Prop. 2.2.11 : E $\mathbb{K}$-ev, $\mathcal{B} = (e_1 \dots e_p)$

F $\mathbb{K}$-ev $\mathcal{C} = (f_1 \dots f_n)$

$u \in \mathcal{L}(E,F)$

$x \in E$. dc $u(x) \in F$.

Alors : $\operatorname{Mat}_{\mathcal{C}} u(x) = (\operatorname{Mat}_{\mathcal{B},\mathcal{C}} u) \times (\operatorname{Mat}_{\mathcal{B}} x)$

$$x \qquad\qquad\qquad\qquad u(x)$$

$$E, \mathcal{B} \xrightarrow{\quad u \quad} F, \mathcal{C}$$

$$\operatorname{Mat}_{\mathcal{B}} x \qquad\qquad \operatorname{Mat}_{\mathcal{B},\mathcal{C}} u \qquad\qquad \operatorname{Mat}_{\mathcal{C}} u(x)$$

Ex : ds les bases canoniques :

$$u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1x + 1y \\ 2x - 1y \end{pmatrix}$$

$\mathcal{B}$: base canonique

$$\mathrm{Mat}_{\mathcal{B}}(u) = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

On a bien : $u\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix}$

Qu : $\mathcal{B}_1 = \left(\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{e_2} \right)$ $\quad \mathcal{B}_2 = \left(\underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{f_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{f_2} \right)$

quelles st les coord de $u(2e_1 + 3e_2)$ ds $\mathcal{B}_2$?

– sans matrice : $u(2e_1 + 3e_2) = u\begin{pmatrix} 5 \\ -3 \end{pmatrix}$ → ds $\mathcal{B}$

ds $\mathcal{B}$

$$= \begin{pmatrix} 2 \\ 13 \end{pmatrix} = 2 f_1 + 11 f_2 .$$

– avec matrice : $u(e_1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = f_1 + f_2 = 1 \times f_1 + 1 \times f_2$

$$u(e_2) = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3 f_2 = 0 \times f_1 + 3 \times f_2$$

dc : $\mathrm{Mat}_{\mathcal{B}_1, \mathcal{B}_2}\, u = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}$

Or :

$$\mathrm{Mat}_{\mathcal{B}_2}\left(u(2e_1+3e_2)\right) = \mathrm{Mat}_{\mathcal{B}_1,\mathcal{B}_2}(u) \times \mathrm{Mat}_{\mathcal{B}_1}(2e_1+3e_2)$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 11 \end{pmatrix} \longrightarrow \text{ds } \mathcal{B}_2.$$

⚠

$$\mathrm{Mat}_{\mathcal{B}}\left(u(2e_1+3e_2)\right) = \begin{pmatrix} 2 \\ 13 \end{pmatrix}$$

le même vecteur !

$$\mathrm{Mat}_{\mathcal{B}_2}\left(u(2e_1+3e_2)\right) = \begin{pmatrix} 2 \\ 11 \end{pmatrix}$$

<u>Cor. 2.2.14</u> : $\mathrm{Mat}_{\mathcal{C}}(u(x_1), u(x_2) \dots u(x_n))$

$$= \left(\mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u\right) \times \left(\mathrm{Mat}_{\mathcal{B}}(x_1, x_2 \dots x_n)\right)$$

en effet $B = M \cdot A$ ssi

la $1^{\text{ère}}$ col. de B $=$ $M \times (1^{\text{e}}$ col de $A)$.

et $\forall j$: ($j^{\text{ème}}$ col de B) $=$ $M \times$ ($j^{\text{ème}}$ col. de A)

Or : $\forall j$, $\mathrm{Mat}_{\mathcal{C}}\, u(x_j) = \mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u \times \mathrm{Mat}_{\mathcal{B}}\, x_j$

on a donc le résultat.

