Strategic Point Processes *

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Abstract

Spatial point processes provide a way for researchers to describe event-based outcomes in a bounded study window. Commonly used processes, such as the Neyman-Scott processes, however, often lack the behavioral frameworks that social scientists need to analyze strategic interactions between cohorts of actors. Strategic point processes address this issue by framing spatial point processes as a fixed set of players who engage in a countably infinite number of sequential games in a bounded study window. In this setup, each player receives a random function as their spatial strategy and their observed joint strategies generate the point pattern formations researchers record from the real world. To introduce the logic of strategic point processes, this paper describes the basic assumptions of Neyman-Scott (NS) spatial games, which model the clustering tendencies of strategic actors. Next, this paper discusses how researchers can reconstruct NS spatial games using observational datasets. Finally, this paper discusses how researchers can use complete spatial randomness (CSR) as an analytic baseline to draw inferences from NS spatial games.

Keywords: Spatial Point Processes, Game Theory, Research Design

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1 Introduction

How can researchers model the joint distribution of two or more non-overlapping point patterns? Consider the following game where two paintballers paint a blank canvas. Player A shoots first by randomly sampling locations according to some (un-) known spatial configuration (i.e., square, star, blob, etc.)[1]. Next, Player B has to shoot Player A's spatial markers within a fixed radius centered at each point. The game concludes once Player B either runs out of spatial markers or hits some proportion of Player A's locations[2]. Figure 1 illustrates a single game round. Here, triangle locations represent Player A's markers, circle locations represent Player B's, and the shaded circles represent the fixed radius that Player B attempts to intersect.

In this example, the strategic interactions between Players A and B produce a simple Neyman-Scott process (Neyman and Scott, 1958) [3] [4]. Specifically, Player A's point allocation process (parent process) conditions Player B's allocation process (offspring process). Player A, however, partially constrains Player B's choices as Player B's imprecise aim generates random noise. Here, Players A and B engage in a countably infinite number of sequential games with four potential joint strategies (see Figure 2) [5].

Researchers can generalize the basic framework of this game to more complex real world interactions, such as residents calling 311/911 and subsequent police responsiveness (Davidson, 2023) and U.S. military strike strategies and subsequent insurgency attacks in warzones (Condra and Shapiro, 2012; Papadogeorgou et al., 2022). In short, researchers can use this game format to describe any social process where two, or more players, engage in a

[1] Player A shoots a finite number of point locations

[2] Player B shoots a finite number of point locations

[3] In this random process design, a random process, $\Lambda(B)$, generates a set of parent points, which then produce a random set of offspring points. The offspring parameter (α) and the radius parameter (μ) distributes these offspring points around their parent's origin site.

[4] Neyman and Scott (1958) originally developed this process so that astrophysicists can draw inferences on clustering tendencies between galaxy star formations within the same neighborhood.

[5] *Outcome # 1:* Players A and B both place markers, *Outcome # 2:* Player A places a marker but Player B does not, *Outcome # 3:* Player B places a marker but Player A does not, *Outcome # 4:* Players A and B do not place markers

non-fixed number of intervention sites in $\mathbb{R}^d$.

By combining game theory with spatial point process theory (Baddeley, Rubak and Turner, 2015), ***strategic point processes*** (SPPs) provide a way for researchers to model and draw inferences on strategic spatial interactions between a fixed set of actors (or groups) inside a bounded study window ($X_i \in \chi \subseteq \mathbb{R}^d$) or a cross-section of the space-time continuum ($p_i \in (\chi, T) \subseteq \mathbb{R}^d \times [0, \infty]$). Specifically, SPPs define strategic spatial interactions as a countably infinite number of sequential games inside a bounded study window $\Gamma_\phi = \{\gamma_\phi(X_i \text{ or } p_i)\}_{i=1}^N$, where $\chi \in \mathbb{R}^d$[6]. These sequential games map a set of marker characteristics (i.e., point presence, point absence, etc.) to a set of players' spatial locations. In solving these game trees, researchers can define a measurement space that describes the underlying stochastic process between two, or more, non-overlapping point patterns across space ($\mathbb{R}^d$)[7].

In addition to introducing the general logic of SPPs, this paper introduces a subclass of SPPs called ***Neyman-Scott (NS) spatial games***, which expand Neyman-Scott processes to a strategic context. These countably infinite games use sequential trees to model the clustering tendencies between two, or more, strategic actors inside a bounded study window. Here, the game nodes represent hypothesized joint strategies between a set of strategic players while the corresponding Nash equilibria represent the point pattern formations found inside observational datasets.

For a 2-player game, NS spatial games can be broken down into a two step process. The first step involves a *Stackelberg competition* between Players A and B. Player A determines how to place spatial markers in strategic locations across the study window (i.e., Player A identifies market locations). Next, Player B must decide on whether to place a spatial marker near Player A's markers (i.e., Player B decides whether to enter the same market as Player A). If Player B decides to place spatial markers in the vicinity of Player A's

[6] This differs from evolutionary spatial models where a variable set of players, indexed in $\mathbb{R}^d$, must compete with their surrounding neighbors (Nowak and May, 1993).

[7] Let $X_i \in \chi$, where $\chi \subset \mathbb{R}^d$. Let $B(\chi)$ be a Borel σ-algebra that defines all compact subsets of χ

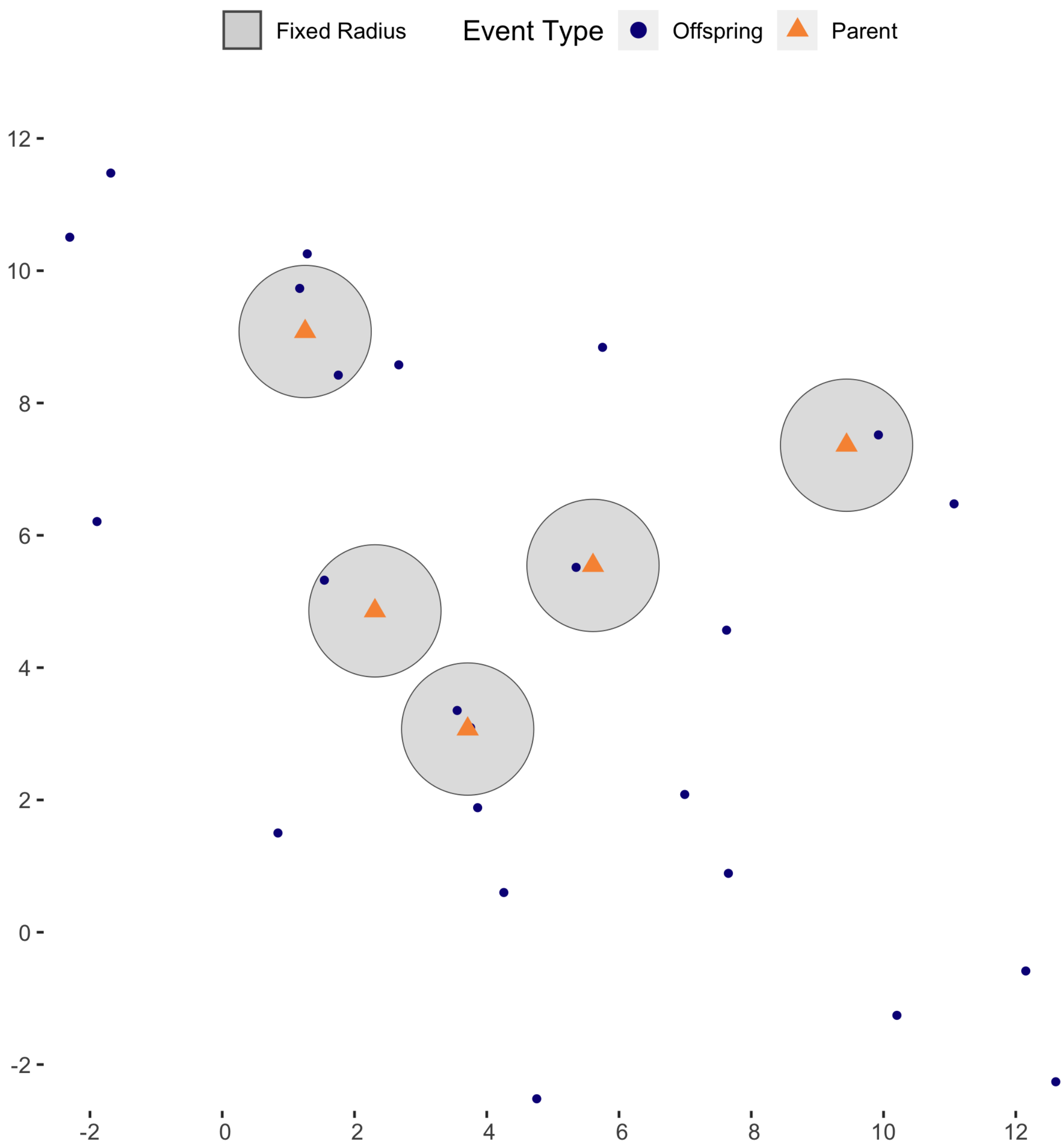

Figure 1: Matern Cluster Process (Special Case of Neyman-Scott Process)

spatial markers, then the second step involves a *Search Game*. Here, Player B must locate Player A's exact locations in the study window. Player A's spatial markers are assumed to be immobile after being distributed; however, Player B incurs costs in their search process.

Researchers can model NS games using a combination of *functional marked point processes* (FMPPs) (Ghorbani, 2021). Ghobrani et al. (2021) define FMPPs as ”marked point processes where the marks are random elements in some (Polish) function space” (p.529)[8]. In the strategic point process context, Player B assigns random functions to Player A's marked point locations. These random functions can include elements of the game, spatial paths, polygons, and functions of time. When Player B chooses to assign random variables instead of random functions, then this multivariate FMPP reduces to a multivariate MPP [cite]. Following Ghobrani et al.'s example Using weighted marked reduced moment measures, researchers can study features of the functional marks and draw inferences from nonparametric estimators.

To describe how NS spatial games work, I rotate between three working examples. First, I build on the intro example of strategic paintballers using simulated data where researchers know, *a priori*, the stationarity and clustering tendencies between Player A and Player B. In addition, I use two real-world examples. The first uses publicly available data from Durham, NC on ”sounds of gunshots” and officer-initiated directed patrols while the second replicates Papadogeorgou et al. (2022) dataset on US airstrikes and insurgency attacks [9].

This paper provides a general overview of the strategic point process framework and discusses how researchers can use its assumptions to model joint strategies between spatial actors. ***First***, I describe why SPPs can help researchers in modelling the strategic selection of different stochastic processes across space-time. Next, I outline the basic setup of a NS spatial game with two players (Player A and Player B). ***Third***, I describe how researchers

[8] A Polish space is a topological space where the metric/distance generates an underlying typology that turns the space into a complete and separable metric space (CSMS) (Ghorbani, 2021, p.533)

[9]

can use extract meaningful information from observational datasets to measure features of the NS spatial games' tree. Typically, datasets encode information on players marginal strategies. Researchers, therefore, have to re-construct the observed joint strategies and assign utility values to them. ***Finally,*** I discuss how researchers can use complete spatial randomness (CSR), or homogeneous Poisson processes, as an inferential baseline for drawing inferences on NS spatial games. In the strategic context, CSR implies any scenario where Player B is strategy indifferent to Player A's spatial marker process.

2 Notation Setup

This section introduces general notation for strategic point processes (SPPs) and functional marked point processes (FMPPs), which are the two basic building blocks for understanding NS spatial games[10]. For a 2-player game, SPPs model the theoretical joint relationships between Player A and B while FMPPs measure their observed joint strategies.

2.1 Spatial Point Process Notation

Let $\chi = (X_1, X_2, ..., X_n)$ be a subset of $\mathbb{R}^d$, which is either compact or given by all of $\mathbb{R}^d$. Let $p = (x, t)$ be a subset of the the space-time continuum $\chi \times T$, where again $\chi \in \mathbb{R}^d$ and $T \in [0, \infty)$[11]. Consider a point process $\Psi_G = \{X_i\}_{i=1}^N$, $N \in \mathbb{N}_0 = \{0, 1, 2, ..., \infty\}$, which represents Player A's unmarked point process. In addition, consider a spatio-temporal point process $\Psi_T = \{p_i\}_{i=1}^N$, $N \in \mathbb{N}_0 = \{0, 1, 2, ..., \infty\}$, $p_i \in \mathbb{R}^2 \times [0, \infty)$ which represents Player A's unmarked spatio-temporal point process.

For NS spatial games, I consider both Ψ_G and Ψ_T to be the ***unmarked parental processes***. For both Ψ_G and Ψ_T, I assume that χ follows a Poisson distribution: $\{X_i\}_{i=1}^N \sim Poisson(\lambda)$, where $\lambda = \{\lambda_1, ..., \lambda_n\}$ represents the intensity function of the process. If $\forall \lambda = c$,

[10] I borrow notation from Ghorbani (2021) previous work on functional marked point processes (FMPPs).

[11] For simplicity, I assume that $\chi \times T$ are completely separable where $\nu(\chi \times T) = \nu(\chi)\nu(T)$ for some function $\nu \in \mathbb{R}^k$

where c is some scalar constant, then Ψ is a homogeneous point process. If not, then Ψ is an inhomogeneous point process.

2.2 Strategic Point Processes (SPPs)

Strategic point processes (SPPs) index a series of sequential games in $\mathbb{R}^D$ or $\mathbb{R}^D \times [0, \infty)$ between a fixed set of players (ϕ). Let $\gamma_\phi(X_i \text{ or } p_i)$ represent a spatio-temporal sequential game between a set of strategic players $\phi \in \{A, B, ..., N\}$ at location X_i or space-time unit p_i. Here, γ_ϕ maps the random unit to a pure-information game tuple $\gamma_\phi = \langle \phi, s_\phi, H, Z, \nabla, \rho, \sigma, \succ_i, u \rangle$, or an imperfect-information game $\gamma_\phi = \langle \phi, s_\phi, H, Z, \nabla, \rho, \sigma, \succ_i, u, f_c, \Im_i \rangle$.

