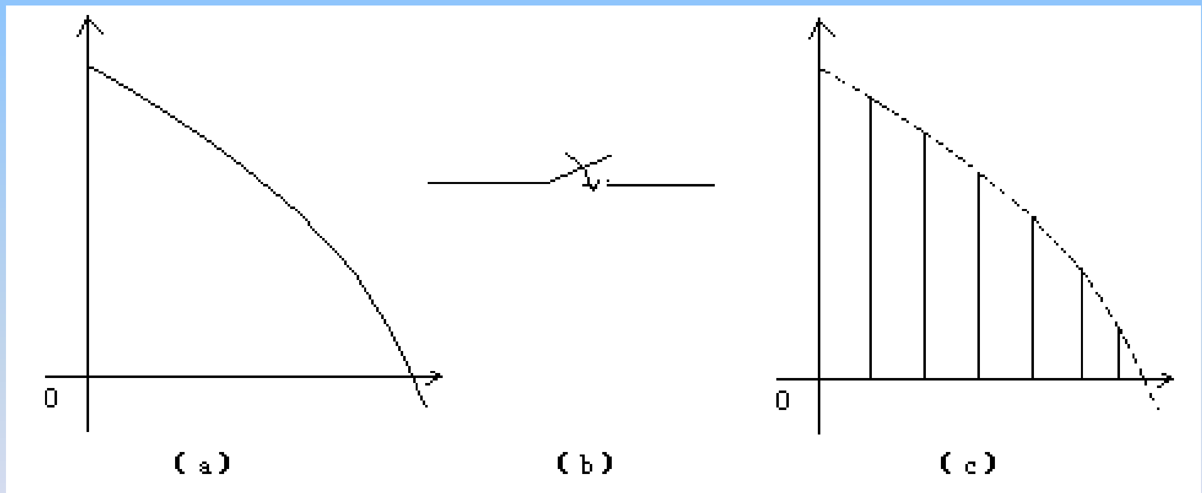
7.1 Discrete-Time Control Systems

Discrete-systems: There is one or more impulse series or digital signals in the system.

Sampled-Data System: a system that is continuous except for one or more sampling operations.

