Real Analysis 1

Forrest Kennedy

March 5, 2025

Contents

Lecture 1: Complete Analysis Theorems List

Tuesday 28 January 2025

1 The Real Numbers

Definition 1 (Definition of a "function"). Given sets A, B a function of $A \to B$ is a mapping that takes ea ch element of A to a single element of B.

Definition 2 (Definition of the "absolute value function"). The **absolute value function"** is defined as $|\cdot| : \mathbb{R} \to \mathbb{R}$ such that:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}.$$

Theorem 1 (The Triangle Inequality). With respect to multiplication and division, the absolute value function satisfies:

1) $|ab| = |a|\,|b|$
2) $|a + b| \leq |a| + |b|$

Proof. We will show the theorem by cases WLOG:

1) $(a = 0)$ $|a + b| = |0 + b| = |b| = |0| + |b| = |a| + |b|$
2) $(a > 0, b > 0)$ By the definition of the absolute value function we have $|a + b| = a + b = |a| + |b|$

3) $(a < 0, b < 0)$ By the definition of the absolute value function we have $|a+b| = -(a+b) = -a + (-b) = |a| + |b|$

4) $(a > 0, b < 0)$ By the definition of the absolute value, we have $|a| = a$ and $|b| = -b$, so $|a|+|b| = a+(-b)$. We want to show that $|a|+|b| = a+(-b) \geq |a+b|$, so again we consider all the possible cases:

   (a) $(a+b=0)$ We have $a + (-b) \overset{?}{\geq} |0| = 0$. Indeed, since $a > 0$ and $b < 0$ we have $a > b$, and our equality holds.

   (b) $(a+b>0)$ We have $a + (-b) \overset{?}{\geq} a + b$. Since $b < 0$, we have $-b > 0$. Comparing the LHS and RHS the equality holds.

   (c) $(a+b<0)$ We have $a+(-b) \overset{?}{\geq} -a+(-b)$. Comparing the LHS and the RHS, the equality holds.

The above considerations exhaust all possible choices for a and b. In all cases. we see that $|a+b| \leq |a| + |b|$ □

Theorem 2 (The ε criteria for equality)**.** Two real numbers a and b are equal if and only if for every real number $\varepsilon > 0$ it follows that $|a-b| < \varepsilon$.

Proof. We will show the theorem in both directions:

- $(\Rightarrow)$ Given $a = b$, we have $a - b = 0 < \varepsilon$ for all $\varepsilon > 0$.
- $(\Leftarrow)$ Assume that for every $\varepsilon > 0$, $|a-b| < \varepsilon$ and, FSOC, that $a \neq b$. Then, let $\varepsilon_0 = a - b$ which we know is nonzero because $a \neq b$. Now, $|a-b| = \varepsilon_0$ and $|a-b| < \varepsilon_0$ by our first assumption. We have reached a contradiction, therefore the reverse implication must hold.

□

Definition 3 (Bounded Above Property of Subsets of $\mathbb{R}$)**.** A set $A \subset \mathbb{R}$ is **bounded above** if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ $\forall a \in A$. The number b is called an **upper bound** for A.

Definition 4 (Bounded Below Property of Subsets of $\mathbb{R}$)**.** A set $A \subset \mathbb{R}$ is **bounded below** if there exists a number $b \in \mathbb{R}$ such that $b \leq a$ $\forall a \in A$. The number b is called a **lower bound** for A.

Definition 5 (The Least Upper Bound)**.** An element $s \in \mathbb{R}$ is called the **least upper bound** for $A \subset \mathbb{R}$ if s meets two conditions:

1) s is an upper bound for A
2) $\forall b$ where b is an upper bound, $s \leq b$.

Definition 6 (The Greatest Lower Bound)**.** An element $l \in \mathbb{R}$ is called the **greatest lower bound** for $A \subset \mathbb{R}$ if l meets two conditions:

1) l is a lower bound for A
2) $\forall b$ where b is an upper bound, $l \geq b$.

Definition 7 (The Maximum is a Set)**.** A real number a_0 is a **maximum** of the set A if a_0 is an elemnt of A and $a_0 \geq a$ for all $a \in A$. Similarly, a number a_1 is a **minimum** of A if $a_1 \in A$ and $a_1 \leq a$ for all $a \in A$.

Theorem 3 (The ε Characterization of the Supremum)**.** Assume $s \in R$ is an upper bound for a set $A \subset \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\varepsilon > 0$, there exists an element $a \in A$ satisfying $s - \varepsilon < a$.

Proof. We will show that both the implication and the inverse implication are true:

- $(\Rightarrow)$If s is the *least* upper bound of A, then $s - \varepsilon$ is not an upper bound for A, thus there exists an $a \in A$ such that $s - \varepsilon < a$.
- $(\Leftarrow)$ Assume s is an upper bound of A and that for every $\varepsilon > 0$, $s - \varepsilon < a$. That is, no number smaller than s is an upper bound of A. Thus for all b where b is an upper bound of A, $s \leq b$. Since we assumed that s is an upper bound, s meets both conditions to be the supremum.

□

Axiom 1 (Axoim of Completeness)**.** Every nonempty set of real numbers that is bounded above has a least upper bound.

Theorem 4 (Nested Interval Property of Subsets of $\mathbb{R}$)**.** For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in R : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supset I_2 \supset I_3 \supset \dots .$$

has a nonempty intersection; that is, $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$, then let $\alpha = \sup A$. From the definition of the supremum, we have $\alpha \geq a_n$ for all $n \in \mathbb{N}$. Because of how we defined our sets, every b_n is an upper bound of A, so we have $\alpha \leq b_n$ for all $n \in \mathbb{N}$. Thus $a_n \leq \alpha \leq b_n$ and $\alpha \in I_n$. Therefore, I_n is nonempty. □

Theorem 5 (Archimedean Property)**.** The theorem has two parts:

1) Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.
2) Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $\frac{1}{n} < y$.

Proof. Statement 1 in the above theorem is equivalent to the statement: $\mathbb{N}$ is not bounded above. FSOC, assume that $\mathbb{N}$ is bounded above, then let $\alpha = \sup \mathbb{N}$. By the definition of the supremum, $\alpha - 1$ is not an upper bound. Thus, $\alpha - 1 < n$ for some $n \in \mathbb{N}$ implies $\alpha < n + 1$, but $n + 1 \in \mathbb{N}$ by definition so α is less than some natural number and cannot be the supremum, a contradiction! Thus $\mathbb{N}$ is not bounded above, and we have proven statement 1. To prove statement 2, let $x = \frac{1}{y}$ and substitute into the expression in statement 1. □

Definition 8 (Sequence)**.** A **sequence** is a function whose domain is $\mathbb{N}$.

Definition 9 (Convergent Property of a Sequence / Limit of a Sequence)**.** A sequence a_n **converges** to a real number a, if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $n \geq N$, we have $|a_n - a| < \varepsilon$. In this case we write

$$\lim_{n \to \infty} a_n = \lim a_n = a \tag{1}$$

Definition 10 (ε-Neighborhood)**.** Given a real number $a \in \mathbb{R}$ and a positive number $\varepsilon > 0$, the set

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\} \tag{2}$$

is called the **ε-neighborhood** of a.

Definition 11 (Topological Definition of the Convergent Property/Limit of a Sequence)**.** A sequence (a_n) converges to a if every ε-neighborhood of a contains all but a finite number of the terms of (a_n).

Theorem 6 (Limit Uniqueness Theorem)**.** The limit of a sequence, when it exists, is unique.

Proof. For the sake of contradiction, let (a_n) be a sequences which converges to both s and t. Then we know that $\exists N_1, N_2 \in \mathbb{N}$ such that for all $\varepsilon > 0$

$$|a_{N_1} - s| < \frac{\varepsilon}{2} \text{ and } |a_{N_2} - t| < \frac{\varepsilon}{2} \tag{3}$$

Now let

$$N = \max\{N_1, N_2\}. \tag{4}$$

And consider $|s-t|$ We can now use the "adding zero" algebraic trick and the triangle inequality:

$$
\begin{align}
|s-t| &= |(s-a_N)+(a_N-t)| \tag{5}\\
&\leq |a_N-s|+|a_N-t| \tag{6}\\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \tag{7}\\
&= \varepsilon \tag{8}
\end{align}
$$

Thus $|s-t|<\varepsilon$ for all $\varepsilon>0$, by the ε-Criteria for Equality, $s=t$. □

Definition 12 (Divergent Property of a Sequence)**.** A sequence that does not converge is said to **diverge**.

Definition 13 (Bounded Property of a Sequence)**.** A sequence (x_n) is **bounded** if there exists a number $M>0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem 7 (Convergence-Boundedness Theorem)**.** Every convergent sequence is bounded.

Proof. Assume that the sequence (x_n) converges to l. Then we can say that for some particular ε, say $\varepsilon=1$, that $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $x_n \in (l-1, l+1)$. We don't know for sure if l is positive or negative, but we can say for sure that

$$|x_n| < |l|+1 \tag{9}$$

For all $n \geq N$. Therefore if we let

$$M = \max\{|x_1|, |x_2|, |x_3|, \ldots, |x_{N-1}|, |l|+1\} \tag{10}$$

it follows that $|x_n| \leq M$ for all $n \in \mathbb{N}$ as desired. □

Theorem 8 (Partial Sequence Gap Theorem)**.** Suppose (a_n) is a convergent sequence with $\lim a_n = L$. If $L \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $\exists \delta > 0$ such that $|a_n| \geq \delta > 0$ for all $n \in \mathbb{N}$.

Proof. As $L \neq 0$, choose $\varepsilon = \frac{|L|}{2} > 0$ $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have

$$
\begin{align}
|a_n - L| < \frac{|L|}{2} \tag{11}\\
\tag{12}
\end{align}
$$

for $n \geq N$ we have

$$|L| = |L - a_n + a_n| \leq |L - a_n| + |a_n| \leq \frac{|L|}{2} + |a_n| \tag{13}$$
$$\tag{14}$$

Therefore, for all $n \geq N$ we have

$$\frac{|L|}{2} \leq |a_n| \tag{15}$$
$$\tag{16}$$

Define $\delta = \min\{|a_1|, |a_2|, \ldots, |a_{N-1}|, \left|\frac{L}{2}\right|\} > 0$. We see that $|a_n| \geq \delta > 0$ $\forall n \in \mathbb{N}$. □

Theorem 9 (Algebraic Limit Theorem)**.** Let $\lim a_n = a$, and $\lim b_n = b$. Then,

a) $\lim ca_n = ca$ for all $c \in \mathbb{R}$;

b) $\lim a_n + b_n = a + b$;

c) $\lim a_n b_n = ab$;

d) $\lim (a_n/b_n) = a/b$, provided $b \neq 0$.

Proof. We will prove each in turn:

a) Consider

$$|ca_n - ca| \tag{17}$$
$$= |c|\,|a_n - a| \tag{18}$$

We know from the given that $\exists N \in \mathbb{N}$ such that if $n \geq N$ we have $|a_n - a| < \varepsilon$ therefore

$$|ca_n - ca| = |c|\,|a_n - a| < |c| \frac{\varepsilon}{|c|} = \varepsilon \tag{19}$$

b) Consider

$$|(a_n + b_n) - (a - b)| \tag{20}$$
$$= |(a_n - a) + (b_n - b)| \tag{21}$$

Apply the triangle inequality

$$\leq |a_n - a| + |b_n - b| \tag{22}$$

Finally apply the fact that a_n and b_n converge to get

$$|(a_n + b_n) - (a - b)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{23}$$

c) Consider

$$|a_n b_n - ab| \tag{24}$$

Use the add-subtract trick and the triangle inequality:

$$\begin{align} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \tag{25} \\ &\leq |b_n| \, |a_n - a| + |a| \, |b_n - b| \tag{26} \end{align}$$

Then use the convergence of a_n and the boundedness of b_n to get

$$\begin{align} &\leq M \, |a_n - a| + |a| \, |b_n - b| \tag{27} \\ &\leq M \frac{\varepsilon}{2M} + |a| \frac{\varepsilon}{2\,|a|} = \varepsilon \tag{28} \end{align}$$

d) The final statement will follow from (c) if we can prove that

$$(b_n) \to b \Rightarrow \left(\frac{1}{b_n}\right) \to \frac{1}{b} \tag{29}$$

Consider

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b - b_n|}{|b| \, |b_n|} \tag{30}$$

Notice that $\exists N_1 \in \mathbb{N}$ such that if $n \geq N_1$ we have $|b_n - b| < \varepsilon$. Before we can continue with our usual strategy, notice that we still have a sequence, b_n, in the denominator. We need to find a number that is *smaller* than every element of the sequence so that we can we a fraction that is always *bigger* than $\frac{|b_n - b|}{|b||b_n|}$. To do this, we will use the fact that $\forall n \in \mathbb{N}$ $|b_n| > \frac{|b|}{2}$ which we used in our proof of the Partial Sum Gap Theorem. Choose N_2 so that $n \geq N_2$ implies

$$|b_n - b| < \frac{\varepsilon \, |b|^2}{2} \tag{31}$$

Now let $N = \max\{N_1, N_2\}$, the $n \geq N$ implies

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = |b - b_n| \frac{1}{|b| \, |b_n|} < \frac{\varepsilon \, |b|^2}{2} \frac{1}{|b| \frac{|b|}{2}} = \varepsilon \tag{32}$$

□

‘

Theorem 10 (Order Limit Theorem)**.** Let $a, b \in \mathbb{R}$ and $\lim a_n = a$ and $\lim b_n = b$.

1) If $a_n \geq 0$ $\forall n \in \mathbb{N}$, then $a \geq 0$
2) If $a_n \leq b_n$ $\forall n \in \mathbb{N}$, then $a \leq b$
3) If $\exists c \in \mathbb{R}$ such that $c \leq b_n$ $\forall n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ $\forall n \in \mathbb{N}$, then $a \leq c$.

Proof. We will show each part in turn. Notice that parts (b) and (c) can be bootstrapped from part (a).

a) By contradiction, assume that $a < 0$, therefore $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have

$$|a_n - a| < \frac{|a|}{2} \Rightarrow a_n - a < \frac{|a|}{2} \tag{33}$$

$$\Rightarrow a_n < a + \frac{|a|}{2} < 0 \tag{34}$$

$$\Rightarrow a_n < 0 \; . \tag{35}$$

A contradiction!

b) The Algebraic Limit THeorem ensures that the sequence $(b_n - a_n)$ converges to $b - a$. Because $b_n - a_n \geq 0$, we can apply part (a) to get that $b - a \geq 0$.

c) Take $a_n = c$ (or $b_n = c$) for all $n \in \mathbb{N}$, and apply (b).

□

Definition 14 (Increasing/Decreasing/Monotone Properties of Sequences)**.** A sequence (a_n) is **increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and **decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is **monotone** if it is either increasing or decreasing.

Theorem 11 (Monotone Convergence Theorem)**.** If a sequence is monotone and bounded, then it converges.

Proof. Assume that (a_n) is a bounded, monotone sequence. Then consider the set $\{a_n | n \in \mathbb{N}\}$. We will proceede by using the definition of convergence, so we will need to guess the exact value so that we can use that definition. Since the sequence is bounded, we can let $s = \sup\{a_n | n \in \mathbb{N}\}$ and be sure that s exists. To see why s is a reasonable choice for a value of the limit, consider $s - \varepsilon < s$. Since s is the least upper bound of the set, $s - \varepsilon$ is not an upper bound and $\exists N \in \mathbb{N}$ such that if $n \geq N$

$$s - \varepsilon < a_N < a_n \leq s < s + \varepsilon \tag{36}$$

$$-\varepsilon < a_n - s < \varepsilon \tag{37}$$

$$|a_n - s| < \varepsilon \tag{38}$$

Note that we needed to know that the sequence is monotonically increasing to write $\forall n > N \; (a_N < a_n)$. Hence proved. □

Definition 15 (Series and the Convergence Property of a Series)**.** Let (b_n) be a sequence. An **infinite series** if a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + b_5 + \dots \tag{39}$$

We define the corresponding **sequence of partial sums** s_m by

$$s_m = b_1 + b_2 + b_3 + \dots + b_m, \tag{40}$$

and say that the series $\sum_{n=1}^{\infty} b_n$ **converges to** B if the sequence (s_m) converges to B. In this case, we write $\sum_{n=1}^{\infty} b_n = B$.

Definition 16 (Subsequence)**.** Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < n_5 < \dots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \dots) \tag{41}$$

is called a **subsequence** of (a_n) and is be denoted by (a_{n_k}), were $k \in \mathbb{N}$ indexes the sequence.

Theorem 12 (Same Limit Theorem of Subsequences)**.** Subsequences of a convergent sequence converge to the same limit as the original sequence.

