2.2 Matrice d'une application linéaire :

Applications, composition

<u>Prop:</u> E $\mathbb{K}$ev de base $\mathcal{B} = (e_1 \ldots e_p)$

F $\mathbb{K}$ev de base $\mathcal{C} = (f_1 \ldots f_n)$.

$u \in \mathcal{L}(E, F)$

$x \in E$.

$$\mathrm{Mat}_{\mathcal{C}}(u(x)) = (\mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u) \times (\mathrm{Mat}_{\mathcal{B}}\, x)$$

<u>Ex</u> : • $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 2x - y \\ -x + y \end{pmatrix}$$

$\mathcal{C}$ = base canonique.

$$\mathrm{Mat}_{\mathcal{C}}(u) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$X = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$: $\mathrm{Mat}_{\mathcal{C}}(X) = \begin{pmatrix} x \\ y \end{pmatrix}$

$$(\mathrm{Mat}_{\mathcal{C}}\, u) \times (\mathrm{Mat}_{\mathcal{C}}\, X) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 2x - y \\ -x + y \end{pmatrix} = \mathrm{Mat}_{\mathcal{C}}\, u(X)$$

$$\mathcal{B}_1 = \left(\overbrace{\begin{pmatrix}1\\0\end{pmatrix}}^{e_1}, \overbrace{\begin{pmatrix}1\\1\end{pmatrix}}^{e_2}\right) \; ; \quad \mathcal{B}_2 = \left(\underbrace{\begin{pmatrix}1\\-1\end{pmatrix}}_{f_1}, \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{f_2}\right).$$

$$u(e_1) = \begin{pmatrix}2\\-1\end{pmatrix}$$

$$= \underline{2} f_1 + \underline{\ } f_2$$

$$u(e_2) = \begin{pmatrix}1\\0\end{pmatrix} = \underline{\ } f_1 + \underline{\ } f_2$$

dc : $\operatorname{Mat}_{\mathcal{B}_1, \mathcal{B}_2}(u) = \begin{pmatrix} \underline{2} & \underline{1} \\ \underline{1} & \underline{1} \end{pmatrix}$

Soit $x = -e_1 + 2e_2 = \begin{pmatrix}1\\2\end{pmatrix}$

$$u(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = f_2$$

dc : $u(-e_1 + 2e_2) = f_2$

ie : $\mathrm{Mat}_{\mathcal{B}_2}\left(u(-e_1 + 2e_2)\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

or : $\mathrm{Mat}_{\mathcal{B}_1}(-e_1 + 2e_2) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

et : $\left(\mathrm{Mat}_{\mathcal{B}_1,\mathcal{B}_2}(u)\right) \times \left(\mathrm{Mat}_{\mathcal{B}_1}(-e_1 + 2e_2)\right)$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathrm{Mat}_{\mathcal{B}_2}(u(x)) .$$

Rq :

$$\begin{array}{ccc} x & & u(x) \\ E, \mathcal{B} & \xrightarrow[M]{u} & F, \mathcal{C} \\ X & & M \cdot X \end{array}$$

$$M = \mathrm{Mat}_{\mathcal{B}, \mathcal{C}}\, u \qquad MX = \mathrm{Mat}_{\mathcal{C}}\, u(x)$$

$$X = \mathrm{Mat}_{\mathcal{B}}\, x$$

$$\mathrm{Mat}_{\mathcal{C}}\, u(x) = \left(\mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u\right) \times \left(\mathrm{Mat}_{\mathcal{B}}\, x\right)$$

<u>Démo :</u>

$$x = \sum_{i=1}^{p} x_i e_i$$

$$u(x) = u\left(\sum_{j=1}^{p} x_j e_j\right)$$

$$= \sum_{j=1}^{p} x_j\, u(e_j)$$

$$M = \mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u = (m_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le p}}$$