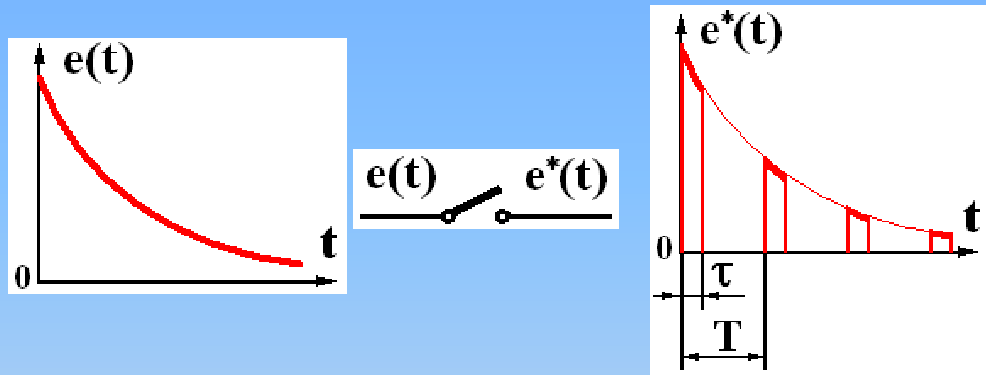


A/D : analog to digital converter

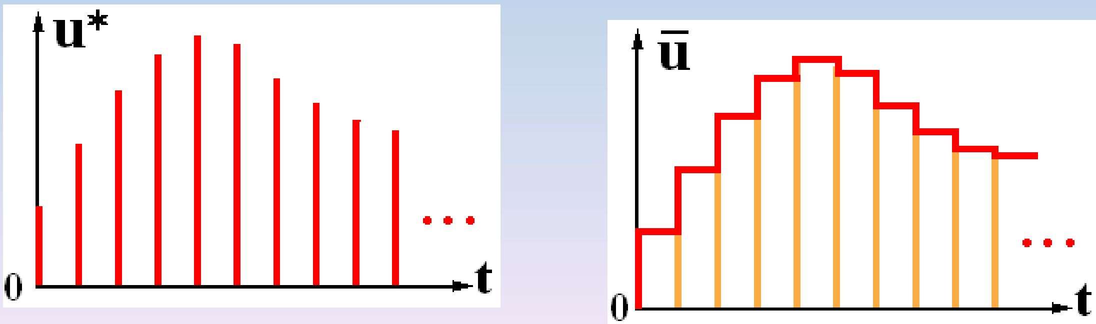
D/A : digital to analog converter

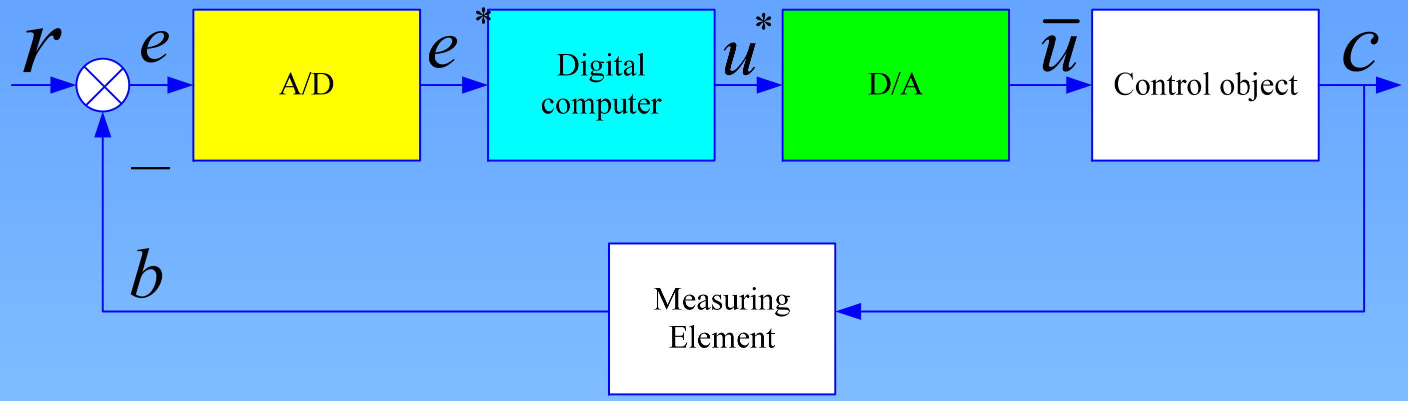
A/D process

- Sampling — Time sampled
- Quantization — Value quantized

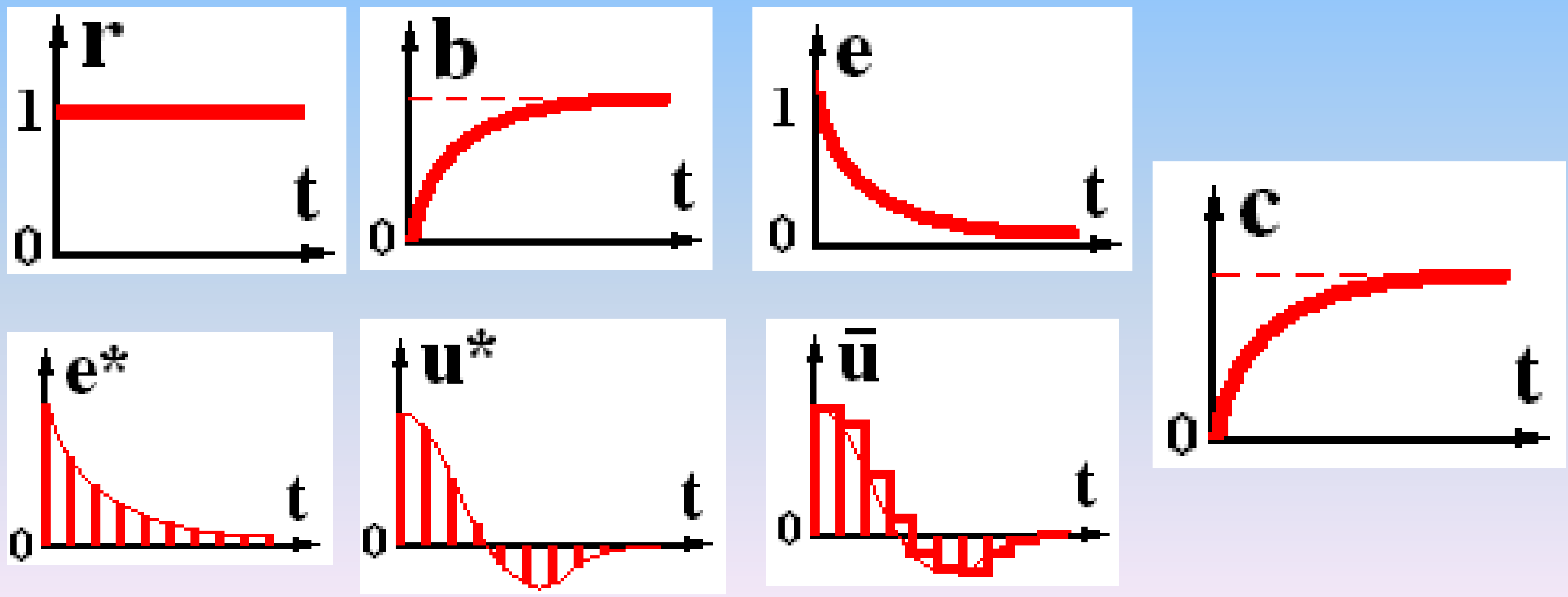


D/A process





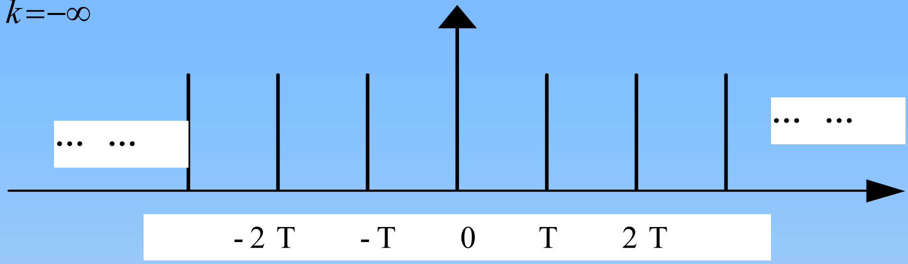
Computer Controlled Systems



7.2 Signal Sampling and Shannon Theorem

3、 Unit Impulse Sequence (Unit Impulse Train)

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT) = \delta(t) + \delta(t-T) + \cdots + \delta(t-kT) + \cdots$$



Unit Impulse sequence

4、 Sampling Signal $\quad \boldsymbol{e^*(t) = \sum_{k=-\infty}^{\infty} e(t)\delta(t-kT)}$

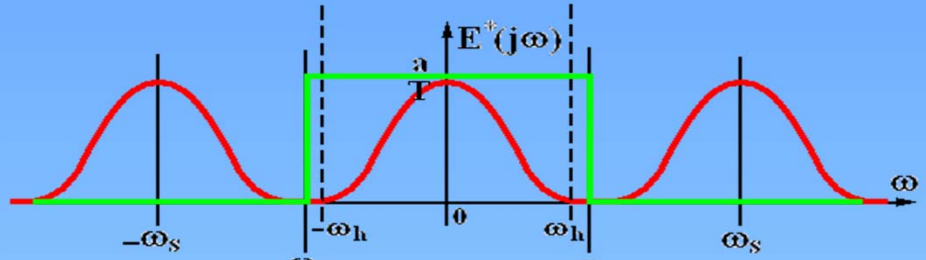
$$E^*(s) = L\left[e^*(t)\right] = \sum_{k=0}^{\infty} e(kT) \cdot e^{-kTs}$$

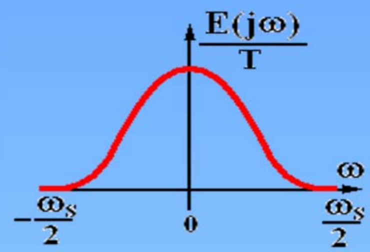
$$E^*(\mathrm{j}\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} E\left[j\left(\omega + k\omega_s\right)\right]$$

Shannon Sampling Theorem—

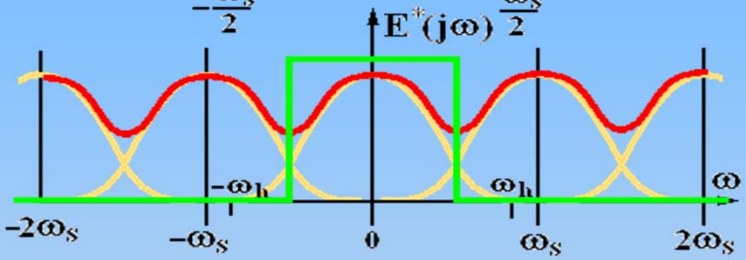
The Necessary Condition for signal recovery

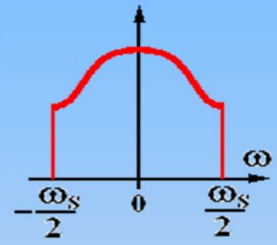
$$\omega_s = \frac{2\pi}{T} > 2\omega_h$$





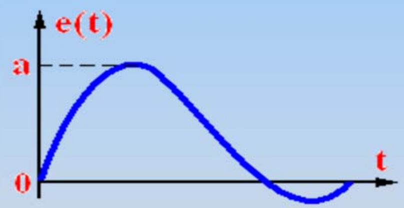
$$\omega_s = \frac{2\pi}{T} < 2\omega_h$$

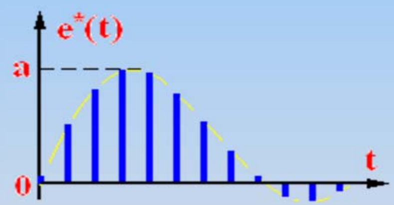


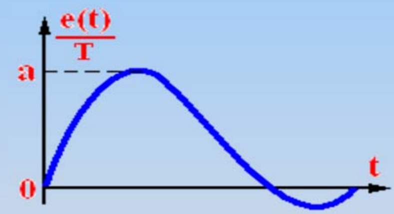


$$\omega_s = \frac{2\pi}{T} > 2\omega_h$$

$$T < \frac{\pi}{\omega_h}$$







$$\omega_s > 2\omega_h$$

Sampling switch **Ideal filter**

Example Consider $e(t)=e^{-t}$, determine the sampling frequency ω_s according to Shannon Sampling Theorem.

