3.1 - Déterminant d'une famille de vecteurs : définition.

Première partie : la "grosse formule"

Objectif : (ii) $\det_{\mathcal{B}}$ est n-linéaire alternée.

(i) $\mathcal{A}_n(E) = \mathrm{Vect}(\det_{\mathcal{B}})$.

(iii) Si $f \in \mathcal{A}_n(E)$, $f = f(\mathcal{B}) \cdot \det_{\mathcal{B}}$.

E : $\mathbb{K}$-ev de dim n, $\mathcal{B} = (e_1 \dots e_n)$.

$X_1 \dots X_n$ des vecteurs de E.

$$X_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} = \sum_{i=1}^{n} x_{i1} e_i$$

$\forall j \in [\![1, n]\!]$: $$X_j = \begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix} = \sum_{i=1}^{n} x_{ij} e_i$$

Rappel : $\mathrm{Mat}_{\mathcal{B}}(X_1 \dots X_n) = (x_{ij})_{1 \leq i,j \leq n}$.

__Déf__ : On appelle déterminant de $(X_1 \dots X_n)$ dans la base $\mathcal{B}$ le scalaire :

$$\det_{\mathcal{B}}(X_1 \dots X_n) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} x_{i\sigma(i)} .$$

__Rq__ : $\det_{\mathcal{B}} : E^n \longrightarrow \mathbb{K}$

$$(X_1 \dots X_n) \longmapsto \det_{\mathcal{B}}(X_1 \dots X_n).$$

__1er point :__ $\displaystyle\prod_{i=1}^{n} x_{i\sigma(i)}$?

Ex: $n = 4$; σ:

$$\begin{aligned} 1 &\longmapsto 2 \\ 2 &\longmapsto 4 \\ 3 &\longmapsto 1 \\ 4 &\longmapsto 3 \end{aligned}$$

$i \setminus \sigma(i)$	1	2	3	4
1		X		
2				X
3	X			
4			X	

Réciproquement :

i \ $\sigma(i)$	1	2	3	4
1	X			
2				X
3		X		
4			X	

correspond à :

$$\sigma : \begin{aligned} 1 &\longmapsto 1 \\ 2 &\longmapsto 4 \\ 3 &\longmapsto 2 \\ 4 &\longmapsto 3 \end{aligned}$$

i \ $\sigma(i)$	1	2	3	4
1		X		
2				X
3	X			
4			X	

$$\begin{pmatrix} x_{11} & \cancel{x_{12}} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & \cancel{x_{24}} \\ \cancel{x_{31}} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & \cancel{x_{43}} & x_{44} \end{pmatrix}$$

$$\sigma : \begin{array}{l} 1 \longmapsto 2 \\ 2 \longmapsto 4 \\ 3 \longmapsto 1 \\ 4 \longmapsto 3 \end{array}$$

$$\prod_{i=1}^{4} x_{i\sigma(i)} = x_{12} \times x_{24} \times x_{31} \times x_{43}$$

$\underline{Ex}$: $\underline{\underline{n=2}}$:

Si $X_1, X_2 \in E$, $\dim E = 2$:

$$\mathrm{Mat}_{\mathcal{B}}(X_1, X_2) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

$$S_2 = \{ id, \tau_{12} \}, \quad \varepsilon(id) = 1, \quad \varepsilon(\tau_{12}) = -1.$$

Ac : $$\det_{\mathcal{B}}(X_1, X_2) = + x_{11} x_{22} - x_{12} \cdot x_{21}$$

Rq : $$\det_{\mathcal{B}}(X_1, X_2) = \det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

__Ex. $n=3$:__ $X_1, X_2, X_3 \in E$, $\dim E = 3$.

$$\mathrm{Mat}_{\mathcal{B}}(X_1, X_2, X_3) = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

$$S_3 = \left\{ id, \tau_{12}, \tau_{13}, \tau_{23}, (1\ 2\ 3), (1\ 3\ 2) \right\}$$

$$\varepsilon(id) = \varepsilon(1\ 2\ 3) = \varepsilon(1\ 3\ 2) = 1$$

$$\varepsilon(\tau_{12}) = \varepsilon(\tau_{13}) = \varepsilon(\tau_{23}) = -1$$

$$\det_{\mathcal{B}}(X_1, X_2, X_3) = x_{11}x_{22}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{32}x_{21}$$

$$- x_{12}x_{21}x_{33} - x_{13}x_{22}x_{31} - x_{11}x_{23}x_{32}$$

C'est la formule de Sarrus.

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{11} & x_{12} \\ x_{21} & x_{22} & x_{23} & x_{21} & x_{22} \\ x_{31} & x_{12} & x_{33} & x_{31} & x_{12} \end{pmatrix}$$

$$\det_{\mathcal{B}}(X_1, X_2, X_3) = x_{11}x_{22}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{12}$$

$$- x_{31}x_{22}x_{13} - x_{12}x_{23}x_{11} - x_{33}x_{21}x_{12}.$$

Dimensions supérieures : trop compliqué.

Rq :

$$\det_{\mathcal{B}}(X_1 \dots X_n) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} x_{i\sigma(i)}$$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} x_{\sigma(i)i} \ .$$