# 8.4 Phase Plane Method

When the nonlinearity of system is ***serious*** or we can not use the describing function method while there are some ***non-periodic*** 非周期 inputs, the phase plane method is still available for these problems.

Phase plane method was first proposed in 1885 by *Jules Henri Poincaré*.

# Scenario

- It is a graphical method for studying ***first-order, second-order*** linear systems or nonlinear systems.
- The phase plane method can be used for analyzing the ***stability, equilibrium position and steady-state accuracy***.
  平衡点
- It also can be used for analyzing the impact on the system motion of ***initial conditions*** and ***parameters*** of this system

# Idea

- The essence of this method is ***visually transforming the motion process of the system into the motion of a point in the phase plane***.
- We can obtain all information regarding the motion patterns of system by studying the motion trajectory of this point.
- Now this method is widely used, because it can intuitively, accurately and comprehensively character the motion states of the system.

# 8.4.1 Basic Concepts of The Phase Plane Method

## (1) Phase plane and Phase portrait

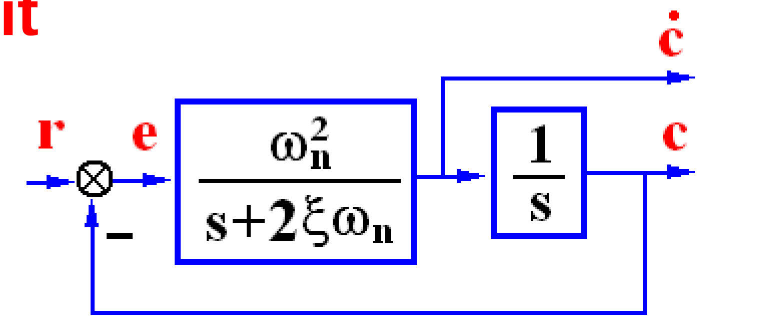
相轨迹

### Phase plane:

The $x_1 - x_2$ plane is called ***Phase Plane***, where $x_1, x_2$ are the system state and its derivative ( $c$, $\dot{c}$ ).

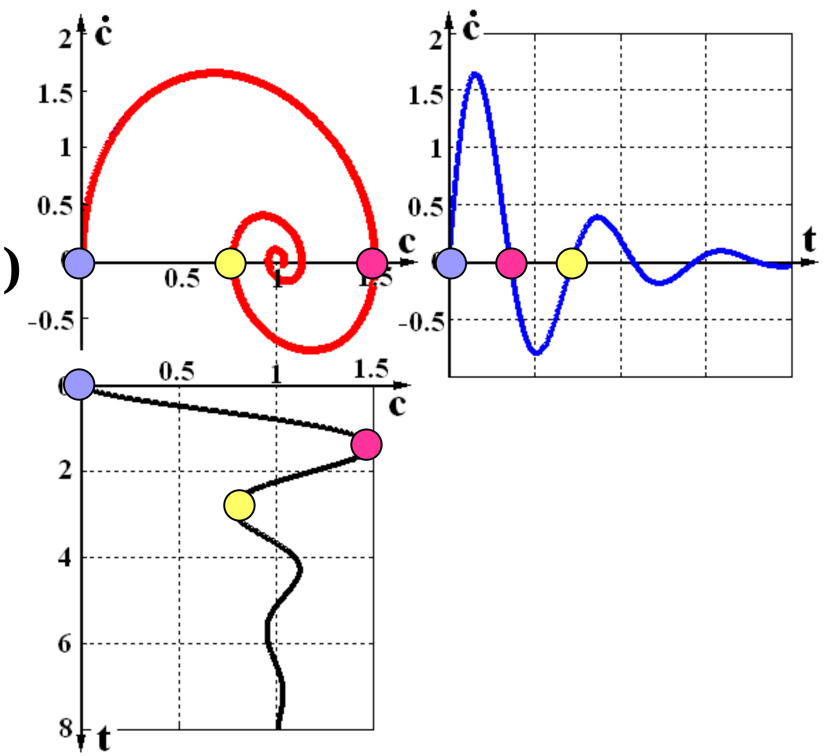
微分

### Phase portrait:




**Example 8.1**: A unit feedback system

$$G(s) = \frac{5}{s(s+1)} \qquad \begin{cases} \omega_n = 2.236 \\ \xi = 0.2236 \end{cases} \qquad r(t) = 1(t)$$




The locus in the x1-x2 plane of the solution x(t) for all t>= 0 is a curve named ***trajectory*** or ***orbit*** that passes through the point x0.

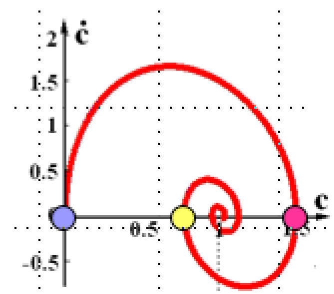
相轨迹

The family of phase plane trajectories corresponding to various initial conditions is called ***Phase Portrait*** of the system.

相平面图

# 8.4.1 Basic Concepts of The Phase Plane Method

## (2) The features of  phase trajectory




任意二阶非线性系统

**System equation:** $\ddot{x} + f(x, \dot{x}) = 0$

**Motion Direction**
- **Upper half-plane** $\dot{x} > 0$ — **move right**
- **Lower half-plane** $\dot{x} < 0$ — **move left**

**clockwise motion**

垂直

**Passing the X-axis** $(\dot{x} = 0)$ **perpendicularly.**

平衡点

**Except the equilibrium points , there is only one phase trajectory passing through any point  in  the phase plane.** It is determined by the existence and uniqueness of solutions of differential equations.

## 8.4.2 Methods of Constructing Phase Plane Trajectories

- **Analytical Method**
- **Isocline Method**
- **Experimental Method**

## ■ Analytical Method

For an arbitrary second-order nonlinear differential equations：

$$\ddot{x} + f(x, \dot{x}) = 0$$

Or

$$\ddot{x} + a_1(x, \dot{x})\dot{x} + a_0(x, \dot{x})x = 0$$

Let $x_1 = x$

$x_2 = \dot{x}_1 = \dot{x}$

Then:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{x} = -a_1(x_1, x_2)x_2 - a_0(x_1, x_2)x_1 \end{cases}$$

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{dx_2}{dx_1} = \frac{-a_1(x_1, x_2)x_2 - a_0(x_1, x_2)x_1}{x_2}$$

Rewrite the system equations in a general form:

$$\begin{cases} \dot{x}_1 = P(x_1, x_2) \\ \dot{x}_2 = Q(x_1, x_2) \end{cases}$$

$$\frac{dx_2}{dx_1} = \frac{Q(x_1, x_2)}{P(x_1, x_2)} \Longrightarrow$$ **The slope of the trajectory at point ($x_1$, $x_2$).**

$$\frac{dx_2}{dx_1} = \frac{Q(x_1, x_2)}{P(x_1, x_2)}$$

- ✓ If $P(x_1, x_2)$ 、$Q(x_1, x_2)$ is analytic，the differential equation can then be solved.
- ✓ Given an initial condition, the solution can be plotted in the phase plane. This curve is named ***Phase trajectory***.
- ✓ The family of phase plane trajectories is called ***Phase portrait***.

例：线性二阶系统的相轨迹 $\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x=0 \qquad \zeta=0$

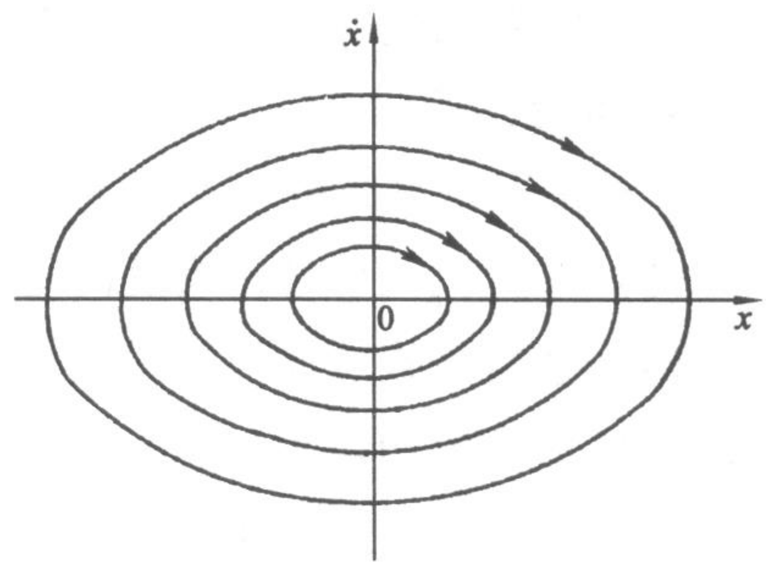
$$\frac{dx}{dt}=\dot{x} \qquad \frac{d\dot{x}}{dt}=\ddot{x}=-2\zeta\omega_n\dot{x}-\omega_n^2x$$

$$\frac{dx_2}{dx_1}=\frac{Q(x_1,x_2)}{P(x_1,x_2)}$$

$$\frac{d\dot{x}}{dx}=-\omega_n^2x$$

$$\dot{x}d\dot{x}=-\omega_n^2xdx$$

$$\therefore\ \frac{\dot{x}^2}{\omega_n^2}+x^2=A^2$$




其中A是初条决定的积分常数，此为同心椭圆。

$$\frac{dx_2}{dx_1}=\frac{Q(x_1,x_2)}{P(x_1,x_2)}$$

- ✓ If $P(x_1,x_2)$ 、$Q(x_1,x_2)$ is analytic，the differential equation can then be solved.
- ✓ Given an initial condition, the solution can be plotted in the phase plane. This curve is named ***Phase trajectory***.
- ✓ The family of phase plane trajectories is called ***Phase portrait***.

Assume $\begin{cases}\dot{x}_1=P(x_1,x_2)=0\\ \dot{x}_2=Q(x_1,x_2)=0\end{cases}$

平衡点

✓ The solutions $(x_{10},x_{20})$ is called the ***equilibrium points*** of system.

✓ 由式 $\frac{dx_2}{dx_1}=\frac{Q}{P}=\frac{0}{0}$ 可知，相轨迹在平衡点附近切线斜率不定，意味着有无穷多根相轨迹到达或离开平衡点。

# Review

- Concept of ***Phase Plane*** and ***Phase Portrait***:
  - $x_1$ – **position**, $x_2$ – **velocity**
  - For a solution of system equation obtained from a given initial condition, there is a ***trajectory*** in the phase plane.

$$\frac{dx_2}{dx_1} = \frac{Q(x_1, x_2)}{P(x_1, x_2)}$$

- Features of phase trajectory:
  - ***Clockwise*** motion.
  - Except the equilibrium points, there is ***only one phase trajectory*** passing through any point in the phase plane.

## 8.4.3 Singular Point and Limit Cycle

奇点 极限环

The purpose of plotting the phase trajectory is to analyze the dynamic characteristics.

- Because there are infinite phase trajectories leaving or arriving at the ***equilibrium point*****,** the phase trajectories near the ***equilibrium point*** reflect the dynamic characteristics of the system. 极限环

  ***Equilibrium points*** is also called ***singular points***.

- ***Limit cycle*** is another phase trajectory which can reflect the dynamic characteristics of the system.

  孤立

  Limit cycle is an ***Isolated and Closed*** phase trajectory, which describes the ***harmonic oscillation*** of a system. It divides the infinite phase plane into two parts. 自激振荡

## Singular Point 奇点

Singular Points are the equilibrium points $(x_{10}, x_{20})$ , which are obtained by solving the following equations.

$$\begin{cases} \dot{x}_1 = P(x_1, x_2) = 0 \\ \dot{x}_2 = Q(x_1, x_2) = 0 \end{cases}$$

* The singular point can only appear on the X-axis.

To study the shape and dynamic characteristics of the phase trajectories near the equilibrium $(x_{10}, x_{20})$ , we expand the function $P(x_1, x_2)$ , $Q(x_1, x_2)$ into Taylor series around it. Ignoring the higher-order terms, without loss of generality we assume that

$$x_{10} = x_{20} = 0$$

Then
$$P(x_1, x_2) = \left.\frac{\partial P(x_1, x_2)}{\partial x_1}\right|_{(0,0)} x_1 + \left.\frac{\partial P(x_1, x_2)}{\partial x_2}\right|_{(0,0)} x_2$$

$$Q(x_1, x_2) = \left.\frac{\partial Q(x_1, x_2)}{\partial x_1}\right|_{(0,0)} x_1 + \left.\frac{\partial Q(x_1, x_2)}{\partial x_2}\right|_{(0,0)} x_2$$

$$P(x_1, x_2) = \left.\frac{\partial P(x_1, x_2)}{\partial x_1}\right|_{(0,0)} x_1 + \left.\frac{\partial P(x_1, x_2)}{\partial x_2}\right|_{(0,0)} x_2$$

$$Q(x_1, x_2) = \left.\frac{\partial Q(x_1, x_2)}{\partial x_1}\right|_{(0,0)} x_1 + \left.\frac{\partial Q(x_1, x_2)}{\partial x_2}\right|_{(0,0)} x_2$$

Assume $a = \left.\frac{\partial P(x_1, x_2)}{\partial x_1}\right|_{(0,0)}$ $b = \left.\frac{\partial P(x_1, x_2)}{\partial x_2}\right|_{(0,0)}$ $c = \left.\frac{\partial Q(x_1, x_2)}{\partial x_1}\right|_{(0,0)}$ $d = \left.\frac{\partial Q(x_1, x_2)}{\partial x_2}\right|_{(0,0)}$

Then $\begin{cases} \dot{x}_1 = ax_1 + bx_2 \\ \dot{x}_2 = cx_1 + dx_2 \end{cases}$

The characteristic equation of system is given by

$$|\lambda I - A| = \lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

The roots of the above equation is

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2}$$

According to the property of these roots, the singular points can be divided into the following classes.