For the perfect information game:

- **Players:** $\phi \cup \{c\}$ represents the players, where $\{c\}$ equals chance or nature
- **Strategies:** s_ϕ represents a player's spatial marker strategy
- **Choice Nodes:** H represents the choice nodes, which includes the empty choice $\varnothing$
- **Terminal Nodes:** Z represents the terminal nodes
- **Player Function:** ρ represents the player function that maps the choice nodes to each player $\rho : H \to N$
- **Action Function:** ∇ is the action function that maps the choice nodes to a set of actions $\nabla : H \to \mathbb{N}_{-0}$ or $\mathbb{R}^+$
- **Successor Function:** σ represents the successor function that identifies the terminal nodes $\sigma : H \times A \to H \cup Z$
- **Preference Relation:** $\succ_i$ represents the preference relation for player k
- **Utility Functions:** $u = (u_1, ..., u_n)$ represents the set of utility functions for each player that maps the terminal nodes to the Real line $u_i : Z \to \mathbb{R}$

For the imperfect information game:

- **Probability Selection:** f_c or $f_c(\cdot|h)$ represents a function that assigns a probability measure to a specific action at a specific choice node $A(h)$
- **Information Set:** $\Im_i$ represents the partition set for player k[12], where $I_i \in \Im_i$ represents the information set for player k

SPPs assign each player a spatial marker strategy ($s_\phi \in (\mathbb{R}^k)^n$) for all X_i or p_i. This set of random functions can range from random constant functions (real-value marks), to simple Bernoulli choices[13], to more complicated functions in $\mathbb{R}^k$. In this setup, the pure strategies for player k at each node equals the cross product of their actions $S = \times_{h \in H, \rho(h)=i} \nabla(h)$. This cross product produces a $(\mathbb{R}^k)^n$ functional space, where $S \subseteq (\mathbb{R}^k)^n$. Finally, let $\theta_\phi = \langle \phi, S, u \rangle$ represent the normal-form notation for the extensive form game γ_ϕ.

Using the above notation, researchers can represent the tuple γ_ϕ at location $X_i \rtimes\!\!\!\ltimes \setminus p_i$ as a marked point process. Let $\Gamma_\phi = \{\gamma_\phi(X_i \text{ or } p_i)\}_{i=1}^N = \{X_i \text{ or } p_i, \gamma_i\}_{i=1}^N$, $N \in \mathbb{N}_0 = \{0, 1, 2, ..., \infty\}$ and $\phi \in \{A, B, ..., N\}$ define the full strategic point process, or the locations where the set of strategic players ϕ engage in an extensive-form game. Here, I assume these marked characteristics exist in a Polish space with a completely separate metric system for the extensive form game γ_i[14]. Across all units, ϕ is assumed to be fixed; game features, however, can exhibit varying degrees of inhomogeneity. A SPP exhibits game homogeneity if $\forall \gamma_i \in \Gamma$ possess the same extensive form game structure.

2.3 Functional Marked Point Process (FMPPs)

In this paper, I use functional marked point processes (FMPPs) to measure the observable joint strategies between the set of players ϕ (Ghorbani, 2021). Ghorbani (2021) define

[12]A partition of the set $\{h \in H : P(h) = i\}$ with the property that $A(h) = A(h^{'})$ whenever h and $h^{'}$ are in the same element of the partition

[13]Point presence (1) or absence (0)

[14]ibid [insert]

an FMPP $\Psi = \{X_i, (L_i, F_i)\}_{i=1}^N$ as a point process $\Psi_G = \{X_i\}_{i=1}^N$ or a spatio-temporal point process $\Psi_T = \{p_i\}_{i=1}^N$ in $\mathbb{R}^d$ and $T \in [0, \infty)$ that is assigned the marked characteristics $\{(L_i, F_i)\}_{i=1}^N$. The collection of elements $\{(X_1, L_1, F_1), ..., (X_n, L_n, F_n)\} \subset \Psi$ or $\{(p_1, L_1, F_1), ..., (p_n, L_n, F_n)\} \subset \Psi$, $n \geq 1$, consists of:

- A collection of random spatial locations $X_1, ..., X_n \in \chi$ or collection of random space-time locations $p_1, ..., p_n \in \chi \times T$
- A collection $L_1, ..., L_n$ of random variables taking values in A
- An n-dimensional random function/stochastic process $\{F_1(t), ..., F_n(t)\}_{t \in \tau} \in (\mathbb{R}^k)^n$, with realizations in F^n[15]

FMPPs record observable joint strategies made by the set of players ϕ. For a 2-player SPP, I rewrite Ψ as $\Psi = \{(X_i, L_i), F_i\}_{i=1}^N$ or $\Psi = \{(p_i, L_i), F_i\}_{i=1}^N$ to emphasize the connection between Player A's (spatio-temporal) locations (X_i, p_i), their strategy choices (L_i), and Player B's strategy choices (F_i).

For each X_i or p_i, Player A assigns a marked characteristic L_i according to some random (function) process, i.e., binomial, multinomial, etc. For simplicity, the marginal distributions of these marked characteristics are assumed to exist in a Polish space[16]. Researchers can, however, model the joint relationship between the intensity of Player A's parental process and its marked characteristics by including the intensity of the parental process as a covariate in the marked characteristic's model (Jiao, Hu and Yan, 2021). Player A's parental process can given by $\Psi_A = \{X_i, L_i\}_{i=1}^N$, $N \in \mathbb{N}_0 = \{0, 1, 2, ..., \infty\}$.

Next, Player B assigns a function given by a k-dimensional random function process $F_i(t) = (F_{i1}(t), ..., F_{i1}(t)), t \in \tau \subset [0, \infty)$ to Player A's point locations X_i or p_i. Player B's offspring process can be given by $\Psi_B = \{X_i, F_i(t)\}_{i=1}^N$, $N \in \mathbb{N}_0 = \{0, 1, 2, ..., \infty\}$ and $t \in \tau \subset [0, \infty)$. This k-dimensional random function can include random constant functions

[15] Formally, this is an unordered collection of n stochastic processes in $\mathbb{R}^k$ with sample paths in $F = U^k \subset \{f | f : \tau \to \mathbb{R}\}^k$ (Ghorbani, 2021, p.535)

[16] ibid [insert]

with real-value outputs to more complicated functions. In this setup, Player A's markers L_i provide a way to control the supports of Player B's functional marks F_i. Similarly, these marks "serve as indicators/labels for different types of points of the point process" (Ghorbani, 2021, p.533).

2.4 Defining a Random Counting Measure over the Set of Outcomes

SPPs define a counting measure over a set of outcomes corresponding utility values. Let O represent the set of outcomes, $O = \{A_1, ..., A_k\}$. Let u_i represent a utility function that maps the set of outcomes to the Real line, $u_i : O \to \mathbb{R}^+$ and let $\Upsilon = u(O) = \{u_1, ..., u_k\}$ represent the set of utility values corresponding to each outcome. Let $N(\cdot)$ represent a random measure that counts the number of points for any subset of $\mathbb{R}^+$. Using Definition D in Appendix [insert], let $B = \Upsilon$. Let N, therefore, be a random measure $N : \Omega \times \Upsilon \to \mathbb{R}$, such that:

- For each $\omega \in \Omega$, $N(\omega, \cdot)$ is a measure on (χ, B)[17]
- For each $u_k \in B$, $N(\cdot, u_k)$ is a real-valued random variable

Next, using Definition F in Appendix [insert], let $(X_n)_{n\geq 0}$ be a collection of random variables taking values in the same measurable space (χ, B), then:

$$N(\omega, u_k) = \sum_{n\geq 0} f_{u_k}(X_n(\omega)), \text{ where } f = \begin{cases} 1 & u_k = \sup(B) \\ 0 & u_k \neq \sup(B) \end{cases} \tag{1}$$

In this setup, the random measure N counts the number of times u_k is the largest utility value in Υ. We can denote $N(\omega, u_k)$ as N_{u_k} to denote the random counting measure for game outcome A_k. According to Theorem [insert] in Appendix [insert], N_{u_k} is a monotonically strictly increasing function, where $N(\cdot) : \mathbb{R} \to \mathbf{N}_0$.

[17] $N(\omega, \cdot)$ is assumed to be finite, σ-finite, and integer-values

Researchers can extend this same logic to simple lotteries. Let L represent the set of all simple lotteries with the form $\langle q, \alpha \rangle$: $L = \{\langle q, \alpha \rangle | q_i \in O$ for all i$\}$ where q is a vector of outcomes $q \in O$ and α is a vector of probabilities that sum to 1. Using this notation, let $N'(\omega, L, \alpha_k)$ represent a random counting measure over the set of simple lotteries. Here, the random measure N' counts the number of times α_k is the largest probability value in L, where α_k corresponds to the outcome $A_k \in q$. N' sums the counts across all simple lotteries $l \in L$, where $A_k \in q$. If, $A_k \notin q$, then N' assigns a zero value:

$$N'(\omega, L, u_k) = \sum_{k=1}^{|L|} \sum_{n \geq 0} f_{u_k}(X_n(\omega)), \text{ where } f = \begin{cases} 1 & \alpha_k = \sup(l) \\ 0 & \alpha_k \neq \sup(l) \\ 0 & \alpha_k \notin l \end{cases} \tag{2}$$

This setup argues that a rational point process for outcome A_k counts the number of times A_k possesses the greatest utility or it possesses the highest probability of selection.

3 Paper Working Examples

Throughout this paper, I will rely on three working examples to help explain the underlying intuition of SPPs and the inferences researchers can draw from them. The first example expands the introductory scenario of strategic paintballers painting a canvas. Using simulated data, this example provides a way to discuss how researchers can analyze fixed sets of players who engage in an infinite series of sequential games inside a bounded study window. The second example uses real-world data to examine the interdependent relationship between residents who make 911 calls about weapon-related crimes (Player A) and neighborhood patrol officers who conduct officer-initiated directed patrols (Player B). The final example uses publicly available data on U.S. military airstrikes and insurgency attacks in Iraq during (Papadogeorgou et al., 2022).

3.1 Simulated: Strategic Paintballers

This simulated data source models the clustering tendencies between two paintballers (Players A and B). Following the intro example setup, Player A places spatial markers in an unconstrained manner across a bounded study window. Following, Player B attempts to place their own spatial markers at the same location of Player A; however, Player B exhibits spatially dependent error in their precision aim. Researchers can frame this hypothetical scenario as a Stackleberg game. Specifically, Player A decides to move into a new subset region of the study window ($\omega \in \mathbb{R}^d$). Next, Player B assesses the choices made by Player A and then it decides whether to move into the same subset region of the study window ($f(\omega) \in (\mathbb{R}^k)^n$, where $(\mathbb{R}^k)^n$ represents a functional space). If Player B decides to move into the same region, it then participates in a Search game with Player A (Alpern and Gal, 2002). Throughout this paper, I will use this simulated data to discuss how spatial trends in preferences and precision error impact modelling approaches. Appendix [insert] explains the setup for this working example.

3.2 Real-World: Effect of Weapon-Related Calls on Directed Patrols

3.3 Real-World: Effect of US Airstrikes on Insurgency Attacks

4 Theory Building with Strategic Point Processes

Spatial point analyses often lack sound theoretical frameworks. Specifically, these analyses struggle to connect microlevel strategies with their respective aggregate spatial distributions. To avoid this hierarchical misalignment[18] (Huckfeldt, 2014), researchers have to directly model the strategic interplay of a set of spatial actors to then draw inferences on aggregate spatial distributions. Several recent research design proposals, however, skip modelling a process' microlevel strategies in favor of modelling its aggregate spatial variation. These

[18] Where aggregate spatial distributions do not reflect their underlying microlevel incentive structures

novel nonparametric designs seek to exploit spatio-temporal variation in a point pattern's unknown spatial structure (Wang et al., 2023; Papadogeorgou et al., 2022).

Papadogeorgou et al. (2022), for example, propose a design that combines inverse weighting with kernel smoothing to estimate the marginal effect of U.S. military strikes on terrorist attacks. These novel designs provide a way to test causal relationships across space; however, their nonparametric understanding on how microlevel actors choose to distribute themselves restrict the ability to analyze spatial actors' microlevel motives[19].

By construction, these designs assume that a singular stochastic process produces set of treatment points (i.e., random realizations of U.S. military airstrikes) and another stochastic process produces the outcome points (i.e., random realizations of terrorist attacks). Across realization pathways, these processes can exhibit spatio-temporal inhomogeneity, which affects the distribution of observable points and their local covariance structure. Confounding spatio-temporal variables, on the other hand, induce endogeneity between the treatment and outcome processes across the indices of the random field.

If spatial actors sample from the same stochastic process across realization pathways, then these designs will accurately capture any underlying strategic dynamics between treatment and outcome processes (i.e., anticipatory behavior between terrorists and the military). If, however, actors sample from multiple stochastic processes, then these research designs will mis-specify the underlying strategic dynamics [20].

This mis-specification typically occurs when actors assign marked characteristics L_i or other random functions F_i to their spatial locations X_i. By framing the underlying stochastic process of non-overlapping point patterns as a countably infinite number of sequential game trees, SPPs help preserve the strategic connection between marker characteristics and spatial location. Specifically, the subgame perfect equilibria (SPE) of the

[19] The authors' spatial smoothing process of treatment-active locations, or the convolution of N-random variables, assumes that each discrete event can be mapped to a common measurement scale.

[20] For example, are the anticipatory behavior of a terrorist organization a function of U.S. military strike campaigns OR did the group sample a realization pathway from a different stochastic process (shifting game strategies)? If game strategy shifts are a function of space, then it becomes difficult to disaggregate anticipatory behaviors from changing game structure.

sequential game trees sample from a larger population of stochastic processes in a Polish space $((\mathbb{R}^k)^d)$[21] and assign these marked characteristics to strategic locations in $\mathbb{R}^d$. These SPEs represent the subset of the ***strategic stochastic space*** where no actor can improve their utility by unilaterally deviating from their chosen strategy in the sequential game tree.

This concept of a strategic stochastic space provides a way for researchers to model how actors strategically sample realization pathways from different stochastic processes. In addition, it provides a way for researchers to incorporate a theoretical component to the growing number of spatial causal research designs. An added layer of theory proves necessary for most spatial analyses as most spatial models tend to be atheoretical. Specifically, researchers regularly import modelling strategies from the physical and natural sciences, such as mining (Krige, 1951), seismology (Ogata and Tanemura, 2003), cosmology (Neyman and Scott, 1958), and ecology (Strong, 1980).

Over time, social scientists have "borrowed freely from the plant ecology literature, adopting techniques that had been used there in the description of spatial patterns and applying them in other contexts" (Gatrell et al., 1996, 256). The implicit behavioral assumptions that motivate popular point process models (PPMs), therefore, rarely reflect the strategic behaviors social scientists care about.

In short, nonparametric research designs avoid modelling the microlevel motives of strategic spatial actors while parametric designs rarely reflect the strategic behaviors that social scientists care about. To develop useful theories of strategic behavior across space, therefore, researchers have to connect strategic motives with their aggregate spatial distributions. Strategic point processes provide a way to overcome this hierarchical misalignment.

4.1 Connection between SPPs and FMPPs

To analyze the game structure of SPPs, researchers can use functional marked point processes, or FMPPs (Ghorbani, 2021). FMPPs expand the general framework of marked point

[21] ibid [insert]

processes so that they can incorporate functions and other stochastic processes. In the context of SPPs, FMPPs provide a way to analyze the N-tuple space $\Gamma_\phi = \{\gamma_\phi(X_i \text{ or } p_i)\}_{i=1}^{N}$, $N \in \mathbb{N}_0 = \{0, 1, 2, ..., \infty\}$ and $\phi \in \{A, B, ..., N\}$, which records the sequential games for the the study window and where N represents the number of game sites.

