$f(x) \xrightarrow[0]{} 0$

Ex.21 : 1) f est continue en 0 et $f(0)=0$.

Soit $\varepsilon>0$, il existe $\alpha>0$ tq. $\forall t\in[0,\alpha[$, $|f(t)|<\varepsilon$.

Si $0<x<\frac{\alpha}{b}$, alors $\forall t\in[ax,bx]$, $|f(t)|<\varepsilon$

donc $$\left|\int_{ax}^{bx}\frac{f(t)}{t}\,dt\right| \leq \int_{ax}^{bx}\frac{|f(t)|}{t}\,dt \quad \text{(i.t.)}$$

$$\leq \int_{ax}^{bx}\frac{\varepsilon}{t}\,dt$$

$$\leq \varepsilon\left(\ln(bx)-\ln(ax)\right)$$

$$\leq \varepsilon\underbrace{(\ln b-\ln a)}_{=C^{te}}$$

donc : $$\left|\int_{ax}^{bx}\frac{f(t)}{t}\,dt\right| \xrightarrow[x\to 0]{} 0$$

2) $f(0) =$ 1 val. quelconque.

On a : $\displaystyle\int_{ax}^{bx} \frac{f(0)}{t}\,dt = f(0)(\ln b - \ln a)$

de : $\displaystyle\int_{ax}^{bx} \frac{f(t) - f(0)}{t}\,dt = \int_{ax}^{bx} \frac{f(t)}{t}\,dt - f(0)\ln\frac{b}{a}$ (A)

On pose $g = f - f(0)$

alors g est continue, et $g(0) = 0$

avec le qu. 1 : $\displaystyle\int_{ax}^{bx} \frac{g(t)}{t}\,dt \xrightarrow[x \to 0]{} 0$

$$\int_{a_n}^{b_n} \frac{f(t)}{t}\,dt = \int_{a_n}^{b_n} \frac{g(t)}{t}\,dt + f(0)\ln\frac{b}{a}$$

$$\xrightarrow[n\to 0]{} f(0)\ln\frac{b}{a}.$$