Real Analysis 1

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Contents

Lecture 13: 02-07-25 Lecture

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Example. Template for a proof of $(x_n \to x)$:

1. Let $\varepsilon > 0$ be given.
2. Choose N (depending on ε in general). This step takes the most amount of work and this work is not shown and is rough work.
3. let $n \geq N$
4. Now prove that $|x_n - x| < \varepsilon$ for all $n \geq N$. Then the proof is complete.

Example. Prove that $\lim \left(\frac{n+1}{n}\right) = 1$

Rough work:

$x_n = \frac{n+1}{n} = 1 + \frac{1}{n}$

$x = 1$

Now we want:

$$|x_n - x| < \varepsilon \tag{1}$$
$$\left|1 + \frac{1}{n} - 1\right| < \varepsilon \tag{2}$$
$$\left|\frac{1}{n}\right| < \varepsilon \tag{3}$$
$$\frac{1}{n} < \varepsilon \tag{4}$$
$$\frac{1}{\varepsilon} < n \tag{5}$$
$$\tag{6}$$

What I really want: Find N so that $\forall n \geq N$, $\frac{1}{\varepsilon} < n$, so choose $N \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < N$, then if $n \geq N \Rightarrow \frac{1}{\varepsilon} < N < n$.

Proof. Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < N$. Let $n \geq N$. This implies that

$$\frac{1}{\varepsilon} < N \leq n \tag{7}$$
$$\frac{1}{n} < \varepsilon \tag{8}$$
$$\left| \left(1 + \frac{1}{n}\right) - 1 \right| < \varepsilon \tag{9}$$
$$\left| \left(\frac{n+1}{n}\right) - 1 \right| < \varepsilon \tag{10}$$
$$\tag{11}$$

Thus we have shown the condition for the proof. □

Example. Prove that $\lim \left(\frac{1}{n^2}\right) = 0$

Proof. Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{\sqrt{\varepsilon}}.$$

Therefore

$$n > \frac{1}{\sqrt{\varepsilon}} \tag{12}$$
$$\frac{1}{n} < \sqrt{\varepsilon} \tag{13}$$
$$\frac{1}{n^2} < \varepsilon \tag{14}$$
$$\left| \frac{1}{n^2} - 0 \right| < \varepsilon \tag{15}$$

□

Example. Prove that $\lim \frac{1}{n^2+576n+100{,}002} = 0$

Proof. If $\frac{1}{n^2} < \varepsilon$

$$\left| \frac{1}{n^2 + 576n + 100{,}002} - 0 \right| < \varepsilon \tag{16}$$

□

Note. Do not try to find an "optimal" N, just find one that works!

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Lecture 14: 02-10-25 Lecture

Theorem 1. The limit of a sequence, when it exists, is unique.

Proof. Let (a_n) be a sequence and assume that $s, t \in \mathbb{R}$ such that $\lim a_n = s$ and $\lim a_n = t$. Let $\varepsilon > 0$ be arbitrary. As $\lim a_n = s$, hence $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, we have $|a_n - s| < \frac{\varepsilon}{2}$. Similarly, as $\lim a_n = t$, $\exists N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$, we have $|a_n - t| < \frac{\varepsilon}{2}$. Let

$$N = \max\{N_1, N_2\} \tag{17}$$

hence $|a_N - s| < \frac{\varepsilon}{2}$ and $|a_N - t| < \frac{\varepsilon}{2}$.

$$\begin{aligned}
|s - t| &= |(s - a_N) + (a_N - t)| && (18)\\
&\leq |s - a_N| + |a_N - t| && (19)\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && (20)\\
&= \varepsilon && (21)\\
&\Rightarrow |s - t| < \varepsilon && (22)
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, this implies that $s = t$ □

Definition 1. A sequence that does not converge is said to diverge.

Example. Prove that the sequence $a_n = (-1)^n$ diverges.

Note. The strategy for these is to assume that it converges, then show that it must converge to two different numbers.

Proof. Suppose by contradiction, let $L \in \mathbb{R}$ be such that $\lim a_n = L$. Therefore, given $\varepsilon = \frac{1}{2}$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$ we have $|a_n - L| < \frac{1}{2}$. Let $n_1 \geq N$ be odd $\Rightarrow |a_n - L| < \frac{1}{2} \Rightarrow |(-1)^{n_1} - L| < \frac{1}{2} \Rightarrow |(-1) - L| < \frac{1}{2}$ as $n_1 + 1 \geq N$ and is even.

$$\Rightarrow |a_{n_1+1} - L| < \frac{1}{2} \Rightarrow |1 - L| < \frac{1}{2} \Rightarrow 2 < 1 \tag{23}$$

a contradiction! □

Example. Prove that $\lim \left(\frac{1}{n}\right) \neq 1$

Proof. By contradiction, assume that $\lim \frac{1}{n} = 1$. Then for $\varepsilon = \frac{1}{2}$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$

$$\left|\frac{1}{n} - 1\right| < \frac{1}{2} \tag{24}$$

$$\Rightarrow 1 - \frac{1}{2} < \frac{1}{n} < 1 + \frac{1}{2} \tag{25}$$

$$\Rightarrow \forall n \geq N \left(\frac{1}{2} < \frac{1}{n}\right) \tag{26}$$

By the Archimedean property, $\exists m \in \mathbb{N}$ such that

$$m \geq N \text{ and } \frac{1}{m} < \frac{1}{2} \tag{27}$$

$$\tag{28}$$

This is a contradiction! □

Definition 2. A sequence (x_n) is **bounded** if there exists $M > 0$ such that $|x_n| \leq M$ $\forall n \in \mathbb{N}$.

Theorem 2. Every convergent sequence is bounded.

This is a standard kind of argument that we will see again and again:

Proof. Let $L \in \mathbb{R}$ be such that $\lim x_n = L$. Hence for $\varepsilon = 1$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have

$$|x_n - L| < 1 \tag{29}$$

Therefore, $\forall n \geq N$

$$|x_n| = |x_n - L + L| \tag{30}$$

$$\leq |x_n - L| + |L| \tag{31}$$

$$< |L| + 1 \tag{32}$$

Let $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |L| + 1\} > 0$. We see that $|x_n| \leq M$ $\forall n \in \mathbb{N}$. Hence (x_n) is bounded. □