Renormalization in Intermediate Asymptotics

Forrest Sheldon   June 11, 2014

Following the discussion in Goldenfeld's text and a series of his papers, I have taken a didactive approach to composing this paper, focusing on the logical progression of ideas allows renormalisation to be applied to dynamic phenomena. Diffusion is used as the 'ising model of dynamic phenomena' to illustrate the emergence of anomalous dimensions and the elucidation of their cause. Lastly renormalisation is used to tame the divergences that arise in singular perturbation theory when a well behaved solution is known to exist.

1 Dimensional Analysis is great!...when it works

The Buckingham Pi Theorem [2] asserts that any equation that completely describes the relation between a collection of physical quantities takes the form,

$$\Pi = f(\Pi_0, \Pi_1, ..., \Pi_n) = 0$$

where the Π_is are all the independent dimensionless products that may be formed from the given quantities. As an example of it's utility, consider the diffusion equation in one dimension,

$$\partial_t u(x,t) = \frac{1}{2}\kappa \partial_{xx} u(x,t)$$

with an initial condition $u(x,0) = \frac{A_0}{\sqrt{2\pi l^2}} e^{-x^2/2l^2}$. By forming the dimensionless quantities,

$$\Pi = \frac{u\sqrt{\kappa t}}{A_0} \quad \Pi_1 = \frac{x}{\sqrt{\kappa t}} \quad \Pi_2 = \frac{l}{\sqrt{\kappa t}}$$

we may immediately infer a solution of the form,

$$u(x,t) = \frac{A_0}{\sqrt{\kappa t}} f\left(\frac{x}{\sqrt{\kappa t}}, \frac{l}{\sqrt{\kappa t}}\right)$$

which forms a starting point for futher inquiry.

1.1 Similarity Solutions

Once the dimensional form of the equation is established, it is often productive to examine similarity solutions. In these functions, the arguments occur such that that length and time scales are interdependent. (As an example $u(x,t) = t^\alpha f(xt^\beta)$.) By reducing the number of arguments of the scaling function f, similarity solutions often allow us to convert PDEs to ODEs greatly facilitating their solution. In addition, the long term behavior of a system is often given by similarity solutions which may be evidence of natural stabilization [1]. We call this regime, where times/distances are large enough that boudary/initial values no longer influence the system but the system is still far from equilibrium **intermediate asymptotics**. (It is worth noting that wave equations $f(x - vt)$ which also often describe long term behavior may be written as a similarity solution. Taking $x = \log X, \quad t = \log T$ the solution becomes $f(\frac{X}{T^v})$.)

As an example of this procedure, we continue with the diffusion equation following Goldenfeld [3]. In the long time limit, we expect the system to be independent of l, the size of the initial state. This suggests a similarity solution,

$$u(x,t) = \frac{A_0}{\sqrt{\kappa t}} f_s\left(\frac{x}{\sqrt{\kappa t}}\right) \propto t^{-1/2} f_s(\xi)$$

. Substitution into the diffusion equation yields the ODE

$$f_s'' + \xi f' + f = 0$$

which is easily solved to give the solution in the intermediate regime, $u(x,t) = \frac{A_0}{\sqrt{2\pi\kappa t}} e^{-x^2/2\kappa t}$.

We observe in this case that the system has no remaining dependence on the initial conditions (other than it's parity and size A_0 which are conserved in its evolution). By observing this equation we cannot determine the initial size of the system, or the time at which the system was begun and thus a locus of initial conditions all will progress to this form in the intermediate limit. This fact is the first hint that renormalization may have a part to play in the analysis of systems of this type. While unnecessary here, we will see that for a variety of systems, renormalization techniques yield valuable insights and numerics on dynamical phenomena evolving in time.

1.2 Intermediate Asymptotics

In taking the limit of our diffusion equation solution as $l \to 0$ we have made the tacit assumption that the limit is regular. In this case we were correct and the similarity solution we obtained gave the correct asymptotic limit. However, for a wide variety of problems this will not be the case and taking the limit will give rise to anamolous dimensions.

Barenblatt [1] distinguishes three cases. Given a relation between dimensionless products $\Pi = f(\Pi_0, \Pi_1, ...)$, as $\Pi_0 \to 0$,

(a) f approaches a finite, nonzero limit. In this case, we may simply replace Π_0 with 0 and proceed with our analysis. We call this **intermediate asymptotics of the first kind**.

