# Real Analysis 1

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February 10, 2025

## Contents

## Lecture 14: 02-10-25 Lecture

Monday 10 February 2025

**Theorem 1.** The limit of a sequence, when it exists, is unique.

*Proof.* Let $(a_n)$ be a sequence and assume that $s, t \in \mathbb{R}$ such that $\lim a_n = s$ and $\lim a_n = t$. Let $\varepsilon > 0$ be arbitrary. As $\lim a_n = s$, hence $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, we have $|a_n - s| < \frac{\varepsilon}{2}$. Similarly, as $\lim a_n = t$, $\exists N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$, we have $|a_n - t| < \frac{\varepsilon}{2}$. Let

$$N = \max\{N_1, N_2\} \tag{1}$$

hence $|a_N - s| < \frac{\varepsilon}{2}$ and $|a_N - t| < \frac{\varepsilon}{2}$.

$$\begin{aligned} |s - t| &= |(s - a_N) + (a_N - t)| && (2)\\ &\leq |s - a_N| + |a_N - t| && (3)\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && (4)\\ &= \varepsilon && (5)\\ &\Rightarrow |s - t| < \varepsilon && (6) \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, this implies that $s = t$ $\square$

**Definition 1.** A sequence that does not converge is said to diverge.

**Example.** Prove that the sequence $a_n = (-1)^n$ diverges.

**Note.** The strategy for these is to assume that it converges, then show that it must converge to two different numbers.

*Proof.* Suppose by contradiction, let $L \in \mathbb{R}$ be such that $\lim a_n = L$. Therefore, given $\varepsilon = \frac{1}{2}$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$ we have $|a_n - L| < \frac{1}{2}$. Let

$n_1 \geq N$ be odd $\Rightarrow |a_n - L| < \frac{1}{2} \Rightarrow |(-1)^{n_1} - L| < \frac{1}{2} \Rightarrow |(-1) - L| < \frac{1}{2}$ as $n_1 + 1 \geq N$ and is even.

$$\Rightarrow |a_{n_1+1} - L| < \frac{1}{2} \Rightarrow |1 - L| < \frac{1}{2} \Rightarrow 2 < 1 \tag{7}$$

a contradiction! □

**Example.** Prove that $\lim \left(\frac{1}{n}\right) \neq 1$

*Proof.* By contradiction, assume that $\lim \frac{1}{n} = 1$. Then for $\varepsilon = \frac{1}{2}$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$

$$\left|\frac{1}{n} - 1\right| < \frac{1}{2} \tag{8}$$

$$\Rightarrow 1 - \frac{1}{2} < \frac{1}{n} < 1 + \frac{1}{2} \tag{9}$$

$$\Rightarrow \forall n \geq N \left(\frac{1}{2} < \frac{1}{n}\right) \tag{10}$$

By the Archimedean property, $\exists m \in \mathbb{N}$ such that

$$m \geq N \text{ and } \frac{1}{m} < \frac{1}{2} \tag{11}$$

$$\tag{12}$$

This is a contradiction! □

**Definition 2.** A sequence $(x_n)$ is **bounded** if there exists $M > 0$ such that $|x_n| \leq M$ $\forall n \in \mathbb{N}$.

**Theorem 2.** Every convergent sequence is bounded.

This is a standard kind of argument that we will see again and again:

*Proof.* Let $L \in \mathbb{R}$ be such that $\lim x_n = L$. Hence for $\varepsilon = 1$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have

$$|x_n - L| < 1 \tag{13}$$

Therefore, $\forall n \geq N$

$$|x_n| = |x_n - L + L| \tag{14}$$

$$\leq |x_n - L| + |L| \tag{15}$$

$$< |L| + 1 \tag{16}$$

Let $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |L| + 1\} > 0$. We see that $|x_n| \leq M$ $\forall n \in \mathbb{N}$. Hence $(x_n)$ is bounded. □

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## Lecture 15: Homework 4

1. Let $C \subset (0,1]$ be uncountable. Show that there exists $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable.

*Proof.* □

2. Prove the following limits:

1. $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$
2. $\lim \frac{5n^2}{n^3+2n^2+3n+4} = 0$
3. $\lim \frac{\sin(n^2)}{\sqrt{n}} = 0$
4. $\lim \frac{1}{n^2-10} = 0$

*Proof.* □

3. Let $(a_n)_{n=1}^{\infty}$ be a sequence and let $L \in \mathbb{R}$. Show that $a_n = L$ if and only if the sequence $(a_n - L)_{n=1}^{\infty}$ converges to zero.

*Proof.* □

4. Prove that $\lim a_n = 0$ if and only if $\lim |a_n| = 0$.

*Proof.* □

5. If $|a| < 1$, then prove that $\lim a^n = 0$. (Hint: Use the inequality proved in HW1 namely that for $x > -1$ we have $(1+x)^n \geq 1 + nx$, for a suitable chosen x).

*Proof.* □