Real Analysis 1

Forrest Kennedy

February 14, 2025

Contents

Lecture 16: 02-12-25 Lecture

Wednesday 12 February 2025

Example. Prove that (a_n) where $a_n = n^2$ is divergent.

Proof. Assume by contradiction that (a_n) is convergent. Therefore (a_n) is a bounded sequence, so $\exists M > 0$ such that

$$\forall n \in \mathbb{N}(|a_n| \leq M) \tag{1}$$
$$\forall n \in \mathbb{N}\left(n^2 \leq M\right) \tag{2}$$
$$\tag{3}$$

Let $N \in \mathbb{N}$ be such that $N > M$ then

$$N^2 > MN \geq M \tag{4}$$
$$N^2 > M \tag{5}$$

this is a contradiction! □

This next theorem is used all the time.

Theorem 1. Suppose (a_n) is a convergent sequence with $\lim a_n = L$. If $L \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $\exists \delta > 0$ such that $|a_n| \geq \delta > 0$ for all $n \in \mathbb{N}$.

Proof. As $L \neq 0$, choose $\varepsilon = \frac{|L|}{2} > 0$ $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have

$$|a_n - L| < \frac{|L|}{2} \tag{6}$$
$$\tag{7}$$

for $n \geq N$ we have

$$|L| \leq |L - a_n| + |a_n| \leq \frac{|L|}{2} + |a_n| \tag{8}$$
$$\tag{9}$$

Therefore, for all $n \geq N$ we have

$$\frac{|L|}{2} \leq |a_n| \tag{10}$$

$$\tag{11}$$

Define $\delta = \min\{|a_1|, |a_2|, \ldots, |a_{N-1}|, \left|\frac{L}{2}\right|\} > 0$. We see that $|a_n| \geq \delta > 0$ $\forall n \in \mathbb{N}$. ☐

Theorem 2 (Algebraic limit theorem)**.** Let $a, b \in \mathbb{R}$ and let $\lim a_n = a$ and $\lim b_n = b$. Then

1. $\lim (ca_n) = ca$ for all $c \in \mathbb{R}$
2. $\lim (a_n + b_n) = a + b$
3. $\lim (a_n b_n) = ab$
4. If $b \neq 0$ and $b_n \neq 0$ $\forall n \in \mathbb{N}$, then $\lim \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$

Example. Given $a_n = \frac{3n^2+5}{n^2+10}$. Prove $\lim a_n = 0$

Example.

$$a_n = \frac{n^2\left(3 + \frac{5}{n^2}\right)}{n^3\left(1 + \frac{10}{n^3}\right)} \tag{12}$$

$$\frac{1}{n} \cdot \frac{3 + \frac{5}{n^2}}{1 + \frac{10}{n^3}} \tag{13}$$

We know that $\lim \frac{1}{n} = 0$

$$\Rightarrow \lim \frac{1}{n^2} = 0 \tag{14}$$

$$\Rightarrow \lim \frac{5}{n^2} = 0 \Rightarrow \lim \left(3 + \frac{5}{n^2}\right) = 3 \tag{15}$$

Proof. We will consider each case in turn:

1. If $c = 0$ then $ca_n = 0$ $\forall n \in \mathbb{N}$. Clearly $(ca_n) \to 0$ in this case. Let $\varepsilon > 0$ be given. Choose $N = 1$. Therefore, $\forall n \geq N$ we have

   $$|ca_n - ca| = |0 - 0| = 0 < \varepsilon \tag{16}$$

   Therefore $(ca_n) \to ca$ in this case

   Let $c \neq 0$ and let $\varepsilon > 0$ be given. Let $N \in \mathbb{N}$ be such that $\forall n \geq N$ we have

   $$|a_n - a| < \frac{\varepsilon}{|c|} \tag{17}$$

Sidebar: we want :

$$ca_n - ca < \varepsilon \tag{18}$$
$$|c|\,|a_n - a| < \varepsilon \tag{19}$$
$$|a_n - a| < \frac{\varepsilon}{|c|} \tag{20}$$

Therefore for $n \geq N$ we have

$$|ca_n - ca| \tag{21}$$
$$= |c|\,|a_n - a| \tag{22}$$
$$< |c|\,\frac{\varepsilon}{|c|} \tag{23}$$
$$= \varepsilon \tag{24}$$

Therefore $|ca_n - ca| < \varepsilon\ \forall n \geq N$ hence proved.

2. Sidebar: WTS

$$|(a_n + b_n) - (a + b)| < \varepsilon \tag{25}$$
$$|(a_n - a) + (b_n - b)| < \varepsilon \tag{26}$$

Now on to the actual proof:

Let $\varepsilon > 0$ be given. Let $N_1 \in \mathbb{N}$ be such that $\forall n \geq N_1$ we have

$$|a_n - a| < \frac{\varepsilon}{2} \tag{27}$$

Let $N_2 \in \mathbb{N}$ be such that $\forall n \geq N_2$ we have

$$|b_n - b| < \frac{\varepsilon}{2} \tag{28}$$

Let $N = \max\{N_1, N_2\}$. Therefore for all $n \geq N$ we have

$$|(a_n + b_n) - (a + b)| \tag{29}$$
$$= |(a_n - a) + (b_n - b)| \tag{30}$$
$$\leq |a_n - a| + |b_n - b| \tag{31}$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \tag{32}$$
$$= \varepsilon \tag{33}$$

Therefore, $|ca_n - ca| < \varepsilon\ \forall n \geq N$ hence proved.

□

Lecture 17: 02-14-25 Lecture

Friday 14 February 2025

Example. $a_n = \frac{1}{n^2+10}$ and $\lim a_n = 0$

$$a_n = \frac{1}{n^2\left(1+\frac{10}{n^2}\right)} \tag{34}$$
$$\left(\frac{1}{n^2}\right)\frac{1}{\left(1+\frac{10}{n^2}\right)} \tag{35}$$

We know that $\lim \frac{1}{n} = 0$ so

$$\text{(By ALT)}\ \lim \frac{1}{n^2} = 0 \tag{36}$$
$$\lim \left(1+\frac{1}{n^2}\right) = 1 \tag{37}$$
$$\lim \frac{1}{1+\frac{10}{n^2}} = 1 \tag{38}$$
$$\lim \frac{1}{n^2}\cdot\frac{1}{\left(1+\frac{10}{n^2}\right)} = 0. \tag{39}$$

Hence proved.

Theorem 3. Let $a, b \in \mathbb{R}$ and $\lim a_n = a$ and $\lim b_n = b$.

1. If $a_n \geq 0\ \forall n \in \mathbb{N}$, then $a \geq 0$
2. If $a_n \leq b_n\ \forall n \in \mathbb{N}$, then $a \leq b$
3. If $\exists c \in \mathbb{R}$ such that $c \leq b_n\ \forall n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ $\forall n \in \mathbb{N}$, then $a \leq c$.

Proof. By contradiction, assume that $a < 0$, therefore $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have

$$|a_n - a| < \frac{|a|}{2} \Rightarrow a_n - a < \frac{|a|}{2} \tag{40}$$
$$\Rightarrow a_n < a + \frac{|a|}{2} < 0 \tag{41}$$
$$\Rightarrow a_n < 0\ \forall n \geq \mathbb{N}. \tag{42}$$

A contradiction! $\square$