Proof. Assume that (a_n) converges to a. Then we know that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$ we have

$$|a_n - a| < \varepsilon \tag{42}$$

Now consider this statement which contains a subsequence of (a_n) :

$$|a_{n_k} - a| \tag{43}$$

We already proved that the we can pick an n such that when $n \geq N$ the former statement is true. Notice that $n_k > k$ for all $k \in \mathbb{N}$ since the point of n_k is to skip values in the sequence. Therefore, for the ladder statement, all we have to do is pick $k \geq N$ and we are guaranteed to have $n_k > N$ and

$$|a_{n_k} - a| < \varepsilon \tag{44}$$

Therefore (a_{n_k}) converges to a as well. □

Theorem 13 (Bolzano-Weirstrass Theorem)**.** Every bounded sequence contains a convergent subsequence.

Proof. To prove the theorem, we will use the definition of convergence. To use that definition, we will need to come up with an accurate guess for the limit. We will use the Nester Interval Property to construct such a number.

Assume that (a_n) is a bounded sequence. Then we know that for all $n \in \mathbb{N}$ we have $|a_n| \leq M$. Now consider the intervals $[-M, 0]$ and $[0, M]$. Because of boundedness, either one or both of these intervals contain an infinite number of points in the sequence. Let I_1 be one of the intervals which contains an infinite number of terms and pick some $a_{n_1} \in I_1$ from the sequence to begin constructing a subsequence of (a_n). Let I_2 be the interval obtained by bisecting I_1 and then picking an interval with infinitely many terms, and then pick a_{n_2} from the initial sequence so that $a_{n_2} \in I_2$ and $n_2 > n_1$.

We can continue this process indefinitely by constructing I_k by bisecting I_{k-1}, picking a half which contains an infinite number of terms, and selecting a_{n_k} with $a_{n_k} \in I_k$ and $n_k > n_{k-1} > n_{k-2} > \ldots > n_2 > n_1$

Notice that the sets

$$I_1 \supset I_2 \supset I_3 \supset \ldots \tag{45}$$

form a nested sequence of closed intervals; therefore, by the Nested Interval Property, there exists at least one point $x \in \mathbb{R}$ such that $x \in I_k$. We will now show that $(a_n) \to x$ which will complete our proof.

Let $\varepsilon > 0$. We constructed I_k to have a length of $M\,(1/2)^{k-1}$. From the Algebraic Limit Theorem, we know that this length converges to 0. Now pick N so that $k \geq N$ implies that the length of I_k is less that ε. Because x and a_{n_k} are both in I_k, and the distance between two points in an interval must be less than or equal to the length of the interval, we have

$$|a_{n_k} - x| < \varepsilon \tag{46}$$

□

Definition 17 (Cauchy Property of Sequences)**.** A sequence (a_n) is called a **Cauchy sequence** if, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ if follows that $|a_n - a_m| < \varepsilon$

Theorem 14. Cauchy sequences are bounded.

Proof. Let (a_n) be a Cauchy sequence, then we know for $\varepsilon = 1$ if $m, n > N$ we have

$$|a_n - a_m| < \varepsilon = 1 \tag{47}$$

If we set $m = N$, we have

$$|a_n - a_N| < 1 \tag{48}$$

Now consider $|a_n|$. We can choose to rewrite this and apply the triangle inequality:

$$|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N| \tag{49}$$

Using the fact that we have $|a_n - a_N| < 1$:

$$|a_n| < 1 + |a_N| \tag{50}$$

Finally let

$$M = \max\{|a_1|, |a_2|, |a_3|, \ldots, |a_{N-1}|, |a_N| + 1\} \tag{51}$$

Then we know that $|a_n| < M$ for all $n \in \mathbb{N}$ and the sequence is bounded. □

Theorem 15 (Cauchy Criterion)**.** A sequence converges if and only if it is a Cauchy sequence.

Proof. We will prove both the forward and backward direction

- ($\Rightarrow$) Assume that (x_n) converges to L. Then consider

$$\begin{aligned} &|x_m - x_n| && (52)\\ &= |x_m - L + L - x_n| && (53)\\ &\leq |x_m - L| + |L - x_n| && (54)\\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon && (55) \end{aligned}$$

  Therefore $|x_m - x_n| \leq \varepsilon$ and the sequence is Cauchy.

- ($\Leftarrow$) Assume that (x_n) is a Cauchy sequence, then we know that (x_n) is bounded. By the Bolzano-Weierstrauss theorem, we know that there exists (x_{n_k}) which converges. Let

$$x = \lim x_{n_k} \tag{56}$$

  Because (x_n) is Cauchy, we know that there exists an $N \in \mathbb{N}$ such that if $n, m > N$

$$|x_n - x_m| < \frac{\varepsilon}{2} \tag{57}$$

  By using the definition of convergence and picking $n_K \geq N$, we can also write

$$|x_{n_K} - x| < \frac{\varepsilon}{2} \tag{58}$$

  To see that N has the desired property, observe that if $n \geq N$, then

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_K} + x_{n_K} - x| && (59)\\ &\leq |x_n - x_{n_k}| + |x_{n_k} - x| && (60)\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon && (61) \end{aligned}$$

□

Theorem 16 (Algebraic Limit Theorem for Series)**.** If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then

i) $\sum_{n=1}^{\infty} ca_n = cA$ for all $c \in \mathbb{R}$ and

ii) $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

Proof. We will prove each part separately.

i) We need to show that $\sum_{n=1}^{\infty} ca_n = cA$, and we know that $\sum_{n=1}^{\infty} a_n = A$ so consider the partial sums

$$t_m = ca_1 + ca_2 + ca_3 + \ldots + ca_m \tag{62}$$
$$s_m = a_1 + a_2 + a_3 + \ldots + a_m \tag{63}$$

Notice that $t_m = cs_m$, therefore by the Algebraic Limit Theorem $\lim t_m = \lim cs_m = cA$. Therefore, since the partial sum converges to cA, $\sum_{n=1}^{\infty} ca_n = A$.

ii) Consider the partial sums:

$$t_m = (a_1 + b_1) + (a_2 + b_2) + \ldots + (a_m + b_m) \tag{64}$$
$$s_m = a_1 + a_2 + a_3 + \ldots + a_m \tag{65}$$

Notice that $t_m = a_m + b_m$, therefore $\lim t_m = \lim (a_m + b_m)$, and by the Algebraic Limit Theorem $\lim t_m = A + B$. Therefore, since the partial sum converges to $A + B$, $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$.

□

Theorem 17 (Cauchy Criterion for Series)**.** The series $\sum_{n=1}^{\infty} a_n$ converges if and only if, given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > m \geq N$ if follows that

$$|a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon \tag{66}$$

Proof. Notice that

$$\begin{aligned} |a_{m+1} + a_{m+2} + \ldots + a_n| &= |s_n - s_m| && (67)\\ &= |s_n - L + L - s_m| && (68)\\ &\leq |s_n - L| + |L - s_m| && (69)\\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon && (70) \end{aligned}$$

Hence proved. □

Theorem 18. If the series $\sum_{n=1}^{\infty} a_n$ converges, then $(a_n) \to 0$.

Proof. If $\sum_{n=1}^{\infty} a_n$ is a converges, then it is a Cauchy sequence, therefore $\exists N \in \mathbb{N}$ such that if $m, n > N$

$$|a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon \tag{71}$$

Pick $n = m + 1$, then

$$|a_{m+1}| < \varepsilon \tag{72}$$

Therefore, if $n > m + 1$

$$|a_n - 0| < \varepsilon \tag{73}$$

Hence proved. □

Theorem 19 (Comparison Test)**.** Assume (a_n) and (b_n) are sequences satisfying $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$.

Proof. □

Theorem 20 (Absolute Convergence Test)**.** If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.

Proof. □

Theorem 21 (Alternating Series Test)**.** Let (a_n) be a sequence satisfying,

i) $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq a_{n+1} \geq \ldots$ and

ii) $(a_n) \to 0$

Proof. □

Definition 18. If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ **converges absolutely**. If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converge, the we say that the original series $\sum_{n=1}^{\infty} a_n$ **converges conditionally**.

Lecture 2: Some historical motivations for Analysis

Monday 13 January 2025

2 The Heat Equation

In 1822, Fourier derived the heat equation. In one dimension:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0.$$

where $u(x,t)$ is the temperature as a function of position and time. A natural problem to solve with the equation is to assume you are given a function $u(x,0)$ which represents the initial temperature distribution of the system which we could measure then ask if it is possible to find a general $u(x,t)$ given $u(x,0)$. Stated another way, if we know the initial temperature distribution, can we find the distribution at an arbitrary time t using only the heat equation. The answer to this question is yes!

If we assume that $u(x,0)$ is a periodic function then:

$$u(x,0) = \sum_{n\in\mathbb{Z}} a_n e^{inx}.$$

Then, rearranging and integrating:

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} u(x,0)\, e^{inx} dx.$$

Then, we will guess that the solution is of the form:

$$u(x,t) = \sum_{n\in\mathbb{Z}} a_n(t)\, e^{inx}.$$

Substituting into the heat equation we get:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sum_{n\in\mathbb{Z}} a_n'(t)\, e^{inx} - a_n(t)\,(in)^2\, e^{inx} = \sum \left(a_n'(t) + a_n(t)\, n^2\right) e^{inx} = 0.$$

So we have the differential equation:

$$a_n'(t) + a_n(t)\, n^2 = 0.$$

which has the solution:

$$a_n(t) = a_n(0)\, e^{-n^2 t}.$$

We can find $a_n(0)$ with the integral above, so we have our solution! If we check the solution experimentally, we see the right behavior, so what's the problem? The issue is:

$$u(x,0) \neq \sum_{n\in\mathbb{Z}} a_n e^{inx}!$$

At least, when we look at the graph for any specific $n \in \mathbb{N}$ we see that at the extreme points of the function, we get oscilations away from the true value of $u(x,0)$. This affect is called the Gibbs phenomenon. This tells us that the function cannot equal the partial sum. On the other hand, the assumption works, so there must be some kind of notion of equality, but we are not in a position to say what that is right now. This gives us a specific example where Newton's calculus fails. The issue in this example has to do with the definition of "convergence" but there is a deeper issue. When we wrote down $u(x,0) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$, we were writing down nonsense, but we didn't know it. In order to know which statements are valid and which are not, we need to develop an axiomatic system that we can use to build up definitions, theorems, and proofs. This is the buisness of Mathematical Analysis: to provide a rigorous base for analysis to rest upon which contains no nonsense!

Lecture 3

Wednesday 15 January 2025

Once we realize that the we don't know how to add up infinitely many functions, it is easy to see that we don't really know how to add infinitely many **numbers** either! Consider:

$$\begin{aligned} 1 &= 1 \\ 1-1 &= 0 \\ 1-1+1 &= 1 \\ 1-1+1-1 &= 0 \\ &\vdots \\ 1-1+1-1+1+\ldots &= \ ? \end{aligned}$$

Already there seems to be a problem! The series does seem to converge to any number. Of course, we don't know what converge means yet, but there is a deeper problem. Consider the rearrangements:

$$\begin{aligned} 1+(-1+1)+(-1+1)+\ldots &= 1 \\ (1+-1)+(1+-1)+(1+\ldots &= 0 \end{aligned}$$

Clearly, these can't both be right. Again, we have been tricked into writing nonsense because we don't have any axioms to tell us which statements are allowed and which are not. Here, the problem has to do with our adding up of an infinite number of things. When we are properly automatized, we will see that we just don't do that. Instead, we will solve this problem with a "limit," **in order to understand the limit, we will need to develop $\mathbb{R}$, the real number system.** This is the goal of Chapter 1. The point is: adding up infinite things, whether they are functions or just numbers, leads to problems, and whatever formal system we come up with will need to be without these problems if we want it to formalize calculus, which is based around the notion of adding up infinitely many things.

Chapter 1

To motivate the definition of $\mathbb{R}$ let's explore ways in which $\mathbb{R}$ is different to other number systems. Why should we expect the definition of $\mathbb{R}$ to be useful and lead us to a notion of a "continuum?"

Lemma 1. For all $m, n \in \mathbb{Z}$, if $n|m^2$ and n is prime, then $n|m$

Proof. Assume for the sake of contradiction that n does *not* divide m, then n cannot be a prime factor of m, so $m = abcd\ldots$ for some prime numbers $a, b, c, d, \ldots$, importantly n **cannot be part of the product since** n **does not divide** m. Then $m^2 = (abcd\ldots)^2 = a^2b^2c^2d^2\ldots$ which does not contain n, so $n \nmid m^2$ which contradicts our assumption. Thus it must be the case that $n|m$. □

Theorem 22. There is no rational number whose square is 2

Proof. Assume for contradiction: $\exists r$ s.t. $r = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $r^2 = 2$. Also assume, WLOG, that p, q **share no common factors** then:

$$r^2 = \left(\frac{p}{q}\right)^2 = 2 \tag{74}$$
$$\Rightarrow p^2 = 2q^2 \tag{75}$$
$$\Rightarrow 2|p^2 \tag{76}$$
$$\Rightarrow 2|p \text{ (by Lemma 1)} \tag{77}$$
$$\Rightarrow p = 2n \text{ where } n \in \mathbb{Z} \tag{78}$$
$$\Rightarrow (2n)^2 = 4n^2 = 2q^2 \text{ (from Eq. 2)} \tag{79}$$
$$\Rightarrow 2|q^2 \Rightarrow 2|q \tag{80}$$

Thus $2|p$ and $2|q$ which violates our assumption that p, q share no common factors! Thus is must be the case that there is no rational number whose square is 2.

□

Theorem 23. If $n \in N$ and n is **not** a perfect square, then there is no $r \in \mathbb{Q}$ such that $r^2 = n$

Proof. Assume for contradiction: $\exists r$ s.t. $r = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $gcd\,(p, q) = 1$, $r^2 = n$, and n is not a perfect square.

$$r^2 = \left(\frac{p}{q}\right)^2 = n \tag{81}$$
$$\Rightarrow p^2 = nq^2 \tag{82}$$

Recall that by the Fundamental Theorem of Arithmetic that we can express any number as a product of prime numbers, so:

$$n = k_1^1 \cdot k_2^2 \cdot k_3^3 \cdot k_4^4 \cdot \ldots.$$

Substitute n into Eq. 9:

$$p^2 = \left(k_1^1 \cdot k_2^2 \cdot k_3^3 \cdot k_4^4 \cdot \ldots\right) q^2.$$

Since n is not a perfect square, we know that $\exists j$ s.t. k_j is odd because if this were not the case, n would be a perfect square. From the above, we can see that $k_j|p^2$. If k_j divides the LHS, it must also divide the RHS, so $k_j|nq^2$. We know that p^2 contains an even number of k_j terms, and we also know that n contains an odd number of k_j terms. For both sides to have the same number of k_j terms, as all equal numbers should, it must be the case that $k_j|q^2$ which implies $k_j|q$ by Lemma 1. Thus, $gcd\,(p,q) = k_j \neq 1$ which contradicts our assumption that $gcd\,(p,q) = 1$. □

After this we talked about "set theory." We did not go into the details.

Theorem 24. The Algebra of Sets exists. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ exist.

Proof. The above is taken as an axiom, but rest assured that their existence can be derived from first-order logic and the ZFC axioms. □

Lecture 4: Homework 1

Tuesday 21 January 2025

1. Prove that there is no rational number, r, such that $r^2 = 8$.

Proof. BWOC, assume there exists a number $r \in \mathbb{Q}$ such that for some $a, b \in \mathbb{Z}$ with $gcd\,(a,b) = 1$:

$$r^2 = \left(\frac{a}{b}\right)^2 = 8.$$

then:

$$\Rightarrow a^2 = 8b^2 \tag{83}$$
$$\Rightarrow 2^2|a^2 \Rightarrow 2|a^2 \Rightarrow 2|a \text{ (by Lemma 1)} \tag{84}$$

so for some $n \in \mathbb{N}$:

$$\left(2^2 n\right)^2 = 8b^2 \tag{85}$$
$$16n^2 = 8b^2 \tag{86}$$
$$2n^2 = b^2 \Rightarrow 2|b \tag{87}$$

Notice $2|a$ and $2|b$ which violates our assumption that $gcd\,(a,b) = 1$. Thus it must be the case that there does not exist a number $r \in \mathbb{Q}$ such that $r^2 = 8$

□

2. Prove that if $a, b \in \mathbb{R}$ then:

$$||\, a \mid - \mid b \,|| \leq |\, a - b \mid .$$

Proof. If $a, b \in \mathbb{R}$, then the triangle inequality holds:

$$\mid a+b \mid \leq \mid a \mid + \mid b \mid .$$

Now consider:

$$\begin{align} \mid a \mid = \mid a-b+b \mid \leq \mid a-b \mid + \mid b \mid \tag{88} \\ \mid b \mid = \mid b-a+a \mid \leq \mid b-a \mid + \mid a \mid \tag{89} \\ \tag{90} \end{align}$$

Rearranging and using the definition of the absolute value function and the fact that $\mid a-b \mid = \mid b-a \mid$:

$$\begin{align} \mid a \mid - \mid b \mid \leq \mid a-b \mid \tag{91} \\ \mid b \mid - \mid a \mid \leq \mid b-a \mid \tag{92} \\ \Rightarrow \mid\mid a \mid - \mid b \mid\mid \leq \mid a-b \mid \tag{93} \end{align}$$

□

3. Let $y_1 = 6$ and for each $n \in \mathbb{N}$ define

$$y_{n+1} = \frac{2}{3} y_n - 2.$$

Prove the following statements:

a. Prove that $y_{n+1} \leq y_n$ for all $n \in \mathbb{N}$.

b. Prove that $y_n > -6$ for all $n \in \mathbb{N}$

Proof. We will show part a by induction. For the base case take $n = 1$, then $y_1 = 6$ and $y_2 = \frac{2}{3}(6) - 2 = 2$, and we have $y_2 \leq y_1$.

