Computational approaches to the diameter of Polytopes

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1 Introduction

Unless explicitly stated, all sets are assumed to be subset of $\mathbb{R}^d$.

1.1 Overview

This report results from a semester project dealing with the diameter of polytopes and the Hirsch conjecture. In particular, the notion of geodesic maps introduced by the disproval of the Hirsch conjecture is studied and used in a C++ code[2] I did to search for special kinds of geodesic maps, giving non-Hirsch prismatoids.

1.2 Polytopes

A *polytope* can be equivalently defined as the convex hull of a finite number of vertices, or a bounded intersection of finitely many half-spaces. A (possibly unbounded) intersection of finitely many half-spaces is a *polyhedron*.
The *dimension* of a polytope is the smallest dimension of an affine space containing it. A d-polytope is a polytope of dimension d.
The *polar* (or *dual*) of a polytope P is defined by $P^\Delta = \{c \in (\mathbb{R}^d)^*, < c, x > \leq 1,\ \forall x \in \mathrm{P}\}$, where $(\mathbb{R}^d)^*$ is the set of all linear maps from $\mathbb{R}^d$ to $\mathbb{R}$.

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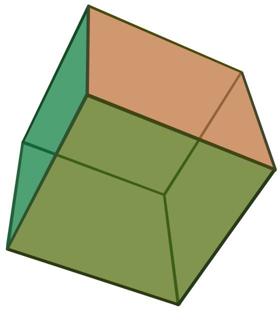
[2] which can be found at http://perso.ens-lyon.fr/quentin.fortier/Hirsch/source.zip



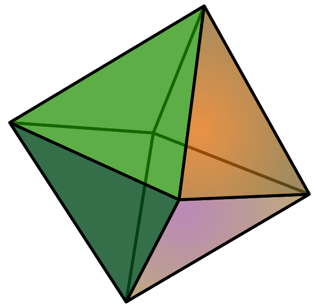


Figure 1: Left: A cube. Right: An octahedron, the polar of the cube.

A *supporting hyperplane* of a d-polytope P is a hyperplane such that P is entirely contained in one of the two half-spaces defined by the hyperplane and at least one point of P is on the hyperplane.
Then we define a *face* of P to be the intersection between P and a supporting hyperplane of P. A face is always a polytope, and if its dimension is k, we say that it is a k-face of P.
Then a 0-face, called *vertex*, is an extremal point of P, a 1-face is an *edge*, a $d-2$ face is a *ridge* and a $(d-1)$ face is a *facet*.
The *face lattice* $\mathrm{L}(P)$ is the set of all faces of P, partially ordered by inclusion.
The set of all vertices and edges of P defines a *polytopal graph* $\mathrm{G}(P)$ and we define the graph distance between two vertices of this graph to be the minimal number of edges required to go from a vertex to the other. The *diameter* of a graph is the maximal distance between two vertices.
We denote by $G(\mathcal{P})$ the set of all graphs of polytopes.
One important theorem that we will often use is the following (the theorem and a proof can be found in [4], for example):

Theorem 1 (Balinski, [7]). *The graph of a d-polytope is d-vertex-connected*

It means that if we remove $d-1$ vertices from a graph of a d-polytope, the resulting graph is still connected.

A d-polytope with $d+1$ vertices is a *d-simplex*. For example a tetrahedron is a 3-simplex.
A d-polytope P is *simplicial* if every proper facet of P is a simplex.

Lemma 1. If P is a simplicial d-polytope, every proper k-face of P has exactly $k+1$ vertex (so it is a k-simplex)

Proof:
Assume a k-face F has at least $k+2$ vertices $v_1, \ldots, v_{k+2}$ and let F' be a facet containing F.
Since $\dim(F')$ - $\dim(F) = d-k-1$, F' must contain at least $d-k-1$ vertices that are different from $v_1, \ldots, v_{k+2}$ and then F' has at least $(d-k-1)+(k+2)$ $= d+1$, which is a contradiction since F' is a $(d-1)$-simplex.

□

A d-polytope P is *simple* if every vertex of P is contained in the minimal number of d facets.

Lemma 2. If P is a simple d-polytope, every k-face of P is contained in exactly $d-k$ facets of P

Proof:
This is the polar version of Lemma 1.

□

A *d-spindle* is a d-polytope P with two vertices u, v (the *end points*) such that every facet of P contains exactly one of them. The length of a spindle is the graph distance between u and v.

A *d-prismatoid* Q is a polytope having two parallel facets Q^+ and Q^- that contain all vertices. We call Q^+ and Q^- the base facets of Q. The width of a prismatoid is the minimum length of a path of facets (crossing ridges) needed to go from Q^+ to Q^-.

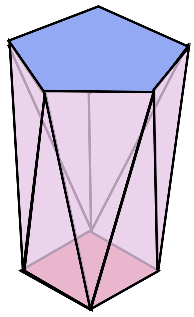


Figure 2: A prismatoid Q with two basis Q^+ and Q^-

Polars of prismatoids are spindle and it is easy to see that the width of a prismatoid equals the length of its dual.

The *Minkowski sum* of two polytopes P_1, P_2 is the polytope $P_1 + P_2 = \{x + y,\ x \in \mathrm{P}_1,\ y \in \mathrm{P}_2\}$. If P_1, P_2 are d dimensional then $P_1 + P_2$ is d dimensional. Every face of $P_1 + P_2$ decomposes uniquely as a sum of one face of P_1 and one of P_2.

1.3 About linear programming

Linear programs can be expressed in the canonical form:

Problem 1 (LP)**.** *Maximize* $< c, x >$*, subject to* $A\ x \leqslant b$

with c, $x \in \mathbb{R}^d$, $b \in \mathbb{R}^n$ and A a real n × d matrix with full rank.
Then the set $\{\ x,\ Ax \leqslant b\ \}$ of feasible solutions define a d-polyhedron P with n facets (one for each inequality).
If P is in fact a polytope, then the linear objective function $x \longmapsto < c, x >$ has

a maximum on P which is obtained for a vertex of P, and every local maximum on P of this function is also a global maximum on P. Hence, the famous *simplex algorithm* takes a vertex of P and then move from vertex to vertex along edges according to a *pivot rule*, until it finds a local maximum.
The choice of a pivot rule consisting in move along the edge which increase the most the objective function may lead to an exponential number of steps, but it doesn't exclude the possibility of another polynomial (in d and n) pivot rule.
Moreover, a randomized pivot rule can be subexponential, and in practice the simplex algorithm is very efficient and more widely used than, for example, the ellipsoid algorithm which has a (weakly) polynomial complexity.
A polynomial pivot rule for the simplex algorithm would prove the LP is *strongly* polynomial (that is polynomial in the input and doesn't depend on the magnitude of the input's value), contrary to the ellipsoid method.
The study of the complexity of the complexity of the simplex algorithm may be best understood if we can know approximatively the diameter of polytopal graphs (for example, the number of step required by any pivot rule is at least the diameter of the graph). We denote by $\Delta_b(d,n)$ (resp. $\Delta(d,n)$) the maximum diameter of a d-polytope (resp. d-polyhedron) with n facets.

