

# Real Analysis 1

Forrest Kennedy

January 26, 2025

## Contents

1 The Heat Equation	1
---------------------	---

### Lecture 1: Some historical motivations for Analysis

Monday 13 January 2025

## 1 The Heat Equation

In 1822, Fourier derived the heat equation. In one dimension:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0.$$

where  $u(x, t)$  is the temperature as a function of position and time. A natural problem to solve with the equation is to assume you are given a function  $u(x, 0)$  which represents the initial temperature distribution of the system which we could measure then ask if it is possible to find a general  $u(x, t)$  given  $u(x, 0)$ . Stated another way, if we know the initial temperature distribution, can we find the distribution at an arbitrary time  $t$  using only the heat equation. The answer to this question is yes!

If we assume that  $u(x, 0)$  is a periodic function then:

$$u(x, 0) = \sum_{n \in \mathbb{Z}} a_n e^{inx}.$$

Then, rearranging and integrating:

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} u(x, 0) e^{-inx} dx.$$

Then, we will guess that the solution is of the form:

$$u(x, t) = \sum_{n \in \mathbb{Z}} a_n(t) e^{inx}.$$

Substituting into the heat equation we get:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sum_{n \in \mathbb{Z}} a'_n(t) e^{inx} - a_n(t) (in)^2 e^{inx} = \sum_{n \in \mathbb{Z}} (a'_n(t) + a_n(t) n^2) e^{inx} = 0.$$

So we have the differential equation:

$$a'_n(t) + a_n(t) n^2 = 0.$$

which has the solution:

$$a_n(t) = a_n(0) e^{-n^2 t}.$$

We can find  $a_n(0)$  with the integral above, so we have our solution! If we check the solution experimentally, we see the right behavior, so what's the problem? The issue is:

$$u(x, 0) \neq \sum_{n \in \mathbb{Z}} a_n e^{inx}$$

At least, when we look at the graph for any specific  $n \in \mathbb{N}$  we see that at the extreme points of the function, we get oscillations away from the true value of  $u(x, 0)$ . This affect is called the Gibbs phenomenon. This tells us that the function cannot equal the partial sum. On the other hand, the assumption works, so there must be some kind of notion of equality, but we are not in a position to say what that is right now. This gives us a specific example where Newton's calculus fails. The issue in this example has to do with the definition of "convergence" but there is a deeper issue. When we wrote down  $u(x, 0) = \sum_{n \in \mathbb{Z}} a_n e^{inx}$ , we were writing down nonsense, but we didn't know it. In order to know which statements are valid and which are not, we need to develop an axiomatic system that we can use to build up definitions, theorems, and proofs. This is the buisness of Mathematical Analysis: to provide a rigorous base for analysis to rest upon which contains no nonsense!

Wednesday 15 January 2025

## Lecture 2

Once we realize that the we don't know how to add up infinitely many functions, it is easy to see that we don't really know how to add infinitely many **numbers** either! Consider:

$$\begin{aligned} 1 &= 1 \\ 1 - 1 &= 0 \\ 1 - 1 + 1 &= 1 \\ 1 - 1 + 1 - 1 &= 0 \\ &\vdots \\ 1 - 1 + 1 - 1 + 1 + \dots &= ? \end{aligned}$$

Already there seems to be a problem! The series does seem to converge to any number. Of course, we don't know what converge means yet, but there is a deeper problem. Consider the rearrangements:

$$1 + (-1 + 1) + (-1 + 1) + \dots = 1$$

$$(1 + -1) + (1 + -1) + (1 + \dots = 0$$

Clearly, these can't both be right. Again, we have been tricked into writing nonsense because we don't have any axioms to tell us which statements are allowed and which are not. Here, the problem has to do with our adding up of an infinite number of things. When we are properly automatized, we will see that we just don't do that. Instead, we will solve this problem with a "limit," **in order to understand the limit, we will need to develop  $\mathbb{R}$ , the real number system.** This is the goal of Chapter 1. The point is: adding up infinite things, whether they are functions or just numbers, leads to problems, and whatever formal system we come up with will need to be without these problems if we want it to formalize calculus, which is based around the notion of adding up infinitely many things.

## Chapter 1

To motivate the definition of  $\mathbb{R}$  let's explore ways in which  $\mathbb{R}$  is different to other number systems. Why should we expect the definition of  $\mathbb{R}$  to be useful and lead us to a notion of a "continuum?"