For the inductive step, assume $y_{n+1} \leq y_n$ then we need to show that $y_{n+2} \leq y_{n+1}$.

$$\begin{align} y_{n+1} \leq y_n \tag{94} \\ \frac{2}{3} y_{n+1} - 2 \leq \frac{2}{3} y_n - 2 \tag{95} \\ \Rightarrow y_{n+2} \leq y_{n+1} \tag{96} \end{align}$$

Thus, we have shown the theorem by induction. □

Proof. We will show part b by induction. For the base case, take $n = 1$, then $y_1 = 6 > -6$.

For the inductive step, assume $y_n > -6$, then $\frac{2}{3} y_n - 2 > \frac{2}{3}(-6) - 2 \Rightarrow y_{n+1} > -6$

Thus, we have shown the theorem by induction. □

4. Prove that if $x \in \mathbb{R}$ and $x > -1$ then for every $n \in \mathbb{N}$ we have $(1+x)^n \geq 1 + nx$

Proof. We will show the theorem by induction. For the base base, take $n = 1$, then $1 + x \geq 1 + x$, which is true.

For the inductive step, assume $(1+x)^n \geq 1 + nx$, then multiply both sides by $(1 + x)$ to get:

$$(1+x)^{n+1} \geq (1+nx)(1+x) = 1 + x + nx + nx^2 = 1 + (n+1)x + nx^2.$$

Since $x > -1$, we know that $nx^2 > 0$, so we have:

$$(1+x)^{n+1} \geq 1 + (n+1)x + nx^2 \geq 1 + (n+1)x.$$

□

5. Prove or give a counterexample for the following statement: Two real numbers satisfy $a < b$ if and only if $a < b + \varepsilon$ for all $\varepsilon > 0$

The statement is false. Take the case where $a = b$ to be the counterexample. We can adjust the statement slightly to make it true.

Theorem 25. Two real numbers satisfy $a \leq b$ if and only if $a \leq b + \varepsilon$ for all $\varepsilon > 0$

1) $(\Rightarrow)$ If $a \leq b$, then $a - b \leq 0$, so $a - b \leq \varepsilon$ for all $\varepsilon > 0$.
2) $(\Leftarrow)$ Assume $a \leq b + \varepsilon$ for all $\varepsilon > 0$. Let $\varepsilon_0 = a - b$, then it must be that $a - b = \varepsilon_0$ and $a - b \leq \varepsilon_0$, a contradiction! Thus, the theorem must be true.

6. Given a function $f\colon C \to D$ and a set $A \subset C$, let $f(A)$ represent the range of f over the set A i.e. $f(A) = \{f(x) | x \in A\}$.

Answer the following questions:

a. Let $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. If $A = [0, 2]$ and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

   $f(A) = [0, 4], f(B) = [1, 16], f(A \cup B) = [0, 16], f(A \cap B) = [1, 4]$. Yes to both.

b. Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

c. Let $g\colon C \to D$ be any function and let $A, B, C \subset C$ be any two subsets of the domain. Prove that $g(A \cup B) = g(A) \cup g(B)$

   Proof. If $x \in g(A \cup B)$, then $x = a^2$ or $x = b^2$ where $a \in A$ and $b \in B$. $A \cup B$ contains all $a \in A$ and $b \in B$, so $x \in g(A \cup B)$ since it contains all a^2, b^2 where $a, b \in A, B$.

   If $x \in g(A) \cup g(B)$, then either $x \in g(A)$ or $x \in g(B)$. In the first scenario, $x = a^2$ for some $a \in A$ which we know is in $g(A \cup B)$. In the second scenario, $x = b^2$ for some $b \in B$ which we know is in $g(A \cup B)$. □

Lecture 5: The Definition of Function

Thursday 23 January 20

Definition 19. Given sets A, B a function of $A \to B$ is a mapping that takes each element of A to a single element of B.

Note:

1) f is the function, $f(x)$ not the function.
2) A is call the **domain**. B is called the **codomain**. $Range(f) = \{y \in B \mid \exists x \in A \text{ and } f(x) = y\}$. $Range(f) \neq$ codomain in general.

Example. Given $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$, the comain is $\mathbb{R}$ the codomain is $\mathbb{R}$ and the range is $[0, \infty]$.

Note:

1) If $f(x) \neq f(y)$ when $x \neq y$ then f is called **injective** or **one-to-one**.
2) If $Range(f) =$ codomain of f then f is called **surjective** or **onto**.
3) If f is both injective and surjective, then it is called **bijective**.

Example (Dirichlet Function 1829)**.** Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q} \end{cases}.$$

The above function definition is important for historical reasons. Dirichlet came up with the definition of a function given above, and it generalizes the concept of a function nicely. Before Dirichlet, function were either thought about as "nice" graphs or as formula, but the new definition generalizes both of these and allows for less traditional function definitions.

Example. The **absolute value function**, $\mid \cdot \mid: \mathbb{R} \to \mathbb{R}$, is given below:

$$\mid x \mid = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

Theorem 26. Given the above definition of the absolute value function, we have:

1) $\mid ab \mid = \mid a \mid \mid b \mid$
2) $\mid a + b \mid \leq \mid a \mid + \mid b \mid$ **(Triangle Inequality)**

Proof. □

Note. A common trick that we will use in Analysis is the "add/subtract" trick. Let $a, b, c \in \mathbb{R}$, then:

$$|a - b| = ||a - c| + |c - b|| \tag{97}$$
$$\Rightarrow |a - b| \leq ||a - c| + |c - b|| \tag{98}$$

Theorem 27. Let $a, b \in R$. Then $a = b \Leftrightarrow |a - b| < \varepsilon$ for all $\varepsilon > 0$.

Proof. ($\Rightarrow$) If $a = b$, then $a - b = 0$ and $a - b < \varepsilon$ for all $\varepsilon > 0$. ($\Leftarrow$) FSOC assume $|a - b| < \varepsilon$ for all $\varepsilon > 0$ and $a \neq b$, then let $\varepsilon_0 = a - b \neq 0$ then we gave $|a - b| < \varepsilon$ and $|a - b| = \varepsilon_0$, a contradiction!

□

Lecture 6: The Axiom of Completeness

Thursday 23 January 2025

We will take an axiomatic approach to Analysis. There are some things which we will just assume are true. Mathematical Formalism is the idea that formal languages with no semantics can serve as the foundation of mathematics. Under this interpretation, the symbols of mathematics do not mean anything at all! They are only symbols and rules for manipulating symbols. Formulating all of mathematics in terms of a formal language allows us to side step assuming the existence of anything. The trade off is that proofs are extraordinarily complex, involve a lot of symbols, and are generally unreadable. For our purposes of writing readable proofs for the most important theorems from Newton's Calculus, we will take a different set of axioms where we do assert the existence of certain mathematical objects. The philosopher should be satisfied with these axioms because they are formally provable within axiomatic set theory. We don't *need* to assume the existence of anything, but we choose to in order to make our lives easier.

Axiom 2 (Algebraic Properties of $\mathbb{R}$)**.** Assume the existence of a set $\mathbb{R}$, called **the Real Numbers**, which is an ordered field.

This axiom gets us most of the way there, however notice that the rational numbers are also an ordered field. We will need to introduce one more axiom to get a unique set for $\mathbb{R}$; but first, we need to define a little bit of mathematical machinery.

Definition 20 (Bounded Above Property of Subsets of $\mathbb{R}$)**.** A set $A \subset \mathbb{R}$ is **bounded above** if there exists a number $b \in \mathbb{R}$ such that $a \leq b \ \forall a \in A$. The number b is called an **upper bound** for A.

Definition 21 (Bounded Below Property of Subsets of $\mathbb{R}$)**.** A set $A \subset \mathbb{R}$ is **bounded below** if there exists a number $b \in \mathbb{R}$ such that $b \leq a \ \forall a \in A$. The number b is called a **lower bound** for A.

Definition 22 (The Least Upper Bound)**.** An element $s \in \mathbb{R}$ is called the **least upper bound** for $A \subset \mathbb{R}$ if s meets two conditions:

1) s is an upper bound for A
2) $\forall b$ where b is an upper bound, $s \leq b$.

Definition 23 (The Greatest Lower Bound)**.** An element $l \in \mathbb{R}$ is called the **greatest lower bound** for $A \subset \mathbb{R}$ if l meets two conditions:

1) l is a lower bound for A
2) $\forall b$ where b is an upper bound, $l \geq b$.

Example. Given the set $A = \{\frac{1}{n} | n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, find: upper bounds, the least upper bound, lower bounds, and the greatest lower bound.

1) some upper bounds: $1, 2, 1.1, 3$
2) least upper bound: 1
3) some lower bounds: $0, -1, -100$
4) greatest lower bound $= 0$

Example. There is no upper bound for $\mathbb{N}$

The above arguments were not very rigorous, so now we will do a slightly more rigorous problem just to prove that we can.

Theorem 28. Given the set $A = \{\frac{1}{n} | n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, then the least upper bound for A is 1.

Proof. We will prove the two conditions one at a time:

1) Observe that $1 \geq \frac{1}{n}$ $\forall n \in \mathbb{N} \Rightarrow 1$ is an upper bound for A.
2) If b is an upper bound, then, because $1 \in A$, $b \geq 1 \Rightarrow 1$ is the upper bound for A.

□

Theorem 29. If some subset of $\mathbb{R}$ has a least upper bound, then it is unique.

Proof. FSOC assume s_1 and s_2 are two distinct greatest upper bounds of some set A, then we have $s_1 \leq s_2$ and $s_2 \leq s_1$ by applying the second condition of the least upper bound property to s_1 and s_2 one at a time. Thus, $s_1 = s_2$. This contradicts our assumption that s_1 and s_2 are distinct. □

Now we are ready to state the Axioms of Completeness:

Axiom 3 (The Axiom of Completeness)**.** Every non-empty set A where $A \subset \mathbb{R}$ which is bounded above has a least upper bound $b \in \mathbb{R}$

Theorem 30. Up to isomorphism, there is one unique complete ordered field.

Proof. The proof of the above theorem is beyond the scope of this course, but it is worth stating because when we work with $\mathbb{R}$ we can be sure that we are working on the right set without having to worry that what we are describing has more interpretations than as real numbers. □

Note. The Axiom of Completeness is not stateable in first-order logic. You can tell because of the "for every nonempty set A." Here, we are quantifying over a set of sets which is not allowed.

Lecture 7: Homework 2

Monday 27 January 2025

1. Compute, without proof, the supremum and infimum (if they exist) of each of the following sets:

2. Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ be two non-empty sets, each of which is bounded above. If $s = \sup A$ and $t = \sup B$, find and prove a formula for $\sup A \cup B$

Proof. We argue that $\sup A \cup B = max\,(s,t)$ by cases:

1) $(s > t)$ WLOG with respect to the $s < t$ case, since $t = \sup B$, we have $t \geq b$ for all $b \in B$, thus by using our case assumption we get $s > t \geq b$ for all $b \in b$ and $s \geq a$ for all $a \in A$ by the definition of the supremum of a set. Therefore, $s \geq u$ for all $u \in A \cup B$ and $\sup A \cup B = s = max\,(s,t)$.

2) $(s = t)$ If $s = t$ by the definition of the supremum we have $s \geq a$ for all $a \in A$ and $t = s \geq b$ for all $b \in B$. Thus $max\,(s,t) = s = t \geq a$ for all $a \in A$ and $max\,(s,t) = s = t \geq b$ for all $b \in B$. Therefore, $\sup A \cup B = max\,(s,t)$.

□

3. Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ be two non-empty sets, each of which is bounded above.

1) If $\sup A < \sup B$, show that there exists $b \in B$ such that b is an upper bound for A.

2) Given an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

Proof. 1) Combining the definition of the supremum of a set and the given, we get

$$a \leq \sup A < \sup B.$$

Thus $\sup B$ is an upper bound for A, but it cannot be the *least* upper bound because we assumed that $\sup A < \sup B$. Then by negating the ε Characterization of the Supremum we see that $\exists \varepsilon > 0\, \forall a \in A\, (\sup B - \varepsilon \geq a)$. Let ε_0 be the ε such that $\forall a \in A\, (\sup B - \varepsilon_0 \geq a)$. Since $\sup B$ is the least upper bound of B, again we can use the ε Characterization of the supremum, $\forall\, \varepsilon > 0 \exists b \in B\, (\sup B - \varepsilon < b)$ thus $\exists b \in B\, (\sup B - \varepsilon_0 < b)$. Thus, $\exists b \in B\, \forall a \in A\, (b > \sup B - \varepsilon_0 \geq a)$. Therefore, there exists $b \in B$ such that $b \geq a$ for all $a \in A$.

2) Take $A = (1, 2)$ and $B = (0, 2)$. In this case, $\sup A = \sup B$, but there is no element of b which is an upper bound of A.

□

5. Let $A \subset \mathbb{R}$ and $c \in \mathbb{R}$. We define the set cA as:

$$cA = \{ca | a \in A\}.$$

If A is non-empty and bounded above and $c \geq 0$, then prove that *sup* cA $=$ $c \cdot$ *sup* A.

Proof. By the definition of the supremum, we have $\sup cA \geq ca \Rightarrow \frac{1}{c}\sup cA \geq a$. Then $\frac{1}{c}\sup cA$ is an upper bound for A. But $\sup A$ is the *least* upper bound of A, so it must be that $\frac{1}{c}\sup cA \geq \sup A$, thus $\sup cA \geq c \sup A$.

By the definition of the supremum, we have $\sup A \geq a$ for all $a \in A \Rightarrow c \sup A \geq ca \ \forall a \in A$. Thus $c \sup A$ is an upper bound of cA. But $\sup cA$ is the *least* upper bound of cA, so it must be that $c \sup A \geq \sup cA$.

Thus, $\sup cA \geq c \sup A$ and $\sup cA \leq c \sup A$; therefore, $\sup cA = c \sup A$. □

Wednesday 29 January 2025

Lecture 8: 1-29-25 Lecture

- Math Club in MATH350

Definition 24. Let A, B be two sets, we say that A has the **same cardinality** as B if there exists $f : A \to B$ which is a bijection. In the case we write $A \sim B$. Note that $A \sim B \Leftrightarrow B \sim A$

Example. $A = \{1, 2\}$, $B = \{apple, banananana\}$. Then $A \sim B$ since we can define $f : A \to B$ such that:

$$f(x) = \begin{cases} f(1) & = \text{apple} \\ f(2) & = \text{banana} \end{cases}.$$

f is a bijection, so $A \sim B$

Example. let $E = \{2, 4, 6, 8, \ldots\}$. Claim: $\mathbb{N} \sim E$. Define $f : \mathbb{N} \to E$ given by:

$$\begin{cases} f(1) & = 2 \\ f(2) & = 4 \\ f(3) & = 6 \\ \ldots \end{cases}.$$

f is a bijection, so $\mathbb{N} \sim E$

Example. $\mathbb{N} \sim \mathbb{Z}$

Proof. $f : \mathbb{N} \to \mathbb{Z}$ is given by

$$f(n) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{-n}{2} & \text{if } n \text{ is even.} \end{cases}.$$

f is a bijection, so $\mathbb{N} \sim \mathbb{Z}$ □

Theorem 31. Let A, B, C be sets. If $A \sim B$ and $B \sim C$, then $A \sim C$.

Proof. As $A \sim B$, hence there exists a bijection $f : A \to B$. As $B \sim C$, there exists a bijection $g : B \to C$. Therefore, $g \circ f : A \to C$ is a bijection □

Theorem 32. Let X, Y be two sets. If there exists an injective function $f : X \to Y$ and an injective function $g : Y \to X$, then there exists a bijection $h : X \to Y$ and hence $X \sim Y$.

The above will make our lives easier. We no longer need to find an explicit function. Notice no need to check either function for surjectivity. We get it for free.

