Percolation in Random Resistor Networks

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1 Introduction

Percolation deals with the properties of clusters created by some probabilistic means. Generally, past a threshold an infinite cluster will form that spans the lattice and we consider this a phase transition. While the process that generates this cluster is simply stated, its properties are quite rich, resisting analytic treatment and exhbiting complex phenomena such as multifractal scaling. The marriage of conceptual simplicity and complex behavior makes percolation a popular introduction to phase transitions, and the generality of the model makes it applicable to a wide variety of physical problems from superconductivity to forest fires.

This coverage aims to give an introduction to percolation theory with particular attention to the problem as a conduction transition in a random resistor network. We will follow our coverage of the Ising model, first introducing percolation and giving brief attention to it's variants and correspondence to physical transitions. Then we will review exact solutions on simple lattices, namely in one dimension and the Bethe lattice. Next we will review mean field theory techniques away from the transition and then scaling relations derivable from assumptions about the form of the infinite cluster. Finally, we will review applications of renormalization, and give a summary of extensions to more complex examples.

1.1 The Basic Percolation Problem

Consider a finite cubic lattice where neighboring sites are not yet connected. At every potential edge, place a bond with a probability, or **concentration** p. As we increase this concentration from 0, at some point instantiations of the lattice will contain clusters that connect from the top to the bottom and we say the lattice **percolates**. As the size of the lattice increases, the point at which a percolating cluster forms becomes more sharply defined and in the infinite limit we can define a critical concentration p_c beyond which an infinite cluster exists.

We may consider a similar problem where bonds are all occupied but *sites* are initially vacant. Each site is occupied with a probability p and clusters are collections of occupied sites connected by bonds. This defines the **site percolation** as opposed to **bond percolation** mentioned previously. Depending on the physical problem we are interested in, we may choose to frame it in terms of sites or bonds but the results given by either will be closely related. Here we will focus on bond percolation with the intention of relating our results to the random resistor network. This coverage is adapted from [3] [1] which both cover site percolation. Results here have been rederived for bond percolation following their arguments.

Lattice	Coord. #	Site	Bond
1d	2	1	1
2d Honeycomb	3	0.6962	$1 - 2\sin\pi/8 \approx 0.65271$
2d Square	4	0.592746	1/2
2d Triangular	6	1/2	$2\sin\pi/18 \approx 0.34729$
3d Diamond	4	0.43	0.388
3d Simple cubic	6	0.3116	0.2488
4d Hypercubic	8	0.197	0.1601
5d Hypercubic	10	0.141	0.1182
Bethe Lattice	z	$1/(z-1)$	$1/(z-1)$

Table 1: Percolation thresholds for various lattices. Note that within a dimension, the threshold decreases with increasing coordination number. This is an abridged version of a table found in [1].

1.2 Important Quantities and Exponents

We will make use of several quantities to characterize the lattice and it's transition. Here we give a brief summary of each, important relations involving them and their critical exponents:

p_c The **percolation threshold** is the concentration at which an infinite cluster will form on an infinite lattice. While easily evaluated on the Bethe lattice, analytical results resisted attempts for 20 years after the problem was initally posed. As with the critical temperature, it is nonuniversal and depends on the details of the lattice structure. Currently accepted values may be viewed in Table 1.

P(p) The **strength** of the infinite cluster gives the probability that an arbirary bond is connected to the infinite cluster. As such, P is 0 below the percolation threshold and goes to 1 as $p \to 1$. In most lattices, at p_c P increases continuously from 0 and so is analogous to our order parameter for a second order phase transition. This allows us to define the critical exponent β,

$$P(p) \propto (p - p_c)^\beta \text{ as } p \to p_c^+$$

$n_s(p)$ The **cluster number distribution** gives the average number of clusters of size s per lattice site (such that in a lattice of size N there will be on average $Nn_s(p)$ clusters of size s at concentration p). As all bonds must belong to a cluster of some size, we have,

$$\sum_s s n_s(p) = p, \quad p < p_c$$

$$\sum_s s n_s(p) + P(p) = p, \quad p > p_c$$

where the sum excludes the infinite cluster. This quantity $sn_s(p)$ is somewhat analogous to a Boltzman weight in that it gives us information on the probabilities of certain configurations of the system.

