Feuille d'exercice n° 18 : **Analyse asymptotique - correction**

Exercice 14

1) 0

2) $\frac{1}{6}$

3) $e^{e^{-1}}$

4) e^{-1}

5) $\frac{2}{3}$

6) $\frac{\sqrt{2}}{8x^3}$

7) $\frac{a^3}{b^3}$

8) -1

9) $-\frac{1}{2\sqrt{2}}$

10) $\frac{x^2}{2}$

11) $-\frac{3}{2}\left(x-\frac{\pi}{4}\right)$

12) $\sqrt{e}$

13) $\frac{1}{\pi}$

14) 1

15) $-x$

Exercice 15 $\lim\limits_{x\to 1}\frac{x^x-1}{\ln x}=1$

$$\lim_{x\to 0}\left(\frac{x^2}{\ln(\cos x)}+\frac{2}{x^2}\sin^2 x\right)=0$$
$$\lim_{x\to\frac{\pi}{4}}\frac{\ln(\sin^2 x)}{(\frac{\pi}{2}-x)^2}=-\frac{16\ln 2}{\pi^2}$$
$$\lim_{x\to\frac{\pi}{2}}\frac{\ln(\sin^2 x)}{(\frac{\pi}{2}-x)^2}=-1$$
$$\lim_{x\to+\infty}\sin\frac{1}{x}\tan\left(\frac{2\pi x}{4x+3}\right)=\frac{8}{3\pi}$$
$$\lim_{x\to 0^+}\ln x\tan(\ln(1+x))=0$$
$$\lim_{x\to e}(\ln x)^{\tan\frac{\pi x}{2e}}=e^{-\frac{2}{\pi}}$$

Exercice 17

1) $\ln(\cos x) = -\dfrac{1}{2}x^2 - \dfrac{1}{12}x^4 - \dfrac{1}{45}x^6 + o\left(x^6\right)$

2) $\tan(x) = x + \dfrac{1}{3}x^3 + \dfrac{2}{15}x^5 + o\left(x^5\right)$

3) $\sin(\tan(x)) = x + \dfrac{1}{6}x^3 - \dfrac{1}{40}x^5 + o\left(x^5\right)$

4) $(\ln(1+x))^2 = x^2 - x^3 + \dfrac{11}{12}x^4 + o\left(x^4\right)$

5) $\exp(\sin(x)) = 1 + x + \dfrac{1}{2}x^2 + o\left(x^3\right)$

6) $\sin^6(x) = x^6 - x^8 + o\left(x^9\right)$

Exercice 18

1) $\sqrt{x+1} = \sqrt{x} + \dfrac{1}{2\sqrt{x}} - \dfrac{1}{8x^{3/2}} + o\left(\dfrac{1}{x^{3/2}}\right)$;

2) $x\ln(x+1) - (x+1)\ln x = 1 - \ln(x) - \dfrac{1}{2x} + \dfrac{1}{3x^2} + o\left(\dfrac{1}{x^2}\right)$;

3) $\left(\dfrac{x+1}{x}\right)^x = \mathrm{e} - \dfrac{1}{2}\dfrac{\mathrm{e}^1}{x} + \dfrac{11}{24}\dfrac{\mathrm{e}}{x^2} + o\left(\dfrac{1}{x^2}\right)$;

4) $\operatorname{Arctan} x = \dfrac{\pi}{2} - \dfrac{1}{x} + \dfrac{1}{3x^3} + o\left(\dfrac{1}{x^4}\right)$.

Exercice 19

1) $\dfrac{\arctan x - x}{\sin x - x} = 2 - \dfrac{11}{10}x^2 + o(x^2)$;

2) $\ln\sin x = \ln(\sqrt{2}/2) + (x-\pi/4) - (x-\pi/4)^2 + \dfrac{2}{3}(x-\pi/4)^3 + o((x-\pi/4)^3)$;

3) $(1+x)^{\frac{1}{x}} = e\left(1 - \dfrac{1}{2}x + \dfrac{11}{24}x^2 - \dfrac{7}{16}x^3\right) + o(x^3)$;

4) $x(\sqrt{x^2+\sqrt{x^4+1}} - x\sqrt{2}) = \dfrac{\sqrt{2}}{8x^2} + o\left(\dfrac{1}{x^2}\right)$.

Exercice 22

1) $\sqrt[3]{x^3+x^2} - \sqrt[3]{x^3-x^2} = x\left(\sqrt{1+\frac{1}{x}} - \sqrt{1-\frac{1}{x}}\right) = 1 + \dfrac{1}{8x^3} + o(1/x^3)$.

2) $\ln\left(\sqrt{1+x}\right) = \frac{1}{2}\ln x + \frac{1}{2x} + o(1/x)$.

Exercice 23

1) **a)** $\dfrac{\cos x}{\sqrt{1+x}} = 1 - \dfrac{x}{2} - \dfrac{x^2}{8} - \dfrac{x^3}{16} + \dfrac{49x^4}{384} + o(x^4)$

b) $\dfrac{\sqrt{1+x}}{\cos x} = 1 + \dfrac{x}{2} + \dfrac{3x^2}{8} + \dfrac{5x^3}{16} + \dfrac{41x^4}{384} + o(x^4)$

c) $\dfrac{\ln(1+x)}{\cos x} = x - \dfrac{x^2}{2} + \dfrac{5x^3}{6} - \dfrac{x^4}{2} + o(x^4)$

d) $\dfrac{1+\cos x}{2+\sin x} = 1 - \dfrac{x}{2} + \dfrac{x^3}{12} - \dfrac{x^4}{16} + o(x^4)$

e) $\dfrac{\sin(x/2)}{\mathrm{e}^{2x}} = \dfrac{x}{2} - x^2 + \dfrac{47x^3}{48} - \dfrac{5x^4}{8} + o(x^4)$

f) $\dfrac{\ln(1+x)}{2-\cos x} = x - \dfrac{x^2}{2} - \dfrac{x^3}{6} + o(x^4)$

2) **a)** $\dfrac{\sin(2x-\pi/4)}{\cos x} = 1 + 3(x-\dfrac{\pi}{4}) + \dfrac{3}{2}(x-\dfrac{\pi}{4})^2 + \dfrac{3}{2}(x-\dfrac{\pi}{4})^3 + \dfrac{19}{8}(x-\dfrac{\pi}{4})^4 + o((x-\dfrac{\pi}{4})^4)$

b) $\dfrac{\cos(x-1)}{\ln(1+x)} = \dfrac{1}{\ln 2} - \dfrac{1}{2(\ln 2)^2}(x-1) + \dfrac{1}{\ln 2}\left(-\dfrac{1}{2} + \dfrac{1}{8\ln 2} + \dfrac{1}{4(\ln 2)^2}\right)(x-1)^2 + o((x-1)^2)$

c) $\dfrac{\mathrm{e}^{x-1}}{\ln x} = \dfrac{1}{x-1} + \dfrac{3}{2} + \dfrac{11}{12}(x-1) + \dfrac{3}{8}(x-1)^2 + \dfrac{71}{720}(x-1)^3 + \dfrac{41}{1440}(x-1)^4 + o((x-1)^4)$

Exercice 24

1) $\mathrm{e}-1$

2) $-\dfrac{1}{2}$

3) $-\dfrac{\mathrm{e}}{2}$

4) $\dfrac{1}{12}$

5) $-\dfrac{1}{6}$

6) $-\dfrac{1}{2}$