**Lemma 1.** For all  $m, n \in \mathbb{Z}$ , if  $n|m^2$  and  $n$  is prime, then  $n|m$

*Proof.* Assume for the sake of contradiction that  $n$  does *not* divide  $m$ , then  $n$  cannot be a prime factor of  $m$ , so  $m = abcd\dots$  for some prime numbers  $a, b, c, d, \dots$ , importantly  $n$  **cannot be part of the product since  $n$  does not divide  $m$ .** Then  $m^2 = (abcd\dots)^2 = a^2b^2c^2d^2\dots$  which does not contain  $n$ , so  $n \nmid m^2$  which contradicts our assumption. Thus it must be the case that  $n|m$ .  $\square$

**Theorem 1.** There is no rational number whose square is 2

*Proof.* Assume for contradiction:  $\exists r$  s.t.  $r = \frac{p}{q}$  where  $p, q \in \mathbb{Z}$  and  $r^2 = 2$ . Also assume, WLOG, that  $p, q$  **share no common factors** then:

$$r^2 = \left(\frac{p}{q}\right)^2 = 2 \tag{1}$$

$$\Rightarrow p^2 = 2q^2 \tag{2}$$

$$\Rightarrow 2|p^2 \tag{3}$$

$$\Rightarrow 2|p \text{ (by Lemma 1)} \tag{4}$$

$$\Rightarrow p = 2n \text{ where } n \in \mathbb{Z} \tag{5}$$

$$\Rightarrow (2n)^2 = 4n^2 = 2q^2 \text{ (from Eq. 2)} \tag{6}$$

$$\Rightarrow 2|q^2 \Rightarrow 2|q \tag{7}$$

Thus  $2|p$  and  $2|q$  which violates our assumption that  $p, q$  share no common factors! Thus it must be the case that there is no rational number whose square is 2.

□

**Theorem 2.** If  $n \in \mathbb{N}$  and  $n$  is **not** a perfect square, then there is no  $r \in \mathbb{Q}$  such that  $r^2 = n$

*Proof.* Assume for contradiction:  $\exists r$  s.t.  $r = \frac{p}{q}$  where  $p, q \in \mathbb{Z}$ ,  $\gcd(p, q) = 1$ ,  $r^2 = n$ , and  $n$  is not a perfect square.

$$r^2 = \left(\frac{p}{q}\right)^2 = n \quad (8)$$

$$\Rightarrow p^2 = nq^2 \quad (9)$$

Recall that by the Fundamental Theorem of Arithmetic that we can express any number as a product of prime numbers, so:

$$n = k_1^1 \cdot k_2^2 \cdot k_3^3 \cdot k_4^4 \cdot \dots$$

Substitute  $n$  into Eq. 9:

$$p^2 = (k_1^1 \cdot k_2^2 \cdot k_3^3 \cdot k_4^4 \cdot \dots) q^2.$$

Since  $n$  is not a perfect square, we know that  $\exists j$  s.t.  $k_j$  is odd because if this were not the case,  $n$  would be a perfect square. From the above, we can see that  $k_j|p^2$ . If  $k_j$  divides the LHS, it must also divide the RHS, so  $k_j|nq^2$ . We know that  $p^2$  contains an even number of  $k_j$  terms, and we also know that  $n$  contains an odd number of  $k_j$  terms. For both sides to have the same number of  $k_j$  terms, as all equal numbers should, it must be the case that  $k_j|q^2$  which implies  $k_j|q$  by Lemma 1. Thus,  $\gcd(p, q) = k_j \neq 1$  which contradicts our assumption that  $\gcd(p, q) = 1$ . □

After this we talked about "set theory." We did not go into the details.

**Theorem 3.** The Algebra of Sets exists.  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$  exist.

*Proof.* The above is taken as an axiom, but rest assured that their existence can be derived from first-order logic and the ZFC axioms. □

Tuesday 21 January 2025

## Lecture Homework 1: Homework 1

1. Prove that there is no rational number,  $r$ , such that  $r^2 = 8$ .

*Proof.* BWOC, assume there exists a number  $r \in \mathbb{Q}$  such that for some  $a, b \in \mathbb{Z}$  with  $\gcd(a, b) = 1$ :

$$r^2 = \left(\frac{a}{b}\right)^2 = 8.$$

then:

$$\Rightarrow a^2 = 8b^2 \quad (10)$$

$$\Rightarrow 2^2|a^2 \Rightarrow 2|a^2 \Rightarrow 2|a \text{ (by Lemma 1)} \quad (11)$$

so for some  $n \in \mathbb{N}$  :

$$(2^2n)^2 = 8b^2 \quad (12)$$

$$16n^2 = 8b^2 \quad (13)$$

$$2n^2 = b^2 \Rightarrow 2|b \quad (14)$$

Notice  $2|a$  and  $2|b$  which violates our assumption that  $\gcd(a, b) = 1$ . Thus it must be the case that there does not exist a number  $r \in \mathbb{Q}$  such that  $r^2 = 8$