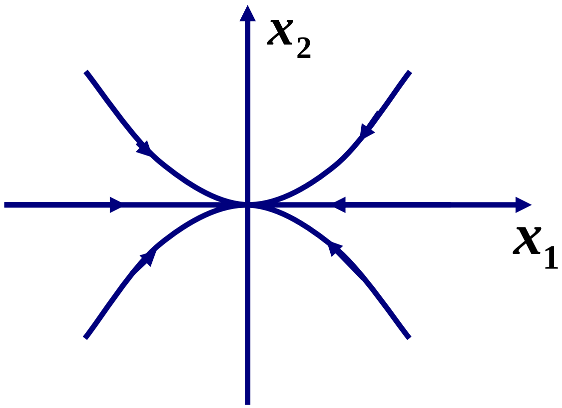
$$\lambda_{1,2}=\frac{a+d\pm\sqrt{(a+d)^2-4(ad-bc)}}{2}$$

## ① Different real roots with the same sign

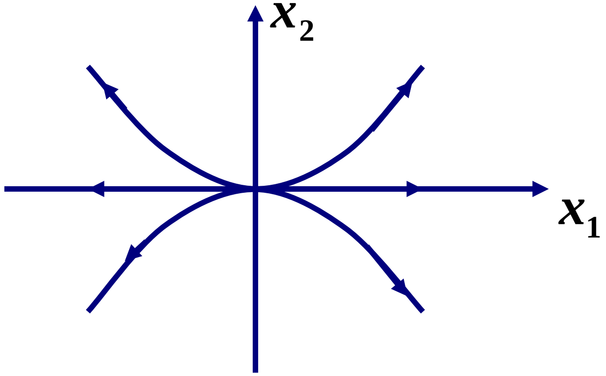
$$(a+d)^2>4(ad-bc)$$

If $a+d<0$ , two roots are all negative, singular point is called **stable node**.

If $a+d>0$ , two roots are all positive, singular point is called **unstable node**.




**(a) stable node**




**(b) unstable node**

**Fig 8-28 Phase trajectory in this case**

$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

## ② Real roots with different signs

$$ad - bc < 0$$

Singular point is called **saddle point** 鞍点

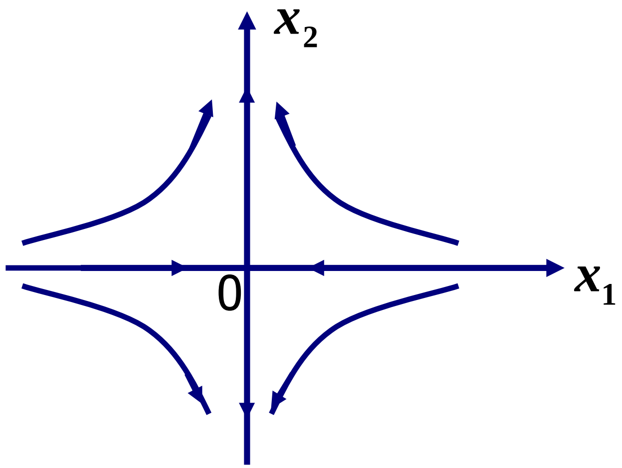



**Fig 8-29 Phase trajectories that are corresponding to a saddle point**

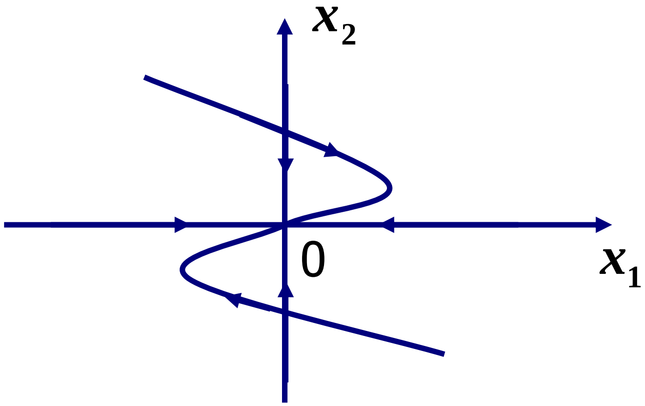
$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

## ③ Double Root

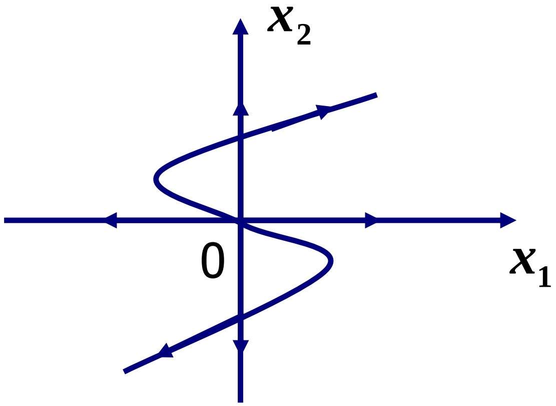
$$(a+d)^2 = 4(ad-bc)$$

If $a+d<0$, there are two equal negative real roots . Singular point is called **degraded stable node**;

If $a+d>0$, there are two equal positive real roots . Singular point is called **degraded unstable node**;




(a) double negative roots




(b) double positive roots

Fig 8-30 Phase trajectories that are corresponding to double point

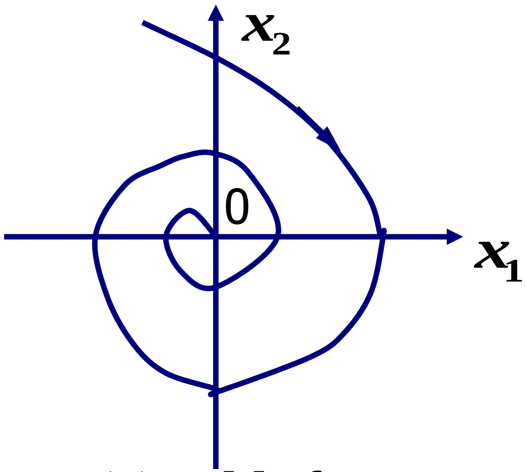
$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

## ④ Complex Conjugate Root 共轭复数根

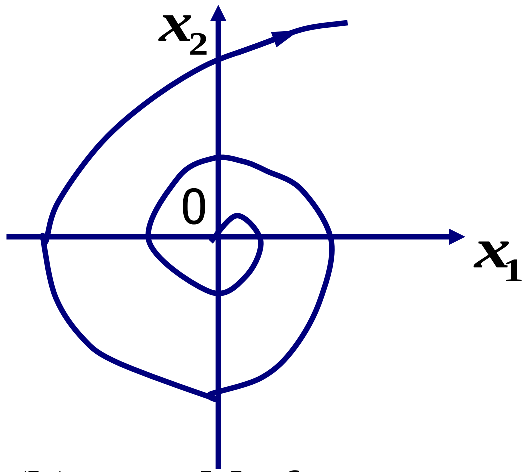
$$(a+d)^2 < 4(ad-bc)$$

If $a+d<0$, there are complex conjugate roots with negative real component. Singular point is called **stable focus**;

If $a+d>0$, there are complex conjugate roots with positive real component. Singular point is called **unstable focus**;




(a)stable focus




(b)unstable focus

Fig 8-31 Phase trajectories that are corresponding to complex conjugate roots

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}$$

## ⑤ Purely imaginary roots

$$(a + d) = 0, \quad ad - bc > 0$$

Singular point is called **center**.

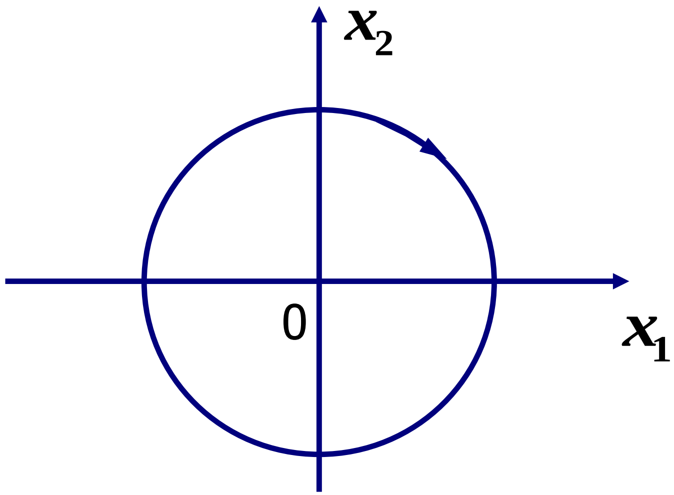



**Fig 8-32** Phase trajectories that are corresponding to Purely imaginary roots

# Limit Cycle 极限环

A *closed and isolated phase trajectory* in the phase plane is called a **limit cycle**. It is corresponding to the *harmonic oscillation state* of a system.

Limit cycle can be easily found in actual physical systems. For example , the response of an unstable linear control system is theoretically a divergent oscillation. Whereas, in reality the amplitude of response may tends to a constant value due to the non-linear characteristics like saturation.

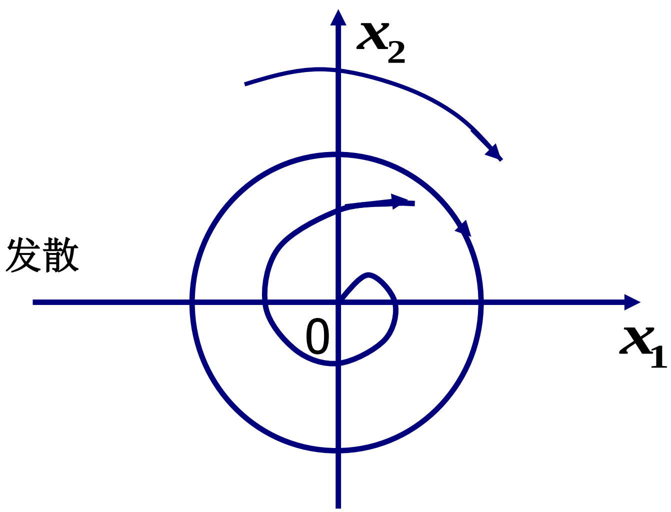
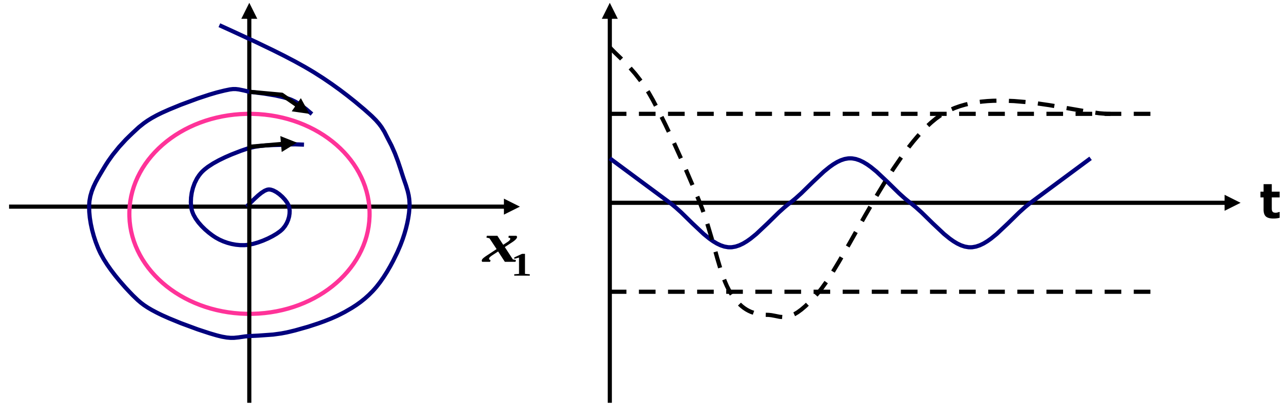



