Real Analysis 1

Forrest Kennedy

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Contents

Lecture 10: 2-3-25 Lecture

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Recall from last time

Theorem 1. If $A \subset B$ and B is countable, then A is countable.

Note. if $B = \mathbb{N}$ and $A = \emptyset$. $\emptyset$ is a finite set and thus countable. $\emptyset \subset \mathbb{N}$.

Theorem 2. A set A is countable if and only if there exists an injective function $f : A \to \mathbb{N}$.

Proof. We will prove the forward and backward direction:

- ($\Rightarrow$) Either A is finite or countably infinite. If $A = \emptyset$, then statement is true vacuously. If A is a nonempty finite set, let $|A| = n$, $n \in \mathbb{N}$. Then clearly there exists a bijection between A and $\{1, 2, \ldots, n\}$. Then we just change the function from being $f : A \to \{1, 2, \ldots, n\}$ to $f : A \to \mathbb{N}$. If A is countably infinite, $\exists f : A \to \mathbb{N}$ a bijection. In particular, it is injective.
- ($\Leftarrow$) Let $f : A \to \mathbb{N}$ be injective. Consider $\text{Range}\,(f) \subset \mathbb{N}$. Observe that $f : A \to \text{Range}\,(f)$ is a bijection. As $\text{Range}\,(f) \subset \mathbb{N} \Rightarrow \text{Range}\,(f)\, is countable$. We also have $A \sim \text{Range}\,(f) \Rightarrow A$ is countable.

$\square$

Theorem 3. If A_n is a countable set for each $n \in \mathbb{N}$, then $\cup_{n=1}^{\infty} A_n$ is also countable, i.e. a countable union of countable sets is countable.

INCLUDE GRID OF $\mathbb{N}^2$.

Note. A_ns may not be disjoint! Consider $A_1 = \{1, 2\}$, $A_2 = \{2, 3\}$, $A_3 = \{3, 4, 5\}$. We will try to make these sets disjoint before we get to the proof.

Proof. Define $B_1 = A_1,\ B_2 = A_2 \setminus A_1, \ldots, B_n = A_n \setminus \{A_1 \cup A_2 \sup \ldots\}$ So we have:

$$B_1 = \{1, 2\} \tag{1}$$
$$B_2 = \{3\} \tag{2}$$
$$B_3 = \{4, 5\} \tag{3}$$

Therefore we have $B_1, B_2, B_3, \ldots$ are all disjoint and $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n$. As $B_n \subset A_n$ and A_n is countable, B_n is countable. Therefore $\exists f_n : B_n \to \mathbb{N}$ which is injective for all $n \in \mathbb{N}$.

Define $g : \cup_{n=1}^{\infty} N_n \to \mathbb{N}^2$ given as follows if $b \subset \cup_{n=1}^{\infty}$, the as B_n's are all disjoint, there exists a unique $N \in \mathbb{N}$ such $b \in B_n$. Define:

$$g\left(b\right) = \left(f_N\left(b\right), N\right).$$

As f_N is injective $\Rightarrow g$ is injective. As $\mathbb{N}^2$ is countably infinite, $\exists h : \mathbb{N}^2 \to \mathbb{N}$ is a bijection. Therefore, $h \circ g : \cup_{n=1}^{\infty} B_n \to \mathbb{N}$ is injective and $\cup_{n=1}^{\infty} B_n$ is countable.

□

Theorem 4. If $m \in \mathbb{N}$, and $A_1, A_2, \ldots, A_n$ are countable, then $A_1 \cup A_2 \cup \ldots \cup A_m$ is also countable.

Proof. Define $A_n = \emptyset$ for $n \geq m + 1$. Therefore each A_n, $n \in \mathbb{N}$ is countable. By previous theorem $\cup_{n=1}^{\infty} A_n$ is countable. But $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n$. □

Theorem 5. Suppose $I = \mathbb{R} \setminus \mathbb{Q}$ is countable which implies $\mathbb{R} = \mathbb{Q} \cup I$ is also countable by the previous corollary, a contradiction!