Real Analysis 1

Forrest Kennedy

March 5, 2025

Contents

Lecture 27: 03-05-25

Tuesday 05 March 2025

1 Basic Topology of $\mathbb{R}$

Definition 1. Given $a \in \mathbb{R}$ and $\varepsilon > 0$, the ε-neighborhood of a is the set $V_\varepsilon(a)$ defined as

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\} \tag{1}$$

i.e. $V_\varepsilon(a) = (a - \varepsilon, a + \varepsilon)$.

Definition 2. A set $O \subset \mathbb{R}$ us called **open** if for every $a \in O$, there exists $\varepsilon > 0$ such that $V_\varepsilon(a) \subset O$

Example. $(0, 1)$ is an open set

$[0, 1]$ is not an open set.

$\mathbb{R}$ is an open set

$(0, 1) \cup (3, 4)$ is an open set

Note. the union of open sets is open

Theorem 1. We have

1) The union of an arbitrary collection of open sets is open.
2) The intersection of a finite collection of open sets is open.

Proof. 1) Let $\{O_\lambda : \lambda \in \Lambda\}$ be a collection of open sets. If $a \in \cup_{\lambda\in\Lambda} O_\lambda$, $\exists \alpha \in \Lambda$ such that $a \in O_\alpha$. Therefore as O_α is open, $\exists \varepsilon > 0$ such that $V_\varepsilon(a) \subset O_\alpha$. Therefore $V_\varepsilon(a) \subset \cup_{x\in\Lambda} O_\lambda$. Hence $\cup_{\lambda\in\Lambda} O_\lambda$ is open.

2) Let $\{O_1, O_2, O_3, \dots, O_N\}$ be an open set. Let $a \in \cap_{i=1}^{N} O_i$. Therefore $a \in O_i$ for each $1 \leq i \leq N$. Then $\exists \varepsilon_i > 0$ such that $V_{\varepsilon_i}(a) \subset O_i$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N\} > 0$. Therefore $V_\varepsilon(a) \subset O_i$ for all $1 \leq i \leq N$. Therefore $V_\varepsilon(a) \subset \cap_{i=1}^{N} O_i$

□

Definition 3. Let $A \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is called a **limit point of the set** A, if every ε-neighborhood, $V_\varepsilon(a)$ of x intersects the set A at some point other than x.

Note. Limit points are also called "cluster points" or "accumulation points" of a set.

Example. $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots\}$

$x = 1$ is not a limit point because I can take an ε small enough to contain no elements in A.

$x = 0$ is a limit point of A

$A = (0, 1)$

Then if $0 \leq x \leq 1$ then x is a limit point.

Example. $\mathbb{Q}$. Every $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$.

Theorem 2. Let $A \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is a limit point of A if and only if there exists a sequence (a_n) such that $a_n \in A$ and $a_n \neq x$ $\forall n \in \mathbb{N}$ and $\lim a_n = x$

- ($\Rightarrow$) Let x be a limit point of A. Consider the set $V_{\frac{1}{n}}(x)$. There exists $a_n \in V_{\frac{1}{n}}(x) \cap A$ such that $a_n \neq x$. Consider (a_n). $|a_n - x| < \frac{1}{n}$, $a_n \in A$, $a_n \neq x$ $\forall n \in \mathbb{N}$. Given some $\varepsilon > 0$, let $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon \Rightarrow \forall n \geq N$, $|a_n - x| < \varepsilon$