Real Analysis 1

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Lecture 1: Some historical motivations for Analysis

Monday 13 January 2025

1 The Heat Equation

In 1822, Fourier derived the heat equation. In one dimension:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0.$$

where $u(x,t)$ is the temperature as a function of position and time. A natural problem to solve with the equation is to assume you are given a function $u(x,0)$ which represents the initial temperature distribution of the system which we could measure then ask if it is possible to find a general $u(x,t)$ given $u(x,0)$. Stated another way, if we know the initial temperature distribution, can we find the distribution at an arbitrary time t using only the heat equation. The answer to this question is yes!

If we assume that $u(x,0)$ is a periodic function then:

$$u(x,0) = \sum_{n\in\mathbb{Z}} a_n e^{inx}.$$

Then, rearranging and integrating:

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} u(x,0)\, e^{inx} dx.$$

Then, we will guess that the solution is of the form:

$$u(x,t) = \sum_{n\in\mathbb{Z}} a_n(t)\, e^{inx}.$$

Substituting into the heat equation we get:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sum_{n\in\mathbb{Z}} a_n'(t)\, e^{inx} - a_n(t)\,(in)^2\, e^{inx} = \sum \left(a_n'(t) + a_n(t)\, n^2\right) e^{inx} = 0.$$

So we have the differential equation:

$$a_n'(t) + a_n(t)\, n^2 = 0.$$

which has the solution:

$$a_n(t) = a_n(0)\, e^{-n^2 t}.$$

We can find $a_n(0)$ with the integral above, so we have our solution! If we check the solution experimentally, we see the right behavior, so what's the problem? The issue is:

$$u(x,0) \neq \sum_{n\in\mathbb{Z}} a_n e^{inx}!$$

At least, when we look at the graph for any specific $n \in \mathbb{N}$ we see that at the extreme points of the function, we get oscillations away from the true value of $u(x,0)$. This affect is called the Gibbs phenomenon. This tells us that the function cannot equal the partial sum. On the other hand, the assumption works, so there must be some kind of notion of equality, but we are not in a position to say what that is right now. This gives us a specific example where Newton's calculus fails. The issue in this example has to do with the definition of "convergence" but there is a deeper issue. When we wrote down $u(x,0) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$, we were writing down nonsense, but we didn't know it. In order to know which statements are valid and which are not, we need to develop an axiomatic system that we can use to build up definitions, theorems, and proofs. This is the buisness of Mathematical Analysis: to provide a rigorous base for analysis to rest upon which contains no nonsense!

Wednesday 15 January 2025

Lecture 2

Once we realize that the we don't know how to add up infinitely many functions, it is easy to see that we don't really know how to add infinitely many **numbers** either! Consider:

$$\begin{aligned}1 &= 1\\ 1-1 &= 0\\ 1-1+1 &= 1\\ 1-1+1-1 &= 0\\ &\vdots\\ 1-1+1-1+1+\ldots &= \ ?\end{aligned}$$

Already there seems to be a problem! The series does seem to converge to any number. Of course, we don't know what converge means yet, but there is a deeper problem. Consider the rearrangements:

$$\begin{aligned}1+(-1+1)+(-1+1)+\ldots &= 1\\ (1+-1)+(1+-1)+(1+\ldots &= 0\end{aligned}$$

Clearly, these can't both be right. Again, we have been tricked into writing nonsense because we don't have any axioms to tell us which statements are allowed and which are not. Here, the problem has to do with our adding up of an infinite number of things. When we are properly automatized, we will see that we just don't do that. Instead, we will solve this problem with a "limit," **in order to understand the limit, we will need to develop $\mathbb{R}$, the real number system.** This is the goal of Chapter 1. The point is: adding up infinite things, whether they are functions or just numbers, leads to problems, and whatever formal system we come up with will need to be without these problems if we want it to formalize calculus, which is based around the notion of adding up infinitely many things.

Chapter 1

To motivate the definition of $\mathbb{R}$ let's explore ways in which $\mathbb{R}$ is different to other number systems. Why should we expect the definition of $\mathbb{R}$ to be useful and lead us to a notion of a "continuum?"

