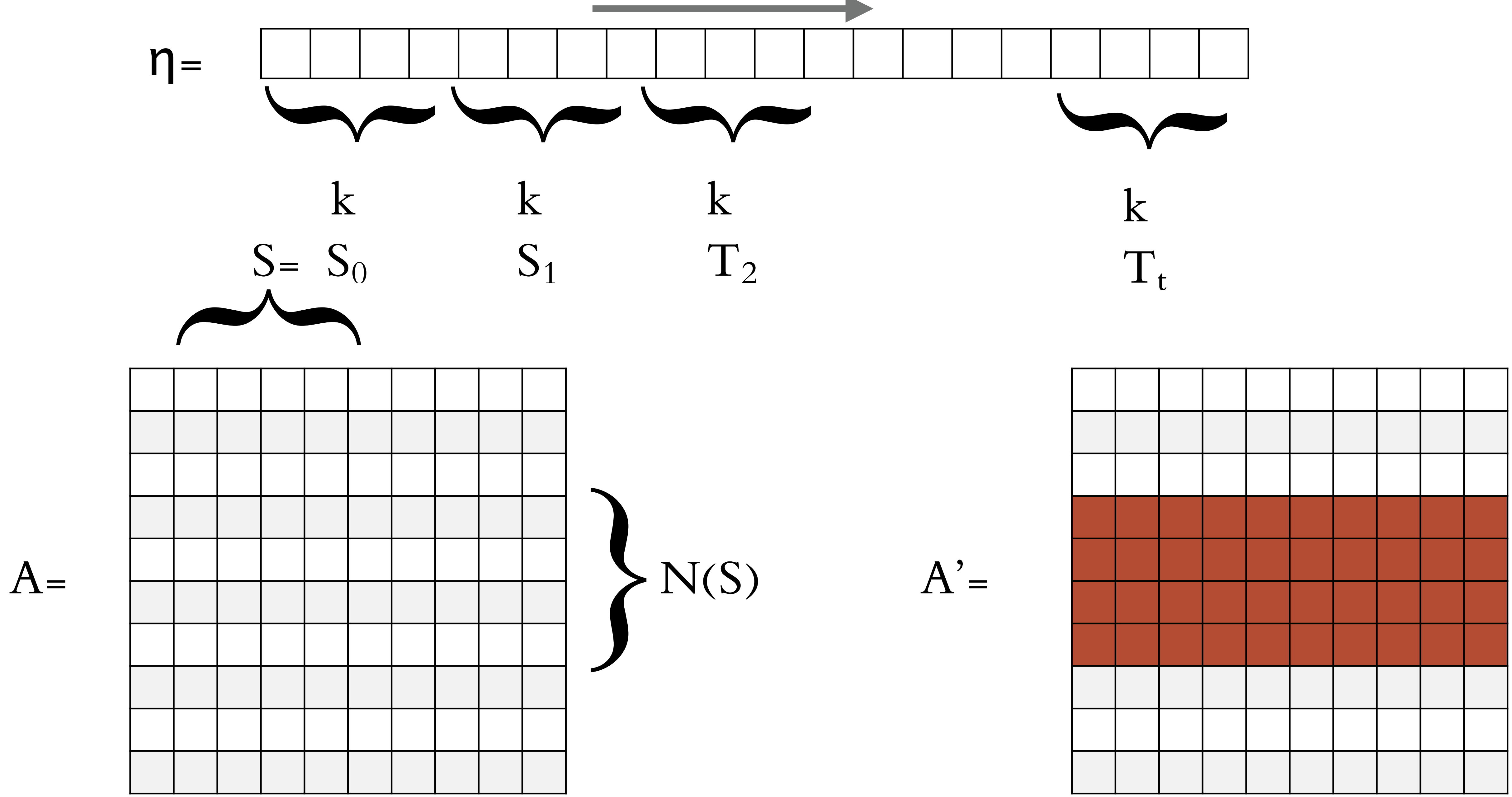


$$\|A'\eta_S\|_1 \;=\; \|A\eta_S\|_1 \geq d(1-2\epsilon)\|\eta_S\|_1$$



$$\|A'\eta_S\|_1 \;=\; \|A\eta_S\|_1 \geq d(1-2\epsilon)\|\eta_S\|_1$$

