3.1 - Déterminant d'une famille de vecteurs : définition.

2ème partie : démonstration du théorème 3.1.2

(i) $\det_{\mathcal{B}}$ est 1 forme n-linéaire alternée :

$X_1 \ldots X_n, Y_1$ des vecteurs de E ; Soit $\lambda \in \mathbb{K}$.

$\forall j : \quad X_j = \sum_{i=1}^{n} x_{ij} e_i$

$Y_1 = \sum_{i=1}^{n} y_i e_i$

- Mq. $\det_{\mathcal{B}}$ est linéaire par rapport à la 1ère variable :

Mq : $\det_{\mathcal{B}}(\overbrace{X_1+\lambda Y_1}^{Z_1}, \underset{\sim}{\overset{Z_2}{X_2}}, \dots, \underset{\sim}{\overset{Z_n}{X_n}})$

$$= \det_{\mathcal{B}}(X_1 \dots X_n) + \lambda \det_{\mathcal{B}}(Y_1, X_2 \dots X_n)$$

$$\forall j, \; Z_j = \sum_{i=1}^{n} z_{ij} \, e_i$$

Si $j=1$: $\quad z_{i1} = x_{i1} + y_i$

Sinon : $\quad z_{ij} = x_{ij}$.

$$d = \det_{\mathcal{B}}(Z_1 \dots Z_n) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} z_{\sigma(i),i}$$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma) \times \left(z_{\sigma(1),1}\right) \times \prod_{i=2}^{n} z_{\sigma(i),i}$$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma) \times \left(x_{\sigma(1),1} + \lambda y_{\sigma(1)}\right) \prod_{i=2}^{n} x_{\sigma(i),i}$$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma)\, x_{\sigma(1)\,1} \prod_{i=2}^{n} x_{\sigma(i),i}$$

$$+ \lambda \sum_{\sigma \in S_n} \varepsilon(\sigma)\, y_{\sigma(1)} \prod_{i=2}^{n} x_{\sigma(i)\,i}$$

$$= \det_{\mathcal{B}}(X_1 \dots X_n) + \lambda \det_{\mathcal{B}}(Y_1, X_2, \dots, X_n).$$

- <u>$\det_{\mathcal{B}}$ est alternée :</u>

Si $X_1 = X_2$, mq. $\det_{\mathcal{B}}(X_1, X_2 \dots X_n) = 0$

$$\det_{\mathcal{B}}(X_1, X_2, \dots X_n) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} x_{i\,\sigma(i)}$$

$$= \sum_{\sigma \in A_n} \prod_{i=1}^{n} x_{i\,\sigma(i)} - \sum_{\sigma \in S_n \setminus A_n} \prod_{i=1}^{n} x_{i\,\sigma(i)}.$$

$\underline{Q}$: τ : transposition qui échange 1 et 2.

$$\varphi : \begin{array}{ccc} S_n \setminus A_n & \longrightarrow & A_n \\ \sigma & \longmapsto & \tau \circ \sigma \end{array}$$

φ est bijective. (son inverse : $\begin{array}{ccc} A_n & \longrightarrow & S_n \setminus A_n \\ \sigma & \longmapsto & \tau \circ \sigma \end{array}$)

dc : $$\sum_{\sigma \in S_n \setminus A_n} \prod_{i=1}^{n} x_{i\,\sigma(i)} = \sum_{\sigma \in A_n} \prod_{i=1}^{n} x_{i,\tau\circ\sigma(i)}$$

Si $\sigma \in A_n$:

$$\prod_{i=1}^{n} x_{i,\tau\circ\sigma(i)} = x_{k,\tau(1)} \times x_{\ell,\tau(2)} \times \prod_{\substack{i=1 \\ i \neq k \\ i \neq \ell}}^{n} x_{i,\tau\circ\sigma(i)}$$

[avec : k tq. $\sigma(k)=1$, ℓ tq. $\sigma(\ell)=2$]

$$= x_{k,2} \times x_{l,1} \times \prod_{\substack{i=1 \\ i\neq k \\ i\neq l}}^{n} x_{i,\sigma(i)}$$

$$= x_{k,1} \times x_{l,2} \times \prod_{\substack{i=1 \\ i\neq k \\ i\neq l}}^{n} x_{i,\gamma(i)}$$

$$= \prod_{i=1}^{n} x_{i,\sigma(i)}$$

d'où : $\det_{\mathcal{B}}(X_1, X_2 \dots X_n) = 0$.