Lemma 1. For all $m, n \in \mathbb{Z}$, if $n|m^2$ and n is prime, then $n|m$

Proof. Assume for the sake of contradiction that n does *not* divide m, then n cannot be a prime factor of m, so $m = abcd\ldots$ for some prime numbers $a, b, c, d, \ldots$, importantly n **cannot be part of the product since** n **does not divide** m. Then $m^2 = (abcd\ldots)^2 = a^2b^2c^2d^2\ldots$ which does not contain n, so $n \nmid m^2$ which contradicts our assumption. Thus it must be the case that $n|m$. $\square$

Theorem 1. There is no rational number whose square is 2

Proof. Assume for contradiction: $\exists r$ s.t. $r = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $r^2 = 2$. Also assume, WLOG, that p, q **share no common factors** then:

$$
\begin{aligned}
r^2 &= \left(\frac{p}{q}\right)^2 = 2 && (1)\\
&\Rightarrow p^2 = 2q^2 && (2)\\
&\Rightarrow 2|p^2 && (3)\\
&\Rightarrow 2|p \text{ (by Lemma 1)} && (4)\\
&\Rightarrow p = 2n \text{ where } n \in \mathbb{Z} && (5)\\
&\Rightarrow (2n)^2 = 4n^2 = 2q^2 \text{ (from Eq. 2)} && (6)\\
&\Rightarrow 2|q^2 \Rightarrow 2|q && (7)
\end{aligned}
$$

Thus $2|p$ and $2|q$ which violates our assumption that p, q share no common factors! Thus is must be the case that there is no rational number whose square is 2.

□

Theorem 2. If $n \in N$ and n is **not** a perfect square, then there is no $r \in \mathbb{Q}$ such that $r^2 = n$

Proof. Assume for contradiction: $\exists r$ s.t. $r = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $gcd\,(p, q) = 1$, $r^2 = n$, and n is not a perfect square.

$$
\begin{aligned}
r^2 &= \left(\frac{p}{q}\right)^2 = n && (8)\\
&\Rightarrow p^2 = nq^2 && (9)
\end{aligned}
$$

Recall that by the Fundamental Theorem of Arithmetic that we can express any number as a product of prime numbers, so:

$$n = k_1^1 \cdot k_2^2 \cdot k_3^3 \cdot k_4^4 \cdot \ldots .$$

Substitute n into Eq. 9:

$$p^2 = \left(k_1^1 \cdot k_2^2 \cdot k_3^3 \cdot k_4^4 \cdot \ldots\right) q^2.$$

Since n is not a perfect square, we know that $\exists j$ s.t. k_j is odd because if this were not the case, n would be a perfect square. From the above, we can see that $k_j|p^2$. If k_j divides the LHS, it must also divide the RHS, so $k_j|nq^2$. We know that p^2 contains an even number of k_j terms, and we also know that n contains an odd number of k_j terms. For both sides to have the same number of k_j terms, as all equal numbers should, it must be the case that $k_j|q^2$ which implies $k_j|q$ by Lemma 1. Thus, $gcd\,(p, q) = k_j \neq 1$ which contradicts our assumption that $gcd\,(p, q) = 1$. □

After this we talked about "set theory." We did not go into the details.

Theorem 3. The Algebra of Sets exists. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ exist.

Proof. The above is taken as an axiom, but rest assured that their existence can be derived from first-order logic and the ZFC axioms. $\square$

Lecture Homework 1: Homework 1

Tuesday 21 January 2025

1. Prove that there is no rational number, r, such that $r^2 = 8$.

Proof. BWOC, assume there exists a number $r \in \mathbb{Q}$ such that for some $a, b \in \mathbb{Z}$ with $gcd\,(a, b) = 1$:

$$r^2 = \left(\frac{a}{b}\right)^2 = 8.$$

then:

$$\Rightarrow a^2 = 8b^2 \tag{10}$$

$$\Rightarrow 2^2|a^2 \Rightarrow 2|a^2 \Rightarrow 2|a \text{ (by Lemma 1)} \tag{11}$$

so for some $n \in \mathbb{N}$:

$$\left(2^2 n\right)^2 = 8b^2 \tag{12}$$

$$16n^2 = 8b^2 \tag{13}$$

$$2n^2 = b^2 \Rightarrow 2|b \tag{14}$$

Notice $2|a$ and $2|b$ which violates our assumption that $gcd\,(a, b) = 1$. Thus it must be the case that there does not exist a number $r \in \mathbb{Q}$ such that $r^2 = 8$