s_ξ We will find that the cluster number distribution away from the percolation threshold, $n_s \propto \exp(s/s_\xi)$. The scaling factor in the denominator is the **cutoff cluster size**. This will generally diverge at the percolation threshold allowing us to define the critical exponent σ

$$s_\xi \propto |p_c - p|^{-\frac{1}{\sigma}} \text{ as } p \to p_c$$

S The cluster number distribution allows us to calculate a mean cluster size which we define as the average cluster size to which a given occupied bond belongs. As the fraction of occupied bonds belonging to s clusters is $\frac{sn_s(p)N}{pN} = \frac{sn_s(p)}{\sum sn_s(p)}$ the average cluster size is

$$S(p) = \frac{\sum s^2 n_s(p)}{\sum sn_s(p)}$$

(Note that in this definition clusters are sampled by *bond* biasing our selection towards larger clusters. We could similarly have defined an average cluster size in which all clusters are sampled with equal probablity $S'(p) = \frac{\sum sn_s(p)}{\sum n_s(p)}$ but we will find that our current definition is more appropriate.) This is analogous to our susceptibility in thermal problems.

The mean cluster size will also diverge at the percolation threshold, allowing us to define the critical exponent γ

$$S \propto |p_c - p|^{-\gamma} \text{ as } p \to p_c$$

$g(r)$ The **correlation function** or **pair connectivity** is the probability that a bond a distance r away from an occupied site belongs to the same cluster. In analogy to the susceptibilty sum rule, we have a relation between the correlation function and the mean cluster size,

$$\sum_{\vec{r}} g(\vec{r}) = S$$

which is our motivation for defining the average cluster size as we have.

ξ The **correlation length** is the rms radius of a typical cluster. We can define it through the normalized correlation function $\frac{g(r)}{S(p)}$ as,

$$\xi^2 = \frac{1}{S(p)} \sum_{\vec{r}} r^2 g(\vec{r}).$$

It will typically diverge at the percolation threshold, allowing us to define the critical exponent ν,

$$\xi \propto |p_c - p|^{-\nu} \text{ as } p \to p_c$$

2 Exact Solutions

In this section we will calculate the preceding quantities in two scenarios in which they may be evaluated exactly: the one dimensional chain and the Bethe Lattice. Again similar coverages for site percolation (which is identical to bond percolation in 1d) may be found in [3] [1]

2.1 One-Dimension

On an infinite lattice, any unoccupied bond will break the infinite cluster and thus we can only have percolation at $p_c = 1$ (in analogy with the 1d Ising model). The strength $P(p)$ is thus zero for $p < 1$ and jumps discontinuously to 1 at percolation.

To find the cluster number distribution, we note that the probability for a cluster of size s to lie at any given position is $p^s(1-p)^2$. As the number of possible locations for such a cluster is equal to the number of bonds we have (from the linearity of the expectation) that the expected number of such clusters *per bond* is

$$n_s(p) = p^s(1-p)^2$$

As a check, we can evaluate,

$$\begin{aligned}\sum_s s n_s(p) &= \sum_s s p^s (1-p)^2 \\ &= (1-p)^2 (p\frac{d}{dp}) \sum_s p^s \\ &= p\frac{(1-p)^2}{(1-p)^2} = p\end{aligned}$$

We may write this quantity as,

$$n_s(p) = (p_c - p)^2 \exp(-s/s_\xi), \quad s_\xi = -\frac{1}{\ln p}$$

and obtain the cutoff cluster size. We thus have

$$s_\xi = -\frac{1}{\ln(1-(p_c-p))} \propto |p_c - p|^{-1}$$

yielding the critical exponent $\sigma = 1$.

To evaluate the average cluster size, we proceed as with our previous check on the cluster number distribution to obtain,

$$S = \frac{\sum_s s^2 n_s(p)}{\sum_s s n_s(p)} = \frac{1+p}{1-p}$$

and we can immediately identify $\gamma = 1$.

In order for two bonds separated by a distance r to be connected, all bonds between them must be connected and thus $g(r) = p^r$. We can confirm the susceptibility sum rule noting that the sum over $\vec{r}$ breaks into a contribution from $r = 0$ and two from bonds to the left and right.

$$\sum_{\vec{r}} g(r) = 1 + 2\sum_{r=1,2..} p^r = 1 + 2p \sum_{s=0,1,2..} p^s = \frac{1+p}{1-p} = S.$$

Lastly, to obtain the correlation length, we evaluate

$$\begin{aligned}\xi^2 =& \frac{1-p}{1+p} \sum_{\vec{r}} r^2 p^r \\ =& \frac{1-p}{1+p} \left(p\frac{d}{dp}\right)^2 2 \sum_{r=1,2...} p^r \\ =& \frac{2p}{(1-p)^2}\end{aligned}$$

and we find $\nu = 1$.

