Real Analysis 1

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January 28, 2025

Contents

Lecture 7: Complete Analysis Theorems List

Tuesday 28 January 2025

1 The Real Numbers

Definition 1 (Definition of a "function"). Given sets A, B a function of $A \to B$ is a mapping that takes ea ch element of A to a single element of B.

Definition 2 (Definition of the "absolute value function"). The **absolute value function"** is defined as $|\cdot| : \mathbb{R} \to \mathbb{R}$ such that:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}.$$

Theorem 1 (The Triangle Inequality). With respect to multiplication and division, the absolute value function satisfies:

1. $|ab| = |a|\,|b|$
2. $|a + b| \leq |a| + |b|$

Proof. We will show the theorem by cases WLOG:

1. $(a = 0)$ $|a + b| = |0 + b| = |b| = |0| + |b| = |a| + |b|$
2. $(a > 0, b > 0)$ By the definition of the absolute value function we have $|a + b| = a + b = |a| + |b|$
3. $(a < 0, b < 0)$ By the definition of the absolute value function we have $|a + b| = -(a + b) = -a + (-b) = |a| + |b|$
4. $(a > 0, b < 0)$ By the definition of the absolute value, we have $|a| = a$ and $|b| = -b$, so $|a| + |b| = a + (-b)$. We want to show that $|a| + |b| = a + (-b) \geq |a + b|$, so again we consider all the possible cases:

(a) $(a + b = 0)$ We have $a + (-b) \overset{?}{\geq} |0| = 0$. Indeed, since $a > 0$ and $b < 0$ we have $a > b$, and our equality holds.

(b) $(a + b > 0)$ We have $a + (-b) \overset{?}{\geq} a + b$. Since $b < 0$, we have $-b > 0$. Comparing the LHS and RHS the equality holds.

(c) $(a + b < 0)$ We have $a + (-b) \overset{?}{\geq} -a + (-b)$. Comparing the LHS and the RHS, the equality holds.

The above considerations exhaust all possible choices for a and b. In all cases. we see that $|a + b| \leq |a| + |b|$ □

Theorem 2 (The ε criteria for equality)**.** Two real numbers a and b are equal if and only if for every real number $\varepsilon > 0$ it follows that $|a - b| < \varepsilon$.

Proof. We will show the theorem in both directions:

- ($\Rightarrow$) Given $a = b$, we have $a - b = 0 < \varepsilon$ for all $\varepsilon > 0$.
- ($\Leftarrow$) Assume that for every $\varepsilon > 0$, $|a - b| < \varepsilon$ and, FSOC, that $a \neq b$. Then, let $\varepsilon_0 = a - b$ which we know is nonzero because $a \neq b$. Now, $|a - b| = \varepsilon_0$ and $|a - b| < \varepsilon_0$ by our first assumption. We have reached a contradiction, therefore the reverse implication must hold.

□

Definition 3 (Bounded Above Property of Subsets of $\mathbb{R}$)**.** A set $A \subset \mathbb{R}$ is **bounded above** if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ $\forall a \in A$. The number b is called an **upper bound** for A.

Definition 4 (Bounded Below Property of Subsets of $\mathbb{R}$)**.** A set $A \subset \mathbb{R}$ is **bounded below** if there exists a number $b \in \mathbb{R}$ such that $b \leq a$ $\forall a \in A$. The number b is called a **lower bound** for A.

Definition 5 (The Least Upper Bound)**.** An element $s \in \mathbb{R}$ is called the **least upper bound** for $A \subset \mathbb{R}$ if s meets two conditions:

1. s is an upper bound for A
2. $\forall b$ where b is an upper bound, $s \leq b$.

Definition 6 (The Greatest Lower Bound)**.** An element $l \in \mathbb{R}$ is called the **greatest lower bound** for $A \subset \mathbb{R}$ if l meets two conditions:

1. l is a lower bound for A
2. $\forall b$ where b is an upper bound, $l \geq b$.

Definition 7. A real number a_0 is a **maximum** of the set A if a_0 is an elemnt of A and $a_0 \geq a$ for all $a \in A$. Similarly, a number a_1 is a **minimum** of A if $a_1 \in A$ and $a_1 \leq a$ for all $a \in A$.

Theorem 3 (The ε Characterization of the Supremum)**.** Assume $s \in R$ is an upper bound for a set $A \subset \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\varepsilon > 0$, there exists an element $a \in A$ satisfying $s - \varepsilon < a$.

Proof. We will show that both the implication and the inverse implication are true:

- ($\Rightarrow$)If s is the *least* upper bound of A, then $s - \varepsilon$ is not an upper bound for A, thus there exists an $a \in A$ such that $s - \varepsilon < a$.
- ($\Leftarrow$) Assume s is an upper bound of A and that for every $\varepsilon > 0$, $s - \varepsilon < a$. That is, no number smaller than s is an upper bound of A. Thus for all b where b is an upper bound of A, $s \leq b$. Since we assumed that s is an upper bound, s meets both conditions to be the supremum.

□

Theorem 4 (Nested Interval Property of Subsets of $\mathbb{R}$)**.** For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in R : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supset I_2 \supset I_3 \supset \dots .$$

has a nonempty intersection; that is, $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$, then let $\alpha = \sup A$. From the definition of the supremum, we have $\alpha \geq a_n$ for all $n \in \mathbb{N}$. Because of how we defined our sets, every b_n is an upper bound of A, so we have $\alpha \leq b_n$ for all $n \in \mathbb{N}$. Thus $a_n \leq \alpha \leq b_n$ and $\alpha \in I_n$. Therefore, I_n is nonempty. □

Theorem 5 (Archimedean Property)**.** The theorem has two parts:

1. Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.
2. Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $\frac{1}{n} < y$.

Proof. Statement 1 in the above theorem is equivalent to the statement: $\mathbb{N}$ is not bounded above. FSOC, assume that $\mathbb{N}$ is bounded above, then let $\alpha = \sup \mathbb{N}$. By the definition of the supremum, $\alpha - 1$ is not an upper bound. Thus, $\alpha - 1 < n$ for some $n \in \mathbb{N}$ implies $\alpha < n + 1$, but $n + 1 \in \mathbb{N}$ by definition so α is less than some natural number and cannot be the supremum, a contradiction! Thus $\mathbb{N}$ is not bounded above, and we have proven statement 1. To prove statement 2, let $x = \frac{1}{y}$ and substitute into the expression in statement 1. □