Solution: L-Transform of e(t) is: $E(s)=\dfrac{1}{s+1}$

Frequency characteristic $E(j\omega)=\dfrac{1}{j\omega+1}$

thus

$$\left|E(j\omega)\right|=\frac{1}{\sqrt{\omega^2+1}}$$

Take $\left|E(j\omega)\right|=0.05\left|E(0)\right|,$

$$\frac{1}{\sqrt{\omega_h^2+1}}=0.05,\quad \omega_{\mathrm{h}}=20rad/s=\omega_{\max}$$

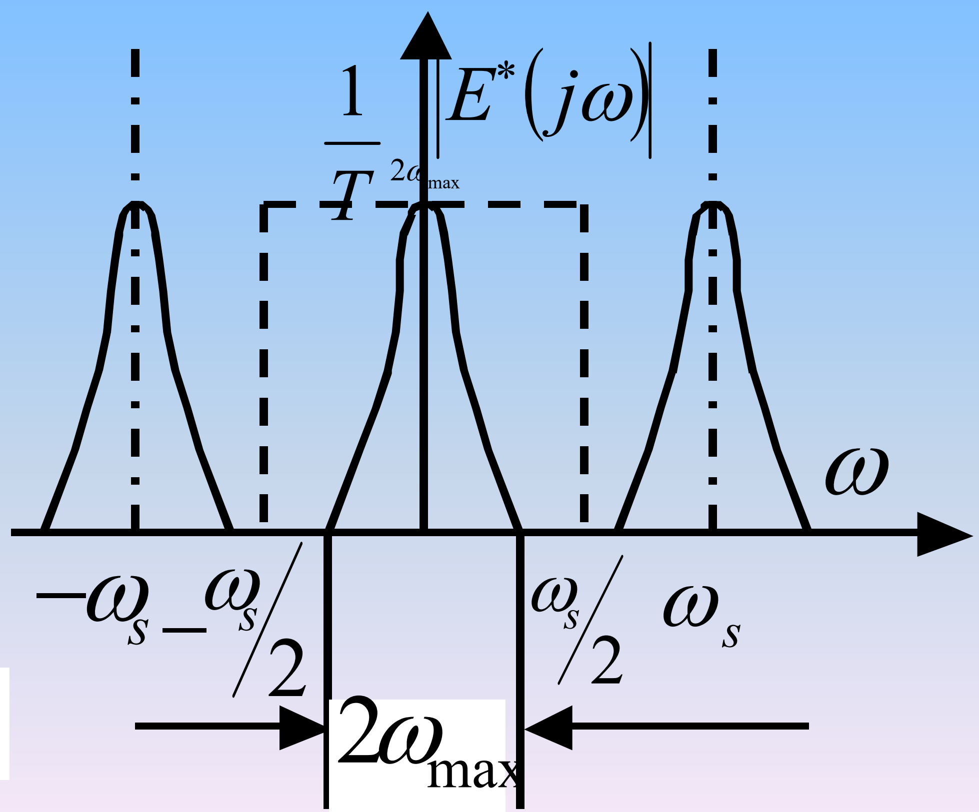
$$\omega_s\geq 2\omega_{\max}=40rad/s$$

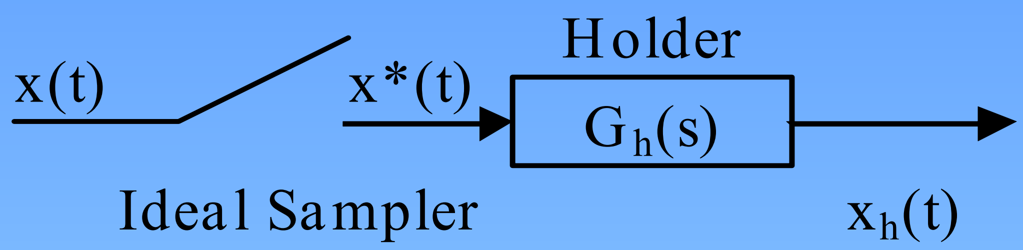
7.3 Signal Recovery and Zero-Order Hold

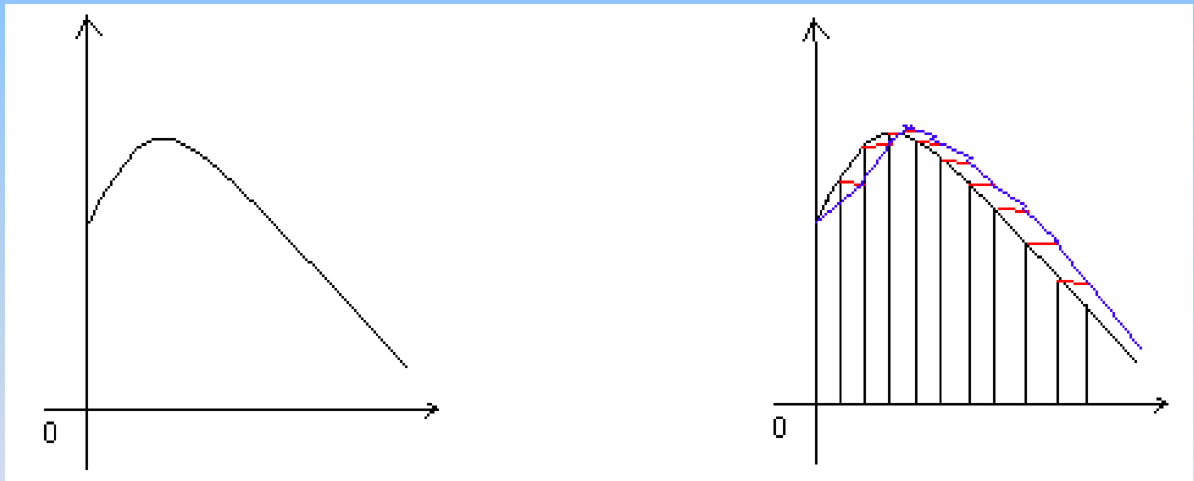
7.3.1 Signal recovery

The ideal filter is illustrated as the dotted line in the figure.

