

Feuille d'exercice n° 17 : Développements limités - fiche d'entraînement - correction

Exercice 1

- 1) $\lim_{n \rightarrow +\infty} \frac{\sqrt{n^2+1}}{n^3} = 0$ donc $n^3 - \sqrt{n^2+1} \underset{n \rightarrow +\infty}{\sim} n^3$. De même, $\ln n - 2n^2 \underset{n \rightarrow +\infty}{\sim} -2n^2$, donc $u_n \underset{n \rightarrow +\infty}{\sim} \frac{n^3}{-2n^2} = -n/2$.
- 2) $\ln(n^2+1) = \ln n^2 + \ln(1+1/n^2) = 2\ln n + \ln(1+1/n^2) \underset{n \rightarrow +\infty}{\sim} 2\ln n$ et $n+1 \underset{n \rightarrow +\infty}{\sim} n$ donc $v_n \underset{n \rightarrow +\infty}{\sim} \frac{2\ln n}{n}$.
- 3) $w_n = \frac{n\sqrt{n+1/n+1/n^2}}{n^{2/3}\sqrt[3]{1-1/n+1/n^2}} \underset{n \rightarrow +\infty}{\sim} \frac{n}{n^{2/3}} = n^{1/3}$.
- 4) $\cos(1/n) = 1 - \frac{1}{2n^2} + o(1/n^2)$ et $e^{1/n} = 1 + 1/n + o(1/n)$, donc :
 $\cos(1/n) - e^{1/n} = -1/n + o(1/n) \underset{n \rightarrow +\infty}{\sim} 1/n$.
- 5) $y_n \underset{n \rightarrow +\infty}{\sim} \frac{1/n^2}{1/n} \underset{n \rightarrow +\infty}{\sim} -1/n$.
- 6) $\ln(1+\sin(1/n)) \underset{n \rightarrow +\infty}{\sim} \sin(1/n)$ car $\sin(1/n) \xrightarrow{n \rightarrow +\infty} 0$, et $\sin(1/n) \underset{n \rightarrow +\infty}{\sim} 1/n$, donc :
 $\ln(1+\sin(1/n)) \underset{n \rightarrow +\infty}{\sim} 1/n$. Enfin, $1 - \sqrt{1+1/n} = 1 - (1 + 1/(2n) + o(1/n)) \underset{n \rightarrow +\infty}{\sim} -1/(2n)$, d'où
 $z_n \underset{n \rightarrow +\infty}{\sim} -2$.

Exercice 2

- 1) $1 - x + \frac{2}{3}x^2 - \frac{11}{24}x^3 + o(x^3)$
- 2) $\frac{1}{3}x + \frac{1}{90}x^3 + o(x^3)$
- 3) $1 + \frac{1}{2}(x-1) - \frac{1}{12}(x-1)^2 + o((x-1)^2)$
- 4) $\frac{2}{3}x + \frac{1}{90}x^3 + o(x^3)$
- 5) $1 - \frac{1}{2}x - \frac{1}{12}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 - \frac{1}{240}x^5 + o(x^5)$
- 6) $\sum_{k=0}^{999} \frac{x^k}{k!} = e^x - \frac{x^{1000}}{1000!} + o(x^{1000})$ d'où :
 $\ln\left(\sum_{k=0}^{999} \frac{x^k}{k!}\right) = \ln(e^x - \frac{x^{1000}}{1000!} + o(x^{1000})) = \ln(e^x) + \ln(1 - \frac{x^{1000}e^{-x}}{1000!} + o(x^{1000})) = x - \frac{x^{1000}}{1000!} + o(x^{1000})$.

Exercice 3

$$6) \ln\left(\frac{1}{\cos x}\right) = \left(\frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + o(x^7)\right)$$

$$7) \ln(1 + \operatorname{ch} x) = (\ln(2) + \frac{1}{4}x^2 - \frac{1}{96}x^4 + o(x^4))$$

$$8) \ln(\tan x) = (2(x - 1/4\pi) + \frac{4}{3}(x - 1/4\pi)^3 + o((x - 1/4\pi)^4))$$

$$9) \arctan(e^x) = (1/4\pi + \frac{1}{2}x - \frac{1}{12}x^3 + o(x^3))$$

$$10) \arctan(2 \sin x) = (1/3\pi + \frac{1}{4}(x - 1/3\pi) - 3/16\sqrt{3}(x - 1/3\pi)^2 + \frac{3}{16}(x - 1/3\pi)^3 + o((x - 1/3\pi)^3))$$