Synthetic Control Methods for Proportions*

Konstantin Bogatyrev[1] and Lukas F. Stoetzer[2]

[1]Bocconi University, Milan, Italy
[2]University of Witten/Herdecke, Germany

July 12, 2024

Abstract

Synthetic control methods are extensively utilized in political science for estimating counterfactual outcomes in case studies and difference-in-differences settings, often applied to model counterfactual proportional data. However, the conventional synthetic control methods are designed for univariate outcomes, leading researchers to model counterfactuals for each proportion separately. This paper introduces an extension, proposing a method to simultaneously handle multivariate proportional outcomes. Our approach establishes constant control comparisons by using the same weights for each proportion, improving comparability while adhering to treatment constraints. Results from a simulation study and the application of our method to data from a recently published article on campaign effects in the 2019 Spanish general election underscore the benefits of accounting for the interplay of proportional outcomes. This advancement extends the validity and reliability of synthetic control estimates to common outcomes in political science.

*The authors thank Tom Pepinsky, Paul Kellstedt, the participants of the 2024 European Political Science Association (EPSA) conference and the PhD seminar at Bocconi University for their helpful comments. The usual disclaimer applies. Konstantin Bogatyrev received funding from the European Research Council (ERC Grant Agreement 864687, PI: Catherine E. de Vries). Contacts: Konstantin Bogatyrev ✉ konstantin.bogatyrev@phd.unibocconi.it, Lukas Stoetzer ✉ lukas.stoetzer@uni-wh.de

1 Introduction

Political and other social scientists are often interested in studying the effects of certain policies, events and interventions. For this purpose, causal inference methods such as the difference-in-differences (DID) have proven incredibly helpful and gained widespread use in analyzing time-series cross-sectional, or long panel data (Liu, Wang, and Xu 2024; Xu 2024; Arkhangelsky and Imbens, Forthcoming). However, the DID-type estimates commonly assume parallel trends between the treatment and control units in the absence of treatment, which proves difficult to evaluate and is often implausible (see e.g. Roth 2022; Chiu et al. 2024; Hassell and Holbein 2024). Following the premise that a convex combination of control units can better replicate the counterfactual trajectory for the treated (Abadie, Diamond, and Hainmueller 2015), synthetic controls have gained recognition as a path-breaking innovation for causal inference with time-series cross-sectional data (Xu 2024). Synthetic control methods (SCMs)[1] found a variety of applications to modeling univariate outcomes in political science, economics, and other social sciences[2].

In many applications, however, political scientists deal with multivariate, compositional data in the form of proportions. As Katz and King noted already 25 years ago: "Compositional data are also common in political science and economics (as in multiparty voting data, the allocation of ministerial portfolios among political parties, trade flows or international conflict directed from each nation to several others, or pro-

1. Hereinafter, we use the term "synthetic control methods" or simply "synthetic controls" to denote the entire family of methods, thus not limiting our discussion to the original method but also including recent advances such as generalized (Xu 2017), penalized (Abadie and L'Hour 2021) and augmented (Ben-Michael et al. 2021) synthetic controls as well as synthetic DID (Arkhangelsky et al. 2021).

2. This includes such outcomes as single-party vote shares (Hilbig and Riaz 2024; Becher, González, and Stegmueller 2023; Thompson 2023; de Kadt, Johnson-Kanu, and Sands 2024), policy adoption (Gilens, Patterson, and Haines 2021; Meserve and Pemstein 2018), welfare outcomes (Eibl and Hertog 2023), crime (Alrababa'h et al. 2021; Pinotti 2015; Trejo and Nieto-Matiz 2023), GDP (Pinotti 2015; Funke, Schularick, and Trebesch 2023; de Kadt and Wittels 2019) and power personalization (Timoneda, Escribà-Folch, and Chin 2023).

portions of budget expenditures in each of several categories)" (Katz and King 1999). In compositional data, each proportion is bound between 0 and 1, and all proportions sum up to 1. Due to these simple yet important properties, compositional data often require separate methods (Philips, Rutherford, and Whitten 2016; Tomz, Tucker, and Wittenberg 2002; Jackson 2002; Lipsmeyer et al. 2019). As we show further on, SCMs are no exception.

When the primary outcomes of interest are proportions of the same whole, the separate application of SCMs to the different proportions violates – what we will call – the constant control comparison for compositional outcomes. Each synthetic control application generates its own set of weights, which results in different control groups for each proportion. This can lead to estimates of treatment effects that do not obey an essential sum constraint. It should hold that the sum of changes in compositional data is zero, because if one proportion increases, another must decrease.

We expand the synthetic control methods framework to align with the constant control comparison for compositional outcomes. This extension introduces a method for estimating models that utilize the same control weights across different proportions. We demonstrate that this model adheres to the sum constraint on treatment effects, thereby offering a more accurate interpretation of treatment effects for proportional outcomes. A freely available R package, `propsdid`, implements the estimation.

A simulation study reveals that the synthetic control methods for proportions reduce bias compared to a DID methods (especially when the parallel trends assumption fails), and are more reliable, compared to the separate application of synthetic control methods that fail to meet the sum constraint of average treatment effects. The application of our method to a recent study in American Political Science Review (Bolet, Green, and González-Eguino 2023) on campaign effects in the 2019 Spanish general election further shows that focusing on multivariate proportional outcomes in the form of voting data is beneficial. The article argues that the social democrats found an electorally successful campaign strategy to convince former coal areas to support their transition away from means of production that are harmful to the climate. Applying the synthetic control methods for proportions shows that not only the Social Democrats won in these areas, but also their main conservative competitor, which questions the main findings from the article.

Our paper makes a set of methodological contributions. We bridge the older and more established literature on compositional data (Aitchison 1982; Katz and King 1999; Philips, Rutherford, and Whitten 2016; Tomz, Tucker, and Wittenberg 2002; Jackson 2002; Lipsmeyer et al. 2019) with the younger and burgeoning field of causal inference in panel data settings (recently reviewed by Xu 2024; Arkhangelsky and Imbens, Forthcoming; Roth et al. 2023). We highlight the relevance of constant control comparison for modeling compositional outcomes in any time-series cross-sectional set-up, regardless of whether SCMs are used. We address the shortcomings of existing synthetic control methods in handling compositional data and develop a novel approach to synthetic controls for proportions. Importantly, our approach can serve as a building block for both the original SC method (Abadie and Gardeazabal 2003; Abadie, Diamond, and Hainmueller 2010, 2015) and its advanced successors (Xu 2017; Ben-Michael et al. 2021; Arkhangelsky et al. 2021). Finally, we provide a freely available R package `propsdid` to facilitate the use of synthetic controls for proportions in applied research.

The paper is organized as follows. In Section 2, we review the theoretical foundations and recent advances in synthetic control methods. Section 3 demonstrates the challenges of applying synthetic controls to compositional data in political and social science. We formally introduce our method extension for proportions in Section 4. Then, Section 5 presents a simulation study to evaluate the performance of our method compared to traditional approaches. In Section 6, we apply our method to the case study of the

2019 Spanish general election. We conclude with a discussion of the implications that our results provide for the research on causal inference with compositional panel data.

2 Synthetic Control Methods

Researchers often seek to evaluate the effects of policies, events, and interventions using panel data, repeated observations of units over time where some units are exposed to these treatments during certain periods and not in others. The key challenge in this regard is the fact that these treatment changes typically are not random across units or time, challenging the assumption of unconfoundedness, even when observed covariates are accounted for. While the DID and the two-way fixed effects have become the dominant approaches to causal inference with panel data in political science and economics, synthetic control methods have established themselves in a smaller niche of applications where DID would not appear valid (Xu 2024; Currie, Kleven, and Zwiers 2020).

These methods create a synthetic version of the treated unit by reweighting control units to match pre-exposure trends, thus compensating for the lack of parallel trends (Xu 2017). While the original synthetic control (SC) method (Abadie and Gardeazabal 2003; Abadie, Diamond, and Hainmueller 2010, 2015) was designed for small-sample single-unit case studies and did not allow for intercept shifts between the treated and control trajectories, recent methodological advancements (e.g., Xu 2017; Ben-Michael, Feller, and Rothstein 2021; Arkhangelsky et al. 2021) have addressed those limitations and shown improved performance, both in small and large panel data settings.

For further discussion of SCMs in general, the synthetic difference in differences (SDID) approach by Arkhangelsky et al. (2021) provides a useful methodological framework, offering a generalization of both DID and SC and combining their strengths. The

SDID method matches and reweights pre-exposure trends, reducing reliance on parallel trend assumptions, akin to SC. It also maintains the invariant properties of DID related to the additive unit-level shifts, allowing for valid inference across large panels. The re-weighted post-treatment control outcomes in both methods then serve as the comparison for the estimation of average treatment effects (on the treated). For a formal description of SCMs following Arkhangelsky et al. (2021), please refer to the SM A. Arkhangelsky et al. (2021) demonstrate both the theoretical robustness of their estimator and its empirical competitiveness with, or superiority to, traditional DID and SC methods in their respective typical applications.

