# Physics 101, Final Project
## Geodesic Motion of Objects in General Relativity, the Newtonian Limit, and Gravitational Time Dilation

Forrest Flesher, Natalia Pacheco-Tallaj, Martin Bernstein

May 3, 2018

## Introduction

Hello, this is the section that contains useless wordy information that makes our project look longer and more professional. (I also added fancy sounding section titles ooooh) Probably should say something about the notion of geodesics in general - distances on spheres, differential geometry, the original notion of geodesy on earth, etc. Should also mention something about how the word "geodesic" comes from ancient Greek "γη" and "διαιρεω" meaning "earth" and "divide" respectively.

## 1 The Geodesic Equation

In this section, we use the fact that

$$\frac{d^2 x'^\mu}{d\lambda^2} = 0$$

to show that in an arbitrary set of coordinates $x^\mu(\lambda)$, we have that in a vanishingly small spacetime region, the trajectory of the particle satisfies the *geodesic equation*, which can be defined as

$$\frac{d^2 x^\mu}{d\lambda^2} = -\Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda}.$$

Here, the $\Gamma^\mu_{\nu\rho}$ are the *Christoffel coefficients*, which can be defined as

$$\Gamma^\mu_{\nu\rho} \equiv \frac{\partial x^\mu}{\partial x'^\sigma} \frac{\partial^2 x'^\sigma}{\partial x^\nu \partial x^\rho}.$$

To derive this geodesic equation, we begin by noting that, by the chain rule, we have

$$\frac{dx'^\mu}{d\lambda} = \frac{\partial x'^\mu}{\partial x^\rho} \frac{dx^\rho}{d\lambda}.$$

Now, we can differentiate with respect to $\lambda$, and use the chain rule once again, to obtain

$$\frac{d^2 x'^\mu}{d\lambda^2} = \frac{\partial x'^\mu}{\partial x^\rho}\frac{d^2 x^\rho}{d\lambda^2} + \frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\nu}\frac{\partial x^\nu}{\partial \lambda}\frac{\partial x^\rho}{\partial \lambda}.$$

But this must be equal to zero, so that we have

$$\frac{\partial x'^\mu}{\partial x^\rho}\frac{d^2 x^\rho}{d\lambda^2} = -\frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\nu}\frac{\partial x^\nu}{\partial \lambda}\frac{\partial x^\rho}{\partial \lambda}.$$

This almost gives us our desired geodesic equation. To finish the derivation, we can multiply both sides of the above equation by the partial derivative

$$\frac{\partial x^\mu}{\partial x'^\mu}$$

to obtain

$$\frac{\partial x^\mu}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\rho}\frac{d^2 x^\rho}{d\lambda^2} = -\frac{\partial x^\mu}{\partial x'^\mu}\frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\nu}\frac{\partial x^\nu}{\partial \lambda}\frac{\partial x^\rho}{\partial \lambda}.$$

By the chain rule (and by noting that the derivative of $x^\rho$ is a total derivative), the left hand side of this above equation simplifies to

$$\frac{\partial^2 x^\mu}{\partial \lambda^2}.$$

Thus, we have that

$$\frac{\partial^2 x^\mu}{\partial \lambda^2} = -\left(\frac{\partial x^\mu}{\partial x'^\mu}\frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\nu}\right)\frac{\partial x^\nu}{\partial \lambda}\frac{\partial x^\rho}{\partial \lambda} = -\Gamma^\mu_{\rho\nu}\frac{\partial x^\nu}{\partial \lambda}\frac{\partial x^\rho}{\partial \lambda}.$$

Thus, we see that in an arbitrary set of coordinates, the particle's trajectory satisfies the geodesic equation, as desired.