$$0 = \|A'\eta\|_1 = \|A'\eta_S + A'\eta_{-S}\|_1$$



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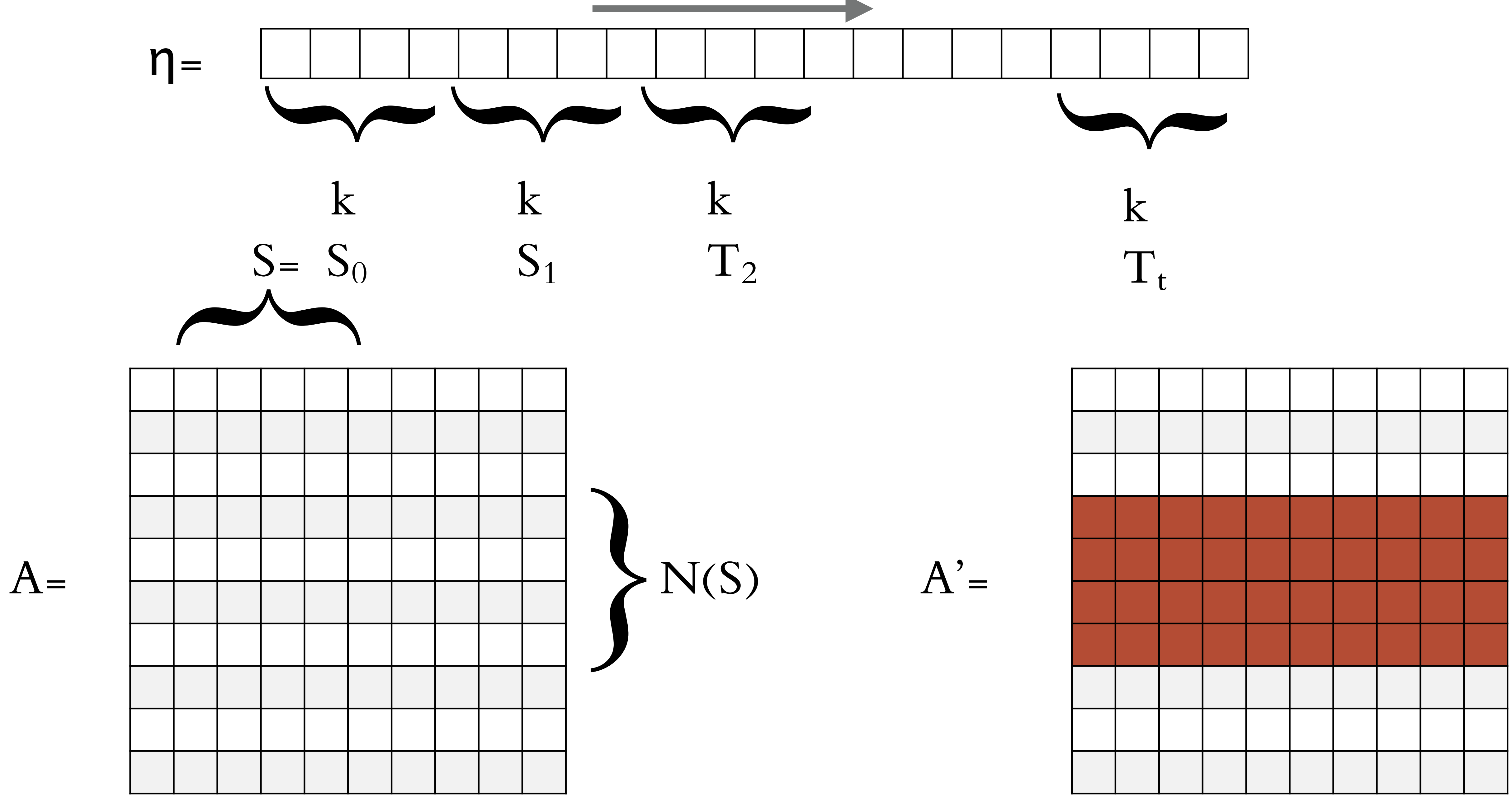