1.4 The Hirsch conjecture

Table 1: Diameter of some polytopes (d is the dimension)

	Number of vertices	**Number of facets**	**Diameter**
Polygone (2-dimensional)	n	n	$\lfloor \frac{n}{2} \rfloor$
Tetrahedron	$d+1$	$d+1$	1
Cube	2^d	$2d$	d
Crosspolytope	$2d$	2^d	d

So, it would be nice if one can prove a "good" upper bound on the diameter of the polytopes.
One may observe that the classical d-polytopes in the Table 1 all have a perimeter bounded by $n-d$, n being the number of facets of the polytope. This observation leads to the important (but false) *Hirsch conjecture*:

Problem 2. *(Hirsch conjecture, 1957 ([8]))* $\Delta_b(d,n) \leq n$ - d

Although stated more than fifty years ago, this conjecture was disproved very recently (June, 2010), by Francisco Santos.
Before describing a counter example of the Hirsch conjecture (in Section 2), it can be useful to review some major results related to this conjecture (all of them can be found in [2] or [4]).

The following statements are equivalent:

- The Hirsch conjecture
- The d-step conjecture: if P is a (simple) d-polytope with $2d$ facets then $\mathrm{diam}(\mathrm{G}(P)) \leq d$

- The nonrevisiting path conjecture: for every pair of vertices of a polytope, there is a path from one vertex to the other which visit at most once every facet

It particular, it would be sufficient to prove a bound on *simple* polytope to have one of these assertions. However, as we will see in Section 2, the counter example crucially uses non-simple polytopes.
There are several results showing that the Hirsch conjecture is plausible:

- The Hirsch conjecture is true for polytopes with at most six facets more than their dimension

- If H(n, d) is the maximal diameter of a d-polytope with n facets then H(n, d) $\leq n^{log_2(d)} + 1$ and H(n, d) $\leq n2^{d-3}$
  [3] proves this bound on a more general set of graphs (i.e *an abstraction*)

1.5 Geodesic maps

This section introduces tools that will be useful for our study of polytopes.
A *cone* cone(F) is a set such that every linear combination, with nonnegative coefficients, of elements of F is in cone(F).
A *fan* is a set $\mathcal{F}$ of cones such that every face of a cone of is a cone of $\mathcal{F}$ and if C_1, $C_2 \in \mathcal{F}$, $C_1 \cap C_2$ is a face of C_1 and of C_2.
A fan is *complete* if the union of its cones equals $\mathbb{R}^d$.
The *refinement* of two fans $\mathcal{F}$ and $\mathcal{G}$ is:

$$\mathcal{F} \wedge \mathcal{G} \ = \ \{C \cap C',\ C \in \mathcal{F},\ C' \in \mathcal{G}\}$$

For simplicity, we assume that every polytope has 0 in its interior and every fan has the cone $\{0\}$.
The *face fan* $\mathcal{F}(P)$ of a polytope P is defined by:

$$\mathcal{F}(P) \ = \ \{cone(F),\ \text{F is a proper face of } P\}$$

The *normal cone* $\mathcal{N}_F$ of a face of a d-polytope P is defined by:

$$\mathcal{N}_F \ = \ \{\ell \in (\mathbb{R}^d)^*, \max\ \ell\mid_P \ \text{ is reached in exactly every point of F}\}$$

Lemma 3. $\dim(\mathcal{N}_F) = d$ - dim(F)

Thanks to the isomorphism

$$\begin{array}{rcl} v & \longmapsto & (x \longmapsto < v, x >) \\ \mathbb{R}^d & \longrightarrow & (\mathbb{R}^d)^* \end{array}$$

we can view $\mathcal{N}_F$ as a subset of $\mathbb{R}^d$.
The *normal fan* $\mathcal{N}$(P) is the set of all cones $\mathcal{N}_F$ with F a proper face of P. If P is full dimensionnal $\mathcal{F}(P)$ and $\mathcal{N}$(P) are complete.

Lemma 4. For all polytope P, $\mathcal{N}(\text{P}) = \mathcal{F}(P^\Delta)$

Lemma 5. For all polytopes P and Q, $\mathcal{N}(\text{P}+Q) = \mathcal{N}(\text{P}) \wedge \mathcal{N}(Q)$

The *Gaussian map* Γ of a d-polytope P is defined by:

$$\begin{array}{rrcl} \Gamma : & F & \longmapsto & \mathbb{S}^{d-1} \cap \mathcal{N}_F \\ & L(P) & \longrightarrow & \mathcal{P}(\mathbb{S}^{d-1}) \end{array}$$

Γ is injective and $\Gamma\ |^{\Gamma(L(P))} : L(P) \longrightarrow \Gamma(L(P))$ is a bijection, still denoted Γ for simplicity.
We call a (polytopal) $(d-1)$-*geodesic map* the polyhedral complex $\Gamma(L(\mathrm{P}))$.

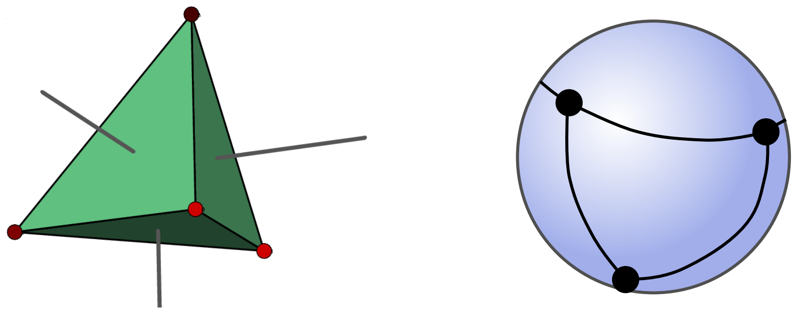


Figure 3: Left: A tetrahedron. Right: The geodesic map associated to this tetrahedron.

