Feuille d'exercice n° 18 : **Analyse asymptotique - correction**

Exercice 5 1) $\dfrac{1}{n\sqrt{n}}$; 2) $\left(\pi - \dfrac{\pi^2}{2}\right)\dfrac{1}{n^2}$; 3) 4 ; 4) $\dfrac{\mathrm{e}^{-n}}{2}$; 5) $\dfrac{\pi}{\sqrt{n}}$; 6) $-\dfrac{1}{n}$; 7) $\dfrac{8}{n^2}$; 8) $n\mathrm{e}^{1-n}$; 9) 1 ; 10) $\dfrac{1}{4n^2}$; 11) e^{1-n}

Exercice 14

1) 0

2) $\dfrac{1}{6}$

3) $\mathrm{e}^{\mathrm{e}^{-1}}$

4) e^{-1}

5) $\dfrac{2}{3}$

6) $\dfrac{a^3}{b^3}$

7) -1

8) $-\dfrac{1}{2\sqrt{2}}$

9) $\sqrt{\mathrm{e}}$

10) $\dfrac{1}{\pi}$

11) 1

12) $\dfrac{\sqrt{2}}{8x^3}$

13) $\dfrac{x^2}{2}$

14) $-\dfrac{3}{2}\left(x - \frac{\pi}{4}\right)$

15) $-x$

Exercice 15 $\lim\limits_{x\to 1} \dfrac{x^x - 1}{\ln x} = 1$

$$\lim_{x\to 0} \left(\frac{x^2}{\ln(\cos x)} + \frac{2}{x^2}\sin^2 x\right) = 0$$

$$\lim_{x\to \frac{\pi}{4}} \frac{\ln(\sin^2 x)}{(\frac{\pi}{2} - x)^2} = -\frac{16\ln 2}{\pi^2}$$

$$\lim_{x\to \frac{\pi}{2}} \frac{\ln(\sin^2 x)}{(\frac{\pi}{2} - x)^2} = -1$$

$$\lim_{x\to +\infty} \sin\frac{1}{x}\tan\left(\frac{2\pi x}{4x+3}\right) = \frac{8}{3\pi}$$

$\lim\limits_{x\to 0^+} \ln x \tan(\ln(1+x)) = 0$

$\lim\limits_{x\to \mathrm{e}} (\ln x)^{\tan \frac{\pi x}{2e}} = \mathrm{e}^{-\frac{2}{\pi}}$

Exercice 17

1) $\ln(\cos x) = -\dfrac{1}{2}x^2 - \dfrac{1}{12}x^4 - \dfrac{1}{45}x^6 + o\left(x^6\right)$

2) $\tan(x) = x + \dfrac{1}{3}x^3 + \dfrac{2}{15}x^5 + o\left(x^5\right)$

3) $\sin(\tan(x)) = x + \dfrac{1}{6}x^3 - \dfrac{1}{40}x^5 + o\left(x^5\right)$

4) $(\ln(1+x))^2 = x^2 - x^3 + \dfrac{11}{12}x^4 + o\left(x^4\right)$

5) $\exp(\sin(x)) = 1 + x + \dfrac{1}{2}x^2 + o\left(x^3\right)$

6) $\sin^6(x) = x^6 - x^8 + o\left(x^9\right)$

Exercice 18

1) $\sqrt{x+1} = \sqrt{x} + \dfrac{1}{2\sqrt{x}} - \dfrac{1}{8x^{3/2}} + o\left(\dfrac{1}{x^{3/2}}\right)$;

2) $x\ln(x+1) - (x+1)\ln x = 1 - \ln(x) - \dfrac{1}{2x} + \dfrac{1}{3x^2} + o\left(\dfrac{1}{x^2}\right)$;

3) $\left(\dfrac{x+1}{x}\right)^x = \mathrm{e} - \dfrac{1}{2}\dfrac{\mathrm{e}^1}{x} + \dfrac{11}{24}\dfrac{\mathrm{e}}{x^2} + o\left(\dfrac{1}{x^2}\right)$;

4) $\operatorname{Arctan} x = \dfrac{\pi}{2} - \dfrac{1}{x} + \dfrac{1}{3x^3} + o\left(\dfrac{1}{x^4}\right)$.

