Percolation in Random Resistor Networks

Forrest Sheldon

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1 Introduction

Percolation deals with the properties of clusters created by some probabilistic means. Generally, past a threshold an infinite cluster will form that spans the lattice and we consider this a phase transition. While the process that generates this cluster is simply stated, its properties are quite rich, resisting analytic treatment and exhbiting complex phenomena such as multifractal scaling. The marriage of conceptual simplicity and complex behavior makes percolation a popular introduction to phase transitions, and the generality of the model makes it applicable to a wide variety of physical problems from superconductivity to forest fires.

This coverage aims to give an introduction to percolation theory with particular attention to the problem as a conduction transition in a random resistor network. We will follow our coverage of the Ising model, first introducing percolation and giving brief attention to it's variants and correspondence to physical transitions. Then we will review exact solutions on simple lattices, namely in one dimension and the Bethe lattice. Next we will review mean field theory techniques away from the transition and then scaling relations derivable from assumptions about the form of the infinite cluster. Finally, we will review applications of renormalization, and give a summary of extensions to more complex examples.

1.1 The Basic Percolation Problem

Consider a finite cubic lattice where neighboring sites are not yet connected. At every potential edge, place a bond with a probability, or **concentration** p. As we increase this concentration from 0, at some point instantiations of the lattice will contain clusters that connect from the top to the bottom and we say the lattice **percolates**. As the size of the lattice increases, the point at which a percolating cluster forms becomes more sharply defined and in the infinite limit we can define a critical concentration p_c beyond which an infinite cluster exists.

We may consider a similar problem where bonds are all occupied but *sites* are initially vacant. Each site is occupied with a probability p and clusters are collections of occupied sites connected by bonds. This defines the **site percolation** as opposed to **bond percolation** mentioned previously. Depending on the physical problem we are interested in, we may choose to frame it in terms of sites or bonds but the results given by either will be closely related. Here we will focus on bond percolation with the intention of relating our results to the random resistor network. This coverage is adapted from [**?**] [**?**] which both cover site percolation. Results here have been rederived for bond percolation following their arguments.

Lattice	Coord. #	Site	Bond
1d	2	1	1
2d Honeycomb	3	0.6962	$1 - 2\sin\pi/8 \approx 0.65271$
2d Square	4	0.592746	1/2
2d Triangular	6	1/2	$2\sin\pi/18 \approx 0.34729$
3d Diamond	4	0.43	0.388
3d Simple cubic	6	0.3116	0.2488
4d Hypercubic	8	0.197	0.1601
5d Hypercubic	10	0.141	0.1182
Bethe Lattice	z	$1/(z-1)$	$1/(z-1)$

Table 1: Percolation thresholds for various lattices. Note that within a dimension, the threshold decreases with increasing coordination number. This is an abridged version of a table found in [**?**].

1.2 Important Quantities and Exponents

We will make use of several quantities to characterize the lattice and it's transition. Here we give a brief summary of each, important relations involving them and their critical exponents:

p_c The **percolation threshold** is the concentration at which an infinite cluster will form on an infinite lattice. While easily evaluated on the Bethe lattice, analytical results resisted attempts for 20 years after the problem was initally posed. As with the critical temperature, it is nonuniversal and depends on the details of the lattice structure. Currently accepted values may be viewed in Table 1.

P(p) The **strength** of the infinite cluster gives the probability that an arbirary bond is connected to the infinite cluster. As such, P is 0 below the percolation threshold and goes to 1 as $p \to 1$. In most lattices, at p_c P increases continuously from 0 and so is analogous to our order parameter for a second order phase transition. This allows us to define the critical exponent β,

$$P(p) \propto (p - p_c)^\beta \text{ as } p \to p_c^+$$

$n_s(p)$ The **cluster number distribution** gives the average number of clusters of size s per lattice site (such that in a lattice of size N there will be on average $Nn_s(p)$ clusters of size s at concentration p). As all bonds must belong to a cluster of some size, we have,

$$\sum_s s n_s(p) = p, \quad p < p_c$$

$$\sum_s s n_s(p) + P(p) = p, \quad p > p_c$$

where the sum excludes the infinite cluster. This quantity $sn_s(p)$ is somewhat analogous to a Boltzman weight in that it gives us information on the probabilities of certain configurations of the system.

s_ξ We will find that the cluster number distribution away from the percolation threshold, $n_s \propto \exp(s/s_\xi)$. The scaling factor in the denominator is the **cutoff cluster size**. This will generally diverge at the percolation threshold allowing us to define the critical exponent σ

$$s_\xi \propto |p_c - p|^{-\frac{1}{\sigma}} \text{ as } p \to p_c$$

S The cluster number distribution allows us to calculate a mean cluster size which we define as the average cluster size to which a given occupied bond belongs. As the fraction of occupied bonds belonging to s clusters is $\frac{sn_s(p)N}{pN} = \frac{sn_s(p)}{\sum sn_s(p)}$ the average cluster size is

$$S(p) = \frac{\sum s^2 n_s(p)}{\sum sn_s(p)}$$

(Note that in this definition clusters are sampled by *bond* biasing our selection towards larger clusters. We could similarly have defined an average cluster size in which all clusters are sampled with equal probablity $S'(p) = \frac{\sum sn_s(p)}{\sum n_s(p)}$ but we will find that our current definition is more appropriate.) This is analogous to our susceptibility in thermal problems.