Here, the researcher assumes that there exists a set of functions $h_i(\cdot) = (h_1(\cdot), ..., h_n(\cdot))$ that maps subsets of the strategic point process Γ_ϕ to the functional marked point process Ψ. For each location X_i or p_i, there exists a function h that maps the set of strategies S to the set of observed marked characteristics $\{L_iF_i(t)\}$ - $\forall p_i, h_i^* : \{S\} \rightarrow \{L_iF_i(t)\}$. In practice, this process requires researchers measure the strategies, utilities, and outcomes for a set of players and assigning these characteristics to a set of discrete locations.

4.2 Drawing Inferences from SPPs

To draw inferences from SPPs, researchers must solve for the (mixed strategy) subgame perfect equilibria using forward induction (Battigalli and Friedenberg, 2012). Forward induction games assumes that all actors acted rationally in the past and will do so in the future. Unlike its more commonly used sibling backward induction, forward induction provides a way to extend multiple game formats into a single sequential game tree. Common examples of forward induction include the "pub hunt" and the "burned battle of the sexes;" in both games, the first player possesses an outside option where they can choose to engage in a game or simply sit it out[22]. The inclusion of this outside option provides a way to identify a unique SPE, which otherwise would not be available using backwards induction.

Forward induction provides a way to make SPPs tractable. Specifically, players can choose to a) ignore a game site ($\varnothing$), b) place zero spatial markers, or c) place more than one spatial marker. Realization pathways of a point process can result in zero or more points: $(X_n)_{x\geq 0}$. Zero points, however, does not mean the absence of a realization pathway. If the first player of an SPP chooses to ignore a specific location in $\mathbb{R}^d$, then conflating zero

[22]In the "pub hunt," Player A can choose to go to the pub or engage in a stage hunt game. In the "burned battle of the sexes," Player A can choose to burn all of his money or engage in a battle of the sexes game.

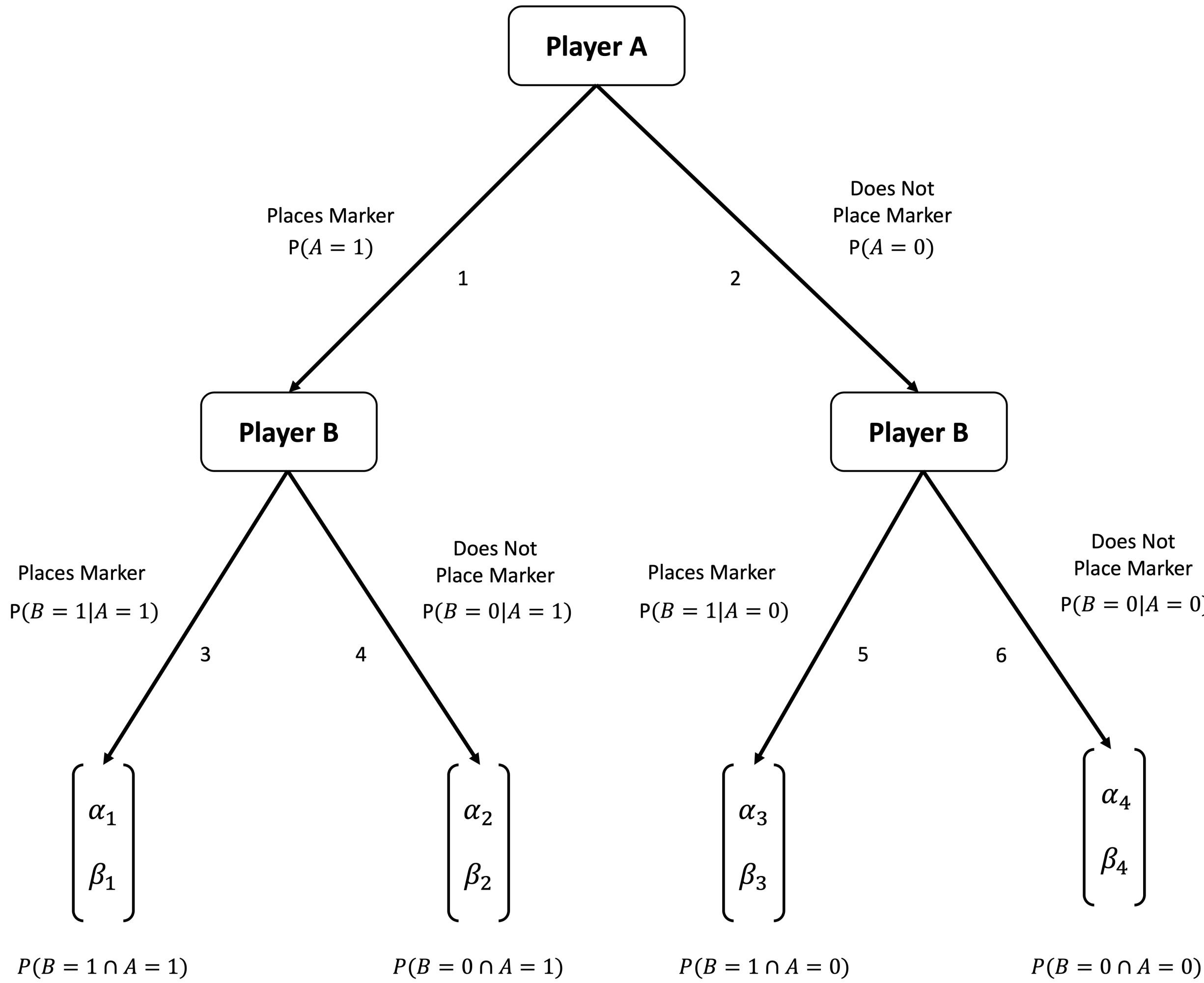

Figure 2: Basic Outline for Neyman-Scott Spatial Game

points with $\varnothing$ will bias the metric structure researchers place on top of the set of strategies: $\{s_\phi(\omega) : \omega \in \Omega\}$. Section 7.1 provides a way for researchers to circumvent this measurement problem by using quadrature schemes to sample the absence of point locations. Following this, researchers can define a reference stochastic process X^S, with sample paths in S, whose distribution ν_S on S acts as a reference measure which researchers can integrates with respect to (Ghorbani, 2021, p.532)[23].

Figure 2 illustrates a hypothetical NS spatial game between two players where the

[23]This closely approximates the Radon-Nikodym theorem, which states that a measure ν on (S, M) is absolutely continuous with respect to measure μ if for all $E \in M$ with $\mu(E) = 0$, we have $\nu(E) = 0$. For a measure space (S, M, μ) and a non-negative function f on S that is measurable with respect to M, the set function ν_S on M defined as $\nu_s(E) = \int_E f d\mu$ is a measure on (S, M) if $E \in M$.

radius parameter (μ) is set to zero (i.e., Players do not play a Search game). The nodes represent four competing theories on how Players A and B jointly interact [24]. Here, Player A can decide (or not) to place a spatial marker at the game site. Unlike NS processes, however, NS spatial games include two additional pathways where Player B can "place" offspring points in the absence of parent points or choose "not to place" offspring points when they are present[25].

With this setup, researchers can draw inferences on SPPs in two distinct ways: (1) direct analysis of the FMPP or (2) analyzing the latent set of functions $\{X_i, h_i^*\}_{i=1}^N$ or $\{p_i, h_i^*\}_{i=1}^N$, $N \in \mathbb{N}_0 = \{0, 1, 2, ..., \infty\}$.

The first method involves ***directly analyzing the FMPP that measures an SPP***, or Player A and B's joint observed outcome. Here, marginal changes in F_i with respect to $(X_i, L_i) \in \Psi_{X \times A}$ provides insights into how Player B conditionally responds to Player A, all else being equal. To draw inferences on Player B's conditional choices, however, researchers must apply a research design to the FMPP that guarantees conditional independent functional marking (Ghorbani, 2021). When this condition holds, the FMPP Ψ possesses a common (marginal) mark distribution for L_i and the underlying ground process $\Psi_G T = \{p_i\}_{i=1}^N$ exhibits (Ghorbani, 2021). (Ghorbani, 2021) define this as each $F_i|\Psi_{\chi x A} \in \Psi|\Psi_{\chi x A}, i = 1, ..., N$ having the same marginal distribution on $(F, B(F))$, which neither depends on its spatial location nor its auxiliary mark (p.543).

The second method involves ***analyzing the latent set of functions*** $\{X_i, h_i^*\}$ or $\{p_i, h_i^*\}$ that map the set of strategies $\{S\}$ to the set of observed marked characteristics $\{L_i F_i(t)\}$. To conduct such an analysis, researchers must specify the set of strategies $\{S\}$ that the FMPP will be compared against (i.e., properly identify the terminal nodes of the game tree). Next, researchers must define a stochastic process over the set of strategies S and

[24] Players A and B can cluster their spatial markers (1-3), Player B can ignore Player A (1-4), Player B can independent act (2-5), or both players can ignore the location site (2-6).

[25] In NS processes, Player A's point locations indexes Player B's strategy set. The offspring process exists only at locations where the parent process assigned points, which researchers assume to be latent. The spatstat R package defines Neyman-Scott processes using this pathway.

solve the SPE given the set of strategies S and available utility functions. Finally, researchers can calculate the expected strategy given the set of strategies S and available utility functions or construct a metric space that measures the distance between the expectation of the set of strategies and the observed marked characteristics $\{L_i F_i(t)\}$.

5 Neyman-Scott Spatial Games

5.1 Spatial Stackelberg Competition

5.2 Search Game

5.3 Forward Induction and Extensive Form Best Response Sets (EF-BRS)

6 Assumptions for Neyman-Scott Spatial Games

Neyman-Scott (NS) spatial games model how two, or more, strategic actors cluster their spatial markers inside a bounded study window. This strategic framing preserves the traditional properties of NS point processes while expanding their definition to include scenarios where offspring points (Players B, ..., N) can select away (or toward) from parent points (Player A's markers). This provides researchers greater flexibility in modelling joint relationships that PPMs and nonparametric designs would otherwise overlook.

6.1 Continuous Game Spaces and Social Flow Patterns

The first mechanic of NS spatial games involve their **continuous game space** found inside a finite **study window**. What is this "continuous game space" and what function assigns players, incentives, strategies, and local interactions to compact sets of the study window? In the strategic point process framework, every finite study window possesses a countably infinite number of game site locations (See Assumption #1) [26]. Each compact subset receives its own set of sequential game trees endowed with their own set of players (ϕ), players strategies (σ_ϕ), player payoffs (δ_ϕ), and probability distributions (mixed strategy) (p_ϕ). Physical structures (i.e., buildings, roadways, etc.) and social flow patterns (i.e., traffic patterns, foot traffic, etc.) map these game features to each compact subset (See Assumption #2).

> **Assumption #1 (Continuous Game Space)**: Finite study windows possess a countably infinite number of disjoint game site locations

> **Technical Definition #1 (Continuous Game Space)**: Let $\chi \subseteq \mathbb{R}^d$ and $(X_i)_{i\geq 0}$ be point locations, or Borel subsets, inside of χ. I will represent the game space as a continuous function γ_ϕ

Lynch, 1960's understanding of how physical spaces structure mircolevel behavior helps explain this second assumption. According to Lynch, people contextualize their surroundings based on immediately available physical, social, and political features (i.e., roads, rivers, landmarks, neighborhoods, businesses, governmental districts and buildings, etc.). Specifically, people structure their daily lives around these features, which generates unique social flow patterns that possess their own sets of incentives, strategies, and payoffs. These patterns can be (non-) stationary and their spatial contours change with the composition of individuals and contextual features inside the study window.

Traffic flow patterns of parents dropping off their kids to school provide an example. Each parent possesses their own set of incentives (i.e., leave before, during, or after rush hour traffic), available strategies (i.e., taking highway versus local roads), and anticipated payoffs (i.e., tardy points for arriving late) when dropping off their kids. Criminal activity around commercial zones provide another example. Specifically, petty criminal theft will converge in locations where personal incentives align (i.e., theft from high-income vs low-income stores),

[26] In keeping with the logic of Poisson processes, integrating across this *set of game sites* returns the area (or volume) of the study window.

available strategies minimize detection by law enforcement (i.e., number of security officers, cameras), and anticipated payoffs are high (i.e., amount of money carried by customers).

> **Assumption #2 (Social Flow Mapping)**: Fixed contextual features and (non-) stationary social flows maps game features to disjoint subsets of a finite study window

Assumptions #1 and #2 guarantee that two study windows will possess identical sets of sequential game trees if, and only if, they possess identical sets of individuals, sets of contextual features, and (non-) stationary social flows. In addition, these two assumptions guarantee that nested study windows will possess identical sets of game features inside their intersecting Borel sets. Finally, Assumption #2 provides necessary conditions that two players who possess identical social flow patterns will possess identical attitudinal and behavioral incentives [27].

6.2 Neyman-Scott Strategies

The second mechanic of NS spatial games involve each players' **Neyman-Scott strategy** (σ_ϕ). To begin, point processes index random functions across compact subsets of a bounded study window (Baddeley, Rubak and Turner, 2015). Homogeneous Poisson processes, for example, uniformly index the same random function across all subsets. Inhomogeneous Poisson processes, on the other hand, index random functions according to some (un-)known spatially varying trend. Neyman-Scott processes build on this by adding a secondary process to these homogeneous and inhomogeneous Poisson processes (Neyman and Scott, 1958). First, a set of (in-)homogeneously distributed random functions produce a set parent points (parent process). These points then produce a set of offspring points (offspring process) with a variable intensity within some distance of the parent point origin sites. Three parameters

[27] More conditions, however, have to be imposed for non-spatially correlated features (i.e., gender, height, etc.)

determine the number and positioning of both parent and offspring points [28]: an intensity (λ, $f : \lambda \rightarrow \mathbb{R}^+$), an offspring multiplier (α, $f : \alpha \rightarrow \mathbb{R}^+$), and a radial ($\mu$, $f : \mu \rightarrow \mathbb{R}^+$) parameter.

NS spatial games draw inspiration from NS processes by assigning each player a random function that corresponds to either a *parental* or *offspring strategy*. The first player in the sequential game tree (i.e., Player A) receives the parental strategy while all subsequent players (i.e., Players B, ..., N) receive an offspring strategy (See Assumption #3). For a single game site, Player A assigns markers with an intensity of $\lambda(\omega)$ while Player B assigns markers with an intensity of $\alpha\lambda(\omega - \mu)$. Player A, therefore, possesses a strategy set of $\{\{\varnothing, \lambda(\omega)\} : \lambda \rightarrow \mathbb{R}^+, \omega \in \Omega\}$ while Player B possesses a strategy set of $\{\{\varnothing, \alpha\lambda(\omega - \mu)\} : \lambda \rightarrow \mathbb{R}^+, \alpha \rightarrow \mathbb{R}^+, \mu \rightarrow \mathbb{R}^+, \omega \subset \Omega\}$ [29]. Here, players can possess pure strategies [30] or mixed strategies [31]. As Section 8 demonstrates, however, mixed strategies provide greater flexibility when modelling joint strategies across space.

