MATRICES

A matrix is a rectangular array of numbers enclosed by a pair of brackets. If a matrix has m rows and n columns, i.e. m x n matrix, then the scalar entry in the i-th row and j-th column of matrix A is denoted by a_{ij} and it is called the i-j entry of A. Each column of A is a list of m real numbers, which identifies a vector in R^m. Often, these columns are denoted by $\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n$, and the matrix A is written as

$$A = \left(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n\right)$$

Types of Matrices

1. Square matrix

A square matrix is a matrix that has the same number of rows as there are columns- i.e. n x n matrix. It has the interesting feature of possessing diagonal elements. Closely related with diagonal elements of a square matrix is its trace, which is the sum of all the diagonal elements.

For example, consider the square matrix A below

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 3 \\ -3 & -2 & -4 \end{pmatrix}$$

The trace of A is 2+(-1)+(-4)=-3.

1. Triangular matrix

One of the forms of square matrices is the triangular matrix which include either of upper or lower triangular matrices.

a. Upper triangular matrix

The elements of an upper triangular matrix is $a_{ij} = 0 \quad for\ i > j$

For example

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$$

b. Lower triangular matrix

The elements of a lower triangular matrix is $a_{ij} = 0 \quad for\ i < j$

For example

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

2. Diagonal matrix

This is a square matrix with elements $a_{ij} = 0 \ if \ i \neq j$ and $a_{ij} \neq 0 \ if \ i = j$.

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & & a_{nn} \end{pmatrix}$$

3. Identity matrix

An identity matrix is a diagonal matrix whose diagonal entries are all equal to 1.

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4. Zero matrix

A zero matrix is a matrix in which every element is zero.

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition of Terms

1. Equal matrices

Two matrices A and B are equal if and only if they have the same order and each element of one is equal to the corresponding element of the other.

2. Similar matrices

Two matrices A and B are similar if their traces are the same.

3. Periodic matrix

A matrix A is periodic if for some positive integer n, $A^{n+1}=A$ then A is said to be of period n.

4. Idempotent matrix

A matrix A is idempotent if $A^2=A$.

5. Nilpotent matrix

A matrix is nilpotent if there exists a positive integer n such that $A^n = 0$. If n is the least positive integer for which $A^n = 0$, then A is nilpotent of index n.

N|B

A matrix A is involuntary if $A^2 = I$ (identity).

<u>Basic Operations on Matrices</u>

1. Sum and scalar multiples

The sum of matrices A and B, given A+B, is defined only when A and B are the same size (order).

The difference A-B or A+(-B) is defined in terms of the corresponding elements of A and B.

Theorem

Let A, B, and C be matrices of the same size, and let α and β be scalars.

a. A+B=B+A
b. (A+B)+C=A+(B+C)
c. A+0=A
d. $\alpha(A+B)=\alpha A+\alpha B$
e. $(\alpha+\beta)A=\alpha A+\beta A$
f. $\alpha(\beta A)=(\alpha\beta)A$

Example

Given the matrices

$$A=\begin{pmatrix} 2 & 3 & 4 \\ 1 & 0 & -2 \\ 3 & 1 & -1 \end{pmatrix}, \qquad B=\begin{pmatrix} -1 & 4 & 2 \\ 1 & -3 & 5 \\ -1 & 3 & 3 \end{pmatrix}$$

Compute the following:

a) A+B
b) A-B
c) 2B-3A

Solution

b.)

$$A=\begin{pmatrix}2&3&4\\1&0&-2\\3&1&-1\end{pmatrix}-\begin{pmatrix}-1&4&2\\1&-3&5\\-1&3&3\end{pmatrix}$$

$$=\begin{pmatrix}3&-1&2\\0&3&-7\\4&-2&-4\end{pmatrix}$$

2. Matrix multiplication

The product of mxr matrix A =(a_{ij}) and rxn matrix B=(b_{ij}) is given as mxn matrix C=(c_{ij}) where

$$c_{ij}=\sum a_{ik}b_{kj}; \qquad i=1,2,\cdots,m \;\; and \;\; j=1,2,\cdots,n$$

$$=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{ir}b_{rj}$$

That is,

$$C=AB=\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1r}\\a_{21}&a_{22}&\cdots&a_{2r}\\\vdots&\vdots&\ddots&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mr}\end{pmatrix}\begin{pmatrix}b_{11}&b_{12}&\cdots&b_{1n}\\b_{21}&b_{22}&\cdots&b_{2n}\\\vdots&\vdots&\ddots&\vdots\\b_{r1}&b_{r2}&\cdots&b_{en}\end{pmatrix}b_{2j}$$

Example

Given the matrices:

$$A=(2,5,3),\quad B=\begin{pmatrix}2\\-3\\-1\end{pmatrix}, \qquad C=\begin{pmatrix}1&2&1\\4&0&2\end{pmatrix},$$

$$D=\begin{pmatrix}4&-1&3&2\\0&-2&5&3\\5&3&-1&-2\end{pmatrix}, \qquad E=\begin{pmatrix}1&2&3\\2&4&1\\4&-5&-2\end{pmatrix}$$

Evaluate the following:

i) AB
ii) BA
iii) -2E
iv) CE
v) ED

Solution

i.)

$$AB = (2,\ 5,\ 3)\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$$
$$= 2\times 2 + 5\times(-3) + 3(-1)$$
$$= -14$$

v.)

$$ED = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 4 & -5 & -2 \end{pmatrix}\begin{pmatrix} 4 & -1 & 3 & 2 \\ 0 & -2 & 5 & 3 \\ 5 & 3 & -1 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} 19 & 4 & 10 & 2 \\ 13 & -7 & 25 & 14 \\ 2 & 1 & -14 & -5 \end{pmatrix}$$

Example (Idempotent matrix)

Show that the matrix below is idempotent

$$H = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$$

Solution

$$H^2 = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}\begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} = H$$

Example (Nilpotent matrix)

Show that the matrix below is nilpotent of index 3

$$G = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$$

Solution

$$G^2 = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{pmatrix}$$

$$G^3 = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, G is nilpotent of order 3.

Classwork

Given the matrices

$$A = \begin{pmatrix} 1 & -3 & 2 \\ 0 & 4 & -1 \\ 5 & -2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ -3 & 5 & -7 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 5 & 3 & 2 \\ 6 & -4 & 4 \end{pmatrix}$$

Show that

i) $A(B+C) = AB + AC$

ii) $A(BC) = (AB)C$

iii) $AB \neq BA$

iv) $BC \neq CB$

Transpose of Matrix

Given an mxn matrix A, the transpose of A is the nxm matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Theorem

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a) $(AT)^T = A$

b.) $(A+B)^T = A^T + B^T$

c.) $(AB)^T = B^T A^T$

d) $(\lambda A)^T = \lambda A^T$, $\quad$ *where λ is a scalar*

Example

Given the matrix below, find its transpose.

$$A=\begin{pmatrix} 2 & 3 & -1 & 4 \\ 1 & 1 & 2 & 2 \\ 4 & -2 & 1 & -3 \end{pmatrix}$$

Solution

$$A^T=\begin{pmatrix} 2 & 1 & 4 \\ 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 2 & -3 \end{pmatrix}$$

Classwork

Verify the above theorem given the following:

$$A=\begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix},\ B=\begin{pmatrix} 1 & -2 \\ 6 & 5 \end{pmatrix},\ \lambda=3$$

Symmetric Matrices

A matrix A is said to be symmetric if A is equal to its transpose, i.e. $A=A^T$.

Example

Show that the matrix below is symmetric

$$A=\begin{pmatrix} 4 & 1 & -4 \\ 1 & -3 & 2 \\ -4 & 2 & 5 \end{pmatrix}$$

Solution

$$A^T=\begin{pmatrix} 4 & 1 & -4 \\ 1 & -3 & 2 \\ -4 & 2 & 5 \end{pmatrix}=A$$

$\therefore A$ is symmetric.

Row Reduction and Echelon Forms

a.) A rectangular matrix is in echelon (or row echelon) form if it has the following three properties:
   i.) All non-zero rows are above any rows of all zeros
   ii.) Each leading (distinguished) entry of a row is in a column to the right of the leading entry of the row above it.
   iii.) All entries in a column below a leading entry are zeros.
b.) If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon (reduced row echelon) form:

i.) The leading entry in each non-zero row is 1.
ii.) Each leading 1 is the only non-zero entry in its column.