Theorem 33. $\mathbb{N} \sim \mathbb{Z}^2$ where

$$\mathbb{Z}^2 = \{(m, n) : m, n \in \mathbb{Z}\}.$$

Informal Proof. Take grid of points down to the number line. □

Proof. Let $f : \mathbb{N} \to \mathbb{Z}^2$ given by

$$f(n) = (n, 0).$$

f is clearly injective.As $\mathbb{Z} \sim \mathbb{N} \Rightarrow$ there exists $g : \mathbb{Z} \to \mathbb{N}$ which is a bijection. Define

$$h : \mathbb{Z}^2 \to \mathbb{N}.$$

where $h(m, n) = 2^{g(m)} \cdot 3^{g(n)}$. Now we will show that h is injective. Assume that $h(m_1, n_1) = h(m_2, n_2)$. We want to show that $m_1 = m_2$ and $n_1 = n_2$:

$$2^{g(m_1)} 3^{g(n_1)} = 2^{g(n_1)} 3^{g(n_2)}.$$

As 2 and 3 are prime numbers, by unique factorization:

$$\Rightarrow g(m_1) = g(m_2) \text{ and } g(n_1) = g(n_2)..$$

But $g : \mathbb{Z} \to \mathbb{N}$ is a bijection, hence $m_1 = m_2$ and $n_1 = n_2 \Rightarrow h$ is injective. Thus, by the Cantor-Schroder-Berstein theorem, there exists $z : \mathbb{N} \to \mathbb{Z}^2$ which is a bijection. □

Theorem 34. Show that $\mathbb{N} \to \mathbb{N}^3$ where $\mathbb{N}^3 = \{(a,b,c) : a,b,c \in \mathbb{N}\}$

Proof. Let $f : \mathbb{N} \to \mathbb{N}^2$ be $f(n) = (n,1,1)$. This f is injective. Let $g : \mathbb{N}^3 \to \mathbb{N}$ where $g(a,b,c) = 2^a 3^b 5^c$. This g is injective by the same logic as before. By CSB, then there exists a bijection $z : \mathbb{N} \to \mathbb{N}^3$. $\square$

Theorem 35. A set S is c called **countably infinite** if $S \sim \mathbb{N}$. A set S is called **countable** if either S is finite or countably infinite. S is called **uncountable** if it is not countable. (This definition's slightly different from the textbook).

Example. $A = \{1,2\}$ is finite and countable. $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$ is countably infinite and countable.

Friday 31 January 2025

Lecture 9: 1-31-25 Lecture

-Recall that a countably infinite set just means that $A \sim \mathbb{N}$.

-Recall that a countable set is either finite or countably infinite.

-Recall that an uncountable set is a set which is not countable. It is not clear a priori that these exist, but they do.

Example. $\mathbb{N} \sim \mathbb{N}^2 \sim \mathbb{Z} \sim Z^2$ are all countable.

Theorem 36. The set $\mathbb{Q}$ is countable (ie it is a countable infinite set).

Proof. Let $f : \mathbb{N} \to \mathbb{Q}$ be $f(n) = \frac{n}{2}$. This is an injective mapping. Every rational number $r \in \mathbb{Q}$ can be uniquely written as $\frac{p}{q}$ where $p \in \mathbb{Z}$, $q \in N$, and $\gcd(p,q) = 1$. Define $g : \mathbb{Q} \to \mathbb{Z}^2$ given by $g(r) = (p,q)$. Clearly this is injective. As $\mathbb{Z}^2 \sim \mathbb{N}$, $\exists h : \mathbb{Z}^2 \to \mathbb{N}$ bijective. Thus $h \circ g : \mathbb{Q} \to \mathbb{N}$ is injective. Therefore, by the Cantor-Schroder-Bernstein theorem. $\mathbb{Q} \sim \mathbb{N}$, and $\mathbb{Q}$ is countable. $\square$

Theorem 37. $\mathbb{R}$ is uncountable.

Proof. We proceed by contradiction. Therefore, $\mathbb{N} \sim \mathbb{R}$ i.e. $f : \mathbb{N} \to \mathbb{R}$ which is a bijection. Therefore, we can write $x_1 = f(1)$, $x_2 = f(2)$, ... We have $\mathbb{R} = \{x_1, x_2, \ldots\}$. Consider a closed interval I_1 which does not contain x_1. Now let I_2 be a closed interval inside I_1 such that $x_2 \notin I_2$. In general, given I_n closed interval, construct a closed interval I_{n+1} such that

1) $I_{n+1} \subset I_n$.
2) $x_{n+1} \notin I_{n+1}$.

Consider the set $\cap_{n=1}^{\infty} I_n$. As $x_n \notin I_n \Rightarrow x_n \notin \cap_{n=1}^{\infty}$. As $f : \mathbb{N} \to \mathbb{R}$ is a bijection (As $\mathbb{R} = \{x_1, x_2, \ldots\}$) we have $\cap_{n=1}^{\infty} I_n = \emptyset$. □

But are there any infinities which are small than the cardinality of $\mathbb{N}$

Theorem 38. If B is a countable set and $A \subset B$, then A is countable.

Proof. We will show the theorem by cases:

- (B is a finite set). As $A \subset B$, A is also a finite set and A is countable.
- (B is countably infinite). If A is finite, then obviously A is countable
- Now we assume that A is an infinite set. As B is countably infinite, we gave a bijection $f : \mathbb{N} \to B$. In particular, we can write

  $$B = \{f(1), f(2), \ldots\}.$$

  $A \subset B$ and is infinite. Let $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$. More generally given n_k we define n_{k+1} as

  $$n_{k+1} = \min\{n \in \mathbb{N}, n > n_k : f(n) \in A\}.$$

  Define $g : \mathbb{N} \to A$ as $g(k) = f(n_k)$. By construction, g is a bijection; therefore, $A \sim \mathbb{N}$ and A is countable.

□

Lecture 10: 2-3-25 Lecture

Monday 03 February 2025

Recall from last time

Theorem 39. If $A \subset B$ and B is countable, then A is countable.

Note. if $B = \mathbb{N}$ and $A = \emptyset$. $\emptyset$ is a finite set and thus countable. $\emptyset \subset \mathbb{N}$.

Theorem 40. A set A is countable if and only if there exists an injective function $f : A \to \mathbb{N}$.

Proof. We will prove the forward and backward direction:

- ($\Rightarrow$) Either A is finite or countably infinite. If $A = \emptyset$, then statement is true vacuously. If A is a nonempty finite set, let $|A| = n$, $n \in \mathbb{N}$. Then clearly there exists a bijection between A and $\{1, 2, \ldots, n\}$. Then we just change the function from being $f : A \to \{1, 2, \ldots, n\}$ to $f : A \to \mathbb{N}$. If A is countably infinite, $\exists f : A \to \mathbb{N}$ a bijection. In particular, it is injective.
- ($\Leftarrow$) Let $f : A \to \mathbb{N}$ be injective. Consider $\text{Range}(f) \subset \mathbb{N}$. Observe that $f : A \to \text{Range}(f)$ is a bijection. As $\text{Range}(f) \subset \mathbb{N} \Rightarrow \text{Range}(f)\, is countable$. We also have $A \sim \text{Range}(f) \Rightarrow A$ is countable.

□

Theorem 41. If A_n is a countable set for each $n \in \mathbb{N}$, then $\cup_{n=1}^{\infty} A_n$ is also countable, i.e. a countable union of countable sets is countable.

INCLUDE GRID OF $\mathbb{N}^2$.

Note. A_ns may not be disjoint! Consider $A_1 = \{1,2\}$, $A_2 = \{2,3\}$, $A_3 = \{3,4,5\}$. We will try to make these sets disjoint before we get to the proof.

Proof. Define $B_1 = A_1$, $B_2 = A_2 \setminus A_1, \ldots, B_n = A_n \setminus \{A_1 \cup A_2 \sup \ldots\}$ So we have:

$$B_1 = \{1,2\} \tag{99}$$
$$B_2 = \{3\} \tag{100}$$
$$B_3 = \{4,5\} \tag{101}$$

Therefore we have $B_1, B_2, B_3, \ldots$ are all disjoint and $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n$. As $B_n \subset A_n$ and A_n is countable, B_n is countable. Therefore $\exists f_n : B_n \to \mathbb{N}$ which is injective for all $n \in \mathbb{N}$.

Define $g : \cup_{n=1}^{\infty} N_n \to \mathbb{N}^2$ given as follows if $b \subset \cup_{n=1}^{\infty}$, the as B_n's are all disjoint, there exists a unique $N \in \mathbb{N}$ such $b \in B_n$. Define:

$$g(b) = (f_N(b), N).$$

As f_N is injective $\Rightarrow g$ is injective. As $\mathbb{N}^2$ is countably infinite, $\exists h : \mathbb{N}^2 \to \mathbb{N}$ is a bijection. Therefore, $h \circ g : \cup_{n=1}^{\infty} B_n \to \mathbb{N}$ is injective and $\cup_{n=1}^{\infty} B_n$ is countable.

□

Theorem 42. If $m \in \mathbb{N}$, and $A_1, A_2, \ldots, A_n$ are countable, then $A_1 \cup A_2 \cup \ldots \cup A_m$ is also countable.

Proof. Define $A_n = \emptyset$ for $n \geq m+1$. Therefore each A_n, $n \in \mathbb{N}$ is countable. By previous theorem $\cup_{n=1}^{\infty} A_n$ is countable. But $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n$. □

Theorem 43. Suppose $I = \mathbb{R} \setminus \mathbb{Q}$ is countable which implies $\mathbb{R} = \mathbb{Q} \cup I$ is also countable by the previous corollary, a contradiction!

Monday 03 February 2025

Lecture 11: Homework 3

Proof. From the definitions, we have $\forall n \in \mathbb{N}, \forall a \in A\left(s + \frac{1}{n} \geq a\right)$ and $\forall n \in \mathbb{N}, \exists a \in A\left(s - \frac{1}{n} < a\right)$ Notice that for all $n \in \mathbb{N}$, $s - \frac{1}{n+1} < s - \frac{1}{n}$, so $\forall n \in \mathbb{N}$ $s - \frac{1}{n}$ is a not the least upper bound. Thus we get

$$s - \frac{1}{n} \leq \sup A \leq s + \frac{1}{n} \tag{102}$$
$$-\frac{1}{n} \leq \sup A - s \leq \frac{1}{n} \tag{103}$$

for all $n \in \mathbb{N}$. From here there are 3 cases and we can immediately eliminate two:

- Assume $\sup A - s > 0$, then $\forall n \in \mathbb{N} \left(\sup A - s \leq \frac{1}{n}\right)$ which contradicts the Archimedean Principle since $\sup A - s > 0$.
- Assume $\sup A - s < 0$, then $\forall n \in \mathbb{N} \left(-\left(\sup A - s\right) \leq \frac{1}{n}\right)$ which contradicts the Archimedean Principle since $-\left(\sup A - s\right) > 0$

Therefore, it must be that $\sup A - s = 0 \Rightarrow s = \sup A$. □

2. Prove that $\cap_{n=1}^{\infty} \left(5, 5 + \frac{1}{n}\right) = \emptyset$.

Proof. Assume for the sake of contradiction, that $x \in \cap_{n=1}^{\infty}$, then $5 < x < 5 + \frac{1}{n} \Rightarrow x = 5 + \varepsilon$ for some $\varepsilon > 0 \in \mathbb{R}$. By the archimedean principle, $\exists n \in \mathbb{N} \left(\varepsilon > \frac{1}{n}\right)$, thus $x = 5 + \varepsilon > 5 + \frac{1}{n}$. But then we have $x > 5 + \frac{1}{n}$ from the previous statement and $x < 5 + \frac{1}{n}$ from the given. This is a contradiction, so it must be that $\nexists x \in \mathbb{R} \left(x \in \cap_{n=1}^{\infty} \left(5, 5 + \frac{1}{n}\right)\right)$. Therefore, $\cap_{n=1}^{\infty} \left(5, 5 + \frac{1}{n}\right) = \emptyset$. □

3. Let $a, b \in \mathbb{R}$ with $a < b$. Let $T = \mathbb{Q} \cap [a, b]$. Prove that $\sup T = b$.

Proof. Notice b is the maximum of $[a, b]$, and $\mathbb{Q} \cap [a, b] \subset [a, b]$. Thus $\forall t \in T, t \in [a, b] \Rightarrow b \geq t$. Thus b is an upper bound of T.

To show that b is the supremum of T, assume FSOC that $\exists s$ such that $s < b$ and $\forall t \in T \left(s \geq t\right)$. By the density of the rationals in $\mathbb{R}$, $\exists r \in \mathbb{Q} \left(s < r < b\right)$. Using the fact that rationals are dense in $\mathbb{R}$ again, we know $\exists q \in \mathbb{Q}$ such that $a < q < s < r < b$, thus $r \in T$. Since $r \in T$ and s is an upper bound of T, we have $s \geq r$. Now we have both $s < r$ by the construction of r and $s \geq r$ by assumption, a contradiction! Therefore there is no upper bound s such that $s < b$.

Therefore, we have shown the two conditions for b to be the supremum of T. □

4. For each $n \in \mathbb{N}$ let I_n be a closed bounded interval (the intervals need not be nested). Assume that for any $N \in \mathbb{N}$ we know that $\cap_{n=1}^{N} I_n = \emptyset$. Prove that $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. □

5. Give an example for each of the following:

1) Two sets A and B with $A \cap B = \emptyset$, $\sup A = \sup B$, $\sup A \notin A$ and $\sup B \notin B$.

   Take $A = \{x : x \in \mathbb{Q}, 0 < x < 1\}$ and $B = \{x : x \in \mathbb{R} \setminus \mathbb{Q}, 0 < x < 1\}$

2) A sequence of nested open intervals $J_1 \supset J_2 \supset J_3 \supset \ldots$ with $\cap_{n=1}^{\infty} J_n$ non-empty but containing only a finite number of elements.

   Take $J_n = (-\frac{1}{n}, \frac{1}{n})$

3) A sequence of nested unbounded closed intervals $L_1 \supset L_2 \supset L_3 \supset \ldots$, where each $L_n = [a_n, \infty)$ for some $a_n \in \mathbb{R}$, such that $\cap_{n=1}^{\infty} L_n = \emptyset$.

   Take $L_n = [n, \infty]$ $\forall n \in \mathbb{Z}$. For any $x \in \mathbb{R}$ we know that $x \notin L_{x+1}$, so $\forall x \in \mathbb{R}$, $x \notin \cap_{n=1}^{\infty} L_n$.

6. If $a, b \in \mathbb{R}$ with $a < b$, show that $[a, b] \sim (a, b)$.

Proof. We will use the Cantor-Schroeder-Bernstein theorem to prove the statement. Defining the injective function $f : (a, b) \to [a, b]$ is trivial; let $f(x) = x$ $\forall x \in (a, b)$. Defining the injective function $g : [a, b] \to (a, b)$ requires a little more doing. Intuitively, we will shrink the set from the range down to any closed set that we want which is contained in $[a, b]$, then we will allow the endpoints of the domain to map to the endpoints of the new closed set. Finally, we linearly map the rest of the uncountably many elements of the domain to the uncountably many elements between the endpoints of the new set which is contained in the range. Formally, we will define a linear function such that $g(a) = \frac{b}{4}$ and $g(b) = \frac{3b}{4}$:

$$g(x) = \begin{cases} \frac{b}{4} & x = a \\ \frac{b}{2(b-a)}x + \frac{b}{4} & a < x < b \\ \frac{3b}{4} & x = b \end{cases}.$$

Thus, $f : (a, b) \to [a, b]$ is injective and $g : [a, b] \to (a, b)$ is injective. Therefore, $\exists h : (a, b) \to [a, b]$ which is a bijective, and $[a, b] \sim (a, b)$.

□

Tuesday 05 February 2025

Lecture 12: 02-05-25

-Sequences and series are the most important part of the class. "If you don't understand this, you are going to fail."

Theorem 44. $\mathbb{R} \setminus \mathbb{Q}$ is uncountable: $\mathbb{R} \setminus \mathbb{Q} \sim \mathbb{R}$

Definition 25. Given a set A, the power set $P(A)$ is the set of all subsets of A.

Theorem 45. If A is a finite set with $|A| = n$ then $|P(A)| = 2^n$

This works even for infinite sets!

Theorem 46. $P(\mathbb{N}) \sim \mathbb{R}$

Theorem 47. Given any set A, there does not exist a surjective function $f : A \to P(A)$.

This means that if A is infinite, then $P(A)$ is a "bigger" infinite than A.

Example. $\mathbb{N} \to P(\mathbb{N}) \sim \mathbb{R} \to P(P(\mathbb{N})) \sim P(\mathbb{R}) \to \dots$

Won't be asking too many questions about this stuff.

3 Sequences and Series

Recall from the example given on day 1 that we cannot sum up infinite stiff. Instead, you add up finitely many things and then take a "limit." Now we define what a limit is.

Definition 26. A sequence is a function whose domain is $\mathbb{N}$.

Example. $2, 4, 8, 16, 32, \dots$ is a sequence of natural numbers. $\pi, \pi^2, \pi^3, \dots$ is a sequence of real numbers.

