5.4 (fin) - Déterminant de Vandermonde

Soit $n \in \mathbb{N}^*$, $x_0 \dots x_n \in \mathbb{K}$.

$$M_n = \begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & & \dots & x_1^n \\ \vdots & & & & & \\ 1 & x_n & x_n^2 & & \dots & x_n^n \end{pmatrix} \in \mathcal{M}_{n+1}(\mathbb{K})$$

$$V(x_0 \dots x_n) = \det(M_n).$$

1ère méthode :

$$-x_0 \cdot \begin{pmatrix} x_0^{j-1} \\ x_1^{j-1} \\ \vdots \\ x_n^{j-1} \end{pmatrix} + \begin{pmatrix} x_0^j \\ x_1^j \\ \vdots \\ x_n^j \end{pmatrix} = \begin{pmatrix} 0 \\ x_1^{j-1}(-x_0 + x_1) \\ \vdots \\ x_n^{j-1}(-x_0 + x_n) \end{pmatrix}$$

$$C_{n+1} \leftarrow C_{n+1} - x_0 C_n$$
$$C_n \leftarrow C_n - x_0 C_{n-1}$$
$$\vdots$$
$$C_2 \leftarrow C_2 - x_0 C_1$$

$$V(x_0 \dots x_n) = \begin{vmatrix} 1 & 0 & 0 & & 0 \\ 1 & x_1 - x_0 & x_1(x_1 - x_0) & & x_1^{n-1}(x_1 - x_0) \\ \vdots & & & & \\ 1 & x_n - x_0 & x_n(x_n - x_0) & \dots & x_n^{n-1}(x_n - x_0) \end{vmatrix}$$

$$= (x_1 - x_0)(x_2 - x_0) \dots (x_n - x_0) \begin{vmatrix} 1 & 0 & - & - & - & 0 \\ 1 & 1 & x_1 & x_1^2 & & x_1^{n-1} \\ \vdots & \vdots & \vdots & & & \\ 1 & 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix}$$

$$= \prod_{i=1}^{n} (x_i - x_0) \cdot V(x_1, \dots, x_n)$$

$$= \prod_{i=1}^{n} (x_i - x_0) \times \prod_{i=2}^{n} (x_i - x_1) \times V(x_2, \dots, x_n)$$

$$= \boxed{\prod_{1 \leq i < j \leq n} (x_j - x_i) = V(x_0, \dots, x_n)}$$

2ème méthode :

$$V(x_0 \dots x_n, X) = \begin{vmatrix} 1 & x_0 & \cdots & x_0^n & x_0^{n+1} \\ 1 & x_1 & \cdots & x_1^n & x_1^{n+2} \\ \vdots & & & & \\ 1 & x_n & \cdots & x_n^n & x_n^{n+2} \\ 1 & X & \cdots & X^n & X^{n+1} \end{vmatrix}$$

$\underset{\text{dvpt. dernière ligne}}{=} \quad 1 \times |\cdots| + X|\cdots| + X^2|\cdots| \ \cdots \ X^{n+1}|\cdots|$

- $V(x_0 \cdots x_n, X) \in \mathbb{K}_{n+1}[X]$.

- $V(\underline{x_0}, \cdots, x_n, \underline{x_0}) = 0$
$$= V(x_0, \cdots, x_n, x_1)$$
$$\vdots$$
$$= V(x_0 \cdots\cdots, x_n)$$

$\forall i \in [\![0, n]\!], \quad V(x_0 \cdots x_n, x_i) = 0$

- $\exists \lambda \in \mathbb{K}, \quad V(x_0, \cdots, x_n, X) = \lambda \prod_{i=0}^{n} (X - x_i)$

le terme en X^{n+1} a pour coefficient :

$$+\begin{vmatrix} 1 & x_0 & \cdots & x_0^n \\ \vdots & & & \\ 1 & x_1 & \cdots & x_1^n \end{vmatrix} = V(x_0, \dots, x_n) = \lambda$$

$$V(x_0, \dots, x_n, X) = V(x_0, \dots x_n) \cdot \prod_{i=0}^{n} (X - x_i).$$

Soit $x_{n+1} \in \mathbb{K}$:

$$V(x_0, \dots, x_{n+1}) = V(x_0, \dots, x_n) \cdot \prod_{i=0}^{n} (x_{n+1} - x_i)$$

par réc : $\boxed{V(x_0 \dots x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i) .}$

Un exemple : le problème d'interpolation polynomiale

Soit $x_0, \dots, x_n, y_0 \dots y_n \in \mathbb{K}$,

tq. les $x_0 \dots x_n$ sont 2 à 2 distincts.

Existe-t-il $P \in \mathbb{K}_n[X]$ tq $\forall i \in [\![0, n]\!]$, $P(x_i) = y_i$?

Soit $P \in \mathbb{K}_n[X]$, $P = \sum_{i=0}^{n} a_i X^i$.

Soit $j \in [\![0, n]\!]$:

$$P(x_j) = \sum_{i=0}^{n} a_i x_j^i = (1 \ x_j \ \dots x_j^n) \times \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}$$

P est sol° du pb.

ssi : $\forall j \in [\![0,n]\!]$, $\begin{pmatrix} 1 & x_j & \cdots & x_j^n \end{pmatrix} \times \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = y_j$

ssi
$$\underbrace{\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}}_{= M_n} \times \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} \quad (S)$$

les x_j sont 2 à 2 distincts :

$$\det M_n = V(x_0 \ldots x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$$
$$\neq 0$$

(S) a 1 unique solution:

$$\begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = M_n^{-1} \times \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}.$$