Despite the recent methodological generalizations of SCMs, the outcomes modeled by these methods are univariate. Although current approaches allow for studying the effects on a particular outcome, often the outcomes of interest are multivariate and inextricably linked to one another. This is particularly true for compositional data, where proportions of the same whole are bound by a linear relationship. Thus, compositional outcome variables present distinct challenges for causal inference (Arnold et al. 2020).

Researchers who want to use SCMs on proportions would usually apply the methods separately to each proportional outcome. To what extent this is problematic has not been discussed in the literature. Researchers have, however, addressed the issue of multivariate outcomes in panel data more generally. A common strategy is to model outcomes jointly, relying on the joint distribution rather than treating each outcome independently (Mullahy 2018; Samartsidis et al. 2021; Negi and Wooldridge 2024). For analyzing the treatment effects on each outcome, scholars proposed to account for multiple testing or to jointly test for effects on any outcomes (Athey and Imbens 2017; Robbins, Saunders, and Kilmer 2017). For SCMs in particular, Sun, Ben-Michael, and Feller (2023) suggest optimizing the pre-treatment fit for concatenated standardized outcomes or the average of standardized outcomes.

Building upon this literature, our paper provides a statistical foundation for using SCMs with compositional data. In the following section, we discuss the issues that arise when researchers apply SCMs to proportions.

3 Compositional Effect Issues with Synthetic Control Estimates

3.1 A Motivating Example

Before discussing the framework in abstract terms, we present a deliberately simplified hypothetical example to illustrate the key concerns that motivated our approach to synthetic methods with compositional outcomes. Consider three countries A, B and C, where country A is affected by treatment at time T, while countries B and C were not treated and constitute the donor pool for creating synthetic controls for outcomes of interest. Suppose that we are interested in the treatment effects on party vote shares in country A, and all the voters in each country choose among three party families: X (e.g., left), Y (e.g., center-right), and Z (e.g., far right). Thus, the synthetic controls for vote shares of each party in country A are obtained by weighting the vote shares of the same party family in countries B and C.

In the approach to synthetic controls currently prevalent in political science, one would apply the method to each outcome variable separately (e.g., Cremaschi et al. 2023), often focusing on one party of interest (e.g., Becher, González, and Stegmueller 2023; Hilbig and Riaz 2024). Following this approach in a synthetic diff-in-diff framework, in our example, a researcher interested in the effects of the treatment would create a synthetic control for each party X, Y and Z separately, trying to approximate that party's pre-treatment trend in A by congenial party trends in B and C. For example, if

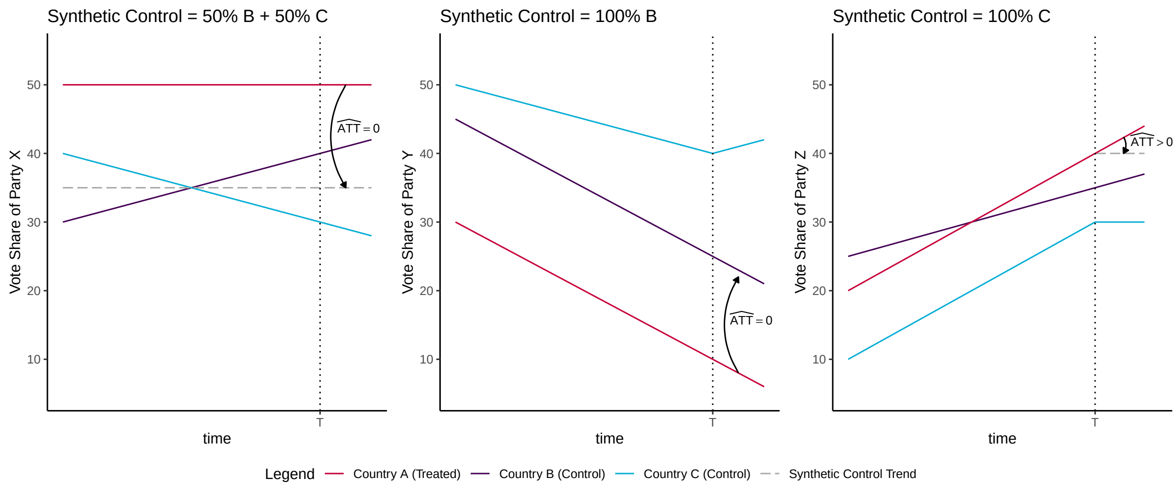

Figure 1: A hypothetical example illustrating the inconsistency in the weighting of control units (countries) for creating synthetic parallel trends for proportional outcomes (party vote shares).

A's pre-treatment trend is a bisector between the trends in B and C, then the optimal synthetic control for A comprises B and C half-and-half (Figure 1, graph 1). However, this does not mean that synthetic controls for the vote shares of Y and Z will assign the same weights to countries B and C. Graphs 2 and 3 in Figure 1 show how this equivalence can easily be violated.

Thus, using the synthetic control methods on each party X, Y and Z separately can lead to effect estimates that do not add up to zero[3] because each estimate is essentially obtained using a different control. Meanwhile, using the same synthetic control weights obtained from analyzing one outcome for estimating the effects on the related proportional outcomes (e.g., vote shares for Y and Z using the weights from analyzing X) would make the effects compatible. However, such an approach could also come with downsides: the same weights that perfectly approximate the pre-treatment trend for X can imply quite imperfect synthetic controls for Y and Z. In what follows, we present this argument formally and propose an approach that takes into account

3. Note that in Figure 1, vote shares of X, Y and Z for each country always add up to 100, as in the example these three party categories obtain all the votes in each country.

proportional outcomes that form a whole.

3.2 Constant Control Comparison

We conceptualize the idea of constant control comparison for compositional outcomes in the following way. Consider K counterfactual proportions for a unit i under the control $y_{ik}(0)$ and under treatment $y_{ik}(1)$ condition with $k \in \{1, \ldots, K\}$.[4] We observe $N - N_{co}$ treated ($D_i = 1$) and N_{co} not-treated, control units ($D_i = 0$), where $i \in \{1, \ldots, N_{co}, N_{co}+1, \ldots, N\}$. This formulation inherits the fundamental problems of causal inference that only one of the counterfactual outcomes for the different proportions can be observed. One of the key interest lies in the formulation of average effect estimates on the proportional outcomes, which depending on the assumptions identifies the average treatment effects $\text{ATE}_k = E[y_{ik}(1)] - E[y_{ik}(0)]$ or the average treatment effects on the treated $\tau_k = E[y_{itk}(1) \mid D_i = 1] - E[y_{itk}(0) \mid D_i = 1]$.

Compositional data implies that the proportions are bound between 0 and 1 ($0 \leq y_{ik}(d) \leq 1$) and sum to one ($\sum_{k=1}^{K} y_{ik}(a) = 1$) for all observations i and the two treatment states $d \in \{0, 1\}$. The first lemma describes the fact that under the proportional constraint, the sum of average treatment effects on the separate proportions should be zero.

Lemma 1 *Constraint on ATTs for proportions. For compositional, proportional outcomes where $0 \leq y_{ik}(d) \leq 1 \;\forall\; k$ and $\sum_{k=1}^{K} y_{itk} = 1$ it holds that*

$$\sum_{k=1}^{K} \tau_k = \sum_{k=1}^{K} \left(E[y_{ik}(1) - y_{ik}(0) \mid D_i = 1]\right) = 0, \tag{1}$$

where $y_{itk}(d)$ is the potential outcome of the proportion under treatment status $d \in$

4. For the simplified description of the results, we neglect the panel structure of the data applications. An extension to panel set-up with never-treated units is straightforward and will be added in later iterations of the paper.

$\{0, 1\}$.[5].

In other words, if one proportion goes up under treatment exposure, another has to go down, leading to a sum constraint over the treatment effects of zero. The same corollary holds for the ATE. Violations of the constraint on the ATTs (or ATEs) lead to contradictory interpretations of the results for different proportions.

To see if this corollary is met by an actual estimation method, it is useful to consider the way most of them evaluate the unobserved counterfactual. For the ATTs, the outcomes for the treated units under treatment $E[y_{ik}(1) \mid D_i = 1]$ is observable in the data and estimable as $E[y_{ik} \mid D_i = 1] \hat{\approx} (N - N_{co})^{-1} \sum_{i=N_{co}+1}^{N} y_{ik}$, and the key challenge lays in evaluating what the outcome would have been without exposure $E[y_{ik}(0) \mid D_i = 1]$.

Different methods can be understood as approximating the counterfactual as a weighing function of the control outcomes $E[y_{ik}(0) \mid D_i = 1] \hat{\approx} \sum_{i=1}^{N_{co}} w_{ik} y_{ik}$, where w_{ik} are the weights. For example, randomization would assign equal weights to the control units. DID also assigns the same weight to all control units, after subtracting the pre-treatment outcome. SCMs, however, estimate weights that are not equal for the control units. This also implies that the separate application to proportional outcomes can lead to non-constant weights $w_{ik} \neq w_{ik} \forall k$. To formulate this idea, we define a constant control comparison for compositional data in the following definition.

Definition 1 *Constant control comparison imposes the same weights on control units for all proportional outcomes* $w_{ik} = w_{ik'} \; \forall \; \{k, k' \in \{1, \ldots, K\} \mid k \neq k'\}$ *in the estimation of the counterfactual expectation* $E[y_{ik}(0) \mid D_i = 1] \hat{\approx} \sum_{i=1}^{N_{co}} w_{ik} y_{ik}$.

