7.5 Mathematical Models of Discrete-Time Systems

7.5.1 Linear Time-Invariant Difference Equations

(1) Definition of difference — ① Forward ② Backward

(2) The difference equation and its solving method — ① Iteration ② Z-transformation

7.5.2 Impulse-Transfer Function

(1) Definition (2) Properties (3) Limitation

7.5.3 Impulse Transfer Function of Open-Loop Systems

(1) Switch between factors

(2) No switch between factors

(3) With ZOH

7.5.4 Impulse Transfer Function of Closed-Loop Systems

(1) General Method

(2) Mason's formula

Chapter 7 Analysis and Design of Linear Discrete-Time System （Sampled-data System）

7.6 Performance Analysis of Discrete-Time Systems

- **Stability**
- **Dynamic Performance**
- **Steady-state Errors**

7.6.1 Stability of Discrete systems

1. Preliminaries

Stability is the most important performance of a system.

When we sampled a continuous systems, we still have a "continuous" system → the same properties hold as before:

A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.

Now, we introduced the variable $z=e^{Ts}$, how does stability look in the new variable?

First, we much understand the relationship between s-domain and z-domain.

2. s-Domain to z-Domain Mapping

Because $z = e^{sT}$, **let** $s = \sigma + j\omega$ **then** $|z| = e^{\sigma T}$

$\angle z = \omega T$, **thus**

$$\begin{cases} \sigma > 0 & |z| > 1 \\ \sigma < 0 & |z| < 1 \\ \sigma = 0 & |z| = 1 \end{cases}$$

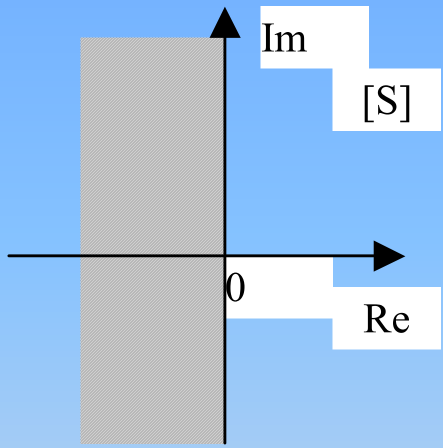
The mapping relationship can be described as in the following figures.

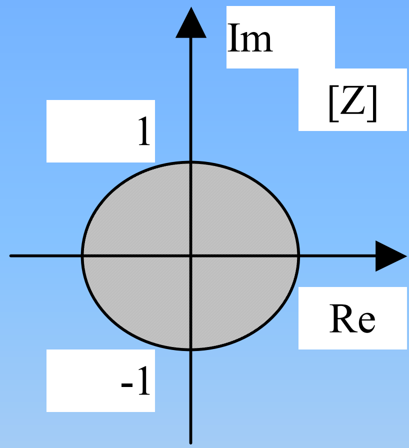
$$z = e^{Ts} = e^{T(\sigma + j\omega)} = e^{T\sigma} e^{j\omega T}$$

$$\sigma < 0 \quad |z| = \frac{1}{e^{-T\sigma}} < 1$$

$$\sigma = 0 \quad |z| = 1$$

$$\sigma > 0 \quad |z| = e^{T\sigma} > 1$$





For a C.L.discrete-time system with unit feedback, the impulse transfer function is:

$$\Phi(z) = \frac{G(z)}{1 + G(z)}$$

Its characteristic function is:

$$1 + G(z) = 0$$

Necessary and Sufficient Condition for Stability of Linear Discrete-Time Systems

——All poles of Φ(z) lie in the unit circle of z plane.