□

2. Prove that if  $a, b \in \mathbb{R}$  then:

$$||a| - |b|| \leq |a - b|.$$

*Proof.* If  $a, b \in \mathbb{R}$ , then the triangle inequality holds:

$$|a + b| \leq |a| + |b|.$$

Now consider:

$$|a| = |a - b + b| \leq |a - b| + |b| \quad (15)$$

$$|b| = |b - a + a| \leq |b - a| + |a| \quad (16)$$

$$(17)$$

Rearranging and using the definition of the absolute value function and the fact that  $|a - b| = |b - a|$ :

$$|a| - |b| \leq |a - b| \quad (18)$$

$$|b| - |a| \leq |b - a| \quad (19)$$

$$\Rightarrow ||a| - |b|| \leq |a - b| \quad (20)$$

□

3. Let  $y_1 = 6$  and for each  $n \in \mathbb{N}$  define

$$y_{n+1} = \frac{2}{3}y_n - 2.$$

Prove the following statements:

- Prove that  $y_{n+1} \leq y_n$  for all  $n \in \mathbb{N}$ .
- Prove that  $y_n > -6$  for all  $n \in \mathbb{N}$

*Proof.* We will show part a by induction. For the base case take  $n = 1$ , then  $y_1 = 6$  and  $y_2 = \frac{2}{3}(6) - 2 = 2$ , and we have  $y_2 \leq y_1$ .

For the inductive step, assume  $y_{n+1} \leq y_n$  then we need to show that  $y_{n+2} \leq y_{n+1}$ .

$$y_{n+1} \leq y_n \quad (21)$$

$$\frac{2}{3}y_{n+1} - 2 \leq \frac{2}{3}y_n - 2 \quad (22)$$

$$\Rightarrow y_{n+2} \leq y_{n+1} \quad (23)$$

Thus, we have shown the theorem by induction.  $\square$

*Proof.* We will show part b by induction. For the base case, take  $n = 1$ , then  $y_1 = 6 > -6$ .

For the inductive step, assume  $y_n > -6$ , then  $\frac{2}{3}y_n - 2 > \frac{2}{3}(-6) - 2 \Rightarrow y_{n+1} > -6$

Thus, we have shown the theorem by induction.  $\square$

4. Prove that if  $x \in \mathbb{R}$  and  $x > -1$  then for every  $n \in \mathbb{N}$  we have  $(1+x)^n \geq 1+nx$

*Proof.* We will show the theorem by induction. For the base case, take  $n = 1$ , then  $1+x \geq 1+x$ , which is true.

For the inductive step, assume  $(1+x)^n \geq 1+nx$ , then multiply both sides by  $(1+x)$  to get:

$$(1+x)^{n+1} \geq (1+nx)(1+x) = 1+x+nx+nx^2 = 1+(n+1)x+nx^2.$$

Since  $x > -1$ , we know that  $nx^2 > 0$ , so we have:

$$(1+x)^{n+1} \geq 1+(n+1)x+nx^2 \geq 1+(n+1)x.$$

$\square$

5. Prove or give a counterexample for the following statement: Two real numbers satisfy  $a < b$  if and only if  $a < b + \varepsilon$  for all  $\varepsilon > 0$

The statement is false. Take the case where  $a = b$  to be the counterexample. We can adjust the statement slightly to make it true.

**Theorem 4.** Two real numbers satisfy  $a \leq b$  if and only if  $a \leq b + \varepsilon$  for all  $\varepsilon > 0$

1.  $(\Rightarrow)$  If  $a \leq b$ , then  $a - b \leq 0$ , so  $a - b \leq \varepsilon$  for all  $\varepsilon > 0$ .
2.  $(\Leftarrow)$  Assume  $a \leq b + \varepsilon$  for all  $\varepsilon > 0$ . Let  $\varepsilon_0 = a - b$ , then it must be that  $a - b = \varepsilon_0$  and  $a - b \leq \varepsilon_0$ , a contradiction! Thus, the theorem must be true.

6. Given a function  $f: C \rightarrow D$  and a set  $A \subset C$ , let  $f(A)$  represent the range of  $f$  over the set  $A$  i.e.  $f(A) = \{f(x) | x \in A\}$ .