(b) f does not approach a finite limit but posseses power law asymptotics. In this case, we may seek a solution of the form

$$\Pi = \Pi_0^{\alpha} g(\frac{\Pi_1}{\Pi_0^{\alpha_1}}, \frac{\Pi_2}{\Pi_0^{\alpha_2}}, ...)$$

where the anomalous dimensions $\alpha, \alpha_1, ...$ must be determined. Systems displaying this behavior posses **intermediate asymptotics of the second kind**.

(c) Neither of these two limits occur.

As illustrated above, dimensional analysis may be capable of tackling problems of the first kind. For problems of the second type, we will need to invoke renormalisation to absorb the singular limit. To motivate this process, we follow another discussion from Goldenfeld [3].

2 Anomalous Dimensions in Problems of the Second Kind

In this section we will examine a particular example of intermediate asymptotics of the second kind. We will see that by allowing a normally conserved quantity to vary, we introduce a singular dependence on the initial conditions in our scaling function. Using an insighful approximation, we will obtain the anomalous dimensions that arise as a consequence without invoking renormalisation.

2.1 Water Flow in Porous Media - Barenblatt's Equation

Consider the half-space $z > 0$ filled with a porous medium. When water moves into a region of this space, it occupies the maximum available space in each pore σ (a fraction of each pore is occupied by gas.). When it exits an occupied pore, adhesion causes it to leave a thin film in the pore such that σ_0 of the pore remains filled. This suggests an asymmetry in our dynamics. When water flows into a space it requires a volume proportional to σ to fill it, but when exiting a space the amount of water that needs to flow out to consider that space empty is proportional to $\sigma - \sigma_0$. This suggests that when the height of the water in a region is rising and thus the water is encountering new pores, it diffuses with a slow diffusion constant D but in regions where the height is falling and less water must flow to empty the space, it diffuses at at some faster rate with $(1+\epsilon)D$. These dynamics are contained in the Barenblatt equation which we write in 1 dimension,

$$\partial_t \, u(x,t) = D\kappa \, \partial_{xx} u(x,t)$$

$$D = \begin{cases} \frac{1}{2} & \partial_t \, u > 0 \\ (1+\epsilon)\frac{1}{2} & \partial_t \, u < 0 \end{cases}$$

We denote the point between these two regimes as the positive x value, $x_0(t,\epsilon)$ such that $\partial_t \, u(x_0, t) = 0$ It has been proved that solutions to this equation exist, are unique, and possess smooth derivatives in x and t.

2.2 Attempting a Similarity Solution

As with the diffusion equation, we form the dimensionless quantities

$$\Pi = \frac{u\sqrt{\kappa t}}{A_0} \quad \Pi_1 = \frac{x}{\sqrt{\kappa t}} \quad \Pi_2 = \frac{l}{\sqrt{\kappa t}} \quad \Pi_3 = \epsilon$$

which suggest a similarity solution of the form,

$$u_s \propto t^{-1/2} f(\xi, \epsilon) \quad \xi = \frac{x}{\sqrt{\kappa t}}$$

with $\xi_0 = \frac{x_0}{\sqrt{\kappa t}}$. Plugging this into the PDE gives two ODEs,

$$\begin{aligned} (1+\epsilon) f_1'' + \xi f_1' + f_1 = 0 \quad & 0 \le \xi < \xi_0 \\ f_3'' + \xi f_2' + f_2 = 0 \quad & \xi_0 < \xi < \infty \end{aligned}$$

both of which may be integrated to give,

$$\begin{aligned} f_1 = & B_1 e^{-\xi^2/2(1+\epsilon)} \\ f_2 = & B_2 e^{-\xi^2/2}. \end{aligned}$$

This seems like a success until we attempt to enforce our matching conditions at ξ_0. In order to match the functions and their first derivatives we are forced to conclude $B_1 = B_2 = 0$. Taken with the existence theorem of the solution, we are forced to conclude that no similarity solution of the form given above exists and this problem belongs to intermediate asymptotics of the second kind.