On the one hand, this constant control comparison, in the form of the same weights on control units for the different proportional outcomes, assures that the constraint on

5. Proof directly follows from the constraint on proportions $\sum_{k=1}^{K} y_{ik}(d) = 1 \; \forall \; d$

ATTs for proportions is met. This leads to the first insight:

Theorem 1 *Constant control comparison assures that the constraint on ATTs for proportions holds.*[6] *Given definition 1 it holds that the sum of the ATT estimates*

$$\sum_{k=1}^{K} \tau_k \hat{\approx} \sum_{k=1}^{K} \left((N - N_{co})^{-1} \sum_{i=N_{co}+1}^{N} y_{ik} - \sum_{i=1}^{N_{co}} w_{ik} y_{ik} \right) = 0 \tag{2}$$

If the constant control comparison is violated, the constraint on ATTs for proportions does not necessarily hold.[7]

This leads to the central insight that SCMs that want to obey the constraint on the ATTs for proportional data need to estimate the same weights for all proportions, such that the constant control comparison holds. The common approach of applying the SCMs to proportional outcomes separately can lead to violations of the constraint on the ATTs.

3.3 Constant Synthetic Control Weights Across Outcomes

Following Definition 1, any weights w_{ik} for control units that are constant among all proportional outcomes $\forall\ k \in \{1, \ldots, K\}$ satisfy the constant control comparison. In other words, the requirement to hold the weights constant alone does not yet imply which weights should be used.

In a DID set-up, constant weights for different outcomes would be imposed implicitly by having a control group pre-selected by the researcher before the effect estimation. For instance, Cremaschi, Bariletto, and De Vries (2024) estimate ATTs of plant disease on party vote shares in the Italian region of Apulia, using uninfected Apulian municipalities as a DID control group across different parties. The choice of the control group for

6. Proof TBA
7. Proof by counter example. TBA.

compositional outcomes can be data-driven: Selb and Munzert (2018) use either all German counties not exposed to Hitler's speeches or only their subsample matched by covariates to estimate the speech on the votes for Nazis and Communists. Regardless of how the control is chosen, the researcher can hold the weights constant using the same control group for the effects on different proportions.

However, SCMs for univariate outcomes cannot guarantee constant control weights across different outcomes[8]. With SCMs, calculating these weights is a part of each estimation. Thus, starting with the same sample of treated and untreated (donor pool) units for different outcomes, one ends up with weights that fit this outcome best.

To ensure constant weights across different outcomes, the synthetic control approaches must be modified. One solution previously adopted by researchers is calculating the weights for just one outcome and using them across outcomes. Indeed, after calculating synthetic control weights based on the regional GDP and its predictors, Pinotti (2015) uses the same weights to estimate the effects of organized crime on the GDP and the homicide rate. Such "incomplete" application of SCMs to all but one outcome resembles matching on covariates with subsequent DID application across outcomes, similar to Selb and Munzert (2018)'s approach to parties in interwar Germany. However, deriving synthetic control weights from only one outcome variable risks over-fitting its pre-treatment trends (Abadie 2021) and neglects the core purpose of SCMs for other outcomes.

We argue that for compositional data, synthetic control weights should maximize the goodness of pre-treatment fit for all proportional outcomes taken together. Indeed, in a DID set-up, one would require the parallel trends assumption to hold for each proportional outcome for valid inference. As violations of parallel trends in the pre-intervention (multivariate) outcomes invalidate the use of DID and make the case

8. Unless the chosen outcomes are perfectly collinear, e.g. shares of "Yes" and "No" in the sum of the two

for synthetic controls, the latter should aim to maximize the pre-treatment fit for all proportional outcomes – simultaneously, i.e. using one constant set of weights. The next section formalizes our approach to calculating synthetic control weights.

4 Synthetic Control Methods for Proportions

Based on the insights and the motivating example, we formulate a synthetic control method for proportions that follows the constant control comparison. The formulation of it closely follows Arkhangelsky et al. (2021), except for the fact that we calculate the same set of pre-treatment weights for all proportions.

We consider a balanced panel with N units and T time periods. Here we denote units by $i \in \{1, \ldots, N\}$ and periods with $t \in \{1, \ldots, T\}$. The outcomes of interest are K proportions, where each proportion value is denoted by $k \in \{1, \ldots, K\}$. The outcome for unit i in period t is a multivariate vector of $Y_{it} = [y_{it1}, \ldots, y_{itk}]$. For compositional data of proportion, it holds that $y_{itk} \in [0, 1] \ \forall \ k$ and $\sum_{k=1}^{K} y_{itk} = 1$, forming a K-dimensional simplex. We also have a binary treatment $D_{it} \in \{0, 1\}$. We assume that there is a share of control units that are never exposed to treatment N_{co}, while there are units N_{tr} that are treated after time T_{pre}. For the analysis, we arrange the i units such that first all control units appear and then all treatment units $i \in \{1, \ldots, N_{co}, N_{co} + 1, \ldots, N\}$.

Synthetic methods calculate a set of weights $\hat{\omega}_{ik}$ that align pre-exposure trends in the control outcomes with the treatment units such that $\sum_{i=1}^{N_{co}} \hat{\omega}_{ik} y_{itk} \approx \frac{1}{N_{tr}} \sum_{i=N_{co}+1}^{N} y_{itk} \ \forall \ t \in \{1, \ldots, T_{pre}\}$. As Arkhangelsky et al. (2021) describe, the idea of weighting control units in a way that to align with the pre-exposure trends of the treated units can be applied to both classic synthetic control units, where weights are calculated such that the pre-trends are matched directly, and synthetic DID methods, where weights are calculated such with a parallel shift and additional time weights. For more details on the models, see SM A.

In our application of synthetic control methods to proportions, we calculate the same set of individual weights for all proportions $\hat{\omega}_i$, and in the case of synthetic DID time weights $\hat{\lambda}_t$. The estimation of the weights has a minimization problem of the following form.

$$\sum_{i=1}^{N_{co}} \hat{\omega}_i y_{itk} \approx \frac{1}{N_{tr}} \sum_{i=N_{co}+1}^{N} y_{itk} \;\forall\; t \in \{1, \ldots, T_{pre}\} \wedge k \in \{1, \ldots, K\} \tag{3}$$

This leads to a minimization problem, where we minimize squared deviations of overall proportions with one set of weights.

$$\begin{aligned}
(\hat{\omega}_0, \hat{\omega}) &= \underset{\hat{\omega}\in\mathbb{R}, \hat{\omega}\in\Omega}{\arg\min}\; \ell_{uni}(\omega_0, \omega_k) \text{ where} \qquad (4)\\
\ell_{uni}(\omega_0, \omega) &= \sum_{k=1}^{K} \left(\sum_{t=1}^{T_{pre}} \left(\omega_0 + \sum_{i=1}^{N_{co}} \omega_i y_{itk} - \frac{1}{N_{tr}} \sum_{i=N_{co}}^{N} y_{itk} \right)^2 \right) + \xi^2 T_{pre} ||\omega||_2^2, \\
\Omega &= \left\{ \omega \in \mathbb{R}_+^N : \sum_{i=1}^{N_{co}} \omega_i = 1, \omega_i = \frac{1}{N_{tr}} \;\forall\; i \in \{N_{co}+1, \ldots, N\} \right\}
\end{aligned}$$

Important here is the summation over the pre-treatment periods and the different proportions Km with respect to the same set of weights for all proportions.

The formulation also includes a regularization parameter that penalizes weights that are too large using a L2 norm. The regularization parameter is pre-specified as ξ:

$$\xi = (N_{tr} T_{post} K)^{1/4} \hat{\sigma} \text{ with } \hat{\sigma}^2 = \frac{1}{N_{co}(T_{pre}-1)K} \sum_{k=1}^{K} \sum_{i=1}^{N_{co}} \sum_{t=1}^{T_{pre}-1} (\Delta_{itk} - \overline{\Delta}_k)^2, \tag{5}$$

$$\text{where } \Delta_{itk} = y_{it(t+1)k} - y_{ittk} \text{ and } \overline{\Delta}_k = \frac{1}{N_{co}(T_{pre}-1)} \sum_{i=1}^{N_{co}} \sum_{t=1}^{T_{pre}-1} \Delta_{itk} \tag{6}$$

Algorithm 1 Compute the Synthetic Difference in Differences (SDID) Estimator

1: Compute regularization parameter ξ
2: Compute unit weights $\hat{\omega}$
3: Compute time weights $\hat{\lambda}$
4: **for** each proportion $k \in \{1, \ldots, K\}$ **do**
5: Compute the SDID estimator via the weighted DID regression:

$$(\hat{\tau}_k^{sdid}, \hat{\mu}_k, \hat{\alpha}_{ik}, \hat{\beta}_{tk}) = \underset{\tau,\mu,\alpha,\beta}{\arg\min} \left\{ \sum_{i=1}^{N} \sum_{t=1}^{T} (y_{itk} - \mu_k - \alpha_{ik} - \beta_{tk} - D_{it}\tau)^2 \, \hat{\omega}_i \hat{\lambda}_t \right\} \quad (10)$$

