بسم الله الرحمن الرحیم

تمرین‌های دوم گراف – درس ریاضیات گسسته نیمسال دوم ۹۲-۹۳ – دانشگاه شریف

تکالیف:

۱. ثابت کنید در یک گراف ساده با حداقل دو راس، دو راس هستند که درجه‌شان برابر است.

۲. ثابت کنید اگر یک گراف دوبخشی k-منتظم باشد(k>0) در این صورت تعداد رئوس این گراف زوج است.

1.2.30. Let G be a simple graph with vertices $v_1, \ldots, v_n$. Let A^k denote the kth power of the adjacency matrix of G under matrix multiplication. Prove that entry i, j of A^k is the number of v_i, v_j-walks of length k in G. Prove that G is bipartite if and only if, for the odd integer r nearest to n, the diagonal entries of A^r are all 0. (Reminder: A walk is an **ordered** list of vertices and edges.)

r یا n است یا n-1. علاوه بر این دقت کنید که در یک walk یک یال می تواند چندبار پیموده شود.

1.3.12. (!) Prove that an even graph has no cut-edge. For each $k \geq 1$, construct a $2k+1$-regular simple graph having a cut-edge.

گراف زوج گرافی است که درجه همه راس های آن زوج باشد.

سوال های امتیازی:

۱. یک صفحه شطرنج ۸ در ۸ داریم که گوشه بالا سمت چپ و پایین سمت راست آن را حذف کرده‌ایم. با کمک گراف های دو بخشی ثابت کنید نمی توان این صفحه را با دومینوهای ۱ در ۲ پوشاند.

1.3.13. (+) A **mountain range** is a polygonal curve from $(a, 0)$ to $(b, 0)$ in the upper half-plane. Hikers A and B begin at $(a, 0)$ and $(b, 0)$, respectively. Prove that A and B can meet by traveling on the mountain range in such a way that at all times their heights above the horizontal axis are the same. (Hint: Define a graph to model the movements, and use Corollary 1.3.5.) (Communicated by D.G. Hoffman.)

سوال برای تمرین بیشتر:

1.1.35. (!) Prove that K_n decomposes into three pairwise-isomorphic subgraphs if and only if $n+1$ is not divisible by 3. (Hint: For the case where n is divisible by 3, split the vertices into three sets of equal size.)

1.2.42. Let G be a connected simple graph that does not have P_4 or C_4 as an induced subgraph. Prove that G has a vertex adjacent to all other vertices. (Hint: Consider a vertex of maximum degree.) (Wolk [1965])

1.2.43. (+) Use induction on k to prove that every connected simple graph with an even number of edges decomposes into paths of length 2. Does the conclusion remain true if the hypothesis of connectedness is omitted?

1.3.16. (+) For $k \geq 2$ and $g \geq 2$, prove that there exists an k-regular graph with girth g. (Hint: To construct such a graph inductively, make use of an $k-1$-regular graph H with girth g and a graph with girth $\lceil g/2 \rceil$ that is $n(H)$-regular. Comment: Such a graph with minimum order is a (k, g)-**cage**.) (Erdős–Sachs [1963])

1.3.31. (!) Use complete graphs and counting arguments (not algebra!) to prove that

a) $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$ for $0 \leq k \leq n$.

b) If $\sum n_i = n$, then $\sum \binom{n_i}{2} \leq \binom{n}{2}$.

1.3.35. (+) Let n and k be integers such that $1 < k < n-1$. Let G be a simple n-vertex graph such that every k-vertex induced subgraph of G has m edges.

a) Let G' be an induced subgraph of G with l vertices, where $l > k$. Prove that $e(G') = m\binom{l}{k}/\binom{l-2}{k-2}$.

b) Use part (a) to prove that $G \in \{K_n, \overline{K}_n\}$. (Hint: Use part (a) to prove that the number of edges with endpoints u, v is independent of the choice of u and v.)

1.3.60. (+) Let d be a list of integers consisting of k copies of a and $n-k$ copies of b, with $a \geq b \geq 0$. Determine necessary and sufficient conditions for d to be graphic.