2.2 Bethe Lattice/Mean-Field Solution

The Bethe lattice is an infinite tree (read no loops) in which each site has z nearest neighbors. If $z = 2$ we regain the 1d chain. The Bethe lattice is so named for the Bethe Peierls approximation whose results are exact when applied to the Bethe lattice. Readers interested in its history and breadth of application in physics can consult Thorpe's charming and seemingly obscure coverage in [4].

Considering entering a site through an occupied bond, we expect that the lattice will percolate when the expected number of bonds continuing onward is 1. This yields the correct percolation threshold of $p_c = \frac{1}{z-1}$ which may be established through more rigorous and complex methods [2].

In order to obtain the strength of the infinite cluster, we consider a arbitrary bond in the lattice. In order for this bond to be connected to the infinite cluster, it must be occupied and at least one neighboring bond must be connected to the cluster. This begs us to introduce the probability Q that a given neighboring bond is not part of the infinite cluster. The structure of the Bethe lattice allows us to obtain a self-consistent equation for the probability Q. If a bond does not connect to infinity, it must be either unoccupied, or occupied and none of its neighboring bonds connects to infinity, $Q = (1-p) + pQ^{z-1}$. Aside from perturbatively, I am not sure how to solve this for arbitrary z (If anyone reading this knows how, I would appreciate an e-mail. While I know that an arbitrary higher degree polynomial is not analyticaly soluble, I am not so pessimistic about the depressed form here.) We will examine the situation in the simpler regime where $z = 3$. Here we can solve, $pQ^2 - Q + (1-p) = 0$ giving the two solutions $Q = 1, \frac{1-p}{p}$. The probability that an arbitrary bond is both occupied and connects to the infinite cluster is then,

$$P(p) = p(1-Q^4) = \frac{p^4 - (1-p)^4}{p^3}.$$

Evaluation at the percolation threshold yields $P(p_c) = 0$ as expected and $P'(p_c) > 0$ which requires $\beta = 1$.

Our argument for the cluster number distribution in 1d applies here as well, but suffers from a complication. In 1d, there is a single possible configuration for a cluster of s bonds: a chain of length l. Such a chain has a perimeter of 2 and so the contribution to the cluster number distribution is of the form $g_{s,t}p^s(1-p)^t$ where $g_{s,t}$ is the multiplicity of shapes of clusters of size s and perimeter t. We can thus write the cluster number distribution for an arbitrary lattice as

$$n_s(p) = \sum_t g_{s,t}p^s(1-p)^t.$$

The Bethe lattice also possesses the attractive property that there is a unique relationship between cluster size and perimeter. If we consider a single bond, it has a perimeter of $t_1 = 2(z-1)$. In adding another bond we will decrease the perimeter by 1 and increase it by $z-1$. This will continue such that we have the general form for a perimeter of $t_s = (s-1)(z-2) + 2z - 2 = s(z-2) + z$ and thus a cluster number distribution of the form

$$n_s(p) = g_{s,t}p^s(1-p)^{s(z-2)+z}.$$

Evaluation of the multiplicity $g_{s,t}$ is somewhat complex and may continue through a pleasing use of generating functions [2] which, time permitting, may be included in an appendix. As we are primarily interested in the form near the critical point, we will bypass this difficulty by considering the ratio,

$$\frac{n_s(p)}{n_s(p_c)} = \frac{(1-p)^z}{(1-p_c)}\left(\frac{p(1-p)^{z-2}}{p_c(1-p_c)^{z-2}}\right)^s \propto \exp\left(-s/s_\xi\right)$$

and we again observe an exponential decay in the cluster size away from the percolation threshold. We thus have for our cutoff cluster size,

$$s_\xi = -\frac{1}{\ln\left(\frac{p(1-p)^{z-2}}{p_c(1-p_c)^{z-2}}\right)}.$$

Observing that the argument of the logarithm is a polynomial with a maximum of 1 at $p_c = \frac{1}{z-1}$, and thus near p_c may be approximated by

$$s_\xi \approx -\frac{1}{\ln\left(1 - a(p-p_c)^2\right)} \propto |p_c - p|^{-2}$$

yielding the critical exponent $\sigma = \frac{1}{2}$.