6: **end for**

The time weights λ pose a similar adapted minimization problem:

$$(\hat{\lambda}_0, \hat{\lambda}) = \underset{\lambda_0 \in \mathbb{R}, \lambda \in \Lambda}{\arg\min} \ell_{time}(\lambda_0, \lambda) \text{ where} \quad (7)$$

$$\ell_{uni}(\lambda_0, \lambda_i) = \sum_{k=1}^{K} \left(\sum_{i=1}^{N_{co}} \left(\lambda_0 + \sum_{t=1}^{T_{pre}} \lambda_t y_{itk} - \frac{1}{T_{post}} \sum_{t=T_{pre}1}^{T} y_{itk} \right)^2 \right),$$

$$\Lambda = \left\{ \lambda \in \mathbb{R}_+^T : \sum_{i=1}^{T_{pre}} \lambda_t = 1, \lambda_t = \frac{1}{T_{post}} \; \forall \; i \in \{T_{pre} + 1, \ldots, T\} \right\} \quad (8)$$

The weights for unit and time can then be used in the two-way fixed effects model to estimate the synthetic DID estimator. The two-way fixed effect regression is of form:

$$(\hat{\tau}_k^{sdid}, \hat{\mu}_k, \hat{\alpha}_{ik}, \hat{\beta}_{tk}) = \underset{\tau,\mu,\alpha,\beta}{\arg\min} \left\{ \sum_{i=1}^{N} \sum_{t=1}^{T} (y_{itk} - \mu_k - \alpha_{ik} - \beta_{tk} - D_{it}\tau)^2 \, \hat{\omega}_i \hat{\lambda}_t \right\}, \quad (9)$$

where α_{ik} are unit fixed effects for each proportion, and β_{tk} are time fixed effects for all proportions.

Algorithm 1 describes the overall procedure. We first calculate the regularization parameter, then compute the unit weights followed by the time weights. Afterward, separately estimate the treatment effects using the weighted two-way regression for each proportion. The result is a set of treatment estimates on the proportions. A similar producer can be used to estimate synthetic control methods models. For this, we skip the time weight estimation, and exclude the time weights from the regression model.

5 Simulation Study

In this section, we evaluate the performance of the synthetic control methods estimator for proportions. The first parts describe a potential data generating process and set up the simulations parameters. The second part describes the results of the simple simulation, comparing the different estimators.

5.1 Data generating process

We use a dynamic latent state model to set up the data generating process for the simulation. We generate balanced panel data with T time points, N units, and K proportions. As mentioned above, $i \in \{1, \ldots, N\}$ refers to units, $t \in \{0, \ldots, T\}$ refers to time points, and $k \in \{1, \ldots, K\}$ refers to proportions. N_1 treatment units are treated from time point T_1 onward.[9]

First, we initialize latent unit factors (state variables) for each unit i and proportion k before the panel starts at $t = 0$ by taking draws from independent normal distributions: $\gamma_{i,k,0} \sim \mathcal{N}(\mu_{\gamma_{k,0}}, \sigma_{\gamma_{k,0}})$, $\upsilon_{i,k,0} \sim \mathcal{N}(\mu_{\upsilon_{k,0}}, \sigma_{\upsilon_{k,0}})$. γ describes the local levels for the different proportions and υ describes the local trends. The hyperparameters $\mu_{\gamma_{k,0}}$, $\sigma_{\gamma_{k,0}}$, $\mu_{\upsilon_{k,0}}$, and $\sigma_{\upsilon_{k,0}}$ allow us to set initial differences between units. As we see later, the local trends model is useful for setting time processes that violate parallel trends

9. There is no sequential exposure to treatment and no treatment reversal, although these processes can easily be integrated.

assumptions.

Units are allocated to treatment and control probabilistically based on a linear combination of the latent factors of one of the proportions j and a parameter δ that governs whether the treatment is selected based on local level differences or local trend differences. We define: $L_i = \psi \gamma_{i,j,0} + (1-\psi)\upsilon_{i,j,0}$. Based on L_i, we calculate treatment assignment probabilities using a logistic function with parameters λ_0 and λ: $\pi_i = \frac{1}{1+\exp(-\lambda_0 - \lambda_1 L_i)}$. We draw treatment assignment from a Bernoulli distribution: $D_i \sim \text{Bernoulli}(\pi_i)$. Here, λ_1 is a hyperparameter that sets the strength of treatment selection based on latent factors. [10] Based on this, we set the treatment indicator $W_{i,t}$ for unit i at time t to 1 if $W_i = 1$ and $t \geq T_1$, otherwise, zero.

To simulate the observed outcomes for different time points, we update the latent unit factors and time factors dynamically over time t:

$$\gamma_{i,k,t+1} = \gamma_{i,k,t} + \upsilon_{i,k,t} + \epsilon_{\gamma_{i,k,t}}, \quad \text{with} \quad \epsilon_{\gamma_{i,k,t}} \sim \mathcal{N}(0, \sigma_{\gamma_k})$$

$$\upsilon_{i,k,t+1} = \upsilon_{i,k,t} + \epsilon_{\upsilon_{i,k,t}}, \quad \text{with} \quad \epsilon_{\upsilon_{i,k,t}} \sim \mathcal{N}(0, \sigma_{\upsilon_k})$$

For each time t and factor k, the outcome $Y^k_{i,t}$ for unit i is generated as:

$$\widetilde{Y}_{i,k,t} = \gamma_{i,k,t} + \tau_k W_{i,t} + \delta_{k,t} + \epsilon_{i,k,t}$$

where $\epsilon_{i,t} \sim \mathcal{N}(0, \sigma_\epsilon)$ are idiosyncratic errors and $\delta_{k,t} \sim \mathcal{N}(0, \sigma_\delta)$ are common time effects. Based on these, we simulate the proportional outcomes using a multinomial

10. To ensure that the number of treated units in our dataset is exactly N_1, we first set $\lambda_0 = \log\left(\frac{N_1}{N}\right)$. Then, we adjust the assignment such that if the number of treatment assignments exceeds the target, we resample:

- If $\sum D_i > N_1$, select N_1 units from the assigned treated units.
- If $\sum D_i < N_1$, select $(N - N_1)$ units from the assigned control units.
- If $\sum D_i = 0$, select N_1 units randomly from all units.

The final assignment ensures exactly N_1 treated units.

link:

$$Y_{i,t,k} = \frac{\exp\left(\widetilde{Y}_{i,t,k}\right)}{\sum_{k=1}^{K} \exp\left(\widetilde{Y}_{i,t,k}\right)}$$

The multinomial link ensures that the sum of outcome proportions for each time point is defined on the simplex, i.e., $\sum_{k=1}^{K} Y_{i,t,k} = 1$.

With this data generating process, we can draw samples of compositional panel data. We can vary the time-horizon, the treatment process, the number of units and the local trend and level dynamics, and the type of treatment selection that induces violation of the parallel trends' assumption.

5.2 Results

For analysis, we use the Data Generating Process (DGP) described above with a local trend and a treatment effect on the first proportion. We analyse a scenario with four proportions and a latent treatment vector of [0.5,0,0,0]. Figure 2 illustrates the average outcomes for treated and control units over time from an example data set with 400 units, of which 200 are treated at the final time point of the five observed. The effect in this figure is not as clear but indicates an increase in proportion one. It also shows that treatment selection based on local trends leads to a violation of the parallel trends assumption and that with compositional data, the effects we induce for proportion one affect the processes for the other proportional outcomes.

For the general analysis, we set up the parameters of the data generation in the following way. We study 5 and 10 time-points with the last period being the treatment period; 200,400, and 600 units; with a treatment proportion of 10%, 20% and 50%; and we select treatment once based on local level and once based on local trends.[11]

11. We further fix the other parameters. All prior local levels parameters are set to $\sigma_{\gamma_{k,0}} = 1$ and $\mu_{\gamma_{k,0}} = 0$. For prior local trends of the first proportion, we impose $\sigma_{\upsilon_{1,0}} = 0.5$ and $\mu_{\upsilon_{1,0}} = 0$. All other

