Real Analysis 1

Forrest Kennedy

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Contents

Lecture 24: 02-26-25

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The Cauchy Criterion. This is very important.

Definition 1. A sequence (a_n) is called a **Cauchy sequence** if for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \varepsilon$.

Theorem 1. A sequence is Cauchy if and only if it is convergent.

Proof. □

Note. If a sequence converges, but we don't know how to find the value of the limit L, then using the above, we can still show that the sequence converges, and still use the convergent limit theorems. This will take some buildup.

Theorem 2. Every convergent sequence is a Cauchy sequence.

Proof. Let $L \in \mathbb{R}$ and let (a_n) be a sequence with $\lim_{n\to\infty} a_n = L$. Let $\varepsilon > 0$ be given. Therefore $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have

$$|a_n - L| < \frac{\varepsilon}{2} \tag{1}$$

If $m \geq N$ we also have

$$|a_m - L| < \frac{\varepsilon}{2} \tag{2}$$

Therefore if $m, n \geq N$ we have

$$\begin{align} |a_n - a_m| &\leq |a_n - L| + |L - a_m| \tag{3} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \tag{4} \\ &= \varepsilon \tag{5} \end{align}$$

□

Theorem 3. Cauchy sequences are bounded.

Theorem 4. Let $\varepsilon = 1$ and let $N \in \mathbb{N}$ such that $\forall m, n \geq N$ we have

$$|a_m - a_n| < 1 \tag{6}$$

Let $m = N$. Therefore for all $n \geq N$ we have

$$|a_N - a_n| < 1 \tag{7}$$

$$\Rightarrow |a_n| \leq |a_n - a_N| + |a_N| < |a_N| + 1 \tag{8}$$

Let $M = \max\{|a_1|, |a_2|, |a_3|, \ldots, |a_{N-1}|, |a_N| + 1\}$. Then we see that

$$|a_n| \leq M \tag{9}$$

for all $n \in \mathbb{N}$. Hence (a_n) is bounded.

Theorem 5. A sequence converges if and only if it is Cauchy.

Proof. Let (x_n) be a Cauchy sequence. Therefore, (x_n) is a bounded sequence. By Bolzano-Weirstrauss theorem, there exists a convergent subsequence (x_{n_k}). Let

$$x = \lim_{k\to\infty} x_{n_k} \tag{10}$$

Now notice that what we want to claim is that $\lim\limits_{n\to\infty} x_n = x$. So let $\varepsilon > 0$ be given. $\exists N \in \mathbb{N}$ such that $\forall m, n \geq N$ we have

$$|x_n - x_n| < \frac{\varepsilon}{2} \tag{11}$$

There also exist $k \in \mathbb{N}$ such that $\forall k \geq K$ with $K \geq N$ such that $\forall n, m \geq N$ we have

$$|x_{n_k} - x| < \frac{\varepsilon}{2} \tag{12}$$

Observe that $n_k \geq K \geq N$. Hence

$$|x_{n_k - x}| < \frac{\varepsilon}{2} \tag{13}$$

and hence $\forall n \geq K$

$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| \tag{14}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \tag{15}$$

$$= \varepsilon \tag{16}$$

Therefore

$$|x_n - x| < \varepsilon \tag{17}$$

For all $n \geq K$. Hence $\lim x_n = x$ □

Now starting back with series. . .

Properties of Infinite Series

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \dots \tag{18}$$

Note. Remember the distinction between sequence and series..

How do we tell if a series if convergent? Construct a sequence of partial sums (s_m)

$$s_m - b_1 + b_2 + \dots + b_m \tag{19}$$

If (s_m) converges, the we say that the series $\sum_{n=1}^{\infty} b_n$ converges.

If $\lim s_m = L$, the $\sum_{n=1}^{\infty} b_n = L$

Theorem 6 (Algebraic Limit Theorem for Series)**.** If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then

1) $\sum_{n=1}^{\infty} ca_n = cA$ for any $c \in \mathbb{R}$
2) $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

Note. The other two don't work!!!

Proof. Let (s_m) and (t_m) be the sequence of partial sums i.e.

$$s_m = a_1 + a_2 + \dots + a_m \tag{20}$$
$$t_m = b_1 + b_2 + \dots + b_m \tag{21}$$

Therefore $\lim\limits_{m\to\infty} s_m = A$ and $\lim\limits_{m\to\infty} t_m = B$. By the Algebraic Limit Theorem

$$\lim_{m\to\infty} cs_m = cA \tag{22}$$
$$\Rightarrow \sum_{n=1}^{\infty} ca_n = cA \tag{23}$$

Note. $ca_1 + ca_2 + \dots + ca_m = cs_n$

Again by ALT

$$\sum_{n=1}^{\infty} (a_n + b_n) = A + B \tag{24}$$

Note. $(a_1 + b_1) + (a_2 + b_2) + \dots + (a_m + b_m) = s_m + t_m$.

$\square$