Real Analysis 1

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March 3, 2025

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Lecture 26: 03-03-25 Lecture

Monday 03 March 2025

Theorem 1 (Absolute converges test)**.** If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} |a_n|$ also converges.

Proof. It is enough to show that $\sum a_n$ satisfies the Cauchy criterion. Let $\varepsilon > 0$ be given. Hence $\exists N \in \mathbb{N}$ such that whenever $n > m \geq N$ we have

$$||a_{m+1}| + |a_{m+2}| + \ldots + |a_n|| < \varepsilon \tag{1}$$

$$\Rightarrow |a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon \tag{2}$$

Hence proved. □

Example.

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2} \tag{3}$$

Observe that

$$\frac{\sin(n)}{n^2} \leq \frac{1}{n^2} \tag{4}$$

As $\sum \frac{1}{n^2}$ converges, by the comparison test $\sum \left|\frac{\sin(n)}{n^2}\right|$ converges. Therefore by the absolute convergence test $\sum \frac{\sin(n)}{n^2}$ also converges.

Definition 1. Consider the series $\sum a_n$

1) If $\sum |a_n|$ converges, then we say that the series $\sum a_n$ converges absolutely.

2) If the series $\sum a_n$ converges, but $\sum |a_n|$ diverges, then we say that $\sum a_n$ converges conditionally.

Example.

$$\sum \frac{\sin(n)}{n^2} \tag{5}$$

By previous logic, the series converges absolutely.

Example.

$$\sum \frac{(-1)^n}{n} \tag{6}$$

converges by the alternating series test but $\sum \frac{1}{n}$ diverges. Therefore $\sum \frac{(-1)^n}{n}$

Note. The conditional convergence is the most painful.

Theorem 2 (Alternating series test)**.** Let (a_n) be a sequence satisfying

1) $a_1 \geq a_2 \geq a_3 \geq \ldots$ and $a_n \geq 0 \; \forall n \in \mathbb{N}$
2) $\lim a_n = 0$

Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof. Consider the partial sum s_m

$$s_m = a_1 + a_2 + \ldots + (-1)^{m+1} a_m \tag{7}$$

Consider the subsequence $(s_{2n})_{n=1}^{\infty} = (s_2, s_4, s_6, \ldots)$

$$s_2 = a_1 - a_2 \tag{8}$$

$$s_4 = a_1 - a_2 + a_3 - a_4 = a_1 - (a_2 - a_3) - a_4 \leq a_1 \tag{9}$$

Therefore

$$s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} \geq 0 \tag{10}$$

Therefore $(s_{2n})_{n=1}^{\infty}$ is an increasing sequence

$$s_{2n} = a_1 - a_2 + a_3 \ldots - a_{2n-2} + a_{2n-1} - a_{2n} \tag{11}$$

$$= a_1 - (a_2 - a_3) - \ldots - (a_{2n-2} - a_{2n-1}) - a_{2n} \tag{12}$$

$$\leq a_1 \tag{13}$$

(s_{2n}) is an increasing sequence which is bounded above and hence by MCT it converges. Let $L \in \mathbb{R}$ such that

$$\lim_{n \to \infty} s_{2n} = L \tag{14}$$

Now I claim that $\lim s_n = L$. To show this, let $\varepsilon > 0$ be given. $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$ we have

$$|s_{2n} - L| < \frac{\varepsilon}{2} \tag{15}$$

As $\lim a_n = 0$, $\exists N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$

$$|a_n| < \frac{\varepsilon}{2} \tag{16}$$

Let $N = \max\{N_2, N_1\}$. Hence $\forall n \geq N$ we have

- (Case 1: n is even)

$$|s_n - L| < \frac{\varepsilon}{2} < \varepsilon \tag{17}$$

- (Case 2: n is odd)

$$\begin{aligned} |s_n - L| &\leq |s_n - s_{n+1}| + |s_{n+1} - L| && (18)\\ &\leq |a_{n+1}| + \frac{\varepsilon}{2} && (19)\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && (20)\\ &= \varepsilon && (21) \end{aligned}$$

□

Example.

$$a_n = \frac{1}{n} \tag{22}$$

By the AST $\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots$ converges.

Theorem 3 (Ratio Test)**.** Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n \neq 0$ $\forall n \in \mathbb{N}$. Assume that $\exists r$ such that $0 \leq r < 1$ and

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r \tag{23}$$

Then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Proof. Given in homework. □

Example. For some $x \in \mathbb{R}$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{24}$$

If $x = 0$, then

$$1 + 0 + 0 + 0 + \dots \tag{25}$$

If $x \neq 0$, the $a_n = \frac{x^n}{n!}$ is non-zero.

$$\left| \frac{a_n + 1}{a_n} \right| = \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \left| \frac{x}{n+1} \right| \tag{26}$$

By ALT $\lim_{x\to\infty} \left|\frac{x}{n+1}\right| = 0 = r$. Therefore, by the ration test, this series converges absolutely $\forall x \in \mathbb{R}$.