Alors : $u(e_j) = \begin{pmatrix} m_{1j} \\ m_{2j} \\ \vdots \\ m_{nj} \end{pmatrix}$ ds $\mathcal{C}$

$$= \sum_{i=1}^{n} m_{ij} f_i$$

donc : $u(x) = \sum_{j=1}^{p} \sum_{i=1}^{n} x_j m_{ij} f_i$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{p} m_{ij} x_j \right) f_i$$

d'où $\mathrm{Mat}_{\mathcal{C}}\, u(x) = \left(\sum_{j=1}^{p} m_{ij} x_j \right)_{1 \leq i \leq n}$

Si $X = (x_i)_{1 \leq i \leq n} = \mathrm{Mat}_{\mathcal{B}}\, x$,

$$MX = \left(\sum_{j=1}^{p} m_{ij} x_j \right)_{1 \leq i \leq n}$$

dc : $\mathrm{Mat}_{\mathcal{C}}\, u(x) = MX$

$= \mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u \times \mathrm{Mat}_{\mathcal{B}}\, x$.

Th : E $\mathbb{K}$-ev. de base $\mathcal{B} = (e_1 \dots e_p)$

F $\mathbb{K}$-ev de base $\mathcal{C} = (f_1 \dots f_n)$

G $\mathbb{K}$-ev de base $\mathcal{D} = (g_1 \dots g_q)$

Soit $u \in \mathcal{L}(E, F)$, $v \in \mathcal{L}(F, G)$

dc : $v \circ u \in \mathcal{L}(E, G)$

Alors : $\boxed{\mathrm{Mat}_{\mathcal{B},\mathcal{D}}(v \circ u) = (\mathrm{Mat}_{\mathcal{C},\mathcal{D}}\, v) \times (\mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u)}$

Rq :

$$\begin{tikzcd}
 & F, \mathcal{C} \arrow[dr, "v", "N"'] & \\
E, \mathcal{B} \arrow[ur, "u", "M"'] \arrow[rr, "v \circ u", "N \times M"'] & & G, \mathcal{D}
\end{tikzcd}$$

$$M = \underset{\mathcal{B},\mathcal{C}}{\mathrm{Mat}}\ u \ , \quad N = \underset{\mathcal{C},\mathcal{D}}{\mathrm{Mat}}\ v$$

$$\underset{\mathcal{B},\mathcal{D}}{\mathrm{Mat}} (v \circ u) = N \times M$$

$$\mathrm{Mat}_{\mathcal{B},\mathcal{D}}\, v \circ u = \left(\mathrm{Mat}_{\mathcal{C},\mathcal{D}}\, v\right) \times \left(\mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u\right)$$

<u>Ex</u> :

$$u : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x-y \\ x+y \\ x \end{pmatrix}$$

$$v : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x-y+z \\ x+z \end{pmatrix}$$

Calculons $v \circ u$:

$$\forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \quad v \circ u \begin{pmatrix} x \\ y \end{pmatrix} = v\left(u\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$= v \begin{pmatrix} x - y \\ x + y \\ x \end{pmatrix}$$

$$= \begin{pmatrix} (x-y) - (x+y) + x \\ (x-y) + x \end{pmatrix}$$

$$= \begin{pmatrix} x - 2y \\ 2x - y \end{pmatrix}.$$

Avec des matrices : $\mathcal{B}$: base canonique de $\mathbb{R}^2$

$\mathcal{C}$: ——— $\mathbb{R}^3$.

$$\mathrm{Mat}_{\mathcal{B},\mathcal{B}}(v \circ u) = \left(\mathrm{Mat}_{\mathcal{C},\mathcal{B}}\, v\right) \times \left(\mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u\right)$$

$$= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

dc : $v \circ u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x - 2y \\ 2x - y \end{pmatrix}$$

Ex : $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x - y \\ y \end{pmatrix}$$