Sequences are not series. Limits apply to sequences, not series.

Example. $\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right)$

Example. $\left(\frac{1+n}{n}{}_{n=1}^{\infty}\right) = \left(2, \frac{3}{2}, \frac{4}{3}, \dots\right)$

Example. $\left(\frac{1+n}{n}\right)$

If you do not write the starting and ending points, it is assume that it is $n = 1$ to ∞.

Example. (a_n), where $a_n = 2^n$ for all $n \in \mathbb{N}$.

Example. (x_n), where $x_1 = 2$ and $\frac{x_n+1}{2}$ for all $n \geq 1$.

Definition 27 (Convergence of a Sequence)**.** A sequence a_n converges to a real number a, if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$, we have $|a_n - a| < \varepsilon$. In this case we write

$$\lim_{n\to\infty} a_n = a \Leftrightarrow \lim a_n = a \Leftrightarrow (a_n) \to a.$$

Definition 28 (Conversgence of a Sequence Topological Definition)**.** A sequence (a_n) converges to a, if every ε-neighborhood of a contains all but a finite number of the terms of (a_n).

Definition 29. Given $a \in \mathbb{R}$ and $\varepsilon > 0$, the set

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}.$$

is called the ε-neighborhood of a

Example. Prove $\lim\left(\frac{1}{\sqrt{n}}\right) = 0$

Proof. 1) Challenge: $\varepsilon = \frac{1}{2}$ Response: let $N = 5$. To confirm, notice $n \geq 5 \Rightarrow \left|\frac{1}{\sqrt{n}} - 0\right| = \frac{1}{\sqrt{n}} < \frac{1}{2}$

2) Challenge: $\varepsilon = \frac{1}{10}$. Response: let $N = 101$. To confirm check $n \geq 101 \Rightarrow \frac{1}{\sqrt{n}} < \frac{1}{10}$

□

Proof. WTS: $\lim\left(\frac{1}{\sqrt{n}}\right) = 0$. If $n \geq N$ we want

$$\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon \tag{104}$$

$$\Leftrightarrow \frac{1}{\sqrt{n}} < \varepsilon \tag{105}$$

$$\Leftrightarrow \frac{1}{\varepsilon^2} < n \tag{106}$$

Choose $N \in \mathbb{N}$ such that $\frac{1}{\varepsilon^2} < N \leq n$ □

Proof. Let $\varepsilon > 0$ be given. Let $N \in \mathbb{N}$ be such that

$$N > \frac{1}{\varepsilon^2}.$$

Let $n \geq N$. Then we observe that

$$n > \frac{1}{\varepsilon^2} \Rightarrow \frac{1}{\sqrt{n}} < \varepsilon \Rightarrow \left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon.$$

Hence, the theorem is proved. □

Friday 07 February 2025

Lecture 13: 02-07-25 Lecture

Example. Template for a proof of $(x_n \to x)$:

1) Let $\varepsilon > 0$ be given.
2) Choose N (depending on ε in general). This step takes the most amount of work and this work is not shown and is rough work.
3) let $n \geq N$
4) Now prove that $|x_n - x| < \varepsilon$ for all $n \geq N$. Then the proof is complete.

Example. Prove that $\lim\left(\frac{n+1}{n}\right) = 1$

Rough work:

$x_n = \frac{n+1}{n} = 1 + \frac{1}{n}$

$x = 1$

Now we want:

$$|x_n - x| < \varepsilon \tag{107}$$
$$\left|1 + \frac{1}{n} - 1\right| < \varepsilon \tag{108}$$
$$\left|\frac{1}{n}\right| < \varepsilon \tag{109}$$
$$\frac{1}{n} < \varepsilon \tag{110}$$
$$\frac{1}{\varepsilon} < n \tag{111}$$
$$\tag{112}$$

What I really want: Find N so that $\forall n \geq N$, $\frac{1}{\varepsilon} < n$, so choose $N \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < N$, then if $n \geq N \Rightarrow \frac{1}{\varepsilon} < N < n$.

Proof. Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < N$. Let $n \geq N$. This implies that

$$\frac{1}{\varepsilon} < N \leq n \tag{113}$$
$$\frac{1}{n} < \varepsilon \tag{114}$$
$$\left|\left(1 + \frac{1}{n}\right) - 1\right| < \varepsilon \tag{115}$$
$$\left|\left(\frac{n+1}{n}\right) - 1\right| < \varepsilon \tag{116}$$
$$\tag{117}$$

Thus we have shown the condition for the proof. $\square$

Example. Prove that $\lim \left(\frac{1}{n^2}\right) = 0$

Proof. Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{\sqrt{\varepsilon}}.$$

Therefore

$$n > \frac{1}{\sqrt{\varepsilon}} \tag{118}$$
$$\frac{1}{n} < \sqrt{\varepsilon} \tag{119}$$
$$\frac{1}{n^2} < \varepsilon \tag{120}$$
$$\left|\frac{1}{n^2} - 0\right| < \varepsilon \tag{121}$$

$\square$

Example. Prove that $\lim \frac{1}{n^2+576n+100{,}002} = 0$

Proof. If $\frac{1}{n^2} < \varepsilon$

$$\left| \frac{1}{n^2 + 576n + 100,002} - 0 \right| < \varepsilon \tag{122}$$

□

Note. Do not try to find an "optimal" N, just find one that works!

Monday 10 February 2025

Lecture 14: 02-10-25 Lecture

Theorem 48. The limit of a sequence, when it exists, is unique.

Proof. Let (a_n) be a sequence and assume that $s, t \in \mathbb{R}$ such that $\lim a_n = s$ and $\lim a_n = t$. Let $\varepsilon > 0$ be arbitrary. As $\lim a_n = s$, hence $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, we have $|a_n - s| < \frac{\varepsilon}{2}$. Similarly, as $\lim a_n = t$, $\exists N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$, we have $|a_n - t| < \frac{\varepsilon}{2}$. Let

$$N = \max\{N_1, N_2\} \tag{123}$$

hence $|a_N - s| < \frac{\varepsilon}{2}$ and $|a_N - t| < \frac{\varepsilon}{2}$.

$$\begin{aligned}
|s - t| &= |(s - a_N) + (a_N - t)| && (124)\\
&\leq |s - a_N| + |a_N - t| && (125)\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && (126)\\
&= \varepsilon && (127)\\
&\Rightarrow |s - t| < \varepsilon && (128)
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, this implies that $s = t$ □

Definition 30. A sequence that does not converge is said to diverge.

Example. Prove that the sequence $a_n = (-1)^n$ diverges.

Note. The strategy for these is to assume that it converges, then show that it must converge to two different numbers.

Proof. Suppose by contradiction, let $L \in \mathbb{R}$ be such that $\lim a_n = L$. Therefore, given $\varepsilon = \frac{1}{2}$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$ we have $|a_n - L| < \frac{1}{2}$. Let $n_1 \geq N$ be odd $\Rightarrow |a_n - L| < \frac{1}{2} \Rightarrow |(-1)^{n_1} - L| < \frac{1}{2} \Rightarrow |(-1) - L| < \frac{1}{2}$ as $n_1 + 1 \geq N$ and is even.

$$\Rightarrow |a_{n_1+1} - L| < \frac{1}{2} \Rightarrow |1 - L| < \frac{1}{2} \Rightarrow 2 < 1 \tag{129}$$

a contradiction! □

Example. Prove that $\lim\left(\frac{1}{n}\right) \neq 1$

Proof. By contradiction, assume that $\lim \frac{1}{n} = 1$. Then for $\varepsilon = \frac{1}{2}$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$

$$\left|\frac{1}{n} - 1\right| < \frac{1}{2} \tag{130}$$

$$\Rightarrow 1 - \frac{1}{2} < \frac{1}{n} < 1 + \frac{1}{2} \tag{131}$$

$$\Rightarrow \forall n \geq N \left(\frac{1}{2} < \frac{1}{n}\right) \tag{132}$$

By the Archimedean property, $\exists m \in \mathbb{N}$ such that

$$m \geq N \text{ and } \frac{1}{m} < \frac{1}{2} \tag{133}$$

$$\tag{134}$$

This is a contradiction! □

Definition 31. A sequence (x_n) is **bounded** if there exists $M > 0$ such that $|x_n| \leq M$ $\forall n \in \mathbb{N}$.

Theorem 49. Every convergent sequence is bounded.

This is a standard kind of argument that we will see again and again:

Proof. Let $L \in \mathbb{R}$ be such that $\lim x_n = L$. Hence for $\varepsilon = 1$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have

$$|x_n - L| < 1 \tag{135}$$

Therefore, $\forall n \geq N$

$$|x_n| = |x_n - L + L| \tag{136}$$

$$\leq |x_n - L| + |L| \tag{137}$$

$$< |L| + 1 \tag{138}$$

Let $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |L| + 1\} > 0$. We see that $|x_n| \leq M$ $\forall n \in \mathbb{N}$. Hence (x_n) is bounded. □

Lecture 15: Homework 4

Monday 10 February 2025

1. Let $C \subset (0, 1]$ be uncountable. Show that there exists $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.

Proof. □

2. Prove the following limits:

1) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$

2) $\lim \frac{5n^2}{n^3+2n^2+3n+4} = 0$

3) $\lim \frac{\sin(n^2)}{\sqrt{n}} = 0$

4) $\lim \frac{1}{n^2-10} = 0$

Proof. □

3. Let $(a_n)_{n=1}^{\infty}$ be a sequence and let $L \in \mathbb{R}$. Show that $a_n = L$ if and only if the sequence $(a_n - L)_{n=1}^{\infty}$ converges to zero.

Proof. □

4. Prove that $\lim a_n = 0$ if and only if $\lim |a_n| = 0$.

Proof. □

5. If $|a| < 1$, then prove that $\lim a^n = 0$. (Hint: Use the inequality proved in HW1 namely that for $x > -1$ we have $(1+x)^n \geq 1 + nx$, for a suitable chosen x).

Proof. □

Tuesday 12 February 2025

Lecture 16: 02-12-25 Lecture

Example. Prove that (a_n) where $a_n = n^2$ is divergent.

Proof. Assume by contradiction that (a_n) is convergent. Therefore (a_n) is a bounded sequence, so $\exists M > 0$ such that

$$\forall n \in \mathbb{N}(|a_n| \leq M) \tag{139}$$
$$\forall n \in \mathbb{N}\left(n^2 \leq M\right) \tag{140}$$
$$\tag{141}$$

Let $N \in \mathbb{N}$ be such that $N > M$ then

$$N^2 > MN \geq M \tag{142}$$
$$N^2 > M \tag{143}$$

this is a contradiction! □

This next theorem is used all the time.

Theorem 50. Suppose (a_n) is a convergent sequence with $\lim a_n = L$. If $L \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $\exists \delta > 0$ such that $|a_n| \geq \delta > 0$ for all $n \in \mathbb{N}$.

Proof. As $L \neq 0$, choose $\varepsilon = \frac{|L|}{2} > 0$ $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have

$$|a_n - L| < \frac{|L|}{2} \tag{144}$$

$$\tag{145}$$

for $n \geq N$ we have

$$|L| \leq |L - a_n| + |a_n| \leq \frac{|L|}{2} + |a_n| \tag{146}$$

$$\tag{147}$$

Therefore, for all $n \geq N$ we have

$$\frac{|L|}{2} \leq |a_n| \tag{148}$$

$$\tag{149}$$

Define $\delta = \min\{|a_1|, |a_2|, \ldots, |a_{N-1}|, \left|\frac{L}{2}\right|\} > 0$. We see that $|a_n| \geq \delta > 0$ $\forall n \in \mathbb{N}$. ☐

Theorem 51 (Algebraic limit theorem)**.** Let $a, b \in \mathbb{R}$ and let $\lim a_n = a$ and $\lim b_n = b$. Then

1) $\lim (ca_n) = ca$ for all $c \in \mathbb{R}$
2) $\lim (a_n + b_n) = a + b$
3) $\lim (a_n b_n) = ab$
4) If $b \neq 0$ and $b_n \neq 0$ $\forall n \in \mathbb{N}$, then $\lim \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$

Example. Given $a_n = \frac{3n^2+5}{n^2+10}$. Prove $\lim a_n = 0$

Example.

$$a_n = \frac{n^2\left(3 + \frac{5}{n^2}\right)}{n^3\left(1 + \frac{10}{n^3}\right)} \tag{150}$$

$$\frac{1}{n} \cdot \frac{3 + \frac{5}{n^2}}{1 + \frac{10}{n^3}} \tag{151}$$

We know that $\lim \frac{1}{n} = 0$

$$\Rightarrow \lim \frac{1}{n^2} = 0 \tag{152}$$

$$\Rightarrow \lim \frac{5}{n^2} = 0 \Rightarrow \lim \left(3 + \frac{5}{n^2}\right) = 3 \tag{153}$$

Proof. We will consider each case in turn:

1) If $c = 0$ then $ca_n = 0\ \forall n \in \mathbb{N}$. Clearly $(ca_n) \to 0$ in this case. Let $\varepsilon > 0$ be given. Choose $N = 1$. Therefore, $\forall n \geq N$ we have

$$|ca_n - ca| = |0 - 0| = 0 < \varepsilon \tag{154}$$

Therefore $(ca_n) \to ca$ in this case

Let $c \neq 0$ and let $\varepsilon > 0$ be given. Let $N \in \mathbb{N}$ be such that $\forall n \geq N$ we have

$$|a_n - a| < \frac{\varepsilon}{|c|} \tag{155}$$

Sidebar: we want :

$$ca_n - ca < \varepsilon \tag{156}$$

$$|c|\,|a_n - a| < \varepsilon \tag{157}$$

$$|a_n - a| < \frac{\varepsilon}{|c|} \tag{158}$$

Therefore for $n \geq N$ we have

$$|ca_n - ca| \tag{159}$$

$$= |c|\,|a_n - a| \tag{160}$$

$$< |c| \frac{\varepsilon}{|c|} \tag{161}$$

$$= \varepsilon \tag{162}$$

Therefore $|ca_n - ca| < \varepsilon\ \forall n \geq N$ hence proved.

2) Sidebar: WTS

$$|(a_n + b_n) - (a + b)| < \varepsilon \tag{163}$$

$$|(a_n - a) + (b_n - b)| < \varepsilon \tag{164}$$

Now on to the actual proof:

Let $\varepsilon > 0$ be given. Let $N_1 \in \mathbb{N}$ be such that $\forall n \geq N_1$ we have

$$|a_n - a| < \frac{\varepsilon}{2} \tag{165}$$

Let $N_2 \in \mathbb{N}$ be such that $\forall n \geq N_2$ we have

$$|b_n - b| < \frac{\varepsilon}{2} \tag{166}$$

Let $N = \max\{N_1, N_2\}$. Therefore for all $n \geq N$ we have

$$\begin{aligned}
&|(a_n + b_n) - (a + b)| && (167)\\
&= |(a_n - a) + (b_n - b)| && (168)\\
&\leq |a_n - a| + |b_n - b| && (169)\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && (170)\\
&= \varepsilon && (171)
\end{aligned}$$

Therefore, $|ca_n - ca| < \varepsilon \; \forall n \geq N$ hence proved.

□

Lecture 17: 02-14-25 Lecture

Friday 14 February 2025

Example. $a_n = \frac{1}{n^2+10}$ and $\lim a_n = 0$

$$a_n = \frac{1}{n^2\left(1 + \frac{10}{n^2}\right)} \tag{172}$$

$$\left(\frac{1}{n^2}\right)\frac{1}{\left(1 + \frac{10}{n^2}\right)} \tag{173}$$

We know that $\lim \frac{1}{n} = 0$ so

$$\text{(By ALT)} \ \lim \frac{1}{n^2} = 0 \tag{174}$$

$$\lim\left(1 + \frac{1}{n^2}\right) = 1 \tag{175}$$

$$\lim \frac{1}{1 + \frac{10}{n^2}} = 1 \tag{176}$$

$$\lim \frac{1}{n^2} \cdot \frac{1}{\left(1 + \frac{10}{n^2}\right)} = 0. \tag{177}$$

Hence proved.

Theorem 52. Let $a, b \in \mathbb{R}$ and $\lim a_n = a$ and $\lim b_n = b$.

1) If $a_n \geq 0 \; \forall n \in \mathbb{N}$, then $a \geq 0$
2) If $a_n \leq b_n \; \forall n \in \mathbb{N}$, then $a \leq b$
3) If $\exists c \in \mathbb{R}$ such that $c \leq b_n \; \forall n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ $\forall n \in \mathbb{N}$, then $a \leq c$.