Th :

$$\begin{tikzcd} & F, \mathcal{C} \arrow[dr, "v"] & \\ E, \mathcal{B} \arrow[ur, "u"] \arrow[rr, "v \circ u"'] & & G, \mathcal{D} \end{tikzcd}$$

$\mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u$

$\mathrm{Mat}_{\mathcal{C},\mathcal{D}}\, v$

$\mathrm{Mat}_{\mathcal{B},\mathcal{D}}\, (v \circ u)$

$$\mathrm{Mat}_{\mathcal{B},\mathcal{D}}\, v \circ u = \left(\mathrm{Mat}_{\mathcal{C},\mathcal{D}}\, v\right) \times \left(\mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u\right)$$

Si $u \in \mathcal{L}(E,F)$, $v \in \mathcal{L}(F,G)$

$\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ st des bases de E, F, G resp.,

alors : $\boxed{\operatorname{Mat}_{\mathcal{B},\mathcal{D}}(v \circ u) = \left(\operatorname{Mat}_{\mathcal{C},\mathcal{D}} v\right) \times \left(\operatorname{Mat}_{\mathcal{B},\mathcal{C}} u\right).}$

<u>Ex</u> : $u : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x - y + z \\ x - z \\ 2y + z \end{pmatrix}$$

$v : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 2x + 3z \\ x + y - z \\ x - 2y + z \end{pmatrix}$$

Donner l'expression de $v \circ u$.

<u>Sans matrice</u> :

$$v \circ u \begin{pmatrix} x \\ y \\ z \end{pmatrix} = v \begin{pmatrix} x - y + z \\ x - z \\ 2y + z \end{pmatrix}$$

$$= \begin{pmatrix} 2(x-y+z) + 3(2y+z) \\ (x-y+z) + (x-z) - (2y+z) \\ (x-y+z) - 2(x-z) + (2y+z) \end{pmatrix}$$

$$= \begin{pmatrix} 2x + 4y + 5z \\ 2x - 3y - z \\ -x + y + 4z \end{pmatrix}$$

Avec matrice : $\mathcal{C}$ base canonique.

$$\mathrm{Mat}_{\mathcal{C}}(u) = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$\mathrm{Mat}_{\mathcal{C}}(v) = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

d'où : $\mathrm{Mat}_{\mathcal{C}}(v \circ u) = \mathrm{Mat}_{\mathcal{C}}(v) \times \mathrm{Mat}_{\mathcal{C}}(u)$

$$= \begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 & 5 \\ 2 & -3 & -1 \\ -1 & 1 & 4 \end{pmatrix}$$

<u>Démo :</u>

$$E, \mathcal{B} \xrightarrow{u} F, \mathcal{C} \xrightarrow{v} G, \mathcal{D}$$

$$\mathcal{B} = (e_1 \dots e_p), \quad \mathcal{C} = (f_1 \dots f_n), \quad \mathcal{D} = (g_1 \dots g_q)$$

$$\mathrm{Mat}_{\mathcal{B},\mathcal{C}}(u) = \mathrm{Mat}_{\mathcal{C}}\big(u(e_1), \dots, u(e_p)\big)$$

$$\mathrm{Mat}_{\mathcal{B},\mathcal{D}}(v \circ u) = \mathrm{Mat}_{\mathcal{D}}\big(v(u(e_1)), \dots, v(u(e_p))\big)$$

$$\underset{2.2.14}{=} \left(\mathrm{Mat}_{\mathcal{C},\mathcal{D}}\, v\right) \times \mathrm{Mat}_{\mathcal{C}}\left(u(e_1), \dots, u(e_p)\right)$$

$$\underset{2.2.14}{=} \left(\mathrm{Mat}_{\mathcal{C},\mathcal{D}}\, v\right) \times \left[\mathrm{Mat}_{\mathcal{B},\mathcal{C}}(u) \times \mathrm{Mat}_{\mathcal{B}}(e_1 \dots e_p)\right]$$

$$= \left(\mathrm{Mat}_{\mathcal{C},\mathcal{D}}\, v\right) \times \left[\left(\mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u\right) \times \underbrace{\left(\mathrm{Mat}_{\mathcal{B}}(e_1 \dots e_p)\right)}_{= I_p}\right]$$