(i) <u>et (iii)</u> : Soit $f \in \mathcal{A}_n(E)$.

<u>cas $n=3$</u>, $\mathcal{B} = (e_1, e_2, e_3)$.

$$\forall j \in [\![1,3]\!] : \quad X_j = \begin{pmatrix} x_{1j} \\ x_{2j} \\ x_{1j} \end{pmatrix}$$

$$\begin{aligned} f(X_1, X_2, X_3) = f(&x_{11}e_1 + x_{21}e_2 + x_{31}e_3, \\ &x_{12}e_1 + x_{22}e_2 + x_{32}e_3, \\ &x_{13}e_1 + x_{23}e_2 + x_{33}e_3) \end{aligned}$$

$$\begin{aligned} &= x_{11} f\underline{(e_1, X_2, X_3)} + x_{21} f(e_2, X_2, X_3) \\ &\quad + x_{31} f(e_3, X_2, X_3). \end{aligned}$$

$$f(e_1, X_2, X_3) = f(e_1, x_{12}e_1 + x_{22}e_2 + x_{32}e_3, X_3)$$

$$= x_{12} \underbrace{f(e_1, e_1, X_3)}_{=0} + x_{22} f(e_1, e_2, X_3)$$

$$+ x_{32} f(e_1, e_3, X_3)$$

Or :

$$x_{22} f(e_1, e_2, X_3)$$

$$= x_{22} f(e_1, e_2, \cancel{x_{13} e_1} + \cancel{x_{23} e_2} + x_{33} e_3)$$

$$= x_{22} x_{33} f(e_1, e_2, e_3)$$

de m̂ : $f(e_1, e_3, X_3) = x_{23} f(e_1, e_3, e_2)$

dc : $f(e_1, X_2, X_3) = x_{22} x_{33} f(e_1, e_2, e_3)$

$$+ x_{32} x_{23} f(e_1, e_3, e_2)$$

$$= x_{22}\, x_{33}\, f(e_1, e_2, e_3) - x_{32}\, x_{23}\, f(e_1, e_2, e_3)$$

$$f(X_1, X_2, X_3) = f(\underbrace{e_1, e_2, e_3}_{\mathcal{B}}) \times \Big[x_{11}\left(x_{22}x_{33} - x_{32}\, x_{23}\right)$$

$$+ x_{21}\left(-x_{12}x_{33} + x_{32}\, x_{13}\right)$$

$$+ x_{31}\left(x_{12}x_{23} - x_{22}\, x_{13}\right)\Big]$$

$$= f(\mathcal{B}) \quad \det_{\mathcal{B}}(X_1, X_2, X_3)$$

dc : $f = \underbrace{f(\mathcal{B})}_{\in \mathbb{K}} \times \det_{\mathcal{B}}$.

$$\cdot \ \det_{\mathcal{B}}(\mathcal{B}) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} x_{i\sigma(i)}$$

où : $\forall j, \quad e_j = \sum_{i=1}^{n} x_{ij} e_i$.

Mais : $e_j = 1 \times e_j + 0 \times$ les autres e_i

donc $\forall i, j$: $\quad x_{ij} = \delta_{ij} = \begin{cases} 0 & \text{si } i \neq j \\ 1 & \text{si } i = j \end{cases}$.

Soit $\sigma \in S_n$, et si $\sigma \neq id$, il existe

$i_0 \in [\![1, n]\!]$, $\quad \sigma(i_0) \neq i_0$.

donc : $\quad \displaystyle\prod_{i=1}^{n} x_{i\sigma(i)} = \underbrace{x_{i_0, \sigma(i_0)}}_{=0} \times \prod_{\substack{i=1 \\ i \neq i_0}}^{n} x_{i\sigma(i)}$

$= 0$

$$\det_{\mathcal{B}} \mathcal{B} = \varepsilon(id) \prod_{i=1}^{n} x_{i,id(i)}$$

$$= \prod_{i=1}^{n} \underbrace{x_{i,i}}_{\neq 0} = 1.$$

dc : $\det_{\mathcal{B}} \neq 0$ dc : $A_n(E) = \mathrm{Vect}\, \det_{\mathcal{B}}$

est de dimension 1.