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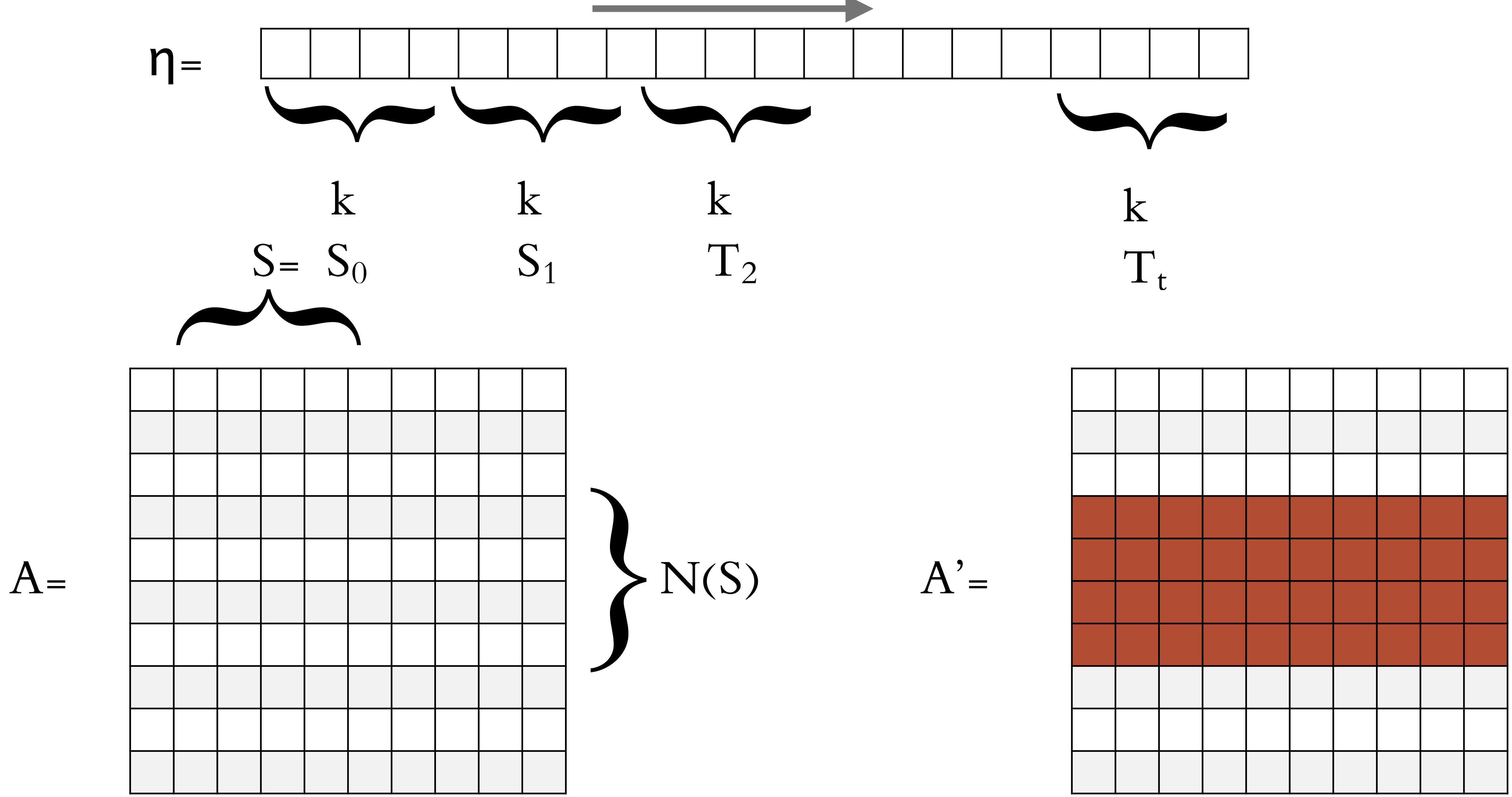

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$$
\begin{aligned}
0 = \|A'\eta\|_1 \quad &\geq \quad \|A'\eta_S\|_1 - \sum_{l\geq 1} \sum_{(i,j)\in E, i\in S_l, j\in N(S)} |\eta_i| \\
&\geq \quad d(1-2\epsilon)\|\eta_S\|_1 - \sum_{l\geq 1} |E(S_l : N(S))| \min_{i\in S_{l-1}} |\eta_i| \\
&\geq \quad d(1-2\epsilon)\|\eta_S\|_1 - \sum_{l\geq 1} |E(S_l : N(S))| \cdot \|\eta_{S_{l-1}}\|_1/k
\end{aligned}
$$

$$
\leq
$$

$$
\boxed{|N(S\cup S_l)| \geq d(1-\epsilon)|S\cup S_l|} \quad d\epsilon 2k
$$

$$
\begin{aligned}
&\geq \quad d(1-2\epsilon)\|\eta_S\|_1 - d\epsilon 2k \sum_{l\geq 1} \|\eta_{S_{l-1}}\|_1/k \\
&\geq \quad d(1-2\epsilon)\|\eta_S\|_1 - 2d\epsilon\|\eta\|_1
\end{aligned}
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$$

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