Real Analysis 1

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Contents

Lecture 25: 02-28-25 Lecture

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-Midterm next Friday!

Theorem 1 (Cauchy Criterion for Series)**.** The series $\sum_{n=1}^{\infty} a_n$ converges if and only if given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$ we have

$$|a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon \tag{1}$$

Proof. Let (s_m) be the sequence of partial sums. Observe that if $n > m$, then

$$\begin{aligned} |s_n - s_m| &= |(a_1 + a_2 + \ldots + a_n) - (a_1 + a_2 + \ldots + a_m)| \quad (2) \\ &= |a_{m+1} + a_{m+2} + \ldots + a_n| \quad (3) \end{aligned}$$

Now the sequence (s_m) converges if and only if it is Cauchy (s_m) is Cauchy if given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n, m \geq N$ we have

$$|s_n - s_m| < \varepsilon \tag{4}$$

we can assume without loss of generality that $n > m$, i.e. $n > m \geq N$ then

$$|s_n - s_m| < \varepsilon \tag{5}$$

□

Theorem 2. If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim a_n = 0$

Proof. As $\sum_{n=1}^{\infty} a_n$ converges, it satisfies the Cauchy criterion. So let $\varepsilon > 0$ be given. By Cauchy criterion, there $\exists N \in \mathbb{N}$ such that if $n > m \geq N$ we have

$$|a_{m+1} + \ldots + a_n| < \varepsilon \tag{6}$$

Choose $n = m + 1$ then

$$|a_{m+1}| < \varepsilon \tag{7}$$

Thus $\forall n \geq N + 1$ we have

$$|a_n - 0| < \varepsilon \tag{8}$$

□

Example. $\sum_{n=1}^{\infty} (-1)^n$ we observe that the sequence $(-1)^n$ does not converge to 0. Therefore, $\sum_{n=1}^{\infty} (-1)^n$ converges.

Theorem 3 (Comparison Test)**.** Assume that (a_n) and (b_n) are sequences satisfying $0 \leq a_n \leq b_n$ $\forall n \in \mathbb{N}$, then

1) If $\sum_{n=1}^{\infty} b_n$ converges, the $\sum_{n=1}^{\infty} a_n$ also converges.
2) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

Note. Notice that (1) and (2) are contrapositives, we only really need one, but both are useful.

Proof. As $\sum_{n=1}^{\infty} b_n$ converges, it satisfies the Cauchy criterion. We **want to show**

$$\sum_{n=1}^{\infty} a_n \text{ satisfies the Cauchy criterion} \tag{9}$$

Let $\varepsilon > 0$ be given. There exist $N \in \mathbb{N}$ such that whenever $n > m \geq N$ we have

$$|b_{m+1} + \ldots + b_n| < \varepsilon \tag{10}$$

But $0 \leq a_k \leq b_k$ so

$$\Rightarrow |a_{m+1} + \ldots + a_n| < \varepsilon \tag{11}$$

Hence proved. □

Example. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \leq 1$

Proof. We have proved that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Observe that for all $n \in \mathbb{N}$ and $p \leq 1$ we have

$$0 < \frac{1}{n} \leq \frac{1}{n^p} \tag{12}$$

Therefore, by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \leq 1$. □

Example (Geometric Series)**.**

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots \tag{13}$$

is called a geometric series

$$s_m = a + ar + \ldots + ar^{m-1} \tag{14}$$

by induction we can prove that for $r \neq 1$

$$s_m = \frac{a\left(1 - r^m\right)}{1 - r} \tag{15}$$

If $|r| < 1$, then $\lim r^n = 0$. By the ALT, $\lim s_m = \frac{a}{1-r}$