# 2 Symmetry in the Christoffel Coefficients

In order to justify calculations made later, we note that the Christoffel coefficients, as defined above, are symmetric on their lower coordinate indices. That is, we have that

$$\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}.$$

This is a consequence of Schwarz theorem in analysis, which asserts equality of mixed partials. This gives us that

$$\frac{\partial^2 x'^\sigma}{\partial x^\nu \partial x^\rho} = \frac{\partial^2 x'^\sigma}{\partial x^\rho \partial x^\nu},$$

which clearly implies that

$$\Gamma^\mu_{\rho\nu} = \Gamma^\mu_{\nu\rho}.$$

## 3 Symmetry of The Metric Tensor

Consider the metric tensor defined as

$$g_{\rho\sigma} = \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} \eta_{\mu\nu}.$$

Here, $\eta_{\mu\nu}$ is the Minkowski metric tensor, defined as

$$\eta_{\mu\nu} \equiv \begin{cases} -1 & \text{for } \mu = \nu = 0 \\ 2 & \text{for } \mu = \nu = 1, 2, 3 \\ 0 & \text{for } \mu \neq \nu \end{cases}.$$

Thus, if $\mu \neq \nu$, we have that

$$g_{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\mu}} \frac{\partial x'^{\nu}}{\partial x^{\nu}} \eta_{\mu\nu} = 0 = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x'^{\nu}}{\partial x^{\mu}} \eta_{\nu\mu} = g_{\nu\mu}.$$

If $\mu = \nu$, then we have that

$$g_{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\mu}} \frac{\partial x'^{\nu}}{\partial x^{\nu}} \eta_{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\mu}} \frac{\partial x'^{\mu}}{\partial x^{\mu}} \eta_{\mu\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x'^{\nu}}{\partial x^{\mu}} \eta_{\nu\mu} = g_{\nu\mu}.$$

Now, the inverse metric tensor $g^{\mu\nu}$ is defined by the condition

$$g^{\mu\rho} g_{\rho\nu} = g_{\nu\rho} g^{\rho\mu} = \delta^{\mu}_{\nu}.$$

Again, if $\mu \neq \nu$, then

$$g^{\mu\nu} = g^{\nu\mu} = 0.$$

If $\mu = \nu$, then we have that in the expression

$$g^{\mu\nu} g_{\mu\nu} = g_{\nu\mu} g^{\nu\mu},$$

then the terms $g_{\mu\nu}$ and $g_{\nu\mu}$ cancel, since they are equal as shown above, which gives us that

$$g^{\mu\nu} = g^{\nu\mu},$$

as desired.

## 4 Alternate Formula for the Christoffel Coefficients

Now, we introduce the notation $\partial_{\mu}$, which is defined as

$$\partial_{\mu} \equiv \left( \frac{1}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)_{\mu}.$$

Using this notation, we now demonstrate that

$$\partial_\rho g_{\rho\nu} = \Gamma^\sigma_{\rho\mu} g_{\sigma\nu} + \Gamma^\sigma_{\rho\nu} g_{\mu\sigma}.$$

To do this, first consider

$$\partial_\rho \left( \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\mu} \eta_{\alpha\beta} \right).$$

We define $m$ to be

$$m \equiv \begin{cases} \frac{1}{c}, & w = t \\ 1, & w = x, y, z \end{cases}$$

Using the product rule, this is

$$m \frac{\partial^2 x'^\alpha}{\partial x^\mu \partial x^\rho} \frac{\partial x'^\beta}{\partial x^\nu} \eta_{\alpha\beta} + m \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial^2 x'^\beta}{\partial x^\nu \partial x^\rho} \eta_{\alpha\beta}$$

Now, note that

$$\partial_\rho \left( \frac{\partial x'^\alpha}{\partial x^\mu} \right) \frac{\partial x'^\beta}{\partial x^\nu} \eta_{\alpha\beta}$$

is equal to

$$\left( m \frac{\partial^2 x'^\alpha}{\partial x^\mu \partial x^\rho} \right) \frac{\partial x'^\beta}{\partial x^\nu} \eta_{\alpha\beta}.$$

Finally, we also have that

$$\partial_\rho \left( \frac{\partial x'^\beta}{\partial x^\nu} \right) \frac{\partial x'^\alpha}{\partial x^\mu} \eta_{\alpha\beta}$$

is equal to

$$\left( m \frac{\partial^2 x'^\beta}{\partial x^\nu \partial x^\rho} \right) \frac{\partial x'^\alpha}{\partial x^\mu} \eta_{\alpha\beta}$$