Exercice 19

1) $\dfrac{\arctan x - x}{\sin x - x} = 2 - \dfrac{11}{10}x^2 + o(x^2)$;

2) $\ln\sin x = \ln(\sqrt{2}/2) + (x-\pi/4) - (x-\pi/4)^2 + \dfrac{2}{3}(x-\pi/4)^3 + o((x-\pi/4)^3)$;

3) $(1+x)^{\frac{1}{x}} = e\left(1 - \dfrac{1}{2}x + \dfrac{11}{24}x^2 - \dfrac{7}{16}x^3\right) + o(x^3)$;

4) $x(\sqrt{x^2+\sqrt{x^4+1}} - x\sqrt{2}) = \dfrac{\sqrt{2}}{8x^2} + o\left(\dfrac{1}{x^2}\right)$.

Exercice 22

1) $\sqrt[3]{x^3+x^2} - \sqrt[3]{x^3-x^2} = x\left(\sqrt{1+\frac{1}{x}} - \sqrt{1-\frac{1}{x}}\right) = 1 + \dfrac{1}{8x^3} + o(1/x^3)$.

2) $\ln\left(\sqrt{1+x}\right) = \frac{1}{2}\ln x + \frac{1}{2x} + o(1/x)$.

Exercice 23

1) **a)** $\dfrac{\cos x}{\sqrt{1+x}} = 1 - \dfrac{x}{2} - \dfrac{x^2}{8} - \dfrac{x^3}{16} + \dfrac{49x^4}{384} + o(x^4)$

b) $\dfrac{\sqrt{1+x}}{\cos x} = 1 + \dfrac{x}{2} + \dfrac{3x^2}{8} + \dfrac{5x^3}{16} + \dfrac{41x^4}{384} + o(x^4)$

c) $\dfrac{\ln(1+x)}{\cos x} = x - \dfrac{x^2}{2} + \dfrac{5x^3}{6} - \dfrac{x^4}{2} + o(x^4)$

d) $\dfrac{1+\cos x}{2+\sin x} = 1 - \dfrac{x}{2} + \dfrac{x^3}{12} - \dfrac{x^4}{16} + o(x^4)$

e) $\dfrac{\sin(x/2)}{\mathrm{e}^{2x}} = \dfrac{x}{2} - x^2 + \dfrac{47x^3}{48} - \dfrac{5x^4}{8} + o(x^4)$

f) $\dfrac{\ln(1+x)}{2-\cos x} = x - \dfrac{x^2}{2} - \dfrac{x^3}{6} + o(x^4)$

2) **a)** $\dfrac{\sin(2x-\pi/4)}{\cos x} = 1 + 3(x-\dfrac{\pi}{4}) + \dfrac{3}{2}(x-\dfrac{\pi}{4})^2 + \dfrac{3}{2}(x-\dfrac{\pi}{4})^3 + \dfrac{19}{8}(x-\dfrac{\pi}{4})^4 + o((x-\dfrac{\pi}{4})^4)$

b) $\dfrac{\cos(x-1)}{\ln(1+x)} = \dfrac{1}{\ln 2} - \dfrac{1}{2(\ln 2)^2}(x-1) + \dfrac{1}{\ln 2}\left(-\dfrac{1}{2} + \dfrac{1}{8\ln 2} + \dfrac{1}{4(\ln 2)^2}\right)(x-1)^2 + o((x-1)^2)$

c) $\dfrac{\mathrm{e}^{x-1}}{\ln x} = \dfrac{1}{x-1} + \dfrac{3}{2} + \dfrac{11}{12}(x-1) + \dfrac{3}{8}(x-1)^2 + \dfrac{71}{720}(x-1)^3 + \dfrac{41}{1440}(x-1)^4 + o((x-1)^4)$

Exercice 24

1) $\mathrm{e}-1$

2) $-\dfrac{1}{2}$

3) $-\dfrac{\mathrm{e}}{2}$

4) $\dfrac{1}{12}$

5) $-\dfrac{1}{6}$

6) $-\dfrac{1}{2}$

Exercice 28 En 0 : $\ln(1+x+x^2) = x + \dfrac{x^2}{2} + o(x^2)$.

En 1 : posons $h = x-1$, donc $1+x+x^2 = 1+(1+h)+(1+h)^2 = 3+3h+h^2 = 3\Big(1+h+\frac{h^2}{3}\Big)$. Et on développe quand $h \to 0$. On obtient :

$\ln(1+x+x^2) = \ln 3 + (x-1) - \dfrac{(x-1)^2}{6}$.

La tangente en 1 pour équation $y = \ln 3 + (x-1)$ et au voisinage de 1 elle se trouve au-dessus du graphe de f.