NB

Note that the reduced echelon form of a matrix is unique.

Definitions

1. A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.
2. The rank of a matrix is the number of non-zero rows when a matrix is in its echelon form.

<u>The Row Reduction Algorithm</u>

The basic assumption of this procedure is that $a_{11} \neq 0$.

Step I

Interchange the first row of the matrix if $a_{11}=0$ with another row whose first entry is non-zero.

Step II

For i > 1, apply the operation $R_i = a_{11}R_i - a_{i1}R_1$

NB

Repeat steps I and II with the submatrix formed by all the row exchanging the first assuming that $a_{ii} \neq 0$

Example

Reduce the matrix below to its echelon form. Hence find its rank.

$$G = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{pmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & -5 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix} R_3 \to 2R_3 + 3R_2$$

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & -4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore Rank \;\; of \;\; G \;\; is \;\; 3$

Example

Use elementary row operations to transform the matrix below to its reduced echelon form.

$$A = \begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix} R_1 \leftrightarrow R_3$$

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix} \begin{matrix} R_1 \to \frac{1}{3} R_1 \\ R_2 \to R_2 - 3R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix} R_3 \to 2R_3 - 3R_2$$

$$\begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 2 & 8 \end{pmatrix} R_2 \to \frac{1}{2} R_2$$

$$\begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 2 & 8 \end{pmatrix} \begin{matrix} R_1 \to R_1 + 3R_2 \\ \\ R_3 \to \frac{1}{2}R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix} \begin{matrix} R_1 \to R_1 - 5R_3 \\ R_2 \to R_2 - R_3 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

Thus, matrix A has been transformed to its reduced echelon form. Its rank is 3.

Classwork

1. Row reduce the following matrices and list the pivot columns:

$$a.)\quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix} \qquad b.)\quad \begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{pmatrix}$$

2. Find the rank of each of the following matrices:

$$a.)\quad \begin{pmatrix} 2 & -1 & 2 & 2 \\ 1 & -2 & 3 & 1 \\ 3 & 1 & 2 & 3 \end{pmatrix} \qquad b.)\quad \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 2 & 3 & 2 & -1 \end{pmatrix}$$

$$c.)\quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 6 & 7 & 4 & 8 \\ 4 & 7 & 10 & 13 & 16 & 9 \\ 5 & 8 & 9 & 10 & 3 & 10 \end{pmatrix}$$

Applications of matrices to real life situations

Example

A man purchased a cupboard at ₦6000, 5 chairs at ₦500 each and 2 tables at ₦1000 each. Determine, using matrix notation, how much the man spent on his purchases,

Solution

$$Total \quad \cos t = \begin{pmatrix} 1 & 5 & 2 \end{pmatrix} \begin{pmatrix} 6000 \\ 500 \\ 1000 \end{pmatrix}$$

$$= 6000 + 5(500) + 2(1000)$$

$$= N10500$$

Determinant

For $n \geq 2$, the determinant of an nxn matrix A=(a_{ij}) is the sum of n terms of the form $\pm a_{ij} \det A_{1j}$, with plus (+) and minus (-) signs alternating, where the entries a_{11}, a_{12}, … , a_{1n} are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-)^{n+1} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-)^{j+1} a_{1j} \det A_{1j}$$

Theorem

The determinant of nxn matrix A can be computed by cofactor expansion across any row or down any column. The expansion across the ith row is given by

$$\det A = a_{i1} C_{i1} - a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

The cofactor expansion down the jth column is given by

$$\det A = a_{1j} C_{1j} - a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

Example

Compute the determinants of the following matrices

1.) $A = \begin{pmatrix} 2 & 3 \\ -4 & 1 \end{pmatrix}$ 2.) $B = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}$

3.) $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & -2 & 1 \\ -1 & 2 & 4 & 3 \\ 3 & 0 & -3 & 4 \end{pmatrix}$

Solution

1.) $|A| = 14$

2.) $|B| = -2$

3.) $|C| = 85$

Determinant of Triangular Matrix

The determinant of a triangular matrix is the product of its diagonal elements.

Note that the identity matrix has determinant of 1.

Example

Find the determinant in each of the following matrices.

1.) $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 2.) $F = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

3.) $G = \begin{pmatrix} 2 & 5 & 6 & -8 \\ 0 & 3 & -1 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ 4.) $H = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 5 & 4 & 0 & 0 & 0 \\ -3 & 8 & -2 & 0 & 0 \\ 0 & 10 & 3 & 1 & 0 \\ 4 & -15 & -3 & 5 & -1 \end{pmatrix}$

Solution

1.) $|I| = 1$

2.) $|F| = 0$

3.) $|G| = -6$

4.) $|H| = 16$

Singular and Non-singular Matrices

A matrix is said to be singular if its determinant is 0; otherwise it is said to be non-singular. A non-singular matrix is invertible, i.e. it possesses an inverse.

Example

Show that the matrix below is singular

$$B=\begin{pmatrix} 2 & 5 & 6 & -8 \\ 0 & 3 & -1 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution

By determinant of triangular matrix, $|B|=0$.

Therefore B is singular.

Classwork

For what values of x is the following matrix singular

$$G=\begin{pmatrix} x-2 & 4 & 3 \\ 1 & x+1 & -2 \\ 0 & 0 & x-4 \end{pmatrix}$$

Solution

X = -2, x = 3 or x = 4

Properties of Determinant

1. If in one row or column of a square matrix A, every element is 0, then $|A|=0$.
2. If two rows of a square matrix A are interchanged to produce B, then $|B|=-|A|$.
3. If a square matrix A has two identical rows or columns, then $|A|=0$.
4. If one row of A is multiplied by a scalar k to produce B, then $|B|=k|A|$.
5. If a multiple of one row or column of A is added to another row or column to produce B, then $|B|=|A|$.
6. The determinant of a product of two matrices A and B is equal to the product of their determinants, i.e. $|AB|=|A||B|$.
7. The determinants of A and its transpose are equal, i.e. $|A|=|A^T|$.

Example

1.) Find the determinants of the following matrices

$$A=\begin{pmatrix} 2 & 5 & 6 & -1 & 2 \\ 4 & -6 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 4 & 3 & 7 & -5 \\ 3 & -7 & -6 & 5 & 2 \end{pmatrix} \qquad B=\begin{pmatrix} 4 & 3 & -1 & 1 & 2 \\ 6 & 5 & -7 & 8 & 4 \\ 2 & -2 & 4 & 3 & 1 \\ 4 & 3 & -1 & 1 & 2 \\ 10 & -8 & 3 & 11 & 5 \end{pmatrix}$$

2.) Evaluate $|Q|$ given that

$$Q = \begin{pmatrix} 4 & 2 & 3 \\ 1 & 2 & 2 \\ 2 & -2 & 3 \end{pmatrix}$$

Hence obtain $|R|$ if $R = \begin{pmatrix} 1 & 2 & 2 \\ 4 & 2 & 3 \\ 2 & -2 & 3 \end{pmatrix}$

3.) Given the matrices below, show that $|PQ| = |P||Q|$.

$$P = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \end{pmatrix} \qquad Q = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & -1 \\ 1 & 2 & 3 \end{pmatrix}$$

Solution

1.) By property 1, $|A| = 0$.

By property 3, $|B| = 0$.

2.) $|Q| = 24$

By property 2, $|R| = -24$.

3.)

$$PQ = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & -1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 5 & 4 \\ 7 & 0 & 2 \\ 2 & -1 & 12 \end{pmatrix}$$

$|PQ| = -450$

But

$|P| = -30$

$|Q| = 15$

$|P||Q| = -30(15) = -450$

$\therefore |PQ| = |P||Q|$

Application of Matrix Determinant to Geometry

1. <u>Areas of Parallelogram and Triangle</u>
   The area od a parallelogram whose vertices are A(x_1, y_1), B(x_2, y_2), C(x_3, y_3), and D(x_4, y_4), is the positive value of $\pm\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Hence, the area of a triangle with vertices A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) is half the area of the parallelogram.

Example

Find the area of a parallelogram whose vertices are A(1,0), B(5,0), C(6,4) and D(2, 4). Hence obtain the area of a triangle of vertices A, B and C.