If $\mathrm{S} = \mathbb{S}^{d-1} \cap \mathcal{N}_F \in \Gamma(L(\mathrm{P}))$ the dimension of S is, by definition, $\dim(\mathcal{N}_F)$ - 1, which is equal to d - dim(F) - 1, and if $\dim(\mathrm{S}) = k$ we say that S is a k-face of $\Gamma(L(\mathrm{P}))$.
Notice that the "dimensionality" of $\Gamma(L(\mathrm{P}))$ is reduced by one, and that the graph of $\Gamma(L(\mathrm{P}))$ is equal to the graph of the dual of P, i.e:

Lemma 6. $\mathrm{G}(\Gamma(L(\mathrm{P}))) = \mathrm{G}(P^{\Delta})$

In particular, a $(d-1)$-geodesic map inherits many of the properties of polytopes: its graph is d-connected according to Theorem 1 applied on P^{Δ}, its faces are connex (in the sphere, meaning that every geodesic curve between two vertices of the same face is included in this face). Moreover, a d-polytopal geodesic map is complete (that is, the union of its faces equal $\mathbb{S}^d$).
The following lemma is an immediate consequence of Lemma 1:

Lemma 7. If P is simplicial d-polytope then, every vertex of $\Gamma(L(\mathrm{P}))$ is contained in d facets of $\Gamma(L(\mathrm{P}))$
In this case we say that $\Gamma(L(\mathrm{P}))$ is *simple*

Similarly to the fans, the *refinement* of two geodesic maps G^+ and G^- is:

$$G^+ \wedge G^- \ = \ \{F \cap F',\ F \in G^+,\ F' \in G^-\}$$

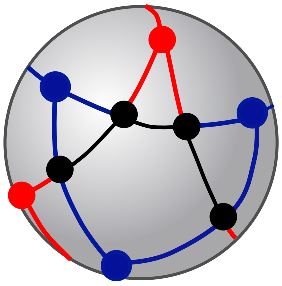


Figure 4: Transversal refinement of two geodesic maps of tetrahedra

In the context of geodesic maps, Lemma 5 translates into the following:

Lemma 8. $\Gamma(L(\mathrm{P}+Q)) = \Gamma(L(\mathrm{P})) \wedge \Gamma(L(Q))$

In words, the geodesic map of the Minkowski sum of two polytopes is equal to the refinement of the geodesic maps of these polytopes.
If G^+, G^- are two d-geodesic maps we say that $G^+ \wedge G^-$ is transversal if $\forall$ $\mathrm{C}^+ \in G^+$, $\mathrm{C}^- \in G^+$:

$$dim(C^+) + dim(C^-) \;=\; d - 1 + dim(C^+ \cap C^i)$$

In words: every pair of faces C^- and C^+ are disjoint or intersect in there interiors.
In particular, if $G^+ \wedge G^-$ is transversal then every face of $G^+ \wedge G^-$ can be written uniquely as the intersection of one face of G^+ and of G^-.

2 The counter example

All this part is described in more details in [1].
Since the width of a prismatoid equals the length of its dual (which is a spindle), to disprove the Hirsch conjecture it is sufficient to show that there is a d-prismatoid with witdh strictly greater than d.
The main goal of this section is to show the existence of such a prismatoid.

2.1 d-step property

Lemma 9. If a d-prismatoid Q with basis Q^+ and Q^- has width l then there is another d-prismatoid with the same number of vertices, with width at least l and such that all its facets except its two basis are simplices.

Proof:
A perturbation sufficiently small of the vertices of Q^+ and Q^- and in the hyperplanes defined by them refine other non-simplicial facets of Q, while decreasing the width.

□

In all this section we will assume that every facet of Q different from Q^+ and Q^- are simplices (otherwise, we can apply a perturbation of the vertices of Q to ensure it). The first important step toward the disproval of the Hirsch conjecture is that we can construct non-Hirsch prismatoids from another prismatoid with low dimension and with the *d-step property*:

Definition 1. A d-prismatoid Q with basis Q^+ and Q^- has the d-step property if its width (i.e the facet-distance between Q^+ and Q^-) is at least $d+1$.

Theorem 2 (Generalized d-step Theorem). *If Q is a prismatoid of dimension d with n vertices and width l, then there is another prismatoid Q' of dimension $n-d$, with $2n-2d$ vertices and width at least $(n-d)$+$(l-d)$. In particular, if $l > d$ then Q' violates the Hirsch conjecture.*

Proof (Sketch):
By induction on $s = n - 2d$, for the induction step (case $s = 0$ is obvious and we assume $s > 0$ so one basis, say Q^-, is not a simplex) we construct the one-point-suspension (Figure 5) $\widehat{Q^+}$ of Q^+. Notice that if $Q^+ \subset \mathbb{R}^p$, $\widehat{Q^+} \subset \mathbb{R}^{p+1}$.

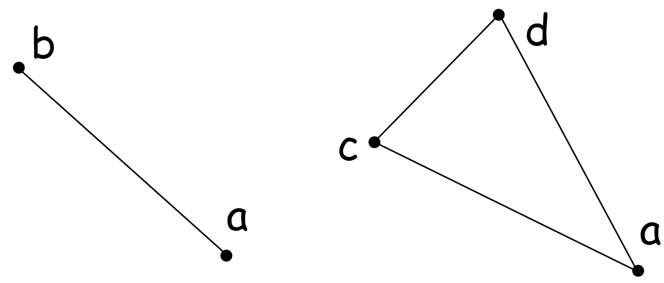


Figure 5: Left: Q^+. Right: $\widehat{Q^+}$. c and d are on the line perpendicular to (ab) and replace b, to increase the dimension.

Since Q^- is not a simplex we can translate his vertices to get a basis $\widehat{Q^-}$ parallel to $\widehat{Q^+}$. The prismatoid $\widehat{Q}$ with basis $\widehat{Q^+}$ and $\widehat{Q^-}$ has one more vertex than Q and a dimension and a width increased by one, so we can apply the induction hypothesis on $\widehat{Q}$ to find the required prismatoid for the induction step.

□

In particular, since the Hirsch conjecture is true for polytopes with at most six facets more than their dimension, this theorem implies that there is no d-prismatoid with n vertices without the d-step property, if $n - d \leq 6$.