2.3 A Heuristic Derivation using the Similarity solution with Anomalous Dimensions

We may define the 'mass' of our solution by $m(t) = \int dxu(x,t)$. The diffusion equation conserves mass and we may thus set it to a constant $m(t) = A_0$ however our discussion of diffusion in porous media makes it clear that the mass should no longer be conserved. To see this explicity we examine,

$$\begin{aligned}\partial_t m(t) &= \int dx D\kappa \partial_{xx} u \\ &= \int dx \kappa (\partial_x D)(\partial_x u)\end{aligned}$$

where the last step follows from integration by parts. A constant D will thus conserve mass. For Barenblatt's equation, $\partial_x D = \frac{\epsilon}{2}(\delta(x+x_0) - \delta(x-x_0))$ giving $\partial_t m(t) = \epsilon\kappa\, \partial_x u|_{x_0,t}$.

If the diffusion is very slow, we might expect that the form of the solution is similar to what we would expect from regular diffusion but with a mass that varies with time,

$$u \propto \frac{m(t)}{\sqrt{2\pi(kt+l^2)}} \exp\left(\frac{-x^2}{2(kt+l^2)}\right).$$

This approximation gives $x_0(t) = \sqrt{kt+l^2}$. Using this to evaluate $\partial_t m(t)$, we obtain

$$\partial_t m(t) = -\frac{\epsilon\kappa}{\sqrt{2\pi e}} \frac{m(t)}{kt+l^2}$$

which we integrate from $t = 0$ to get

$$m(t) = \frac{m(0) l^{2\alpha}}{(kt+l^2)^\alpha} \quad \alpha = \frac{\epsilon}{\sqrt{2\pi e}}.$$

Inserting this into our supposed asymptotic form and setting all ls that do not give a singular constribution to 0 yields the intermediate asymptotic form

$$u \propto \frac{m(0) l^{2\alpha}}{(kt)^{1/2+\alpha}} e^{\frac{-x^2}{2kt}}.$$

It is interesting to note that while α is a consequence of the presence of an initial width, it's value is independent of that width confirming that all initial conditions will evolve towards this form. We also note that unlike diffusion, the Barenblatt equation possesses no solution for δ function initial conditions for whom $l = 0$.

3 Renormalisation

Taking the last section as a parable, we will attempt to recreate the result we found previously without assuming the explicit form of a solution. The renormalisation we apply here will consist of two parts: first, by inserting a parameter in our scaling function we will move the singular behavior out of the function. This will imply a relationship between the scaling of our parameter, and the behavior of the solution. By approximating the solution, we will be able to extract the scaling behavior and calculate the anomalous dimensions. This coverage roughly follows that found in Goldenfeld [3] and [4], both of which discuss Barenblatt's equation.

3.1 Mass Renormalization

We begin with our scaling assumption from dimensional analysis,

$$u(x,t) = \frac{m_l}{\sqrt{\kappa t}} f(\xi, \eta, \epsilon)$$

where we have called $\xi = \frac{x}{\sqrt{\kappa t}}$ and $\eta = \frac{l}{\sqrt{\kappa t}}$. We are interested in the limit $\eta \to 0$ in which we know f is singular. In the last section we touched on the scaling in the intermediate regime $m_l(0) \propto \eta^{-\alpha} m(t)$. According to this relationship, there is no unique value $m_l(0)$ that leads to the currently observed value $m(t)$. Instead, for every l there is a separate $m_l(0)$ that leads to the the currently observed value. It thus follows both that we may redefine the value of $m_l(0)$ however we wish so long as we retain $m(t)$, and that the intermediate asymptotic solution corresponds to the $\eta \to 0$ $m_l(0) \to \infty$ limit in which the initial mass will absorb the singular behavior in f.

To formalize this insight, we reason that as $m(t)$ and $m_l(0)$ have the same units, they must be proportional to one another and introduce the constant of proportionality $m(t) = \frac{m_l}{Z}$ (where from now one we will write $m_l(0) = m_l$). As $m(t)$ in the intermediate regime is independent of l, Z must cancel the dependence and thus $Z = Z(\frac{l}{\mu})$ where μ is an arbitrary length scale chosen to make the ratio dimensionless. Inserting this into our solution gives

$$u = \frac{m(t) Z}{\sqrt{\kappa t}} f\left(\xi, \eta, \epsilon\right).$$

where Z is defined such that for small l,

$$Z\left(\frac{l}{\mu}\right) f\left(\frac{x}{\sqrt{\kappa t}}, \frac{l}{\sqrt{\kappa t}}, \epsilon\right) = F\left(\frac{x}{\sqrt{\kappa t}}, \frac{\mu}{\sqrt{\kappa t}}, \epsilon\right) + O(l) \quad l \to 0.$$