$\square$

2. Prove that if $a, b \in \mathbb{R}$ then:

$$\mid\mid a \mid - \mid b \mid\mid \leq \mid a - b \mid .$$

Proof. If $a, b \in \mathbb{R}$, then the triangle inequality holds:

$$\mid a + b \mid \leq \mid a \mid + \mid b \mid .$$

Now consider:

$$\mid a \mid = \mid a - b + b \mid \leq \mid a - b \mid + \mid b \mid \tag{15}$$

$$\mid b \mid = \mid b - a + a \mid \leq \mid b - a \mid + \mid a \mid \tag{16}$$

$$\tag{17}$$

Rearranging and using the definition of the absolute value function and the fact that $\mid a - b \mid = \mid b - a \mid$:

$$\mid a \mid - \mid b \mid \leq \mid a - b \mid \tag{18}$$

$$\mid b \mid - \mid a \mid \leq \mid b - a \mid \tag{19}$$

$$\Rightarrow \mid\mid a \mid - \mid b \mid\mid \leq \mid a - b \mid \tag{20}$$

□

3. Let $y_1 = 6$ and for each $n \in \mathbb{N}$ define

$$y_{n+1} = \frac{2}{3} y_n - 2.$$

Prove the following statements:

a. Prove that $y_{n+1} \leq y_n$ for all $n \in \mathbb{N}$.

b. Prove that $y_n > -6$ for all $n \in \mathbb{N}$

Proof. We will show part a by induction. For the base case take $n = 1$, then $y_1 = 6$ and $y_2 = \frac{2}{3}(6) - 2 = 2$, and we have $y_2 \leq y_1$.

For the inductive step, assume $y_{n+1} \leq y_n$ then we need to show that $y_{n+2} \leq y_{n+1}$.

$$y_{n+1} \leq y_n \tag{21}$$

$$\frac{2}{3} y_{n+1} - 2 \leq \frac{2}{3} y_n - 2 \tag{22}$$

$$\Rightarrow y_{n+2} \leq y_{n+1} \tag{23}$$

Thus, we have shown the theorem by induction. □

Proof. We will show part b by induction. For the base case, take $n = 1$, then $y_1 = 6 > -6$.

For the inductive step, assume $y_n > -6$, then $\frac{2}{3} y_n - 2 > \frac{2}{3}(-6) - 2 \Rightarrow y_{n+1} > -6$

Thus, we have shown the theorem by induction. □

4. Prove that if $x \in \mathbb{R}$ and $x > -1$ then for every $n \in \mathbb{N}$ we have $(1+x)^n \geq 1 + nx$

Proof. We will show the theorem by induction. For the base base, take $n = 1$, then $1 + x \geq 1 + x$, which is true.

For the inductive step, assume $(1+x)^n \geq 1 + nx$, then multiply both sides by $(1+x)$ to get:

$$(1+x)^{n+1} \geq (1+nx)(1+x) = 1 + x + nx + nx^2 = 1 + (n+1)x + nx^2.$$

Since $x > -1$, we know that $nx^2 > 0$, so we have:

$$(1+x)^{n+1} \geq 1 + (n+1)x + nx^2 \geq 1 + (n+1)x.$$

□

5. Prove or give a counterexample for the following statement: Two real numbers satisfy $a < b$ if and only if $a < b + \varepsilon$ for all $\varepsilon > 0$

The statement is false. Take the case where $a = b$ to be the counterexample. We can adjust the statement slightly to make it true.

Theorem 4. Two real numbers satisfy $a \leq b$ if and only if $a \leq b + \varepsilon$ for all $\varepsilon > 0$

1. ($\Rightarrow$) If $a \leq b$, then $a - b \leq 0$, so $a - b \leq \varepsilon$ for all $\varepsilon > 0$.
2. ($\Leftarrow$) Assume $a \leq b + \varepsilon$ for all $\varepsilon > 0$. Let $\varepsilon_0 = a - b$, then it must be that $a - b = \varepsilon_0$ and $a - b \leq \varepsilon_0$, a contradiction! Thus, the theorem must be true.

6. Given a function $f\colon C \to D$ and a set $A \subset C$, let $f(A)$ represent the range of f over the set A i.e. $f(A) = \{f(x) \,|x \in A\}$.

Answer the following questions:

a. Let $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. If $A = [0, 2]$ and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

   $f(A) = [0, 4], f(B) = [1, 16], f(A \cup B) = [0, 16], f(A \cap B) = [1, 4]$. Yes to both.

b. Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

c. Let $g\colon C \to D$ be any function and let $A, B, C \subset C$ be any two subsets of the domain. Prove that $g(A \cup B) = g(A) \cup g(B)$

   Proof. If $x \in g(A \cup B)$, then $x = a^2$ or $x = b^2$ where $a \in A$ and $b \in B$. $A \cup B$ contains all $a \in A$ and $b \in B$, so $x \in g(A \cup B)$ since it contains all a^2, b^2 where $a, b \in A, B$.

