Real Analysis 1

Forrest Kennedy

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Contents

Lecture 20: 02-19-25 Lecture

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Last time we showed that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{1}$$

converges. Now we will do a slightly different problem.

Example. $\sum_{n=1}^{\infty} \frac{1}{n}$

The partial sums are

$$S_m = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m} \tag{2}$$

Observe that (S_m) is an increasing sequence. To prove the statement, we will show that (S_m) is *not* bounded.

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) \tag{3}$$

$$S_8 \tag{4}$$

$$S_{16} \tag{5}$$

$$S_{32} \tag{6}$$

$$S_{2^k} \tag{7}$$

for $k \in \mathbb{N}$ we have

$$S_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \ldots + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{k-1}+1} + \ldots + \frac{1}{2^k}\right) \tag{8}$$

$$S_{2^k} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \ldots + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^k} + \ldots + \frac{1}{2^k}\right) \tag{9}$$

$$= 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \ldots + 2^{k-1}\frac{1}{2^k} \tag{10}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} \tag{11}$$

$$= 1 + k\left(\frac{1}{2}\right) \tag{12}$$

$$\Rightarrow S_{2^k} > 1 + \frac{k}{2} \tag{13}$$

As the sequence $\left(1 + \frac{k}{2}\right)_{k=1}^{\infty}$ is not bounded. Therefore $(S_m)_{m=1}^{\infty}$ is not bounded. Therefore $(S_m)_{m=1}^{\infty}$ is not convergent. Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Theorem 1. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{14}$$

converges for $p > 1$ and diverges for $p \leq 1$

Proof. See textbook. ∎

Bolzano was a priest who first came up with the definition of the limit that we have been using. We will see some theorems named after him in this section.

Definition 1. Let (a_n) j be a sequence of real numbers and let $n_1 < n_2 < n_3 < \ldots$ be an increasing sequence of natural numbers, then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, \ldots) \tag{15}$$

is called a **subsequence** of (a_n) and is denoted by (a_{n_k}) where $k \in \mathbb{N}$ indexes the subsequence.

Example. Let $a_n = n^2$ i.e.

$$(a_n) = (1, 4, 9, 16, 25, \ldots) \tag{16}$$

Let $a_{n_k} = (2k)^2$

$$(a_{n_k}) = (4, 16, 36, \ldots) \tag{17}$$

(a_{n_k}) is a subsequence of (a_n). Here $n_k = 2k$

$$\left(6^2, 11^2, 16^2, 21^2, \ldots\right) \text{ is also a subsequence} \tag{18}$$

$$\left(2^2, 2^2, 2^2, 3^2, 4^2, 5^2, \ldots\right) \text{ is } \textit{not} \text{ a subsequence.} \tag{19}$$

The original sequence is

$$(1, 2, 2, 2, 3, 4, 5, 6, \ldots) \tag{20}$$

then

$$(2, 2, 2, 3, 4, 5, 6, \ldots) \tag{21}$$

is a subsequence.

Theorem 2. All sub sequences of a convergent sequence converge to the same limit as the original sequence.

Proof. Assume that $\lim a_n = a$ and let (a_{n_k}) be a subsequence. Let $\varepsilon > 0$ be given, then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have

$$|a_n - a| < \varepsilon \tag{22}$$

For $k \geq N$ we observe that $n_k \geq k \geq N$. Therefore

$$|a_{n_k} - a| < \varepsilon \tag{23}$$

$\square$

Note. The crucial thing to realize in the above is that a_{n_k} is indexed by k the n_k is just there for emphasis.

Example. Let $0 < b < 1$, then $\lim b^n = 0$

Proof. Observe that

$$b > b^2 > b^3 > \ldots > 0 \tag{24}$$

Therefore, (b^n) is a decreasing sequence which is bounded. Then, by MCT, this sequence converges. Let $L \in \mathbb{R}$ such that $\lim_{n\to\infty} b_n = L$ Observe that for all $n \in \mathbb{N}\,(b \geq b^n)$. Therefore, by order limit theorem

$$1 > b \geq L \tag{25}$$

Similarly, $b^n \geq 0\ \forall n \in \mathbb{N}$. Therefore $L \geq 0$. Therefore $0 \leq L \leq 1$ Look at the subsequence $\left(b^{2n}\right) = \left(b^2, b^4, b^6, \ldots\right)$. Therefore, $\lim_{n\to\infty} b^{2n} = L$. Notice

$$b^n b^n = b^{2n} \tag{26}$$

$$a_n \cdot b_n \tag{27}$$

Therefore, by the ALT we have

$$L \cdot L = L \tag{28}$$

$$L^2 = L \tag{29}$$

$$L = 0 \text{ or } L = 1 \tag{30}$$

But as $0 \leq L < 1$. Therefore, $L = 0$ and $\lim_{n\to\infty} b^n = 0$ $\square$