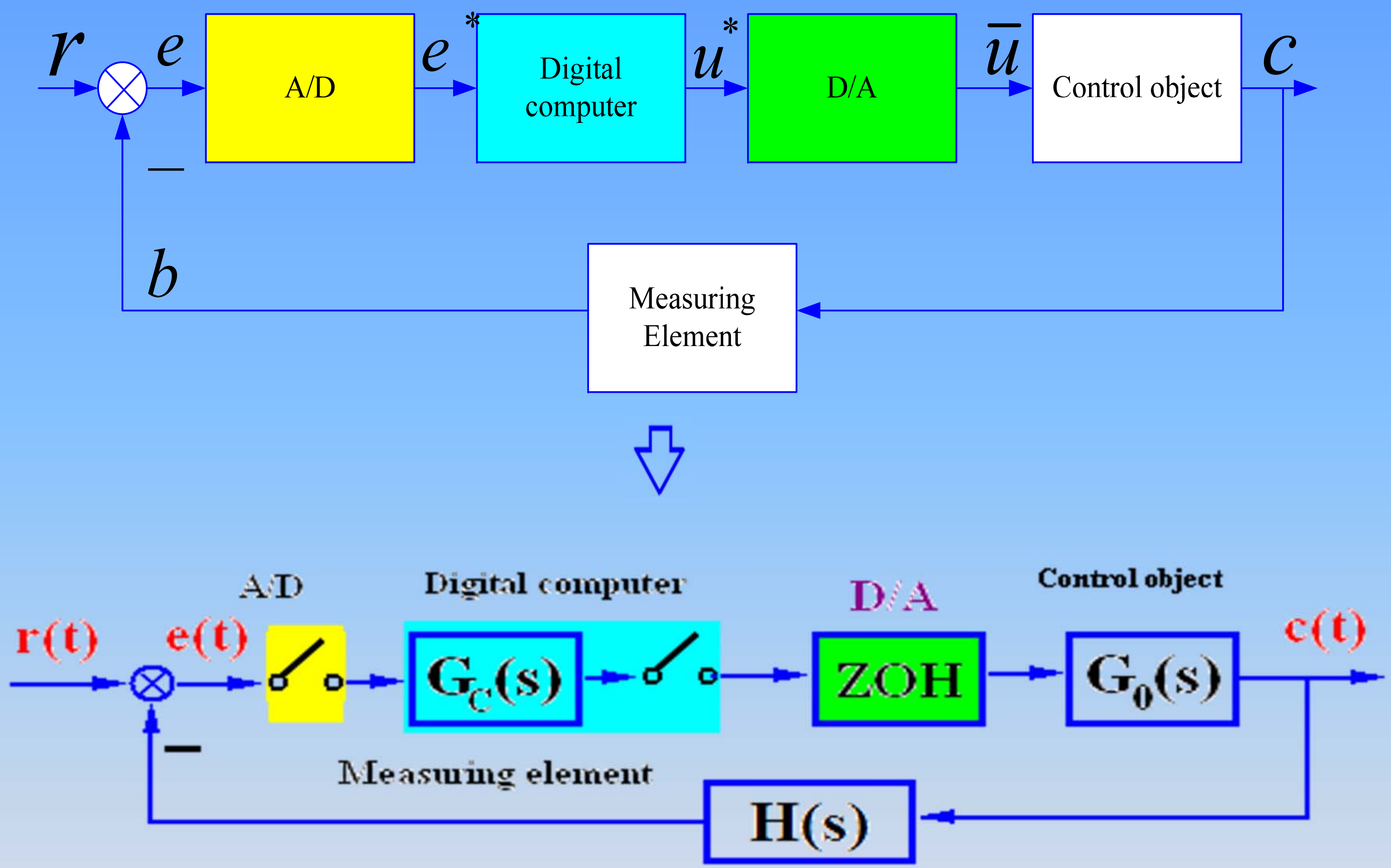
0







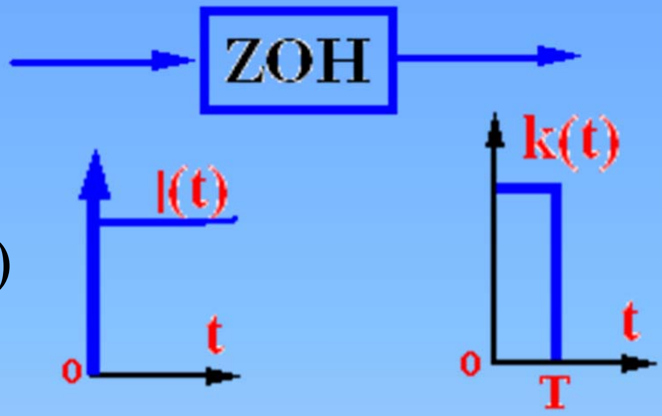
原连续信号　　　　零阶保持器的恢复信号



Computer Control System

7.3.2 Zero-Order Hold

$$k(t) = 1(t) - 1(t-T)$$



$$x_h(t) = \sum_{k=0}^{\infty} x(kT)(1(t-kT) - 1(t-kT-T))$$

Using L-Transform，we get

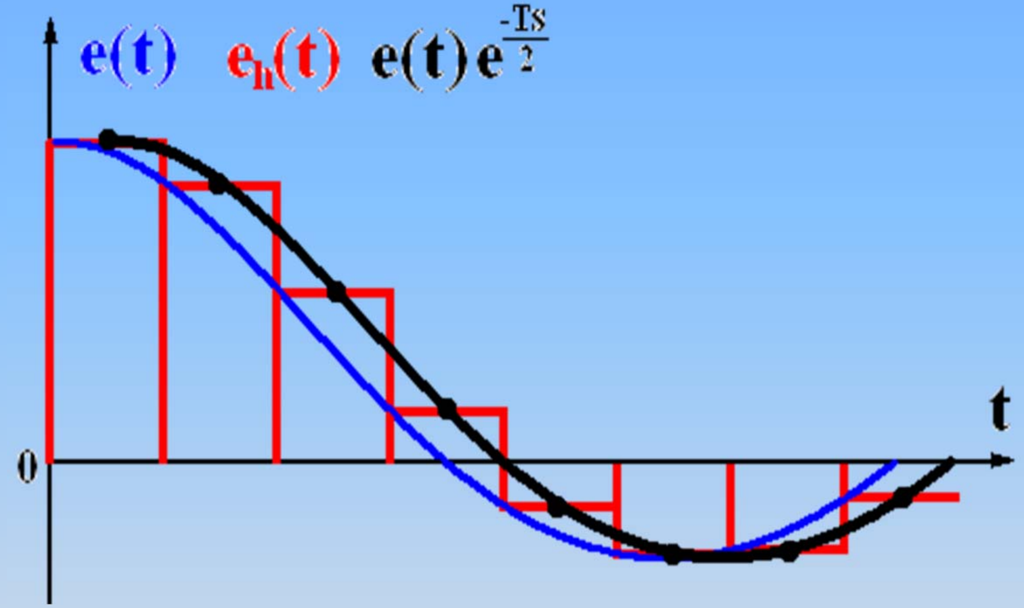
$$x_h(s) = \underbrace{\sum_{k=0}^{\infty} x(kT)e^{-kTs}}_{x^*(s)} \left[\frac{1}{s} - \frac{1}{s}e^{-Ts}\right]$$