Proof. By contradiction, assume that $a < 0$, therefore $\exists N \in \mathbb{N}$ such that $\forall n \geq N$

we have

$$|a_n - a| < \frac{|a|}{2} \Rightarrow a_n - a < \frac{|a|}{2} \tag{178}$$

$$\Rightarrow a_n < a + \frac{|a|}{2} < 0 \tag{179}$$

$$\Rightarrow a_n < 0 \ \forall n \geq \mathbb{N}. \tag{180}$$

A contradiction! □

Monday 17 February 2025

Lecture 18: 02-17-25 Lecture

- DeLong Lecture today 3:30 - 4:30pm in Kitt Multipurpose room. Speaker: Prof. Laura DeMarco (Harvard)

Definition 32. A sequence (a_n) is called **increasing** if $a_n \leq a_{n+1}$ for all $n \in N$. It is called **decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is called monotone if it is either increasing or decreasing.

Example. $1, 1, 2, 2, 3, 3, 4, 4, \ldots$ is increasing.

Example. $1, 1, 0, 0, -1, -1, \ldots$ is decreasing

Example. $1, 1, 1, 1, 1, \ldots$ is constant and monotone.

Example. $1, 0, 1, 0, 1, 0, 1, \ldots$ is *not* monotone.

Theorem 53 (Monotone Convergence Theorem)**.** If a sequence is monotone and bounded, then it converges.

Note. There are two enemies of convergence:

1) Oscillations (killed by monotone)
2) Growth (killed by boundedness)

Proof. Let (a_n) be monotone and bounded. Let us assume that (a_n) is increasing (the case for decreasing is proved similarly). Define the set

$$S = \{a_n | n \in \mathbb{N}\} \tag{181}$$

As (a_n) is bounded, this means that the set S is bounded above. Let $x = \sup S$ Now we just need to show that $\lim a_n = x$ to prove the statement. Let $\varepsilon > 0$ be given. As x is the least upper bound, $x - \varepsilon$ is not an upper bound for S. Then there exists $n \in \mathbb{N}$ such that $x - \varepsilon < a_N$. Therefore, for all $n \geq N$ we have

$$x - \varepsilon < a_N \leq a_n \leq x \tag{182}$$

$$x - \varepsilon < a_n < x + \varepsilon \tag{183}$$

$$|a_n - x| < \varepsilon \tag{184}$$

Hence proved. □

Definition 33. Let (b_n) be a sequence. An **infinite series** is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \dots \tag{185}$$

We define the corresponding **sequence of partial sums**, (S_m) by

$$S_m = b_1 + b_2 + \dots + b_m \tag{186}$$

we say that the series $\sum_{n=1}^{\infty} b_n$ **converges to B** if the sequence (S_m) converges to B. In this case, we write $\sum_{n=1}^{\infty} = B$.

Note. When we write the first sum, we are literally just writing symbols. If we want to assign meaning to this, we need to construct a sequence of partial sums $b_1, b_1 + b_2, b_1 + b_2 + b_3, \dots$.

Example. Recall from day 1:

$$\begin{aligned}
&b_n = (-1)^n && (187)\\
&S_1 = b_1 = -1 && (188)\\
&S_2 = b_1 + b_2 = 0 && (189)\\
&S_2 = b_1 + b_2 + b_3 = -1 && (190)\\
&\dots && (191)
\end{aligned}$$

Then construct the sequence:

$$(S_1, S_2, S_3, \dots) = (-1, 0, -1, 0, -1, \dots) \tag{192}$$

The sequence does not converge, therefore the series doesn't converge.

Example. Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \tag{193}$$

As all the terms in the series are positive, we observe that the sequence (S_m) is an increasing sequence. Now we will apply a trick

$$\begin{aligned}
S_m &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{m^2} && (194)\\
&< 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(m-1)\, m} && (195)\\
&= 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{m-1} - \frac{1}{m}\right) && (196)\\
&= 1 + 1 - \frac{1}{m} && (197)\\
&< 2 && (198)
\end{aligned}$$

Therefore $S_m < 2$ for all $M \in \mathbb{N}$. Hence the sequence (S_m) is bounded. As (S_m) is an increasing bounded sequence, by the monotone convergence theorem, it converges.

Note. The above is the Basel Problem. The value that it converges to was found by Euler in 1734 and surprisingly is $\frac{\pi^2}{6}$. This is connected to the Riemann Zeta function.

Monday 17 February 2025

Lecture 19: Homework 5

1. Let $(a_n) \to 0$. Use the Algebraic limit theorem to compute each of the following limits (assuming the functions are always defined). Justify all of your actions.

1) $\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2} \right)$

$$= \frac{\lim (1 + 2a_n)}{\lim (1 + 3a_n - 4a_n^2)} \tag{199}$$

$$= \frac{\lim 1 + \lim 2 \cdot \lim a_n}{\lim 1 + \lim 3 \cdot \lim a_n - \lim 4 \cdot \lim a_n \cdot \lim a_n} \tag{200}$$

$$= 1 \tag{201}$$

2) $\lim \left(\frac{(a_n+2)^2-4}{a_n} \right)$

$$= \frac{\lim (a_n + 2) \cdot \lim (a_n + 2) - \lim 4}{\lim a_n} \tag{202}$$

$$= \frac{\Big(\lim a_n + \lim 2\Big) \Big(\lim a_n + \lim 2\Big) - \lim 4}{\lim a_n} \tag{203}$$

$$= \frac{\Big(\lim a_n + 2\Big) \Big(\lim a_n + 2\Big) - 4}{\lim a_n} \tag{204}$$

$$= \frac{\Big(\lim a_n\Big)^2 + 4 \lim a_n + 4 - 4}{\lim a_n} \tag{205}$$

$$= \lim a_n + 4 = 4 \tag{206}$$

3) $\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5} \right)$

$$= \frac{\lim\left(\frac{2}{a_n}+3\right)}{\lim\left(\frac{1}{a_n}+5\right)} \tag{207}$$

$$= \frac{\lim \frac{2}{a_n} + \lim 3}{\lim \frac{1}{a_n} + \lim 5} \tag{208}$$

$$= \frac{\frac{\lim 2}{\lim a_n} + 3}{\frac{\lim 1}{\lim a_n} + 5} \cdot \frac{\lim a_n}{\lim a_n} \tag{209}$$

$$= \frac{\lim 2 + 3 \lim a_n}{\lim 1 + 5 \lim a_n} = 2 \tag{210}$$

2. Prove that the following sequences diverge:

1) The sequence (a_n) where

$$a_n = (-1)^n n^2 + 1 \tag{211}$$

Proof. Assume for the sake of contradiction that (a_n) converges. Since (a_n) converges, it is bounded. Therefore $\exists M > 0$ such that

$$\forall n \in \mathbb{N} \left(|a_n| \leq M\right) \tag{212}$$

$$\forall n \in \mathbb{N} \left(\left|(-1)^n n^2 + 1\right| \leq M\right) \tag{213}$$

If we force n to be even, then

$$\forall n \in \mathbb{N} \left(n^2 + 1 \leq M\right) \tag{214}$$

But by the Archimedean principle, we can always pick N such that N is even and $N > \sqrt{M-1}$. Then we have

$$N > \sqrt{M-1} \Rightarrow N^2 + 1 > M \tag{215}$$

Which contradicts our assumption that (a_n) converges. Thus (a_n) must diverge.

□

2) The sequence (a_n) where

$$a_n = (-1)^n + \frac{1}{n} \tag{216}$$

Proof. Assume for the sake of contradiction that (a_n) converges. Let $a =$

$\lim a_n$. Then pick $\varepsilon < a$; and, by the definition of convergence, we have:

$$|a_n - a| \leq \varepsilon \tag{217}$$

$$\left|(-1)^n + \frac{1}{n} - a\right| \leq \varepsilon \tag{218}$$

$$-\varepsilon \leq (-1)^n + \frac{1}{n} - a \leq \varepsilon \tag{219}$$

$$a - \varepsilon \leq (-1)^n + \frac{1}{n} \leq \varepsilon + a \tag{220}$$

If we force n to be even and consider the left side of the inequality, then for all even n we have

$$a - \varepsilon \leq \frac{1}{n} \tag{221}$$

But notice that because of how we picked ε, we know that $a - \varepsilon > 0$. Then, by the Archimedean property of $\mathbb{N}$, there exists an $N \in \mathbb{N}$ such that N is even and

$$a - \varepsilon > \frac{1}{N}. \tag{222}$$

Thus, we have reached a contradiction. Therefore, the sequence must diverge. □

3. (Squeeze Theorem). Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Proof. Take the first given statement, and subtract l from everything:

$$x_n - l \leq y_n - l \leq z_n - l \tag{223}$$

Since we are given that $\lim x_n = \lim z_n = l$, we can say that for all $\varepsilon > 0$ there exists $n_1, n_2 \in \mathbb{N}$ such that, if $N_1 \geq n_1$ and $N_2 \geq n_2$

$$|x_{N_1} - l| < \varepsilon \Rightarrow -\varepsilon < x_{N_1} - l < \varepsilon \tag{224}$$

$$|z_{N_2} - l| < \varepsilon \Rightarrow -\varepsilon < z_{N_2} - l < \varepsilon \tag{225}$$

Therefore, if we let $p = \max\{n_1, n_2\}$ we can say that for all $P > p$

$$-\varepsilon < x_P - l \leq y_P - l \leq z_P - l < \varepsilon \tag{226}$$

Thus for all $\varepsilon > 0$ there exists a $P \in \mathbb{N}$ such that

$$|y_P - l| < \varepsilon \tag{227}$$

Therefore, $\lim y_n = l$

□

4. Let $x_n \geq 0$ for all $n \in \mathbb{N}$.

1) If $(x_n) \to 0$, show that $\sqrt{x_n} \to 0$.

Proof. We know that if $\lim x_n = 0$ that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n > N$ then $|x_n - 0| < \varepsilon$. Therefore $x_n < \varepsilon^2 \Rightarrow \left|\sqrt{x_n} - 0\right| < \varepsilon$. □

2) If $(x_n) \to x$, show that $\left(\sqrt{x_n}\right) \to \sqrt{x}$.

Proof. Consider the following equation and apply the fact that $x_n \geq 0$ and $\sqrt{x_n} + \sqrt{x} > \sqrt{x}$, and then the definition of convergence to show the identity:

$$\left|\sqrt{x_n} - \sqrt{x}\right| \tag{228}$$

$$= \left|\sqrt{x_n} - \sqrt{x}\right| \cdot \frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} \tag{229}$$

$$= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} < \frac{|x_n - x|}{\sqrt{x}} < \frac{\varepsilon}{\sqrt{x}} \tag{230}$$

Therefore, $|x_n - x| < \varepsilon$, hence proved. □

5. Consider the sequence (b_n) where $b_n = \sqrt{n^2 + 2n} - n$. Prove that (b_n) is convergent and find its limit.

6. Give an example of each of the following:

1) Sequences (a_n) and (b_n), which both diverge, but whose sum $(a_n + b_n)$ converges.

   Consider $(a_n) = n$ and $(b_n) = -n$

2) Sequences (a_n) and (b_n), which both diverge, but whose product $(a_n b_n)$ converges.

   Consider $(a_n) = (-1)^n$ and $(b_n) = (-1)^{n+2}$

3) Convergent sequences (a_n) and (b_n) with $a_n < b_n$ for all $n \in \mathbb{N}$ such that $\lim(a_n) = \lim(b_n)$.

   Consider $(a_n) = \left(\frac{1}{4}\right)^n$ and $(b_n) = \left(\frac{1}{2}\right)^n$

4) A convergent sequence (b_n) with $b_n \neq 0$ for all $n \in \mathbb{N}$, such that $(1/b_n)$ diverges.

   Consider $(b_n) = \frac{1}{n}$

5) Two sequence (a_n) and (b_n) so that (a_n) is unbounded, (b_n) is bounded, and $(a_n b_n)$ converges.

   Consider $(a_n) = n$ and $(b_n) = \frac{1}{n}$

6) Two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

   Consider $(a_n) = \frac{1}{n}$ and $(b_n) = n$

7. Let (a_n) be a bounded (not necessarily convergent) sequence, and assume that $\lim b_n = 0$. Show that $\lim a_n b_n = 0$. Why are we not allowed to use the Algebraic limit theorem to prove this?

Proof. From the triangle inequality, we have

$$|a_n b_n - 0| = |a_n|\,|b_n| = |a_n|\,|b_n - 0| \tag{231}$$

Since a_n is bounded, we know that there exists an M such that $a_n \leq M$ for all $n \in \mathbb{N}$. Therefore

$$|a_n b_n - 0| = |a_n|\,|b_n - 0| < M\,|b_n - 0| \tag{232}$$

Finally, we know that for all $\varepsilon > 0$, $|b_n - 0| < \varepsilon$, so, from the definition of convergence, $|b_n - 0| < \frac{\varepsilon}{M}$ and

$$|a_n b_n - 0| = |a_n|\,|b_n - 0| < M\,|b_n - 0| < M\frac{\varepsilon}{M} = \varepsilon \tag{233}$$

Therefore, $\lim a_n b_n = 0$. Notice that we could not use the Algebraic limit theorem because that theorem requires both a_n and b_n to converge. We know that b_n converges, but we are only given that a_n is bounded, so it is not necessarily convergent. □

Tuesday 19 February 2025

Lecture 20: 02-19-25 Lecture

Last time we showed that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{234}$$

converges. Now we will do a slightly different problem.

Example. $\sum_{n=1}^{\infty} \frac{1}{n}$

The partial sums are

$$S_m = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m} \tag{235}$$

Observe that (S_m) is an increasing sequence. To prove the statement, we will show that (S_m) is *not* bounded.

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) \tag{236}$$

$$S_8 \tag{237}$$

$$S_{16} \tag{238}$$

$$S_{32} \tag{239}$$

$$S_{2^k} \tag{240}$$

for $k \in \mathbb{N}$ we have

$$S_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \ldots + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{k-1}+1} + \ldots + \frac{1}{2^k}\right) \tag{241}$$

$$S_{2^k} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \ldots + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^k} + \ldots + \frac{1}{2^k}\right) \tag{242}$$

$$= 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \ldots + 2^{k-1}\frac{1}{2^k} \tag{243}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} \tag{244}$$

$$= 1 + k\left(\frac{1}{2}\right) \tag{245}$$

$$\Rightarrow S_{2^k} > 1 + \frac{k}{2} \tag{246}$$

As the sequence $\left(1 + \frac{k}{2}\right)_{k=1}^{\infty}$ is not bounded. Therefore $(S_m)_{m=1}^{\infty}$ is not bounded. Therefore $(S_m)_{m=1}^{\infty}$ is not convergent. Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Theorem 54. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{247}$$

converges for $p > 1$ and diverges for $p \leq 1$

Proof. See textbook. □

Bolzano was a priest who first came up with the definition of the limit that we have been using. We will see some theorems named after him in this section.

Definition 34. Let (a_n) j be a sequence of real numbers and let $n_1 < n_2 < n_3 < \ldots$ be an increasing sequence of natural numbers, then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, \ldots) \tag{248}$$

is called a **subsequence** of (a_n) and is denoted by (a_{n_k}) where $k \in \mathbb{N}$ indexes the subsequence.

Example. Let $a_n = n^2$ i.e.

$$(a_n) = (1, 4, 9, 16, 25, \ldots) \tag{249}$$

Let $a_{n_k} = (2k)^2$

$$(a_{n_k}) = (4, 16, 36, \ldots) \tag{250}$$

(a_{n_k}) is a subsequence of (a_n). Here $n_k = 2k$

$$\left(6^2, 11^2, 16^2, 21^2, \ldots\right) \text{ is also a subsequence} \tag{251}$$

$$\left(2^2, 2^2, 2^2, 3^2, 4^2, 5^2, \ldots\right) \text{ is } \textit{not} \text{ a subsequence.} \tag{252}$$

The original sequence is

$$(1, 2, 2, 2, 3, 4, 5, 6, \ldots) \tag{253}$$

then

$$(2, 2, 2, 3, 4, 5, 6, \ldots) \tag{254}$$

is a subsequence.

Theorem 55. All sub sequences of a convergent sequence converge to the same limit as the original sequence.

Proof. Assume that $\lim a_n = a$ and let (a_{n_k}) be a subsequence. Let $\varepsilon > 0$ be given, then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have

$$|a_n - a| < \varepsilon \tag{255}$$

For $k \geq N$ we observe that $n_k \geq k \geq N$. Therefore

$$|a_{n_k} - a| < \varepsilon \tag{256}$$

$\square$

Note. The crucial thing to realize in the above is that a_{n_k} is indexed by k the n_k is just there for emphasis.