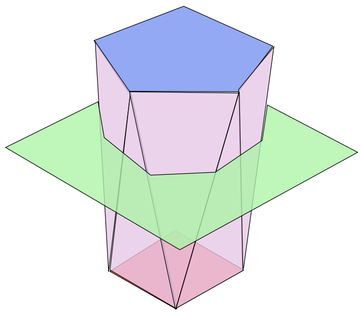
2.2 Translation into the language of geodesic maps

[1] gives explicit coordinates (that we will not reproduce here) of a 5-prismatoid without the d-step property and with 48 vertices (24 vertices for Q^+ and Q^-)[1].

[1]In fact, the "best" (i.e. with as few vertices as possible) 5-prismatoid without the d-step property found so far has 25 vertices (20/01/2011)

This implies, by Theorem 2, that there is a 43-spindle with 86 facets and diameter at least 44, violating the Hirsch conjecture. Furthermore, it gives two proofs of the fact that this prismatoid has not the d-step property: we will not describe the first one, giving the links between (orbits) between facets and taking advantage of the symmetries of the prismatoid, but we will review the main arguments of the second proof which use geodesic maps.
For now on, assume Q is a d-prismatoid with two parallel basis Q^+ and Q^-.
We will show how to translate the d-step property on a property of geodesic maps. One major remark is that knowing $Q^+ + Q^-$, we can deduce the whole combinatorics of Q:

Lemma 10. Let H be a hyperplane parallel to Q^+ and Q^- such that d(H, Q^+) = d(H, Q^-).
Then H = $\frac{1}{2}$ $(Q^+ + Q^-)$



Then, to every facet f of a d-prismatoid Q different from Q^+ and Q^- we can associate a unique facet of $Q^+ + Q^-$, corresponding to the intersection of f and H. Moreover if $\mathrm{f} \cap \mathrm{H} = \mathrm{f}^+ + \mathrm{f}^-$ with $\mathrm{f}^+ \in Q^+$ and $\mathrm{f}^- \in Q^-$ then $\dim(\mathrm{f}^+) = \dim(\mathrm{f} \cap Q^+)$ and $\dim(\mathrm{f}^-) = \dim(\mathrm{f} \cap Q^-)$ (I call these quantites the Q^+(resp. Q^-)-dimension of $\mathrm{f} \cap \mathrm{H}$). In particular, f is adjacent to (i. e share a ridge with) Q^+ if and only if $\dim(\mathrm{f}^+) = d-2$ and $\dim(\mathrm{f}^-) = 0$.
Moreover, it is equivalent to say that there is a facet-path $\mathrm{f}_1, \ldots, \mathrm{f}_p$ in Q from Q^+ to Q^- and that there is a path $\mathrm{f'}_1, \ldots, \mathrm{f'}_{p-2}$ of facets of $Q^+ + Q^-$ such that the Q^+ dimension of $\mathrm{f'}_1$ is $d-1$ and the Q^- dimension of $\mathrm{f'}_{p-2}$ is $d-1$. We need two less steps because we "forget" the first step from Q^+ to an adjacent facet and the last step to Q^-.
Now we turn to geodesic maps:

Definition 2. The refinement $G^+ \wedge G^-$ of two d-geodesic maps G^+ and G^- has the d-step property if it is possible to go from a vertex of G^+ to a vertex of G^- with at most d edges.

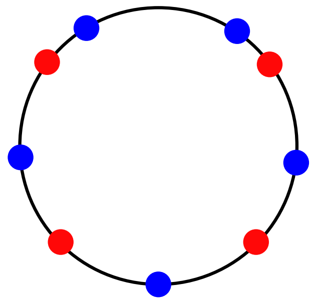


Figure 6: Refinement of the geodesic maps defined by the basis of the prismatoid of Figure 2.2

For example the refinement of Figure 4 and of Figure 6 has the d-step property.
Since $G^+ \wedge G^-$ is combinatorially the polar of $Q^+ + Q^-$:

Theorem 3. *Let Q be a $(d+2)$-prismatoid with basis Q^+ and Q^-.
Q has the d-step property if and only if the refinement $G^+ \wedge G^-$ of the two d-geodesic maps G^+ and G^- of Q^+ and Q^- has the d-step property.*

According to Lemma 9, we can assume that G^+ and G^- are simple. Moreover, we can assume that all refinements of geodesic maps are transversal (if not, we can rotate them slighty so that it becomes true).
The transversality implies that we can't have "jumps" in dimension on a path:

Lemma 11. If $G^+ \wedge G^-$ is transversal, $\mathrm{F}^+ \cap \mathrm{F}^- \in G^+ \wedge G^-$ is a vertex and $\mathrm{E}^+ \cap \mathrm{E}^-$ is an edge incident to $\mathrm{F}^+ \cap \mathrm{F}^-$ then $\dim(\mathrm{E}^+) = \dim(\mathrm{F}^+) + 1$ and $\dim(\mathrm{E}^-) = \dim(\mathrm{F}^-)$ (or the opposite case).

Lemma 12. If $G^+ \wedge G^-$ is transversal, and if there is a path $\mathrm{F}_1^+ \cap \mathrm{F}_1^-, \ldots, \mathrm{F}_p^+ \cap \mathrm{F}_p^-$ of vertices in $G^+ \wedge G^-$ then $\forall i, \max(\mid F_i^+ - F_{i+1}^+ \mid, \mid F_i^- - F_{i+1}^- \mid) \leq 1$

As a consequence, the diameter of a d-refinement is always at least d, so the d-step property is equivalent to claim that every pair of vertices (one in G^+, the other in G^-) are at distance exactly d.
This gives us a condition for $G^+ \wedge G^-$ to have the d-step property:

Theorem 4. *If $G^+ \wedge G^-$ is transversal d-refinement of geodesic maps and if there is a path of length d from a vertex v^+ of G^+ and v^- of G^- then the facet of G^- containing v^+ has v^- as a vertex and the facet of G^+ containing v^- has v^+ as a vertex. (cf. Figure 7)*

As a corollary, we get one (important) condition, which is useful to find (and prove) prismatoids without the d-step property.

Definition 3. We say two vertices u of G^+, v of G^- are *separated* if there is no facet of G^- with vertex v containing u or there is no facet of G^+ with vertex u containing v.
We say that $G^+ \wedge G^-$ is *separated* if every pair of vertices u of G^+ and v of G^- is separated, and that a prismatoid is *separated* if its corresponding refinement is separated.