The mean cluster size will also diverge at the percolation threshold, allowing us to define the critical exponent γ

$$S \propto |p_c - p|^{-\gamma} \text{ as } p \to p_c$$

$g(r)$ The **correlation function** or **pair connectivity** is the probability that a bond a distance r away from an occupied site belongs to the same cluster. In analogy to the susceptibilty sum rule, we have a relation between the correlation function and the mean cluster size,

$$\sum_{\vec{r}} g(\vec{r}) = S$$

which is our motivation for defining the average cluster size as we have.

ξ We will find that we can typically write the correlation function as $g(r) \propto \exp(r/\xi)$ where the scaling factor ξ is the **correlation length**. It will typically diverge at the percolation threshold, allowing us to define the critical exponent ν,

$$\xi \propto |p_c - p|^{-\nu} \text{ as } p \to p_c$$

2 Exact Solutions

In this section we will calculate the preceding quantities in two scenarios in which they may be evaluated exactly: the one dimensional chain and the Bethe Lattice. Again similar coverages for site percolation (which is identical to bond percolation in 1d) may be found in [**?**][**?**]

2.1 One-Dimension

On an infinite lattice, any unoccupied bond will break the infinite cluster and thus we can only have percolation at $p_c = 1$ (in analogy with the 1d Ising model). The strength $P(p)$ is thus zero for $p < 1$ and jumps discontinuously to 1 at percolation.

To find the cluster number distribution, we note that the probability for a cluster of size s to lie at any given position is $p^s(1-p)^2$. As the number of possible locations for such a cluster is equal to the number of bonds we have (from the linearity of the expectation) that the expected number of such clusters *per bond* is

$$n_s(p) = p^s(1-p)^2$$

As a check, we can evaluate,

$$\begin{aligned}\sum_s s n_s(p) &= \sum_s s p^s (1-p)^2 \\ &= (1-p)^2 (p\frac{d}{dp}) \sum_s p^s \\ &= p\frac{(1-p)^2}{(1-p)^2} = p\end{aligned}$$

We may write this quantity as,

$$n_s(p) = (p_c - p)^2 \exp(-s/s_\xi), \quad s_\xi = -\frac{1}{\ln p}$$

and obtain the cutoff cluster size. We thus have

$$s_\xi = -\frac{1}{\ln(1-(p_c-p))} \propto |p_c - p|^{-1}$$

yielding the critical exponent $\sigma = 1$.

To evaluate the average cluster size, we proceed as with our previous check on the cluster number distribution to obtain,

$$S = \frac{\sum_s s^2 n_s(p)}{\sum_s s n_s(p)} = \frac{1+p}{1-p}$$

and we can immediately identify $\gamma = 1$.

Lastly, in order for two bonds separated by a distance r to be connected, all bonds between them must be connected and thus $g(r) = p^r = \exp(-r/\xi)$ giving

$$\xi = -\frac{1}{\ln(1-(p_c-p))} \propto |p_c - p|^{-1}$$

and $\nu = 1$. We can confirm the susceptibility sum rule noting that the sum over $\vec{r}$ breaks into a contribution from $r = 0$ and two from bonds to the left and right.

$$\sum_{\vec{r}} g(r) = 1 + 2\sum_{r=1,2..} p^r = 1 + 2p \sum_{s=0,1,2..} p^s = \frac{1+p}{1-p} = S$$

2.2 Bethe Lattice/Mean-Field Solution

The Bethe lattice is an infinite tree (read no loops) in which each site has z nearest neighbors. If $z = 2$ we regain the 1d chain. The Bethe lattice is so named for the Bethe Peierls approximation which is exact upon it and the ease of solution on the Bethe lattice stems from the fact that every pair of sites possesses a single path joining them and its translation symmetry may be used to obtain self consistency equations for many quantities. Readers interested in its history and breadth of application in physics can consult Thorpe's charming and seemingly obscure coverage in [**?**].

Considering entering a site through an occupied bond, we expect that the lattice will percolate when the expected number of bonds continuing onward is 1. This yields the correct percolation threshold of $p_c = \frac{1}{z-1}$ which may be established rigorously [**?**].

Our argument for the cluster number distribution in 1d applies here as well, but suffers from a complication. In 1d, there is a single possible configuration for a cluster of s bonds: a chain of length

l. Such a chain has a perimeter of 2 and so the contribution to the cluster number distribution is of the form $g_{s,t}p^s(1-p)^t$ where $g_{s,t}$ is the multiplicity of shapes of clusters of size s and perimeter t. We can thus write the cluster number distribution as

$$n_s(p) = \sum_t g_{s,t}p^s(1-p)^t.$$

3 Scaling Relations

4 Renormalization

5 Extensions