Example. Let $0 < b < 1$, then $\lim b^n = 0$

Proof. Observe that

$$b > b^2 > b^3 > \ldots > 0 \tag{257}$$

Therefore, (b^n) is a decreasing sequence which is bounded. Then, by MCT, this sequence converges. Let $L \in \mathbb{R}$ such that $\lim_{n\to\infty} b_n = L$ Observe that for all $n \in \mathbb{N}\,(b \geq b^n)$. Therefore, by order limit theorem

$$1 > b \geq L \tag{258}$$

Similarly, $b^n \geq 0 \;\forall n \in \mathbb{N}$. Therefore $L \geq 0$. Therefore $0 \leq L \leq 1$ Look at the subsequence $\left(b^{2n}\right) = \left(b^2, b^4, b^6, \ldots\right)$. Therefore, $\lim_{n\to\infty} b^{2n} = L$. Notice

$$b^n b^n = b^{2n} \tag{259}$$

$$a_n \cdot b_n \tag{260}$$

Therefore, by the ALT we have

$$L \cdot L = L \tag{261}$$
$$L^2 = L \tag{262}$$
$$L = 0 \text{ or } L = 1 \tag{263}$$

But as $0 \leq L < 1$. Therefore, $L = 0$ and $\lim_{n \to \infty} b^n = 0$ □

Lecture 21: 02-21-25 Lecture

Friday 21 February 2025

Example. Let $a_n = (-1)^n$. Prove that (a_n) diverges.

Proof. Already done once in class. Now we present a second proof.

$$(a_n) = (-1, 1, -1, 1, -1, 1, \ldots). \tag{264}$$

Observe that $(1, -1, 1, -1, 1, -1, \ldots)$ is a subsequence of (a_n) and this subsequence has a limit of 1. We also observe that $(-1, 1, -1, 1, -1, \ldots)$ is a subsequence of (a_n) and this subsequence has limit -1. We have found that the sequence has subsequences which converge to different limits, therefore (a_n) diverges. □

Very fundamental theorem:

Theorem 56 (Bolzano-Weirstrass Theorem)**.** Every bounded sequence contains a convergent subsequence.

Proof. Let (a_n) be a bounded sequence. Hence there exists $M > 0$ such that $|a_n| \leq M \ \forall n \in \mathbb{N}$. Let $a_{n_1} = a_1$ and let $I_1 = [-M, M]$. Now, we sketch the plan for the rest of the proof:

1) We divinde I_1 into two intervals $[-M, 0]$ and $[0, M]$
2) At least one of these closed intervals must contain an infinite number of terms in the sequence (a_n). Call this interval I_2.
3) Let a_{n_2} be such that $a_{n_2} \in I_2$ and $n_2 > n_1 = 1$

We repeat this procedure inductively, so if $a_{n_k} \in I_k$, then

1) Divide the interval I_k into two equal closed intervals.
2) Let I_{k+1} be a closed interval such that it contains infinite number of terms of the sequence (a_n).
3) Let $a_{n_{k+1}}$ be such that $a_{n_{k+1}} \in I_{k+1}$ and $n_{k+1} > n_k$

This gives us a subsequence a_{n_k} with $a_{n_k} \in I_k$ and

$$I_1 \subset I_2 \subset I_3 \subset \ldots \text{(FIX DIRECTION OF SUBSET)} \tag{265}$$

By the nester interval property, $\exists x \in \cap_{k=1}^{\infty} I_k$

Claim: $\lim_{k \to \infty} a_{n_k} = x$

Proof. Let $\varepsilon > 0$ be given. Observe from construction the length of I_k is $(2M) \cdot 2^{-(k-1)}$. We know that $\lim_{k\to\infty} (2M)\, 2^{-(k+1)} = 0$ by the ALT since

$$\lim_{k\to\infty} \left(\frac{2M}{2^k \cdot 2^{-1}} \right) \tag{266}$$

$$\frac{1}{2^n} \to 0 \tag{267}$$

□

So choose $N \in \mathbb{N}$ such that the lenth of I_N is less than ε. Therefore, for all $k \geq N$ we observe that $a_{n_k} \in I_N$ and hence $x \in I_N$. Therefore for all $k \geq N$

$$|a_{n_k} - x| < \varepsilon \tag{268}$$

Hence proved. □

Friday 21 February 2025

Lecture 22: Homework 6

1. Give an example of each of the following:

   a) A monotone sequence that diverges.

      $(x_n) = n$

   b) A sequence that has no convergent subsequence.

      $(x_n) = n$

   c) A divergent sequence that has a convergent subsequence.

      We need a sequence which is bounded, but diverges:

      Consider $(x_n) = (-1)^n$ and notice that (x_{n_k}) converges when $n_k = 2k$.

   d) A sequence that does not contain 0 or 1 as a term, but contains subsequences converging to each of these values.

      $(x_n)_{n=2}^{\infty} = 1/n$

   e) A sequence containing subsequences converging to every point in $\mathbb{R}$

      We need a sequence which visits every point in $\mathbb{R}$ infinitely many times so that we can construct our sequence. Take

      $$(x_n) = \tan(n) \tag{269}$$

      Now if we define the subsequence (x_{n_k}) where $n_k = \frac{1}{k} + \tan^{-1}(\alpha) + 2\pi k$. Then we have that $\lim x_{n_k} = \alpha$ for all $\alpha \in \mathbb{R}$.

2. Consider the sequence (a_n) defined as follows: Let $a_1 = \sqrt{2}$ and for $n \geq 1$ let

$$a_{n+1} = \sqrt{2 + a_n} \tag{270}$$

Hence the sequence is

$$\left(\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots \right) \tag{271}$$

Prove that (a_n) converges and find its limit. (Hint: First using induction prove that $1 < a_n < 2$ for all $n \in \mathbb{N}$. Then show that (a_n) is a monotonic bounded sequences and then find its limit)

Proof. We will show that (a_n) converges by showing that it is monotonic and bounded. We will show each inductively starting with monotonicity:

- (Base Case) when $n = 1$ we have $\sqrt{2} \overset{?}{\leq} \sqrt{2+\sqrt{2}}$ which is true.
- (Inductive Step) Assume that $a_{n+1} \geq a_n$, then $\sqrt{2+a_{n+1}} \geq \sqrt{2+a_n} \Rightarrow a_{n+2} \geq a_{n+1}$.

Now we will inductively prove that the sequence is bounded by 2:

- (Base Case) when $n = 1$, we have $1 \overset{?}{\leq} \sqrt{2} \overset{?}{\leq} 2$ which is true.
- (Inductive Step) Assume that $1 < a_n < 2$, then $3 < a_n + 2 < 4$ and $\sqrt{3} < \sqrt{2+a_n} < 2$. Therefore $a_{n+1} < 2$.

We have shown that (a_n) is both monotone and bounded, therefore it converges. □

Proof. To show that $\lim (a_n) = 2$, we use a little trick. Consider

$$a_{n+1} = \sqrt{2+a_n}. \tag{272}$$

Since we already proved that $\lim (a_n)$ exists, we are justified in saying that:

$$\lim (a_{n+1}) = \lim \left(\sqrt{2+a_n}\right). \tag{273}$$

Notice that $\lim (a_{n+1}) = \lim (a_n)$, so let $l = \lim (a_{n+1}) = \lim (a_n)$. Then we have

$$l = \sqrt{2+l} \tag{274}$$

$$\Rightarrow l^2 - l - 2 = 0 \tag{275}$$

Thus, since we already proved that $a_n > \sqrt{3}$ for all $n \in \mathbb{N}$, we know that $l = \lim (a_n) = 2$. □

3. (AM-GM inequality) Prove the Arithmetic-Geometric mean inequality: For $x, y \geq 0$ show that $\frac{x+y}{2} \geq \sqrt{xy}$.

Proof. We are given that $x, y \geq 0$, so

$$x + y \geq 0 \tag{276}$$

$$\frac{x+y}{2} \geq 0 \tag{277}$$

$$\frac{x+y}{2} + \sqrt{xy} - \sqrt{xy} \geq 0 \tag{278}$$

$$\frac{x+y}{2} + \sqrt{xy} \geq \sqrt{xy} \tag{279}$$

$$\left(\frac{x+y}{2} + \sqrt{xy}\right)\left(\frac{x+y}{2} - \sqrt{xy}\right) \geq \sqrt{xy}\left(\frac{x+y}{2} - \sqrt{xy}\right) \tag{280}$$

$$\frac{(x+y)^2}{4} - xy \geq \sqrt{xy}\left(\frac{x+y}{2}\right) - xy \tag{281}$$

$$(x+y)(x+y) \geq 2\sqrt{xy}\,(x+y) \tag{282}$$

By examining the final equation, we see that it must be that $(x+y) \geq 2\sqrt{xy}$ so that the inequality holds. Hence proved. □

4. (Calculating square roots) Let $a_1 = 2$ and for $n \geq 1$ define

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \tag{283}$$

Prove that (a_n) converges and then show that $\lim (a_n) = \sqrt{2}$. (Hint: First using induction and the AM-GM inequality prove that $\sqrt{2} \leq a_n \leq 2$ for all $n \in \mathbb{N}$. Then show that (a_n) is a monotonic bounded sequence and then find its limit)

Remark. This gives us an algorithm to compute $\sqrt{2}$. This can be generalized to compute $\sqrt{x}$ and more generally $\sqrt[n]{x}$ for any $x > 0$.

Proof. First we will prove that the sequence converges by showing that it is monotonic and bounded inductively. We will start with monotonicity:

- (Base Case) When $n = 1$ we have $a_1 = 2$ and $a_2 = 3/2$. $3/2 > 2$ holds.
- (Inductive Step) Assume that the statement holds for a_n, then

$$a_n \leq a_{n+1} \tag{284}$$

$$\frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \leq \frac{1}{2}\left(a_{n+1} + \frac{2}{a_n}\right) \leq \frac{1}{2}\left(a_{n+1} + \frac{2}{a_{n+1}}\right) \tag{285}$$

$$a_{n+1} \leq a_{n+2} \tag{286}$$

Now we will show boundedness using the AM-GM inequality which we proved in problem 3. Consider:

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \tag{287}$$

Now we will show boundedness using the AM-GM inequality which we proved in problem 3. Consider $a_{n+1} = \frac{1}{2}(a_n) + \frac{2}{a_n}$, then we know by AM-GM Ineqiality that:

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \geq \sqrt{a_n \cdot \frac{2}{a_n}} = \sqrt{2} \tag{288}$$

(a_n) is monotonic and bounded, therefore it converges. ☐

Proof. We will use the same trick to find this limit. Since we know that the limit exists and is non-zero, we can feel confident in writing $l = \lim(a_n) = \lim a_{n+1}$, then

$$\lim a_{n+1} = \frac{1}{2}\left(\lim a_n + \frac{2}{\lim a_n}\right) \tag{289}$$

$$\Rightarrow l = \frac{1}{2}\left(l + \frac{2}{l}\right) \tag{290}$$

Solving the above gives $l = \lim(a_n) = \sqrt{2}$

☐

5. Consider a sequence $(a_n)_{n=1}^{\infty}$ defined by

$$a_n = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \tag{291}$$

Prove that $(a_n)_{n=1}^{\infty}$ is a convergent sequence. (Hint: prove that the sequence is bounded and monotonic).

6. Let (a_n) be a bounded sequence and let $a \in \mathbb{R}$. Show that if every converges subsequence of (a_n) converges to a, then the sequence (a_n) itself converges and $\lim a_n = a$

Proof. If every (a_{n_k}) converges to a, then the "trivial subsequence" where $n_k = k$ also converges to a. Now, assume for the sake of contradiction that $\lim(a_n) \neq a$, then $\exists \varepsilon > 0$ such that

$$|a_n - a| > \varepsilon \tag{292}$$

But that implies that there exists a subsequence of (a_n) which does not converge to a, i.e. the trivial subsequence, which contradicts our assumption that all subsequences of (a_n) converge to a. ☐

7. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \tag{293}$$

Prove that this series converges and find the limit. (See the proof of convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ done in class)

8. (Bonus problem. Will not be asked in the quiz or exams) Suppose the sequence $(a_n)_{n=1}^{\infty}$ converges to $a \in \mathbb{R}$. Prove that the sequence $(c_N)_{N=1}^{\infty}$ defined as $c_N = \frac{1}{N}\sum_{n=1}^{N} a_n$, satisfies $\lim\limits_{N\to\infty} c_N = a$

Monday 24 February 2025

Lecture 23: 02-24-25 Lecture

-Midterm: Next Friday in class.

Limsup and Liminf

Definition 35. Let (a_n) be a bounded sequence. Define the sequence (y_n) as

$$y_n = \sup\{a_k | k \geq n\} \tag{294}$$

Then we have that $y_1, \geq y_2 \geq y_3 \geq \ldots$ and (y_n) is a decreasing bounded sequence. Thus it cconverges by MCT. Define

$$\limsup a_n = \lim y_n \tag{295}$$

Similarly define the sequence (z_n) where

$$z_n = \inf\{a_k | k \geq n\} \tag{296}$$

Therefore $z_1 \leq z_2 \leq z_3 < \ldots$ and (z_n) is an increasing bounded sequence. Therefore it converges my MCT. Define

$$\lim\ \inf a_n = \lim z_n \tag{297}$$

Example. $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$

$$a_1 = (-1)\left(1 + \frac{1}{1}\right) = -1 - 1 \tag{298}$$

$$a_2 = 1\left(1 + \frac{1}{2}\right) = 1 + \frac{1}{2} \tag{299}$$

$$a_3 = -1\left(1 + \frac{1}{3}\right) = -1 - \frac{1}{3} \tag{300}$$

$$a_4 = 1\left(1 + \frac{1}{4}\right) = 1 + \frac{1}{4} \tag{301}$$

$$a_5 = -1\left(1 + \frac{1}{5}\right) = -1 - \frac{1}{5} \tag{302}$$

So we have

$$y_1 = 1 + \frac{1}{2} \tag{303}$$
$$y_2 = 1 + \frac{1}{2} \tag{304}$$
$$y_3 = 1 + \frac{1}{4} \tag{305}$$
$$y_4 = 1 + \frac{1}{4} \tag{306}$$
$$y_5 = 1 + \frac{1}{6} \tag{307}$$
$$y_6 = 1 + \frac{1}{6} \tag{308}$$

Therefore $\lim y_n = 1$, thus $\limsup a_n = 1$

$$z_1 = 1 - 1 \tag{309}$$
$$z_2 = -1 - \frac{1}{3} \tag{310}$$
$$z_3 = -1 - \frac{1}{3} \tag{311}$$
$$z_4 = -1 - \frac{1}{5} \tag{312}$$
$$z_5 = -1 - \frac{1}{5} \tag{313}$$

Therefore $\lim z_n = -1$ and $\lim\ \inf a_n = -1$

Example. $a_n = (-1)^n = -1, 1, -1, 1, \ldots$

$$\limsup = 1 \tag{314}$$
$$\lim\ \inf\ = -1 \tag{315}$$
$$\tag{316}$$

Theorem 57. Let (a_n) be a bounded sequence. The $\lim\ \inf a_n \leq \limsup a_n$. Moreover, $\lim\ \inf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case we have $\lim\ \inf a_n = \limsup a_n = \lim a_n$

Proof. 1) Observe that

$$\inf\{a_k | k \geq 0\} \leq \sup\{a_k | k \geq n\} \tag{317}$$

Therefore $z_n \leq y_n$ because $\lim z_n = \liminf a_n$ and $\lim y_n = \limsup a_n$, therefore $\liminf a_n \leq \limsup a_n$.

2) Assume that $\liminf a_n = \limsup a_n = a$. We want to show that $\lim a_n = a$. Let $\varepsilon > 0$ be given $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$

$$|z_n - a| < \varepsilon \Rightarrow a - \varepsilon < z_n < a + \varepsilon \tag{318}$$

Similarly, $\exists N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$

$$|y_n - a| < \varepsilon \Rightarrow a - \varepsilon < y_n < a + \varepsilon \tag{319}$$

Let $N = \max\{N_1, N_2\}$. Therefore for all $n \geq N$ we have

$$a - \varepsilon < z_n \leq a_n \leq y_n \leq a + \varepsilon \tag{320}$$
$$\Rightarrow a - \varepsilon < a_n < a + \varepsilon \tag{321}$$
$$\Rightarrow |a_n - a| < \varepsilon \tag{322}$$

Hence $\lim a_n = a$. We want to show that $\liminf a_n = \limsup a_n = a$. Let $\varepsilon > 0$ be given. Therefore $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have

$$|a_n - a| < \frac{\varepsilon}{2} \Rightarrow a - \frac{\varepsilon}{2} < a_n < a + \frac{\varepsilon}{2} \tag{323}$$

Now for all $n \geq N$ we have

$$y_n = \sup\{a_k | k \geq n\} \Rightarrow a - \frac{\varepsilon}{2} \leq y_n \leq a + \frac{\varepsilon}{2} \tag{324}$$
$$\Rightarrow |y_n - a| < \varepsilon \tag{325}$$

Therefore, $\limsup a_n = a$. Similarly for all $n \geq N$

$$z_n = \inf\ \{a_k | k \geq n\} \Rightarrow a - \frac{\varepsilon}{2} \leq z_n \leq a + \frac{\varepsilon}{2} \Rightarrow |z_n - a| < \varepsilon \Rightarrow \liminf a_n = a \tag{326}$$

□

Definition 36. Let (a_n) be a seqience we say that $\lim a_n = \infty$, if for any $M > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have

$$a_n > M \tag{327}$$

(we define $\lim a_n = -\infty$ in a similar fashion).

