Real Analysis 1

Forrest Kennedy

January 29, 2025

Contents

Lecture 7: Complete Analysis Theorems List

Tuesday 28 January 2025

1 The Real Numbers

Definition 1 (Definition of a "function")**.** Given sets A, B a function of $A \to B$ is a mapping that takes ea ch element of A to a single element of B.

Definition 2 (Definition of the "absolute value function")**.** The **absolute value function"** is defined as $|\cdot| : \mathbb{R} \to \mathbb{R}$ such that:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}.$$

Theorem 1 (The Triangle Inequality)**.** With respect to multiplication and division, the absolute value function satisfies:

1. $|ab| = |a|\,|b|$
2. $|a+b| \leq |a| + |b|$

Proof. We will show the theorem by cases WLOG:

1. $(a = 0)$ $|a+b| = |0+b| = |b| = |0| + |b| = |a| + |b|$
2. $(a > 0, b > 0)$ By the definition of the absolute value function we have $|a+b| = a+b = |a| + |b|$
3. $(a < 0, b < 0)$ By the definition of the absolute value function we have $|a+b| = -(a+b) = -a + (-b) = |a| + |b|$
4. $(a > 0, b < 0)$ By the definition of the absolute value, we have $|a| = a$ and $|b| = -b$, so $|a|+|b| = a+(-b)$. We want to show that $|a|+|b| = a+(-b) \geq |a+b|$, so again we consider all the possible cases:

(a) $(a+b=0)$ We have $a+(-b) \overset{?}{\geq} |0| = 0$. Indeed, since $a > 0$ and $b < 0$ we have $a > b$, and our equality holds.

(b) $(a+b>0)$ We have $a+(-b) \overset{?}{\geq} a+b$. Since $b < 0$, we have $-b > 0$. Comparing the LHS and RHS the equality holds.

(c) $(a+b<0)$ We have $a+(-b) \overset{?}{\geq} -a+(-b)$. Comparing the LHS and the RHS, the equality holds.

The above considerations exhaust all possible choices for a and b. In all cases. we see that $|a+b| \leq |a| + |b|$ □

Theorem 2 (The ε criteria for equality)**.** Two real numbers a and b are equal if and only if for every real number $\varepsilon > 0$ it follows that $|a-b| < \varepsilon$.

Proof. We will show the theorem in both directions:

- $(\Rightarrow)$ Given $a = b$, we have $a - b = 0 < \varepsilon$ for all $\varepsilon > 0$.
- $(\Leftarrow)$ Assume that for every $\varepsilon > 0$, $|a-b| < \varepsilon$ and, FSOC, that $a \neq b$. Then, let $\varepsilon_0 = a - b$ which we know is nonzero because $a \neq b$. Now, $|a-b| = \varepsilon_0$ and $|a-b| < \varepsilon_0$ by our first assumption. We have reached a contradiction, therefore the reverse implication must hold.

□

Definition 3 (Bounded Above Property of Subsets of $\mathbb{R}$)**.** A set $A \subset \mathbb{R}$ is **bounded above** if there exists a number $b \in \mathbb{R}$ such that $a \leq b \ \forall a \in A$. The number b is called an **upper bound** for A.

Definition 4 (Bounded Below Property of Subsets of $\mathbb{R}$)**.** A set $A \subset \mathbb{R}$ is **bounded below** if there exists a number $b \in \mathbb{R}$ such that $b \leq a \ \forall a \in A$. The number b is called a **lower bound** for A.

Definition 5 (The Least Upper Bound)**.** An element $s \in \mathbb{R}$ is called the **least upper bound** for $A \subset \mathbb{R}$ if s meets two conditions:

1. s is an upper bound for A
2. $\forall b$ where b is an upper bound, $s \leq b$.

Definition 6 (The Greatest Lower Bound)**.** An element $l \in \mathbb{R}$ is called the **greatest lower bound** for $A \subset \mathbb{R}$ if l meets two conditions:

1. l is a lower bound for A
2. $\forall b$ where b is an upper bound, $l \geq b$.

Definition 7. A real number a_0 is a **maximum** of the set A if a_0 is an elemnt of A and $a_0 \geq a$ for all $a \in A$. Similarly, a number a_1 is a **minimum** of A if $a_1 \in A$ and $a_1 \leq a$ for all $a \in A$.

Theorem 3 (The ε Characterization of the Supremum)**.** Assume $s \in R$ is an upper bound for a set $A \subset \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\varepsilon > 0$, there exists an element $a \in A$ satisfying $s - \varepsilon < a$.

Proof. We will show that both the implication and the inverse implication are true:

- ($\Rightarrow$)If s is the *least* upper bound of A, then $s - \varepsilon$ is not an upper bound for A, thus there exists an $a \in A$ such that $s - \varepsilon < a$.
- ($\Leftarrow$) Assume s is an upper bound of A and that for every $\varepsilon > 0$, $s - \varepsilon < a$. That is, no number smaller than s is an upper bound of A. Thus for all b where b is an upper bound of A, $s \leq b$. Since we assumed that s is an upper bound, s meets both conditions to be the supremum.