If this tranformation is possible, we then reason that the solution cannot depend on the arbitrary length scale mu that we introduced. This gives the renormalization group equation,

$$\mu \frac{du}{d\mu} = 0$$

which, eliminating m in favor of Z is,

$$\mu \frac{d}{d\mu} \frac{m_l}{Z\sqrt{\kappa t}} F(\xi, \sigma, \epsilon) = 0, \quad \sigma = \frac{\mu}{\sqrt{\kappa t}}.$$
$$\frac{d \log Z}{d \log \mu} F - \sigma \frac{dF}{d\sigma} = 0$$

As m_l diverges with $l^{-2\alpha}$ the assumption that our renormalization has absorbed the divergence is

$$\frac{d \log Z}{d \log \mu} = 2\alpha$$

using which we can solve the full equation to give,

$$u(x,t) = \frac{m_l}{\sqrt{\kappa t}} \left(\frac{l}{\mu}\right)^{2\alpha} \left(\frac{\mu}{\sqrt{\kappa t}}\right)^{2\alpha} G(\xi, \epsilon)$$

3.2 Perturbative Renormalization

We now undertake to perform actually use the RG we have defined to calculate the anomalous dimension of Barenblatt's equation. To do so we perform a perturbative expansion on Barenblatt's equation and use our approximation in the scheme above. In doing so we will find that although the solution to the equation is guaranteed to be smooth, the terms in the perturbation series will diverge. By use of the arbitrary length scale μ we introduced above, we will be able to continually push these divergences to higher order, signaling that our actual solutions are well behaved. This is an example of a singular perturbation, in which even small additions to the PDE drastically change the behavior of the solution. As this calculation involves many steps, we will split it up into subsections.

3.2.1 Perturbative expansion of the solution

We may write Barenblatt's equation as a diffusion equation with a small perturbing term (and rescaling κ to 1 for convenience),

$$\left(\partial_t - \frac{1}{2}\partial_{xx}\right) u(x,t) = \frac{\epsilon}{2}\Theta(x_0(t) - |x|)\partial_{xx}u(x,t).$$

We attempt an asymptotic expansion of the solution,

$$u(x,t) = u_0 + u_1\epsilon + u_2\epsilon^2 + ...$$

Writing the equation abstractly as $L_D u = \epsilon L_P u$ where L_D is the diffusion operator and L_P is the perturbing operator, we obtain equations for the first two terms,

$$\begin{aligned} L_D u_0 &= 0 \\ L_D u_1 &= L_P u_0. \end{aligned}$$

With a gaussian initial condition with standard deviation l, u_0 is just,

$$u_0(x,t) = \frac{m_0}{\sqrt{2\pi(t+l^2)}} \exp\left(\frac{-x^2}{2(t+l^2)}\right).$$

and the first order term follows from the green's function,

$$u_1(x,t) = \frac{m_0}{2}\int_0^t ds \int_{-\infty}^{\infty} dy\, G(x-y,t-s)\Theta(y_0(t)-|y|)\partial_{yy}u(y,t)$$

$$u_1(x,t) = \frac{m_0}{2}\int_0^t \frac{ds}{s+l^2}\int_{-\sqrt{s+l^2}}^{\sqrt{s+l^2}} dy \frac{\exp(-(x-y)^2/(2(t-s)))}{\sqrt{2\pi(t-s)}} \frac{\exp(-y^2/(2(s+l^2)))}{\sqrt{2\pi(s+l^2)}} \left(\frac{y^2}{t+l^2} - 1\right)$$

3.2.2 Leading Divergence in u_1

This expression can be greatly cleaned up by the subsitution, $w = \frac{y}{\sqrt{s+l^2}}$ giving

$$u_1(x,t) = \frac{m_0}{4\pi}\int_0^t \frac{ds}{s+l^2}\int_{-1}^{1} dw \frac{e^{-\frac{(x-\sqrt{s+l^2}w)^2}{2(t-s)}}}{\sqrt{t-s}} e^{-w^2/2}\left(w^2-1\right).$$

