example values a deferred American swaption in a 20-factor string model where each point on the interest-rate curve is a separate factor. We also show how the algorithm can be used in a risk-management context by computing the sensitivity of swaption values to each point along the curve.

A number of other recent articles also address the pricing of American options by simulation. In an important early contribution to this literature, Bossaerts (1989) solves for the exercise strategy that maximizes the simulated value of the option. Other important examples of this literature include Tilley (1993), Barraquand and Martineau (1995), Averbukh (1997), Broadie and Glasserman (1997a,b,c), Broadie, Glasserman, and Jain (1997), Raymar and Zwecher (1997), Broadie et al. (1998), Carr (1998), Ibanez and Zapatero (1998), and Garcia (1999). These articles use various stratification or parameterization techniques to approximate the transitional density function or the early exercise boundary. This article takes a fundamentally different approach by focusing directly on the conditional expectation function.

Several recent articles that use an approach similar to ours include Carriere (1996) and Tsitsiklis and Van Roy (1999). Our work, however, differs in a number of ways. For example, neither of these articles take the approach to the level of practical implementation we do in this article. Furthermore, we include in the regression only paths for which the option is in the money. This significantly increases the efficiency of the algorithm and decreases the computational time. In addition, we demonstrate the application of the methodology to complex derivatives with many underlying factors and evaluate the accuracy of the algorithm by comparing our solutions to finite difference approximations.[3]

The remainder of this article is organized as follows. Section 1 presents a simple numerical example of the simulation approach. Section 2 describes the underlying theoretical framework. Sections 3–7 provide specific examples of the application of this approach. Section 8 discusses a number of numerical and implementation issues. Section 9 summarizes the results and contains concluding remarks.

1. A Numerical Example

At the final exercise date, the optimal exercise strategy for an American option is to exercise the option if it is in the money. Prior to the final date, however, the optimal strategy is to compare the immediate exercise value with the expected cash flows from continuing, and then exercise if immediate exercise is more valuable. Thus, the key to optimally exercising an American option is identifying "the conditional expected value of continuation." In this approach, we use the cross-sectional information in the simulated paths to

[3] Another related article is Keane and Wolpin (1994), which uses regression in a simulation context to solve discrete choice dynamic programming problems.

identify the conditional expectation function. This is done by regressing the subsequent realized cash flows from continuation on a set of basis functions of the values of the relevant state variables. The fitted value of this regression is an efficient unbiased estimate of the conditional expectation function and allows us to accurately estimate the optimal stopping rule for the option.

Perhaps the best way to convey the intuition of the LSM approach is through a simple numerical example. Consider an American put option on a share of non-dividend-paying stock. The put option is exercisable at a strike price of 1.10 at times 1, 2, and 3, where time three is the final expiration date of the option. The riskless rate is 6%. For simplicity, we illustrate the algorithm using only eight sample paths for the price of the stock. These sample paths are generated under the risk-neutral measure and are shown in the following matrix.

Stock price paths

Path	$t=0$	$t=1$	$t=2$	$t=3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	.93	.97	.92
5	1.00	1.11	1.56	1.52
6	1.00	.76	.77	.90
7	1.00	.92	.84	1.01
8	1.00	.88	1.22	1.34

Our objective is to solve for the stopping rule that maximizes the value of the option at each point along each path. Since the algorithm is recursive, however, we first need to compute a number of intermediate matrices. Conditional on not exercising the option before the final expiration date at time 3, the cash flows realized by the optionholder from following the optimal strategy at time 3 are given below.

Cash-flow matrix at time 3

Path	$t=1$	$t=2$	$t=3$
1	—	—	.00
2	—	—	.00
3	—	—	.07 = 1.10 − 1.03
4	—	—	.18
5	—	—	.00
6	—	—	.20
7	—	—	.09
8	—	—	.00

OTM

These cash flows are identical to the cash flows that would be received if the option were European rather than American.

If the put is in the money at time 2, the optionholder must then decide whether to exercise the option immediately or continue the option's life until the final expiration date at time 3. From the stock-price matrix, there are only five paths for which the option is in the money at time 2. Let X denote the stock prices at time 2 for these five paths and Y denote the corresponding discounted cash flows received at time 3 if the put is not exercised at time 2. We use only in-the-money paths since it allows us to better estimate the conditional expectation function in the region where exercise is relevant and significantly improves the efficiency of the algorithm. The vectors X and Y are given by the nondashed entries below.

$e^{-6\% \times 1yr}$

Regression at time 2

Path	Y	X
1	.00 × .94176	1.08
2	—	—
3	.07 × .94176	1.07
4	.18 × .94176	.97
5	—	—
6	.20 × .94176	.77
7	.09 × .94176	.84
8	—	

3yr ITM額

t=2の株価

To estimate the expected cash flow from continuing the option's life conditional on the stock price at time 2, we regress Y on a constant, X, and X^2. This specification is one of the simplest possible; more general specifications are considered later in the article. The resulting conditional expectation function is $E[\, Y \mid X \,] = -1.070 + 2.983X - 1.813X^2$.

Xの条件の下のYの期待値
＝
継続価値

With this conditional expectation function, we now compare the value of immediate exercise at time 2, given in the first column below, with the value from continuation, given in the second column below.

