Computational approaches to the diameter of Polytopes

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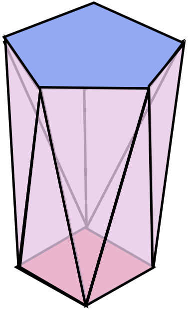
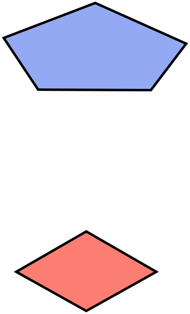
Prismatoids (duals of spindles)



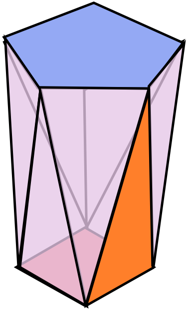
Prismatoids

All vertices are contained in two parallel basis:



Prismatoids

Its width is the facet-distance between the two basis:



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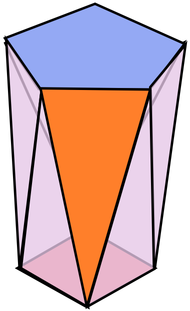

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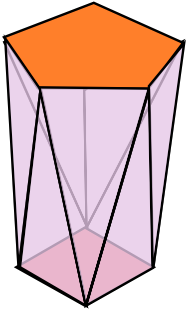

Hirsch conjecture

Hirsch conjecture, 1957

Every d-polytope with n facets has diameter at most $n - d$.

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F. Santos, May 2010

Hirsch conjecture is false.

d-step property

Definition

A prismatoid has the d-step property if its width is less than its dimension.

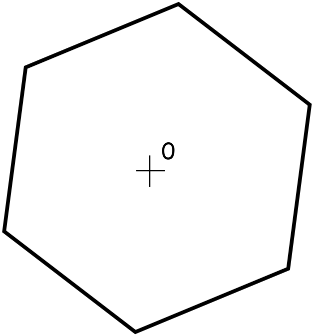
Theorem

If one can find a prismatoid with n vertices and dimension d without the d-step property, one can construct (another) non-Hirsch prismatoid in dimension $n - d$.

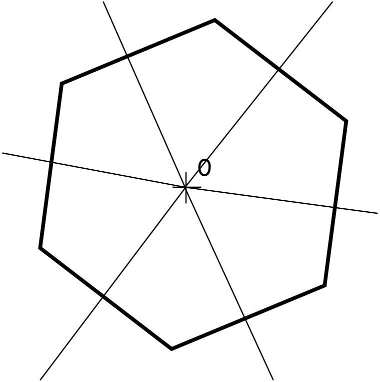
Main goal of this project

Find prismatoids without the d-step property and with as few vertices as possible (or prove that they don't exist).

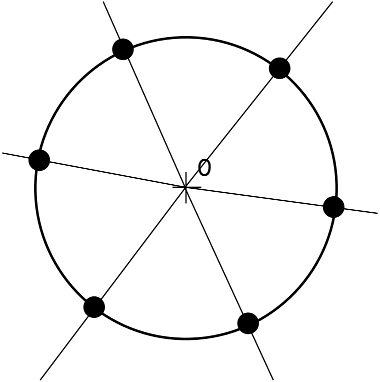
Geodesic maps : Construction



Geodesic maps : Normal fan



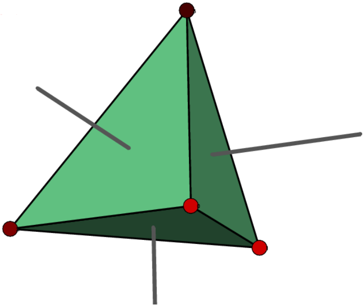
Geodesic maps : Intersection with $\mathbb{S}^{d-1}$

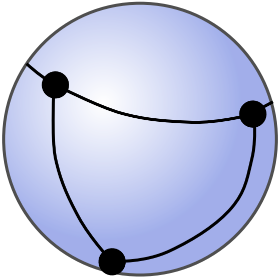


Geodesic maps

Definition

A geodesic map of a d-polytope is the intersection of its normal fan with the sphere $\mathbb{S}^{d-1}$.



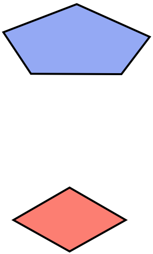


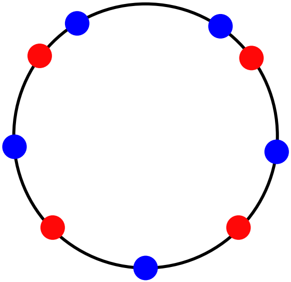
Geodesic maps of prismatoids

Let Q be a prismatoid with basis Q^+ and Q^-, and G^+, G^- the geodesic maps of Q^+ and Q^-:

Refinement

The *refinement* $G^+ \wedge G^-$ is the set of the intersections of faces of G^+ with faces of G^-.