**Fig 8-33 limit cycle**

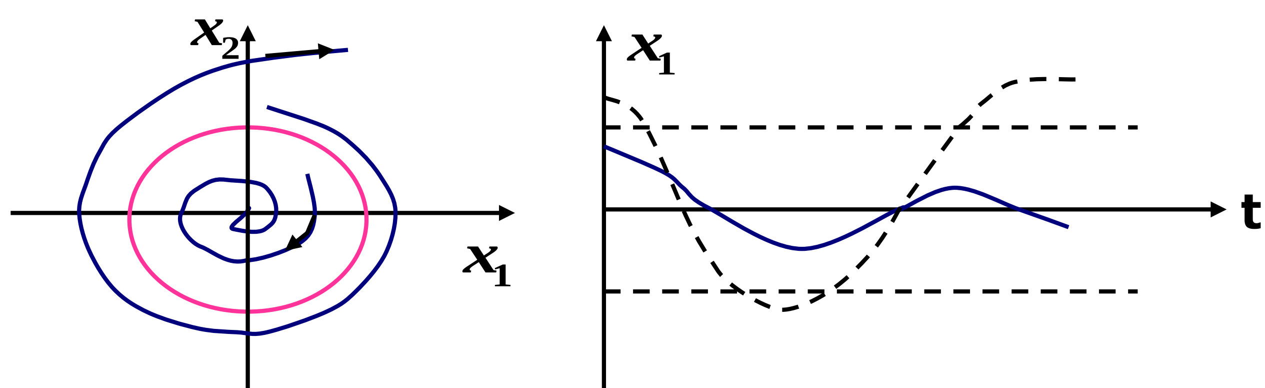
- Limit cycle divides the phase-plane into two parts : the part inside the limit cycle and the part outside the limit cycle. ***Any phase trajectory can not enter one part from the other*.**

- Something should be pointed out, not all the closed curves in phase-plane are limit cycles. (Think about the trajectories corresponding to a center.) This kind of curves are not limit cycle, for they are not isolated.

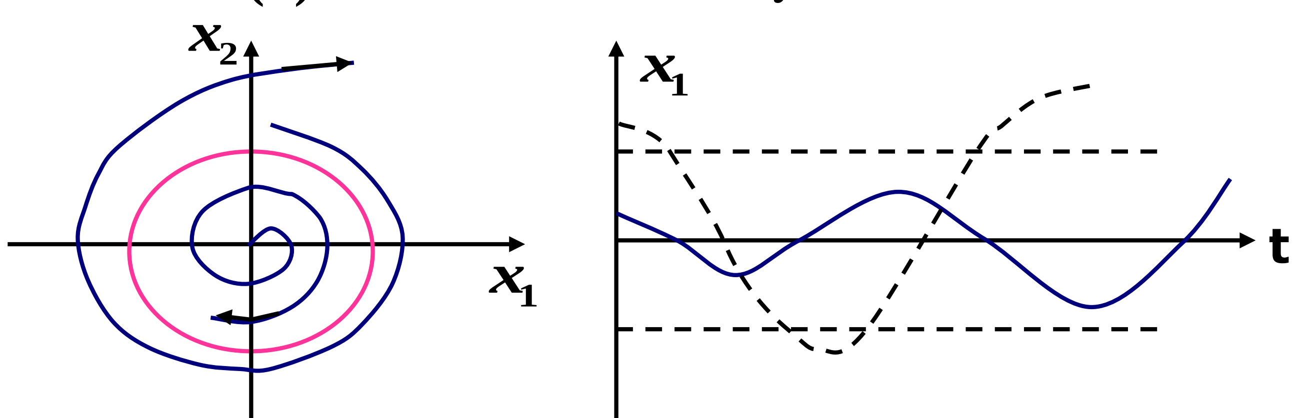
- Limit cycle is a special phenomena which only exists in non-conservation systems. It is caused by nonlinearity of systems, not the non-damping feature of linear systems.



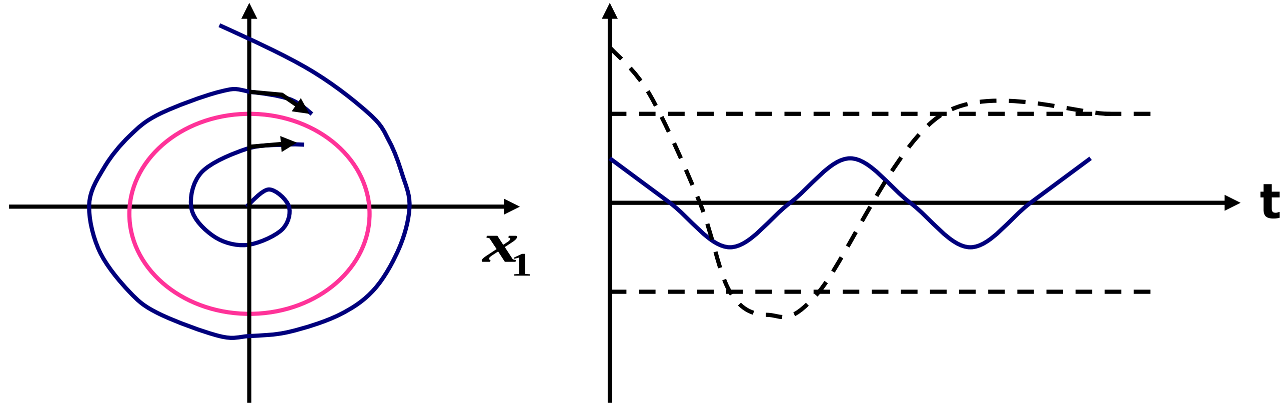

**(a)** Stable limit cycle




**(b) Un**stable limit cycle

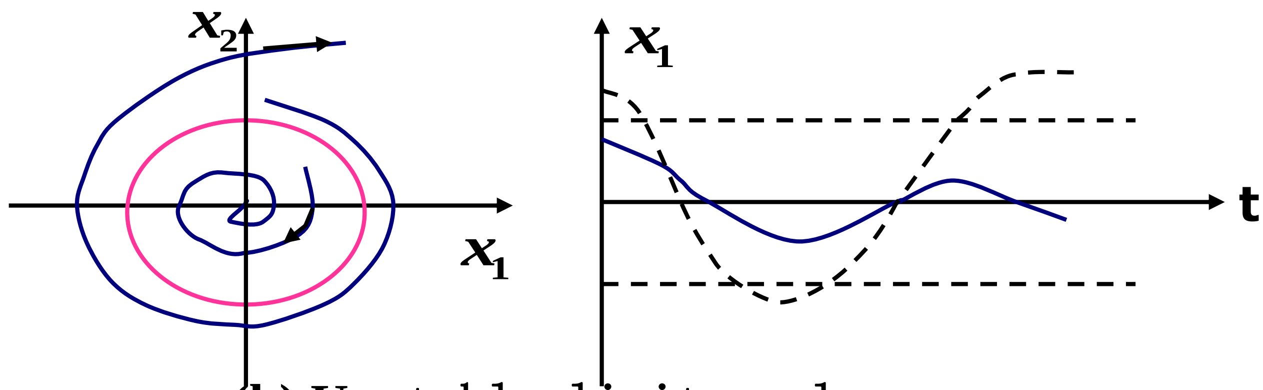



**(c) Semi-stable limit Cycle**



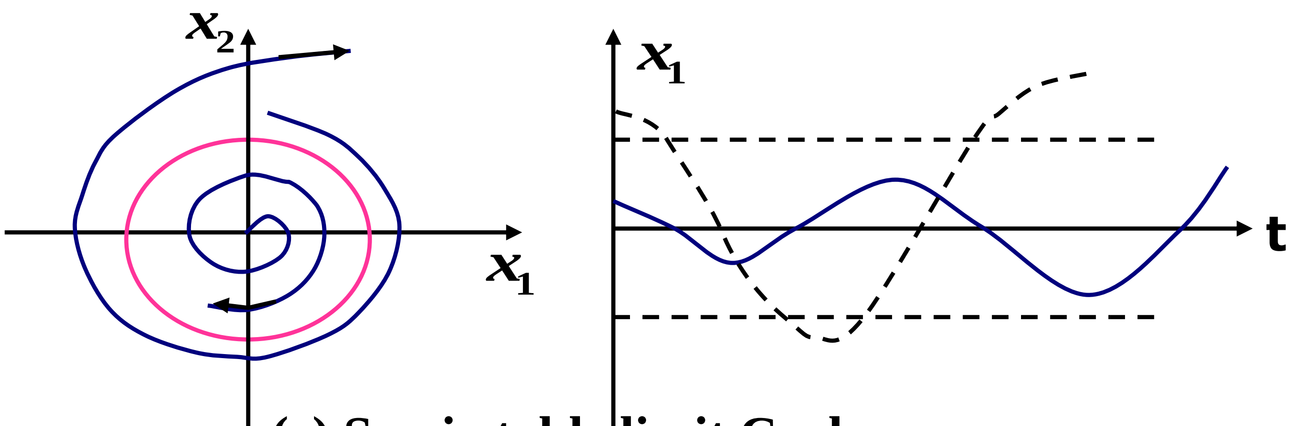

(a) Stable limit cycle

- ✓ 从系统的运动状态来看，这种稳定极限环表示系统具有固定周期和幅值的稳定振荡状态，即自振荡。
- ✓ 从相平面图上看，稳定极限环把相平面图划分成两个区域。由于在极限环内部的相轨迹随时间的增加是发散的，故内部区域为不稳定区。而在极限环外部的相轨迹随时间的增加收敛于这个极限环，因此外部区域为稳定区。
- ✓ 通常在设计系统时，则应尽量减小这种极限环，以满足对稳态误差的要求。

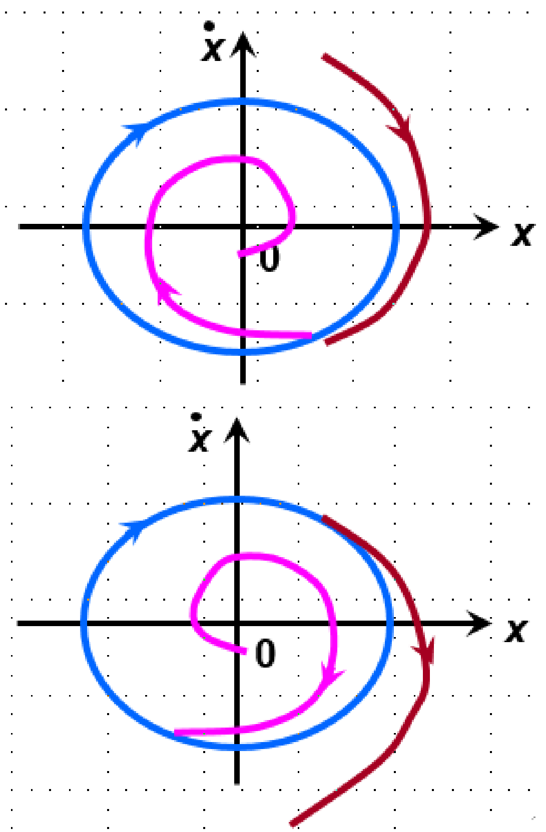



(b) Unstable limit cycle

- ✓ 系统的运动状态与初始条件有关，若初始状态在环内，则系统状态将趋于平衡点（坐标原点）。反之，系统状态将远离平衡点。
- ✓ 具有不稳定极限环的系统，其平衡状态是小范围稳定的，大范围是不稳定的。
- ✓ 通常在设计系统时，应尽量增大极限环。




(c) Semi-stable limit Cycle




- ✓ 对于上图来说，两个区域都是稳定的，所以系统的运动状态最终将趋于环内的平衡点，不会产生自振荡。
- ✓ 而对于下图来说，由于被极限环所划分的两个区域都是不稳定的，因此系统将具有振荡发散状态。

**Example 8.2** The equation of a non-linear system is given by

$$\dot{x}_1 = x_2 + x_1(1 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = -x_1 + x_2(1 - x_1^2 - x_2^2)$$

Analyze the stability of this system.