Solution

Area of parallelogram = $\pm\begin{vmatrix} 1 & 0 & 1 \\ 5 & 0 & 1 \\ 2 & 4 & 1 \end{vmatrix} = 16\ units^2$

Area of triangle = $\frac{1}{2}\times 16 = 8 units^2$

2. <u>Equation of Straight Line</u>

The equation of a straight line through two given points A(x_1, y_1), B(x_2, y_2) is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Where (x, y) is the variable point.

Example

Use the determinant method to find the equation of a line that passes through the points (2, 4) and (-1, 3).

Solution

$$\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = x\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} - y\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix}$$

$$x - 3y + 10 = 0$$

Inverse of Matrix

For any square non-singular matrix A, the inverse of A, denoted by A^{-1}, is given by

$$A^{-1} = \frac{adj\ A}{|A|}, \qquad |A| \neq 0$$

*adj*A is the adjoint of A.

NB

$$AA^{-1} = A^{-1}A = I\ (identity\ \ matrix)$$

Example

Given the matrix

$$G = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & 3 \end{pmatrix}$$

Find

a.) The minors
b.) The cofactors
c.) The adjoint
d.) The determinant
e.) The inverse

Solution

Method I

a.)

$$|G_{11}| = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 4, \qquad |G_{12}| = \begin{vmatrix} 0 & 1 \\ 5 & 3 \end{vmatrix} = -5, \qquad |G_{13}| = \begin{vmatrix} 0 & 2 \\ 5 & 2 \end{vmatrix} = -10,$$

$$|G_{21}| = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5, \qquad |G_{22}| = \begin{vmatrix} 2 & -1 \\ 5 & 3 \end{vmatrix} = 11, \qquad |G_{23}| = \begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} = -1$$

$$|G_{31}| = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3, \qquad |G_{32}| = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2, \qquad |G_{33}| = \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = 4$$

b.) Using the signs (+ and -),

$$|G_{11}| = 4, \qquad |G_{12}| = 5, \qquad |G_{13}| = -10,$$

$$|G_{21}| = -5, \qquad |G_{22}| = 11, \qquad |G_{23}| = 1$$

$$|G_{31}| = 3, \qquad |G_{32}| = -2, \qquad |G_{33}| = 4$$

Therefore, the matrix of cofactors is

$$\begin{pmatrix} 4 & 5 & -10 \\ -5 & 11 & 1 \\ 3 & -2 & 4 \end{pmatrix}$$

c.) Adjoint of G is given by

$$G^T = \begin{pmatrix} 4 & 5 & -10 \\ -5 & 11 & 1 \\ 3 & -2 & 4 \end{pmatrix}^T = \begin{pmatrix} 4 & -5 & -10 \\ 5 & 11 & -2 \\ -10 & 1 & 4 \end{pmatrix}$$

d.) Determinant

$$|G| = \begin{vmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & 3 \end{vmatrix} = 23$$

e.) Inverse

$$G^{-1} = \frac{1}{23}\begin{pmatrix} 4 & -5 & -10 \\ 5 & 11 & -2 \\ -10 & 1 & 4 \end{pmatrix} = \begin{pmatrix} \frac{4}{23} & -\frac{5}{23} & \frac{3}{23} \\ \frac{5}{23} & \frac{11}{23} & -\frac{2}{23} \\ -\frac{10}{23} & \frac{1}{23} & \frac{4}{23} \end{pmatrix}$$

Method II

Since $GG^{-1} = I$, set $G^{-1} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$ then

$$GG^{-1} = I$$

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & 3 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Using matrix multiplication, we have 3 different systems of equations in 3 variables. Solving these systems, we have

$$a = \frac{4}{23}, \qquad d = \frac{5}{23}, \qquad g = -\frac{10}{23}$$

$$b = -\frac{5}{23}, \qquad e = \frac{11}{23}, \qquad h = \frac{1}{23}$$

$$c = \frac{3}{23}, \qquad f = -\frac{2}{23}, \qquad k = \frac{4}{23}$$

Method III

From an augmented matrix of the form (G|I), a transformation using row reduction to the form ($I|G^{-1}$) gives the inverse of G.

Classwork

Determine whether or not the matrices below are invertible. Hence obtain their inverses where possible.

$$A=\begin{pmatrix} 2 & 4 & -5 \\ 1 & 1 & 2 \\ -4 & -8 & 10 \end{pmatrix}, \quad B=\begin{pmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{pmatrix}$$

Systems of Equations

Let $\mathbb{R}$ be the real field. An expression of the form $a_1x_1+a_2x_2+\cdots+a_nx_n=b_1$ is called a linear equation, where the $x_i's$ are the unknowns. The scalars $a_i's\in\mathbb{R}$ are called the coefficients of $x_i's$ and b_1 the constant of the linear equation.

Two or more of such linear equations constitute a system of linear equations.

A system of m linear equations in n unknowns is given as

$$\begin{aligned} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n&=b_1 \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n&=b_2 \\ &\vdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n&=b_m \end{aligned}$$

This can be written in matrix equation form as $A\vec{x}=\vec{b}$

Homogeneous and Non-homogeneous Systems

A system of linear equations is said to be homogeneous if the constants $b_1,\ b_2,\ \cdots b_n$ are all zeros; otherwise it is said to be non-homogeneous.

Solution of System of Linear Equations

A set of values $(s_1,\ s_2,\ \cdots s_n)$ is a solution of a system if it makes each equation a true statement when the values are substituted for $x_1,\ x_2,\ \cdots x_n$ respectively in the equations.

Methods of Solution

1. Gaussian Elimination

This involves a system of equations being reduced to echelon form to make it amenable for back substitution.

Theorem

A system of linear equations with $a_{11} \neq 0$ and a system obtained by the operation $L_i \to a_{11}L_i - a_{i1}L_1$ are equivalent, i.e. they have the same solution set.

Example

Using elimination method, solve the system of equations

$$\begin{aligned} 2x + y - 2z &= 10 \\ 3x + 2y + 2z &= 1 \\ 5x + 4y + 3z &= 4 \end{aligned}$$

Solution

$$\left.\begin{aligned} 2x + y - 2z &= 10 \\ 3x + 2y + 2z &= 1 \\ 5x + 4y + 3z &= 4 \end{aligned}\right\} \begin{aligned} & \\ &L_2 \to 2L_2 - 3L_1 \\ &L_3 \to 2L_3 - 5L_1 \end{aligned}$$

$$\left.\begin{aligned} 2x + y - 2z &= 10 \\ y + 2z &= -28 \\ 3y + 16z &= -42 \end{aligned}\right\} \begin{aligned} & \\ & \\ &L_3 \to L_3 - 3L_2 \end{aligned}$$

$$\begin{aligned} 2x + y - 2z &= 10 \\ y + 2z &= -28 \\ 14z &= -42 \end{aligned}$$

Using back substitution, starting from (iii), we have

Z = -3, y = 2, x = 1

Classwork

Using elimination method, solve the following systems of equations

1.
$$\begin{aligned} x + 2z &= -1 \\ 2x - y + 3z &= -6 \\ 4x + y + 8z &= 2 \end{aligned}$$

2.
$$\begin{aligned} 2x + 3y - 5z + 2w &= 3 \\ 3x + 2y - 7z + 4w &= 1 \\ x + 2y - 3z + 6w &= 5 \\ x + 4y + z - 2w &= 2 \end{aligned}$$

3.

$$A=\begin{pmatrix}-1 & 2 & -3\\ 2 & 1 & 0\\ 4 & -2 & 5\end{pmatrix},\quad b=\begin{pmatrix}0\\ 2\\ 3\end{pmatrix}$$

Classification of Systems of Linear Equations

There are mainly two types of systems- consistent and inconsistent. A consistent system is one whose solution exists; whereas an inconsistent system does not have a solution.

Consistent Systems

Consistent systems are distinguished into two classes- systems with unique solutions and those with more than one solution.

i.) A non-homogeneous system with as many equations as there are unknowns has unique, non-zero or non-trivial solution. For example, the system below has unique non-zero solution.

$$2x+y-2z=10$$
$$3x+2y+2z=1$$
$$5x+4y+3z=4$$

ii.) A homogeneous system with as many equations as there are unknowns have unique zero or trivial solution. For example, the system below has unique zero solution.

$$2x+3y-5z-2w=0$$
$$3x-y+7z+4w=0$$
$$x+2y-3z+6w=0$$
$$-5x+4y+z-2w=0$$

iii.) A non-homogeneous or homogeneous system with more unknowns than there are equations are expected to have infinitely many solutions. For example, these systems have infinitely many solutions.