From this, we see that the following formula holds:

$$\partial_\rho \left( \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\mu} \eta_{\alpha\beta} \right) - \partial_\rho \left( \frac{\partial x'^\alpha}{\partial x^\mu} \right) \frac{\partial x'^\beta}{\partial x^\nu} \eta_{\alpha\beta} - \partial_\rho \left( \frac{\partial x'^\beta}{\partial x^\nu} \right) \frac{\partial x'^\alpha}{\partial x^\mu} \eta_{\alpha\beta} = 0$$

Now, let $\alpha = \mu$, and $\beta = \nu$. Then we have

$$\frac{1}{m} \partial_\rho \left( \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\mu} \eta_{\alpha\beta} \right) = \frac{1}{m} \partial_\rho \left( \frac{\partial x'^\mu}{\partial x^\mu} \frac{\partial x'^\nu}{\partial x^\mu} \eta_{\mu\nu} \right) = \partial_\rho g_{\mu\nu}$$

Now, we have that

$$\partial_\rho \left( \frac{\partial x'^\mu}{\partial x^\mu} \right) \frac{\partial x'^\nu}{\partial x^\nu} \eta_{\mu\nu} = m \frac{\partial^2 x'^\mu}{\partial x^\mu \partial x^\rho} \frac{\partial x'^\nu}{\partial x^\nu} \eta_{\mu\nu}$$

and

$$\partial_\rho \left( \frac{\partial x'^\mu}{\partial x^\nu} \right) \frac{\partial x'^\nu}{\partial x^\mu} \eta_{\mu\nu} = m \frac{\partial^2 x'^\nu}{\partial x^\nu \partial x^\rho} \frac{\partial x'^\mu}{\partial x^\mu} \eta_{\mu\nu}$$

But also, note that

$$\begin{aligned}
\Gamma^{\sigma}_{\rho\mu} &= \frac{\partial x^{\sigma}}{\partial x'^{\nu}}\frac{\partial^2 x'^{\nu}}{\partial x^{\rho}\partial x^{\mu}} \\
g_{\sigma\nu} &= \frac{\partial x'^{\mu}}{\partial x^{\sigma}}\frac{\partial x'^{\nu}}{\partial x^{\nu}}\eta_{\mu\nu} \\
\Gamma^{\sigma}_{\rho\nu} &= \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial^2 x'^{\mu}}{\partial x^{\rho}\partial x^{\nu}} \\
g_{\mu\sigma} &= \frac{\partial x'^{\mu}}{\partial x^{\mu}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}\eta_{\mu\nu}
\end{aligned}$$

Then we have that

$$\begin{aligned}
\Gamma^{\sigma}_{\rho\mu}g_{\sigma\nu} &= \frac{\partial x^{\sigma}}{\partial x'^{\nu}}\frac{\partial^2 x'^{\mu}}{\partial x^{\rho}\partial x^{\mu}}\frac{\partial x'^{\mu}}{\partial x^{\sigma}}\frac{\partial x'^{\nu}}{\partial x^{\nu}}\eta_{\mu\nu} \\
\Gamma^{\sigma}_{\rho\nu}g_{\mu\sigma} &= \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial^2 x'^{\nu}}{\partial x^{\rho}\partial^{\nu}}\frac{\partial^{x'^{\mu}}}{\partial x^{\mu}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}\eta_{\mu\nu}
\end{aligned}$$

Now, note that by the chain rule, we have that

$$\frac{\partial x'^{\mu}}{\partial x^{\sigma}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} = \frac{\partial x^{\sigma}}{\partial x'^{nu}}\frac{\partial x'^{\mu}}{\partial x^{\sigma}} = \delta^{\mu}_{\nu}.$$

Thus, noting that we don't need the Kronecker delta here, since the $g_{\mu\nu}$ are already zero for $\mu \neq \nu$, we can simplify our formulas above:

$$\begin{aligned}
\Gamma^{\sigma}_{\rho\mu}g_{\sigma\nu} &= \frac{\partial^2 x'^{\mu}}{\partial x^{\rho}\partial x^{\mu}}\frac{\partial x'^{\nu}}{\partial x^{\nu}}\eta_{\mu\nu} \\
\Gamma^{\sigma}_{\rho\nu}g_{\mu\sigma} &= \frac{\partial^2 x'^{\nu}}{\partial x^{\rho}\partial^{\nu}}\frac{\partial x'^{\mu}}{\partial x^{\mu}}\eta_{\mu\nu}
\end{aligned}$$