$v : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} y \\ 2x + y \end{pmatrix}$$

$$\mathcal{B}_1 = \Big(\underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{e_2}\Big) \; ; \; \mathcal{B}_2 = \Big(\underbrace{\begin{pmatrix}1\\-1\end{pmatrix}}_{f_1}, \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{f_2}\Big)$$

$$\mathcal{B}_3 = \Big(\underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{g_1}, \underbrace{\begin{pmatrix}-1\\1\end{pmatrix}}_{g_2}\Big)$$

$$u(e_1) = u\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = 0\cdot f_1 + 1\cdot f_2$$

$$u(e_2) = u\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = f_1 + f_2$$

d.h. $\mathrm{Mat}_{\mathcal{B}_1, \mathcal{B}_2}(u) = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$

$$v(f_1) = v\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} = 0\cdot g_1 + 1\cdot g_2$$

$$v(f_2) = v\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix} = 2\cdot g_1 + 0\cdot g_2$$

$$\text{dc}: \quad \mathrm{Mat}_{\mathcal{B}_2,\mathcal{B}_3}(v) = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

$$v \circ u(e_1) = v(f_2) = 2g_1 + 0 \cdot g_2$$

$$v \circ u(e_2) = v(f_1 + f_2)$$

$$= v(f_1) + v(f_2) = 2g_1 + g_2$$

$$\text{dc}: \quad \mathrm{Mat}_{\mathcal{B}_1,\mathcal{B}_3}(v \circ u) = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\left(\mathrm{Mat}_{\mathcal{B}_2,\mathcal{B}_3} v\right) \times \left(\mathrm{Mat}_{\mathcal{B}_1,\mathcal{B}_2} u\right) = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} = \mathrm{Mat}_{\mathcal{B}_1, \mathcal{B}_3} v \circ u$$

Cor. 2.2.14 : Si $v_1 \dots v_q$ sont des vecteurs de E, alors :

$$\mathrm{Mat}_{\mathcal{C}}(u(v_1) \dots u(v_p)) = \mathrm{Mat}_{\mathcal{B},\mathcal{C}} u \times \mathrm{Mat}_{\mathcal{B}} (v_1 \dots v_p).$$

Démo du th. 2.2.15 :

$$\mathrm{Mat}_{\mathcal{B},\mathcal{D}} (v \circ u) = \mathrm{Mat}_{\mathcal{D}} \left(v(u(e_1)), \dots, v(u(e_p)) \right)$$

$$\underset{2.2.14}{=} \left(\mathrm{Mat}_{\mathcal{C},\mathcal{D}}\, v\right) \times \left(\mathrm{Mat}_{\mathcal{C}}\left(u(e_1) \cdots u(e_p)\right)\right)$$

$$\underset{2.2.14}{=} \left(\mathrm{Mat}_{\mathcal{C},\mathcal{D}}\, v\right) \times \left[\left(\mathrm{Mat}_{\mathcal{B},\mathcal{C}}(u)\right) \times \mathrm{Mat}_{\mathcal{D}}(e_1 \cdots e_p)\right]$$

Mais : $\mathrm{Mat}_{\mathcal{D}}(e_1 \cdots e_p) = \mathrm{Mat}_{\mathcal{B}}(\mathcal{D})$

Or : $e_1 = 1 \times e_1 + 0 \times e_2 + \cdots + 0 \times e_p$

$e_2 = 0 \times e_1 + 1 \times e_2 + \cdots + 0 \times e_p$

$\vdots$

d'où $\mathrm{Mat}_{\mathcal{D}}(\mathcal{D}) = I_p$

donc : $\mathrm{Mat}_{\mathcal{B},\mathcal{D}}(v \circ u) = \mathrm{Mat}_{\mathcal{C},\mathcal{D}}\, v \times \mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u \times I_p$

$= \mathrm{Mat}_{\mathcal{C},\mathcal{D}}\, v \times \mathrm{Mat}_{\mathcal{B},\mathcal{C}}\, u.$

$\square$