Definition

$G^+ \wedge G^-$ has the d-step property if $d-2$ steps are sufficient to go from a vertex of G^+ to a vertex of G^- (using edges)

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Lemma

Q has the d-step property if and only if $G^+ \wedge G^-$ has the d-step property

Dimension 3

Lemma

There is no 3-prismatoid without the d-step property.

Proof:
Every refinement $G^+ \wedge G^-$ in $\mathbb{S}^1$ is such that one vertex of G^+ is adjacent to one vertex of G^-.

$\square$

Dimension 4

Lemma

There is no 4-prismatoid without the d-step property.

Proof

Assume by contradiction that a refinement of two geodesic maps has width > 2.

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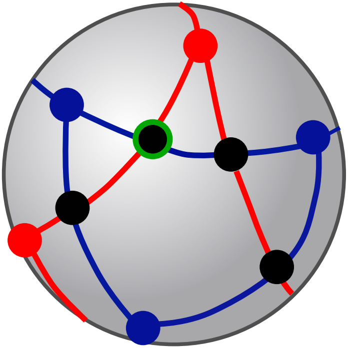
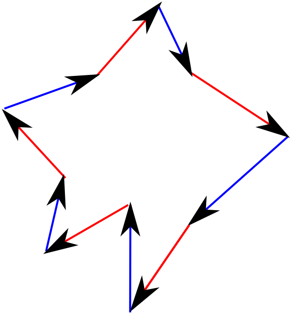
For example we can't have one non terminal vertex adjacent to both a red vertex and blue vertex:



Zig-zag

Definition

A zig-zag is a loop of color alternating non terminal edges which turns to the right from red to blue edges and turns to the left from blue to red edges except maybe for its base point.



Proof

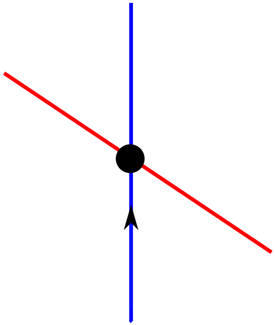
To get a contradiction, we will show that we can construct an infinite decreasing (for $\subset$) sequence of zig-zag.

First zig-zag

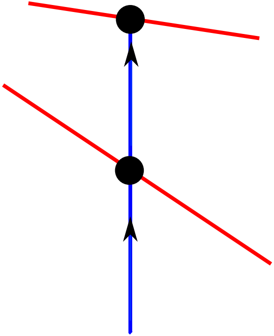
To construct a first zig-zag, take a non terminal edge and always continue in a direction that satisfy the zig zag property.

Proof

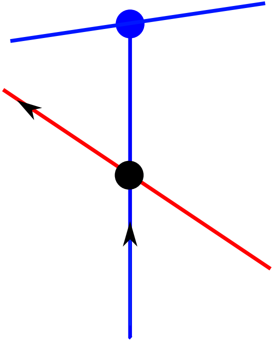
For example, assume we come to a vertex by a non terminal edge:



First possibility



Second possibility



First zig-zag

When we reach a previously visited vertex, this gives us a zig-zag.

Second zig-zag

Clearly, this zig-zag is made of at least four edges, so at least one of them is "concave".

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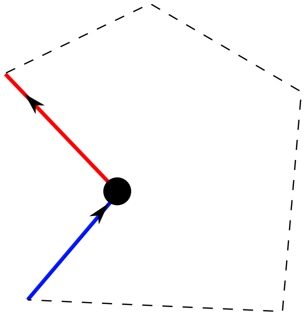
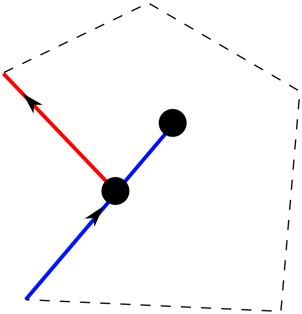


Second zig-zag

This vertex is the intersection of 4 edges, so at least one more edge must be non-terminal (by the initial assumption).



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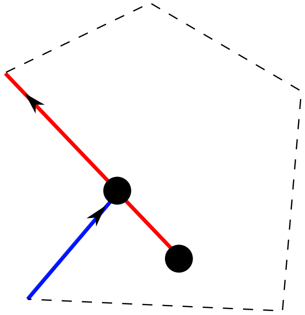
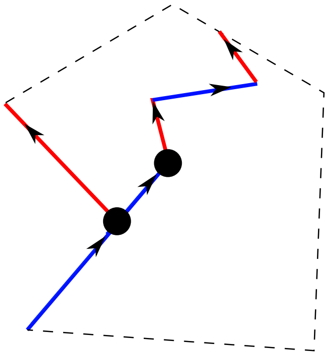


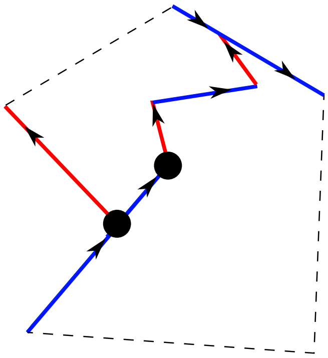
Second zig-zag

We start a new zig-zag from this edge.