   If $x \in g(A) \cup g(B)$, then either $x \in g(A)$ or $x \in g(B)$. In the first scenario, $x = a^2$ for some $a \in A$ which we know is in $g(A \cup B)$. In the second scenario, $x = b^2$ for some $b \in B$ which we know is in $g(A \cup B)$. $\square$

Lecture 6: The Definition of Function

Thursday 23 January 2025

Definition 1. Given sets A, B a function of $A \to B$ is a mapping that takes each element of A to a single element of B.

Note:

1. f is the function, $f(x)$ not the function.
2. A is call the **domain**. B is called the **codomain**. $\mathit{Range}(f) = \{y \in B \mid \exists x \in A \text{ and } f(x) = y\}$. $\mathit{Range}(f) \neq$ codomain in general.

Example. Given $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$, the comain is $\mathbb{R}$ the codomain is $\mathbb{R}$ and the range is $[0, \infty]$.

Note:

1. If $f(x) \neq f(y)$ when $x \neq y$ then f is called **injective** or **one-to-one**.
2. If $\mathit{Range}(f) =$ codomain of f then f is called **surjective** or **onto**.
3. If f is both injective and surjective, then it is called **bijective**.

Example (Dirichlet Function 1829)**.** Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q} \end{cases}.$$

The above function definition is important for historical reasons. Dirichlet came up with the definition of a function given above, and it generalizes the concept of a function nicely. Before Dirichlet, function were either thought about as "nice" graphs or as formula, but the new definition generalizes both of these and allows for less traditional function definitions.

Example. The **absolute value function**, $|\cdot|: \mathbb{R} \to \mathbb{R}$, is given below:

$$| x |= \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

Theorem 5. Given the above definition of the absolute value function, we have:

1. $| ab |=| a || b |$
2. $| a + b |\leq| a | + | b |$ **(Triangle Inequality)**

Proof. □

Note. A common trick that we will use in Analysis is the "add/subtract" trick. Let $a, b, c \in \mathbb{R}$, then:

$$|a - b| = ||a - c| + |c - b|| \tag{24}$$

$$\Rightarrow |a - b| \leq ||a - c| + |c - b|| \tag{25}$$

Theorem 6. Let $a, b \in R$. Then $a = b \Leftrightarrow |a - b| < \varepsilon$ for all $\varepsilon > 0$.

Proof. ($\Rightarrow$) If $a = b$, then $a - b = 0$ and $a - b < \varepsilon$ for all $\varepsilon > 0$. ($\Leftarrow$) FSOC assume $|a - b| < \varepsilon$ for all $\varepsilon > 0$ and $a \neq b$, then let $\varepsilon_0 = a - b \neq 0$ then we gave $|a - b| < \varepsilon$ and $|a - b| = \varepsilon_0$, a contradiction!

□

Thursday 23 January 2025

Lecture 7: The Axiom of Completeness

We will take an axiomatic approach to Analysis. There are some things which we will just assume are true. Mathematical Formalism is the idea that formal languages with no semantics can serve as the foundation of mathematics. Under this interpretation, the symbols of mathematics do not mean anything at all! They are only symbols and rules for manipulating symbols. Formulating all of mathematics in terms of a formal language allows us to side step assuming the existence of anything. The trade off is that proofs are extraordinarily complex, involve a lot of symbols, and are generally unreadable. For our purposes of writing readable proofs for the most important theorems from Newton's Calculus, we will take a different set of axioms where we do assert the existence of certain mathematical objects. The philosopher should be satisfied with these axioms because they are formally provable within axiomatic set theory. We don't *need* to assume the existence of anything, but we choose to in order to make our lives easier.

Axiom 1 (Algebraic Properties of $\mathbb{R}$)**.** Assume the existence of a set $\mathbb{R}$, called **the Real Numbers**, which is an ordered field.

This axiom gets us most of the way there, however notice that the rational numbers are also an ordered field. We will need to introduce one more axiom to get a unique set for $\mathbb{R}$; but first, we need to define a little bit of mathematical machinery.