Answer the following questions:

- a. Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^2$ . If  $A = [0, 2]$  and  $B = [1, 4]$ , find  $f(A)$  and  $f(B)$ . Does  $f(A \cap B) = f(A) \cap f(B)$  in this case? Does  $f(A \cup B) = f(A) \cup f(B)$ ?

$f(A) = [0, 4]$ ,  $f(B) = [1, 16]$ ,  $f(A \cup B) = [0, 16]$ ,  $f(A \cap B) = [1, 4]$ . Yes to both.

- b. Find two sets  $A$  and  $B$  for which  $f(A \cap B) \neq f(A) \cap f(B)$ .

- c. Let  $g: C \rightarrow D$  be any function and let  $A, B, C \subset C$  be any two subsets of the domain. Prove that  $g(A \cup B) = g(A) \cup g(B)$

*Proof.* If  $x \in g(A \cup B)$ , then  $x = a^2$  or  $x = b^2$  where  $a \in A$  and  $b \in B$ .  $A \cup B$  contains all  $a \in A$  and  $b \in B$ , so  $x \in g(A \cup B)$  since it contains all  $a^2, b^2$  where  $a, b \in A, B$ .

If  $x \in g(A) \cup g(B)$ , then either  $x \in g(A)$  or  $x \in g(B)$ . In the first scenario,  $x = a^2$  for some  $a \in A$  which we know is in  $g(A \cup B)$ . In the second scenario,  $x = b^2$  for some  $b \in B$  which we know is in  $g(A \cup B)$ .  $\square$

## Lecture 6: The Definition of Function

Thursday 23 January 2025

**Definition 1.** Given sets  $A, B$  a function of  $A \rightarrow B$  is a mapping that takes each element of  $A$  to a single element of  $B$ .

Note:

1.  $f$  is the function,  $f(x)$  not the function.
2.  $A$  is called the **domain**.  $B$  is called the **codomain**.  $\text{Range}(f) = \{y \in B | \exists x \in A \text{ and } f(x) = y\}$ .  $\text{Range}(f) \neq \text{codomain}$  in general.

**Example.** Given  $f: \mathbb{R} \rightarrow \mathbb{R}$  where  $f(x) = x^2$ , the domain is  $\mathbb{R}$  the codomain is  $\mathbb{R}$  and the range is  $[0, \infty]$ .

Note:

1. If  $f(x) \neq f(y)$  when  $x \neq y$  then  $f$  is called **injective** or **one-to-one**.
2. If  $\text{Range}(f) = \text{codomain of } f$  then  $f$  is called **surjective** or **onto**.
3. If  $f$  is both injective and surjective, then it is called **bijective**.

**Example** (Dirichlet Function 1829). Define  $g: \mathbb{R} \rightarrow \mathbb{R}$  by

$$g(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q} \end{cases}.$$

The above function definition is important for historical reasons. Dirichlet came up with the definition of a function given above, and it generalizes the concept of a function nicely. Before Dirichlet, functions were either thought about as

"nice" graphs or as formula, but the new definition generalizes both of these and allows for less traditional function definitions.

**Example.** The **absolute value function**,  $|\cdot|: \mathbb{R} \rightarrow \mathbb{R}$ , is given below:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

**Theorem 5.** Given the above definition of the absolute value function, we have:

1.  $|ab| = |a| |b|$
2.  $|a + b| \leq |a| + |b|$  (**Triangle Inequality**)

*Proof.*

□

**Note.** A common trick that we will use in Analysis is the "add/subtract" trick. Let  $a, b, c \in \mathbb{R}$ , then:

$$|a - b| = ||a - c| + |c - b|| \quad (24)$$

$$\Rightarrow |a - b| \leq |a - c| + |c - b| \quad (25)$$

**Theorem 6.** Let  $a, b \in \mathbb{R}$ . Then  $a = b \Leftrightarrow |a - b| < \varepsilon$  for all  $\varepsilon > 0$ .

*Proof.* ( $\Rightarrow$ ) If  $a = b$ , then  $a - b = 0$  and  $a - b < \varepsilon$  for all  $\varepsilon > 0$ . ( $\Leftarrow$ ) FSOC assume  $|a - b| < \varepsilon$  for all  $\varepsilon > 0$  and  $a \neq b$ , then let  $\varepsilon_0 = a - b \neq 0$  then we gave  $|a - b| < \varepsilon$  and  $|a - b| = \varepsilon_0$ , a contradiction!