**Solution :**

The Cartesian coordinate is transformed to the polar one as following:

Assume $x_1 = r\cos\theta, \quad x_2 = r\sin\theta$

Then $\dot{x}_1 = \dot{r}\cos\theta - r\sin\theta \cdot \dot{\theta}$

$$\dot{x}_2 = \dot{r}\sin\theta + r\cos\theta \cdot \dot{\theta}$$

Substituting the above equations into the system equations.

$$\dot{r}\cos\theta - r\sin\theta\cdot\dot{\theta} = r\sin\theta + r\cos\theta(1-r^2) \qquad \cdots(1)$$

$$\dot{r}\sin\theta + r\cos\theta\cdot\dot{\theta} = -r\cos\theta + r\sin\theta(1-r^2) \qquad \cdots(2)$$

It follows from (2) that

$$\dot{\theta} = \frac{-r\cos\theta + r\sin\theta(1-r^2) - \dot{r}\sin\theta}{r\cos\theta} \qquad \cdots(3)$$

Substituting (3) into (1), we have

$$\boxed{\dot{r} = r(1-r^2)}$$

It follows from (1) that

$$\dot{r} = \frac{r\sin\theta + r\cos\theta(1-r^2) + r\sin\theta\cdot\dot{\theta}}{\cos\theta} \qquad \cdots(4)$$

Substituting (4) into (2), we have $\boxed{\dot{\theta} = -1}$

$$\therefore \begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = -1 \end{cases}$$

$$P(x_1, x_2) = x_2 + x_1(1 - x_1^2 - x_2^2)$$
$$Q(x_1, x_2) = -x_1 + x_2(1 - x_1^2 - x_2^2)$$

There are two cases: $r = 0$ and $1 - r^2 = 0$

**(1) $r = 0$** $\quad x_1 = 0,\ x_2 = 0$ is the **singular point**.

$$a = \left.\frac{\partial P(x_1, x_2)}{\partial x_1}\right|_{(0,0)} = 1 \qquad b = \left.\frac{\partial P(x_1, x_2)}{\partial x_2}\right|_{(0,0)} = 1$$

$$c = \left.\frac{\partial Q(x_1, x_2)}{\partial x_1}\right|_{(0,0)} = -1 \qquad d = \left.\frac{\partial Q(x_1, x_2)}{\partial x_2}\right|_{(0,0)} = 1$$

The roots of characteristic equation are:

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2} = 1 \pm j$$

Singular point (0,0) is an ***unstable focus***, corresponding phase trajectories nearby are all divergent oscillations.

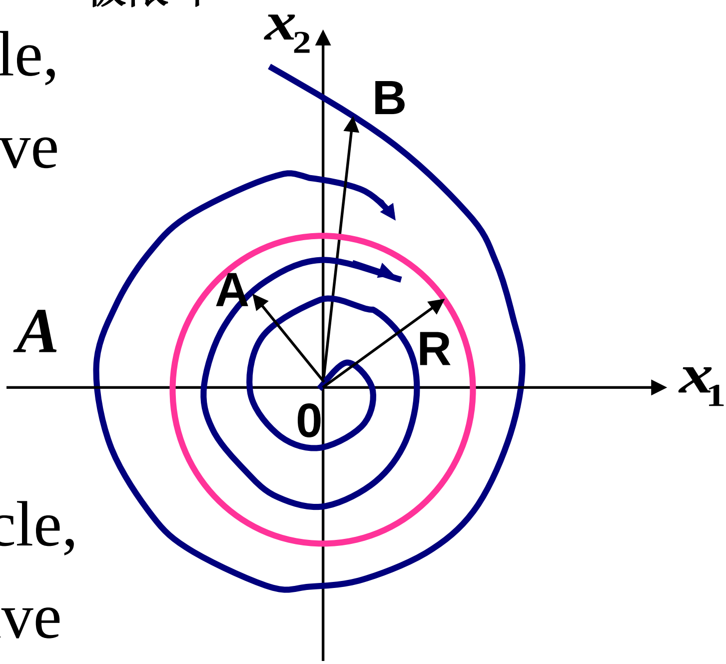
震荡发散

(2) $r = 1$ $x_1^2 + x_2^2 = 1$, unit circle (limit cycle of systems)

极限环

For an arbitrary point $A$ inside the unit circle, 单位圆 inequality $\dot{r} = r(1 - r^2) > 0$ holds since we have $OA = r < 1$.

Then the phase trajectory crossing point $A$ 相轨迹 will finally approach to the unit circle.




For an arbitrary point $B$ outside the unit circle, inequality $\dot{r} = r(1 - r^2) < 0$ holds since we have $OB = r > 1$.

Then the phase trajectory crossing point $B$ will also finally approach to the unit circle.

$\therefore\ x_1^2 + x_2^2 = 1$ is a **stable limit cycle**.

**Example 8.3** 若非线性系统的微分方程为 $\ddot{x}+x\dot{x}+x=0$

试求系统的奇点。

$$\begin{cases}\dot{x}_1=P(x_1,x_2)=0\\ \dot{x}_2=Q(x_1,x_2)=0\end{cases}$$

解：令 $\dot{x}=\ddot{x}=0$ 有 $x=0$ 则奇点为 $(0,0)$

由 $f(x,\dot{x})=\ddot{x}=-x\dot{x}-x$

$$P(x_1,x_2)=\left.\frac{\partial P(x_1,x_2)}{\partial x_1}\right|_{(0,0)}x_1+\left.\frac{\partial P(x_1,x_2)}{\partial x_2}\right|_{(0,0)}x_2$$

$$Q(x_1,x_2)=\left.\frac{\partial Q(x_1,x_2)}{\partial x_1}\right|_{(0,0)}x_1+\left.\frac{\partial Q(x_1,x_2)}{\partial x_2}\right|_{(0,0)}x_2$$

得 $\dfrac{\partial f}{\partial x}=-\dot{x}-1$ $\quad\dfrac{\partial f}{\partial \dot{x}}=-x$

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=0\\ \dot{x}=0}}=-1\qquad \left.\frac{\partial f}{\partial \dot{x}}\right|_{\substack{x=0\\ \dot{x}=0}}=0\qquad \ddot{x}=\left.\frac{\partial f}{\partial x}\right|_{x=0}\cdot x+\left.\frac{\partial f}{\partial \dot{x}}\right|_{\dot{x}=0}\cdot\dot{x}=-x$$

所以奇点的附近特征方程为 $\lambda^2+1=0$ $\quad\lambda=\pm j$

故该奇点为中心点。

## Singular Point 奇点

$$\begin{cases} \dot{x}_1 = P(x_1, x_2) = 0 \\ \dot{x}_2 = Q(x_1, x_2) = 0 \end{cases}$$

Singular Point $(x_{10}, x_{20})$

$$x_{10} = x_{20} = 0$$

$$P(x_1, x_2) = \left.\frac{\partial P(x_1, x_2)}{\partial x_1}\right|_{(0,0)} x_1 + \left.\frac{\partial P(x_1, x_2)}{\partial x_2}\right|_{(0,0)} x_2$$

$$Q(x_1, x_2) = \left.\frac{\partial Q(x_1, x_2)}{\partial x_1}\right|_{(0,0)} x_1 + \left.\frac{\partial Q(x_1, x_2)}{\partial x_2}\right|_{(0,0)} x_2$$

$$\begin{cases} \dot{x}_1 = a x_1 + b x_2 \\ \dot{x}_2 = c x_1 + d x_2 \end{cases}$$

The characteristic equation of system is given by

$$\left|\lambda I - A\right| = \lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

The roots of the above equation is

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}$$

## <u>Singular Point</u> 奇点

$\ddot{x} = f(x, \dot{x})$ Singular Point $\boldsymbol{(x_{10}, x_{20})}$

$\boldsymbol{x_{10} = x_{20} = 0}$

$$\ddot{x} = \frac{\partial f}{\partial x}\Big|_{x=0} \cdot x + \frac{\partial f}{\partial \dot{x}}\Big|_{\dot{x}=0} \cdot \dot{x}$$

The characteristic equation of system is given by

$$\lambda^2 = \frac{\partial f}{\partial x}\Big|_{x=0} + \frac{\partial f}{\partial \dot{x}}\Big|_{\dot{x}=0} \lambda$$

**Ex** 若非线性系统的为微分方为 $\ddot{x}+(3\dot{x}-0.5)\dot{x}+x+x^{2}=0$

试求系统的奇点。

**Ex** 若非线性系统的微分方为 $\ddot{x}+(3\dot{x}-0.5)\dot{x}+x+x^2=0$

试求系统的奇点。

解：令 $\dot{x}=\ddot{x}=0$ 有 $x+x^2=0$ 则 $x=0$ $x=-1$ 奇点为 $(0,0)$ $(-1,0)$

由 $\ddot{x}=-3\dot{x}^2+0.5\dot{x}-x-x^2=f$ 得 $\dfrac{\partial f}{\partial x}=-1-2x$ $\dfrac{\partial f}{\partial \dot{x}}=-6\dot{x}+0.5$

在 $(0,0)$ $\ddot{x}=\dfrac{\partial f}{\partial x}\Big|_{x=0}\cdot x+\dfrac{\partial f}{\partial \dot{x}}\Big|_{\dot{x}=0}\cdot \dot{x}=-x+0.5\dot{x}$

即 $\ddot{x}-0.5\dot{x}+x=0$ 故特征方程为 $\lambda^2-0.5\lambda+1=0$

解特征方程得 $\lambda_{1,2}=\dfrac{0.5\pm j\sqrt{3.75}}{2}$ 故该奇点为不稳定的焦点。

在 $(-1,0)$ $\ddot{x}=\dfrac{\partial f}{\partial x}\Big|_{x=-1}\cdot x+\dfrac{\partial f}{\partial \dot{x}}\Big|_{\dot{x}=0}\cdot \dot{x}=x+0.5\dot{x}$

即 $\ddot{x}-0.5\dot{x}-x=0$ 故特征方程为 $\lambda^2-0.5\lambda-1=0$

解特征方程得 $\lambda_1=1.28$ $\lambda_2=-0.78$ 故该奇点为鞍点。

## 8.4.4 Nonlinear Systems Using Phase Plane Analysis

**Idea：**

根据非线性特性，将相平面划分为若干区域，每个区域均可用一个二阶线性微分方程描述，求解这个微分方程，绘制这个区域的相轨迹，平滑的将不同区域的相轨迹连接起来，得到整个系统的相轨迹。.

## Algorithm of phase plane analysis：

1. Select ***appropriate coordinate axis*** during the analysis.

2. Divide the phase plane into ***several areas*** according to nonlinear characteristics. Establish ***linear differential equations*** for each area.

非线性特性

线性微分方程

切换线

3. Establishing equations for the ***switching lines*** in the phase plane according to different nonlinear characteristics.

4. ***Solve the differential equations*** of each area and then draw phase trajectory.

相轨迹

5. The phase trajectory of the whole system can be obtained by ***connecting all the partial trajectories*** in different areas.

阶跃输入

**Example 8.4** **The following nonlinear system is excited with a step input signal of amplitude 6. If the initial state of the system is** $e(0)=6,\ \dot{e}(0)=0$ **, how many seconds will it take for the system state to reach the origin.**

初始状态

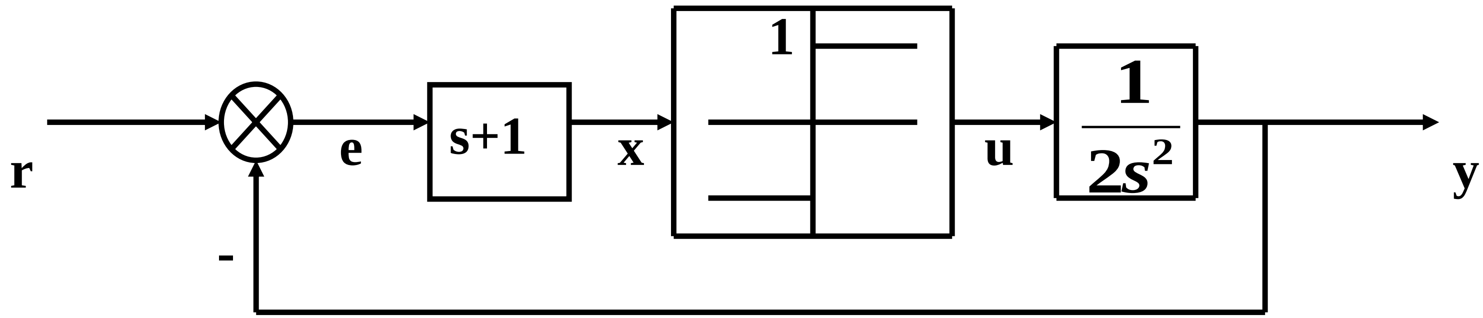


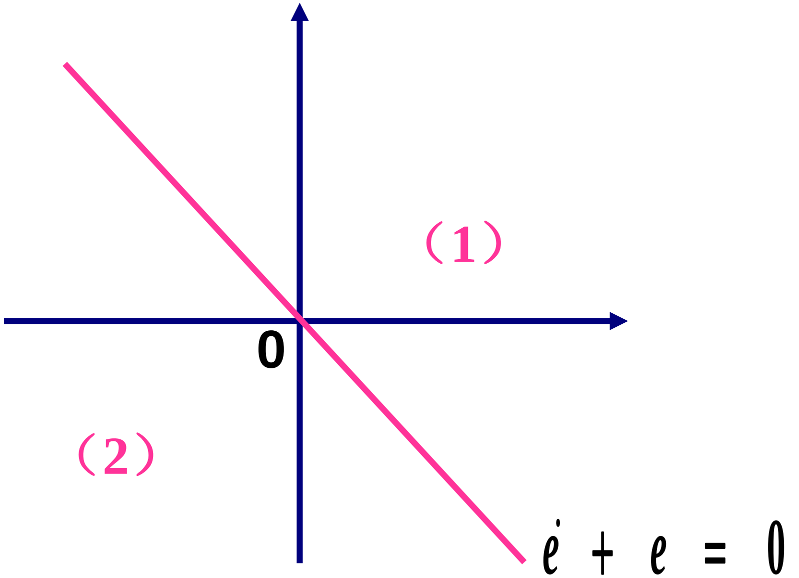

Fig 8-36 control system with a relay module

**Solution: The dynamic equation is:**

$$\begin{cases} e = r - y \\ x = \dot{e} + e \\ u = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \\ 2\ddot{y} = u \end{cases}$$

