On the Mittag-Leffler Function connection to Spiral like functions

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Abstract
In this study, we evoke the connection between a Mittag-Leffler function with a subclass of spiral like functions. Hence, we derive several results and conclusions.

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INTRODUCTION

We consider here the Mittag- Leffler function

$$ML^{\theta}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{(\theta)_n}{\Gamma(\alpha n+\beta)} \cdot \frac{z^n}{n!}, \quad z, \beta, \theta \in \mathbb{C}\,; \ \mathrm{Re}\,\alpha > 0.$$

Let $\mathcal{A}$ refers to the group of operations whose members fall under a given type

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in \mathbb{D},$$

$$\mathbb{E}\mathrm{L}^{\theta}_{\alpha,\beta}(z) := z\Gamma(\beta) E^{\theta}_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)\,(\theta)_n}{n!\,\Gamma(\alpha(n-1)+\beta)} z^n,$$

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n, \ z \in \mathbb{D}.$$

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \ z \in \mathbb{D}$$

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma, \ z \in \mathbb{D}$$

A function $f \in \mathcal{A}$ is said to be spiral-like if

$$Re\left(e^{-i\xi} \frac{zf'(z)}{f(z)}\right) > 0, \ z \in \mathbb{D},$$

Definition 1.

$$\mathcal{S}(\xi, \gamma, \rho) := \left\{ f \in \mathcal{A}: \mathrm{Re}\left(e^{i\xi} \frac{zf'(z)}{(1-\rho)f(z) + \rho z f'(z)}\right) > \gamma \cos\xi, z \in \mathbb{D} \right\}$$

Definition 2.

$$\mathcal{K}(\xi,\gamma,\rho): \;=\left\{f\in\mathcal{A}:\operatorname{Re}\left(e^{i\xi}\,\frac{zf''(z)+f'(z)}{f'(z)+\rho zf''(z)}\right)\;>\gamma\;\cos\xi, z\in\mathbb{D}\right\}$$

Definition 3.

$$\left|\frac{(1-\vartheta)\dfrac{f(z)}{z}+\vartheta f'(z)-1}{2\tau(1-\delta)\;\;+(1-\vartheta)\dfrac{f(z)}{z}+\vartheta f^{\mathrm{i}\prime}(z)-1}\right|<1,\; z\in\mathbb{D}.$$

$$\Lambda_\beta^\alpha f(z):=f(z)*\mathbb{E}_{\alpha,\beta}^\theta(z\,)=z\;+\sum_{n=2}^{\infty}\frac{(\theta)_n\Gamma(\beta)}{n!\Gamma(\alpha(n-1)+\beta)}\;a_n z^n,\; z\in\mathbb{D}.$$

Remark

$$ML\mathbb{E}_{\alpha,\beta}^\theta(1)-1=\;\sum_{n=2}^{\infty}\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)\;\;+\beta)}$$

$$ML\left(\mathbb{E}_{\alpha,\beta}^\theta\right)'(1)-1=\;\sum_{n=2}^{\infty}\frac{(\theta)_n\Gamma(\beta)}{(n-1)!\,\Gamma(\alpha(n-\;\;1)+\beta)},$$

$$ML\left(\mathbb{E}_{\alpha,\beta}^\theta\right)\;''(1)\;=\;\sum_{n=2}^{\infty}\frac{(\theta)_n\Gamma(\beta)}{(n-2)\;\,!\,\Gamma(\alpha(n-1)+\beta)}.$$

Theorem 1

$$[(1-\rho)sec\,\xi+\rho(1-\gamma)]\left(\mathbb{E}_{\alpha,\beta}^\theta\right)'(1)+(1-\rho)(1-\gamma-sec\,\xi)\mathbb{E}_{\alpha,\beta}^\theta(1\leq 2(1-\gamma),$$

then $\mathbb{E}ML_{\alpha,\beta}^\theta\in\mathcal{S}(\xi,\gamma,\rho)$.

Theorem 2. *If*

$$[(1-\rho)\sec\xi\;\;+\rho(1-\gamma)]\left(\mathbb{E}_{\alpha,\beta}^\theta\right)''(1)+(1-\gamma)\left(\mathbb{E}_{\alpha,\beta}^\theta\right)'(1)\leq 2(1-\gamma),$$

then $ML\mathbb{E}_{\alpha,\beta}^\theta\;\;\in\mathcal{K}(\xi,\gamma,\rho)$.