□

Theorem 4 (Nested Interval Property of Subsets of $\mathbb{R}$)**.** For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in R : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supset I_2 \supset I_3 \supset \ldots.$$

has a nonempty intersection; that is, $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$, then let $\alpha = \sup A$. From the definition of the supremum, we have $\alpha \geq a_n$ for all $n \in \mathbb{N}$. Because of how we defined our sets, every b_n is an upper bound of A, so we have $\alpha \leq b_n$ for all $n \in \mathbb{N}$. Thus $a_n \leq \alpha \leq b_n$ and $\alpha \in I_n$. Therefore, I_n is nonempty. □

Theorem 5 (Archimedean Property)**.** The theorem has two parts:

1. Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.
2. Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $\frac{1}{n} < y$.

Proof. Statement 1 in the above theorem is equivalent to the statement: $\mathbb{N}$ is not bounded above. FSOC, assume that $\mathbb{N}$ is bounded above, then let $\alpha = \sup \mathbb{N}$. By the definition of the supremum, $\alpha - 1$ is not an upper bound. Thus, $\alpha - 1 < n$ for some $n \in \mathbb{N}$ implies $\alpha < n + 1$, but $n + 1 \in \mathbb{N}$ by definition so α is less than some natural number and cannot be the supremum, a contradiction! Thus $\mathbb{N}$ is not bounded above, and we have proven statement 1. To prove statement 2, let $x = \frac{1}{y}$ and substitute into the expression in statement 1. □

Wednesday 29 January 2025

Lecture 8

- Math Club in MATH350

Definition 8. Let A, B be two sets, we say that A has the **same cardinality** as B if there exists $f : A \to B$ which is a bijection. In the case we write $A \sim B$. Note that $A \sim B \Leftrightarrow B \sim A$

Example. $A = \{1, 2\}$, $B = \{apple, banana na\}$. Then $A \sim B$ since we can define $f : A \to B$ such that:

$$f(x) = \begin{cases} f(1) & = \text{apple} \\ f(2) & = \text{banana} \end{cases}.$$

f is a bijection, so $A \sim B$

Example. let $E = \{2, 4, 6, 8, \ldots\}$. Claim: $\mathbb{N} \sim E$. Define $f : \mathbb{N} \to E$ given by:

$$\begin{cases} f(1) & = 2 \\ f(2) & = 4 \\ f(3) & = 6 \\ \ldots \end{cases}.$$

f is a bijection, so $\mathbb{N} \sim E$

Example. $\mathbb{N} \sim \mathbb{Z}$

Proof. $f : \mathbb{N} \to \mathbb{Z}$ is given by

$$f(n) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{-n}{2} & \text{if } n \text{ is even.} \end{cases}.$$

f is a bijection, so $\mathbb{N} \sim \mathbb{Z}$ □

Theorem 6. Let A, B, C be sets. If $A \sim B$ and $B \sim C$, then $A \sim C$.

Proof. As $A \sim B$, hence there exists a bijection $f : A \to B$. As $B \sim C$, there exists a bijection $g : B \to C$. Therefore, $g \circ f : A \to C$ is a bijection □

Theorem 7. Let X, Y be two sets. If there exists an injective function $f : X \to Y$ and an injective function $g : Y \to X$, then there exists a bijection $h : X \to Y$ and hence $X \sim Y$.

The above will make our lives easier. We no longer need to find an explicit function. Notice no need to check either function for surjectivity. We get it for free.

Theorem 8. $\mathbb{N} \sim \mathbb{Z}^2$ where

$$\mathbb{Z}^2 = \{(m, n) : m, n \in \mathbb{Z}\}.$$

Informal Proof. Take grid of points down to the number line. □

Proof. Let $f : \mathbb{N} \to \mathbb{Z}^2$ given by

$$f(n) = (n, 0).$$

f is clearly injective.As $\mathbb{Z} \sim \mathbb{N} \Rightarrow$ there exists $g : \mathbb{Z} \to \mathbb{N}$ which is a bijection. Define

$$h : \mathbb{Z}^2 \to \mathbb{N}.$$

where $h(m, n) = 2^{g(m)} \cdot 3^{g(n)}$. Now we will show that h is injective. Assume that $h(m_1, n_1) = h(m_2, n_2)$. We want to show that $m_1 = m_2$ and $n_1 = n_2$:

$$2^{g(m_1)} 3^{g(n_1)} = 2^{g(n_1)} 3^{g(n_2)}.$$

As 2 and 3 are prime numbers, by unique factorization:

$$\Rightarrow g(m_1) = g(m_2) \text{ and } g(n_1) = g(n_2)..$$

But $g : \mathbb{Z} \to \mathbb{N}$ is a bijection, hence $m_1 = m_2$ and $n_1 = n_2 \Rightarrow h$ is injective. Thus, by the Cantor-Schroder-Berstein theorem, there exists $z : \mathbb{N} \to \mathbb{Z}^2$ which is a bijection. □

Theorem 9. Show that $\mathbb{N} \to \mathbb{N}^3$ where $\mathbb{N}^3 = \{(a, b, c) : a, b, c \in \mathbb{N}\}$

Proof. Let $f : \mathbb{N} \to \mathbb{N}^2$ be $f(n) = (n, 1, 1)$. This f is injective. Let $g : \mathbb{N}^3 \to \mathbb{N}$ where $g(a, b, c) = 2^a 3^b 5^c$. This g is injective by the same logic as before. By CSB, then there exists a bijection $z : \mathbb{N} \to \mathbb{N}^3$. □

Theorem 10. A set S is c called **countably infinite** if $S \sim \mathbb{N}$. A set S is called **countable** if either S is finite or countably infinite. S is called **uncountable** if it is not countable. (This definition's slightly different from the textbook).

Example. $A = \{1, 2\}$ is finite and countable. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is countably infinite and countable.