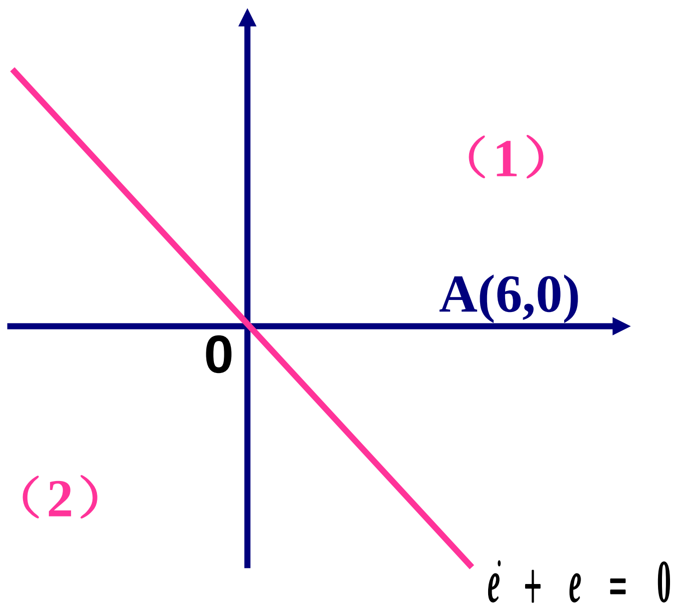
$$\begin{cases} e = r - y \\ x = \dot{e} + e \\ u = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \\ 2\ddot{y} = u \end{cases}$$

$$\therefore \quad \dot{e} = -\dot{y} \quad \therefore \quad \ddot{e} = -\ddot{y} = -\frac{1}{2}u = \begin{cases} -0.5 & \dot{e} + e > 0 \\ 0.5 & \dot{e} + e < 0 \end{cases} \quad \begin{matrix} \textbf{Area（1）} \\ \textbf{Area（2）} \end{matrix}$$



**Area（1）：** $\therefore\ \ddot{e} = -\ddot{y} = -\dfrac{1}{2}u = \begin{cases} -0.5 & \dot{e}+e>0 \\ 0.5 & \dot{e}+e<0 \end{cases}$ **Area（1）** **Area（2）**

$$\begin{cases} \ddot{e} = -0.5 \\ \dot{e} = -0.5t + c_1 \\ e = -0.25t^2 + c_1 t + c_2 \end{cases}$$




**The initial conditions is** $e(0)=6,\quad \dot{e}(0)=0$

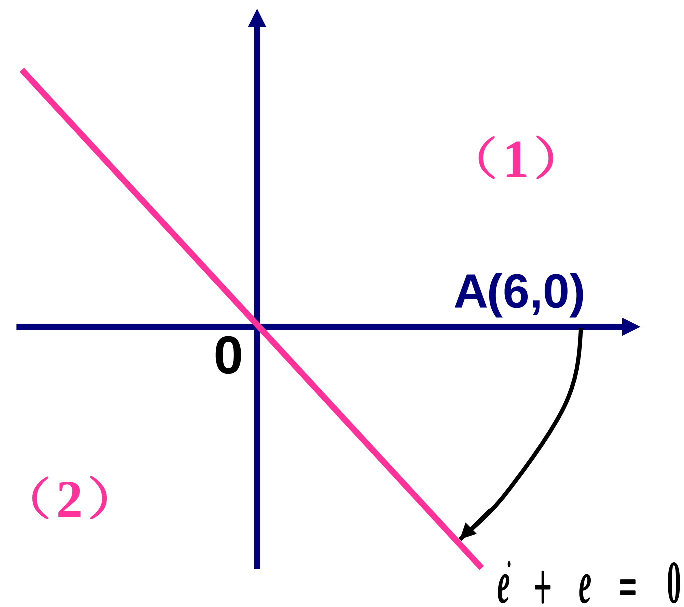
**Then** $c_1 = 0, c_2 = 6$

$$\therefore\ \begin{cases} \dot{e} = -0.5t \\ e = -0.25t^2 + 6 \end{cases}$$

$\rightarrow\ e = -\dot{e}^2 + 6$ ——**This is the phase trajectory in area (1)**

**Area（1）:** $\therefore\ \ddot{e} = -\ddot{y} = -\frac{1}{2}u = \begin{cases} -0.5 & \dot{e}+e>0 \\ 0.5 & \dot{e}+e<0 \end{cases}$ **Area（1）** **Area（2）**

$$\begin{cases} \ddot{e} = -0.5 \\ \dot{e} = -0.5t + c_1 \\ e = -0.25t^2 + c_1 t + c_2 \end{cases}$$




**The initial conditions is** $e(0)=6,\quad \dot{e}(0)=0$

**Then** $c_1 = 0, c_2 = 6$

$$\therefore \begin{cases} \dot{e} = -0.5t \\ e = -0.25t^2 + 6 \end{cases}$$

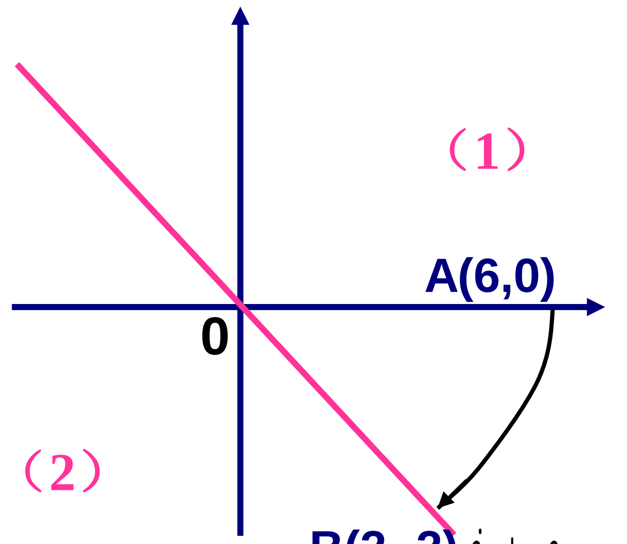
$\rightarrow\ e = -\dot{e}^2 + 6$ ——**This is the phase trajectory in area (1)**

抛物线

**The phase trajectory is a *parabola*. From point A , the system state reaches point B and enter area (2).**

$$e = -\dot{e}^2 + 6$$

$$\begin{cases} e_B = -\dot{e}_B^2 + 6 \\ \dot{e}_B + e_B = 0 \end{cases}$$




**Solution** $e_B = 2, \dot{e}_B = -2$

**Area（2）：** $\therefore\ \ddot{e} = -\ddot{y} = -\frac{1}{2}u = \begin{cases} -0.5 & \dot{e} + e > 0 \quad \text{Area（1）} \\ 0.5 & \dot{e} + e < 0 \quad \text{Area（2）} \end{cases}$

$$\begin{cases} \ddot{e} = 0.5 \\ \dot{e} = 0.5t + c_3 \\ e = 0.25t^2 + c_3 t + c_4 \end{cases}$$

**Consider the initial condition** $e_B = 2, \dot{e}_B = -2$

**We obtain** $c_3 = -2, c_4 = 2$

$$\therefore \begin{cases} \dot{e} = 0.5t - 2 \\ e = 0.25t^2 - 2t + 2 \end{cases}$$

**Eliminating *t*， we have**

$$e = \dot{e}^2 - 2$$ ——**This is the trajectory in area (2)**

抛物线

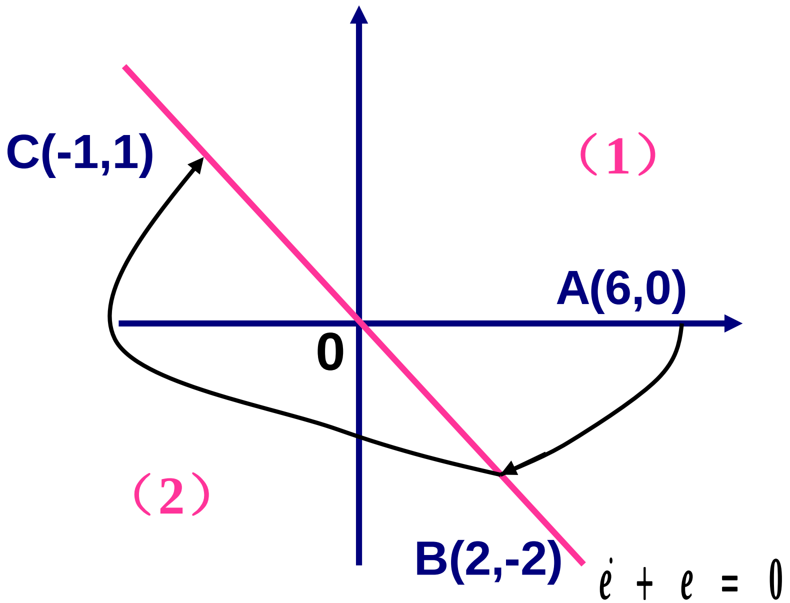
**The system state moves along the parabola from point B to point C, then enter area (1).**

**The coordinate of point C satisfies**

$$\begin{cases} e_C = \dot{e}_C^2 - 2 \\ \dot{e}_C + e_C = 0 \end{cases}$$

**We can obtain**

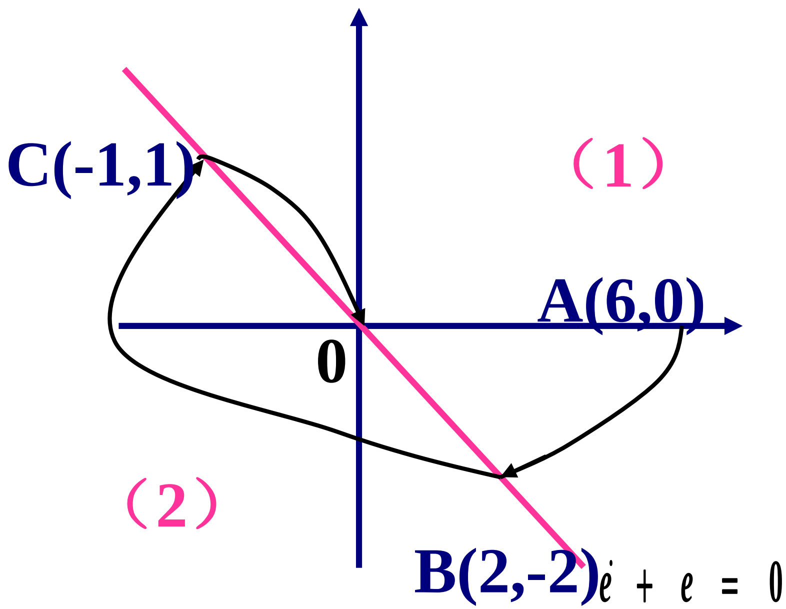
$$e_C = -1, \dot{e}_C = 1$$



**Area（1）：** $\therefore\ \ddot{e}=-\ddot{y}=-\frac{1}{2}u=\begin{cases}-0.5 & \dot{e}+e>0 \\ 0.5 & \dot{e}+e<0\end{cases}$ **Area（1）** **Area（2）**

$$\begin{cases}\ddot{e}=-0.5 \\ \dot{e}=-0.5t+c_5 \\ e=-0.25t^2+c_5t+c_6\end{cases}$$

**In terms of the initial condition C（—1，1）, we have**




$$c_5=1,\ c_6=-1$$

$$\therefore\ \begin{cases}\dot{e}=-0.5t+1 \\ e=-0.25t^2+t-1\end{cases}$$

**Eliminating *t* we have** $e=-\dot{e}^2$

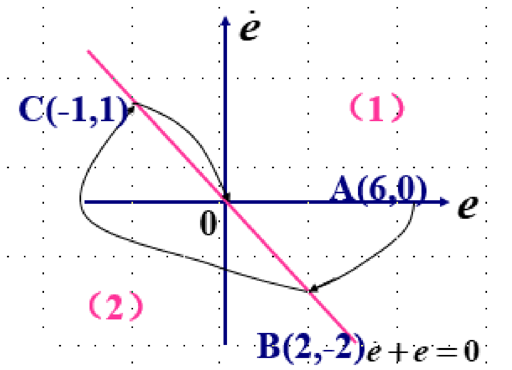
**In area（1）, the system state moves along the parabola from point C to the origin.**

**The time $t_{AO}$ of moving to the origin from point A can be obtained by the following equation:**

$$t_{AO} = t_{AB} + t_{BC} + t_{CO}$$

**Based on the dynamic equations in different areas：**

$$t_{AB}: \quad \dot{e} = -0.5t \qquad -2 = -0.5t_{AB} \qquad \therefore \quad t_{AB} = 4$$




$$t_{BC}: \quad \dot{e} = 0.5t - 2 \qquad 3 = 0.5t_{BC} \qquad \therefore \quad t_{BC} = 6$$

$$t_{CO}: \quad \dot{e} = -0.5t + 1 \qquad -1 = -0.5t_{CO} \qquad \therefore \quad t_{CO} = 2$$

$$\therefore \quad t_{AO} = t_{AB} + t_{BC} + t_{CO} = 12 \text{ Seconds}$$

**Example 8.5** **The structure of a nonlinear system as shown in the following figure**, where $a = 1$， $tg\alpha_1 = 1$, $tg\alpha_2 = 1/2$.