Theorem 3.

For $f(z)\;\in\mathcal{A}.$ *If*

$$\frac{2|\tau|(1-\delta)}{\vartheta}[(1-\rho)\sec\xi+\rho(1-\gamma)]\left[\mathbb{E}_{\alpha,\beta}^{\theta}(1)-1\right]$$
$$+(1-\rho)(\ 1-\gamma-\sec\xi)\int_0^1\ \left(\frac{\mathbb{E}_{\alpha,\beta}^{\theta}(t)}{t}-1\right)dt\leq 1-\gamma$$

then

$$\Lambda_{\beta}^{\alpha}(\mathcal{R}^{\tau}(\vartheta,\delta))\subset\mathcal{S}(\xi,\gamma,\rho).$$

Theorem 4. *For* $f(z)\ \in\mathcal{A}$. *If*

$$\frac{2|\tau|(1-\delta)}{\vartheta}\Big\{[(1-\rho)\ \ sec\,\xi+\rho(1-\gamma)]\left(\mathbb{E}_{\alpha,\beta}^{\theta}\right)'(1)$$
$$+(1-\rho)(1-\gamma-sec\ \ \xi)\mathbb{E}_{\alpha,\beta}^{\theta}(1)-(1-\gamma)\Big\}\leq 1-\gamma$$

Theorem 5. *If*

$$[(1-\rho)\sec\xi+\rho(1-\gamma)]\left(\mathbb{E}_{\alpha,\beta}^{\theta}\right)'(1)+(1-\rho)(1-\gamma-\sec\xi)\mathbb{E}_{\alpha,\beta}^{\theta}(1)\leq 2(1-\gamma),$$

then $\Psi_{\beta}^{\alpha}\in\mathcal{K}(\xi,\gamma,\rho)$.

Theorem 7. *If*

$$[(1-\rho)\sec\xi+\rho(1-\gamma)]\left(\mathbb{E}_{\alpha,\beta}^{\theta}(1)-1\right)$$
$$+(1-\rho)(1-\gamma-\sec\xi)\int_0^1\left(\frac{\mathbb{E}_{\alpha,\beta}^{\theta}(t)}{t}-1\right)d\ \ t\leq 1-\gamma,$$

then $\Psi_{\beta}^{\alpha}\in\mathcal{S}(\xi,\gamma,\rho)$.

Theorem 7. *If*

$$[(1-\rho)sec\,\xi+\rho(1-\gamma)]\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)}+(1-\rho)(1-\gamma-sec\,\xi)\sum_{n=1}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}$$
$$<2(1-\gamma)$$

then $j\ \ \in\mathcal{S}(\xi,\gamma,\rho)$.

Theorem 8.. *If*

$$[(1-\rho)sec\,\xi+\rho(1-\gamma)]\sum_{n=2}^{\infty}\frac{n(n-1)}{\Gamma\left(\frac{n+1}{2}\right)}+(1-\gamma)\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)}\leq 2(1-\gamma),$$

then $j\in\mathcal{K}(\xi,\gamma,\rho)$.

Theorem 9.. *If*

$$\frac{2|\tau|(1-\delta)}{\vartheta}[(1-\rho\ \ )sec\,\xi+\rho(1-\gamma)]\sum_{n=2}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}$$
$$+(1-\rho)(1-\gamma-\ \ sec\,\xi)\sum_{n=2}^{\infty}\frac{1}{n\Gamma\left(\frac{n+1}{2}\right)}\leq 1-\gamma,$$

then

$$h(\mathcal{R}^{\tau}(\vartheta,\delta))\subset\mathcal{S}(\xi,\gamma,\rho),$$

Theorem 10. *If*

$$\frac{2|\tau|(1-\delta)}{\vartheta}\left\{[(1-\rho)sec\,\xi+\rho(1-\gamma)]\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)}\right.$$
$$\left.+(1-\rho)(1-\gamma-sec\,\xi)\sum_{n=1}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}-(1-\gamma)\right\}\leq 1-\gamma$$

then

$$h(\mathcal{R}^{\tau}(\vartheta,\delta))\subset\mathcal{K}(\xi,\gamma,\rho)$$

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