Optimal early exercise decision at time 2

Path	Exercise	Continuation
1	.02	.0369
2	—	—
3	.03	.0461
4	.13	.1176
5	—	—
6	.33	.1520
7	.26	.1565
8	—	—

1.10−1.08 (t=2)

Exercise となるので

The value of immediate exercise equals the intrinsic value $1.10 - X$ for the in-the-money paths, while the value from continuation is given by substituting X into the conditional expectation function. This comparison implies that it is optimal to exercise the option at time 2 for the fourth, sixth, and seventh paths. This leads to the following matrix, which shows the cash flows received by the optionholder conditional on not exercising prior to time 2.

Cash-flow matrix at time 2

Path	$t=1$	$t=2$	$t=3$
1	—	.00	.00
2	—	.00	.00
3	—	.00	.07
4	—	.13	.00
5	—	.00	.00
6	—	.33	.00
7	—	.26	.00
8	—	.00	.00

Observe that when the option is exercised at time 2, the cash flow in the final column becomes zero. This is because once the option is exercised there are no further cash flows since the option can only be exercised once.

Proceeding recursively, we next examine whether the option should be exercised at time 1. From the stock price matrix, there are again five paths where the option is in the money at time 1. For these paths, we again define Y as the discounted value of subsequent option cash flows. Note that in defining Y, we use actual realized cash flows along each path; we do not use the conditional expected value of Y estimated at time 2 in defining Y at time 1. As is discussed later, discounting back the conditional expected value rather than actual cash flows can lead to an upward bias in the value of the option.

Since the option can only be exercised once, future cash flows occur at either time 2 or time 3, but not both. Cash flows received at time 2 are discounted back one period to time 1, and any cash flows received at time 3 are discounted back two periods to time 1. Similarly X represents the stock prices at time 1 for the paths where the option is in the money. The vectors X and Y are given by the nondashed elements in the following matrix.

Regression at time 1

Path	Y	X
1	.00 × .94176	1.09
2	—	—
3	—	—
4	.13 × .94176	.93
5	—	—
6	.33 × .94176	.76
7	.26 × .94176	.92
8	.00 × .94176	.88

t=1 株価

t=1 株価から見た t=2 株価の
期待値 = E[Y | X]

The conditional expectation function at time 1 is estimated by again regressing Y on a constant, X and X^2. The estimated conditional expectation function is $E[\ Y \mid X\ ] = 2.038 - 3.335X + 1.356X^2$. Substituting the values of X into this regression gives the estimated conditional expectation function. These estimated continuation values and immediate exercise values at time 1 are given in the first and second columns below. Comparing the two columns shows that exercise at time 1 is optimal for the fourth, sixth, seventh, and eighth paths.

Optimal early exercise decision at time 1

Path	Exercise	Continuation
1	.01	.0139
2	—	—
3	—	—
4	.17	.1092
5	—	—
6	.34	.2866
7	.18	.1175
8	.22	.1533

Exercise

1.10 − 0.88

Having identified the exercise strategy at times 1, 2, and 3, the stopping rule can now be represented by the following matrix, where the ones denote exercise dates at which the option is exercised.

Stopping rule

Path	$t = 1$	$t = 2$	$t = 3$
1	0	0	0
2	0	0	0
3	0	0	1
4	1	0	0
5	0	0	0
6	1	0	0
7	1	0	0
8	1	0	0

With this specification of the stopping rule, it is now straightforward to determine the cash flows realized by following this stopping rule. This is done by simply exercising the option at the exercise dates where there is a one in the above matrix. This leads to the following option cash flow matrix.

Option cash flow matrix

Path	$t = 1$	$t = 2$	$t = 3$
1	.00	.00	.00
2	.00	.00	.00
3	.00	.00	.07
4	.17	.00	.00
5	.00	.00	.00
6	.34	.00	.00
7	.18	.00	.00
8	.22	.00	.00

$e^{-6\%}$

$.07 \times e^{-6\% \times 3}$

合计 0.9154

$\div 8 = 0.1144$

Having identified the cash flows generated by the American put at each date along each path, the option can now be valued by discounting each cash flow in the option cash flow matrix back to time zero, and averaging over all paths. Applying this procedure results in a value of .1144 for the American put. This is roughly twice the value of .0564 for the European put obtained by discounting back the cash flows at time 3 from the first cash flow matrix.

Although very stylized, this example illustrates how least squares can use the cross-sectional information in the simulated paths to estimate the conditional expectation function. In turn, the conditional expectation function is used to identify the exercise decision that maximizes the value of the option at each date along each path. As shown by this example, the LSM approach is easily implemented since nothing more than simple regression is involved.

2. The Valuation Algorithm

In this section we describe the general LSM algorithm. The valuation framework underlying the LSM algorithm is based on the general derivative pricing paradigm of Black and Scholes (1973), Merton (1973), Harrison and Kreps (1979), Harrison and Pliska (1981), Cox, Ingersoll, and Ross (1985), Heath, Jarrow, and Morton (1992), and others. We also present several convergence results for the algorithm.

2.1 The valuation framework

We assume an underlying complete probability space $(\Omega, \mathcal{F}, P)$ and finite time horizon $[0, T]$, where the state space Ω is the set of all possible realizations of the stochastic economy between time 0 and T and has typical element ω representing a sample path, $\mathcal{F}$ is the sigma field of distinguishable events at time T, and P is a probability measure defined on the elements of $\mathcal{F}$. We define $F = \{\mathcal{F}_t; t \in [0, T]\}$ to be the augmented filtration generated by the relevant price processes for the securities in the economy, and assume that $\mathcal{F}_T = \mathcal{F}$. Consistent with the no-arbitrage paradigm, we assume the existence of an equivalent martingale measure Q for this economy.

We are interested in valuing American-style derivative securities with random cash flows which may occur during $[0, T]$. We restrict our attention to