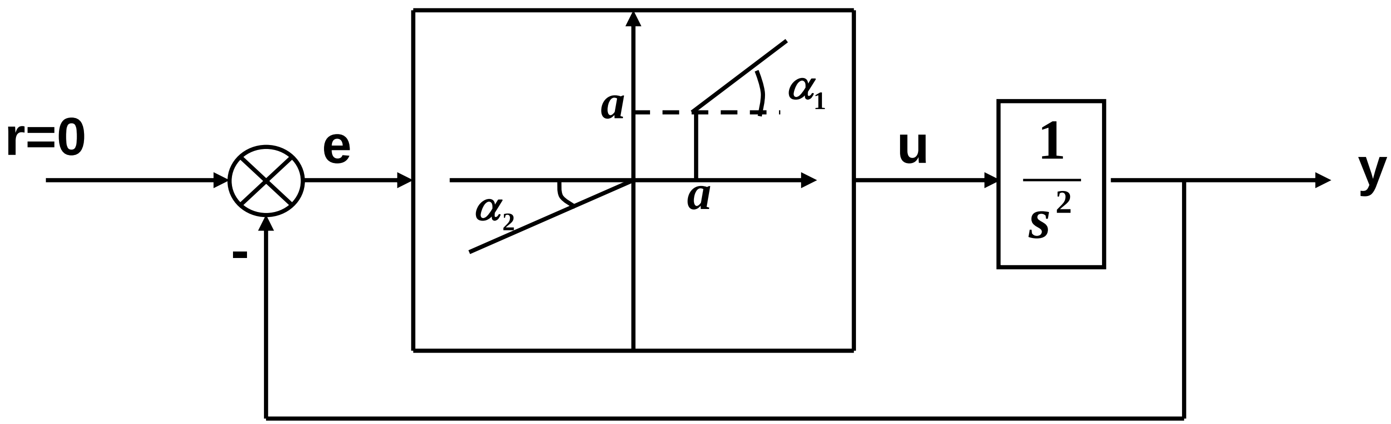
相轨迹

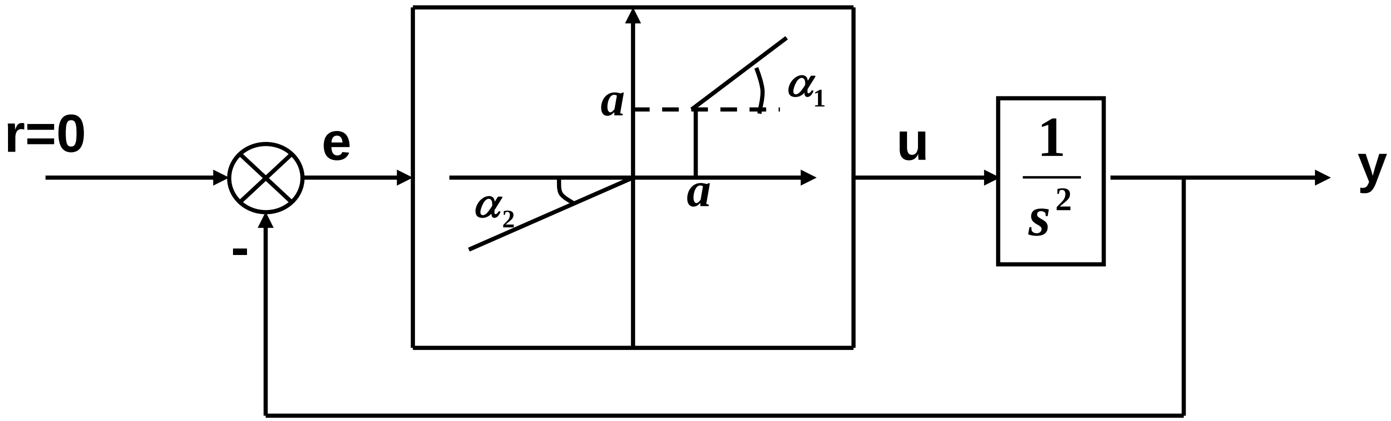
(1) Plot the phase trajectory of this system with the initial state $y(0) = -1, \dot{y}(0) = -1$

(2) Draw briefly the corresponding curve $y(t)$. Try to Obtain the values of $t$ when $y(t)$= 0.

(3) If $y(t)$ is periodic, obtain the value of this period.

周期性






Solution : Dynamic equations $\ddot{y} = u$

$$u = \begin{cases} a + (e-a)tg\alpha_1 & e > a \\ 0 & 0 \le e \le a \\ etg\alpha_2 & e < 0 \end{cases}$$

$a = 1$

$tg\alpha_1 = 1,$

$tg\alpha_2 = 1/2.$

When $r = 0$ , $e = -y$ . Substituting the known conditions into the above equation, we obtain

$$\ddot{y} = u = \begin{cases} -y & y < -1 \\ 0 & -1 \le y \le 0 \\ -0.5y & y > 0 \end{cases}$$

$$y(0)=-1, \dot{y}(0)=-1$$

$$\ddot{y}=u=\begin{cases}-y & y<-1\\ 0 & -1\le y\le 0\\ -0.5y & y>0\end{cases}$$

**Area（1）：** $\ddot{y}=-y$

$$\ddot{y}+y=0 \rightarrow \lambda^2+1=0, \lambda=\pm j$$

$$\therefore\ y=c_1\cos t+c_2\sin t$$

$$\dot{y}=-c_1\sin t+c_2\cos t$$

Substitute the initial conditions into the above equations. We obtain

$$c_1=-1, c_2=-1$$

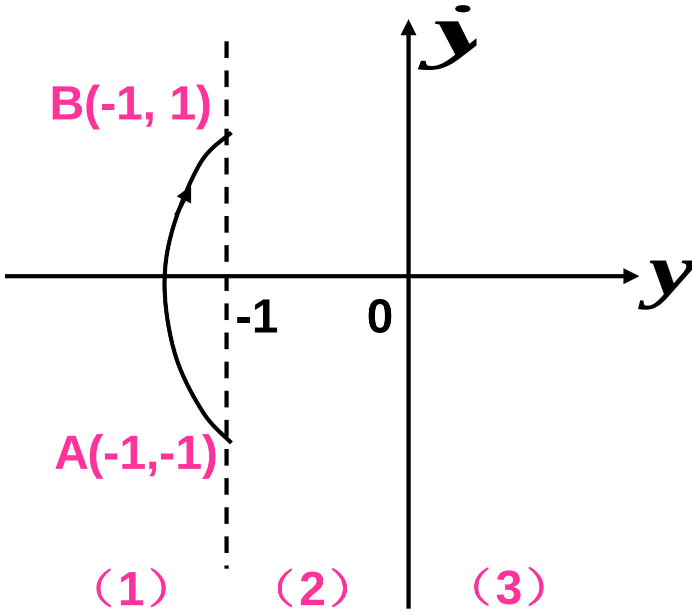
$$\therefore\ y=-\cos t-\sin t=-\sqrt{2}\sin(t+\frac{\pi}{4})$$

$$\dot{y}=\sin t-\cos t$$

Eliminating $t$ we have

$$\dot{y}^2+y^2=2$$

B(-1,1)




The system state moves in area(1) along the arc from point A to point B, and then enter area（2）.

$$\ddot{y} = u = \begin{cases} -y & y < -1 \\ 0 & -1 \le y \le 0 \\ -0.5y & y > 0 \end{cases}$$

**Area（2）：** $\ddot{y} = 0$

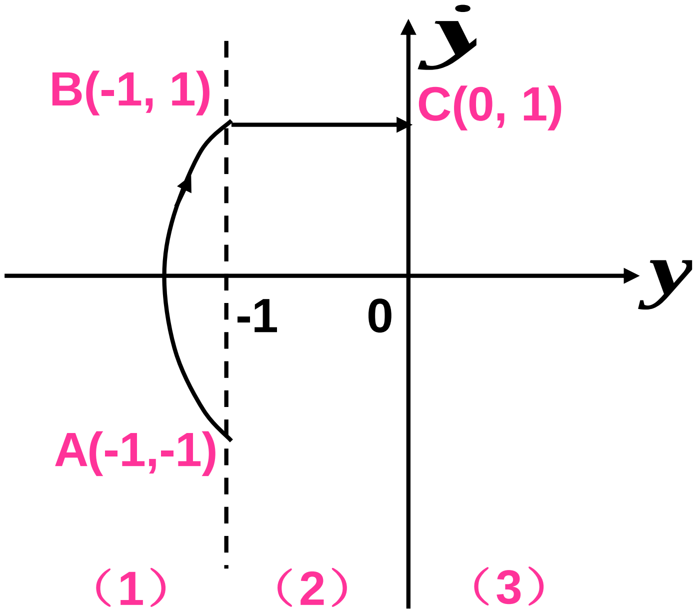
$$\dot{y} = c_3, \quad y = c_3 t + c_4$$

Substituting the initial conditions B(-1,1) into above equations, we obtain

$$c_3 = 1, c_4 = -1$$

$$\therefore \quad y = t - 1, \quad \dot{y} = 1 \qquad \text{C(0,1)}$$

The system state moves in area(2) along the line $\dot{y} = 1$ from point B to point C, and then enter area(3).




**Area（3）：** $\ddot{y} = -0.5y$

$$\ddot{y} + \frac{1}{2}y = 0 \;\rightarrow\; \lambda^2 + \frac{1}{2} = 0, \;\; \lambda = \pm\frac{\sqrt{2}}{2}j$$

$$\therefore \quad y = c_5 \cos\frac{\sqrt{2}}{2}t + c_6 \sin\frac{\sqrt{2}}{2}t$$

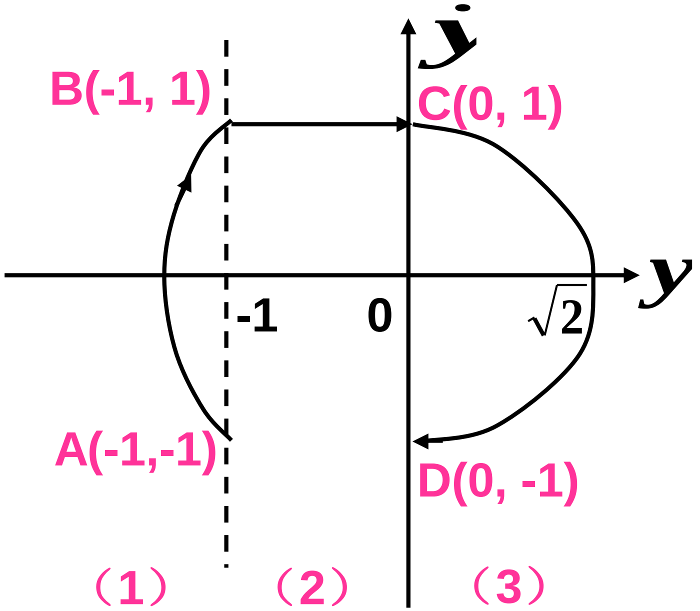
$$\dot{y} = -\frac{\sqrt{2}}{2}c_5 \sin\frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}c_6 \cos\frac{\sqrt{2}}{2}t$$

Substituting the initial conditions into above equations, we obtain

$$c_5 = 0, c_6 = \sqrt{2}$$

$$\therefore \begin{cases} y = \sqrt{2}\sin\frac{\sqrt{2}}{2}t \\ \dot{y} = \cos\frac{\sqrt{2}}{2}t \end{cases}$$

then $\left(\frac{y}{\sqrt{2}}\right)^2 + \dot{y}^2 = 1$ C(0,-1)