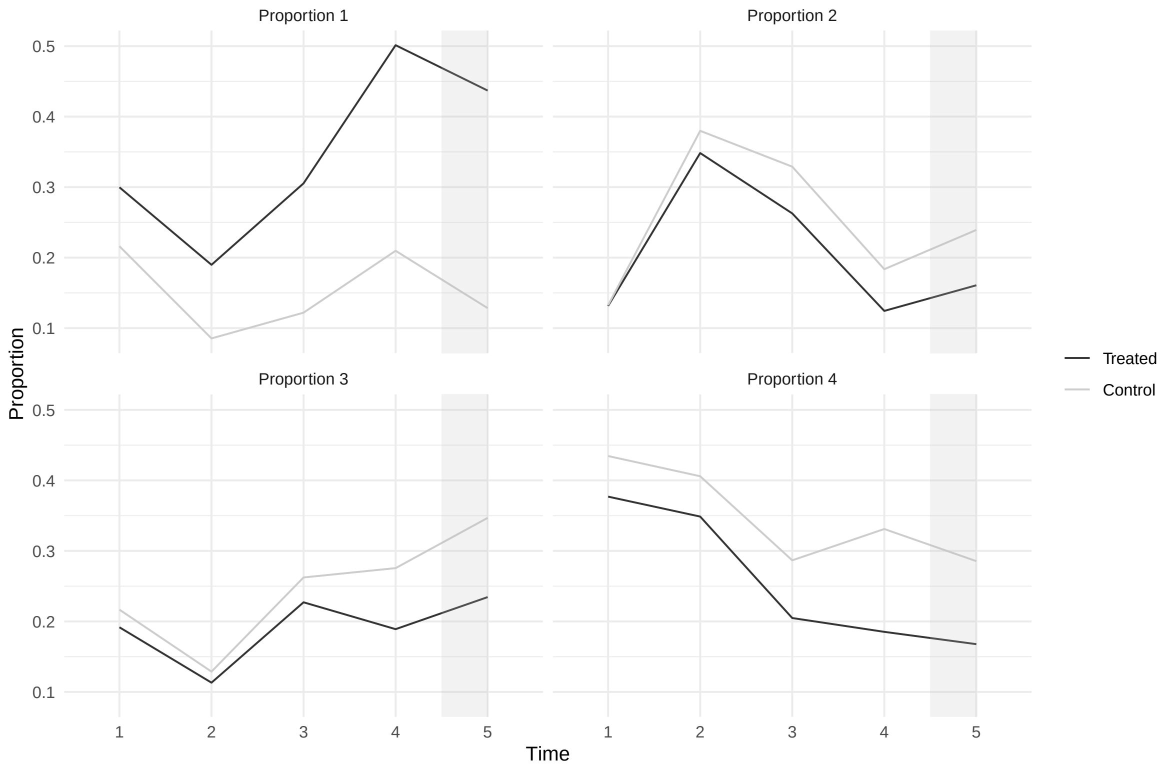

Figure 2: Example Simulation data from the data generating process (DGP). We analyse a situation with 4 proportions, and a latent treatment vector of $[0.5, 0, 0, 0]$. The example is for 400 units (of which 200 are treated) and 5 times points, where in the last time points the treatment is introduced. The first proportion exhibits a local trend, and selection is based on the local trend.

We draw 1000 independent sample for each combination and estimate a difference-in-difference (DID) model using two-way fixed effects, synthetic difference-in-differences (SDID) model separately for all outcomes and our proposed synthetic difference-in-differences model for proportions (PropSDID). We compare the estimates based on standard deviation (S.D.), root-mean-square error (RMSE) and absolute valued bias (abs. bias). We also calculate the absolute deviation of the sum of the treatment effect estimates on the separate proportions, which under the constant control comparisons

prior local trends are set to zero. The variability of the dynamics is set to $\sigma_{\gamma_k} = 0.1$ and $\sigma_{\upsilon_k} = 0.2$. The standard deviation of the errors is set to $\sigma_\epsilon = 0.1$ and $\sigma_\delta = 0.2$. The treatment selection parameter λ_1 is set to 2.

Method	S.D.	RMSE	abs Bias	Sum const.
Treatment selection: local levels				
DID	0.060	0.040	0.032	**0.000**
SDID	0.033	**0.021**	**0.016**	0.008
PropSDID	**0.031**	0.022	0.017	**0.000**
Treatment selection: local trends				
DID	0.060	0.060	0.047	**0.000**
SDID	0.033	0.026	0.019	0.018
PropSDID	**0.031**	**0.022**	**0.017**	**0.000**

Table 1: Monte-Carlo evaluation of difference-in-differences (DID), synthetic difference-in-differences (SDID) and synthetic difference-in-differences model for proportions (PropSDID) for Monte Carlo simulations of data based on the data generating process.

should be zero[12].

Table 1 describes the results. We aggregate the results for all combinations of parameter setups but differentiate between the treatment selection based on local trends and local levels, aggregating the metrics over all kk proportions. First, the table shows that the constant comparison group indeed leads to the sum constraint holding. For both the Difference-in-Differences (DID) and the Proportional Synthetic Difference-in-Differences (PropSDID), the sum of the treatment effects on the different proportions is always zero, while for the separate application of the Synthetic Difference-in-Differences (SDID), there are deviations on average. For the simulations with treatment selection based on local levels, the average deviation is 0.8 percentage points, and for the selection based on local trends, it is 1.8 percentage points. Given that the average treatment effect on the treated (ATT) in our simulation is 1.7 percentage points, this is a substantial deviation, which can make it difficult to interpret the results from the separate application of DID accurately.

12. For all the calculations, we calculate the true ATTs in the sample by comparing the counterfactuals for treated units in treatment time points.

Second, the SCMs are superior to DID, which has a considerable bias of 3.2 percentage points when treatment selection is based on local levels, and an even higher bias of 4.7 percentage points when based on local trends. This bias is also reflected in high root mean square error (RMSE). Synthetic control methods are generally preferable with our data-generating process. The absolute error is below 2 percentage points for both SDID and PropSDID. In the case of PropSDID, we find a slightly smaller RMSE and absolute bias for treatment selection based on local trends but a slightly higher RMSE and absolute bias for selection based on local levels. These differences are almost negligible in practice.[13]

Overall, the simulations confirm the central role the constant control comparison plays in ensuring the sum constraint of the ATT on proportional outcomes. They also highlight that the synthetic control estimation, which ensures that the control comparison holds, is on par with the separate application of the method in our simulations.

6 Application: Climate Policy and Voting Outcomes

In this section, we apply our proportional synthetic control methods framework to reevaluate the findings presented by Bolet, Green, and González-Eguino (2023) in the *American Political Science Review*. We use this example for three purposes. First, we demonstrate the importance of considering multivariate outcomes instead of focusing on one proportion, as previously discussed by political scientists analyzing compositional data (Katz and King 1999; Philips, Rutherford, and Whitten 2015, 2016; Lipsmeyer et al. 2019). Second, we demonstrate the imbalance of the effect estimates obtained from applying SCMs to compositional variables separately. Third, we show how this issue can be resolved by using the proportional method described in the current paper.

13. We also find that that the bias decreases with larger sample size, partciluar in the number of control units.

Bolet, Green, and González-Eguino (2023) examine the electoral effects of a compensatory "Just Transition Agreement" negotiated by the Spanish government in coalmining areas transitioning away from environmentally harmful production. Using electoral data from 2008 to 2019 and focusing on the 2019 electoral result of the governing Spanish Socialist Workers' Party (PSOE) as the outcome of interest, the study applies a DID framework to compare changes in voting between mining and non-mining municipalities in three Spanish provinces. With a few DID specifications arriving at similar point estimates of plus 1.8 percentage point effect on PSOE's vote share, the authors interpret that as the electoral reward from the voters to the incumbent party for the Just Transition Agreement.

	psoe	pp	podem	cs	vox	others
2019 × Coalmines	1.825*	1.731+	0.447	0.669	−4.593***	−0.071
	(0.771)	(0.940)	(0.398)	(0.814)	(0.482)	(0.531)
Num.Obs.	2455	2455	2455	2455	1473	2455
R2	0.364	0.488	0.520	0.650	0.752	0.214

+ $p < 0.1$, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 2: The DID estimates of the effects of the Just Transition Agreement on party vote shares in 2019 Spanish elections.

To reassess these findings, we first replicate the analysis by Bolet, Green, and González-Eguino (2023) for a complete set of party vote shares that sum up to 100: PSOE (the aforementioned center-left governing party), PP (People's Party, center-right opposition), PODEMOS (left opposition), Citizens (right-liberal opposition), VOX (far-right opposition) and others (the residual category including all remaining small parties). For this, we downloaded the electoral results for 2008-2019 national elections from the Spanish Electoral Archive (Pérez, Aybar, and Pavía 2021). Our replication confirms the reproducibility of the findings by Bolet, Green, and González-Eguino (2023) for PSOE vote shares[14]. Now, we estimate the effect of the Just Transition Agreement on a complete set of parties using the DID specification from the original study that also includes province and year fixed effects as well as covariates such as the log population, unemployment growth, the shares of immigrants, people with primary education and men over 50 in the population, all provided by the authors in their replication package. The results of this estimation, presented in Table 2, paint an interesting picture: most notably, the radical right VOX has suffered a large negative effect of minus 4.6 percentage points. The votes lost by VOX did not only benefit PSOE by about 1.8 percentage points, as was already established by Bolet, Green, and González-Eguino (2023). The results also indicate an almost equally large positive effect, significant at 0.1 level, on the vote share of the center-right opposition PP. As always with the DID, the sum of the effects for an exhaustive set of compositional outcomes is equal to zero, i.e. the effects on different parties are consistent with each other and some parties' losses are translatable to other parties' gains.