> **Assumption #3 (Neyman-Scott Strategies)**: Player A distributes spatial markers according to a parental strategy while Players B, ..., N distribute spatial markers according to an offspring strategy.

Taking the outer product of each players' spatial strategy generates a *N-dimensional strategy space*, where N equals the number of players in the sequence tree. This strategy space can exhibit homogeneous of inhomogeneous properties across compact subsets. In homogeneous games, each game site possesses the same N-dimensional strategy space as players interact with the same strategy set. In inhomogeneous games, players' strategy sets depend on location. In these games, researchers cannot directly compare strategy spaces

[28] The following equation summarizes the simplest NS process: $\alpha\lambda(\omega - \mu)$. Here, the parent process sets the offspring multiplier equal to one ($\alpha = 1$) and the radial parameter to zero ($\mu = 0$). For the offspring process, the offspring multiplier and radial parameters can exhibit fixed $\{\alpha, \mu\}$ or spatially varying $\{\alpha(\omega), \mu(\omega) : \omega \in \Omega\}$ properties, where parameter values depend on location ($\omega \in \Omega$).

[29] A discrete version of the game assigns Player A with the strategy set $\{\gamma : \gamma \rightarrow \mathbb{N}_0^+\}$ and subsequent players with $\{\{\alpha\gamma(\omega - \mu)\} : \gamma \rightarrow \mathbb{N}_0^+, \alpha \rightarrow \mathbb{R}^+, \mu \rightarrow \mathbb{R}^+, \omega \subset \Omega\}$

[30] Player selects a random function based on the available payoffs

[31] Player assigns a probability distribution to each random function

across compact subsets without employing an observational research design [32] that makes strategy spaces comparable.

6.3 Player Payoffs and Local versus Neighborhood Monotonicity

The third mechanic of NS spatial games involve **player payoffs**. Specifically, each game site possesses its own set of player payoffs (δ_ϕ), which determine how players select their spatial strategies. This, in turn, determines how players allocate spatial markers across game sites. To model this behavior, researchers have to place constraints on how players should allocate points locally and across compact neighborhoods. Whereas *local monotone preferences* ignore neighboring game sites, *neighborhood monotone preferences* factor in neighborhood effects according to Tobler's first law of geography (Tobler, 2004) (see Figure 3 for illustration using a queen's neighborhood structure). Understanding the interplay between these preferences can help researchers overcome the hierarchical misalignment problem stated in the paper's introduction.

Local monotonicity governs how players select spatial strategies at specific game sites. By definition, it ignores the influence of neighboring game sites. In general, players select the strategy with the highest available payoff (Assumption #4) [33]; however, this local rationality depends on the strategic process under investigation [34]. In NS spatial games, locally rational actors choose to place spatial markers when the payoff for placing a marker exceeds the Null choice. In Figure 2, Players A and B possess two choices; however, players in other sequential games can possess more than a binary choice.

> **Assumption #4 (Local Monotonicity)**: Players place spatial markers when that strategy option maximizes their payoffs

Neighborhood monotonicity factors in neighborhood influence according to Tobler's

[32]The underlying logic of randomized control trials does not easily extend to point processes

[33]Players who exhibit this preference relationship are considered to be locally rational actors.

[34]For example, detrimental spatial games demand that strategic actors remove points for the highest available payoff

first law. According to this first law of geography, "everything is related to everything else but near things are more related than distant things" (Tobler, 2004, 304). In applied form, this law translates into a spatial weight matrix with a pre-defined neighborhood structure (Franzese and Hays, 2007). In clustering games, such as NS spatial games, players select strategies that maximize their payoffs across compact neighborhoods (Assumption #5) [35]. Specifically, players exhibit an auto-regressive trend when maximizing their payoffs. This trend exhibits rational behavior so long as players continually maximize their utility across compact neighborhoods.

> **Assumption #5 (Neighborhood Monotonicity)**: Players place spatial markers in compact neighborhoods whose local neighborhood effect maximizes their payoffs

Local monotonicity provides neither sufficient nor necessary conditions for neighborhood monotonicity. Neighborhood monotonicity, on the other hand, can be a necessary condition for local monotonicity if, and only if, the neighborhood spatial structure includes the focal game site's payoffs and zero weights are assigned to the payoffs of neighboring sites. Similar to the spatial autorgressive coefficient (ρ) in spatial lagged models (Franzese and Hays, 2007), however, players possess a weighted preference relationship with variable contributions from local and neighborhood influence. Unfortunately, under certain conditions, a weighted preference relationship can produce non-rational strategy selection as local and neighborhood preferences diverge sharply.

> **Assumption #6 (Focal Dominance)**: Players assign greater weight to their local preference structure versus their neighborhood preference structure

To avoid this hierarchical misalignment, NS spatial games, and clustering games more generally, require the focal game site to dominate the strategy choice (Assumption #6). In short, players place greater weight on their local preference structure versus their

[35] Spacing focused strategic point processes will demand different neighborhood monotone preferences.

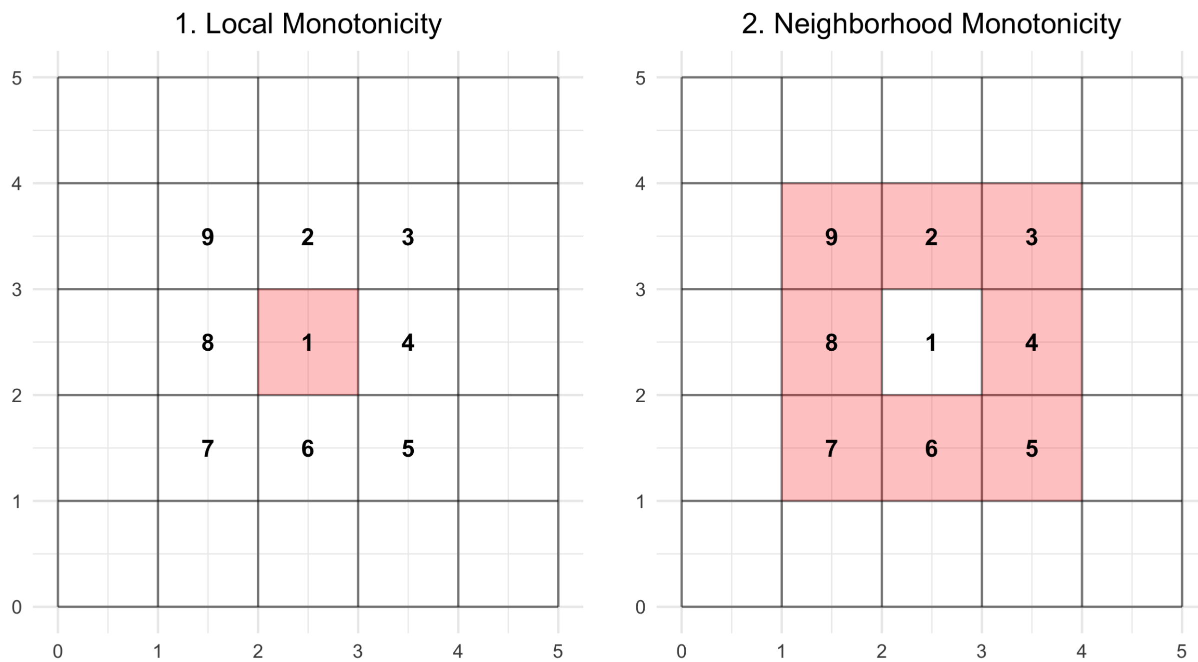

Figure 3: Local versus Neighborhood Monotonicity

neighborhood preference structure. This assumption, however, will change depending on the strategic point process under consideration[36].

6.4 Local Interactions and Nash Equilibrium Sampling

The fourth mechanic of NS spatial games involve the **local interactions** between players (Nash equilibria) and the Nash equilibria sampling process from the N-dimensional strategy space. In NS spatial games, and strategic point processes more generally, *local interactions* refer to the set of of Nash equilibria, or the sequential game tree nodes where no player increases their own payoff by changing their strategy (Rasmusen, 2006). This set samples from the N-dimensional strategy space as each draw produces a unique point pattern that encodes information on marginal and conditional strategies (Assumption #7).

Assumption #7 (Nash Equilibria Sampling): The set of (mixed strategy)

[36]For example, spacing games assume the opposite as players should place greater weight on their neighborhood preference structure than their local preference structure.

Nash equilibria samples the subset of the N-dimensional strategy space that researchers can observe

Given the four possible outcomes in Figure 2, the Nash equilibrium represents the outcome researchers can directly observe at this specific game site. In a pure strategy game, the set of player payoffs - $\delta_A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\delta_B = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ - fully determine this Nash equilibrium. Similar to traditional cooperation/defection games, however, pure strategies ignore anticipation strategies between players who frequently interact (Axelrod, 1984). Patrols officers, for example, can anticipate the neighborhoods they should patrol based on how (in-)frequently residents submit specific 911 calls (Davidson, 2023). Researchers can model this uncertainty by assigning a probability distribution - $p_\phi \in \{p_1, p_2, ..., p_n\}$ - to each item in a players' strategy set. Across all compact subsets, this generates a set of spatially varying probability distributions that determine a players' indifference strategy, or the threshold where they choose to randomize over their strategy choices irrespective of other players' choices.

In this setup, players are strategy indifferent when each strategy set item receives the same probability distribution. This local indifference, however, does not translate to neighborhood indifference. Neighborhood indifference involves scenarios where players assign the same probability distribution to each strategy set item inside a neighborhood manifold. Similar to local and neighborhood preference structures, local indifference does not guarantee neighborhood indifference. Neighborhood indifference guarantees local indifference if, and only if, the neighborhood manifold includes the focal game site and all units receive the same probability. Figure 4 illustrates this concept using queen's neighborhood structure.

6.5 Study Window Ignorance and Generational Learning

Assumptions 1-7 outline a basic structure for NS spatial games. Researchers, however, can generalize these assumptions to *game generations*, which index the study window and its game features across temporal cross sections: [insert notation]. Each generation possesses

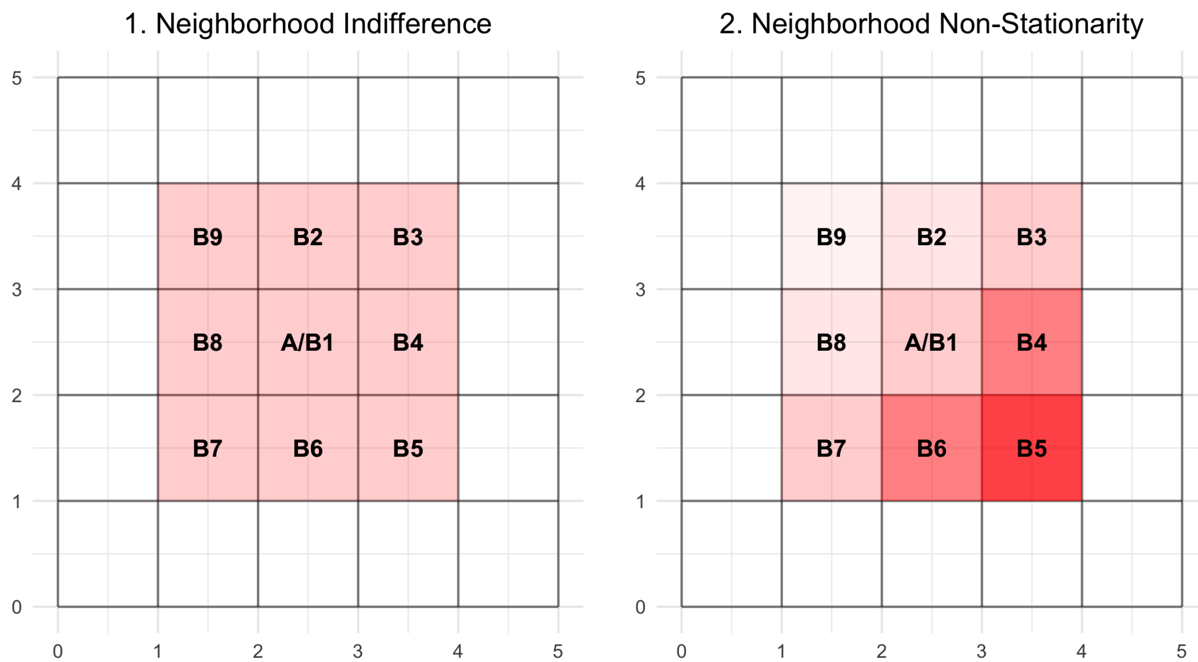

Figure 4: Neighborhood Indifference versus Non-Stationarity

the same finite number of players; each of which have access to the full study window. Similar to evolutionary game theory, game features can develop across generations (Nowak and May, 1993). Each generation, however, can differ in their payoffs, spatial strategies, and Nash equilibria. Unfortunately, having a fixed set of players across multiple time units with full study window access introduces several complications to NS spatial games. This section briefly addresses these concerns and introduces some basic assumptions for NS spatio-temporal games.

To begin, game generations introduce space-time interactions to NS spatial games, which, in turn, introduce the *space-time separability problem* (Diggle, 2013). This problem involves, in part, the arbitrary ways researchers can partition spatial processes into temporal cross-sections (i.e., hour, day, month, year, etc.). This problem does not impact NS spatial games where payoffs and strategies are temporally homogeneous across multiple generations. Instead, it largely affects non-stationary games where features vary spatially and temporally. The flow of traffic, for example, will differ depending on the temporal cross-section (i.e., Mon, Fri, Sun) and bandwidth (i.e., 12-6PM, 4-8AM).

Since players in NS spatial games have access to the entire study window, space-time interactions introduce the possibility of information spillovers across game generations. In short, players can forecast each others local and neighborhood strategies based on their access to the entire study window. This occurs as researchers pool too many time periods together. Unfortunately, there is no way to verify whether a chosen set of temporal bandwidths are appropriate. To restrict information spillovers, therefore, NS games assume that players are study window ignorant within a generation (Assumption #8) and adhere to conditional tree branching (Assumption #9).

> **Assumption #8 (Study Window Ignorance)**: Players do not learn about game features when moving across game site locations inside a bounded study window

> **Assumption #9 (Conditional Tree Branching)**: A player's future strategy choices at location ω_i are uncorrelated with past strategy choices conditional on the location's game tree structure $\gamma(\phi, \sigma_\phi, \delta_\phi, p_\phi, \eta_\omega, \omega_i)$

Local law enforcement solving crime helps explain how these two assumptions can address the issue of information spillovers. Let us assume the police want to solve a series of home intrusions in a neighborhood. When a new robbery occurs, the affected homeowners call 911 to report the incident. Law enforcement, unfortunately can only send one response team to the neighborhood. If two homeowners call 911 simultaneously, then study window ignorance prevents the response team from gathering information about both crime scenes when responding only to one location.