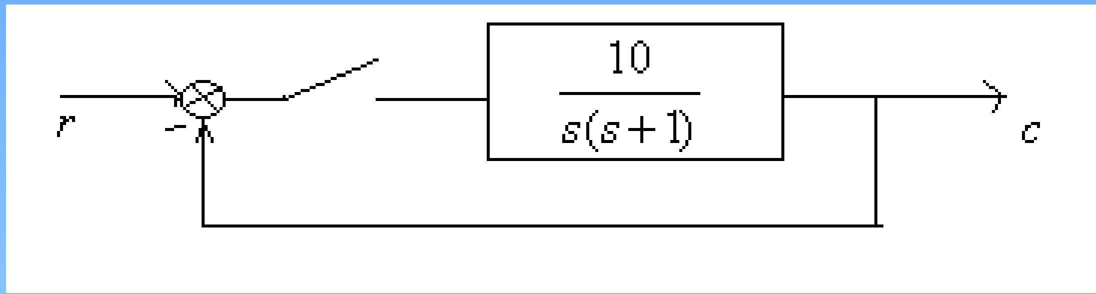
Prove:

$$\Phi(z)=\frac{M(z)}{D(z)}=\frac{\prod_{i=1}^{m}(z-\alpha_i)}{\prod_{j=1}^{n}(z-\beta_j)}=\sum_{j=1}^{n}\frac{C_j z}{z-\beta_j}=K(z)$$

$$c(k)=\sum_{j=1}^{n}C_j\beta_j^k \overset{k\to\infty}{=} 0 \qquad \left|\beta_j\right|<1 \qquad \text{— Necessity}$$

$$c^*(t)=\sum_{k=0}^{\infty}\left(\sum_{j=1}^{n}C_j\beta_j^k\right)\cdot\delta(t-kT) \qquad \text{— Sufficiency}$$

Example **The discrete-time system is shown as the following figure， suppose T=1， is the system stable?**



Solution:

$$G(z)=Z[\frac{10}{s(s+1)}]=\frac{6.32z}{(z-1)(z-0.368)}$$

$$1+G(z)=0\Rightarrow z^2+4.952z+0.368=0$$

$$\Rightarrow z_1=-0.076 \quad z_2=-4.876$$

$\because |z_2|>1$ So the system is unstable.

3. The Stability Criterion of Discrete-Time Systems

- **For continuous-time systems，we can use Routh criterion to determine the stability of the system, where the stable area is on LHP (left-hand-plane) of [s]-domain.**
- **Unfortunately, for discrete-time systems, the stable area is unit circle, not LHP of [z]-domain, we cannot directly apply the Routh criterion as we have to test on something else than LHP.**

w-transformation and Routh criterion in w-domain

We find a transformation that maps the unit circle back onto the LHP while maintaining the algebraic structure of rational functions.

A particular transformation that will accomplish this would be the bilinear transformation:

$$z = \frac{w+1}{w-1}, \quad w = \frac{z+1}{z-1}$$

Suppose $\boldsymbol{z = x + j\,y \quad w = u + j\,v}$

Then

$$w = \frac{z+1}{z-1} = \frac{(x+1) + jy}{(x-1) + jy}$$

$$u + jv = \frac{(x^2 + y^2) - 1}{(x-1)^2 + y^2} + j\frac{2y}{(x-1)^2 + y^2}$$

$$\therefore u = \frac{x^2 + y^2 - 1}{(x-1)^2 + y^2}$$

$$w = \frac{z+1}{z-1} = \frac{x+1+jy}{x-1+jy} = \frac{x^2-1+y^2-j2y}{(x-1)^2+y^2} = u+jv$$