Theorem 5. *If $G^+ \wedge G^-$ is separated, it has not the d-step property.*

Proof:
We prove the contrapositive: assume there is a path $p = u \longrightarrow w_1 \ldots w_d \longrightarrow v$ of length d from $u \in G^-$ to $v \in G^+$.
Let $f_1^+, \ldots f_{d+1}^+ \in G^-$ and $f-$ be the facets containg u. Since $G^+ \wedge G^-$ is transversal, two adjacent vertices of p must be contained in the same facets, except for one of them. Since u is contained in $d+1$ facets of G^-, v is contained in 1 facet of G^- and p is of length d, each step in p reduces one adjacent facet of G^- (and thus add one adjacent facet of G^+).
Thus the unique facet of G^- containing v also contains u (but since u is a vertex of G^- it means that this facet has u as a vertex). If we use the same technique for v, we deduce that u and v are not separated.

□

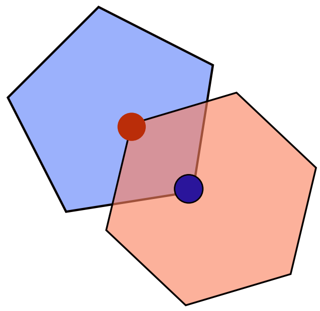


Figure 7: There is a path of length 2 from the red to the blue vertex

However, if there is a vertex v^+ of G^+ and a vertex v^- of G^- such that the facet of G^- containing v^+ has v^- as a vertex and the facet of G^+ containing v^- has v$^+$ as a vertex, there is no necesssarily a path of length d from v^+ to v^- as illustrated in Figure 2.2. Moreover the converce statement of 5 is likely to be false, although it is an open problem to know if there is a prismatoid without the d-step property but not separated. Indeed *every* 5-prismatoid without the d-step property found so far are separated.

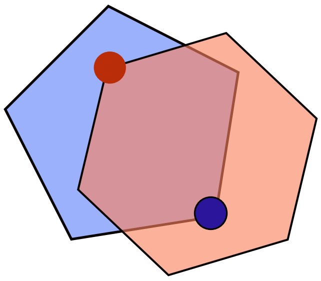


Figure 8: There is no path of length 2 from the red to the blue vertex

2.3 Non-existence

So we know that a 5-prismatoid without the d-step property exist. What about other dimensions?
The following easy Lemma show that there is no such prismatoid in dimension 3:

Lemma 13. Every refinement $G^+ \wedge G^-$ of two 1-geodesic maps has the d-step property

Proof:
Every vertex belongs to G^+ or G^- in $\mathbb{S}^1$. If we take one vertex u in G^+ and v in G^-, there is a path $u = u_0, u_1, \ldots, u_p = v$ from u to v and if $t = \max(q, u_q \in G^+)$ then we can go from $u_t \in G^+$ to $u_{t+1} \in G^-$ in one step.

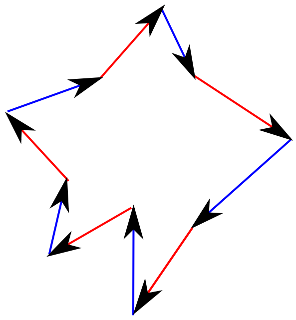
□

It is less immediate for 4-prismatoids:

Lemma 14 ([9]). Every refinement $G^+ \wedge G^-$ of two 2-geodesic maps has the d-step property.

Proof:
Assume by contradiction that a refinement of two geodesic maps has width $>$ 2.
It means that every vertex is adjacent to two non-terminal red vertices or two non-terminal blue vertices.

Definition 4. A zig-zag is a loop of color alternating non terminal edges which turns to the right from red to blue edges and turns to the left from blue to red edges except maybe for its base point.

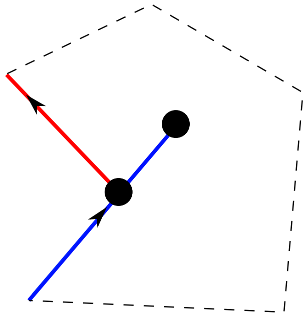


To get a contradiction, we will show that we can construct an infinite decreasing (for $\subset$) sequence of zig-zag. To construct a first zig-zag, take a non terminal edge and always continue in a direction that satisfy the zig zag property.

Figure 9: Two possibilities to continue the first zig-zag. Left: the next edge in a straight line is not terminal, so we take it. Right: the next edge in a straight line is terminal, so we take the one to the left (which is not terminal, since the refinement is of width > 2)

When we reach a previously visited vertex, this gives us a zig-zag. Clearly, this zig-zag is made of at least four edges, so at we must have the situation in Figure 10.

Figure 10:



This vertex is the intersection of 4 edges, so at least one more edge must be non-terminal (by the initial assumption). We start a new zig-zag from this edge and this gives us a new zig-zag, included in the first. To construct explicitly this zig-zag, we must distinguish the case where the new path hits the boundary of the previous zig-zag first (and we concatenate this path with one side of the previous zig-zag) and the case where this path hits itself first giving directly a new zig-zag.

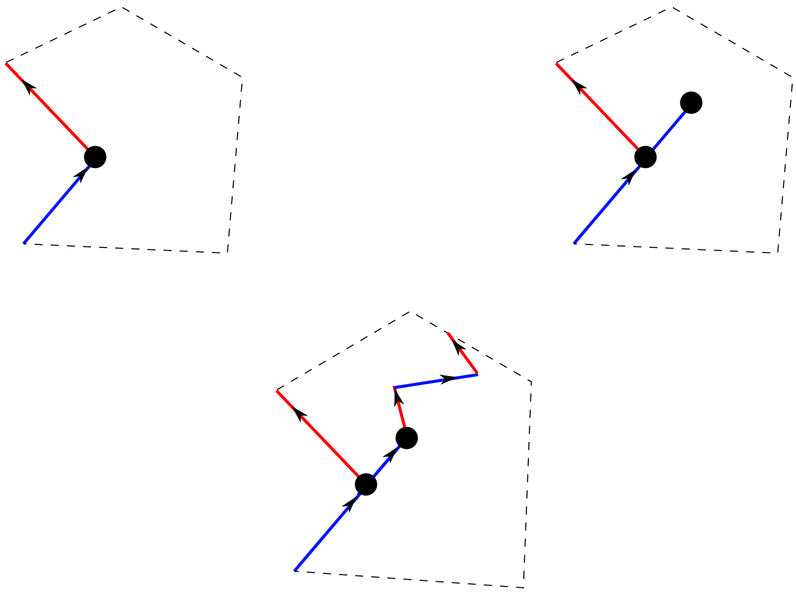


Figure 11: Steps for the construction of the next zig-zag

Since we can continue this indefinitely, this is a contradiction.