□

Thursday 23 January 2025

## Lecture 7: The Axiom of Completeness

We will take an axiomatic approach to Analysis. There are some things which we will just assume are true. Mathematical Formalism is the idea that formal languages with no semantics can serve as the foundation of mathematics. Under this interpretation, the symbols of mathematics do not mean anything at all! They are only symbols and rules for manipulating symbols. Formulating all of mathematics in terms of a formal language allows us to side step assuming the existence of anything. The trade off is that proofs are extraordinarily complex, involve a lot of symbols, and are generally unreadable. For our purposes of writing readable proofs for the most important theorems from Newton's Calculus, we will take a different set of axioms where we do assert the existence of certain mathematical objects. The philosopher should be satisfied with these axioms because they are formally provable within axiomatic set theory. We don't *need* to assume the existence of anything, but we choose to in order to make our lives easier.



**Axiom 1** (Algebraic Properties of  $\mathbb{R}$ ). Assume the existence of a set  $\mathbb{R}$ , called **the Real Numbers**, which is an ordered field.

This axiom gets us most of the way there, however notice that the rational numbers are also an ordered field. We will need to introduce one more axiom to get a unique set for  $\mathbb{R}$ ; but first, we need to define a little bit of mathematical machinery.

**Definition 2** (Bounded Above Property of Subsets of  $\mathbb{R}$ ). A set  $A \subset \mathbb{R}$  is **bounded above** if there exists a number  $b \in \mathbb{R}$  such that  $a \leq b \forall a \in A$ . The number  $b$  is called an **upper bound** for  $A$ .

**Definition 3** (Bounded Below Property of Subsets of  $\mathbb{R}$ ). A set  $A \subset \mathbb{R}$  is **bounded below** if there exists a number  $b \in \mathbb{R}$  such that  $b \leq a \forall a \in A$ . The number  $b$  is called a **lower bound** for  $A$ .

**Definition 4** (The Least Upper Bound). An element  $s \in \mathbb{R}$  is called the **least upper bound** for  $A \subset \mathbb{R}$  if  $s$  meets two conditions:

1.  $s$  is an upper bound for  $A$
2.  $\forall b$  where  $b$  is an upper bound,  $s \leq b$ .

**Definition 5** (The Greatest Lower Bound). An element  $l \in \mathbb{R}$  is called the **greatest lower bound** for  $A \subset \mathbb{R}$  if  $l$  meets two conditions:

1.  $l$  is a lower bound for  $A$
2.  $\forall b$  where  $b$  is an upper bound,  $l \geq b$ .

**Example.** Given the set  $A = \{\frac{1}{n} | n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ , find: upper bounds, the least upper bound, lower bounds, and the greatest lower bound.

1. some upper bounds: 1, 2, 1.1, 3
2. least upper bound: 1
3. some lower bounds: 0, -1, -100
4. greatest lower bound = 0

**Example.** There is no upper bound for  $\mathbb{N}$

The above arguments were not very rigorous, so now we will do a slightly more rigorous problem just to prove that we can.

**Theorem 7.** Given the set  $A = \{\frac{1}{n} | n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ , then the least upper bound for  $A$  is 1.

*Proof.* We will prove the two conditions one at a time:

1. Observe that  $1 \geq \frac{1}{n} \forall n \in \mathbb{N} \Rightarrow 1$  is an upper bound for  $A$ .
2. If  $b$  is an upper bound, then, because  $1 \in A$ ,  $b \geq 1 \Rightarrow 1$  is the upper bound for  $A$ .

□

**Theorem 8.** If some subset of  $\mathbb{R}$  has a least upper bound, then it is unique.

*Proof.* FSOC assume  $s_1$  and  $s_2$  are two distinct greatest upper bounds of some set  $A$ , then we have  $s_1 \leq s_2$  and  $s_2 \leq s_1$  by applying the second condition of the least upper bound property to  $s_1$  and  $s_2$  one at a time. Thus,  $s_1 = s_2$ . This contradicts our assumption that  $s_1$  and  $s_2$  are distinct. □

Now we are ready to state the Axioms of Completeness:

**Axiom 2** (The Axiom of Completeness). Every non-empty set  $A$  where  $A \subset \mathbb{R}$  which is bounded above has a least upper bound  $b \in \mathbb{R}$

**Theorem 9.** Up to isomorphism, there is one unique complete ordered field.

*Proof.* The proof of the above theorem is beyond the scope of this course, but it is worth stating because when we work with  $\mathbb{R}$  we can be sure that we are working on the right set without having to worry that what we are describing has more interpretations than as real numbers. □

**Note.** The Axiom of Completeness is not stateable in first-order logic. You can tell because of the "for every nonempty set  $A$ ." Here, we are quantifying over a set of sets which is not allowed.