Or : $\mathcal{B} = (e_1 \dots e_p)$ dc :

$$\mathrm{Mat}_{\mathcal{B}}(e_1 \dots e_p) = \mathrm{Mat}_{\mathcal{B}}(\mathcal{B}) = I_p$$

dc : $\mathrm{Mat}_{\mathcal{B},\mathcal{D}}(v \circ u) = \left(\mathrm{Mat}_{\mathcal{C},\mathcal{D}}\, v\right)\left(\mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u\right)$.

Rq. 2.2.16 : souvenir : le produit matriciel est associatif, on l'a démontré dans le chapitre II, la démo était assez pénible.

Autre démo : A, B, C 3 matrices.

$A \in \mathcal{M}_{q,p}(\mathbb{K})$, $B \in \mathcal{M}_{p,n}(\mathbb{K})$, $C \in \mathcal{M}_{n,m}(\mathbb{K})$

Mq. $(A \times B) \times C = A \times (B \times C)$.

Soit E, F, G, H 4 ev

$\dim E = m$, $\dim F = n$, $\dim G = p$, $\dim H = q$

$\mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ des bases de E, F, G, H resp.

il existe $u \in \mathcal{L}(E, F)$ tq. $\mathrm{Mat}_{\mathcal{B}\mathcal{C}}\, u = C$

$v \in \mathcal{L}(F, G)$ tq. $\mathrm{Mat}_{\mathcal{C},\mathcal{D}}\, v = B$

$w \in \mathcal{L}(G, H)$ tq. $\mathrm{Mat}_{\mathcal{D},\mathcal{E}}\, w = A$

$$\text{dc} : (A \times B) \times C = \left(\mathrm{Mat}_{\mathcal{D},\mathcal{E}}\, w \times \mathrm{Mat}_{\mathcal{C},\mathcal{D}}\, v\right) \times \mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u$$

$$= \left(\mathrm{Mat}_{\mathcal{C},\mathcal{E}}\, w \circ v\right) \times \mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u$$

$$= \mathrm{Mat}_{\mathcal{B},\mathcal{E}}\, (w \circ v) \circ u$$

$$= \mathrm{Mat}_{\mathcal{B},\mathcal{E}}\, w \circ (v \circ u)$$

$$= \mathrm{Mat}_{\mathcal{D},\mathcal{E}}\, w \times \mathrm{Mat}_{\mathcal{B},\mathcal{D}}\, v \circ u$$

$$= \mathrm{Mat}_{\mathcal{D},\mathcal{E}}\, w \times \left(\mathrm{Mat}_{\mathcal{C},\mathcal{D}}\, v \times \mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u \right)$$

$$= A \times (B \times C).$$

<u>Rq. 2.2.21</u> : $(A_1, +, \times)$, $(A_2, +, \times)$

$$\varphi(x+y) = \varphi(x) + \varphi(y)$$

$$\varphi(x \times y) = \varphi(x) \times \varphi(y)$$

$$\varphi(1_{A_1}) = 1_{A_2}$$

$$\varphi : (\mathcal{L}(E), +, \circ) \longrightarrow (\mathcal{M}_n(\mathbb{K}), +, \times)$$

$$u \longmapsto \mathrm{Mat}_{\mathcal{B}}(u)$$

est 1 morphisme d'anneaux.

→ $\mathrm{Mat}_{\mathcal{B}}(u+v) = \mathrm{Mat}_{\mathcal{B}}\, u + \mathrm{Mat}_{\mathcal{B}}\, v$

→ $\mathrm{Mat}_{\mathcal{B}}(u \circ v) = (\mathrm{Mat}_{\mathcal{B}}\, u) \times (\mathrm{Mat}_{\mathcal{B}}\, v)$

→ $\mathrm{Mat}_{\mathcal{B}}(\mathrm{id}_E) = I_n$.