□

3 Computational point of view

3.1 Abstractions

So far, we were interested in the diameter of graphs in $G(\mathcal{P})$. An approach to study this diameter is to use an *abstraction* of them, i.e we study the diameter of a larger set of graphs characterized by purely combinatorial properties. All graphs of $G(\mathcal{P})$ must therefore belong to the abstraction. Obviously if we prove an upper bound on the diameter on an abstraction it would give us a bound on $\Delta_b(d, n)$.
A purely combinatorial abstraction has the advantage to be more "computable", and one main goal of this project was to encode the properties of such an abstraction in a SAT solver (this will be explained in more details in the next section).
We assume d and n are fixed integers. An "extension" of the abstraction studied in [3], consists in the set $\mathcal{B}_{d,n}$ of all connected graphs G = (V, E) such that a vertex is a subset of [n] = $\{1, \ldots, n\}$ of cardinality d and with the following conditions:

a) (Convexity) $\forall u, v \in$ V there is a path from u to v whose intermediate vertices all contain $u \cap v$

b) $(u, v) \in$ E $\Longleftrightarrow$ | $u \cap v$ | = d - 1

c) $\forall$ e subset of [n] of cardinality $d-1$, $|\{e \cup f \in V, f \in [n]\}\ | \in \{0, 2\}$

Lemma 15. $\mathcal{B}_{d,n}$ is an abstraction of simple d-polytopes with n facets

Proof:
If P is a simple d-polytope with n facets f_1, ..., f_n and graph (V, E), we define:

$$\begin{array}{cccc} \phi_P : & F & \longmapsto & \{\ f_i \text{ containing F }\} \\ & L(P) & \longrightarrow & \mathcal{P}([n]) \end{array}$$

We define a similar map for geodesic maps.
By convention, we denote $\phi_P((\mathrm{F}_1, \mathrm{F}_2)) = (\phi_P(\mathrm{F}_1), \phi_P(\mathrm{F}_2))$. We may also use "$\phi$" instead of ϕ_P.
Some basic facts about ϕ that we will use:

a) ϕ_P is strictly decreasing with respect to $\subset$ ($\mathrm{F}_1 \subsetneq \mathrm{F}_2 \Longleftrightarrow \phi_P(\mathrm{F}_1) \supsetneq \phi_P(\mathrm{F}_2)$)

b) | ϕ_P(F) | = d - dim(F) (This is Lemma 2)

Let G = $(\phi_P(V), \phi_P(E))$.
In this case, a) says that if we take two vertices u and v of V, there is a path from u to v which is included in the facets containing both u and v (to show it: take the intersection of all facets containing both u and v, this is a face with u and v as vertices, and we can find a path in this face from u to v according to Theorem 1), b) that $(u, v) \in$ E if and only if they have $d-1$ facets in common (this is Lemma 2) and then c) says that for every u, v vertices, there is either one or zero edge between them.

□

Now we define an abstraction $\mathcal{G}_{d,n,p}$ of the transversal refinements of two polytopal simple geodesic maps:
$\mathcal{G}_{d,n,p}$ is the set of all graphs G = (V, E) such that a vertex v is a pair (F^+, F^-), F^+ being a k-subset of $[n]$ of and F^- a $(d+2-k)$-subset of $[p]$, and such that the following conditions hold:

a) There is an edge between two vertices (F_1^+, F_1^-) and (F_2^+, F_2^-) if and only if $\mid F_1^+ \cap F_2^+ \mid + \mid F_1^- \cap F_2^- \mid = d+1$

b) $\forall$ $\mathrm{F}^+ \subset [n]$ not empty, $\mathrm{F}^- \subset [p]$ not empty such that $\mid F^+ \mid + \mid F^- \mid = d+1$, among all pairs ($\mathrm{F}^+ \cup \{j\}$, F^-), j $\in [n]$ and (F^+, $\mathrm{F}^- \cup \{j\}$), j $\in [p]$ there are exactly 0 or 2 vertices in V

c) (Convexity) $\forall$ (F_1^+, F_1^-), (F_2^+, F_2^-) $\in$ V there is a path from u to v whose intermediate vertices are all of the form (F^+, F^-) with $F_1^+ \cap F_2^+ \subset F^+$ and $F_1^- \cap F_2^- \subset F^-$

Now if G^+, G^- are simple transversal d-geodesic maps with respectively n and p facets and such that their refinement has the graph (V, E), let G = ($\phi(V)$, ϕ(E)), where $\phi(\mathrm{F}^+ \cap \mathrm{F}^-)$ is the pair ($\phi(\mathrm{F}^+)$, $\phi(\mathrm{F}^-)$).
Notice that, by definition of $G^+ \wedge G^-$, neither $\phi(\mathrm{F}^+)$ nor $\phi(\mathrm{F}^-)$ is empty if $\mathrm{F}^+ \cap \mathrm{F}^- \in G^+ \wedge G^-$.
Since $G^+ \wedge G^-$ is transversal:

Lemma 16. If $\mathrm{F}^+ \cap \mathrm{F}^-$ is a non empty face of $G^+ \wedge G^-$ then $\mid \phi_{G^+}(F^+) \mid + \mid \phi_{G^-}(F^-) \mid = d+2$ - dim($\mathrm{F}^+ \cap \mathrm{F}^-$).

The previous conditions become, for simple transversal d-geodesic maps:

a) There is an edge between $\mathrm{F}_1^+ \cap \mathrm{F}_1^-$ and $\mathrm{F}_2^+ \cap \mathrm{F}_2^-$ if and only if $\mid \phi_{G^+}(F_1^+) \cap \phi_{G^+}(F_2^+) \mid + \mid \phi_{G^-}(F_1^-) \cap \phi_{G^-}(F_2^-) \mid = d+1$

b) $\forall$ $\phi_{G^+}(\mathrm{e}^+) \subset [n]$ not empty, $\phi_{G^-}(\mathrm{e}^-) \subset [p]$ not empty such that $\mid \phi_{G^+}(e^+) \mid + \mid \phi_{G^-}(e^-) \mid = d+1$, among all pairs ($\phi_{G^+}(\mathrm{e}^+) \cup \{f\}$, $\phi_{G^-}(\mathrm{e}^-)$), f $\in [n]$ and ($\phi_{G^+}(\mathrm{e}^+)$, $\phi_{G^-}(\mathrm{e}^-) \cup \{f\}$), f $\in [p]$ there are exactly 0 or 2 vertices in V

c) (Convexity) $\forall$ $u = \mathrm{F}_1^+ \cap \mathrm{F}_1^-$, $v = \mathrm{F}_2^+ \cap \mathrm{F}_2^- \in$ V there is a path from u to v whose intermediate vertices are all contained in $\mathrm{F}_1^+ \cap \mathrm{F}_2^+$ and $\mathrm{F}_1^- \cap \mathrm{F}_2^-$

d) (Degree) $\forall$ $\mathrm{F}^+ \cap \mathrm{F}^- \in V$: if $\mathrm{F}^+ = \mathrm{H}^+ \cup \{f^+\}$ and if f^+ is a facet then $\mathrm{H}^+ \cap \mathrm{F}^-$ is an edge (and the symmetric case)