Now, we apply the synthetic difference-in-differences method to the same case. As SDID is a generalization of DID methods that relaxes the assumptions about the raw data and typically outperforms DID (Arkhangelsky et al. 2021), SDID is generally applicable to any DID setting. The pre-treatment trends in the voting outcomes between treated and control units are virtually never perfectly parallel, nor are they in our case,

14. We base the DID replication on the way the models are specified in the APSR replication package. While the article text states that model 1 includes municipal fixed effects, the code is missing them, thus we reproduce the code and not the paper description. The numerical results of Bolet, Green, and González-Eguino (2023) can only be reproduced, provided we also deliberately introduce a data error from the replication package into our independently cleaned electoral data, understating PSOE's result in one of the municipalities in 2008 by about 15 percentage points. Without this error, all point estimates from Bolet, Green, and González-Eguino (2023) become slightly smaller but remain close to the original results. Hereinafter, we use the data without that error. Furthermore, it appears that the authors defined PSOE vote share as a ratio of PSOE votes to the number of all ballots cast, including blank and null (invalid) votes. This differs from the way vote shares are commonly calculated by political scientists and electoral statistics, dividing the votes by party only by valid votes (excluding null ballots). We use that latter approach to ensure a more intuitive interpretation of our estimates and their direct applicability to the official electoral results calculated by the Spanish authorities.

as can be seen in Figure 2 in Bolet, Green, and González-Eguino (2023). Thus, applying SDID can ensure the best pre-treatment fit in vote shares of the synthetic control municipalities and the coalmine municipalities for each party. The results of this estimation, presented in Table 3, paint a picture similar to the DID results in terms of size and sign of the coefficients. The 1.8 p.p. effects on PSOE is very close to the DID estimate, while PP has gained 1.5 p.p. and VOX has lost 3.9 p.p. from the treatment, according to the SDID estimates. However, as anticipated in Section 3, these effect estimates should be interpreted carefully as they are not consistent with each other: instead of summing up to zero, these effects sum up to $1.82 + 1.52 + 0.39 + 0.69 - 3.89 - 0.04 = 0.89$, which contradicts to the nature of proportions in compositional data.

	PSOE	PP	PODEMOS	Citizens	VOX	Others
coalXpost	1.82	1.52	0.39	0.69	-3.89	-0.04
	(0.65)	(0.76)	(0.34)	(0.65)	(0.47)	(0.29)

Table 3: The SDID estimates of the effects of the Just Transition Agreement on party vote shares in 2019 Spanish elections.

To resolve this issue, we now estimate the same effects with our novel synthetic control methods tailored for multivariate proportional outcomes, using the `propsdid` package accompanying this paper. Thus, we ensure a constant synthetic control group for the coalmine municipalities, aligning the effects with the essential sum constraint where any increase in vote shares for one party necessitates a decrease in another. The results of this estimation, presented in Table 4, are generally more conservative than any of the previous estimates, pointing that those could overestimate the effects. According to the proportional SDID, the overall negative effects on the far right VOX was about -3.4 percentage points. Though the positive effects on other parties lose significance due to their small sizes, the point estimates for the largest parties PSOE and PP remain positive and larger than for the remaining parties, suggesting that PSOE and PP were the main beneficiaries of the VOX losses.

	PSOE	PP	PODEMOS	Citizens	VOX	Others
coalXpost	1.33	1.08	0.26	0.77	-3.41	-0.03
	(0.69)	(0.82)	(0.36)	(0.69)	(0.53)	(0.26)

Table 4: The proportional SDID estimates of the effects of the Just Transition Agreement on party vote shares in 2019 Spanish elections.

Our analysis demonstrates the applicability of proportional synthetic control methods for causal inference with panel data on compositional outcomes. Moreover, in that case, the estimates provided by proportional SDID are more reliable than both DID and SDID, as our approach maximizes the fit in parallel pre-treatment trends while keeping the effects balanced with each other. The evidence obtained from this application complements and challenges the initial conclusions drawn by Bolet, Green, and González-Eguino (2023), showing that the Just Transition Agreement did not just draw voters to reward to governing party. Rather, it had a deradicalizing effect, making the citizens vote less for the far right and more for the large mainstream parties both in government and opposition. Besides, applying the synthetic DID estimator shows that those effects appear to be smaller than their estimates obtained by the DID methods.

7 Discussion

In this paper, we extend the synthetic control methods framework to handle proportions as outcomes. Our proposed approach, which ensures constant control comparison across different proportions, adheres to the essential sum constraint inherent in compositional data. This not only enhances the validity and reliability of the treatment effect estimates but also provides a more coherent interpretation of the results.

Our study builds on a long methodological tradition to take compositional outcomes seriously (Aitchison 1982; Katz and King 1999; Jackson 2002; Tomz, Tucker, and Wittenberg 2002; Philips, Rutherford, and Whitten 2016; Lipsmeyer et al. 2019; Arnold et al. 2020; Cohen and Hanretty 2024). The formulated framework demonstrates a way to include compositional outcomes in causal inference with time-series cross-sectional data, central to many recent methodological advances (Liu, Wang, and Xu 2024; Xu 2024; Arkhangelsky and Imbens, Forthcoming; Roth et al. 2023; Imai, Kim, and Wang 2023; Imai and Kim 2019; Blackwell and Glynn 2018). Whenever proportions are studied, our approach can serve as a building block for a whole range of synthetic control methods - from the standard SC (Abadie and Gardeazabal 2003; Abadie, Diamond, and Hainmueller 2010, 2015) to generalized (Xu 2017), penalized (Abadie and L'Hour 2021) and augmented (Ben-Michael et al. 2021) synthetic controls as well as SDID (Arkhangelsky et al. 2021).

Our approach inherits the data requirements and assumptions of the synthetic control method implemented by the researcher. However, compared to separate outcome modeling, our approach to proportions reduces the necessity for extensive pre-treatment data for each outcome (Abadie 2021). Relying on the deterministic linear relationship among the proportions and the constant control comparison, one can effectively construct synthetic controls even for outcomes with limited pre-exposure data, such as a vote share for a recently formed political party. This capability is akin to using strong predictors of post-intervention values to mitigate the challenges posed by the lack of pre-exposure periods (Abadie 2021). By leveraging the interconnected nature of compositional data, our approach strengthens the basis for robust estimates, reducing the risk of over-fitting and enhancing the reliability of the synthetic control for when outcomes are unbalanced across time.

The implications of our research extend beyond synthetic control methods. The constant control comparison principle we outlined can be vital for other causal inference techniques that model or impute counterfactual outcomes, such as matching (Imai, Kim, and Wang 2023; Kellogg et al. 2021), trajectory balancing (Hazlett and Xu 2018) or imputation-based methods (Athey et al. 2021; Borusyak, Jaravel, and Spiess 2024; Liu, Wang, and Xu 2024). For instance, in matching techniques where researchers study the effect of covariates on voting decisions, applying separate matching weights for binary representations of these decisions can lead to estimates that violate the sum constraint, rendering the results difficult to interpret. We highlight the importance of adapting causal inference approaches for producing valid and meaningful results with compositional data.

References

Abadie, A. 2021. "Using Synthetic Controls: Feasibility, Data Requirements, and Methodological Aspects." *Journal of Economic Literature* 59 (2): 391–425. doi:10.1257/jel.20191450.

Abadie, A., A. Diamond, and J. Hainmueller. 2010. "Synthetic Control Methods for Comparative Case Studies: Estimating the Effect of California's Tobacco Control Program." *Journal of the American Statistical Association* 105 (490): 493–505. doi:10.1198/jasa.2009.ap08746.

———. 2015. "Comparative Politics and the Synthetic Control Method." *American Journal of Political Science* 59 (2): 495–510. doi:10.1111/ajps.12116.

Abadie, A., and J. Gardeazabal. 2003. "The Economic Costs of Conflict: A Case Study of the Basque Country." *American Economic Review* 93 (1): 113–132. doi:10.1257/000282803321455188.

Abadie, A., and J. L'Hour. 2021. "A Penalized Synthetic Control Estimator for Disaggregated Data." *Journal of the American Statistical Association,* 1–18. doi:10.1080/01621459.2021.1971535.

Aitchison, J. 1982. "The Statistical Analysis of Compositional Data." *Journal of the Royal Statistical Society. Series B (Methodological)* 44 (2): 139–177. doi:10.1111/j.2517-6161.1982.tb01195.x. JSTOR: 2345821.

Alrababa'h, A., W. Marble, S. Mousa, and A. A. Siegel. 2021. "Can Exposure to Celebrities Reduce Prejudice? The Effect of Mohamed Salah on Islamophobic Behaviors and Attitudes." *American Political Science Review* 115 (4): 1111–1128. doi:10.1017/S0003055421000423.

Arkhangelsky, D., S. Athey, D. A. Hirshberg, G. W. Imbens, and S. Wager. 2021. "Synthetic Difference-in-Differences." *American Economic Review* 111 (12): 4088–4118. doi:10.1257/aer.20190159.

Arkhangelsky, D., and G. Imbens. Forthcoming. "Causal Models for Longitudinal and Panel Data: A Survey." *The Econometrics Journal,* doi:10.48550/arXiv.2311.15458. arXiv: 2311.15458 [econ].

Arnold, K. F., L. Berrie, P. W. G. Tennant, and M. S. Gilthorpe. 2020. "A Causal Inference Perspective on the Analysis of Compositional Data." *International Journal of Epidemiology* 49 (4): 1307–1313. doi:10.1093/ije/dyaa021.

Athey, S., and G. W. Imbens. 2017. "The Econometrics of Randomized Experiments." In *Handbook of Economic Field Experiments,* edited by A. V. Banerjee and E. Duflo, 1:73–140. Handbook of Field Experiments. North-Holland. doi:10.1016/bs.hefe.2016.10.003.