Second zig-zag

This gives us a new zig-zag, included in the first.



End of the proof

Since we can continue this indefinitely, this is a contradiction.

Dimension 5

To prove that a given 5-prismatoid (i.e. a refinement of geodesic maps in $\mathbb{S}^3$) doesn't have the *d*-step property, it is convenient to show that it is separated.

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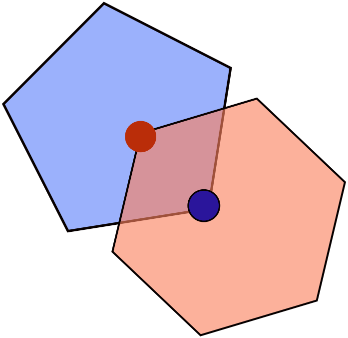
Separation

We say two vertices u of G^+, v of G^- are *separated* if there is no facet of G^- with vertex v containing u or there is no facet of G^+ with vertex u containing v.
We say that $G^+ \wedge G^-$ is *separated* if every pair of vertices u of G^+ and v of G^- is separated, and that a prismatoid is *separated* if its corresponding refinement is separated.

Two vertices *not* separated:



Lemma

If $G^{+} \wedge G^{-}$ is separated, it has not the d-step property

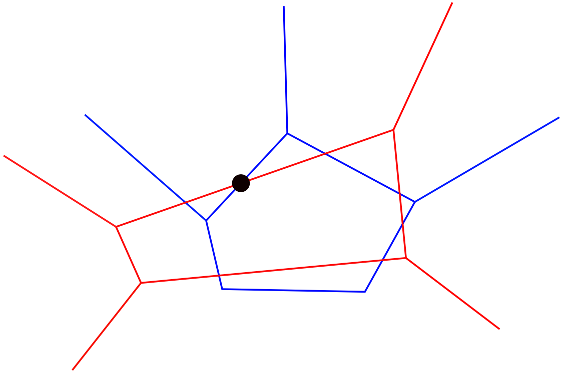
Abstraction used

Graphs such that a vertex v is a pair (F^+, F^-), F^+ being a k-subset of $[n]$ of and F^- a $(d+2-k)$-subset of $[p]$.

Abstraction used

If $G^+ \wedge G^-$ is a refinement of geodesic maps in $\mathbb{S}^d$, we can construct an abstract graph G by putting, for every vertex w of $G^+ \wedge G^-$, a vertex (F^+, F^-) in G such that F^+ **is the set of all facets of** G^+ **containing** w and F^- **is the set of all facets of** G^- **containing** w.

Abstraction



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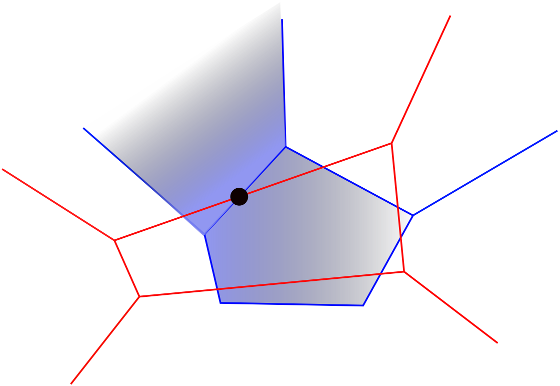

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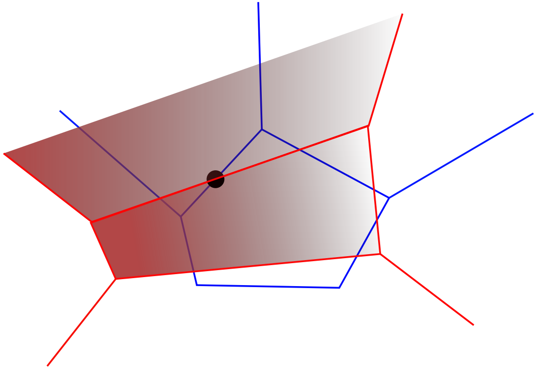

Conditions

- Separation
- "Convexity"
- Every edge has 0 or 2 vertices
- $(d+1)$-connectivity
- ...

Conditions

Strategy used: if a condition is easy to encode in the SMT solver, we do it.

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Otherwise we only check if the condition is verified when the SMT solver outputs a solution. If it is not verified, we add a "minimal" condition to reject this solution, and ask for the next solution.
Example: $(d+1)$-connectivity, convexity.

I didn't find any geodesic maps without the d-step property with less than 12 facets (it doesn't mean that they don't exist since I check separation instead of d-step property).

Total number of facets	4+4	4+5	5+5	5+6	6+6
Time (seconds)	0,01	0,03	0,41	10,1	297451 ($\sim$ 3 days)

However the main problem is the memory: with 6+6 facets the program uses about 10 GO memory.

The number of potential solutions rejected because of non-convexity grows very fast.

Thank you for your attention.

Do you have any questions?