Lemma 17. $\mathcal{G}_{d,n,p}$ is an abstraction of simple refinements of two geodesic maps of d-polytopes with n and p facets

Proof:

a) Let ϕ(e) = ($\phi_{G^+}(\mathrm{e}^+)$, $\phi_{G^-}(\mathrm{e}^-)$) be an edge between ($\phi_{G^+}(F_1^+)$, $\phi_{G^-}(F_1^-)$) and ($\phi_{G^+}(F_2^+)$, $\phi_{G^-}(F_2^-)$), so that e = $\mathrm{e}^+ \cap \mathrm{e}^-$ is an edge in $G^+ \wedge G^-$ between $v_1 = F_1^+ \cap F_1^-$ and $v_2 = F_2^+ \cap F_2^-$. Since the facets of G^+ (resp. G^-) containing v_1 and v_2 are exactly those containing e, ϕ(e) = ($\phi_{G^+}(F_1^+) \cap \phi_{G^+}(F_2^+)$, $\phi_{G^-}(F_1^-) \cap \phi_{G^-}(F_2^-)$). Then if we use Lemma 16 we get $\mid \phi(F_1^+) \cap \phi(F_2^+) \mid + \mid \phi(F_1^-) \cap \phi(F_2^-) \mid = d+1$.
Conversely, if $\mid \phi(F_1^+) \cap \phi(F_2^+) \mid + \mid \phi(F_1^-) \cap \phi(F_2^-) \mid = d+1$, then there

is a total of $d+1$ facets in $\phi(F_1^-) \cup \phi(F_1^+)$ that are also facets of $\phi(F_2^-) \cup \phi(F_2^+)$.
The intersection of these $d+1$ facets is a non empty face of $G^+ \wedge G^-$, so it is an edge (by 16) and it contains v_1 and v_2.

b) We show that if there is at least one vertex, say $\phi(\mathrm{v}) = (\phi_{G^+}(\mathrm{e}^+) \cup \{f\}, \phi_{G^-}(\mathrm{e}^-))$, among the set Pa of all pairs $(\phi_{G^+}(\mathrm{e}^+) \cup \{f\}, \phi_{G^-}(\mathrm{e}^-))$, $\mathrm{f} \in [n]$ and $(\phi_{G^+}(\mathrm{e}^+), \phi_{G^-}(\mathrm{e}^-) \cup \{f\})$, $\mathrm{f} \in [p]$, there are in fact exactly 2 such vertices.
In this case, $\mathrm{e} = \mathrm{e}^+ \cap \mathrm{e}^-$ is a face of $G^+ \wedge G^-$, not empty since it contains v so it is an edge and the extremity of this edge different from v gives us a second vertex in Pa.
Moreover any vertex $\phi(w)$ in Pa gives a vertex $\mathrm{w} \in G^+ \wedge G^-$ which is on e, but there are only two such vertices.

c) Let $\mathrm{f}_1^+, \ldots \mathrm{f}_k^+$ be the facets of G^+ and $\mathrm{f}_1^-, \ldots \mathrm{f}_l^-$ the facets of G^- containing both u and v, with possibly k=0 (resp. l=0), meaning that there is no facet of G^+ (resp. G^-) containing both u and v.
Assume first that $k \neq 0$ and $l \neq 0$: then let $\mathrm{f} = (\mathrm{f}_1^+ \cap \ldots \mathrm{f}_k^+) \cap (\mathrm{f}_1^- \cap \ldots \mathrm{f}_l^-)$. $\mathrm{f}_1^+ \cap \ldots \mathrm{f}_k^+$ (resp. $\mathrm{f}_1^- \cap \ldots \mathrm{f}_l^-$) is a face of G^+ (resp. G^-) so f is a face of $G^+ \wedge G^-$. Moreover u and v belong to f and are in fact vertices of f (no vertex can be in the interior of a face). Since f is combinatorially equivalent to a polytope, according to Theorem 1 there is a path from u to v in f with the convexity condition.
Now assume that, for example, $l = 0$ and $k \neq 0$ (in the case k=0, l=0 we only have to prove that the graph is connected and this is true according to Lemma 6).
The proof for the case $k \neq 0$, $l \neq 0$ doesn't work for this case because the intersection of facets of G^+ is not, *a priori*, a face of $G^+ \wedge G^-$ (so u and v can be in the interior of this intersection).
Let $\mathrm{f}^+ = \mathrm{f}_1^+ \cap \ldots \mathrm{f}_k^+$: f^+ is a non empty face of G^+ containing u and v.
Now let $\mathrm{S} = \{ \mathrm{f}^+ \cap \mathrm{f}^-, \mathrm{f}^-$ is a face of G^- $\}$: every face in S is a face of $G^+ \wedge G^-$ and since G^- is complete (i.e the union of all faces of G^- equals $\mathbb{S}^d$), $\bigcup_{f \in S} f = \mathrm{f}^+$.
In the following, we call every $\dim(\mathrm{f}^+)$-face of S a facet of S.

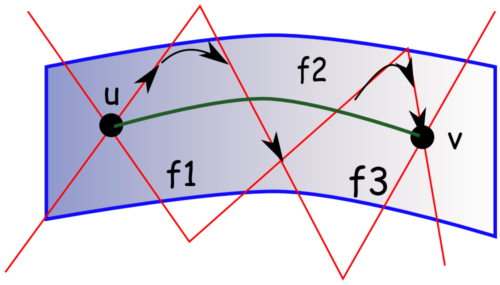


Figure 12: Illustration of the proof of convexity, when one (blue) face f^+ of G^+ contain u and v and no (red) face of G^- contain both u and v. γ is green.