$a.)$

$$x+2y-3z+w=0$$
$$2x-y+4z+3w=0$$
$$3x+2y-z+2w=0$$

$b.)$

$$x_1+2x_2-4x_3+2x_4=3$$
$$3x_1+6x_2-8x_3-x_4=5$$
$$2x_1+5x_2+2x_3+3x_4=1$$

Inconsistent Systems

Inconsistent systems are systems with no solutions. For example, these systems have no solutions.

$$
\begin{aligned}
&a.) \\
&x-2y=-1 \\
&-x+2y=3 \\
&b.) \\
&3x_1+2x_2-x_3+2x_4=4 \\
&2x_1+x_2-2x_3+3x_4=1 \\
&3x_1+3x_2+3x_3-3x_4=5
\end{aligned}
$$

Deceptive systems

There are systems in which the number of linear equations decreases on reduction to echelon form. Actually, when a system is reduced to echelon form, sometimes two or more of its identical equations, which ordinarily would be hidden are made manifest. Such a system id referred to as a deceptive system. For example, these are deceptive systems.

$$
\begin{aligned}
&a.) \\
&x_1+2x_2-3x_3=6 \\
&2x_1-x_2+4x_3=2 \\
&4x_1+3x_2-2x_3=14 \\
&b.) \\
&x+2y-2z+3w=2 \\
&2x+4y-3z+4w=5 \\
&5x+10y-8z+11w=12 \\
&c.) \\
&2x_1+5x_2-4x_3=13 \\
&x_1+2x_2-3x_3=4 \\
&2x_1+6x_2+2x_3=22 \\
&x_1+3x_2+x_3=11
\end{aligned}
$$

VECTOR SPACES

Field

A field F is a non-empty set of scalars a, b, c such that the following properties hold:

1. For all scalars a, b, c in F,
   A1: a+b=b+a (Commutativity w.r.t. addition)
   A2: a+(b+c)=(a+b)+c (Associativity w.r.t. addition)
   A3: there exists 0 in F such that a+0=0+a=a (Additive identity)

A4: there exists –a in F such that a+(-a)=-a+a=0 (Additive inverse)

2. For all a, b, c in F,
   M1: a.b=b.a (Commutativity w.r.t. multiplication)
   M2: a(b.c)=(a.b)c (Associativity w.r.t. multiplication)
   M3: there exists 1 in F such that a.1=1.a=a (Multiplicative identity)
   M4: there exists a^{-1} in F such that a. a^{-1}= a^{-1}.a=1 (Multiplicative inverse)
3. For all a, b, c in F,
   D1: a.(b+c)=a.b+a.c (left distributivity)
   D2: (a+b).c=a.c+b.c (Right distributivity)

Examples of Field

The following sets are examples of field.

1. The set of real numbers $\mathbb{R}$
2. The set of rational numbers $\mathbb{Q}$
3. The set of complex numbers $\mathbb{C}$

Vector spaces

Let F be a given field and V a non-empty set closed under vector addition and scalar multiplication. Then V is called a vector space over the field F if the following axioms are hold:

1. For all vectors u, v, w in V,
   A1: u+v=v+u (Commutativity w.r.t. addition)
   A2: u+(v+w)=(u+v)+w (Associativity w.r.t. addition)
   A3: there exists 0 in V such that u+0=0+u=u (Additive identity)
   A4: there exists –u in V such that u+(-u)=-u+u=0 (Additive inverse)
2. For all u, v in V and a, b in F,
   M1: a(u + v)=au+av
   M2: (a+b)u=au+bu (Associativity w.r.t. multiplication)
   M3: (ab)u=a(bu)
   M4: there exists 1 in F such that 1.u=u.1=u

Examples of Vector Spaces

1. The usual picture of directed line segments in a plane, using the parallelogram law of addition.
2. The set of n-tuples of real numbers $(u_1, u_2, \cdots u_n)$ where addition and scalar multiplication are defined by
   i.) $(u_1, u_2, \cdots u_n) + (v_1, v_2, \cdots v_n) = (u_1 + v_1, u_2 + v_2, \cdots u_n + v_n)$
   ii.) $\alpha(u_1, u_2, \cdots, u_n) = (\alpha u_1, \alpha u_2, \cdots, \alpha u_n)$
3. The set of all polynomials
   $P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$
   Over a field F.
4. The set of all functions f(x) from a non-empty set X into a field F.
5. The set of all m x n matrices $M_{m\times n}(\mathbb{R})$ over the real field, $\mathbb{R}$.

6. The set of solution of the second order homogeneous differential equation with constant coefficients

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by-0$$

<u>Verification of the Examples</u>

<u>Number (2.)</u>

Let F be an arbitrary field. We consider the three dimensional Euclidean space $\mathbb{R}^3$ (i.e. 3-tuple of real numbers).

$$\mathbb{R}^3=\{(u_1,u_2,u_3):u_1,u_2,u_3\in\mathbb{R}\}$$

Let $u=(u_1,u_2,u_3),\ v=(v_1,v_2,v_3),\ and\ \ w=(w_1,w_2,w_3)$ be in $\mathbb{R}^3$.

A1:

$$\begin{aligned}u+v&=(u_1,u_2,u_3)+(v_1,v_2,v_3)\\&=(u_1+v_1,\ u_2+v_2,\ u_3+v_3)\\&=(v_1+u_1,\ v_2+u_2,\ v_3+u_3)\\&=(v_1,v_2,v_3)+(u_1,u_2,u_3)\\&=v+u\end{aligned}$$

A2:

$$\begin{aligned}u+(v+w)&=(u_1,u_2,u_3)+\left[(v_1,v_2,v_3)+(w_1,w_2,w_3)\right]\\&=(u_1,u_2,u_3)+\left[(v_1+w_1,\ v_2+w_2,\ v_3+w_3)\right]\\&=(u_1+v_1+w_1,\ u_2+v_2+w_2,\ u_3+v_3+w_3)\\&=\left[(u_1+v_1,\ u_2+v_2,\ u_3+v_3)\right]+(w_1,w_2,w_3)\\&=\left[(u_1,u_2,u_3)+(v_1,v_2,v_3)\right]+(w_1,w_2,w_3)\\&=[u+v]+w\end{aligned}$$

A3:

Let there exist 0 = (0,0,0) such that

$$\begin{aligned}u+0&=(u_1,u_2,u_3)+(0,0,0)\\&=(u_1+0,\ u_2+0,\ u_3+0)\\&=(0+u_1,\ 0+u_2,\ 0+u_3)\\&=(0,0,0)+(u_1,u_2,u_3)\\&=0+u\\&=u\end{aligned}$$

A4:

For $u=(u_1,u_2,u_3),\ let\ -u=-(u_1,u_2,u_3)=(-u_1,-u_2,-u_3)$ such that

$$
\begin{aligned}
u+(-u) &= (u_1,u_2,u_3)+(-u_1,-u_2,-u_3) \\
&= (u_1+(-u_1),\ u_2+(-u_2),\ u_3+(-u_3)) \\
&= (u_1-u_1,\ u_2-u_2,\ u_3-u_3) \\
&= (0,0,0) \\
&= 0
\end{aligned}
$$

M1:

$$
\begin{aligned}
a(u+v) &= a\left[(u_1,u_2,u_3)+(v_1,v_2,v_3)\right] \\
&= a\left[(u_1+v_1,\ u_2+v_2,\ u_3+v_3)\right] \\
&= \left[a(u_1+v_1),\ a(u_2+v_2),\ a(u_3+v_3)\right] \\
&= (au_1+av_1),\ (au_2+av_2),\ (au_3+av_3) \\
&= (au_1,au_2,au_3)+(av_1,av_2,av_3) \\
&= a(u_1,u_2,u_3)+a(v_1,v_2,v_3) \\
&= au+av
\end{aligned}
$$

M2:

$$
\begin{aligned}
(a+b)u &= (a+b)(u_1,u_2,u_3) \\
&= (a+b)u_1,\ (a+b)u_2,\ (a+b)u_3 \\
&= (au_1+bu_1,\ au_2+bu_2,\ au_3+bu_3) \\
&= (au_1,au_2,au_3)+(bu_1,bu_2,bu_3) \\
&= a(u_1,u_2,u_3)+b(u_1,u_2,u_3) \\
&= au+bu
\end{aligned}
$$

M3:

$$
\begin{aligned}
(ab)u &= (ab)(u_1,u_2,u_3) \\
&= (ab)u_1,\ (ab)u_2,\ (ab)u_3 \\
&= a(bu_1),\ a(bu_2),\ a(bu_3) \\
&= a(bu_1,bu_2,bu_3) \\
&= a(bu)
\end{aligned}
$$

M4:

Let there exist 1 = (1, 1, 1) such that

$$\begin{aligned}
1.u &= (1,1,1)(u_1,u_2,u_3)\\
&= (1.u_1,\ 1.u_2,\ 1.u_3)\\
&= (u_1.1,\ u_2.1,\ u_3.1)\\
&= (u_1,u_2,u_3)(1,1,1)\\
&= u.1\\
&= u
\end{aligned}$$

The three dimensional Euclidean space $\mathbb{R}^3$ satisfies all the axioms of a vector space. Hence $\mathbb{R}^3$ is a vector space.