Athey, S., M. Bayati, N. Doudchenko, G. Imbens, and K. Khosravi. 2021. "Matrix Completion Methods for Causal Panel Data Models." *Journal of the American Statistical Association* 116 (536): 1716–1730. doi:10.1080/01621459.2021.1891924.

Becher, M., I. M. González, and D. Stegmueller. 2023. "Proportional Representation and Right-Wing Populism: Evidence from Electoral System Change in Europe." *British Journal of Political Science* 53 (1): 261–268. doi:10.1017/S0007123421000703.

Ben-Michael, E., A. Feller, D. A. Hirshberg, and J. R. Zubizarreta. 2021. *The Balancing Act in Causal Inference.* doi:10.48550/arXiv.2110.14831. arXiv: 2110.14831 [stat].

Ben-Michael, E., A. Feller, and J. Rothstein. 2021. "The Augmented Synthetic Control Method." *Journal of the American Statistical Association* 116 (536): 1789–1803. doi:10.1080/01621459.2021.1929245.

Blackwell, M., and A. N. Glynn. 2018. "How to Make Causal Inferences with Time-Series Cross-Sectional Data under Selection on Observables." *American Political Science Review* 112 (4): 1067–1082. doi:10.1017/S0003055418000357.

Bolet, D., F. Green, and M. González-Eguino. 2023. "How to Get Coal Country to Vote for Climate Policy: The Effect of a "Just Transition Agreement" on Spanish Election Results." *American Political Science Review,* 1–16. doi:10.1017/S0003055423001235.

Borusyak, K., X. Jaravel, and J. Spiess. 2024. "Revisiting Event-Study Designs: Robust and Efficient Estimation." *The Review of Economic Studies,* rdae007. doi:10.1093/restud/rdae007.

Chiu, A., X. Lan, Z. Liu, and Y. Xu. 2024. *What To Do (and Not to Do) with Causal Panel Analysis under Parallel Trends: Lessons from A Large Reanalysis Study *.* SSRN Scholarly Paper. Rochester, NY. doi:10.2139/ssrn.4490035.

Cohen, D., and C. Hanretty. 2024. "Simulating Party Shares." *Political Analysis* 32 (1): 140–147. doi:10.1017/pan.2023.13.

Cremaschi, S., N. Bariletto, and C. E. De Vries. 2024. "Without Roots: The Political Consequences of Plant Disease Epidemics."

Cremaschi, S., P. Rettl, M. Cappelluti, and C. E. De Vries. 2023. *Geographies of Discontent: Public Service Deprivation and the Rise of the Far Right in Italy.* HBS Working Paper. doi:10.31219/osf.io/5s2cu.

Currie, J., H. Kleven, and E. Zwiers. 2020. "Technology and Big Data Are Changing Economics: Mining Text to Track Methods." *AEA Papers and Proceedings* 110:42–48. doi:10.1257/pandp.20201058.

de Kadt, D., A. Johnson-Kanu, and M. L. Sands. 2024. "State Violence, Party Formation, and Electoral Accountability: The Political Legacy of the Marikana Massacre." *American Political Science Review* 118 (2): 563–583. doi:10.1017/S0003055423000448.

de Kadt, D., and S. B. Wittels. 2019. "Democratization and Economic Output in Sub-Saharan Africa." *Political Science Research and Methods* 7 (1): 63–84. doi:10.1017/psrm.2016.15.

Eibl, F., and S. Hertog. 2023. "From Rents to Welfare: Why Are Some Oil-Rich States Generous to Their People?" *American Political Science Review,* 1–20. doi:10.1017/S0003055423000977.

Funke, M., M. Schularick, and C. Trebesch. 2023. "Populist Leaders and the Economy." *American Economic Review* 113 (12): 3249–3288. doi:10.1257/aer.20202045.

Gilens, M., S. Patterson, and P. Haines. 2021. "Campaign Finance Regulations and Public Policy." *American Political Science Review* 115 (3): 1074–1081. doi:10.1017/S0003055421000149.

Hassell, H. J. G., and J. B. Holbein. 2024. "Navigating Potential Pitfalls in Difference-in-Differences Designs: Reconciling Conflicting Findings on Mass Shootings' Effect on Electoral Outcomes." *American Political Science Review,* 1–21. doi:10.1017/S0003055424000108.

Hazlett, C., and Y. Xu. 2018. *Trajectory Balancing: A General Reweighting Approach to Causal Inference With Time-Series Cross-Sectional Data.* SSRN Scholarly Paper. Rochester, NY. doi:10.2139/ssrn.3214231.

Hilbig, H., and S. Riaz. 2024. "Natural Disasters and Green Party Support." *The Journal of Politics* 86 (1): 241–256. doi:10.1086/726917.

Imai, K., and I. S. Kim. 2019. "When Should We Use Unit Fixed Effects Regression Models for Causal Inference with Longitudinal Data?" *American Journal of Political Science* 63 (2): 467–490. doi:10.1111/ajps.12417.

Imai, K., I. S. Kim, and E. H. Wang. 2023. "Matching Methods for Causal Inference with Time-Series Cross-Sectional Data." *American Journal of Political Science* 67 (3): 587–605. doi:10.1111/ajps.12685.

Jackson, J. E. 2002. "A Seemingly Unrelated Regression Model for Analyzing Multiparty Elections." *Political Analysis* 10 (1): 49–65. doi:10.1093/pan/10.1.49. JSTOR: 25791664.

Katz, J. N., and G. King. 1999. "A Statistical Model for Multiparty Electoral Data." *The American Political Science Review* 93 (1): 15–32. doi:10.2307/2585758.

Kellogg, M., M. Mogstad, G. A. Pouliot, and A. Torgovitsky. 2021. "Combining Matching and Synthetic Control to Tradeoff Biases From Extrapolation and Interpolation." *Journal of the American Statistical Association* 116 (536): 1804–1816. doi:10.1080/01621459.2021.1979562.

Lipsmeyer, C. S., A. Q. Philips, A. Rutherford, and G. D. Whitten. 2019. "Comparing Dynamic Pies: A Strategy for Modeling Compositional Variables in Time and Space." *Political Science Research and Methods* 7 (3): 523–540. doi:10.1017/psrm.2017.39.

Liu, L., Y. Wang, and Y. Xu. 2024. "A Practical Guide to Counterfactual Estimators for Causal Inference with Time-Series Cross-Sectional Data." *American Journal of Political Science* 68 (1): 160–176. doi:10.1111/ajps.12723.

Meserve, S. A., and D. Pemstein. 2018. "Google Politics: The Political Determinants of Internet Censorship in Democracies." *Political Science Research and Methods* 6 (2): 245–263. doi:10.1017/psrm.2017.1.

Mullahy, J. 2018. *Treatment Effects with Multiple Outcomes.* Working Paper. doi:10.3386/w25307. National Bureau of Economic Research: 25307.

Negi, A., and J. M. Wooldridge. 2024. "Doubly Robust Estimation of Multivariate Fractional Outcome Means with Multivalued Treatments." *Econometric Reviews* 43 (2-4): 175–196. doi:10.1080/07474938.2024.2310987.

Pérez, V., C. Aybar, and J. M. Pavía. 2021. "Spanish Electoral Archive. SEA Database." *Scientific Data* 8 (1): 193. doi:10.1038/s41597-021-00975-y.

Philips, A. Q., A. Rutherford, and G. D. Whitten. 2015. "The Dynamic Battle for Pieces of Pie—Modeling Party Support in Multi-Party Nations." *Electoral Studies* 39:264–274. doi:10.1016/j.electstud.2015.03.019.

———. 2016. "Dynamic Pie: A Strategy for Modeling Trade-Offs in Compositional Variables over Time." *American Journal of Political Science* 60 (1): 268–283. doi:10.1111/ajps.12204.

Pinotti, P. 2015. "The Economic Costs of Organised Crime: Evidence from Southern Italy." *The Economic Journal* 125 (586): F203–F232. doi:10.1111/ecoj.12235.

Robbins, M. W., J. Saunders, and B. Kilmer. 2017. "A Framework for Synthetic Control Methods With High-Dimensional, Micro-Level Data: Evaluating a Neighborhood-Specific Crime Intervention." *Journal of the American Statistical Association* 112 (517): 109–126. doi:10.1080/01621459.2016.1213634.

Roth, J. 2022. "Pretest with Caution: Event-Study Estimates after Testing for Parallel Trends." *American Economic Review: Insights* 4 (3): 305–322. doi:10.1257/aeri.20210236.

Roth, J., P. H. C. Sant'Anna, A. Bilinski, and J. Poe. 2023. "What's Trending in Difference-in-Differences? A Synthesis of the Recent Econometrics Literature." *Journal of Econometrics* 235 (2): 2218–2244. doi:10.1016/j.jeconom.2023.03.008.

Samartsidis, P., S. R. Seaman, S. Montagna, A. Charlett, M. Hickman, and D. De Angelis. 2021. "A Bayesian Multivariate Factor Analysis Model for Evaluating an Intervention by Using Observational Time Series Data on Multiple Outcomes." *Journal of the Royal Statistical Society. Series A, (Statistics in Society)* 28 (1): 155–166. doi:10.1111/rssa.12569.