[w] imaginary axis $u = 0 = \dfrac{x^2+y^2-1}{(x-1)^2+y^2} = 0$

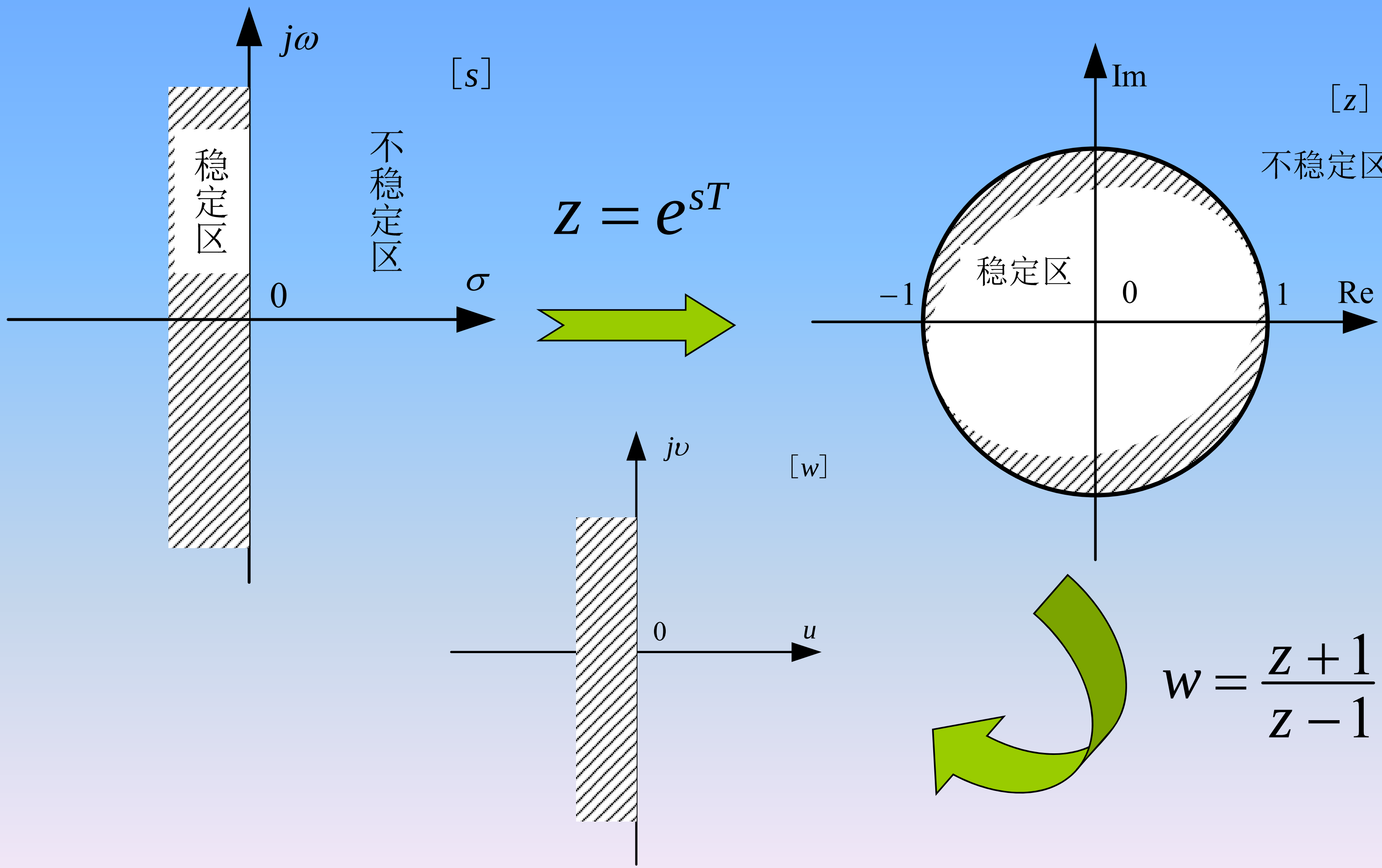
$\Rightarrow \; x^2+y^2=1$ **[z] unit circle**

Points $\begin{cases} \text{inside} \\ \text{outside} \end{cases}$ of the unit circle

$\Rightarrow \; x^2+y^2 \begin{cases} <1 \\ >1 \end{cases}$ in z-plane

$\Rightarrow \begin{cases} u<0 \\ u>0 \end{cases}$ in s-plane

we've learned three methods to determine the stability of a discrete-time systems.