Definition 2 (Bounded Above Property of Subsets of $\mathbb{R}$)**.** A set $A \subset \mathbb{R}$ is **bounded above** if there exists a number $b \in \mathbb{R}$ such that $a \leq b \ \forall a \in A$. The number b is called an **upper bound** for A.

Definition 3 (Bounded Below Property of Subsets of $\mathbb{R}$)**.** A set $A \subset \mathbb{R}$ is **bounded below** if there exists a number $b \in \mathbb{R}$ such that $b \leq a \ \forall a \in A$. The number b is called a **lower bound** for A.

Definition 4 (The Least Upper Bound)**.** An element $s \in \mathbb{R}$ is called the **least upper bound** for $A \subset \mathbb{R}$ if s meets two conditions:

1. s is an upper bound for A
2. $\forall b$ where b is an upper bound, $s \leq b$.

Definition 5 (The Greatest Lower Bound)**.** An element $l \in \mathbb{R}$ is called the **greatest lower bound** for $A \subset \mathbb{R}$ if l meets two conditions:

1. l is a lower bound for A
2. $\forall b$ where b is an upper bound, $l \geq b$.

Example. Given the set $A = \{\frac{1}{n} | n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, find: upper bounds, the least upper bound, lower bounds, and the greatest lower bound.

1. some upper bounds: $1, 2, 1.1, 3$
2. least upper bound: 1
3. some lower bounds: $0, -1, -100$
4. greatest lower bound $= 0$

Example. There is no upper bound for $\mathbb{N}$

The above arguments were not very rigorous, so now we will do a slightly more rigorous problem just to prove that we can.

Theorem 7. Given the set $A = \{\frac{1}{n} | n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, then the least upper bound for A is 1.

Proof. We will prove the two conditions one at a time:

1. Observe that $1 \geq \frac{1}{n}\ \forall n \in \mathbb{N} \Rightarrow 1$ is an upper bound for A.
2. If b is an upper bound, then, because $1 \in A$, $b \geq 1 \Rightarrow 1$ is the upper bound for A.

□

Theorem 8. If some subset of $\mathbb{R}$ has a least upper bound, then it is unique.

Proof. FSOC assume s_1 and s_2 are two distinct greatest upper bounds of some set A, then we have $s_1 \leq s_2$ and $s_2 \leq s_1$ by applying the second condition of the least upper bound property to s_1 and s_2 one at a time. Thus, $s_1 = s_2$. This contradicts our assumption that s_1 and s_2 are distinct. □

Now we are ready to state the Axioms of Completeness:

Axiom 2 (The Axiom of Completeness)**.** Every non-empty set A where $A \subset \mathbb{R}$ which is bounded above has a least upper bound $b \in \mathbb{R}$

Theorem 9. Up to isomorphism, there is one unique complete ordered field.

Proof. The proof of the above theorem is beyond the scope of this course, but it is worth stating because when we work with $\mathbb{R}$ we can be sure that we are working on the right set without having to worry that what we are describing has more interpretations than as real numbers. □

Note. The Axiom of Completeness is not stateable in first-order logic. You can tell because of the "for every nonempty set A." Here, we are quantifying over a set of sets which is not allowed.

Monday 27 January 2025

Lecture 6: Homework 2

1. Compute, without proof, the supremum and infimum (if they exist) of each of the following sets:

2. Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ be two non-empty sets, each of which is bounded above. If $s = \sup A$ and $t = \sup B$, find and prove a formula for $\sup A \cup B$

Proof. We argue that $\sup A \cup B = max\,(s,t)$ by cases:

1. $(s > t)$ WLOG with respect to the $s < t$ case, since $t = \sup B$, we have $t \geq b$ for all $b \in B$, thus by using our case assumption we get $s > t \geq b$ for all $b \in b$ and $s \geq a$ for all $a \in A$ by the definition of the supremum of a set. Therefore, $s \geq u$ for all $u \in A \cup B$ and $\sup A \cup B = s = max\,(s,t)$.

2. $(s = t)$ If $s = t$ by the definition of the supremum we have $s \geq a$ for all $a \in A$ and $t = s \geq b$ for all $b \in B$. Thus $max\,(s,t) = s = t \geq a$ for all $a \in A$ and $max\,(s,t) = s = t \geq b$ for all $b \in B$. Therefore, $\sup A \cup B = max\,(s,t)$.

□

3. Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ be two non-empty sets, each of which is bounded above.

