Real Analysis 1

Forrest Kennedy

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Contents

Lecture 18: 02-17-25 Lecture

Monday 17 February 2025

- DeLong Lecture today 3:30 - 4:30pm in Kitt Multipurpose room. Speaker: Prof. Laura DeMarco (Harvard)

> **Definition 1.** A sequence (a_n) is called **increasing** if $a_n \leq a_{n+1}$ for all $n \in N$. It is called **decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is called monotone if it is either increasing or decreasing.

Example. $1, 1, 2, 2, 3, 3, 4, 4, \ldots$ is increasing.

Example. $1, 1, 0, 0, -1, -1, \ldots$ is decreasing

Example. $1, 1, 1, 1, 1, \ldots$ is constant and monotone.

Example. $1, 0, 1, 0, 1, 0, 1, \ldots$ is *not* monotone.

> **Theorem 1** (Monotone Convergence Theorem)**.** If a sequence is monotone and bounded, then it converges.

Note. There are two enemies of convergence:

1. Oscillations (killed by monotone)
2. Growth (killed by boundedness)

Proof. Let (a_n) be monotone and bounded. Let us assume that (a_n) is increasing (the case for decreasing is proved similarly). Define the set

$$S = \{a_n | n \in \mathbb{N}\} \tag{1}$$

As (a_n) is bounded, this means that the set S is bounded above. Let $x = \sup S$ Now we just need to show that $\lim a_n = x$ to prove the statement. Let $\varepsilon > 0$ be given. As x is the least upper bound, $x - \varepsilon$ is not an upper bound for S. Then there exists $n \in \mathbb{N}$ such that $x - \varepsilon < a_N$. Therefore, for all $n \geq N$ we have

$$x - \varepsilon < a_N \leq a_n \leq x \tag{2}$$
$$x - \varepsilon < a_n < x + \varepsilon \tag{3}$$
$$|a_n - x| < \varepsilon \tag{4}$$

Hence proved. □

Definition 2. Let (b_n) be a sequence. An **infinite series** is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \dots \tag{5}$$

We define the corresponding **sequence of partial sums**, (S_m) by

$$S_m = b_1 + b_2 + \dots + b_m \tag{6}$$

we say that the series $\sum_{n=1}^{\infty} b_n$ **converges to B** if the sequence (S_m) converges to B. In this case, we write $\sum_{n=1}^{\infty} = B$.

Note. When we write the first sum, we are literally just writing symbols. If we want to assign meaning to this, we need to construct a sequence of partial sums $b_1, b_1 + b_2, b_1 + b_2 + b_3, \dots$.

Example. Recall from day 1:

$$b_n = (-1)^n \tag{7}$$
$$S_1 = b_1 = -1 \tag{8}$$
$$S_2 = b_1 + b_2 = 0 \tag{9}$$
$$S_2 = b_1 + b_2 + b_3 = -1 \tag{10}$$
$$\dots \tag{11}$$

Then construct the sequence:

$$(S_1, S_2, S_3, \dots) = (-1, 0, -1, 0, -1, \dots) \tag{12}$$

The sequence does not converge, therefore the series doesn't converge.

Example. Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \tag{13}$$

As all the terms in the series are positive, we observe that the sequence (S_m) is an increasing sequence. Now we will apply a trick

$$\begin{aligned}
S_m &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{m^2} && (14)\\
&< 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{(m-1)\, m} && (15)\\
&= 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{m-1} - \frac{1}{m}\right) && (16)\\
&= 1 + 1 - \frac{1}{m} && (17)\\
&< 2 && (18)
\end{aligned}$$

Therefore $S_m < 2$ for all $M \in \mathbb{N}$. Hence the sequence (S_m) is bounded. As (S_m) is an increasing bounded sequence, by the monotone convergence theorem, it converges.

Note. The above is the Basel Problem. The value that it converges to was found by Euler in 1734 and surprisingly is $\frac{\pi^2}{6}$. This is connected to the Riemann Zeta function.

Lecture 19: Homework 5

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1. Let $(a_n) \to 0$. Use the Algebraic limit theorem to compute each of the following limits (assuming the functions are always defined). Justify all of your actions.

   1. $\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$
   2. $\lim \left(\frac{(a_n+2)^2-4}{a_n}\right)$
   3. $\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$

2. Prove that the following sequences diverge:

   1. The sequence (a_n) where
   $$a_n = (-1)^n n^2 + 1 \tag{19}$$
   2. The sequence (a_n) where
   $$(-1)^n + \frac{1}{n} \tag{20}$$

3. (Squeeze Theorem). Show that if $x_n \le y_n \le z_n$ for all $n \in \mathbb{N}$ and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

4. Let $x_n \ge 0$ for all $n \in \mathbb{N}$.

   1. If $(x_n) \to 0$, show that $\sqrt{s_n} \to 0$.
   2. If $(x_n) \to x$, show that $\left(\sqrt{x_n}\right) \to \sqrt{x}$.

5. Consider the sequence (b_n) where $b_n = \sqrt{n^2 + 2n} - n$. Prove that (b_n) is convergent and find its limit.

6. Give an example of each of the following:

   1. Sequences (a_n) and (b_n), which both diverge, but whose sum $(a_n + b_n)$ converges.
   2. Sequences (a_n) and (b_n), which both diverge, but whose product $(a_n b_n)$ converges.
   3. Convergent sequences (a_n) and (b_n) with $a_n < b_n$ for all $n \in \mathbb{N}$ such that $\lim (a_n) = \lim (b_n)$.
   4. A convergent sequence (b_n) with $b_n \neq 0$ for all $n \in \mathbb{N}$, such that $(1/b_n)$ diverges.
   5. Two sequence (a_n) and (b_n) so that (a_n) is unbounded, (b_n) is bounded, and $(a_n b_n)$ converges.
   6. Two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

7. Let (a_n) be a bounded (not necessarily convergent) sequence, and assume that $\lim b_n = 0$. Show that $\lim\limits_{a_n b_n} = 0$. Why are we not allowed to use the Algebraic limit theorem to prove this?