Now we claim that there is a sequence A of facets $f_1, \ldots, f_q$ of S such that u (resp. v) is a vertex of f_1 (resp. f_q) and $\forall$ i $\in [q-1]$, f_i has at least one vertex in common with f_{i+1}.
If we find such a set A, then we can start at u, move to a vertex of $f_1 \cap f_2$ with a path of edges of f_1, and so on until we reach v, staying in f^+ since it is connex, so it proves the convexity.
Let $\gamma : [0, 1] \longrightarrow \mathbb{S}^d$ a geodesic of $\mathbb{S}^d$ such that $\gamma(0) = u$ and $\gamma(1) = \mathrm{v}$.
We can take for A the set of all facets of S that intersect $\gamma([0, 1])$: since every facet of S is convex and compact, there are $t_1 < \ldots < t_{q-1}$ such that $\gamma([0, t_1])$ is included in one facet f_1 of A, $\gamma([t_1, t_2])$ is included in one facet f_2 of A, …, $\gamma([t_{q-1}, 1])$ is included in one facet f_q of A (for example, to define t_1: we know that for ϵ sufficiently small, $\gamma([0, \epsilon])$ is included in one facet, say f_1, of A and we take $t_1 = \sup(t, \gamma([0, t])$ is included in $f_1)$).
$\forall$ i $\in [q-1]$, f_i has at least one vertex in common with f_{i+1} since f_i and f_{i+1} has the point $\gamma(t_i)$ in common.

d) $H^+ \cap F^-$ contains a vertex and it is a face so it is an edge, according to Lemma 16.

□

One problem we faced was to encode the $(d+1)$-connectivity and I instead decided to use the (Degree) condition: it doesn't implies necessarily $(d+1)$-connectivity but is decreases the probability of having a not $(d+1)$-connected graph.
Moreover, when the SMT solver ouput a graph I check if it is $(d+1)$-vertex-connected and if not, I add another constraint and search for the next graph.

To ensure that we find a prismatoid without the $d-$step property, we add the (stronger) property of separation: $\forall$ ({ f_1^+, ... f_{d-1}^+ }, f^-), (f^+, { f_1^-, ... f_{d-1}^- }) vertices, $\neg$ ($\exists$ i, $f^- = f_i^- \wedge \exists$ j, $f^+ = f_i^+$).
It is important to notice that the fact that we don't find any prismatoid with this property doesn't mean that there is no prismatoid without the $d-$step property with a given number of vertices (and, a fortiori, it doesn't mean that there is no non-Hirsch prismatoid with given dimension and number of facets).

3.2 SAT modulo theories

Instead of dealing directly with polytopes I encode the conditions defining an abstraction in a SAT solver, and either no solution is found (and then there is no such polytope) or one solution is found and we try to know if it comes from a polytope.
In fact, we used *Satisfiability Modulo Theories* (SMT) solver, allowing first-order logic formulas such as addition, equality, quantifiers ....
Instead of using directly the API of one particular SMT solver, I used the SMT-LIB format, which is a standard language for SMT problems (and is usable by almost all SMT solvers) and I wrote a C++ code to output a file in this format.
Then, I also wrote a code C++ with the API of the Z3 SMT solver [5], which:

- reads this file
- finds a solution
- checks if the output graph G is $(d + 1)$-vertex-connected
  For this part, I construct a network R from G the following way: if v is a vertex of G then I put two vertices v_1, v_2 in R and I link them with one directed edge from v_1 to v_2 and with capacity 1. If (u, v) is an edge of G I put two edges (u_2, v_1) and (v_2, u_1) with infinite capacity.
  Then I compute, for each pair of vertices of R of the form (u_2, v_1), the maximum $(u_2$-$v_1)$-flow (which is equal to the minimum vertex cut between u_2 and v_1). Finally the vertex-connectivity of the graph is the minimum of all these maximum flow, and I just check if it is at least $d + 1$.
- checks if the output graph G has the convexity condition:
  For each pair u, v of vertices I compute all reachable vertices $C(u)$ from u such that the faces defining them contain the faces defining $u \cap v$. If v is not in $C(u)$, the graph has the not convexity property and then we add the following constraint: either there is no vertex of $C(u)$ or no vertex of $C(v)$ or there is an edge that go out of $C(u)$ and both of its end points are defined by faces containing $u \cap v$.
  This constraints ensures that the graph will not be found again by the SMT solver and it removes some other bad configurations.
- outputs the resulting graph as a picture, to analyze it more easily.

  For the three last points I used OGDF [1] which is a C++ class library for the automatic layout of graph drawings.

[1]Open Graph Drawing Framework: http://www.ogdf.net/doku.php/start

3.3 Variables

I used one variable $F_{(S^+,S^-)}$ for each $S^+ \in \binom{n}{k}$, $S^- \in \binom{p}{d+2-k}$, $k = 1, \ldots d+1$ (these are the "vertices") and for each $S^+ \in \binom{n}{k}$, $S^- \in \binom{p}{d+1-k}$, $k = 1, \ldots d$ ("edges"). Each variable is a bit: either one (meaning that the vertex is present) or zero.

Thus we have $\sum_{k=1}^{d+1} \binom{n}{k}\binom{p}{d+2-k} = \binom{n+p}{d+2} - \binom{p}{d+2} - \binom{n}{d+2} \sim \binom{n+p}{d+2}$ vertex variables (according to Vandermonde formula) and $\binom{n+p}{d+1} - \binom{p}{d+1} - \binom{n}{d+1} \sim \binom{n+p}{d+1}$ edge variables. The number of variables grows quickly.

4 Results

I searched for separated prismatoid with dimension 5 only (we know that there is no such a prismatoid in dimension strictly less than 5 and searching in dimension 6 or greater is, *a priori*, more difficult) and with various number of facets of the geodesic maps, using a SMT solver as explained above. Table 4 gives some examples of running time for simulations: the running time grows very fast. This results from the number of variables being exponential in the number of vertices. Similarly, the memory used grows exponentially in the number of facets used, especially because the number of potential solutions rejected (because of non-convexity) becomes very high and every condition added this way may be polynomial in the number of vertices of the geodesic maps. Unfortunately, I found no prismatoid meeting all the conditions and I was not able to do simulations for prismatoid with more than a total of 13 vertices (the main problem is memory).

Total number of facets	4+4	4+5	5+5	5+6	6+6
Time (seconds)	0,01	0,03	0,41	10,1	297451 ($\sim$ 3 days)

Table 2: Runtime for a given number of vertices

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