$$\frac{x_h(s)}{x^*(s)} = \frac{1-e^{-Ts}}{s} = G_h(s)$$

Effect of zero-order holder on the system

$$G_h(s) = \frac{1 - e^{-Ts}}{s}$$

$$\approx e^{-Ts/2}$$



Frequency characteristics:

$$G_h(j\omega)=\frac{1-e^{-j\omega T}}{j\omega}=\frac{e^{-\frac{1}{2}j\omega T}(e^{j\frac{1}{2}\omega T}-e^{-j\frac{1}{2}\omega T})}{j\omega}=\frac{2e^{-\frac{1}{2}j\omega T}\sin(\frac{1}{2}\omega T)}{\omega}$$

$$G_h(j\omega)=T\frac{\sin(\frac{\omega T}{2})}{\frac{\omega T}{2}}e^{-\frac{1}{2}j\omega T}$$

$$\because T=\frac{2\pi}{\omega_s}$$

setting $S_a(x)=\sin x/x$

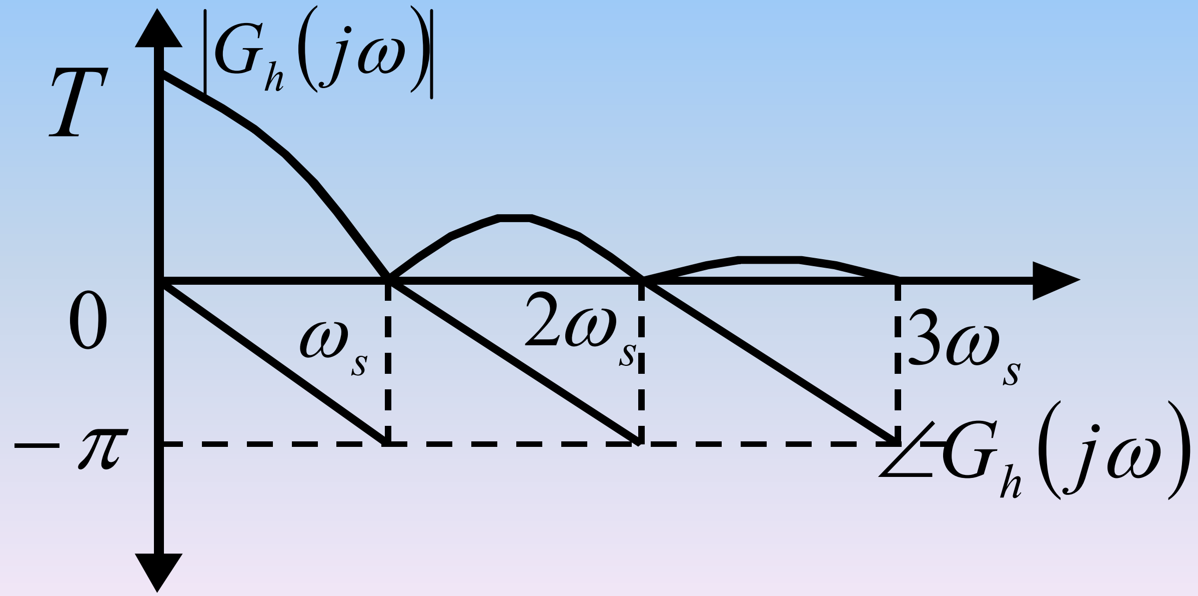
so the

$$G_h(j\omega)=\frac{2\pi}{\omega_s}\cdot S_a(\pi\omega/\omega_s)\cdot e^{-j\frac{\pi\omega}{\omega_s}}$$

the amplitude $|G_h(j\omega)| = \dfrac{2\pi}{\omega_s} \cdot |S_a(\pi\omega/\omega_s)|$

phase angle $\angle G_h(j\omega) = -\dfrac{\pi\omega}{\omega_s} + \angle S_a(\pi\omega/\omega_s)$

where $\angle S_a(\pi\omega/\omega_s) = \begin{cases} 0, & 2n\omega_s < \omega < (2n+1)\omega_s \\ \pi, & (2n+1)\omega_s < \omega < (2n+2)\omega_s \end{cases}$



Chapter 7 Analysis and Design of Linear Discrete-Time System （Sampled-data System）

7.1 Introduction

7.2 The Sampling Process and Sampling Theorem

7.3 Signal Recovery and Zero-Order Hold

7.4 Z-Transform and Inverse Z Transform

7.5 Mathematical Models of Discrete-Time Systems

7.6 Performance Analysis of Discrete-Time Systems

7.7 Digital Control Design for Discrete-Time Systems

7.4 Z-Transform and Inverse Z Transform

7.4.1. Z-transform

Definition:

$$\because E^*(s) = \sum_{k=0}^{+\infty} e(kT) \cdot e^{-kTs}$$

Set

$$z = e^{Ts} \qquad s = \frac{1}{T} \ln z$$

$$E(z) = \sum_{k=0}^{\infty} e(kT) \cdot z^{-k}$$

$$E(z) = E^*(s)\big|_{z=e^{Ts}}$$

$$E^*(s) = \sum_{k=0}^{\infty} e(kT) \cdot e^{-kTs}$$

$$z^{-1} = e^{-Ts}$$

$$E(z) = Z[e^*(t)] = E^*(s)\Big|_{z=e^{Ts}} = \sum_{n=0}^{\infty} e(kT) \cdot z^{-k}$$

Rmk:

$$E(z) = Z[e^*(t)] = Z[E(s)] = Z[E^*(s)] = Z[e(t)]$$

The z-transform is only for discrete signal.

E(z) is only mapping to a unique e*(t), but not a unique e(t).

7.4.2 Methods of z-Transform

By the definition.

Partial fraction expansion.