The system state moves in area(3) along the ellipse 椭圆 from point C to point D, and then enter area(2).

$$\ddot{y} = u = \begin{cases} -y & y < -1 \\ 0 & -1 \le y \le 0 \\ -0.5y & y > 0 \end{cases}$$

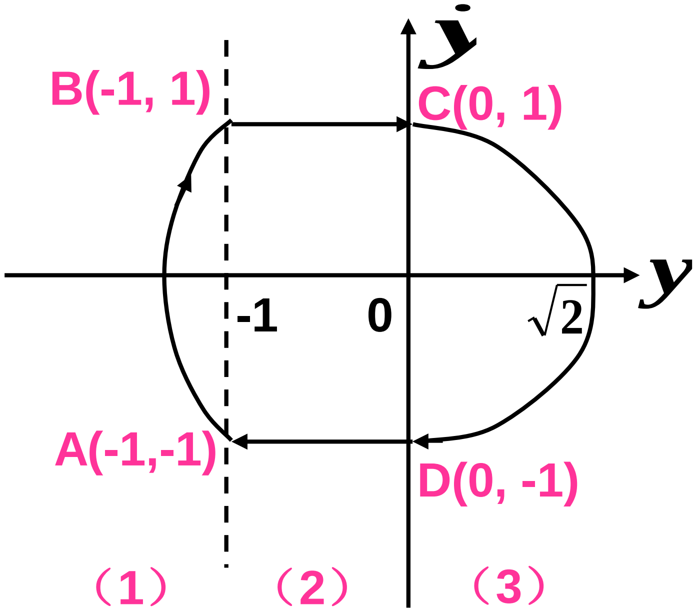
**Area（2）：** $\ddot{y} = 0$

$$\dot{y} = c_7, \quad y = c_7 t + c_8$$

Substituting the initial conditions into above equations, we obtain

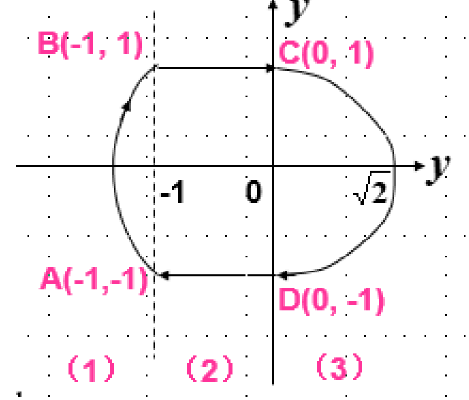
$$c_7 = -1, c_8 = 0$$

$$\therefore \begin{cases} y = -t \\ \dot{y} = -1 \end{cases} \qquad \text{A(-1,-1)}$$




The system state moves in area(2) along the line $\dot{y} = -1$ from point D to point A, and then enter area(1). Because it is a closed phase trajectory, the motion of system is periodic 。

Based on the dynamic equations in different areas ,




$$t_{AB}: \quad y = -\sqrt{2}\sin(t+\frac{\pi}{4}) \qquad \sin(t_B+\frac{\pi}{4})=\frac{\sqrt{2}}{2} \quad \therefore\ t_B=\frac{\pi}{2}$$

$$t_{BC}: \quad y = t-1 \qquad 1=t_C \quad \therefore\ t_C=1$$

$$t_{CD}: \quad y=\sqrt{2}\sin\frac{\sqrt{2}}{2}t \qquad \frac{\sqrt{2}}{2}t_D=\pi \quad \therefore\ t_D=\sqrt{2}\pi$$

$$t_{DA}: \quad y=-t \qquad -1=-t_A \quad \therefore\ t_A=1$$

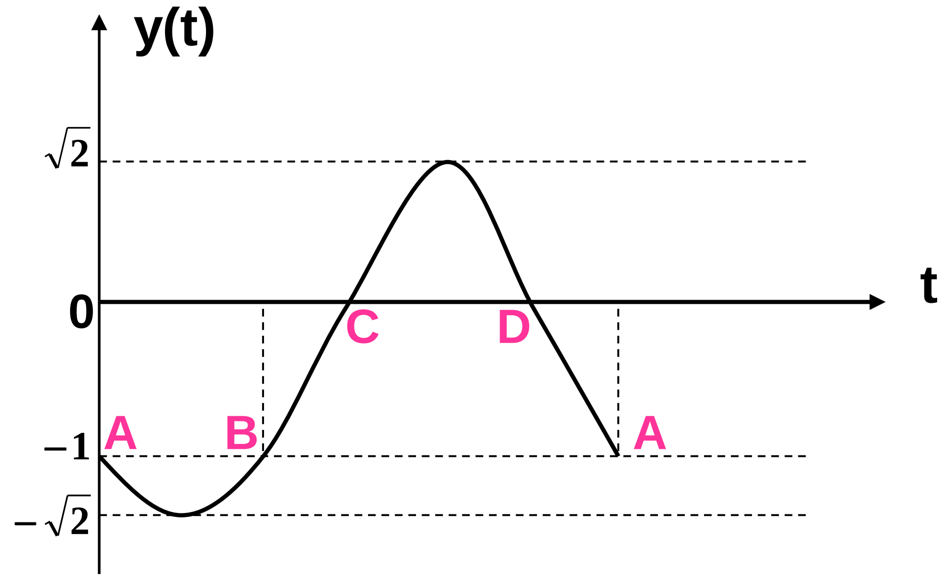
The period of system motion $\quad T=2+\frac{\pi}{2}+\sqrt{2}\pi$

$$t_{AB}: \quad y = -\sqrt{2}\sin(t + \frac{\pi}{4})$$

$$t_{BC}: \quad y = t - 1$$

$$t_{CD}: \quad y = \sqrt{2}\sin\frac{\sqrt{2}}{2}t$$

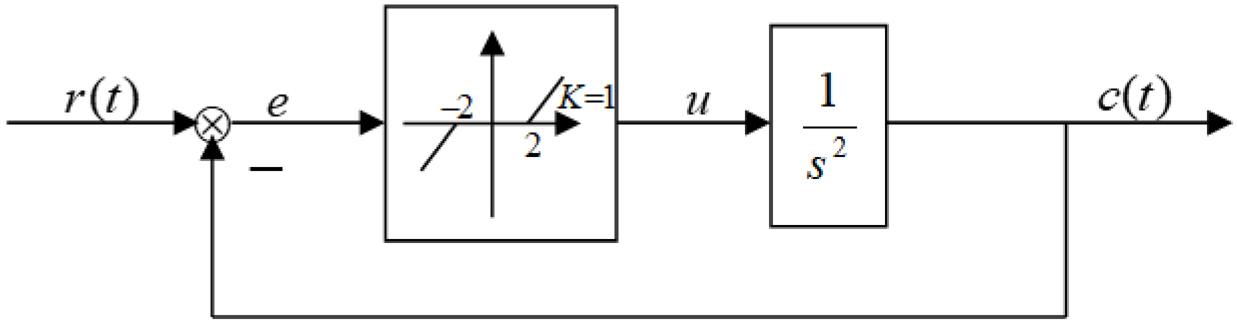
$$t_{DA}: \quad y = -t$$




Where,

$$t_B = \frac{\pi}{2} \qquad t_C = 1 + \frac{\pi}{2} \qquad t_D = 1 + \frac{\pi}{2} + \sqrt{2}\pi \qquad t_A = 2 + \frac{\pi}{2} + \sqrt{2}\pi$$

**EX** 非线性系统的结构图如图所示。系统开始是静止的，输入信号 $r(t)=4I(t)$，试写出开关线方程，做系统相面图，并分析系统运动的特点。




初始状态为 $e(0)=4 \quad \dot{e}(t)=0$

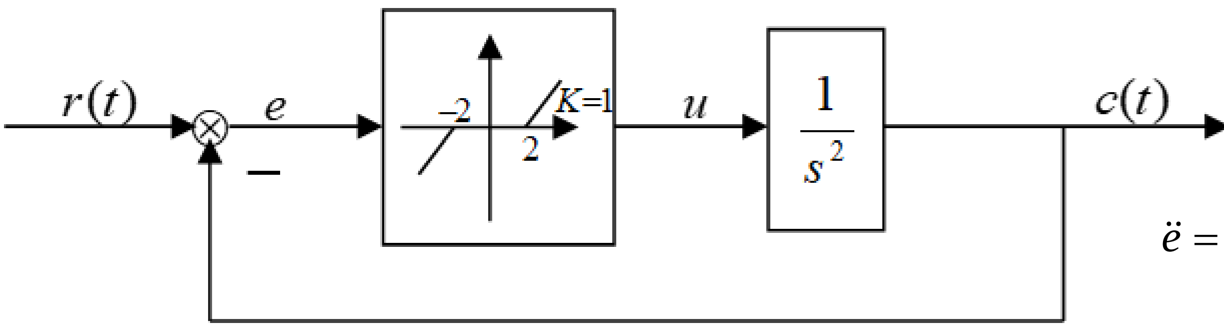
解：由结构图可得 $e=r-c=4-c(t) \qquad \dot{e}=-\dot{c} \qquad \ddot{e}=-\ddot{c} \qquad u=\ddot{c}=-\ddot{e}$

由 $u=\begin{cases} e-2 & e>2 \\ 0 & -2<e<2 \\ e+2 & e<-2 \end{cases}$ 得 $\ddot{e}=\begin{cases} -e+2 & e>2 & [1] \\ 0 & -2<e<2 & [2] \\ -e-2 & e<-2 & [3] \end{cases}$

由结构图可得开关线为 $e=2 \qquad e=-2$

$\dot{e}$

**EX** 非线性系统的结构图如图所示。系统开始是静止的，输入信号 $r(t)=4I(t)$，试写出开关线方程，做系统相面图，并分析系统运动的特点。

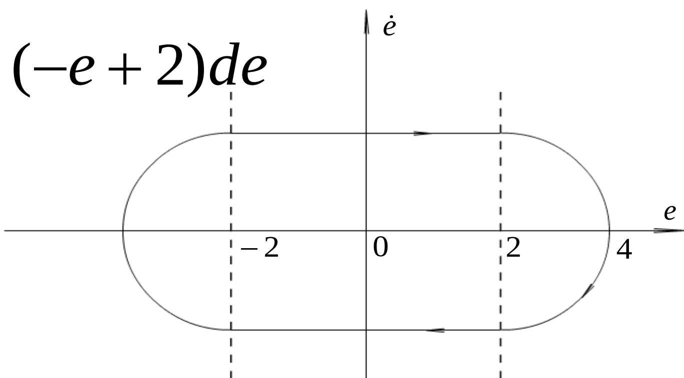



$$\ddot{e}=\begin{cases}-e+2 & e>2 & [1]\\ 0 & -2<e<2 & [2]\\ -e-2 & e<-2 & [3]\end{cases}$$

初始状态为 $e(0)=4 \quad \dot{e}(t)=0$

在[1]区域内，由 $\ddot{e}=\dot{e}\dfrac{d\dot{e}}{de}=-e+2$ 得 $\dot{e}d\dot{e}=(-e+2)de$