But now we see that these are equal to

$$\frac{1}{m}\partial_{\rho}\left(\frac{\partial x'^{\mu}}{\partial x^{\mu}}\right)\frac{\partial x'^{\nu}}{\partial x^{\nu}}\eta_{\mu\nu}$$

and

$$\frac{1}{m}\partial_{\rho}\left(\frac{\partial x'^{\nu}}{\partial x^{\nu}}\right)\frac{\partial x'^{\mu}}{\partial x^{\mu}}\eta_{\mu\nu}$$

from above. Now, by our equation

$$\partial_{\rho}\left(\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\mu}}\eta_{\alpha\beta}\right) - \partial_{\rho}\left(\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\right)\frac{\partial x'^{\beta}}{\partial x^{\nu}}\eta_{\alpha\beta} - \partial_{\rho}\left(\frac{\partial x'^{\beta}}{\partial x^{\nu}}\right)\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\eta_{\alpha\beta} = 0,$$

we see that the $m$ terms cancel, so that by the equalities shown above, we have that

$$\partial_{\rho}g_{\mu\nu} = \Gamma^{\sigma}_{\rho\mu}g_{\sigma\nu} + \Gamma^{\sigma}_{\rho\nu}g_{\mu\sigma},$$

as desired.

Now, note that from the above expression

$$\partial_\rho g_{\mu\nu} = \Gamma^\sigma_{\rho\mu} g_{\sigma\nu} + \Gamma^\sigma_{\rho\nu} g_{\mu\sigma}$$

we obtain the following formulas by plugging in $(\rho, \sigma) = (\mu, \rho), (\rho, \sigma) = (\nu, \rho)$, and $(\rho, \sigma) = (\sigma, \rho)$ respectively:

$$\begin{aligned}
\partial_\mu g_{\nu\sigma} &= \Gamma^\rho_{\mu\nu} g_{\rho\sigma} + \Gamma^\rho_{\mu\sigma} g_{\nu\rho} \\
\partial_\nu g_{\mu\sigma} &= \Gamma^\rho_{\nu\mu} g_{\rho\sigma} + \Gamma^\rho_{\nu\sigma} g_{\mu\rho} \\
\partial_\sigma g_{\mu\nu} &= \Gamma^\rho_{\sigma\mu} g_{\rho\nu} + \Gamma^\rho_{\sigma\nu} g_{\mu\rho}
\end{aligned}$$

Using these equations, and the symmetry of the Christoffel coefficients and metric tensors, we have that

$$\begin{aligned}
&\frac{1}{2} g^{\rho\sigma} \left( \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) = \\
&\quad \frac{1}{2} g^{\rho\sigma} \Big( \Gamma^\rho_{\mu\nu} g_{\rho\sigma} + \Gamma^\rho_{\mu\sigma} g_{\nu\rho} + \Gamma^\rho_{\nu\mu} g_{\rho\sigma} + \Gamma^\rho_{\nu\sigma} g_{\mu\rho} - \left( \Gamma^\rho_{\sigma\mu} g_{\rho\nu} + \Gamma^\rho_{\sigma\nu} g_{\mu\rho} \right) \Big) \\
&\quad = \frac{1}{2} \left( \Gamma^\rho_{\mu\nu} g_{\rho\sigma} + \Gamma^\rho_{\nu\mu} g_{\rho\sigma} \right) \\
&\quad = \Gamma^\rho_{\mu\nu}.
\end{aligned}$$

Thus, gives us a formula for the Christoffel coefficients in terms of the metric tensor and its inverse, with no reference to locally intertial coordinates:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} \left( \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)$$

# 5 The Newtonian Limit

# 6 Gravitational Fields and Time Dilation

# 7 An Application to Spherically Symmetric Objects