Selb, P., and S. Munzert. 2018. "Examining a Most Likely Case for Strong Campaign Effects: Hitler's Speeches and the Rise of the Nazi Party, 1927–1933." *American Political Science Review* 112 (4): 1050–1066. doi:10.1017/S0003055418000424.

Sun, L., E. Ben-Michael, and A. Feller. 2023. *Using Multiple Outcomes to Improve the Synthetic Control Method.* doi:10.48550/arXiv.2311.16260. arXiv: 2311.16260 [econ, stat].

Thompson, D. M. 2023. "Do Citizens Vote against Incumbents Who Permit Local Immigration? Evidence from the Mariel Boatlift." *Political Science Research and Methods* 11 (4): 728–745. doi:10.1017/psrm.2021.76.

Timoneda, J. C., A. Escribà-Folch, and J. Chin. 2023. "The Rush to Personalize: Power Concentration after Failed Coups in Dictatorships." *British Journal of Political Science* 53 (3): 878–901. doi:10.1017/S0007123422000655.

Tomz, M., J. A. Tucker, and J. Wittenberg. 2002. "An Easy and Accurate Regression Model for Multiparty Electoral Data." *Political Analysis* 10 (1): 66–83. doi:10.1093/pan/10.1.66. JSTOR: 25791665.

Trejo, G., and C. Nieto-Matiz. 2023. "Containing Large-Scale Criminal Violence Through Internationalized Prosecution: How the Collaboration Between the CICIG and Guatemala's Law Enforcement Contributed to a Sustained Reduction in the Murder Rate." *Comparative Political Studies* 56 (9): 1328–1364. doi:10.1177/00104140221139386.

Xu, Y. 2017. "Generalized Synthetic Control Method: Causal Inference with Interactive Fixed Effects Models." *Political Analysis* 25 (1): 57–76. doi:10.1017/pan.2016.2.

———. 2024. "Causal Inference with Time-Series Cross-Sectional Data: A Reflection." In *Oxford Handbook of Engaged Methodological Pluralism in Political Science (Vol 1),* edited by J. M. Box-Steffensmeier, D. P. Christenson, and V. Sinclair-Chapman, 0. Oxford University Press. doi:10.1093/oxfordhb/9780192868282.013.30.

Supplementary Material

Synthetic Control Methods for Proportions

Table of Contents

A Formalization of Synthetic Control Methods

We consider a balanced panel with N units and T time periods. Where we denote units with $i \in \{1, \dots, N\}$ and periods with $t \in \{1, \dots, T\}$. The outcome of interest are K proportions, where each proportion value is denoted with $k \in \{1, \dots, K\}$. The outcome for unit i in period t is a multivariate vector of $Y_{it} = [y_{it1}, \dots, y_{itk}]$. With a proportion, it holds that $y_{itk} \in [0, 1] \ \forall \ k$ and $\sum_{k=1}^{K} y_{itk} = 1$, forming a K-dimensional simplex.

We further have a binary treatment $D_{it} \in \{0, 1\}$. For (now) we assume that there is a share of control units that are never exposed to the treatment N_{co}, while there are units N_{tr} that are treated after time T_{pre}. For the analysis, we arrange the i units such that first all control units appear and then all treatment units $i \in \{1, \dots, N_{co}, N_{co} + 1, \dots, N\}$.

A DID method would estimate separate average treatment effects of exposure for each proportion, τ_k^{did} essentially solving a two-way fixed effects regression problem:

$$(\hat{\tau}_k^{did}, \hat{\mu}_k, \hat{\alpha}_{ik}, \hat{\beta}_{tk}) = \underset{\tau,\mu,\alpha,\beta}{\arg\min} \left\{ \sum_{i=1}^{N} \sum_{i=t}^{T} \left(y_{itk} - \mu_k - \alpha_{ik} - \beta_{tk} - D_{it}\tau \right)^2 \right\} \tag{1}$$

for this specification, the (unweighted) average of the control units essentially serves as the counterfactual outcome for the treatment units after treatment exposure. The key identification parallel trend assumption makes this (unweighted) average a valid counterfactual outcome.

Synthetic methods instead calculate weights $\hat{\omega}_{ik}$ that align pre-exposure trends in the outcomes of the control with the treatment units, such that $\sum_{i=1}^{N_{co}} \hat{\omega}_{ik} y_{itk} \approx \frac{1}{N_{tr}} \sum_{i=N_{co}+1}^{N} y_{itk} \ \forall \ t \in \{1, \dots, T_{pre}\}$. The approach further specifies time-weights $\hat{\lambda}_{tk}$ that balance pre- and post-exposure periods for the different shares. The estimation then proceeds with finding a solution to the minimization problem.

$$(\hat{\tau}_k^{sdid}, \hat{\mu}_k, \hat{\alpha}_{ik}, \hat{\beta}_{tk}) = \underset{\tau,\mu,\alpha,\beta}{\arg\min} \left\{ \sum_{i=1}^{N} \sum_{i=t}^{T} \left(y_{itk} - \mu_k - \alpha_{ik} - \beta_{tk} - D_{it}\tau \right)^2 \hat{\omega}_{ik} \hat{\lambda}_{tk} \right\} \tag{2}$$

As Arkhangelsky et al. point out, the synthetic difference-in-differences estimator can also be applied to classic synthetic method approaches (Arkhangelsky et al. 2021). Here the minimization problem excludes the time-weights and unit fixed effects.

Of central importance for SCMs is the calculation of the weights. In the application to proportions, these would be calculated for each proportion separably. Arkhangelsky et al. approach for this is an optimization problem with regularization:

$$
\begin{aligned}
(\hat{\omega}_{0k}, \hat{\omega}_k) &= \underset{\omega_{0k}\in\mathbb{R}, \omega_k\in\Omega}{\arg\min}\ \ell_{uni}(\omega_{0k}, \omega_k) \text{ where} \qquad (3)\\
\ell_{uni}(\omega_{0k}, \omega_k) &= \sum_{t=1}^{T_{pre}} \left(\omega_{0k} + \sum_{i=1}^{N_{co}} \omega_{ik} y_{itk} - \frac{1}{N_{tr}} \sum_{i=N_{co}1}^{N} y_{itk} \right)^2 + \xi_k^2 T_{pre} ||\omega_k||_2^2, \\
\Omega &= \left\{ \omega_k \in \mathbb{R}_+^N : \sum_{i=1}^{N_{co}} \omega_{ik} = 1, \omega_{ik} = \frac{1}{N_{tr}}\ \forall\ i \in \{N_{co}+1, \ldots, N\} \right\}
\end{aligned}
$$

where the regularization parameter is pre-specified as ξ

$$
\xi_k = (N_{tr} T_{post})^{1/4} \hat{\sigma} \text{ with } \hat{\sigma}^2 = \frac{1}{N_{co}(T_{pre}-1)} \sum_{i=1}^{N_{co}} \sum_{t=1}^{T_{pre}-1} (\Delta_{itk} - \overline{\Delta}_k)^2, \qquad (4)
$$

$$
\text{where } \Delta_{itk} = y_{it(t+1)k} - y_{ittk} \text{ and } \overline{\Delta}_k = \frac{1}{N_{co}(T_{pre}-1)} \sum_{i=1}^{N_{co}} \sum_{t=1}^{T_{pre}-1} \Delta_{itk} \qquad (5)
$$

The time weights are analogously implemented by solving:

$$
\begin{aligned}
(\hat{\lambda}_{0k}, \hat{\lambda}_k) &= \underset{\lambda_{0k}\in\mathbb{R}, \lambda_k\in\Lambda}{\arg\min}\ \ell_{time}(\lambda_{0k}, \lambda_k) \text{ where} \qquad (6)\\
\ell_{uni}(\lambda_{0k}, \lambda_{ik}) &= \sum_{i=1}^{N_{co}} \left(\lambda_{0k} + \sum_{t=1}^{T_{pre}} \lambda_{tk} y_{itk} - \frac{1}{T_{post}} \sum_{t=T_{pre}1}^{T} y_{itk} \right)^2, \\
\Lambda &= \left\{ \lambda_k \in \mathbb{R}_+^T : \sum_{i=1}^{T_{pre}} \lambda_t = 1, \lambda_t = \frac{1}{T_{post}}\ \forall\ i \in \{T_{pre}+1, \ldots, T\} \right\}
\end{aligned}
$$

The algorithm for synthetic Difference in Differences proceeds as follows:

- For each proportion $k \in \{1, \ldots, K\}$:
  1. Compute regularization parameter ξ_k
  2. Compute unit weights $\hat{\omega}_k$
  3. Compute time weights $\hat{\lambda}_k$
  4. Compute the SDID estimator via the weighted DID regression

$$
(\hat{\tau}_k^{sdid}, \hat{\mu}_k, \hat{\alpha}_{ik}, \hat{\beta}_{tk}) = \underset{\tau,\mu,\alpha,\beta}{\arg\min} \left\{ \sum_{i=1}^{N} \sum_{i=t}^{T} (y_{itk} - \mu_k - \alpha_{ik} - \beta_{tk} - D_{it}\tau)^2\, \hat{\omega}_{ik} \hat{\lambda}_{tk} \right\} \qquad (7)
$$