An argument similar to our calculation of the strength yields the average cluster size. Given an occupied site, the average cluster size will be $S = 1 + 2(z-1)T$ where T is the average number of sites connected to a single branch. The average contribution is 0 with probability $1-p$ and $1 + (z-1)T$ with probability p. This yields an average contribution per branch of $T = \frac{p}{1-(z-1)p}$ and an average cluster size of $S = \frac{1+(z-1)p}{1-(z-1)p}$ from which we can read off the critical exponent $\gamma = 1$

In the same fashion as in 1d, in order for two bonds separated by r to be connected, all bonds along the unique path joining them must be occupied. This gives $g(\vec{r}) = p^r$ and we may again confirm our susceptibility sum rule: beginning at a single bond, the $r = 1$ level has $2(z-1)$ bonds, the $r = 2$, $2(z-1)^2$ and so on such that,

$$\begin{aligned}\sum_{\vec{r}} g(\vec{r}) =& 1 + 2\sum_{r=1,2,\ldots} (z-1)^r p^r \\ =& 1 + \frac{2(z-1)p}{1-(z-1)p} = \frac{1+(z-1)p}{1-(z-1)p} = S\end{aligned}$$

Lastly, to obtain the correlation length, we evaluate

$$\begin{aligned}\xi^2 =& \frac{1-(z-1)p}{1+(z-1)p} \sum_{\vec{r}} r^2 (z-1)^r p^r \\ =& \frac{1-(z-1)p}{1+(z-1)p} \left(p\frac{d}{dp}\right)^2 2\sum_{r=1,2\ldots} (z-1)^r p^r \\ =& \frac{2(z-1)p}{(1-(z-1)p)^2}\end{aligned}$$

and we find $\nu = 1$.

3 Scaling Relations

The exponential decay we have observed in the cluster number distribution is far from a complete picture of its behavior. As with magnetic systems, there are many choices for a scaling hypothesis including the finite cluster probability, correlation function, and cluster number distribution. These are reviewed and compared in [**?**]. In the interest of conserving space and not introducing more new quantities I have chosen to follow Stauffer and Christensen in examining the cluster number distribution scaling.

To facilitate the discussion, in figure **??** we have displayed the cluster number distribution in 1d taken from [1]. On the left, we observe two regimes of behavior. For $s < s_\xi$ the distribution decreases with a power law behavior while clusters larger than s_ξ are exponentially suppressed. The similarity of these curves suggests that if we rescale by the cutoff such that the dotted lines coincide and introduce a scaling on the vertical axis, these may all be instantiations of the same curve. Indeed, on the right we find that when the proper scaling factors are selected, all distributions collapse onto a single curve $x^2 e^{-x}$. This suggests that cluster number distribution scales as a power

of s mulitiplied by a function of the single variable $s/s_\xi = (p-p_c)^{\frac{1}{\sigma}}s$. For 1d this is apparently $n_s(p) = s^{-2}f(s/s_\xi)$ with $f(x) = x^2e^{-x}$. A similar argument for the Bethe lattice suggests the form $n_s(p) \propto s^{-5/2}f(s/s_\xi)$ with $f(x) = e^{-x}$. We would like to posit the general form $n_s(p) \propto s^{-\tau}f(s/s_\xi)$ but we encounter a problem in the argument of f. From the perimeter expansion for $n_s(p)$ we know $n_s(p)$ is polynomial in p and thus all derivatives are smooth. However if the argument of the function is of the form $(p-p_c)^{\frac{1}{\sigma}}s$ with noninteger $\frac{1}{\sigma}$, derivatives will generate negative powers of $(p-p_c)$ and thus singularities. To sidestep this difficulty, we modify the argument of the function $(p-p_c)^{\frac{1}{\sigma}}s \Rightarrow (p-p_c)s^\sigma$ and require that all derivatives of our function are also smooth.

Our final scaling form is thus,

$$n_s(p) \propto s^{-\tau}f((p-p_c)s^\sigma)$$

for large s and $p \to p_c$. Generally $f(x)$ is slowly varying for $x \ll 1$ and decays exponentially for $x \gg 1$. The exponents τ and σ and the function f are independent of the lattice details and depend only on dimensionality. In analogy to thermal phase transitions we may use this relationship to derive scaling laws relating our critical exponents. For a few examples:

- The divergence in the average cluster size as $p \to p_c^-$ is driven by the numerator $\sum_s s^2 n_s(p)$. As this sum diverges due to contributions at large s, we may extract the scaling of the divergence by converting the sum to an integral,

$$\begin{aligned}\sum_s s^{2-\tau}f((p-p_c)s^\sigma) \approx &\int_1^\infty ds\, s^{2-\tau}f((p-p_c)s^\sigma) \\ \approx &(p-p_c)^{\frac{\tau-3}{\sigma}}\int_{(p-p_c)}^\infty dz\frac{1}{\sigma}|z|^{\frac{3-\sigma-\tau}{\sigma}}f(z).\end{aligned}$$

  As the divergence was at large s we may set the lower limit to 0 and identify $\gamma = \frac{3-\tau}{\sigma}$.