Conditional tree branching, on the other hand, prevents the response team from "solving a future crime that *will be reported*" in the past. Here, the key phrase is "will be reported." If two crime scenes are correlated across time (i.e., the same criminal breaks into different homes at different times), then the response teams actions in time $t-1$ should be uncorrelated with their actions in time t. In short, a player's future strategy choices at location ω are uncorrelated with past strategy choices conditional on the location's game tree structure.

6.6 Considerations when using NS Spatial Games

Although NS spatial games draw inspiration from NS processes, they differ in several important ways. First, NS spatial games assume Player B can directly observe Player A's spatial markers. NS processes, on the other hand, typically assume that a latent set of parent points generate an observable set of offspring points. Current algorithms that generate NS processes reflect this assumption. For example, the spatstat R package function rNeymanScott conducts a two-step process where a) "the algorithm generates a Poisson point process of "parent" points" and then b) "the algorithm replaces each parent point with a random cluster of points" (Baddeley and Turner, 2005, 2024). By default, the parent points are latent.

Second, NS spatial games explicitly define the absence of Player A's spatial markers (Pathways 2-5 and 2-6 in Figure 2). Defining these pathways provide the opportunity to examine Player B's selective behaviors. NS processes, on the other hand, adopt a parthenogenic approach towards point pattern formation. Specifically, Player B, or offspring points, can only spawn in locations where Player A, or parent points, chooses to place spatial markers. Finally, NS spatial games allow players to choose more than two spatial strategies at each game site. So far, Figure 2 restricted Players A and B to a binary choice set. In theory, however, Players A and B can receive $\{\varnothing, \mathbb{R}^+\}$ in continuous games and $\mathbb{N}_0^+$ spatial strategies in discrete games.

7 Constructing Joint Strategies for NS Spatial Games

NS spatial games provide a useful framework for modelling joint strategies. Unfortunately, observational datasets rarely reflect the joint structure that this framework demands. Specifically, these datasets encode information on players' marginal strategies but do not provide information on their joint strategies. To draw inferences on conditional strategies, therefore, researchers have to reconstruct the observed joint strategies, or the set of Nash equilibria, of Players A and B using observational data. This section describes how researchers can: a) define the absence of Player A's spatial markers, b) measure Player B's strategy profile, and c) reconstruct a set of observed joint strategies (Nash equilibria).

7.1 Sampling the Absence of Player A's Spatial Markers

In NS spatial games, researchers can measure the presence of Player A's spatial markers (i.e., longitude/latitude); however, they cannot directly measure their absence. This presents a measurement dilemma because "in a point pattern data set, the observed information does not consist solely of the locations of the observed points ... the absence of points at other locations is also informative" (Baddeley, 2005, 626). Researchers, therefore, have to measure Player A's Null choice, or the absence of spatial markers. This requirement, unfortunately, generates two measurement problems. First, points possess zero mass, meaning their area (or volume) are undefined. In strategic terms, Player A's observed spatial markers occupy zero area (volume) coverage of the study window. Second, NS spatial games have a countably infinite number of games. This means Player A possesses a countably infinite number of absent spatial markers. Researchers, therefore, have to find a way to sample this countably infinite set while preserving the spatial structure of Player A's strategy choices.

Quadrature schemes, such as the Berman-Turner (BT) device, help overcome this measurement and sampling problem (Baddeley and Turner, 1998) [37]. Here, I re-purpose

[37] Baddeley and Turner originally developed this method as a way to circumvent the integration problem of PPMs. Spatial point processes possess an integral term that is analytically intractable within a maximum

the BT device as an approximation to Player A's true strategy set. Specifically, this device reduces the countably infinite number of sequential games into a finite number of *strategy tiles*, or neighborhood manifolds[38]. First, this scheme generates a set of dummy points, or a set of uniformly (or randomly) spaced points in the study window. Researchers can use this set as a proxy for the absence of Player A's spatial markers [39]. Next, researchers can merge this set of *dummy points* with the observed point pattern - Player A's spatial markers. Figure 5 illustrates the BT device using simulated data, where the orange triangles represent present markers and the blue circles represent absent markers.

In general, the uniformly sampled dummy tiles provide an unbiased set of strategy tiles for Player A. Their uniform structure preserves the spatial (in-) homogeneity and clustering tendencies of Player A's spatial markers. Inferential problems emerge, however, when large regions of "true zeroes," or regions where Player A will never place a spatial marker, exist in the study window (i.e, large bodies of water, dense forests, etc.). These regions negatively bias offspring estimates and positively bias radial estimates. Researchers can reduce this bias by increasing the number of dummy tiles, which reduces neighborhood area coverage, and sub-setting the window with external shapefile features (i.e., rivers, lakes, etc.)[40].

In addition to receiving a binary tag (observed or absent), each strategy tile receives two numeric scores: the total area of the strategy tile and its inverse, which Baddeley and Turner, 1998 use as an intensity measure[41]. In the strategic context, this inverse score represents Player A's continuous strategy choices. Whereas the binary tag denotes only the presence or absence of a spatial marker, the continuous strategy choice encodes information

likelihood context. To circumvent this issue, the authors convert the integration problem into a weighted summation problem. This discretizes the region into n non-overlapping Borel sets with associated areas: α. Summing across these Borel sets returns the total area of the study window: $\sum_{i=1}^{n} \alpha = area_{\Omega}$.

[38]Here, we use the Dirichlet tessellation algorithm to generate these strategy tiles Green and Sibson (1978).

[39]In its most basic form, the BT device uniformly samples the study window with a square tiles. Researchers can use more complicated sampling schemes, including random samples of the study window.

[40]Increasing the number of points will increase the precision of estimates; however, this comes at the expense of increased computational demands

[41]the inverse of the total area serves as the dependent variable for PPMs (Baddeley, Rubak and Turner, 2015). These quasi-likelihood PPMs weight the likelihood function by the total area of the strategy tile

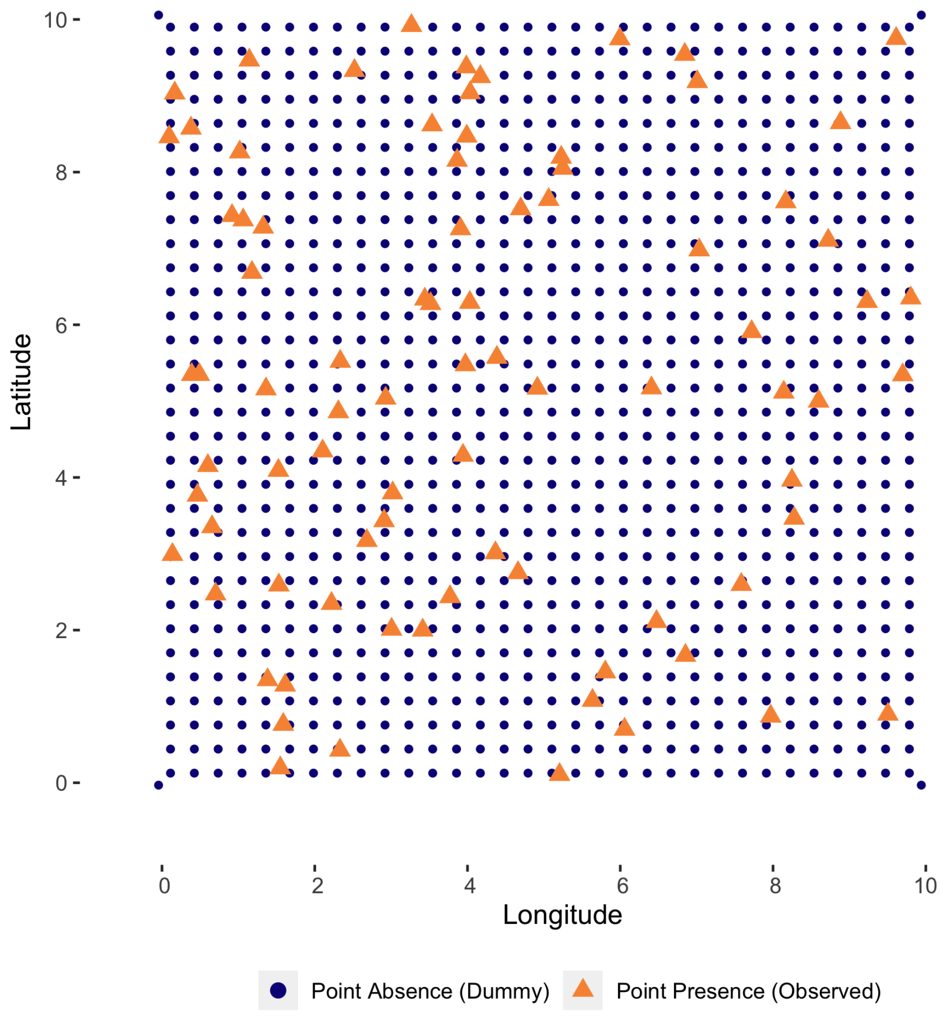

Figure 5: Defining Player A's Strategies with the Berman-Turner Device

on the spatial intensity and clustering tendencies of Player A's choices. For example, if Player A clusters their spatial markers in a specific neighborhood, then its spatial intensity will increase as the neighborhood manifold decreases.

Discretizing the study window into neighborhood manifolds generates N + D conditionally independent strategy tiles, where N is the number of observed points and D is the number of dummy points. If points exhibit interpoint correlations, then these conditionally independent tiles do not encode info on Player A's clustering tendencies. Researchers, therefore, need to incorporate this information into their spatial intensity measures by separately estimating interpoint correlations through the method of minimum contrast[42] (Pfanzagl, 1969; Waagepetersen, 2007) or by extracting this information from a fitted model's residuals (Matheron, 1969).

7.2 Measuring Player B's Strategy Profile

In NS spatial games, Player B places down points with an average intensity of $\alpha\lambda(\omega - \mu)$. Researchers can quantify Player B's strategy profile by measuring their offspring multiplier (α) and their radial (μ) estimates from observational datasets and assigning this information to Player A's strategy tiles [43]. Intersecting the two data sources reconstructs the set of Nash equilibria, or Player A and B's observable joint strategies.

To measure Player B's offspring multiplier parameter, researchers first smooth Player A and B's spatial points (both observed and absent) with kernel intensity estimation Diggle (1985). Converting these discrete event sites into a continuous surface allows researchers to take the ratio. Their ratio produces a naive set of offspring parameters across all compact subsets.

Figure 6 illustrates this ratio process using simulated data. Here, the figure's left

[42]Researchers need to specify the correlation structure and assume the interpoint correlations are uniformly distributed in the study window

[43]NS processes assume the λ parameter is constant across parent and offspring points (Neyman and Scott, 1958). The same applies for NS spatial games; however, researchers can relax this assumption where inappropriate.

side illustrates the parent and offspring intensities, which represent the smoothed surfaces of Player A and B's strategies respectively. The figure's top right corner illustrates the naive offspring parameters in $\mathbb{R}^2$. Finally, the figure's bottom right corner illustrates these parameters in $\mathbb{R}$ [44]. We include two lines to mark the expected and empirical average for the study window's offspring parameter.

To measure Player B's radial distance parameter, researchers first estimate the vector (or linear network) distance between Player A strategy choices (point presence and absence) and Player B's spatial markers. This generates two empirical distance distributions: *clustering distance*, or the distance between Player A and B's spatial markers, and *spacing distance*, or the distance between Player A's absent markers and Player B's spatial markers. These empirical distance distributions provide a way to draw inferences on how Player B chooses to cluster their spatial markers around Player A's choice set.

7.3 Basic Data Structure for NS Spatial Games

Each strategy tile receives the following empirical features. First, each tile receives a matrix, or set of matrices, whose (co-) domains represent the bounds of the N-dimensional strategy space. NS spatial games result in three matrices: an *offspring strategy matrix*[45], a *clustering strategy matrix*[46], and *Player B's strategy matrix*[47] [48]. Next, each tile receives a data point identifying the observed joint strategy (Nash equilibrium) for each matrix. Finally, in keeping with Baddeley and Turner (1998) quasi-likelihood PPM model setup, researchers weight the observed joint strategies with their corresponding probability mass, which is the total area of the strategy tile. Singletons provide little information about Player A and B's joint strategies; however, pooling information across all strategy tiles increases information signals to draw

[44] Projecting the offspring parameters from $\mathbb{R}^2$ to $\mathbb{R}$ produces a unique distribution and point estimate (Cramér and Wold, 1936)

[45] Bivariate graph between Player A's spatial intensity (λ) and Player B's offspring multiplier (α)

[46] Bivariate graph between Player A's spatial intensity (λ) and Player B's radial parameter (μ)

[47] Bivariate graph between Player B's offspring multiplier (α) and Player B's radial parameter (μ)

[48] If researchers relax the assumption that Player A and B share the same spatial intensity (λ), then they must include a bivariate graph between Player A's spatial intensity(λ) and Player B's spatial intensity (λ).