<u>Number (3.)</u>

Let

$$\begin{aligned}
P_1(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\\
P_2(x) &= b_0 + b_1x + b_2x^2 + \cdots + b_nx^n\\
P_3(x) &= c_0 + c_1x + c_2x^2 + \cdots + c_nx^n
\end{aligned}$$

A1:

$$\begin{aligned}
P_1 + P_2 &= \left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right) + \left(b_0 + b_1x + b_2x^2 + \cdots + b_nx^n\right)\\
&= (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots + (a_n+b_n)x^n\\
&= (b_0+a_0) + (b_1+a_1)x + (b_2+a_2)x^2 + \cdots + (b_n+a_n)x^n\\
&= b_0 + a_0 + b_1x + a_1x + b_2x^2 + a_2x^2 + \cdots + b_nx^n + a_nx^n\\
&= \left(b_0 + b_1x + b_2x^2 + \cdots + b_nx^n\right) + \left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right)\\
&= P_2 + P_1
\end{aligned}$$

A2:

$$\begin{aligned}
P_1 + P_2 + P_3 &= \left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right) + \left[\left(b_0 + b_1x + b_2x^2 + \cdots + b_nx^n\right) + \left(c_0 + c_1x + c_2x^2 + \cdots + c_nx^n\right)\right]\\
&= \left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right) + \left[(b_0+c_0) + (b_1+c_1)x + (b_2+c_2)x^2 + \cdots + (b_n+c_n)x^n\right]\\
&= (a_0+b_0+c_0) + (a_1+b_1+c_1)x + (a_2+b_2+c_2)x^2 + \cdots + (a_n+b_n+c_n)x^n\\
&= \left[(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots + (a_n+b_n)x^n\right] + (c_0 + c_1x + c_2x^2 + \cdots + c_n)x^n\\
&= \left[\left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right) + \left(b_0 + b_1x + b_2x^2 + \cdots + b_nx^n\right)\right] + (c_0 + c_1x + c_2x^2 + \cdots + c_n)x^n\\
&= (P_1 + P_2) + P_3
\end{aligned}$$

A3: Let there exist $0 = 0 + 0x + 0x^2 + \cdots + 0x^n$ such that:

$$
\begin{aligned}
P_1 + 0 &= \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\right) + \left(0 + 0x + 0x^2 + \cdots + 0x^n\right) \\
&= (a_0 + 0) + (a_1 + 0)x + (a_2 + 0)x^2 + \cdots + (a_n + 0)x^n \\
&= (0 + a_0) + (0 + a_1)x + (0 + a_2)x^2 + \cdots + (0 + a_n)x^n \\
&= 0 + a_0 + 0x + a_1 x + 0x^2 + a_2 x^2 + \cdots + 0x^n + a_n x^n \\
&= \left(0 + 0x + 0x^2 + \cdots + 0x^n\right) + \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\right) \\
&= 0 + P_1 = P_1
\end{aligned}
$$

A4: Let there exist $-P_1 = -\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\right) = -a_0 - a_1 x - a_2 x^2 - \cdots - a_n x^n$ such that:

$$
\begin{aligned}
P_1 + (-P_1) &= \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\right) + \left(-a_0 - a_1 x - a_2 x^2 - \cdots - a_n x^n\right) \\
&= (a_0 + (-a_0)) + (a_1 + (-a_1))x + (a_2 + (-a_2))x^2 + \cdots + (a_n + (-a_n))x^n \\
&= (a_0 - a_0) + (a_1 - a_1)x + (a_2 - a_2)x^2 + \cdots + (a_n - a_n)x^n \\
&= 0 + 0x + 0x^2 + \cdots + 0x^n \\
&= 0
\end{aligned}
$$

M1: For $\alpha \; and \; \beta \; in \; F; \; and \; P_1 \; and \; P_2 \; in \; P_n(x)$,

$$
\begin{aligned}
\alpha\left(P_1 + P_2\right) &= \alpha\left[\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\right) + \left(b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n\right)\right] \\
&= \alpha\left[(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n\right] \\
&= \alpha(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n \\
&= \alpha a_0 + \alpha b_0 + \alpha a_1 x + \alpha b_1 x + \alpha a_2 x^2 + \alpha b_2 x^2 + \cdots + \alpha a_n x^n + \alpha b_n x^n \\
&= \alpha\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\right) + \alpha\left(b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n\right) \\
&= \alpha P_1 + \alpha P_2
\end{aligned}
$$

M2: For $\alpha \; and \; \beta \; in \; F; \; and \; P_1 \; in \; P_n(x)$,

$$
\begin{aligned}
(\alpha + \beta)P_1 &= (\alpha + \beta)\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\right) \\
&= (\alpha + \beta)a_0 + (\alpha + \beta)a_1 x + (\alpha + \beta)a_2 x^2 + \cdots + (\alpha + \beta)a_n x^n \\
&= \alpha a_0 + \beta a_0 + \alpha a_1 x + \beta a_1 x + \alpha a_2 x^2 + \beta a_2 x^2 + \cdots + \alpha a_n x^n + \beta a_n x^n \\
&= \alpha\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\right) + \beta\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\right) \\
&= \alpha\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\right) + \alpha\left(b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n\right) \\
&= \alpha P_1 + \beta P_1
\end{aligned}
$$

M3: For $\alpha \; and \; \beta \; in \; F; \; and \; P_1 \; in \; P_n(x)$,

$$
\begin{aligned}
(\alpha\beta)P_1 &= (\alpha\beta)\left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right) \\
&= (\alpha\beta)a_0 + (\alpha\beta)a_1x + (\alpha\beta)a_2x^2 + \cdots + (\alpha\beta)a_nx^n \\
&= \alpha(\beta a_0) + \alpha(\beta a_1)x + \alpha(\beta a_2)x^2 + \cdots + \alpha(\beta a_n)x^n \\
&= \alpha\left(\beta a_0 + \beta a_1x + \beta a_2x^2 + \cdots + \beta a_nx^n\right) \\
&= \alpha\left[\beta\left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right)\right] \\
&= \alpha(\beta P_1)
\end{aligned}
$$

M4:

Let there exists $1 = (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)^0$ such that:

$$
\begin{aligned}
1.P_1 &= \left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right)^0\left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right) \\
&= \left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right)^{0+1} \\
&= \left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right)^{1+0} \\
&= \left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right)\left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right)^0 \\
&= P_1.1 = P_1
\end{aligned}
$$

Classwork

Show that the set of all matrices $M_{2\times 2}(\mathbb{R})$ is a vector space.

Subspaces

Let V be a vector space over a field F. A subset S of V is said to be a subspace of V if S is itself a vector space with the same operations of addition and scalar multiplication as in V.

Theorem

Let S be a subset of a vector space V over a field F. S is a subspace of V if and only if the following conditions hold:

i.) S is non-empty, i.e. $0 \in S$

ii.) S is closed under vector addition, i.e. for every $u, v \in S,\ u + v \in S$

iii.) S is closed under scalar multiplication, i.e. for $u \in S$ *and* $\alpha \in F$, *then* $\alpha u \in S$

Examples of Subspaces

1. The singleton set $\{0\}$ consisting of the zero vector alone.
2. The whole vector space, V.

These are referred to as improper subspaces.