由 $\int_0^{\dot{e}}\dot{e}d\dot{e}=\int_4^{e}(-e+2)de$ 得 $(e-2)^2+\dot{e}^2=4$



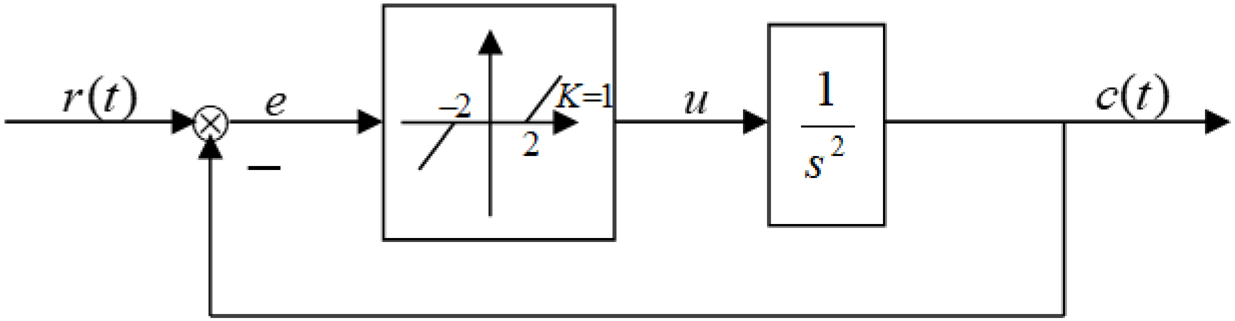

这是一个圆，相轨迹沿该圆进入区域[2]。

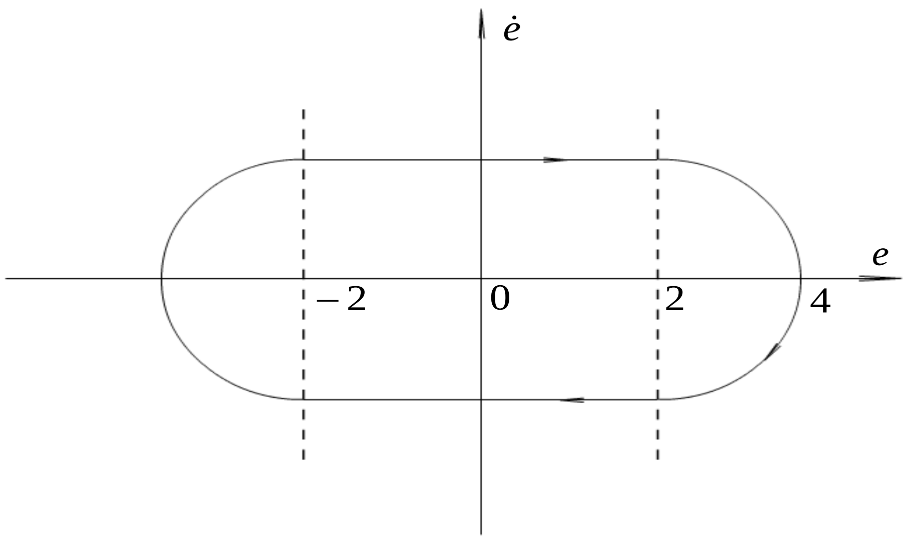
在[2]区域内 $\ddot{e}=0$ 则为 $\dot{e}$ 常数，相轨迹为平行于 $e$ 轴的直线，且沿该轨迹入区域[3]。

依次下去，得非线性系统的相轨迹如图。

$\dot{e}$

**EX** 非线性系统的结构图如图所示。系统开始是静止的，输入信号 $r(t)=4I(t)$，试写出开关线方程，做系统相面图，并分析系统运动的特点。




从图可知，$e$ 为周期运动，由 $e=4-c$ ，得知系统输出 $c$ 也为周期振动。

作业：P266  8-10   8-11

# 第八章 总结

## 1. Typical Nonlinear Characteristics and Mathematical Description

四种非线性特性（了解）

**Saturation characteristics**

饱和特性

**Dead-zone characteristics**

死区特性

**Gap characteristics**

间隙特性

**Relay characteristics**

继电特性

## 2. Describing Function Approach

### 2.1 描述函数的定义

复数比

The *complex ratio* of the fundamental component of the output $y(t)$ and the sinusoidal input $e(t)$, that is:

$$N(A) = \frac{x_1 e^{j\phi_1}}{A} = \frac{B_1}{A} + j\frac{A_1}{A}$$

$$x_1(t) = A_1 \cos\omega t + B_1 \sin\omega t = x_1 \sin(\omega t + \phi_1)$$

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos\omega t \, d(\omega t) \qquad B_1 = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin\omega t \, d(\omega t)$$

$$x_1 = \sqrt{A_1^2 + B_1^2} \qquad \phi_1 = arctg\frac{A_1}{B_1}$$

## 2.2 获取描述函数的方法

① 给定正弦输入信号作用下，获取非线性环节输出**x(t)**

② 计算基波分量的**A1**和**B1**

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos\omega t \, d(\omega t) \quad B_1 = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin\omega t \, d(\omega t)$$

③ 按照描述函数定义计算**N(A)**

$$N(A) = \frac{x_1 e^{j\phi_1}}{A} = \frac{B_1}{A} + j\frac{A_1}{A}$$

## 2.3 根据描述函数分析系统的稳定性

① 绘制系统线性环节的奈氏曲线

② 绘制系统非线性环节的 $-\frac{1}{N(A)}$

③ 计算二者的交点

④ 根奈氏判据判断系统稳定性

# 3. Phase Plane Analysis

## 3.1 相平面，相轨迹，相平面图的定义（了解）

### Phase plane ：

**The** $x_1 - x_2$ **plane is called Phase Plane, where** $x_1, x_2$ **are the system state and its derivative (** $c$, $\dot{c}$ **).**

微分

### Phase portrait ：

The locus in the x1-x2 plane of the solution x(t) for all t>= 0 is a curve named ***trajectory*** or ***orbit*** that passes through the point x0.

相轨迹

The family of phase plane trajectories corresponding to various initial conditions is called ***Phase Portrait*** of the system.

相平面图

## 3.2 奇点和极限环的定义及分析

Singular Points are the equilibrium points $(x_{10}, x_{20})$ ,
奇点 平衡点
which are obtained by solving the following equations.

$$\begin{cases} \dot{x}_1 = P(x_1, x_2) = 0 \\ \dot{x}_2 = Q(x_1, x_2) = 0 \end{cases}$$

Assume

$$a = \left. \frac{\partial P(x_1, x_2)}{\partial x_1} \right|_{(0,0)} \qquad b = \left. \frac{\partial P(x_1, x_2)}{\partial x_2} \right|_{(0,0)}$$

$$c = \left. \frac{\partial Q(x_1, x_2)}{\partial x_1} \right|_{(0,0)} \qquad d = \left. \frac{\partial Q(x_1, x_2)}{\partial x_2} \right|_{(0,0)}$$

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2}$$

# 3.2 奇点和极限环的定义及分析

## Singular Point 奇点

$$\begin{cases} \dot{x}_1 = P(x_1, x_2) = 0 \\ \dot{x}_2 = Q(x_1, x_2) = 0 \end{cases}$$

Singular Point $(x_{10}, x_{20})$

$$x_{10} = x_{20} = 0$$

$$P(x_1, x_2) = \left.\frac{\partial P(x_1, x_2)}{\partial x_1}\right|_{(0,0)} x_1 + \left.\frac{\partial P(x_1, x_2)}{\partial x_2}\right|_{(0,0)} x_2$$

$$Q(x_1, x_2) = \left.\frac{\partial Q(x_1, x_2)}{\partial x_1}\right|_{(0,0)} x_1 + \left.\frac{\partial Q(x_1, x_2)}{\partial x_2}\right|_{(0,0)} x_2$$

$$\begin{cases} \dot{x}_1 = ax_1 + bx_2 \\ \dot{x}_2 = cx_1 + dx_2 \end{cases}$$

The characteristic equation of system is given by

$$\left|\lambda I - A\right| = \lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

The roots of the above equation is

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2}$$

## Singular Point 奇点

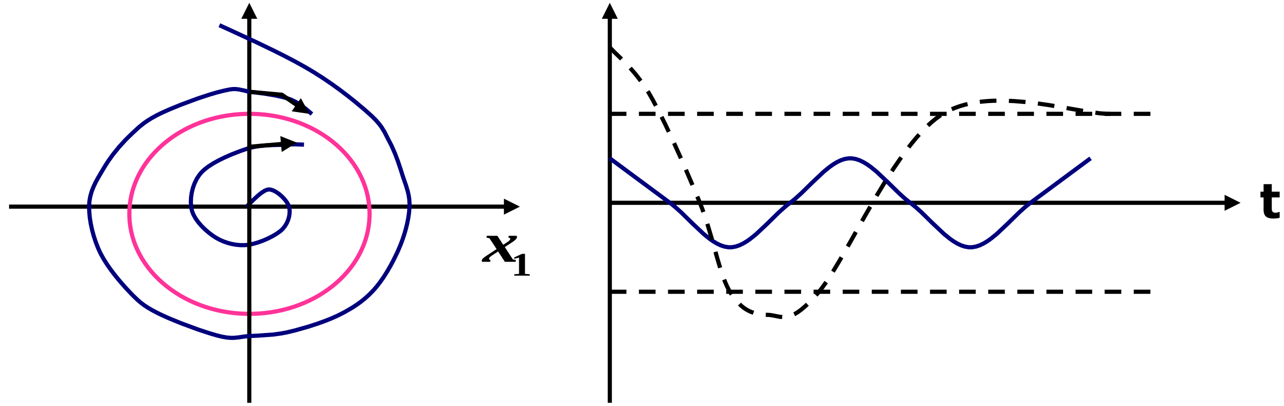
$\ddot{x} = f(x, \dot{x})$ Singular Point $(x_{10}, x_{20})$

$\boldsymbol{x_{10} = x_{20} = 0}$

$$\ddot{x} = \frac{\partial f}{\partial x}\Big|_{x=0} \cdot x + \frac{\partial f}{\partial \dot{x}}\Big|_{\dot{x}=0} \cdot \dot{x}$$

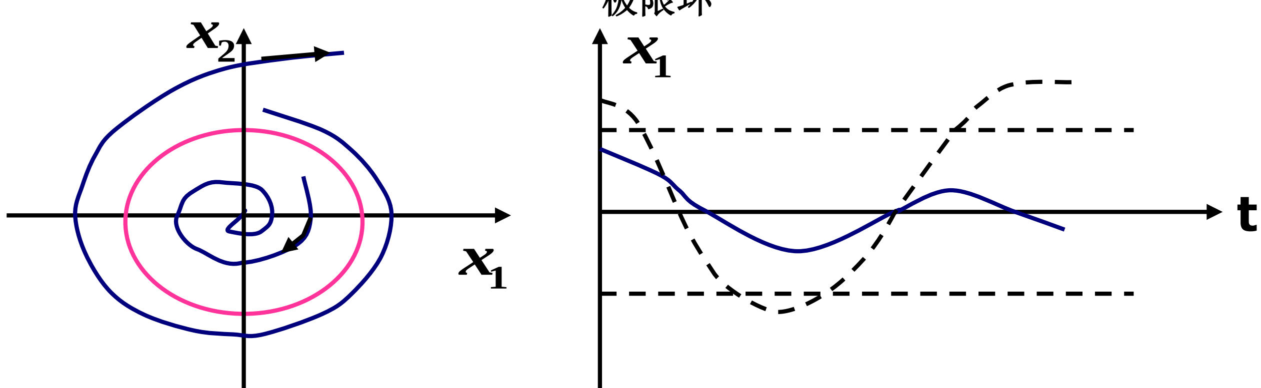
The characteristic equation of system is given by

$$\lambda^2 = \frac{\partial f}{\partial x}\Big|_{x=0} + \frac{\partial f}{\partial \dot{x}}\Big|_{\dot{x}=0} \lambda$$

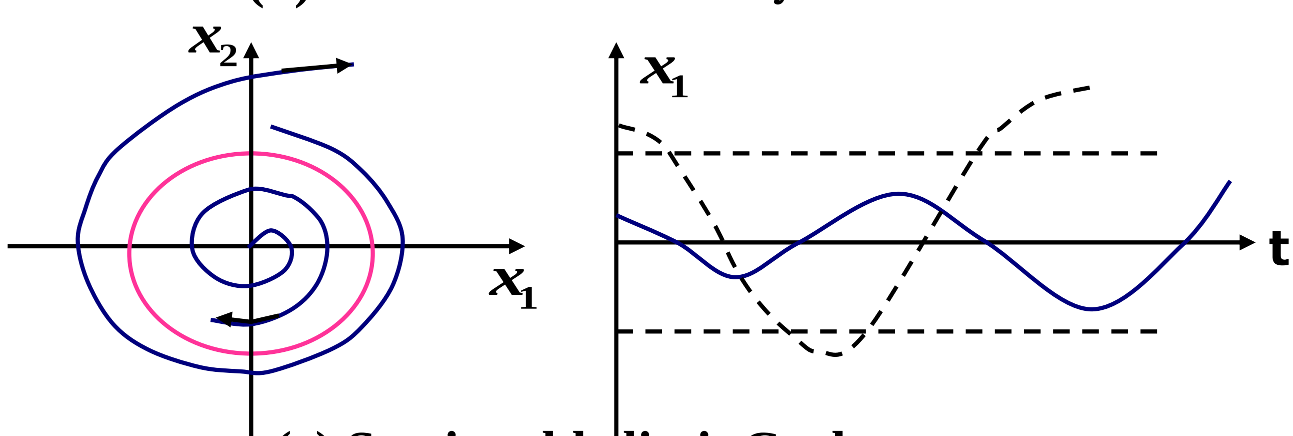



**(a)** Stable limit cycle

极限环




**(b) Unstable limit cycle**




**(c) Semi-stable limit Cycle**

## 3.3 相平面法分析非线性系统

1. Divide the phase plane into ***several areas*** according to nonlinear characteristics. Establish ***linear differential equations*** for each area.
线性微分方程

2. Select ***appropriate coordinate axis*** during the analysis.
坐标轴

切换线
3. Establishing equations for the ***switching lines*** in the phase plane according to different nonlinear characteristics.

4. ***Solve the differential equations*** of each area and then draw phase trajectory. 切换线

5. The phase trajectory of the whole system can be obtained by ***connecting all the partial trajectories*** in different areas.