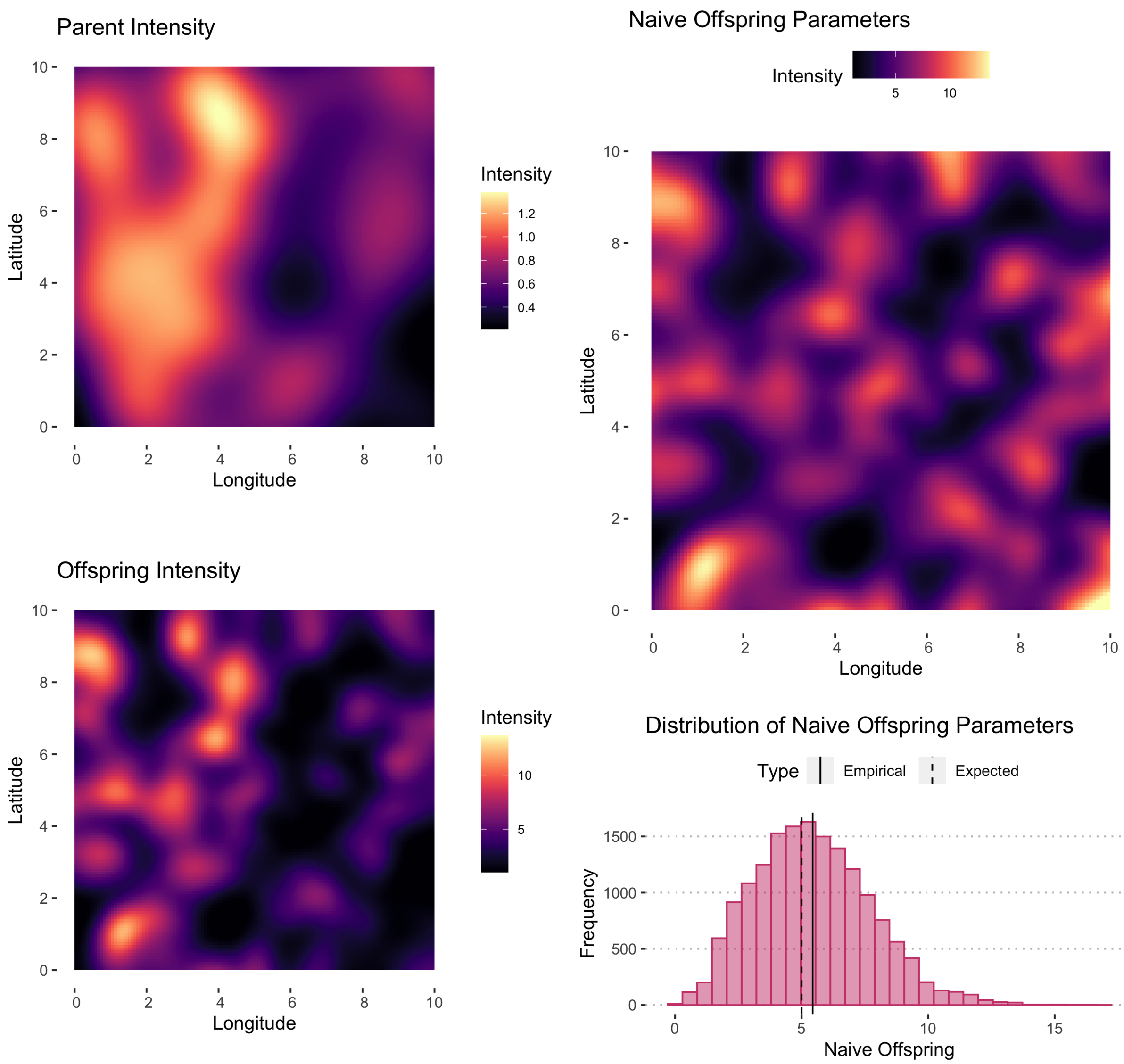

Figure 6: Measuring Player B's Offspring Parameter

inferences on joint and conditional relationships.

When Player A and B's spatial markers possess temporal tags, researchers have to separate Player A's spatial markers into appropriate cross-sections and generate a set of dummy points for each cross-section. In certain situations, this requires assigning a set of dummy points to each of Player A's spatial markers. For each temporal cross-section, researchers can estimate the offspring parameters and empirical distance distributions. Researchers, however, have to specify the temporal bandwidth for mapping Player B's strategy choices to Player A's choices. For example, a researcher might want to examine the five-day change in offspring parameters across each of Player A's spatial markers. In this setup, each of Player's spatial markers receives its own set of dummy points [49]. In addition, each spatial marker receives a vector identifying which of Player B's points spawned within the specified time bandwidth. With this information, researchers conduct the same steps outlined in the previous section.

8 Drawing Inferences from NS Spatial Games

NS spatial games provide a way for researchers to draw inferences on Player B's conditional strategies. For exampple, researchers can use this framework to investigate what effect Player A's strategy choices have on Player B's strategy choices across the study window (or some smaller compact subset)? Answering this question, however, requires an analytic baseline where Player A is strategy indifferent to Player B's potential choices (see Figure 4). Fortunately, complete spatial randomness (CSR) provides an appropriate inferential baseline (Ripley, 1979).

CSR involves scenarios where "events are distributed uniformly and independently" in the study window (Besag and Diggle, 1977). In the spatial point process literature, researchers regularly use CSR as a null hypothesis for Monte Carlo based simulation envelopes (Baddeley and Nair, 2014) or as a baseline for PPM model diagnostics (Baddeley, 2005).

[49] Researchers should use the same set of dummy points across each unit

In the potential outcomes framework (Rubin, 2005), CSR translates into a scenario where Player A randomly assigns treatment (point presence) and control (point absence) units across available units (compact subsets of the study window).

This section provides a general overview of how researchers can apply CSR to strategic point processes. Specifically, researchers can draw inferences from Player B's conditional strategy choices when Player A exhibits CSR. For NS processes, for example, if the parent process exhibits CSR, then researchers can unbiasedly estimate the offspring process' hyper-parameters when marginalizing across the study window [50]. In discussing the necessary conditions for CSR in a strategic context, I forego discussion on which research design best guarantees CSR.

8.1 CSR in a Strategic Context

In a strategic context, CSR requires a set of mixed strategy Nash equilibria where Player A randomizes their strategies across game sites and this randomization scheme does not change Player B's expected payoff. To demonstrate, let us assume that Player A and B engage in a uni-dimensional NS spatial game across seven game sites $(G_1, ..., G_7) \in \mathbb{R}$ (Figure 7). At each game site, Player A and B receive a strategy set - a vector of spatial marker strategies, a vector of corresponding payoffs, and a vector of probability distributions. Across all game sites, each player possesses three matrices: a matrix of strategies, a matrix of payoffs, and a matrix of probability distributions.

Solving for Player A's matrix of probability distributions makes Player B strategy indifferent across all game sites. Simply solving for Player A's matrix of probability distributions, however, does not guarantee CSR. For CSR conditions to hold in a stratgeic context, several other conditions have to be met. First, Player A's strategy set has to be homogeneous across all compact subsets of the study window (**Necessary Condition #1 - Strategy Set Homogeneity**). Second, Player A's expected strategy has to be uniformly

[50] Assuming that the offspring hyper-parameters are a single statistic

distributed across all compact subsets (**Necessary Condition #2 - Homogeneous Expectation**). Finally, Player A's payoffs have to be uniformly distributed across strategy choices (**Necessary Condition #3 - Uniform Strategy Payoff**). **Weak CSR conditions** apply when conditions #1 and #2 hold and **strong CSR conditions** apply when all conditions are met.

The biggest challenge for point pattern analyses involves defining a probability measure over the study window's Borel σ-algebra (Baddeley, Rubak and Turner, 2015) [51]. This challenge extends to strategic point processes as well. At a minimum, when drawing inferences from NS spatial games, researchers need to make sure their chosen research designs satisfy weak CSR conditions. Under weak CSR, Player A's strategy set and its expectation are uniformly distributed across all compact subsets. Researchers, however, rarely have direct access to player payoffs. Therefore, they cannot directly test condition #3 and verify whether conditions for strong CSR are met. Although researchers lack info on corresponding payoffs for each of Player A's strategy set items, they can model the sets expectation with a convex combination of spatial and temporal covariates.

In modelling the strategy set's expectation, researchers can account for violations of weak CSR's first condition [52]. Specifically, researchers can model the strategy set's inhomogeneity. Figure 7 demonstrates this issue where the shaded region illustrates an NS spatial game that violates weak CSR's first order condition. Instead of having a uniform strategy set across the seven game sites $(G_1, ..., G_7) \in \mathbb{R}$, Player A possesses an inhomogeneous strategy set that shrinks until the seventh game site. The expectation of Player A's strategy set, therefore, will reflect this inhomogeneity.

[51] Researchers have to map each compact subset's intensity measure to the unit interval scale [0,1] while preserving their underlying spatial structure. For example, homogeneous Poisson processes assign the same probability density to each compact subset because the intensity measure is uniformly distributed inside the study window.

[52] Assuming the convex combination of spatial covariates equals, or is proportional, to the true strategy set.

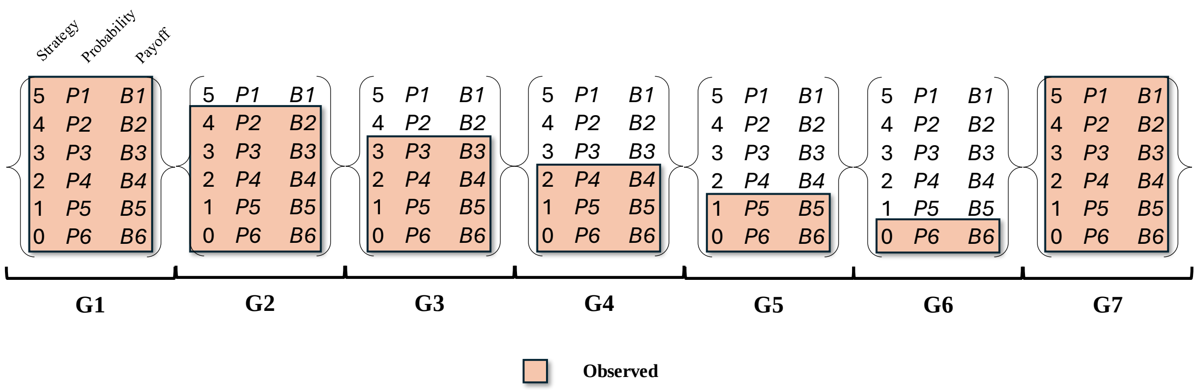

Figure 7: Uni-Dimensional NS Spatial Game

8.2 Modelling Weak CSR Conditions

To satisfy weak CSR's first and second conditions, researchers can directly model Player A's expected strategy set and apply a research design that converts the inhomogeneous process into CSR. Spatial Bernoulli processes, which researchers can use as a statistical proxy for Player A's latent probability distribution [53], help demonstrate this approach. These processes assign treatment (point presence) or control (point absence) across compact subsets according to a spatially varying probability distribution[54]. When a spatial Bernoulli process exhibits weak CSR, its probability distribution equals some constant, p. This constant need not be a coin flip, where $p = 0.5$, so long as each compact receives the same score.

When a spatial Bernoulli process fails to satisfy weak CSR, then its probability distribution exhibits inhomogeneity. Specifically, its probability levels depend on a convex combination of observable spatial covariates, S, observable temporal covariates, T, spatio-temporal interactions, S_T, and unobserved regionalized confounders, U[55] (see Equation 3). When modelling the expectation of Player A's strategy set, researchers do not model Player

[53]The same approach used here applies to settings where Player A possesses more than two strategy choices - Binomial or Poisson processes

[54]This framing generalizes Wang et al., 2023 randomization scheme - a spatial Bernoulli process that assigns treatment status {0,1} - from a fixed set of spatial units with an unknown spatial structure to all compact subsets of a bounded study window. Papadogeorgou et al., 2022 use a similar treatment assignment process as the one proposed here. The authors, however, focus on *treatment-active locations* and do not include *control-active locations* in their collection of treatments W_t

[55]See Appendix B for overview of concept

B's payoffs nor do they model the probability distribution over Player A's strategy set items. It is an ecological fallacy to assume otherwise.

$$logit(p_\omega) = \alpha_\omega + \beta S_\omega + \gamma T_\omega + \eta S_{T\omega} + \zeta U_\omega + \varepsilon_\omega \tag{3}$$

Under weak CSR conditions, Player B's payoffs and the probabilities assigned to Player A's strategy set items can possess any distribution. Let us assume that the expectation of Player A's strategy set equals 2.5 and this value is uniformly distributed across the seven game sites. Under weak CSR conditions, any convex combination that results in 2.5 is valid. Under strong CSR conditions, however, Player B's payoffs and Player A's probabilities must be uniformly distributed across strategy set items and their expected values must equal 2.5.

In general, selection-on-observable researcher designs can convert Player A's imhomogeneous strategy expectation into a process that satisfies weak CSR conditions. For example, researchers can model Player A's expected strategy and then generate a set of inverse weights that capture Player A's spatial inhomogeneity and interpoint correlations (Robins, Hernan and Brumback, 2000; Papadogeorgou et al., 2022). If properly specified, then these weights generate a pseudo spatial process that can satisfy weak CSR's conditions when applied to Player A's observed strategy choices.

To assess the validity of the modelling strategy, researchers can examine the spatial residuals of the fitted model and perform spatial diagnostics (Baddeley, 2005; Ripley, 1979). In addition, researchers can simulate points using the inverse weighted model and conduct a series of χ^2 quadrant tests to test whether the expectation of the fitted model produces randomly distributed points (Davidson, 2023).

9 Discussion and Conclusion

Strategic point processes provide a way for researchers to discuss the interactions of strategic actors across space without falling into the debate of whether spatial context matters (King, 1996). By framing the local interactions of strategic actors as a countably infinite number of sequential games inside a bounded study window, researchers can develop more complex spatial theories that prioritize strategic behaviors instead of spatial location. NS spatial games represent only a single class of strategic point processes. Using the basic framework proposed here, researchers can hopefully frame other point processes in similar strategic terms.

A Definitions and Terms

Definition A (Preference Relationship): A preference relation of player i over a set of outcomes O is a binary relation denoted by $\geq_i$. I assume that this preference relationship is complete, reflexive, and transitive.

Definition B (Utility Function): Let O be a set of outcomes and $\geq$ be a complete, reflexive, and transitive preference relation over O. A function $u : O \to \mathbb{R}$ is called a utility function representing $\geq$ if for all $x, y \in O$,

$$x \geq y \Leftrightarrow u(x) \geq u(y) \tag{4}$$

Definition C (Counting Measure): A counting measure ν on a space S is a measure with the following properties:

- $\nu(A)$ is a non-negative integer values for any measurable A
- $\nu(A) < \infty$ for any bounded, measurable A

$$\nu = \sum_i k_i \delta_{xi} \tag{5}$$

for a countable collection of positive integers k_i and points $x_i \in S$, where δ_x is a point-mass at x.

Definition D (Stochastic Process): Let (χ, B) be some measurable space and let (Ω, F, E) be a probability space. Let M, then, be a random measure $M : \Omega \times B \to \mathbb{R}$, such that:

- For each $\omega \in \Omega$, $M(\omega, \cdot)$ is a measure on (χ, B)[56]

- For each $A \in B$, $M(\cdot, A)$ is a real-valued random variable

Definition E (Mean Measure): If M is a random measure on (χ, B), then $m(A) = E[M(A)]$ is a measure on (χ, B) called the mean measure

Definition F (Point Process): Let $(X_n)_{n \geq 0}$ be a collection of random variables taking values in the same measurable space (χ, B), then:

$$M(\omega, A) = \sum_{n \geq 0} 1_A(X_n(\omega)) \tag{6}$$

B Basic Overview of Spatial Point Processes

Spatial point processes, which often derive from social (i.e., cellphone location, foot traffic, etc.) and environmental factors (i.e., species distributions), generate non-randomly distributed point patterns inside bounded study windows (Baddeley, Rubak and Turner, 2015). Given their relative unfamiliarity in political science, this section provides a brief overview of concepts I use in this article.

B.1 Complete Spatial Randomness and Spatial Inhomogeneity

Dozens of spatial point processes, such as the Neyman-Scott process, currently exist (Baddeley, Rubak and Turner, 2015). To understand how these processes work, researchers need to be aware of two concepts: *complete spatial randomness* and *spatial inhomogeneity*. When a point pattern exhibits complete spatial randomness (CSR), its underlying process generates independent and identically distributed points inside the study window. When a point pattern exhibits inhomogeneity, its underlying process non-randomly distributes points according to a spatially varying intensity function. Figure 12 illustrates two simulated point patterns along with their underlying processes.

