Percolation in Random Resistor Networks

Forrest Sheldon June 5, 2014

1 Introduction

Percolation deals with the properties of clusters created by some probabilistic means. Generally, past a threshold an infinite cluster will form that spans the lattice and we consider this a phase transition. While the process that generates this cluster is simply stated, its properties are quite rich, resisting analytic treatment and exhbiting complex phenomena such as multifractal scaling. The marriage of conceptual simplicity and complex behavior makes percolation a popular introduction to phase transitions, and the generality of the model makes it applicable to a wide variety of physical problems from superconductivity to forest fires.

This coverage aims to give an introduction to percolation theory with particular attention to the problem as a conduction transition in a random resistor network. We will follow our coverage of the Ising model, first introducing percolation and giving brief attention to it's variants and correspondence to physical transitions. Then we will review exact solutions on simple lattices, namely in one dimension and the Bethe lattice. Next we will review mean field theory techniques away from the transition and then scaling relations derivable from assumptions about the form of the infinite cluster. Finally, we will review applications of renormalization, and give a summary of extensions to more complex examples.

1.1 The Basic Percolation Problem

Consider a finite cubic lattice where neighboring sites are not yet connected. At every potential edge, place a bond with a probability, or **concentration** p. As we increase this concentration from 0, at some point instantiations of the lattice will contain clusters that connect from the top to the bottom and we say the lattice **percolates**. As the size of the lattice increases, the point at which a percolating cluster forms becomes more sharply defined and in the infinite limit we can define a critical concentration p_c beyond which an infinite cluster exists.

We may consider a similar problem where bonds are all occupied but *sites* are initially vacant. Each site is occupied with a probability p and clusters are collections of occupied sites connected by bonds. This defines the **site percolation** as opposed to **bond percolation** mentioned previously. Depending on the physical problem we are interested in, we may choose to frame it in terms of sites or bonds but the results given by either will be closely related. Here we will focus on bond percolation with the intention of relating our results to the random resistor network. This coverage is adapted from [**?**] [**?**] which both cover site percolation. Results here have been rederived for bond percolation following their arguments.

1.2 Important Quantities and Exponents

We will make use of several quantities to characterize the lattice and it's transition. Here we give a brief summary of each, important relations involving them and their critical exponents:

- p_c The **percolation threshold** is the concentration at which an infinite cluster will form on an infinite lattice. While easily evaluated on the Bethe lattice, analytical results resisted attempts for 20 years after the problem was initally posed. As with the critical temperature, it is nonuniversal and depends on the details of the lattice structure. Currently accepted values may be viewed in Table **??**.

- $n_s(p)$ The **cluster number distribution** gives the average number of clusters of size s per lattice site (such that in a lattice of size N there will be on average $Nn_s(p)$ clusters of size s at concentration p). As all bonds must belong to a cluster of some size (at least below the percolation threshold) we have $\sum_s sn_s(p) = p$. This quantity $sn_s(p)$ is somewhat analogous to a Boltzman weight in that it gives us information on the probabilities of certain configurations of the system.

Lattice	Coord. #	Site	Bond
1d	2	1	1
2d Honeycomb	3	0.6962	$1 - 2\sin\pi/8 \approx 0.65271$
2d Square	4	0.592746	1/2
2d Triangular	6	1/2	$2\sin\pi/18 \approx 0.34729$
3d Diamond	4	0.43	0.388
3d Simple cubic	6	0.3116	0.2488
4d Hypercubic	8	0.197	0.1601
5d Hypercubic	10	0.141	0.1182
Bethe Lattice	z	1 / (z-1)	1 / (z-1)

Table 1: Percolation thresholds for various lattices. Note that within a dimension, the threshold decreases with increasing coordination number. This is an abridged version of a table found in KRISTENSON.

s_ξ We will find that the cluster number distribution away from the percolation threshold, $n_s \propto \exp(s/s_\xi)$. The scaling factor in the denominator is the **cutoff cluster size**. This will generally diverge at the percolation threshold allowing us to define the critical exponent σ

$$s_\xi \propto |p_c - p|^{-\frac{1}{\sigma}} \text{ as } p \to p_c$$

S The cluster number distribution allows us to calculate a mean cluster size which we define as the average cluster size to which a given occupied bond belongs. As the probability of occupied sites belonging to s clusters is $\frac{sn_s(p)N}{pN} = \frac{sn_s(p)}{\sum sn_s(p)}$ the average cluster size is

$$S(p) = \frac{\sum s^2 n_s(p)}{\sum sn_s(p)}$$

(Note that in this definition clusters are sampled by *site* biasing our selection towards larger clusters. We could similarly have defined an average cluster size in which all clusters are sampled with equal probablity $S'(p) = \frac{\sum sn_s(p)}{\sum n_s(p)}$ but we will find that our current definition is more appropriate.)

The mean cluster size will also diverge at the percolation threshold, allowing us to define the critical exponent γ

$$S \propto |p_c - p|^{-\gamma} \text{ as } p \to p_c$$

1.3 Critical Exponents

1.4 Physical Correspondences

2 Exact Solutions

2.1 One-Dimension

2.2 Bethe Lattice

3 Mean-Field Techniques

3.1 Mean Field Theory

3.2 Effective Medium Theory

4 Scaling Relations

5 Renormalization

6 Extensions