1. By the Definition

Example 1 $x_1(t) = 1(t)$ **and** $x_2(t) = \sum_{k=0}^{\infty} \delta(t - kT)$, **obtain** $X_1(z)$ and $X_2(z)$。

Solution:

$$X_1(z) = \sum_{k=0}^{\infty} x_1(kT) z^{-k} = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

$$X_2(z) = \sum_{k=0}^{\infty} x_2(kT) z^{-k} = 1 + z^{-1} + z^{-2} + \cdots = \frac{z}{z - 1}$$

Tips: Though $x_1(t)$ and $x_2(t)$ are not same, they may have the same Z-transform.

$$E(z)=\sum_{k=0}^{\infty} e(kT)\cdot z^{-k}$$

Example 2 $e(t)=\sin\omega t=\frac{1}{2j}\left[e^{j\omega t}-e^{-j\omega t}\right]$

$$E(z)=\sum_{k=0}^{\infty}\frac{1}{2j}\left[e^{j\omega kT}-e^{-j\omega kT}\right]\cdot z^{-k} \quad =\frac{1}{2j}\sum_{k=0}^{\infty}\left[(e^{j\omega T}z^{-1})^{k}-(e^{-j\omega T}z^{-1})^{k}\right]$$

$$=\frac{1}{2j}\left[\frac{1}{1-e^{j\omega T}z^{-1}}-\frac{1}{1-e^{-j\omega T}z^{-1}}\right]=\frac{1}{2j}\left[\frac{z}{z-e^{j\omega T}}-\frac{z}{z-e^{-j\omega T}}\right]$$

$$=\frac{1}{2j}\cdot\frac{z(e^{j\omega T}-e^{-j\omega T})}{z^{2}-(e^{j\omega T}+e^{-j\omega T})z+1}=\frac{z\sin\omega T}{z^{2}-2\cos\omega T\cdot z+1}$$

$$E(\mathrm{z}) = \sum_{k=0}^{\infty} e(kT) \cdot z^{-k}$$

Example 3 $\boldsymbol{e(t) = t}$

Solution. $E(z) = \sum_{k=0}^{\infty} kT \cdot z^{-k} = T\left[z^{-1} + 2z^{-2} + 3z^{-3} + \ \cdots\right]$

$$= Tz\left[z^{-2} + 2z^{-3} + 3z^{-4} + \ \cdots\right]$$

$$= -Tz\left[\frac{d}{dz} z^{-1} + \frac{d}{dz} z^{-2} + \frac{d}{dz} z^{-3} + \ \cdots\right]$$

$$= -Tz \frac{d}{dz}\left[z^{-1} + z^{-2} + z^{-3} + \ \cdots\right]$$

$$= -Tz \frac{d}{dz} z^{-1}\left[1 + z^{-1} + z^{-2} + \ \cdots\right]$$

$$= -Tz \frac{d}{dz}\left[\frac{1}{z} \cdot \frac{1}{1 - z^{-1}}\right] = -Tz \frac{d}{dz}\left[\frac{1}{z - 1}\right] \boxed{= \frac{Tz}{(z-1)^2}}$$

$$E(z)=\sum_{k=0}^{\infty} e(kT)\cdot z^{-k}$$

2. Partial Fraction Expansion

Example 4 $E(s)=\dfrac{1}{(s+a)(s+b)}$ **Obtain $E(z)$=?**

Solution：

$$E(s)=\frac{1}{a-b}\cdot\frac{(s+a)-(s+b)}{(s+a)(s+b)}=\frac{1}{a-b}\left[\frac{1}{s+b}-\frac{1}{s+a}\right]$$

$$e(t)=\frac{1}{a-b}\left[e^{-bt}-e^{-at}\right]$$

$$E(z)=\frac{1}{a-b}\sum_{k=0}^{\infty}\left[e^{-bkT}-e^{-akT}\right]\cdot z^{-k}$$

$$=\frac{1}{a-b}\left[\sum_{k=0}^{\infty}(e^{-bT}\cdot z^{-1})^{k}-\sum_{k=0}^{\infty}(e^{-aT}\cdot z^{-1})^{k}\right]$$

$$=\frac{1}{a-b}\left[\frac{1}{1-e^{-bT}z^{-1}}-\frac{1}{1-e^{-aT}z^{-1}}\right]=\frac{1}{a-b}\left[\frac{z}{z-e^{-bT}}-\frac{z}{z-e^{-aT}}\right]$$

The z-transform of typical functions

$f(t)$	$F(s)$	$F(z)$
$\delta(t)$	1	1
$f(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{zT}{(z-1)^2}$
$t^2/2$	$\frac{1}{s^3}$	$\frac{z(z+1)T^2}{2(z-1)^3}$

e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\frac{zTe^{-aT}}{(z-e^{-aT})^2}$
$a^{t/T}$	$\frac{1}{s-(1/T)\ln a}$	$\frac{z}{z-a}$ $(a>0)$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z\sin\omega T}{z^2-2z\cos\omega T+1}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z^2-z\cos\omega T}{z^2-2z\cos\omega T+1}$
$1-e^{-at}$	$\frac{a}{s(s+a)}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-at}\sin\omega T}{z^2-2ze^{-at}\cos\omega T+e^{-2aT}}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{z(z-e^{-aT}\cos\omega T)}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$

7.4.3 Properties of z-Transform

1. linear property $$Z\left[a\cdot e_1^*(t)\pm b\cdot e_2^*(t)\right]=a\cdot E_1(z)\pm b\cdot E_2(z)$$

2. Real shifting theorem 实位移定理

① Lag 延时定理 $$Z\left[e(t-nT)\right]=z^{-n}E(z)$$

Proof. $$\text{LHS}=\sum_{K=0}^{\infty}e(kT-nT)\cdot z^{-k}$$

$$j=k-n$$

$$=\sum_{j=-n}^{\infty}e(jT)\cdot z^{-(j+n)}=z^{-n}\sum_{j=0}^{\infty}e(jT)\cdot z^{-j}$$

$$=z^{-n}E(z)=\text{RHS}$$

2. Real shifting theorem 实位移定理

② Lead 超前定理

$$Z\left[e(t+nT)\right]=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]$$

Proof.

$$\text{LHS}=\sum_{k=0}^{\infty}e(kT+nT)\cdot z^{-k}=z^{n}\sum_{k=0}^{\infty}e(kT+nT)\cdot z^{-(k+n)}$$

$$\Big\downarrow\; j=k+n$$

$$=z^{n}\sum_{j=n}^{\infty}e(jT)\cdot z^{-j}=z^{n}\left[\sum_{j=0}^{\infty}e(jT)\cdot z^{-j}-\sum_{j=0}^{n-1}e(jT)\cdot z^{-j}\right]$$