[56] $M(\omega, \cdot)$ is assumed to be finite, σ-finite, and integer-values

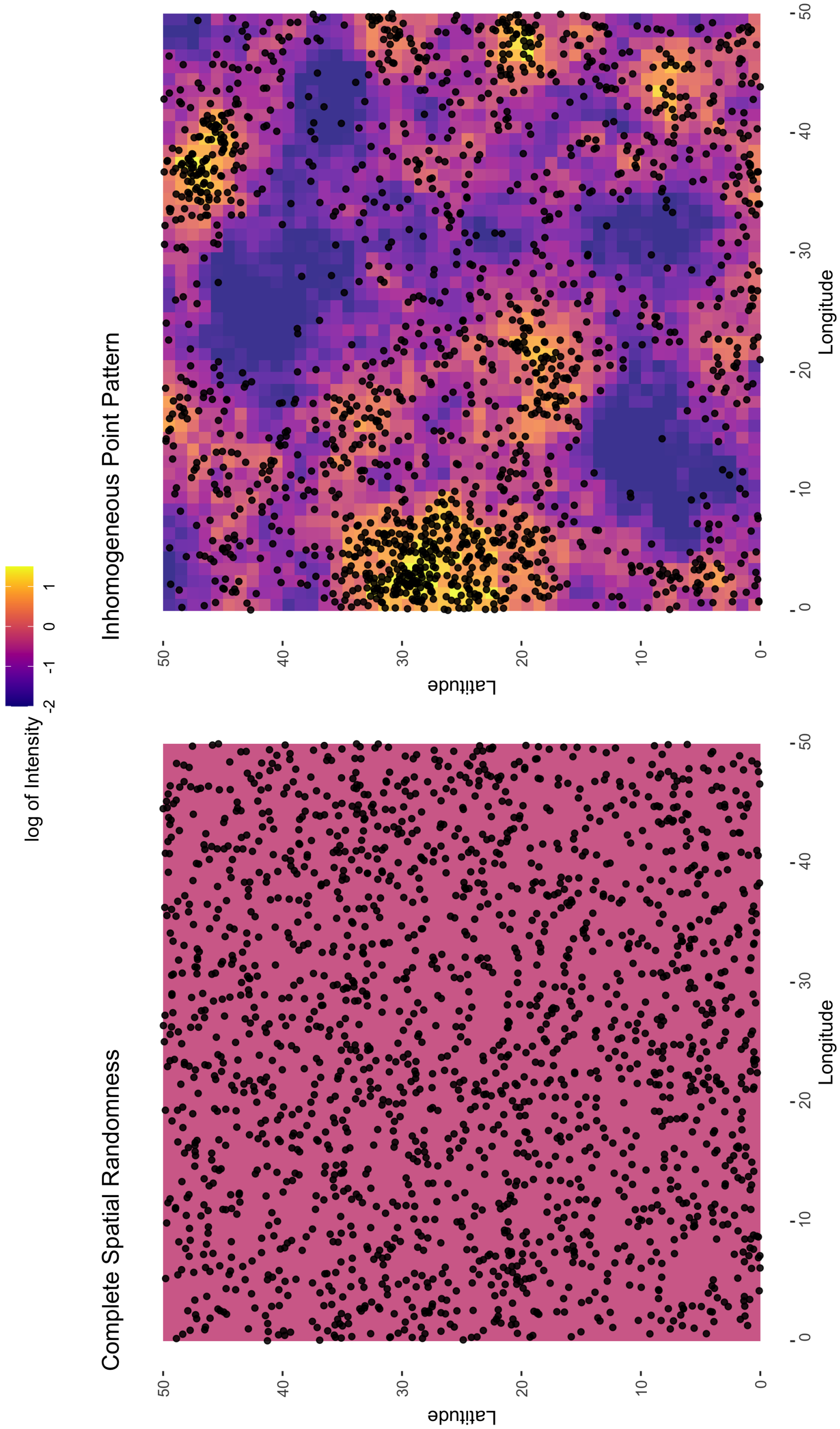

Figure 8: Visual Comparison of CSR and Inhomogeneous Point Processes

B.2 Random Components of Observed Point Patterns

Spatial point processes possess two random components (see Figure 9). The first involves the *total number of points* in the study window, which researchers assume is drawn from a Poisson distribution in $\mathbb{R}$. Researchers calculate this number with the surface integral function $\Lambda(\Omega)$, which counts the total number of points across the study window. The second component involves the placement of points inside the study window. Specifically, the spatial intensity function ($\lambda(\omega)$) distributes the total number of points across a Borel σ-algebra defined over the study window.

The total number of points over several realizations of the process generates a Poisson distribution [57]. This is not the case for the intensity function defined over the Borel σ-algebra. Even though the spatial intensity maps to the Real positive line ($f : \lambda(\omega) \rightarrow \mathbb{R}^{+}$, where $\omega \in \Omega$), researchers have to define an appropriate probability measure for all compact subsets. Researchers can circumvent this by using the empirical distributions of the process' previous realizations.

B.3 Estimating Features of the Second Random Component

Spatial point processes' second random component possesses two attributes: (a) the intensity function – the expected number of events per square kilometer – and (b) interpoint correlations – i.e., how much do points cluster together in a local vicinity. When estimating the intensity function, researchers can use either (a) kernel intensity estimation (Diggle, 1985) or (b) point process models (Baddeley and Turner, 1998). If a pattern exhibits interpoint clustering, then researchers typically use the method of minimum contrast to estimate the interpoint correlations between event sites (Pfanzagl, 1969; Waagepetersen, 2007).

[57] Assuming realizations are IID and do not exhibit temporal auto-correlation or some other unit root that shifts the distributions mean over time.

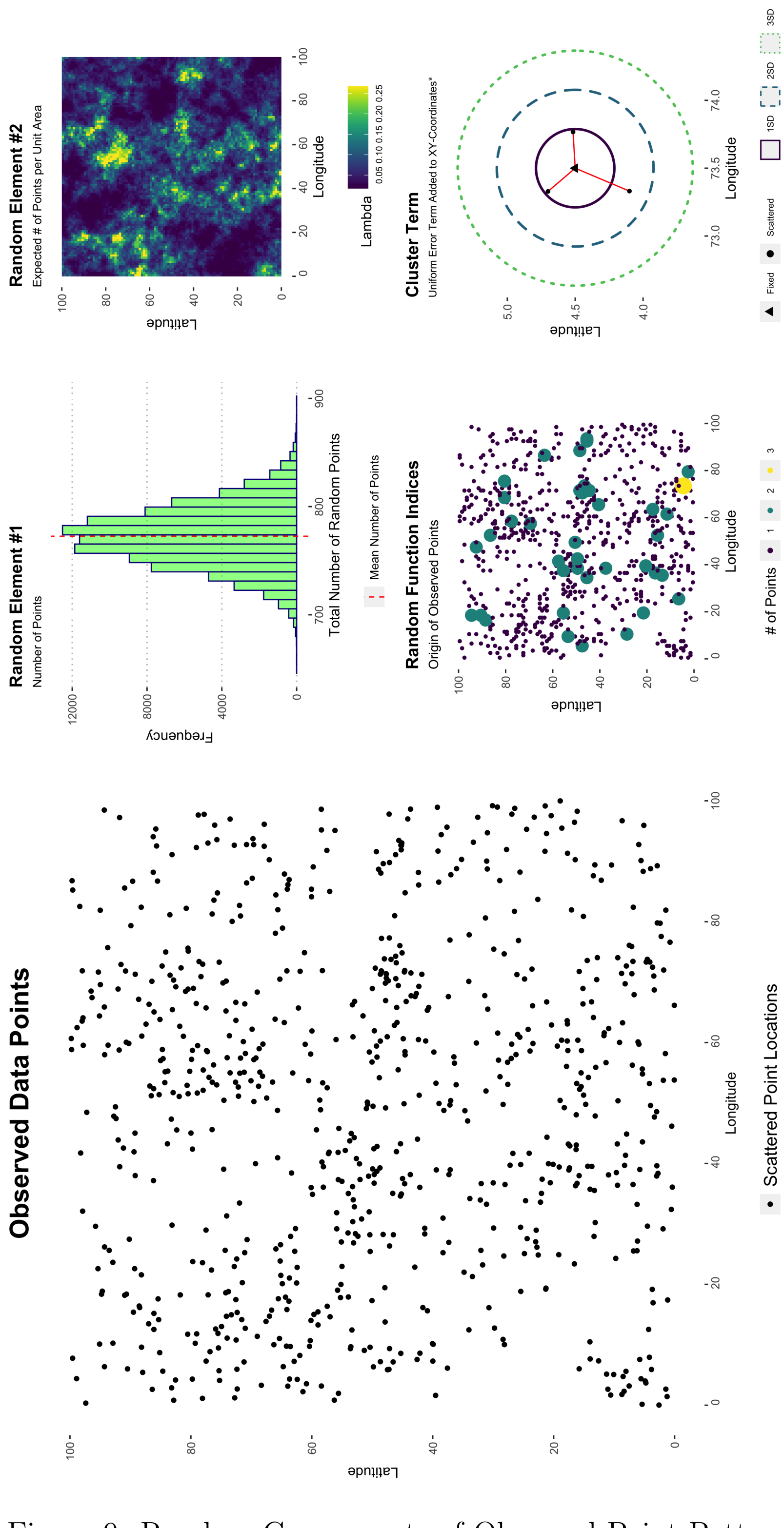

Figure 9: Random Components of Observed Point Pattern

B.3.1 Non-Parametric Kernel Intensity Estimation

The first approach, kernel intensity estimation, provides a fully non-parametric approach for estimating the intensity function (Diggle, 1985). This approach constructs an empirical intensity function using the convolution of N random variables, where each random variable equals a specific point. Each point, therefore, receives a Gaussian radial basis function with a fixed, or variable, kernel bandwidth. For (in-)homogeneous processes, the average empirical intensity function of T sequential realizations of the process converges to the true intensity function as $T \to \infty$ (Papadogeorgou et al., 2022). For homogeneous processes, the intensity function converges to a single number. For inhomogeneous processes, the intensity function converges to a set of number indexed over the study window's Borel -algebra. Equation 7 summarizes the general function for kernel intensity estimation at location ω.

$$\widehat{\lambda(\omega)} = \sum_{i=1}^{N} w_i \varphi(\|\omega - \omega_i\|) \tag{7}$$

B.3.2 Parametric Point Process Models

The second approach involves point process models (Baddeley and Turner, 1998). This process is akin to fitting a Poisson model, where the researcher assumes the logarithm of the mean is a linear combination of unknown parameters. In this conditional setting, researchers can apply asymptotic approximations to likelihood inference (Diggle and Rowlingson, 1994). These models differ from traditional Poisson models, however, as their likelihood function includes an integral term that sums information across the study window.

Researchers can approximate this integral term using a quadrature scheme (Baddeley and Turner, 1998). This weighted summation approximates the total area size of the study region by discretizing the window into a finite set of square tiles with areas equal to α. These tiles serve two purposes. First, summing these tiles returns the total area of the geographic study window: $\Omega = \sum_{i=1}^{N} \alpha$. Researchers can better approximate the total area by increasing the number of square tiles . Second, researchers use the centroid of these tiles to model the absence of point locations.

B.3.3 Method of Minimum Contrast

When estimating interpoint correlations, researchers have to make certain assumptions about the clustering tendencies between points. For example, the Poisson point process assumes points are independent and identically distributed across the study window (Diggle, 1985). Contrast this with the Matern cluster processes [58], which assume a set of offspring points are correlated within a fixed radius around a set of parent points (Baddeley, Rubak and Turner, 2015). If interpoint correlations exist, then researchers must (i) specify its correlation structure a priori (i.e., Poisson, Matern, etc.) and then (b) estimate its parameters by fitting a cluster process to an observed point pattern via the Method of Minimum Contrast (Pfanzagl, 1969; Waagepetersen, 2007).

B.4 Spatial Covariates and Unobserved Regionalized Confounders

When researchers use point process models to estimate the intensity function, they will often draw upon a set of spatial covariates. These spatial covariates can include social variables, such as proportion of minority residents, economic variables, such as the proportion of commercial buildings, or environmental variables, such as the proportion of land area covered by water. In addition, these spatial covariates can include unobserved regionalized confounders (or omitted variables) (Matheron, 1971), which distort marginal and conditional inferences. For CSR processes, the intercept of the PPM estimates the intensity function. Figure 10 illustrates some spatial covariates on the intensity of 911 calls (left) and how these spatial covariates can confound each other. Confounding can still exist in locations even when the location lacks observable points due to the continuous underlying process pattern. Figure 11 illustrates confounding introduced by location thinning. Here, an unobserved regionalized confounder removes points generated by the underlying spatial process.

