Feuille d'exercice n° 18 : Développements limités - fiche d'entraînement - correction

Exercice 1

1)
$$\lim_{n \to +\infty} \frac{\sqrt{n^2 + 1}}{n^3} = 0 \text{ donc } n^3 - \sqrt{n^2 + 1} \underset{n \to +\infty}{\sim} n^3$$
. De même, $\ln n - 2n^2 \underset{n \to +\infty}{\sim} -2n^2$, donc $u_n \underset{n \to +\infty}{\sim} \frac{n^3}{-2n^2} = -n/2$.

2)
$$\ln(n^2+1) = \ln n^2 + \ln(1+1/n^2) = 2\ln n + \ln(1+1/n^2) \underset{n\to+\infty}{\sim} 2\ln n \text{ et } n+1 \underset{n\to+\infty}{\sim} n \text{ donc}$$

$$v_n \underset{n\to+\infty}{\sim} \frac{2\ln n}{n}.$$

3)
$$w_n = \frac{n\sqrt{n+1/n+1/n^2}}{n^{2/3}\sqrt[3]{1-1/n+1/n^2}} \underset{n \to +\infty}{\sim} \frac{n}{n^{2/3}} = n^{1/3}.$$

4)
$$\cos(1/n) = 1 - \frac{1}{2n^2} + o(1/n^2)$$
 et $e^{1/n} = 1 + 1/n + o(1/n)$, donc : $\cos(1/n) - e^{1/n} = -1/n + o(1/n) \underset{n \to +\infty}{\sim} 1/n$.

5)
$$y_n \underset{n \to +\infty}{\sim} \frac{1/n^2}{1/n} \underset{n \to +\infty}{\sim} -1/n$$
.

6)
$$\ln(1+\sin(1/n)) \underset{n\to+\infty}{\sim} \sin(1/n) \cot \sin(1/n) \xrightarrow[n\to+\infty]{} n+\infty 0$$
, et $\sin(1/n) \underset{n\to+\infty}{\sim} 1/n$, donc : $\ln(1+\sin(1/n)) \underset{n\to+\infty}{\sim} 1/n$. Enfin, $1-\sqrt{1+1/n}=1-1-1/(2n)+o(1/n) \underset{n\to+\infty}{\sim} -1/(2n)$, d'où $z_n \underset{n\to+\infty}{\sim} -2$.

Exercice 2

1)
$$1 - x + \frac{2}{3}x^2 - \frac{11}{24}x^3 + o(x^3)$$

2)
$$\frac{1}{3}x + \frac{1}{90}x^3 + o(x^3)$$

3)
$$1 + \frac{1}{2}(x-1) - \frac{1}{12}(x-1)^2 + o((x-1)^2)$$

4)
$$\frac{2}{3}x + \frac{1}{90}x^3 + o(x^3)$$

5)
$$1 - \frac{1}{2}x - \frac{1}{12}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 - \frac{1}{240}x^5 + o(x^5)$$

6)
$$\sum_{k=0}^{999} \frac{x^k}{k!} = e^x - \frac{x^{1000}}{1000!} + o(x^{1000}) \text{ d'où} :$$

$$\ln\left(\sum_{k=0}^{999} \frac{x^k}{k!}\right) = \ln(e^x - \frac{x^{1000}}{1000!} + o(x^{1000})) = \ln(e^x) + \ln(1 - \frac{x^{1000}e^{-x}}{1000!} + o(x^{1000})) = x - \frac{x^{1000}}{1000!} + o(x^{1000}).$$

Exercice 3

6)
$$\ln\left(\frac{1}{\cos x}\right) = \left(\frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + o\left(x^7\right)\right)$$

$$7) \ln (1 + \operatorname{ch} x) = (\ln (2) + \frac{1}{4}x^2 - \frac{1}{96}x^4 + o\left(x^4\right))$$

$$8) \ln (\tan x) = (2 (x - 1/4\pi) + \frac{4}{3}(x - 1/4\pi)^3 + o\left((x - 1/4\pi)^4\right))$$

$$9) \arctan (e^x) = (1/4\pi + \frac{1}{2}x - \frac{1}{12}x^3 + o\left(x^3\right))$$

$$10) \arctan (2\sin x) = (1/3\pi + \frac{1}{4}(x - 1/3\pi) - 3/16\sqrt{3}(x - 1/3\pi)^2 + \frac{3}{16}(x - 1/3\pi)^3 + o\left((x - 1/3\pi)^3\right))$$