Example 1 Determine the stability from the characteristic equation of a discrete system.

$$D(z) = 45z^3 - 117z^2 + 119z - 39 = 0$$

$$\downarrow \quad z = (w+1)/(w-1)$$

$$= 45(\frac{w+1}{w-1})^3 - 117(\frac{w+1}{w-1})^2 + 119(\frac{w+1}{w-1}) - 39 = 0$$

$$D(w) = 45(w+1)^3 - 117(w+1)^2(w-1) + 119(w+1)(w-1)^2 - 39(w-1)^3 = 0$$

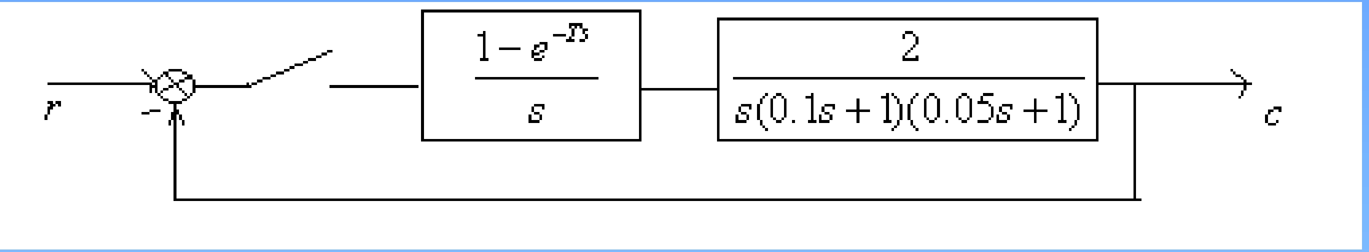
$$D(w) = w^3 + 2w^2 + 2w + 40 = 0$$

Routh

w^3	1	2
w^2	2	40
w^1	-18	
w^0	40	

Unstable!

Example 2 Consider the discrete-time system as shown in the figure, if T=0.1, determine the stability of the system.



Solution:

$$G(z)=\frac{z-1}{z}Z\left[\frac{2}{s^2(0.1s+1)(0.05s+1)}\right]$$

$$=\frac{z-1}{z}\left[-\frac{0.3z}{z-1}+\frac{0.4z}{(z-1)^2}+\frac{0.4z}{z-e^{-10T}}-\frac{0.1z}{z-e^{-20T}}\right]$$

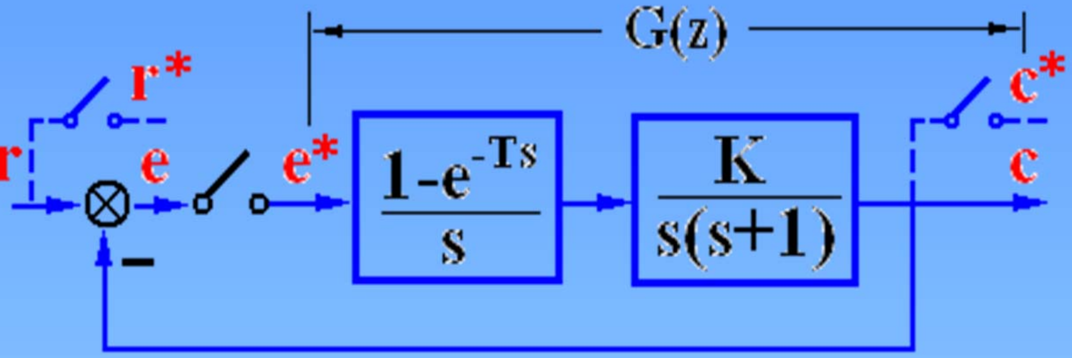
$\boxed{1+G(z)=0}$ $$z^3-1.001z^2+0.3356z+0.0535=0$$

$\boxed{z=\frac{w+1}{w-1}}$ $$2.33w^3+3.68w^2+1.65w+0.34=0$$

w^3	2.33	1.65
w^2	3.68	0.34
w	1.43	0
w^0	0.34	

The elements in the first column are all positive, the system is stable.

Example 3 Consider the system shown in the figure (T=1). Determine the stable range of K.