1. If $\sup A < \sup B$, show that there exists $b \in B$ such that b is an upper bound for A.

2. Given an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

Proof. 1. Combining the definition of the supremum of a set and the given, we get

$$a \leq \sup A < \sup B.$$

Thus $\sup B$ is an upper bound for A, but it cannot be the *least* upper bound because we assumed that $\sup A < \sup B$. Then by negating the ε Characterization of the Supremum we see that $\exists \varepsilon > 0\, \forall a \in A\,(\sup B - \varepsilon \geq a)$. Let ε_0 be the ε such that $\forall a \in A\,(\sup B - \varepsilon_0 \geq a)$. Since $\sup B$ is the least upper bound of B, again we can use the ε Characterization of the supremum, $\forall\, \varepsilon > 0 \exists b \in B\ (\sup B - \varepsilon < b)$ thus $\exists b \in B\,(\sup B - \varepsilon_0 < b)$. Thus, $\exists b \in B\ \forall a \in A\,(b > \sup B - \varepsilon_0 \geq a)$. Therefore, there exists $b \in B$ such that $b \geq a$ for all $a \in A$.

2. Take $A = (1,2)$ and $B = (0,2)$. In this case, $\sup A = \sup B$, but there is no element of b which is an upper bound of A.

□

5. Let $A \subset \mathbb{R}$ and $c \in \mathbb{R}$. We define the set cA as:

$$cA = \{ca | a \in A\}.$$

If A is non-empty and bounded above and $c \geq 0$, then prove that $sup\ cA = c \cdot sup\ A$.

Proof. By the definition of the supremum, we have $\sup cA \geq ca \Rightarrow \frac{1}{c} \sup cA \geq a$. Then $\frac{1}{c} \sup cA$ is an upper bound for A. But $\sup A$ is the *least* upper bound of A, so it must be that $\frac{1}{c} \sup cA \geq \sup A$, thus $\sup cA \geq c \sup A$.

By the definition of the supremum, we have $\sup A \geq a$ for all $a \in A \Rightarrow c \sup A \geq ca\ \forall a \in A$. Thus $c \sup A$ is an upper bound of cA. But $\sup cA$ is the *least* upper bound of cA, so it must be that $c \sup A \geq \sup cA$.

Thus, $\sup cA \geq c \sup A$ and $\sup cA \leq c \sup A$; therefore, $\sup cA = c \sup A$. □

Tuesday 28 January 2025

Lecture 7: Complete Analysis Theorems List

2 The Real Numbers

Definition 6 (Definition of a "function")**.** Given sets A, B a function of $A \to B$ is a mapping that takes ea ch element of A to a single element of B.

Definition 7 (Definition of the "absolute value function")**.** The **absolute value function"** is defined as $|\cdot| : \mathbb{R} \to \mathbb{R}$ such that:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}.$$

Theorem 10 (The Triangle Inequality)**.** With respect to multiplication and division, the absolute value function satisfies:

1. $|ab| = |a|\,|b|$
2. $|a + b| \leq |a| + |b|$

Proof. We will show the theorem by cases WLOG:

1. $(a = 0)$ $|a + b| = |0 + b| = |b| = |0| + |b| = |a| + |b|$
2. $(a > 0, b > 0)$ By the definition of the absolute value function we have $|a + b| = a + b = |a| + |b|$
3. $(a < 0, b < 0)$ By the definition of the absolute value function we have $|a + b| = -(a + b) = -a + (-b) = |a| + |b|$
4. $(a > 0, b < 0)$ By the definition of the absolute value, we have $|a| = a$ and $|b| = -b$, so $|a|+|b| = a+(-b)$. We want to show that $|a|+|b| = a+(-b) \geq |a + b|$, so again we consider all the possible cases:
   (a) $(a + b = 0)$ We have $a + (-b) \overset{?}{\geq} |0| = 0$. Indeed, since $a > 0$ and $b < 0$ we have $a > b$, and our equality holds.
   (b) $(a + b > 0)$ We have $a + (-b) \overset{?}{\geq} a + b$. Since $b < 0$, we have $-b > 0$. Comparing the LHS and RHS the equality holds.
   (c) $(a + b < 0)$ We have $a + (-b) \overset{?}{\geq} -a + (-b)$. Comparing the LHS and the RHS, the equality holds.

The above considerations exhaust all possible choices for a and b. In all cases. we see that $|a+b| \leq |a| + |b|$ □

Theorem 11 (The ε criteria for equality)**.** Two real numbers a and b are equal if and only if for every real number $\varepsilon > 0$ it follows that $|a-b| < \varepsilon$.