Examining this term for divergences, it appears that for $l \to 0$ the only divergence is a logarithmic one at small s. We may thus set l to 0 in all terms past the $\frac{1}{s+l^2}$,

$$u_1(x,t) \approx \frac{m_0}{4\pi} \int_0^t \frac{ds}{s+l^2} \int_{-1}^1 dw \, \frac{e^{-\frac{(x-\sqrt{s}w)^2}{2(t-s)}}}{\sqrt{t-s}} e^{-w^2/2} \left(w^2 - 1\right).$$

Similarly, as the divergence occurs for $s \to 0$, we may set all s terms to the right of the logarithmic divergence to 0, leaving the leading singular contribution,

$$u_1(x,t) \approx \frac{e^{-\frac{x^2}{2t}}}{\sqrt{t}} \frac{m_0}{4\pi} \int_0^t \frac{ds}{s+l^2} \int_{-1}^1 dw \, e^{-w^2/2} \left(w^2 - 1\right).$$

A substitution of $w^2 = t$ and integration by parts on the last integral yields

$$2 \int_0^1 dw \, e^{-w^2/2} \left(w^2 - 1\right) = -\frac{2}{\sqrt{e}}$$

and the integral over s in the limit $l \to 0$ gives the logarithmic term,

$$\int_0^t \frac{ds}{s+l^2} = \log\left(\frac{t}{l^2}\right).$$

To first order in ϵ our solution is thus,

$$u(x,t) = \frac{m_0}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}} \left[1 - \frac{\epsilon}{\sqrt{2\pi e}} \log\left(\frac{t}{l^2}\right) + O(\epsilon^2)\right] + O(l, \epsilon)$$

3.2.3 Perturbative Renormalisation

We expand Z in ϵ, $Z = 1 + \sum_n z_n(l/\mu)\epsilon^n$. Recalling $m(0) = Zm(t)$, we insert this into our solution to obtain, keeping terms to order ϵ,

$$\begin{aligned} u(x,t) =& \frac{m(t)}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}} \left[1 + \epsilon z_1 + ...\right] \times \left[1 - \frac{\epsilon}{\sqrt{2\pi e}} \log\left(\frac{t}{l^2}\right) + O(\epsilon^2)\right] \\ =& \frac{m(t)}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}} \left[1 + \epsilon\left(z_1 - \frac{1}{\sqrt{2\pi e}} \log\left(\frac{t}{l^2}\right)\right) + ...\right] \end{aligned}$$

Evidently, by choosing $z_1 = \frac{1}{\sqrt{2\pi e}} \log\left(C_1 \frac{\mu^2}{l^2}\right)$ we will cancel the singular dependence on l in our solution.

3.2.4 Calculating the Anomalous Dimension

At this point we can use the previously defined relationship, $\frac{d \log Z}{d \log \mu} = 2\alpha$ to calculate the anomalous dimension.

$$\log Z = \log\left(1 + \frac{\epsilon}{\sqrt{2\pi e}} \log\left(C_1 \frac{\mu^2}{l^2}\right) + ...\right) \approx \frac{2\epsilon}{\sqrt{2\pi e}} \log \mu + ..$$

and we have confirmed our earlier heuristic derivation $\alpha = \frac{\epsilon}{\sqrt{2\pi e}} + O(\epsilon^2)$ without the use of an ad hoc solution form.

4 Wrapping Up

After seeing traditional dimensional analysis fail to give appropriate forms for solutions of certain problems, we discovered that the problem was a singular limit in our scaling form. By introducing a renormalization parameter and using it to absorb the dependence of our troublesome parameter we are able to calculate the degree of the singularity and extract the long term behavior of the system. Here we see that the flow in time which takes many initial conditions to single 'fixed point' asymptotic solutions is exactly analogous to the application of renormalization to thermal problems.

References

[1] GI Barenblatt. *Scaling, self-similarity, and intermediate asymptotics.* CUP, New York, 1996.

[2] E Buckingham. On physically similar systems; illustrations of the use of dimensional equations. *Physical Review*, 1914.

[3] Nigel Goldenfeld. *Lectures on Phase Transitions and the Renormalization Group.* Westview Press, 1992.

[4] Nigel Goldenfeld, Olivier Martin, and Y. Oono. Intermediate asymptotics and renormalization group theory. *Journal of Scientific Computing*, 4(4):355–372, December 1989.