[58] A special case of the Neyman-Scott process (Neyman and Scott, 1958)

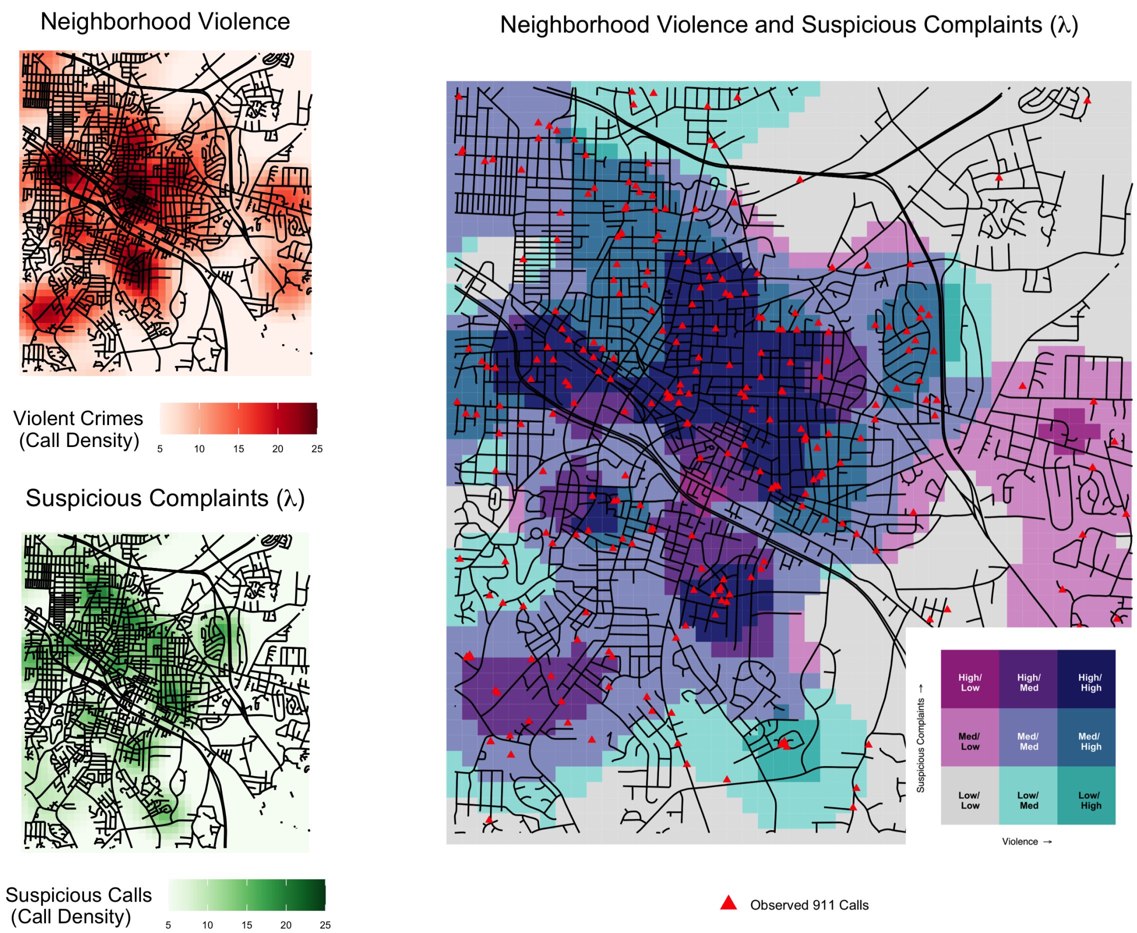

Figure 10: Regionalized Confounding of Spatial Covariates

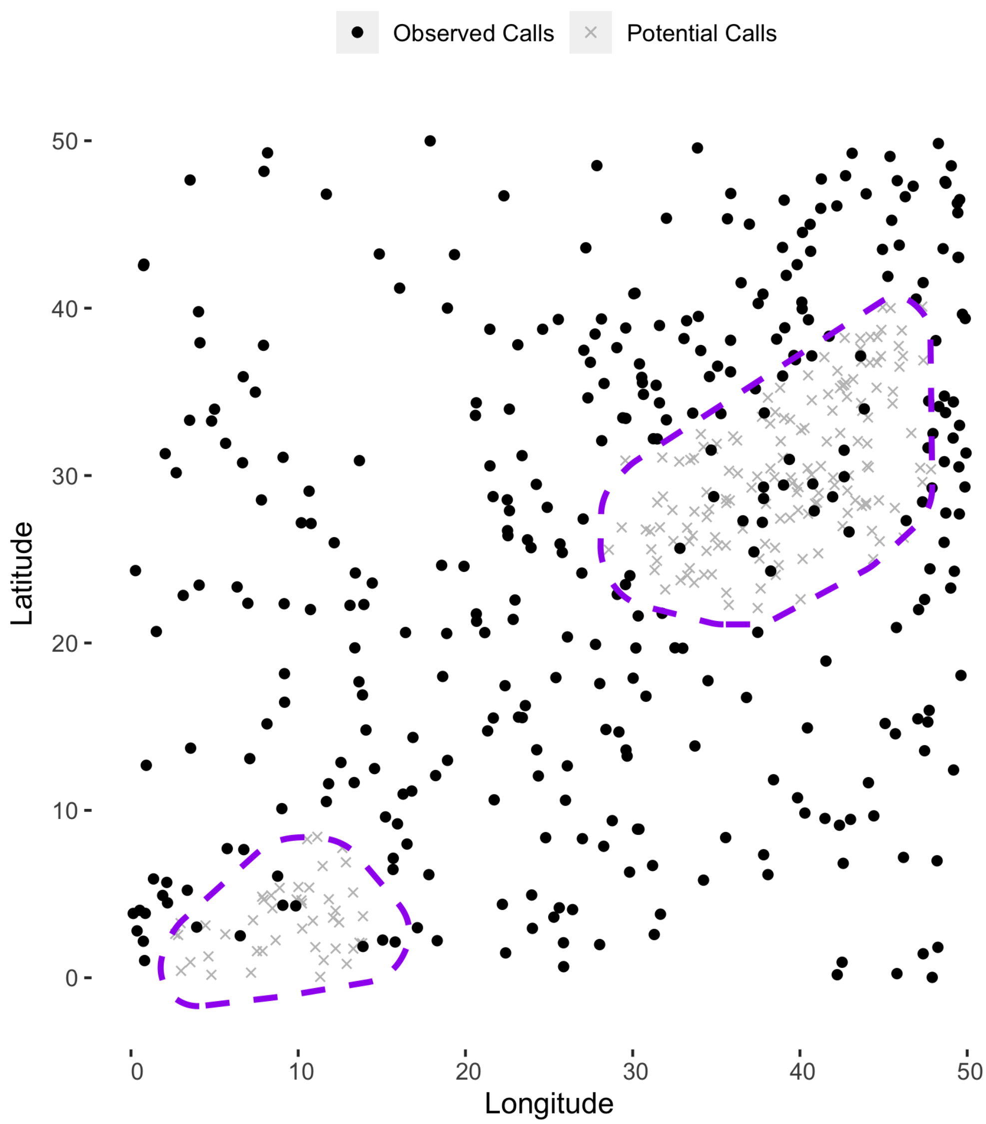

Figure 11: Regionalized Confounding - Location Thinning

Concept	**Poisson Point Process**	**Kriging (Gaussian Process)**
Number of Points	Random	Fixed
Location of Points	Randomly Assigned (CSR) Conditionally Assigned (Inhomogeneous)	Randomly Sampled
Trend of Points	Intensity Function	Zero (Simple Kriging) Linear Trend Function (Universal Kriging)
Clustering of Points	IID (Poisson Process) Other Point Process Models	IID (By Construction)
Prediction Sample Size	Complete Census of Points	Greater than one (In Theory) Greater than 20 (In Practice)

Table 1: Comparison of Point Processes and Gaussian Kriging

B.5 Handling Marked Characteristics in Spatial Point Analyses

Point patterns can possess a set of marked characteristics, or microlevel covariates, which can include demographic information (i.e., the race or gender of a 911 caller), the timing of the event, etc. These marked characteristics can possess regionalized properties, meaning factor levels depend on location. Similar to the spatial intensity function, researchers have to assign probability densities to each marked characteristic to draw proper inferences. If marked characteristics are uncorrelated with space, then researchers have to use external information to generate these densities. If correlated, then researchers can assume that the total area coverage for each factor level is proportional to their respective population level density estimates.

B.6 Comparison with Spatial Kriging

The following table summarizes key differences between Poisson point processes and spatial kriging.

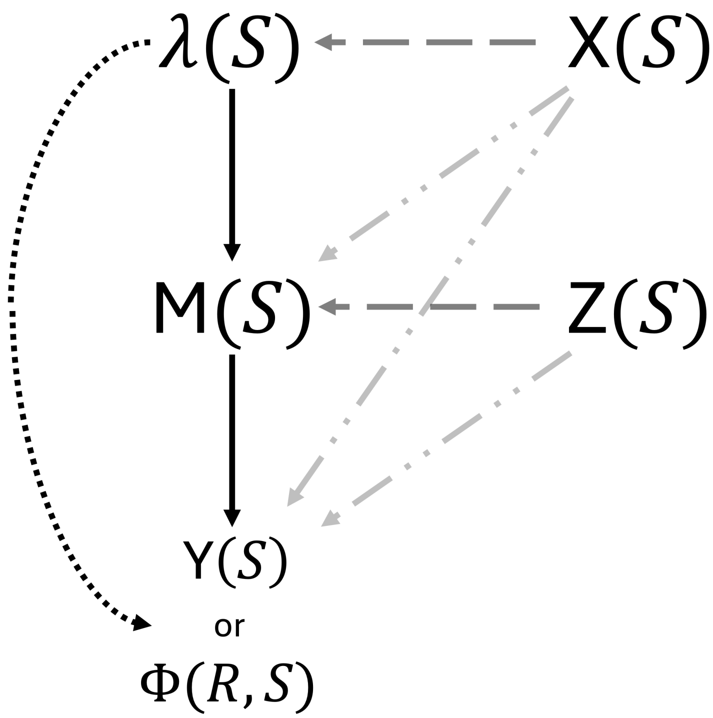

Figure 12: XXXX

C Basic Overview of Spatial Point Processes

D Sampling Properties of Random Point Processes

This appendix compares the sampling logic of point processes with longitudinal research designs (both complete panel and repeated cross-section panel). These designs both follow similar sampling logics, with some important caveats. Table 2 summarizes these key differences. To demonstrate the underlying logic, however, let us imagine a scenario where a homeowner lives in a town that celebrates Halloween every night. Each night, a large group of trick-or-treaters appear at their front door and demands candy. If the homeowner has none, then the kids will egg their house. To avoid being egged, the homeowner decides to do a random experiment where they ask the trick-or-treaters about their candy preferences (i.e., how much candy they would prefer to receive). By performing an experiment each day, the homeowner can calculate the minimum number of candies required to avoid hav-

	Longitudinal	**Repeated Cross-Section**	**Random Point Process**
Number of Cohort Samples	One	Multiple	Multiple
Across Cohort Correlations	Correlated	Independent and identically distributed (If statistical properties of known population are stationary)	Independent and identically distributed (If information recorded across same indices and exhibits temporal stationarity)
Within Cohort Correlations	Independent and identically distributed	Independent and identically distributed	Correlated

Table 2: Comparison of Point Processes, Longitudinal, and Repeated Cross-Section

ing their house egged. The homeowner can choose two different research designs: *repeated cross-sectional* and *spatial point process* designs. These two designs differ in how the owner samples preferences from trick-or-treaters.

In the random cross-sectional design, the owner randomly samples a subset of trick-or-treaters to understand their candy preferences. Using this sample, they then calculate the average number of candies to hand each trick-or-treater. In the point process design, the homeowner determines a priori the total number of candies they want to hand out. Once they select this total, they then pass out candy to the trick-or-treaters based on their queuing order at the front door. Whereas the former ignores the queuing order of trick-or-treaters, the later depends entirely on the queue order.

Let us assume the homeowner wants to conduct a repeated cross-sectional design first. In this setup, they randomly sample a portion of the trick-or-treaters, ask them about their candy preferences, record their responses, and calculate a set of statistics. They repeat this process each day to learn more about temporal trends in candy preferences. To estimate the total candy required, the homeowner multiplies this average statistic with the average number of trick-or-treaters (daily) to predict how much candy they should have in-reserve. Inference validity, however, will depend on (a) the sampling strategy – sampling people at the front door – and (b) the sampling pool. Random samples of the sample pool produce unbiased estimates if, and only if, temporal cross-sections are uncorrelated.

Now let us assume the homeowner wants to conduct a point process design. Unlike the repeated cross-section, which possesses one random component – the random samples of trick-or-treaters, the point process design possesses two random components. This includes (a) the homeowner's candy preferences (i.e., the total number of candies to distribute) and (b) the candy preferences of trick-or-treaters, which are organized into a queue. First, the homeowner samples, from their chosen distribution, the total number of candies they want to hand out [59]. Next, they hand out candies to trick-or-treaters based on queue order. As the homeowner walks, each trick-or-treaters announces their candy preference, which can be fixed or random. The homeowner continually hands out candy until they run out (i.e., sampling without replacement).

Space-time interactions complicate the point process design. Specifically, researchers have to make simplifying assumptions on the inter-dependencies between spatial and temporal trends. When modeling these interactions, researchers typically invoke the space-time separability assumption, which assumes spatial and temporal dependencies are uncorrelated. Although this rarely holds in practice, this assumption provides a way to marginalize over a point process' random components. A third-party observer, for example, can recover the homeowner's marginal preferences by integrating over the trick-or-treaters queuing order when this assumption holds. Similarly, this observer can recover the expected candy preference for each trick-or-treater in the queuing order. If trick-or-treaters exhibit complete spatial randomness [60], then the expected candy preference will be uniformly distributed across the queue. If, however, trick-or-treaters exhibit inhomogeneity [61], then the expected value depends on the location in the queue.

[59] We can frame this number as a sample draw from a Poisson distribution with mean that equals the homeowner's candy preferences.

[60] Preferences do not depend on location in the queue

[61] Preferences depend on location in the queue

E Comparison with Papadogeorgou et al. 2022

Papadogeorgou et al., 2022 estimate the causal effect of a spatio-temporal treatment process (U.S. military strikes) on an outcome process (acts of terrorism). The authors introduce an inverse probability of treatment weighting (IPTW) research design. This design leverages temporal variation in spatial distributions (historical treatment patterns) to remove confounding between treatment and outcome processes across compact subsets of the study window. The strategic point process framework can accommodate this innovative research design; however, researchers must recognize several important caveats. This section briefly discusses these caveats using the language introduced by Papadogeorgou et al., 2022. Here, I will refer to Player A's spatial strategy choices as the treatment process and Player B's spatial strategy as the outcome process.

First, both designs assume that a spatial Bernoulli process assigns treatment points across the study window. They differ, however, in the information they choose to record. For Papadogeorgou et al., 2022, this Bernoulli process generates treatment-active locations, which assigns values of 1 to the study window [62]. In a 2-person NS spatial game, Player A's discrete choices equals the Bernoulli process. Here, Player A can assign both treatment-active (1) and control-active (0) locations, which does not exist in Papadogeorgou et al., 2022's framework. In short, strategic point processes analyze the full support of the spatial Bernoulli process.

Second, both designs differ in their counterfactual comparisons. The strategic framework places greater emphasis on spatial comparisons (Player B's choices conditional on Player A) while Papadogeorgou et al., 2022 emphasizes temporal comparisons (difference between observed realization and historical treatment patterns). The strategic framework makes counterfactual comparisons between treatment and control-active locations. For these comparisons to be valid, the underlying process has to satisfy weak CSR conditions. Pa-

[62] Papadogerogu et al.'s notation: $W_t(\cdot)$ assigns a [0,1] treatment status to each subset location ($\omega = s$) in the study window ($\omega \in \Omega$)

padogeorgou et al., 2022, on the other hand, make counterfactual comparisons across temporally ordered spatial treatment patterns. The authors draw inferences by comparing an observed treatment pattern against a set of historical treatment patterns. These deviations from historical trends allow the authors to assess strike effectiveness while factoring in spatio-temproal changes in the treatment's baseline. The validity of the authors' estimation strategy hinges on their spatial smoothing technique of treatment-active locations.

Finally, the designs differ in how they interpret the spatial correlation between treatment and outcome processes. Papadogeorgou et al., 2022 define this correlation across compact subsets of the study window. Within each compact subset, there exists an (un-) observable set of confounders that induce a spurious relationship between treatment and outcome process. Accounting for all these potential confounders provides a way to estimate the true causal effect. The strategic framework, on the other hand, builds in correlation between the treatment and outcome process through the sequential game tree. Researchers can use spatial covariates to model the expectation of Player A's strategy set and satisfy weak CSR conditions.

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