Routh criterion in w domain

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s}\cdot\frac{K}{s(s+1)}\right] = (1-z^{-1})K\cdot Z\left[\frac{1}{s^2(s+1)}\right]$$

$$= \frac{(z-1)K}{z}\cdot Z\left[\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}\right] = \frac{(z-1)K}{z}\cdot\left[\frac{Tz}{(z-1)^2}-\frac{z}{z-1}+\frac{z}{z-e^{-T}}\right]$$

$$= K\left[\frac{(T-1+e^{-T})z+(1-e^{-T}-Te^{-T})}{(z-1)(z-e^{-T})}\right] \overset{T=1}{=} \frac{0.368K(z+0.718)}{(z-1)(z-0.368)}$$

$$\Phi(z) = \frac{G(z)}{1+G(z)} = \frac{0.368K(z+0.718)}{z^2+(0.368K-1.368)z+(0.264K+0.368)}$$

$$\Phi(z)=\frac{G(z)}{1+G(z)}=\frac{0.368K(z+0.718)}{z^2+(0.368K-1.368)z+(0.264K+0.368)}$$

$$D(z)=z^2+(0.368K-1.368)z+(0.264K+0.368)=0$$

$$\Big\downarrow\; z=\frac{w+1}{w-1}$$

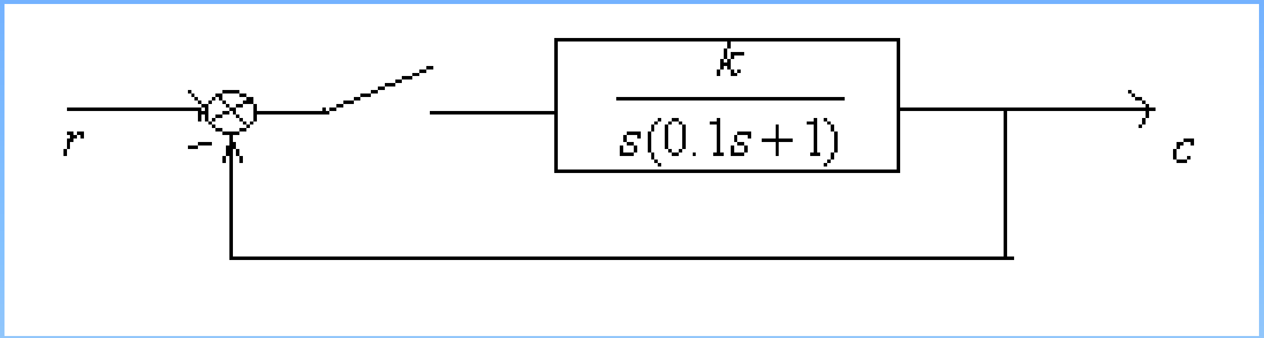
$$=(\frac{w+1}{w-1})^2+(0.368K-1.368)(\frac{w+1}{w-1})+(0.264K+0.368)=0$$

$$(w+1)^2+(0.368K-1.368)(w+1)(w-1)+(0.264K+0.368)(w-1)^2=0$$

$$D(w)=0.632Kw^2+(1.264-0.528K)w+(2.736-0.104K)=0$$

$$\begin{cases} K>0 \\ 1.264-0.528K>0 \\ 2.736-0.104K>0 \end{cases} \qquad \begin{cases} K>0 \\ K<2.394 \\ K<26.3 \end{cases} \qquad \boxed{0<K<2.394}$$

Exercise Consider the system shown in the figure (T=0.1). Determine the stable range of K. If it is a continuous-time system, Determine the stable range of K



Solution:

$$G(z) = Z\left[\frac{k}{s(0.1s+1)}\right] = \frac{0.632kz}{z^2 - 1.368z + 0.368}$$

$$1 + G(z) = 0 \qquad z^2 + (0.632k - 1.368)z + 0.368 = 0$$

$$z = \frac{w+1}{w-1} \qquad 0.632kw^2 + 1.264w + 2.736 - 0.632k = 0$$

For continuous-time system, $0 < k$, then system is stable;

For discrete-time system, $0 < k < 4.33$, stable.

Summary

s-Domain to z-Domain Mapping

Necessary and Sufficient Condition for Stability of Linear Discrete-Time Systems

— All poles of Φ(z) lie in the unit circle of z plane

Routh criterion in w domain (Generalized Routh Criterion)