Verification

1. To show that the singleton set $\{0\}$ is a subspace, let $S=\{0\}$.
   i.) $0\in S$, so S is non-empty
   ii.) $0+0=0\in S$. So S is closed under addition.
   iii.) For $\alpha\in F$ and $0\in S$, then $\alpha 0\in S$. Thus S is closed under scalar multiplication.
   $\therefore$ $\{0\}$ is a subspace.
2. To show that V is a subspace of itself,
   i.) $0\in V$; V is non-empty. V being the entire vector space itself.
   ii.) For every $u,\ v\in V,\ u+v\in V$
   iii.) For $\alpha\in F$ and $u\in V$, then $\alpha u\in V$.
   $\therefore$ V is a subspace.

Example

Let V be the vector space $\mathbb{R}^3$. Show that the set $S=\{(a,b,c):a+b+c=0\}$ is a subspace of $\mathbb{R}^3$.

Solution

i.) Let $0=(0,0,0)$ be in S. Then 0+0+0=0. Thus $0\in S$. Hence S is non-empty.

ii.) Let $u=(a_1,b_1,c_1)$ and $v=(a_2,b_2,c_2)$ be in S.

$$u+v=(a_1,b_1,c_1)+(a_2,b_2,c_2)$$
$$=(a_1+a_2,\ b_1+b_2,\ c_1+c_2)$$

Now, to show that $(a_1+a_2,\ b_1+b_2,\ c_1+c_2)\in S$,

$$(a_1+a_2)+(b_1+b_2)+(c_1+c_2)=(a_1+b_1+c_1)+(a_2+b_2+c_2)$$
$$=\quad 0 \quad + \quad 0 \quad =0$$

iii.) Let $u=(a,b,c)$ be in S and $\alpha\in F$.

$$\alpha u=\alpha(a,b,c)=(\alpha a,\ \alpha b,\ \alpha c)$$

Now to show that $(\alpha a,\ \alpha b,\ \alpha c)\in S$,

$$\alpha a+\alpha b+\alpha c=\alpha(a+b+c)$$
$$=\alpha 0$$
$$=0$$

Thus, $S=\{(a,b,c):a+b+c=0\}$ is subspace of $\mathbb{R}^3$.

Theorem

Let S_1 and S_2 be two subspaces of a vector space V, then their intersection $S_1\cap S_2$ is also a subspace of V.

Proof

i.) Since S_1 and S_2 are subspaces, then $0 \in S_1$ *and* $0 \in S_2$. So $0 \in S_1 \cap S_2$. Hence $S_1 \cap S_2$ is non-empty.

ii.) Let $x, y \in S_1 \cap S_2$, then $x \in S_1$ *and* $x \in S_2$. *Also* $y \in S_1$ *and* $y \in S_2$.

$x + y \in S_1$ since S_1 is a subspace.

$x + y \in S_2$ since S_2 is a subspace.

$\therefore\ x + y \in S_1 \cap S_2$.

This establishes that $S_1 \cap S_2$ is closed under vector addition.

iii.) Let $\alpha \in F$ *and* $x \in S_1 \cap S_2$.

$x \in S_1$ *and* $x \in S_2$.

$\therefore\ \alpha x \in S_1$ *and* $\alpha x \in S_2$ (since S_1 *and* S_2 are subspaces).

Now $\alpha x \in S_1 \cap S_2$.

The intersection $S_1 \cap S_2$ is closed under scalar multiplication.

Thus $S_1 \cap S_2$ is a subspace of V.

Theorem

Let S_1 and S_2 be two subspaces of a vector space V, then their sum $S_1 + S_2$ is also a subspace of V.

Proof

Let S_1 and S_2 be two subspaces of V.

Define $S_1 + S_2 = \{x = y + z : y \in S_1 \text{ and } z \in S_2\}$

i.) $S_1 + S_2$ is not empty

ii.) Let $x_1, x_2 \in S_1 + S_2$ then there exist $y_1, y_2 \in S_1$ and $z_1, z_2 \in S_2$ such that

$$x_1 = y_1 + z_1$$

$$x_2 = y_2 + z_2$$

$$\begin{aligned} x_1 + x_2 &= (y_1 + z_1) + (y_2 + z_2) \\ &= (y_1 + y_2) + (z_1 + z_2) \end{aligned}$$

$y_1 + y_2 \in S_1$ since S_1 is a subspace

$z_1 + z_2 \in S_2$ since S_2 is a subspace

iii.) Let $\alpha \in F$ *and* $x \in S_1 + S_2$ such that $\alpha x = \alpha(y + z) = \alpha y + \alpha z$

$\alpha y \in S_1$ *and* $\alpha z \in S_2$ (since S_1 *and* S_2 are subspaces).

Thus $\alpha x \in S_1 + S_2$.

The intersection $S_1 + S_2$ is closed under scalar multiplication.

Thus $S_1 + S_2$ is a subspace of V.

Linear Combination

Let V be a vector over a field F and let $v_1, v_2, \cdots, v_n \in V$. Any vector $y \in V$ of the form $y = a_1v_1 + a_2v_2 + \cdots + a_nv_n$, where the $a_i 's \in F$, is called a linear combination of $v_1, v_2, \cdots, v_n$.

Example

Express the vector $y = (1, -2, 5)$ as a linear combination of the vectors $v_1 = (1,1,1),\ v_2 = (1,2,3),\ v_3 = (2,-1,1)$.

Solution

The vector y must be written in the form

$$y = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

$$(1,-2,5) = a_1(1,1,1) + a_2(1,2,3) + a_3(2,-1,1)$$

$$a_1 + a_2 + 2a_3 = 1$$
$$a_1 + 2a_2 - a_3 = -2$$
$$a_1 + 3a_2 + a_3 = 5$$

Solving the system of equations, $a_1 = -6,\ a_2 = 3,\ a_3 = 2$

$$\therefore\ y = -6v_1 + 3v_2 + 2v_3$$

Classwork

1. Express the polynomial $P = x^2 + 4x - 3$ as a linear combination of the polynomials $P_1 = x^2 - 2x + 5,\ P_2 = 2x^2 - 3x\ and\ P_3 = x + 3$
2. Write matrix A below as a linear combination of matrices B, C, D and E.
$A = \begin{bmatrix} 2 & 4 \\ -5 & 3 \end{bmatrix},\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\quad D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\quad E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Solution

1. $P = -3P_1 + 2P_2 + 4P_3$
2. $A = 2B + 4C - 5D + 3E$

Homework

Determine whether or not the vector $v = (2,-5,3)$ in $\mathbb{R}^3$ can be expressed as a linear combination of $v_1 = (1,-3,2),\ v_2 = (2,-4,-1),\ v_3 = (1,-5,7)$

Solution

The resulting system is inconsistent.

No. The vector cannot be expressed as linear combination of the vectors.

Linear Span

Let V be a vector over a field F and let $X = (x_1, x_2, \cdots, x_n)$ be a subset of V. Then the linear span of X, denoted by SpanX or L(X), is the set of all vectors in V which are linear combinations of elements of X.

SpanX is the smallest subspace of X containing X. It is the subspace spanned or generated by X.

In other words, $y \in SpanX$ if and only if there exist scalars $a_1, a_2, \cdots, a_n \in F$ such that $y = a_1x_1 + a_2x_2 + \cdots + a_nx_n$.

Example

Let $V = \mathbb{R}^2$ and X = {(1,0)}.

Show that SpanX is the x-axis.

Solution

$$\begin{aligned} y &= a(1,\ 0) \\ &= (a,\ 0),\ for\ all\ a\ in\ \mathbb{R} \\ \therefore SpanX &= \{(a,0)\} \end{aligned}$$

Example

Let $V = \mathbb{R}^3$ and X = {(2,1,0),(1,-1,2),(0,3,-4)}.

Determine the conditions on x_1, x_2, x_3 so that the vector $y = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ belongs to the span of X.

Solution

$$\begin{aligned} (x_1, x_2, x_3) &= a_1(2,1,0) + a_2(1,-1,2) + a_3(0,3,-4) \\ x_1 &= 2a_1 + a_2 \\ x_2 &= a_1 - a_2 + 3a_3 \\ x_3 &= 2a_2 - 4a_3 \end{aligned}$$

$y = (x_1, x_2, x_3)$ can only be in spanX if the system of equations is consistent.

