Real Analysis 1

Forrest Kennedy

February 7, 2025

Contents

Lecture 13: 02-07-25 Lecture

Friday 07 February 2025

Example. Template for a proof of $(x_n \to x)$:

1. Let $\varepsilon > 0$ be given.
2. Choose N (depending on ε in general). This step takes the most amount of work and this work is not shown and is rough work.
3. let $n \geq N$
4. Now prove that $|x_n - x| < \varepsilon$ for all $n \geq N$. Then the proof is complete.

Example. Prove that $\lim \left(\frac{n+1}{n}\right) = 1$

Rough work:

$x_n = \frac{n+1}{n} = 1 + \frac{1}{n}$

$x = 1$

Now we want:

$$|x_n - x| < \varepsilon \tag{1}$$
$$\left|1 + \frac{1}{n} - 1\right| < \varepsilon \tag{2}$$
$$\left|\frac{1}{n}\right| < \varepsilon \tag{3}$$
$$\frac{1}{n} < \varepsilon \tag{4}$$
$$\frac{1}{\varepsilon} < n \tag{5}$$
$$\tag{6}$$

What I really want: Find N so that $\forall n \geq N$, $\frac{1}{\varepsilon} < n$, so choose $N \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < N$, then if $n \geq N \Rightarrow \frac{1}{\varepsilon} < N < n$.

Proof. Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < N$. Let $n \geq N$. This implies that

$$\frac{1}{\varepsilon} < N \leq n \tag{7}$$
$$\frac{1}{n} < \varepsilon \tag{8}$$
$$\left| \left(1 + \frac{1}{n} \right) - 1 \right| < \varepsilon \tag{9}$$
$$\left| \left(\frac{n+1}{n} \right) - 1 \right| < \varepsilon \tag{10}$$
$$\tag{11}$$

Thus we have shown the condition for the proof. □

Example. Prove that $\lim \left(\frac{1}{n^2} \right) = 0$

Proof. Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{\sqrt{\varepsilon}}.$$

Therefore

$$n > \frac{1}{\sqrt{\varepsilon}} \tag{12}$$
$$\frac{1}{n} < \sqrt{\varepsilon} \tag{13}$$
$$\frac{1}{n^2} < \varepsilon \tag{14}$$
$$\left| \frac{1}{n^2} - 0 \right| < \varepsilon \tag{15}$$

□

Example. Prove that $\lim \frac{1}{n^2+576n+100{,}002} = 0$

Proof. If $\frac{1}{n^2} < \varepsilon$

$$\left| \frac{1}{n^2 + 576n + 100,002} - 0 \right| < \varepsilon \tag{16}$$

□

Note. Do not try to find an "optimal" N, just find one that works!