$$=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]=\text{RHS}$$

$$Z\left[e(t+nT)\right]=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]$$

Example 5 $\quad e(t)=t-T$

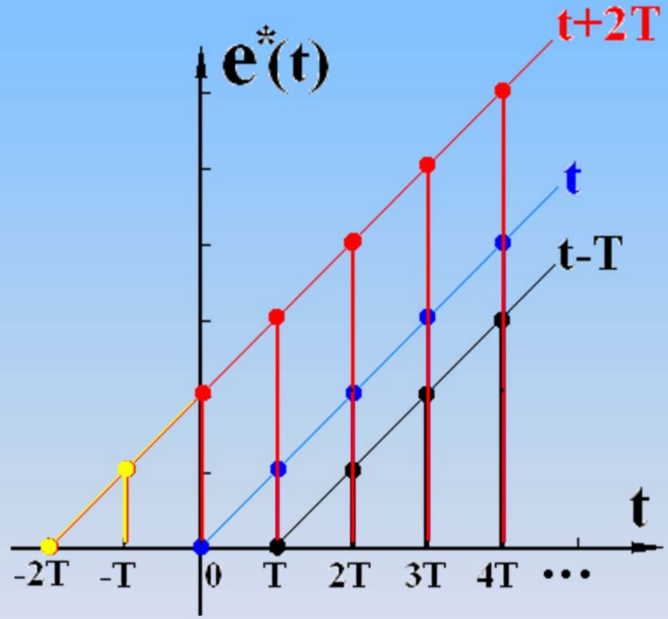
$$E(z)=Z\left[t-T\right]=z^{-1}Z[t]=z^{-1}\frac{Tz}{(z-1)^{2}}=\frac{T}{(z-1)^{2}}$$

Example 6 $\quad e(t)=t+2T$

$$E(z)=Z\left[t+2T\right]$$

$$=z^{2}\left\{Z\left[t\right]-\sum_{k=0}^{1}kT\cdot z^{-k}\right\}$$

$$=z^{2}\left[\frac{Tz}{(z-1)^{2}}-0-Tz^{-1}\right]$$



3. Complex shifting theorem

复位移定理

$$\boldsymbol{Z}\left[e(t)\cdot e^{\mp at}\right]=\boldsymbol{E}\left(z\cdot e^{\pm aT}\right)$$

Proof.

$$\mathrm{LHS}=\sum_{k=0}^{\infty}e(kT)\cdot e^{\mp akT}z^{-k}\quad=\sum_{k=0}^{\infty}e(kT)\cdot\left(z\cdot e^{\pm aT}\right)^{-k}$$

$$\Bigg\downarrow\quad z_1=z\cdot e^{\pm aT}$$

$$=\sum_{k=0}^{\infty}e(kT)\cdot\left(z\cdot e^{\pm aT}\right)^{-k}=\boldsymbol{E}(z_1)=E\left(z\cdot e^{\pm akT}\right)=\mathrm{RHS}$$

Example 7 $\boldsymbol{e(t)=t\cdot e^{-at}}$

$$\boldsymbol{E(z_1)=Z[t]\big|_{z_1=z\cdot e^{aT}}=\frac{Tz_1}{(z_1-1)^2}=\frac{T(z\cdot e^{aT})}{(z\cdot e^{aT}-1)^2}=\frac{Tz\cdot e^{-aT}}{(z-e^{-aT})^2}}$$

4. Initial-value Theorem

$$\lim_{n\to 0} e(nT) = \lim_{z\to\infty} E(z)$$

Proof:

$$E(z) = \sum_{n=0}^{\infty} e(nT)\cdot z^{-n}$$

$$= \left[e(0) + e(1)\cdot z^{-1} + e(2)\cdot z^{-2} + e(3)\cdot z^{-3} + \cdots\right]$$

$$\lim_{z\to\infty} E(z) = e(0)$$

<u>Example 8</u> $E(z) = \dfrac{0.792\cdot z^2}{(z-1)[z^2 - 0.416z + 0.208]}$

$$e(0) = \lim_{z\to\infty} E(z) = 0$$

$$Z\left[e(t+nT)\right]=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]$$

5. Final value theorem $\lim_{n\to\infty} e(nT) = \lim_{z\to 1}(z-1)\cdot E(z)$

Proof： $Z\left[e(t+T)-e(t)\right]=z\left[E(z)-e(0)\right]-E(z)$

$$=(z-1)E(z)-z\cdot e(0)$$

$$(z-1)E(z)=z\cdot e(0)+Z\left[e(t+T)-e(t)\right]$$

$$\lim_{z\to 1}(z-1)E(z)=\lim_{z\to 1}\left\{z\cdot e(0)+\sum_{n=0}^{\infty}\left[e[(n+1)T]-e(nT)\right]\cdot z^{-n}\right\}$$

$$=e(0)+[e(1)-e(0)]+[e(2)-e(1)]+[e(3)-e(2)]+\cdots$$

$$=e(\infty T)$$

Example 9 $E(z) = \dfrac{0.792 \cdot z^2}{(z-1)[z^2 - 0.416z + 0.208]}$

$$e(\infty T) = \lim_{z \to 1}(z-1)E(z) = \lim_{z \to 1} \frac{0.792 \cdot z^2}{[z^2 - 0.416z + 0.208]} = \mathbf{1}$$

6. Convolution theorem

If $c^*(t) = e^*(t) * g^*(t) = \sum_{k=0}^{\infty} e(kT) \cdot g[(n-k)T]$

We have: $\boldsymbol{C(z) = E(z) \cdot G(z)}$

(See the text book for proof)

Properties of z-Transform

1. linear property $$Z\left[a\cdot e_1^*(t)\pm b\cdot e_2^*(t)\right]=a\cdot E_1(z)\pm b\cdot E_2(z)$$

2. Real shifting theorem
$$\begin{cases}\text{Lag} & Z\left[e(t-nT)\right]=z^{-n}E(z)\\ \text{Lead} & Z\left[e(t+nT)\right]=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]\end{cases}$$

3. Complex shifting theorem $$Z\left[e(t)\cdot e^{\mp at}\right]=E\left(z\cdot e^{\pm aT}\right)$$

4. Initial-value theorem $$\lim_{n\to 0}e(nT)=\lim_{z\to\infty}E(z)$$

5. Final-value theorem $$\lim_{n\to\infty}e(nT)=\lim_{z\to 1}(z-1)\cdot E(z)$$

6. Convolution theorem $$c^*(t)=e^*(t)*g^*(t)\quad\Rightarrow\quad C(z)=E(z)\cdot G(z)$$

7.4.4 Inverse z-Transform

$$Z^{-1}[X(z)] = x(nT)$$

Tips:

Inverse Z-transform can only provide discrete-time signal $x^*(t)$, instead of continuous signal $x(t)$。

- Long Division（长除法）
- Partial-Fraction expansion — Expansion of $\dfrac{E(z)}{z}$
- Residue（留数法） — $e(nT) = \sum \mathrm{Res}\left[E(z) \cdot z^{n-1}\right]$

1. Long Division（长除法）

$$E(z)=\frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}$$

Numerator is divided by denominator ,we get

$$E(z)=c_0+c_1 z^{-1}+\cdots+c_k z^{-k}+\cdots=\sum_{k=0}^{\infty} c_k z^{-k}=\sum_{k=0}^{\infty} e(kT) z^{-k}$$

$$e^*(t)=c_0\delta(t)+c_1\delta(t-T)+\cdots+c_k\delta(t-kT)+\cdots$$

Example 10 $F(z)=\dfrac{z}{(z-2)(z-3)}$, **obtain f*(t).**

Solution:

Because $F(z)=\dfrac{z}{z^2-5z+6}=\dfrac{z^{-1}}{1-5z^{-1}+6z^{-2}}$

By long-division, we get that

$$F(z)=z^{-1}+5z^{-2}+19z^{-3}+65z^{-4}+\cdots$$

Thus

$$f(0)=0,\ \ f(T)=1,\ \ f(2T)=5,\ \ f(3T)=19,\ \ f(4T)=65,\cdots$$

Then

$$f^*(t)=\delta(t-T)+5\delta(t-2T)+19\delta(t-3T)+65\delta(t-4T)+\cdots$$

2. Partial fraction expansion

Note: here, we expand $\frac{X(z)}{z}$, instead of z。

$$\frac{X(z)}{z} = \sum_{i=1}^{n} \frac{A_i}{z - z_i}$$

Consider

$$X(z)=\frac{b_0z^m+b_1z^{m-1}+\cdots+b_{m-1}z+b_m}{a_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n}$$

Then

$$X(z)=\frac{b_0z^m+b_1z^{m-1}+\cdots+b_{m-1}z+b_m}{a_0\prod_{i=1}^{n}(z-z_i)}$$

If there is no repeated root for the denominator，it generates

$$X(z)=z\left(\frac{A_1}{z-z_1}+\frac{A_2}{z-z_2}+\cdots+\frac{A_n}{z-z_n}\right)$$

其中系数A_i可由式决定：

$$A_i=\left[(z-z_i)\frac{X(z)}{z}\right]\Bigg|_{z=z_i}$$

Example 12 $E(z)=\dfrac{z^2}{(z-0.8)(z-0.1)}$ Obtain e*(t). (PFE & Residue)

PFE: $$\frac{E(z)}{z}=\frac{z}{(z-0.8)(z-0.1)}=\frac{C_1}{(z-0.8)}+\frac{C_2}{(z-0.1)}$$

$$C_1=\lim_{z\to 0.8}\frac{z}{(z-0.1)}=\frac{8}{7} \qquad C_2=\lim_{z\to 0.1}\frac{z}{(z-0.8)}=\frac{-1}{7}$$

$$=\frac{8/7}{(z-0.8)}-\frac{1/7}{(z-0.1)}$$

$$E(z)=\frac{8}{7}\cdot\frac{z}{(z-0.8)}-\frac{1}{7}\cdot\frac{z}{(z-0.1)}$$

$$e(t)=(8\times 0.8^{\frac{t}{T}}-0.1^{\frac{t}{T}})/7 \qquad e(nT)=(8\times 0.8^n-0.1^n)/7$$

$$e^*(t)=\sum_{n=0}^{\infty}\left[(8\times 0.8^n-0.1^n)/7\right]\cdot\delta(t-nT)$$

Example 13 **Consider**

$$F(z)=\frac{z}{(z-1)(z-e^{-T})}$$

Obtain f*(t)。

Solution：

$$\frac{F(z)}{z}=\frac{K_1}{z-1}+\frac{K_2}{z-e^{-T}}$$

$$K_1=\lim_{z\to 1}\left(\frac{z-1}{z}\right)F(z)=\frac{1}{1-e^{-T}}$$

$$K_2=\lim_{z\to e^{-T}}\left(\frac{z-e^{-T}}{z}\right)F(z)=-\frac{1}{1-e^{-T}}$$

$$F(z)=\frac{1}{1-e^{-T}}\left(\frac{z}{z-1}-\frac{z}{z-e^{-T}}\right)$$

$$\Rightarrow\quad f(nT)=\frac{1}{1-e^{-T}}\left(1-e^{-nT}\right)$$

$$f^*(t)=\frac{1}{1-e^{-T}}\sum_{k=0}^{+\infty}(1-e^{-kT})\delta(t-kT)$$

3、 Residue(留数法)

$$F(z)=\sum_{k=0}^{+\infty} f(kT)z^{-k}$$

$$F(z)z^{m-1}=\sum_{k=0}^{+\infty} f(kT)z^{m-k-1}$$

Γ **Encircle all the poles of** $F(z)z^{k-1}$

$$\oint_{\Gamma} F(z)z^{m-1}dz=\oint_{\Gamma}\left[\sum_{k=0}^{+\infty} f(kT)z^{m-k-1}\right]dz$$

$$\oint_{\Gamma} F(z)z^{m-1}dz=\sum_{k=0}^{+\infty} f(kT)\oint_{\Gamma} z^{m-k-1}dz$$

$$f(kT)=\sum_{i=1}^{n} res[F(z)z^{k-1},z_i]$$

$z_i, i=1,2,\cdots,n$ **are all the poles of** $F(z)z^{k-1}$

$$Res\left[z^{(k-1)}x(z)\right]=\lim_{z\to z_i}\frac{1}{(r-1)!}\frac{d^{r-1}}{dz^{r-1}}\left[(z-z_i)^r z^{k-1}x(z)\right]$$

其中***Res*[]**表示函数的留数。

Example 14 For

$$F(z)=\frac{10z}{(z-1)(z-2)}$$

Obtain its inverse z-transform by residue Method.

Solution： $$F(z)z^{k-1} = \frac{10z^k}{(z-1)(z-2)}$$

Poles $z_1 = 1$ and $z_2 = 2$ ， and

$$res[F(z)z^{k-1},1] = \lim_{z\to 1}(z-1)F(z)z^{k-1} = -10$$

$$res[F(z)z^{k-1},2] = \lim_{z\to 2}(z-2)F(z)z^{k-1} = 10\cdot 2^k$$

Then $$f(kT) = 10(2^k - 1) \qquad (k = 0,1,2,\cdots)$$

Example 15 $$E(z)=\frac{5}{(z-a)^2}$$ Obtain e*(t). (Residue)

Solution.

$$e(nT)=\sum \mathrm{Res}\left[E(z)\cdot z^{n-1}\right]=\mathrm{Res}_{z=a}\left[\frac{5}{(z-a)^2}\cdot z^{n-1}\right]$$

$$e(nT)=\frac{1}{(2-1)!}\lim_{z\to a}\frac{d}{dz}\left[(z-a)^2\frac{5\cdot z^{n-1}}{(z-a)^2}\right]$$

$$=\lim_{z\to a}\frac{d}{dz}\left[5\cdot z^{n-1}\right]$$

$$=5\cdot\lim_{z\to a}\left[(n-1)\cdot z^{n-2}\right]$$

$$=5\cdot(n-1)\cdot a^{n-2}$$

$$e^*(t)=\sum_{n=0}^{\infty}\left(5(n-1)\cdot a^{n-2}\right)\cdot\delta(t-nT)$$

7.4.5 Limitations of z-Transform

(1) only shows the information of samples;

(2) In some cases, the continuous signal may jump on the sampling point.