Linear Dependence and Independence

Let V be a vector space over a field F. The vectors $v_1, v_2, \cdots, v_n$ in V are said to be linearly dependent if there exist scalars $a_1, a_2, \cdots, a_n$ in F, not all of them being zero, such that $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$.

The vectors $v_1, v_2, \cdots, v_3$ in V are said to be linearly independent if $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$ holds only if $a_1 = a_2 = \cdots = a_n = 0$.

<u>NB</u>

Two vectors are linearly dependent if one is a scalar multiple of the other.

Example

Show that the vectors $v_1 = (1,2,8),\ v_2 = (1,-3,-7),\ v_3 = (-1,1,1)$ are linearly dependent.

Solution

<u>Method I</u>

$$a_1(1,2,8) + a_2(1,-3,-7) + a_3(-1,1,1) = 0$$

$$\begin{aligned} a_1 + a_2 - a_3 &= 0 \\ 2a_1 - 3a_2 + a_3 &= 0 \\ 8a_1 - 7a_2 + a_3 &= 0 \end{aligned}$$

Solving the system, we have $a_1 = 2,\ a_2 = 3\ \textit{and}\ a_3 = 5$.

Since the scalars are not all zeros, then the vectors are linearly dependent.

<u>Method II</u>

Use the vectors to form a matrix. Reduce the matrix to echelon form. If the matrix has a zero row, then the vectors are linearly dependent; otherwise they are linearly independent. Thus,

$$\begin{bmatrix} 1 & 2 & 8 \\ 1 & -3 & -7 \\ -1 & 1 & 1 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -15 \\ 0 & 3 & 9 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to -5R_3 - 3R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -15 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a zero row, then the vectors are linearly dependent.

Method III

Use the vectors to form a matrix. If the matrix is singular (i.e. if determinant is 0), then the vectors are linearly dependent; otherwise they are linearly independent.

NB This is only applicable if the vectors form a square matrix.

$$\begin{vmatrix} 1 & 2 & 8 \\ 1 & -3 & -7 \\ -1 & 1 & 1 \end{vmatrix} = 0$$

Since the determinant of the formed matrix is zero, then the vectors are linearly dependent.

Classwork

Determine whether the following set of vectors are linearly dependent or otherwise.

1. $v_1 = (1,-2,1),\ v_2 = (2,1,-1),\ v_3 = (7,-4,1)$
2. $v_1 = (2,-3,1),\ v_2 = (-6,9,-3)$
3. $\{(1,-2,-3),(2,3,-1),(3,2,1)\}$

   Solution
   1. Dependent
   2. Dependent
   3. Independent

Basis and Dimension

Definition

A basis for a vector space V is a set of linearly independent vectors $v_1, v_2, \cdots, v_3$ in V with the property that for each v in V, there exist scalars $a_1, a_2, \cdots, a_n$ such that $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$.

It is evident from this definition that if a set X is a basis of a vector space V, then:

i. X is linearly independent
ii. Every element of V can be expressed as a linear combination of the elements of X

Definition

The dimension of a vector space V over a field F is the number of elements in the basis of V.

NB

The standard (usual) basis $X = \{e_1, e_2, \cdots, e_n\}$ for $\mathbb{R}^2$ is {(1, 0), (0,1)}; for $\mathbb{R}^3$ it is {(1,0,0),(0,1,0),(0,0,1)} and so on.

Example

Let $V = \{(x,y) : x, y \in \mathbb{R}\}$ and $X = \{(1,0),(0,1)\}$.

Show that X is a basis for V. Hence find the dimension of V.

Solution

i.) We show that (1, 0) and (0, 1) are linearly independent.

$a_1(1,0) + a_2(0,1) = (0,0)$

$a_1 = 0$

$a_2 = 0$

Since $a_1 = a_2 = 0$, then (1, 0) and (0, 1) are linearly independent.

ii.) We show that every element in V can be expressed as a linear combination of (1, 0) and (0, 1).

Let (x_1, x_2) be in V.

$(x_1, x_2) = a_1(1,0) + a_2(0,1)$

$x_1 = a_1$

$x_2 = a_2$

Since the scalars a_1 and a_2 are well-defined, every element of V can be written as a linear combination of (1, 0) and (0, 1).

Since (i) and (ii) are satisfied, then X is a basis for V.

The set X has only two elements, (1,0) and (0, 1).

Thus, the dimension of V is 2.

Example

Let $V = \mathbb{R}^3$ and $X = \{(1,1,1),(0,1,1),(0,1,-1)\}$.

Show that X is a basis for V.

Solution

i.) We show that (1, 0) and (0, 1) are linearly independent.

$a_1(1,1,1) + a_2(0,1,1) + a_3(0,1,-1) = (0,0,0)$

$a_1 = 0$

$a_1 + a_2 + a_3 = 0$

$a_1 + a_2 - a_3 = 0$

Solving this system of equations using Gaussian elimination, $a_1 = 0,\ a_2 = 0,\ a_3 = 0$.

Thus, $(1,1,1),(0,1,1),(0,1,-1)$ are linearly independent.

ii.) We show that every element in V can be expressed as a linear combination of $(1,1,1),(0,1,1),(0,1,-1)$.

Let (x_1, x_2, x_3) be in V.

$$(x_1, x_2, x_3) = a_1(1,1,1) + a_2(0,1,1) + a_3(0,1,-1)$$

$$a_1 = x_1$$

$$a_1 + a_2 + a_3 = x_2$$

$$a_1 + a_2 - a_3 = x_3$$

Solving this system for $a_1,\ a_2,\ a_3$,

$$a_3 = -\frac{(x_3 - x_2)}{2}$$

$$a_2 = \frac{x_2 + x_3 - 2x_1}{2}$$

$$a_3 = x_2 - x_3$$

The scalars are well-defined. Thus every element of V can be written as a linear combination of $(1,1,1),(0,1,1),(0,1,-1)$.

Since (i) and (ii) are satisfied, then X is a basis for V.

Classwork

Let $V = \mathbb{R}^3$ and $X = \{(1,-1,1),(1,2,-2),(-1,1,1)\}$.

Show that X is not a basis for $\mathbb{R}^3$.

Coordinate Vectors

Let $X = \{e_1, e_2, \cdots, e_n\}$ be a basis of an n dimensional vector space V over a field F. Let $v \in V$. Since X spans V, v is a linear combination of elements of X. That is, we can write

$$v = a_1e_1 + a_2e_2 + \cdots + a_ne_n$$

The scalars $a_1,\ a_2, \cdots, a_n$ are called the coordinates of v relative to X, and they form a vector $(a_1,\ a_2, \cdots, a_n)$, denoted by $[v]_e$.

Example

Let $v_1 = (1,-1,0),\ v_2 = (1,1,0)\ and\ v_3 = (0,1,1)$ be a basis. Obtain the coordinate vector of $v = (5,3,4)$ relative to the basis.

Solution

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3$$
$$(5,3,4) = a_1(1,-1,0) + a_2(1,1,0) + a_3(0,1,1)$$
$$a_1 + a_2 = 5$$
$$-a_1 + a_2 + a_3 = 3$$
$$a_3 = 4$$

$$a_1 = 3,\ a_2 = 3,\ a_3 = 4$$
$$v = 3v_1 + 2v_2 + 4v_3$$
$$\therefore [v]_e = (3,2,4)$$

Classwork

Obtain the coordinate vector of A relative to the given basis X, $[A]_e$

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 4 \end{bmatrix}$$
$$X = \left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

Solution

$$[A]_e = (\frac{5}{8}, \frac{9}{8}, -\frac{17}{8})$$

Linear Transformation

Let V and W be vector spaces over the same field, F. A transformation (mapping or homomorphism), $T: V \to W$, is a rule that assigns to each vector $v \in V$ a unique vector T(v) in W.