Example. Let $a_n = n^2 + 1$. Prove that $\lim a_n = \infty$

Proof. Let $M > 0$ be given. Choose $N \in \mathbb{N}$ such that

$$N > M \tag{328}$$

Then for all $n \geq N$ we have

$$n \geq N \Rightarrow n^2 + 1 \geq N > M \tag{329}$$
$$\Rightarrow n^2 + 1 > M \tag{330}$$

Hence proved. □

day 26 February 2025

Lecture 24: 02-26-25 Lecture

The Cauchy Criterion. This is very important.

Definition 37. A sequence (a_n) is called a **Cauchy sequence** if for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \varepsilon$.

In this lecture, we will build up to proving the following theorem:

Theorem 58. A sequence is Cauchy if and only if it is convergent.

Note. If a sequence converges, but we don't know how to find the value of the limit L, then using the above, we can still show that the sequence converges, and still use the convergent limit theorems. This will take some buildup.

Theorem 59. Every convergent sequence is a Cauchy sequence.

Proof. Let $L \in \mathbb{R}$ and let (a_n) be a sequence with $\lim_{n\to\infty} a_n = L$. Let $\varepsilon > 0$ be given. Therefore $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have

$$|a_n - L| < \frac{\varepsilon}{2} \tag{331}$$

If $m \geq N$ we also have

$$|a_m - L| < \frac{\varepsilon}{2} \tag{332}$$

Therefore if $m, n \geq N$ we have

$$\begin{aligned} |a_n - a_m| &\leq |a_n - L| + |L - a_m| && (333)\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && (334)\\ &= \varepsilon && (335) \end{aligned}$$

□

Theorem 60. Cauchy sequences are bounded.

Theorem 61. Let $\varepsilon = 1$ and let $N \in \mathbb{N}$ such that $\forall m, n \geq N$ we have

$$|a_m - a_n| < 1 \tag{336}$$

Let $m = N$. Therefore for all $n \geq N$ we have

$$|a_N - a_n| < 1 \tag{337}$$
$$\Rightarrow |a_n| \leq |a_n - a_N| + |a_N| < |a_N| + 1 \tag{338}$$

Let $M = \max\{|a_1|, |a_2|, |a_3|, \ldots, |a_{N-1}|, |a_N| + 1\}$. Then we see that

$$|a_n| \leq M \tag{339}$$

for all $n \in \mathbb{N}$. Hence (a_n) is bounded.

Theorem 62. A sequence converges if and only if it is Cauchy.

Proof. Let (x_n) be a Cauchy sequence. Therefore, (x_n) is a bounded sequence. By Bolzano-Weirstrauss theorem, there exists a convergent subsequence (x_{n_k}). Let

$$x = \lim_{k\to\infty} x_{n_k} \tag{340}$$

Now notice that what we want to claim is that $\lim\limits_{n\to\infty} x_n = x$. So let $\varepsilon > 0$ be given. $\exists N \in \mathbb{N}$ such that $\forall m, n \geq N$ we have

$$|x_n - x_n| < \frac{\varepsilon}{2} \tag{341}$$

There also exist $k \in \mathbb{N}$ such that $\forall k \geq K$ with $K \geq N$ such that $\forall n, m \geq N$ we have

$$|x_{n_k} - x| < \frac{\varepsilon}{2} \tag{342}$$

Observe that $n_k \geq K \geq N$. Hence

$$|x_{n_k - x}| < \frac{\varepsilon}{2} \tag{343}$$

and hence $\forall n \geq K$

$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| \tag{344}$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \tag{345}$$
$$= \varepsilon \tag{346}$$

Therefore

$$|x_n - x| < \varepsilon \tag{347}$$

For all $n \geq K$. Hence $\lim x_n = x$ ☐

Now starting back with series. . .

Properties of Infinite Series

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \ldots \tag{348}$$

Note. Remember the distinction between sequence and series..

How do we tell if a series if convergent? Construct a sequence of partial sums (s_m)

$$s_m - b_1 + b_2 + \ldots + b_m \tag{349}$$

If (s_m) converges, the we say that the series $\sum_{n=1}^{\infty} b_n$ converges.

If $\lim s_m = L$, the $\sum_{n=1}^{\infty} b_n = L$

Theorem 63 (Algebraic Limit Theorem for Series)**.** If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then

1) $\sum_{n=1}^{\infty} ca_n = cA$ for any $c \in \mathbb{R}$
2) $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

Note. The other two don't work!!!

Proof. Let (s_m) and (t_m) be the sequence of partial sums i.e.

$$s_m = a_1 + a_2 + \ldots + a_m \tag{350}$$
$$t_m = b_1 + b_2 + \ldots + b_m \tag{351}$$

Therefore $\lim\limits_{m\to\infty} s_m = A$ and $\lim\limits_{m\to\infty} t_m = B$. By the Algebraic Limit Theorem

$$\lim_{m\to\infty} cs_m = cA \tag{352}$$
$$\Rightarrow \sum_{n=1}^{\infty} ca_n = cA \tag{353}$$

Note. $ca_1 + ca_2 + \ldots + ca_m = cs_n$

Again by ALT

$$\sum_{n=1}^{\infty} (a_n + b_n) = A + B \tag{354}$$

Note. $(a_1 + b_1) + (a_2 + b_2) + \ldots + (a_m + b_m) = s_m + t_m$.

□

Lecture 25: 02-28-25 Lecture

Friday 28 February 2025

-Midterm next Friday!

Theorem 64 (Cauchy Criterion for Series)**.** The series $\sum_{n=1}^{\infty} a_n$ converges if and only if given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$ we have

$$|a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon \tag{355}$$

Proof. Let (s_m) be the sequence of partial sums. Observe that if $n > m$, then

$$|s_n - s_m| = |(a_1 + a_2 + \ldots + a_n) - (a_1 + a_2 + \ldots + a_m)| \tag{356}$$
$$= |a_{m+1} + a_{m+2} + \ldots + a_n| \tag{357}$$

Now the sequence (s_m) converges if and only if it is Cauchy (s_m) is Cauchy if given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n, m \geq N$ we have

$$|s_n - s_m| < \varepsilon \tag{358}$$

we can assume without loss of generality that $n > m$, i.e. $n > m \geq N$ then

$$|s_n - s_m| < \varepsilon \tag{359}$$

□

Theorem 65. If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim a_n = 0$

Proof. As $\sum_{n=1}^{\infty} a_n$ converges, it satisfies the Cauchy criterion. So let $\varepsilon > 0$ be given. By Cauchy criterion, there $\exists N \in \mathbb{N}$ such that if $n > m \geq N$ we have

$$|a_{m+1} + \ldots + a_n| < \varepsilon \tag{360}$$

Choose $n = m + 1$ then

$$|a_{m+1}| < \varepsilon \tag{361}$$

Thus $\forall n \geq N + 1$ we have

$$|a_n - 0| < \varepsilon \tag{362}$$

□

Example. $\sum_{n=1}^{\infty} (-1)^n$ we observe that the sequence $(-1)^n$ does not converge to 0. Therefore, $\sum_{n=1}^{\infty} (-1)^n$ converges.

Theorem 66 (Comparison Test)**.** Assume that (a_n) and (b_n) are sequences satisfying $0 \leq a_n \leq b_n$ $\forall n \in \mathbb{N}$, then

1) If $\sum_{n=1}^{\infty} b_n$ converges, the $\sum_{n=1}^{\infty} a_n$ also converges.
2) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

Note. Notice that (1) and (2) are contrapositives, we only really need one, but both are useful.

Proof. As $\sum_{n=1}^{\infty} b_n$ converges, it satisfies the Cauchy criterion. We **want to show**

$$\sum_{n=1}^{\infty} a_n \text{ satisfies the Cauchy criterion} \tag{363}$$

Let $\varepsilon > 0$ be given. There exist $N \in \mathbb{N}$ such that whenever $n > m \geq N$ we have

$$|b_{m+1} + \ldots + b_n| < \varepsilon \tag{364}$$

But $0 \leq a_k \leq b_k$ so

$$\Rightarrow |a_{m+1} + \ldots + a_n| < \varepsilon \tag{365}$$

Hence proved. □

Example. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \leq 1$

Proof. We have proved that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Observe that for all $n \in \mathbb{N}$ and $p \leq 1$ we have

$$0 < \frac{1}{n} \leq \frac{1}{n^p} \tag{366}$$

Therefore, by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \leq 1$. □

Example (Geometric Series)**.**

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots \tag{367}$$

is called a geometric series

$$s_m = a + ar + \ldots + ar^{m-1} \tag{368}$$

by induction we can prove that for $r \neq 1$

$$s_m = \frac{a\left(1 - r^m\right)}{1 - r} \tag{369}$$

If $|r| < 1$, then $\lim r^n = 0$. By the ALT, $\lim s_m = \frac{a}{1-r}$

Lecture 26: 03-03-25 Lecture

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Theorem 67 (Absolute converges test)**.** If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} |a_n|$ also converges.

Proof. It is enough to show that $\sum a_n$ satisfies the Cauchy criterion. Let $\varepsilon > 0$ be given. Hence $\exists N \in \mathbb{N}$ such that whenever $n > m \geq N$ we have

$$||a_{m+1}| + |a_{m+2}| + \ldots + |a_n|| < \varepsilon \tag{370}$$
$$\Rightarrow |a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon \tag{371}$$

Hence proved. □

Example.

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2} \tag{372}$$

Observe that

$$\frac{\sin(n)}{n^2} \leq \frac{1}{n^2} \tag{373}$$

As $\sum \frac{1}{n^2}$ converges, by the comparison test $\sum \left|\frac{\sin(n)}{n^2}\right|$ converges. Therefore by the absolute convergence test $\sum \frac{\sin(n)}{n^2}$ also converges.

Definition 38. Consider the series $\sum a_n$

1) If $\sum |a_n|$ converges, then we say that the series $\sum a_n$ converges absolutely.
2) If the series $\sum a_n$ converges, but $\sum |a_n|$ diverges, then we say that $\sum a_n$ converges conditionally.

Example.

$$\sum \frac{\sin(n)}{n^2} \tag{374}$$

By previous logic, the series converges absolutely.

Example.

$$\sum \frac{(-1)^n}{n} \tag{375}$$

converges by the alternating series test but $\sum \frac{1}{n}$ diverges. Therefore $\sum \frac{(-1)^n}{n}$

Note. The conditional convergence is the most painful.

Theorem 68 (Alternating series test)**.** Let (a_n) be a sequence satisfying

1) $a_1 \geq a_2 \geq a_3 \geq \ldots$ and $a_n \geq 0 \; \forall n \in \mathbb{N}$
2) $\lim a_n = 0$

Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof. Consider the partial sum s_m

$$s_m = a_1 + a_2 + \ldots + (-1)^{m+1} a_m \tag{376}$$

Consider the subsequence $(s_{2n})_{n=1}^{\infty} = (s_2, s_4, s_6, \ldots)$

$$s_2 = a_1 - a_2 \tag{377}$$

$$s_4 = a_1 - a_2 + a_3 - a_4 = a_1 - (a_2 - a_3) - a_4 \leq a_1 \tag{378}$$

Therefore

$$s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} \geq 0 \tag{379}$$

Therefore $(s_{2n})_{n=1}^{\infty}$ is an increasing sequence

$$s_{2n} = a_1 - a_2 + a_3 \ldots - a_{2n-2} + a_{2n-1} - a_{2n} \tag{380}$$

$$= a_1 - (a_2 - a_3) - \ldots - (a_{2n-2} - a_{2n-1}) - a_{2n} \tag{381}$$

$$\leq a_1 \tag{382}$$

(s_{2n}) is an increasing sequence which is bounded above and hence by MCT it converges. Let $L \in \mathbb{R}$ such that

$$\lim_{n \to \infty} s_{2n} = L \tag{383}$$

Now I claim that $\lim s_n = L$. To show this, let $\varepsilon > 0$ be given. $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$ we have

$$|s_{2n} - L| < \frac{\varepsilon}{2} \tag{384}$$

As $\lim a_n = 0$, $\exists N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$

$$|a_n| < \frac{\varepsilon}{2} \tag{385}$$

Let $N = \max\{N_2, N_1\}$. Hence $\forall n \geq N$ we have

- (Case 1: n is even)

$$|s_n - L| < \frac{\varepsilon}{2} < \varepsilon \tag{386}$$

- (Case 2: n is odd)

$$|s_n - L| \leq |s_n - s_{n+1}| + |s_{n+1} - L| \tag{387}$$

$$\leq |a_{n+1}| + \frac{\varepsilon}{2} \tag{388}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \tag{389}$$

$$= \varepsilon \tag{390}$$

□

Example.

$$a_n = \frac{1}{n} \tag{391}$$

By the AST $\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots$ converges.

Theorem 69 (Ratio Test)**.** Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n \neq 0$ $\forall n \in \mathbb{N}$. Assume that $\exists r$ such that $0 \leq r < 1$ and

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r \tag{392}$$

Then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Proof. Given in homework. □

Example. For some $x \in \mathbb{R}$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{393}$$

If $x = 0$, then

$$1 + 0 + 0 + 0 + \dots \tag{394}$$

If $x \neq 0$, the $a_n = \frac{x^n}{n!}$ is non-zero.

$$\left| \frac{a_n + 1}{a_n} \right| = \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \left| \frac{x}{n+1} \right| \tag{395}$$

By ALT $\lim\limits_{x\to\infty} \left| \frac{x}{n+1} \right| = 0 = r$. Therefore, by the ration test, this series converges absolutely $\forall x \in \mathbb{R}$.

Tuesday 05 March 2025

Lecture 27: 03-05-25

4 Basic Topology of $\mathbb{R}$

Definition 39. Given $a \in \mathbb{R}$ and $\varepsilon > 0$, the ε-neighborhood of a is the set $V_\varepsilon(a)$ defined as

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\} \tag{396}$$

i.e. $V_\varepsilon(a) = (a - \varepsilon, a + \varepsilon)$.

Definition 40. A set $O \subset \mathbb{R}$ us called **open** if for every $a \in O$, there exists $\varepsilon > 0$ such that $V_\varepsilon(a) \subset O$

Example. $(0, 1)$ is an open set

$[0, 1]$ is not an open set.

$\mathbb{R}$ is an open set

$(0, 1) \cup (3, 4)$ is an open set

Note. the union of open sets is open

Theorem 70. We have

1) The union of an arbitrary collection of open sets is open.
2) The intersection of a finite collection of open sets is open.

Proof.

1) Let $\{O_\lambda : \lambda \in \Lambda\}$ be a collection of open sets. If $a \in \cup_{\lambda\in\Lambda} O_\lambda$, $\exists \alpha \in \Lambda$ such that $a \in O_\alpha$. Therefore as O_α is open, $\exists\varepsilon > 0$ such that $V_\varepsilon(a) \subset O_\alpha$. Therefore $V_\varepsilon(a) \subset \cup_{x\in\Lambda} O_\lambda$. Hence $\cup_{\lambda\in\Lambda} O_\lambda$ is open.
2) Let $\{O_1, O_2, O_3, \ldots, O_N\}$ be an open set. Let $a \in \cap_{i=1}^N O_i$. Therefore $a \in O_i$ for each $1 \le i \le N$. Then $\exists\varepsilon_i > 0$ such that $V_{\varepsilon_i}(a) \subset O_i$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N\} > 0$. Therefore $V_\varepsilon(a) \subset O_i$ for all $1 \le i \le N$. Therefore $V_\varepsilon(a) \subset \cap_{i=1}^N O_i$

□

Definition 41. Let $A \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is called a **limit point of the set** A, if every ε-neighborhood, $V_\varepsilon(a)$ of x intersects the set A at some point other than x.

Note. Limit points are also called "cluster points" or "accumulation points" of a set.

Example. $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\}$

$x = 1$ is not a limit point because I can take an ε small enough to contain no elements in A.

$x = 0$ is a limit point of A

$A = (0, 1)$

Then if $0 \le x \le 1$ then x is a limit point.

Example. $\mathbb{Q}$. Every $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$.

Theorem 71. Let $A \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is a limit point of A if and only if there exists a sequence (a_n) such that $a_n \in A$ and $a_n \neq x\ \forall n \in \mathbb{N}$ and $\lim a_n = x$

- $(\Rightarrow\ )$ Let x be a limit point of A. Consider the set $V_{\frac{1}{n}}(x)$. There exists $a_n \in V_{\frac{1}{n}}(x) \cap A$ such that $a_n \neq x$. Consider (a_n). $|a_n - x| < \frac{1}{n}$, $a_n \in A$, $a_n \neq x\ \forall n \in \mathbb{N}$. Given some $\varepsilon > 0$, let $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon \Rightarrow \forall n \geq N$, $|a_n - x| < \varepsilon$
- $(\Leftarrow)$ Let (a_n) be a sequence with $a_n \in A$,