Proof. We will show the theorem in both directions:

- ($\Rightarrow$) Given $a = b$, we have $a - b = 0 < \varepsilon$ for all $\varepsilon > 0$.
- ($\Leftarrow$) Assume that for every $\varepsilon > 0$, $|a-b| < \varepsilon$ and, FSOC, that $a \neq b$. Then, let $\varepsilon_0 = a - b$ which we know is nonzero because $a \neq b$. Now, $|a-b| = \varepsilon_0$ and $|a-b| < \varepsilon_0$ by our first assumption. We have reached a contradiction, therefore the reverse implication must hold.

□

Definition 8 (Bounded Above Property of Subsets of $\mathbb{R}$)**.** A set $A \subset \mathbb{R}$ is **bounded above** if there exists a number $b \in \mathbb{R}$ such that $a \leq b\ \forall a \in A$. The number b is called an **upper bound** for A.

Definition 9 (Bounded Below Property of Subsets of $\mathbb{R}$)**.** A set $A \subset \mathbb{R}$ is **bounded below** if there exists a number $b \in \mathbb{R}$ such that $b \leq a\ \forall a \in A$. The number b is called a **lower bound** for A.

Definition 10 (The Least Upper Bound)**.** An element $s \in \mathbb{R}$ is called the **least upper bound** for $A \subset \mathbb{R}$ if s meets two conditions:

1. s is an upper bound for A
2. $\forall b$ where b is an upper bound, $s \leq b$.

Definition 11 (The Greatest Lower Bound)**.** An element $l \in \mathbb{R}$ is called the **greatest lower bound** for $A \subset \mathbb{R}$ if l meets two conditions:

1. l is a lower bound for A
2. $\forall b$ where b is an upper bound, $l \geq b$.

Definition 12. A real number a_0 is a **maximum** of the set A if a_0 is an elemnt of A and $a_0 \geq a$ for all $a \in A$. Similarly, a number a_1 is a **minimum** of A if $a_1 \in A$ and $a_1 \leq a$ for all $a \in A$.

Theorem 12 (The ε Characterization of the Supremum)**.** Assume $s \in R$ is an upper bound for a set $A \subset \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\varepsilon > 0$, there exists an element $a \in A$ satisfying $s - \varepsilon < a$.

Proof. We will show that both the implication and the inverse implication are true:

- ($\Rightarrow$)If s is the *least* upper bound of A, then $s - \varepsilon$ is not an upper bound for A, thus there exists an $a \in A$ such that $s - \varepsilon < a$.
- ($\Leftarrow$) Assume s is an upper bound of A and that for every $\varepsilon > 0$, $s - \varepsilon < a$. That is, no number smaller than s is an upper bound of A. Thus for all b where b is an upper bound of A, $s \leq b$. Since we assumed that s is an upper bound, s meets both conditions to be the supremum.

$\square$

Theorem 13 (Nested Interval Property of Subsets of $\mathbb{R}$)**.** For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in R : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supset I_2 \supset I_3 \supset \dots.$$

has a nonempty intersection; that is, $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$, then let $\alpha = \sup A$. From the definition of the supremum, we have $\alpha \geq a_n$ for all $n \in \mathbb{N}$. Because of how we defined our sets, every b_n is an upper bound of A, so we have $\alpha \leq b_n$ for all $n \in \mathbb{N}$. Thus $a_n \leq \alpha \leq b_n$ and $\alpha \in I_n$. Therefore, I_n is nonempty. $\square$

Theorem 14 (Archimedean Property)**.** The theorem has two parts:

1. Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.
2. Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $\frac{1}{n} < y$.

Proof. Statement 1 in the above theorem is equivalent to the statement: $\mathbb{N}$ is not bounded above. FSOC, assume that $\mathbb{N}$ is bounded above, then let $\alpha = \sup \mathbb{N}$. By the definition of the supremum, $\alpha - 1$ is not an upper bound. Thus, $\alpha - 1 < n$ for some $n \in \mathbb{N}$ implies $\alpha < n + 1$, but $n + 1 \in \mathbb{N}$ by definition so α is less than some natural number and cannot be the supremum, a contradiction! Thus $\mathbb{N}$ is not bounded above, and we have proven statement 1. To prove statement 2, let $x = \frac{1}{y}$ and substitute into the expression in statement 1. $\square$