- When we attempt to calculate the power, $P = p - \sum_s sn_s(p)$ the sum is finite and conversion to an integral will not work. Furthermore, the difference with p does not allow us to extract the scaling. We circumvent this by taking two derivatives with respect to p, yielding the relation,

$$\begin{aligned}(p-p_c)^{\beta-2} \propto &\int_1^\infty ds\, s^{1-\tau+2\sigma}f((p-p_c)s^\sigma) \\ \approx &(p-p_c)^{\frac{2-\tau+2\sigma}{\sigma}}\int_{(p-p_c)}^\infty dz\frac{1}{\sigma}|z|^{\frac{2-\tau+\sigma}{\sigma}}f(z).\end{aligned}$$

  Which yields the relation $\beta = \frac{\tau-2}{\sigma}$.

- We may also define a specific heat exponent in analogy to the scaling of the free energy in thermal transitions. The number of finite clusters per lattice site $M_0 = \sum_s n_s(p) \propto (p-p_c)^{2-\alpha}$. An argument similar to the last two yields $2-\alpha = \frac{\tau-1}{\sigma}$.

Combining these three relationships to eliminate τ and σ gives the familiar Rushbrooke scaling law, $\alpha + 2\beta + \gamma = 2$.

4 Renormalization

The percolation problem is in some ways a more accessible environment in which to apply renormalization methods. As we are able to renormalize the probabilites directly and in real space, the procedure is simplified and thus more conceptually clear. To avoid a digression into a mapping to the Potts model in order to apply momentum space RG, only real-space methods will be discussed here. As the lattice tranformation in site percolation is somewhat more transparent, we will walk through the RG steps using it as an example.

I We begin by replacing the lattice with a 'superlattice' of blocks of linear size l. As in thermal transitions, the new lattice should carry the same symmetry as the old such that renormalization may be repeated.

On the triangular lattice in site percolation, we may select triangle of three sites which will become a single site on the 'superlattice.' This gives a linear size of $l = \sqrt{3}$.

II Blocks in the old lattice are averaged to determine a new probability $p' = R_l(p)$.

The RG will shift the probability of our latticeWe use a majority rule to determine the state of the 'superblock.' The probability that two or more sites are occupied is,

$$p' = R_l(p) = p^3 + 3p^2(1-p) = 3p^2 - 2p^3.$$

This gives the fixed points $R_l(p^*) = p^* = 0, 1, \frac{1}{2}$.

III We restore the original lattice dimensions by rescaling by l.

The correlation length in our new model is $\xi' = \frac{\xi}{l}$. Near the percolation threshold we have $|R_l(p) - p_c|^{-\nu} = \frac{|p-p_c|^{-\nu}}{l}$. A bit of rearrangement yields a formula for ν,

$$\nu = \frac{\log(l)}{\log\left[\frac{|R_l(p)-R_l(p_c)|}{|p-p_c|}\right]} = \frac{\log(l)}{\log\left[\left.\frac{dR_l(p)}{dp}\right|_{p^*}\right]}$$

Evaluating this on the triangular lattice gives

$$\nu = \frac{\log(\sqrt{3})}{\log(3/2)} = 1.355$$

which when compared to the exact values of $p_c = \frac{1}{2}$ and $\nu = \frac{4}{3}$ is quite impressive for the investment.

For bond percolation, we are required to choose a slightly more complicated transformation. We consider the square lattice in 2 dimensions and choose a $2x2$ block of bonds to renormalize upon (see figure FIGURE). Bonds along the bottom and right sides of the block are left to neighboring blocks and the block will be replaced by two bonds, one along the top and another along the left side which will be placed with a probability p' in the superlattice. The cluster spanning rule stipulates that the horizontal superbond will be occupied if the original block percolates from left to right, and the vertical superbond will be placed if it percolates vertically. The symmetry of the block requires that these two events occur with equal probability and we thus only consider horizontal percolation. The possible percolating pond structures are displayed in figure FIGURE. These give the RG transformation,

$$R_2(p) = 2p^5 - 5p^4 + 2p^3 + 2p^2$$

with the fixed points,

$$R_2(p^*) = p^* = 0, 1, \frac{1}{2}.$$

The RG approximation for bond percolation on the square lattice thus gives the exact percolation threshold $p_c = \frac{1}{2}$ and approximate critical exponent $\nu = \frac{\log(2)}{\log(\frac{13}{8})} = 1.428$ for the exact value $\nu = \frac{4}{3}$.

5 Extensions

References

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