A transformation $T: V \to W$ is linear if:

$$i.)\ T(u+v) = T(u) + T(v), \qquad \forall u, v \in V$$
$$ii.)\ T(\alpha v) = \alpha T(v), \qquad \forall u, v \in V \ and \ \alpha \in F$$

Definitions

1. Image of T, (Im T)
   Let $T: V \to W$ be a linear mapping. The image of T is the vector T(v) in W, for v in V.
2. Range of T, (Range T)
   Let $T: V \to W$ be a linear mapping. The range of T is the set of all images.
   $$Range\ T = \{w \in W : T(v) = w \quad for\ v \in V\}$$

3. Kernel (Null space) of T, (Ker T)
   Let $T:V \to W$ be a linear mapping. The kernel of T is the set of all v in V such that T(0)=0.
   $Ker\ T = \{v \in V : T(v) = 0\}$
4. Rank and Nullity of T
   Let $T:V \to W$ be a linear mapping.
   The rank of T is the dimension of the range of T. Rank = dim(Range T).
   The nullity of T is the dimension of the kernel of T. Nullity = dim(ker T).

<u>Theorem</u>

Let V be of finite dimension and let $T:V \to W$ be a linear mapping. Then the dimension of V is equal to the sum of the rank and nullity of T. That is,

Dim V = dim(Range T) + dim(ker T)

= Rank + Nullity

Example

Show that for $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by T(x,y,z) = (0,x,y) is a linear transformation, where $v = (x, y, z) \in \mathbb{R}^3$.

Solution

Let $u = (x_1, y_1, z_1)\ and\ v = (x_2, y_2, z_2) \in \mathbb{R}^3$. We want to show that:

$i.)\ T(u+v) = T(u) + T(v)$
$ii.)\ T(\alpha v) = \alpha T(v)$

i.) From definition,

$$T(u) = (0, x_1, y_1)$$
$$T(v) = (0, x_2, y_2)$$
$$u + v = (x_1, y_1, z_1) + (x_2, y_2, z_2)$$
$$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

From definition,

$$T(u+v) = (0, x_1 + x_2, y_1 + y_2)$$
$$= (0, x_1, y_1) + (0, x_2, y_2)$$
$$= T(u) + T(v)$$
$$\therefore T(u+v) = T(u) + T(v)$$

ii.) $T(\alpha u) = T(\alpha(x_1, y_1, z_1)) = T(\alpha x_1, \alpha y_1, \alpha z_1)$

By definition,

$$T(\alpha x_1, \alpha y_1, \alpha z_1) = (0, \alpha x_1, \alpha y_1)$$
$$= \alpha(0, x_1, y_1)$$
$$= \alpha T(u)$$

Thus, $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear.

<u>Example</u>

Let the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y, z) = (x + y, x)$. Show that T is linear.

Solution

Let $v = (x_1, y_1)$ *and* $w = (x_2, y_2)$.

a.) $T(v) = (x_1 + y_1, x_1)$

$T(w) = (x_2 + y_2, x_2)$

$v + w = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

$T(v + w) = T(x_1 + x_2, y_1 + y_2)$

$= (x_1 + x_2 + y_1 + y_2,\ x_1 + x_2)$

$= \{[(x_1 + y_1) + (x_2 + y_2)],\ x_1 + x_2\}$

$= (x_1 + y_1, x_1) + (x_2 + y_2, y_2)$

$= \quad T(v) \quad + \quad T(w)$

b.) $T(\alpha v) = T(\alpha(x, y))$

$T(\alpha v) = T(\alpha x_1, \alpha y_1)$

$= (\alpha x_1 + \alpha y_1, \alpha x_1)$

$= \alpha(x_1 + y_1, x_1)$

$= \alpha T(v)$

$\therefore T \text{ is linear}$

<u>Example/Classwork</u>

Verify, in each case, whether the following mappings are linear or not:

1. $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = e^x$
2. $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (2x, -y, 4z)$
3. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x+1, 2y, x + y)$
4. $T : X \to X$ defined by $T(f(x)) = \dfrac{df}{dx}$, where is a vector space and $\dfrac{df}{dx}$ is the derivative of the function f.

Solution

1. Non-linear
2. Linear
3. Non-linear
4. Linear

Example on Rank and Nullity of T

Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y) = (x+2y, y-z, x+2z)$.

a.) Find the basis and dimension of its range (i.e. its rank)
b.) Find the basis and dimension of its kernel (i.e. its nullity)

Solution

a.) Using the standard basis in $\mathbb{R}^3$, $e_1 = (1,0,0),\ e_2 = (0,1,0)\ and\ e_3 = (0,0,1)$, we find the range of T.

$T(e_1) = T(1,0,0) = (1,0,1)$

$T(e_2) = T(0,1,0) = (2,1,0)$

$T(e_3) = T(0,0,1) = (0,-1,2)$

We now form the matrix of the images and reduce to echelon form to find the basis and dimension

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore Basis = \{(1,0,1), (0,1,-2)\}$

$Rank = Dim(Range\,T) = 2$

b.) To find Nullity, we find v such that T(v) = 0

$$\left.\begin{aligned} x + 2y \quad &= 0 \\ y - z &= 0 \\ x \quad + z &= 0 \end{aligned}\right\}$$

Solving this system of equations by elimination, x = -2, y = 1 and z = 1.

$\therefore Basis = (-2,1,1)$

$Nullity = Dim(Ker\,T) = 1$

Classwork

Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y, z) = (3x - y + 4z, x + y + 2z, 2x + 4y - z)$. Determine the rank and nullity of T.

Definition

1. A linear mapping $T:V \to W$ is said to be singular if the image of some non-zero vector under T is zero. That is, for non-zero v in V, T(v) = 0.
2. A linear mapping $T:V \to W$ is said to be non-singular if and only if $0 \in V$ maps into $0 \in W$. Equivalently, T is non-singular if its kernel consists only of the zero vector.

Matrix representations of linear transformations

Let V and W be vector spaces of dimension m and n respectively. Let $X = \{x_1, x_2, \cdots, x_m\}$ be a basis of V; and $Y = \{y_1, y_2, \cdots, y_n\}$ be a basis of W. Let $T:V \to W$ be a linear mapping. $T(x_i)$ is a linear combination of the basis of $Y = \{y_1, y_2, \cdots, y_n\}$.

$$T(x_i) = \sum_{j=1}^{n} a_{ij} y_i, \qquad for\ i = 1, 2, \cdots, m\ \ and\ \ j = 1, 2, \cdots, n$$

Expanding this, we have

$$\begin{aligned} T(x_1) &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ T(x_2) &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ &\vdots \\ T(x_m) &= a_{m1}y_1 + a_{m2}y_2 + \cdots + a_{mn}y_n \end{aligned}$$

If we define the coefficient matrix of the above system as A, then A^T is the matrix representation of the linear mapping T relative to the bases X and Y.

Example

Let $T:V \to W$ be a linear mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by $T(x, y, z) = (x, y + z)$. Compute the matrix of T relative to:

a.) The usual bases of $\mathbb{R}^3$ to $\mathbb{R}^2$
b.) The bases $X = \{(1,1,0), (0,1,1), (1,0,1)\}$ and $Y = \{(1,0), (0,1)\}$.

Solution

a.) The usual bases are $X = \{(1,0,0), (0,1,0), (0,0,1)\}$ and $Y = \{(1,0), (0,1)\}$

$$T(e_1) = T(1,0,0) = (1,0)$$
$$T(e_2) = T(0,1,0) = (0,1)$$
$$T(e_3) = T(0,0,1) = (0,1)$$

$$(1,0) = a_{11}(1,0) + a_{12}(0,1)$$
$$(0,1) = a_{21}(1,0) + a_{22}(0,1)$$
$$(0,1) = a_{31}(1,0) + a_{32}(0,1)$$

$$a_{11} = 1,\ \ a_{12} = 0$$
$$a_{21} = 0,\ \ a_{22} = 1$$
$$a_{31} = 0,\ \ a_{32} = 1$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$
$$\therefore A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

b.) The bases are $X = \{(1,1,0),(1,0,1),(0,0,1)\}$ and $Y = \{(1,0),(0,1)\}$

$$T(x_1) = T(1,1,0) = (1,1)$$
$$T(x_2) = T(1,0,1) = (0,2)$$
$$T(x_3) = T(0,0,1) = (1,1)$$

$$(1,1) = a_{11}(1,0) + a_{12}(0,1)$$
$$(0,2) = a_{21}(1,0) + a_{22}(0,1)$$
$$(1,1) = a_{31}(1,0) + a_{32}(0,1)$$

$$a_{11} = 1,\ \ a_{12} = 1$$
$$a_{21} = 0,\ \ a_{22} = 2$$
$$a_{31} = 1,\ \ a_{32} = 1$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$$
$$\therefore A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$