Wednesday 29 January 2025

Lecture 8

- Math Club in MATH350

Definition 13. Let A, B be two sets, we say that A has the **same cardinality** as B if there exists $f : A \to B$ which is a bijection. In the case we write $A \sim B$. Note that $A \sim B \Leftrightarrow B \sim A$

Example. $A = \{1, 2\}$, $B = \{apple, bananana\}$. Then $A \sim B$ since we can define $f : A \to B$ such that:

$$f(x) = \begin{cases} f(1) & = \text{apple} \\ f(2) & = \text{banana} \end{cases}.$$

f is a bijection, so $A \sim B$

Example. let $E = \{2, 4, 6, 8, \ldots\}$. Claim: $\mathbb{N} \sim E$. Define $f : \mathbb{N} \to E$ given by:

$$\begin{cases} f(1) & = 2 \\ f(2) & = 4 \\ f(3) & = 6 \\ \ldots \end{cases}.$$

f is a bijection, so $\mathbb{N} \sim E$

Example. $\mathbb{N} \sim \mathbb{Z}$

Proof. $f : \mathbb{N} \to \mathbb{Z}$ is given by

$$f(n) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{-n}{2} & \text{if } n \text{ is even.} \end{cases}.$$

f is a bijection, so $\mathbb{N} \sim \mathbb{Z}$ □

Theorem 15. Let A, B, C be sets. If $A \sim B$ and $B \sim C$, then $A \sim C$.

Proof. As $A \sim B$, hence there exists a bijection $f : A \to B$. As $B \sim C$, there exists a bijection $g : B \to C$. Therefore, $g \circ f : A \to C$ is a bijection □

Theorem 16. Let X, Y be two sets. If there exists an injective function $f : X \to Y$ and an injective function $g : Y \to X$, then there exists a bijection $h : X \to Y$ and hence $X \sim Y$.

The above will make our lives easier. We no longer need to find an explicit function. Notice no need to check either function for surjectivity. We get it for free.

Theorem 17. $\mathbb{N} \sim \mathbb{Z}^2$ where

$$\mathbb{Z}^2 = \{(m, n) : m, n \in \mathbb{Z}\}.$$

Informal Proof. Take grid of points down to the number line. □

Proof. Let $f : \mathbb{N} \to \mathbb{Z}^2$ given by

$$f(n) = (n, 0).$$

f is clearly injective.As $\mathbb{Z} \sim \mathbb{N} \Rightarrow$ there exists $g : \mathbb{Z} \to \mathbb{N}$ which is a bijection. Define

$$h : \mathbb{Z}^2 \to \mathbb{N}.$$

where $h(m, n) = 2^{g(m)} \cdot 3^{g(n)}$. Now we will show that h is injective. Assume that $h(m_1, n_1) = h(m_2, n_2)$. We want to show that $m_1 = m_2$ and $n_1 = n_2$:

$$2^{g(m_1)} 3^{g(n_1)} = 2^{g(n_1)} 3^{g(n_2)}.$$

As 2 and 3 are prime numbers, by unique factorization:

$$\Rightarrow g(m_1) = g(m_2) \text{ and } g(n_1) = g(n_2)\,..$$

But $g : \mathbb{Z} \to \mathbb{N}$ is a bijection, hence $m_1 = m_2$ and $n_1 = n_2 \Rightarrow h$ is injective. Thus, by the Cantor-Schroder-Berstein theorem, there exists $z : \mathbb{N} \to \mathbb{Z}^2$ which is a bijection. ☐

Theorem 18. Show that $\mathbb{N} \to \mathbb{N}^3$ where $\mathbb{N}^3 = \{(a, b, c) : a, b, c \in \mathbb{N}\}$

Proof. Let $f : \mathbb{N} \to \mathbb{N}^2$ be $f(n) = (n, 1, 1)$. This f is injective. Let $g : \mathbb{N}^3 \to \mathbb{N}$ where $g(a, b, c) = 2^a 3^b 5^c$. This g is injective by the same logic as before. By CSB, then there exists a bijection $z : \mathbb{N} \to \mathbb{N}^3$. ☐

Theorem 19. A set S is c called **countably infinite** if $S \sim \mathbb{N}$. A set S is called **countable** if either S is finite or countably infinite. S is called **uncountable** if it is not countable. (This definition's slightly different from the textbook).

Example. $A = \{1, 2\}$ is finite and countable. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is countably infinite and countable.

Thursday 30 January 2025

Lecture 9