9.4 The solution of linear time-invariant system state equation

Mathematical Representation

Quantitative(定量) Analysis

Qualitative(定性) Analysis

Solving the dynamic mathematical model equations;
The solutions analysis.

Transfer function

State Space – modern control theory

Matrices Operation (State transition Matrix)

Dynamic Response

State equation (Model) $\Longrightarrow$ Dynamic analysis (Solve state equation)

Insuring the existence and uniqueness of the solution: the elements in A and B are bounded.

9.4.1 Solution of Linear Time-invariant Continual System

1. The solution of homogeneous state equation(齐次状态方程)

$\dot{x} = Ax$ is homogeneous state equation, and there are general 2 solutions:

- Power Series Method (幂级数法)

Assume the solution of above equation is a vector power series (幂级数) of $\boldsymbol{t}$

$$\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{b}_0 + \boldsymbol{b}_1\boldsymbol{t} + \boldsymbol{b}_2\boldsymbol{t}^2 + \cdots + \boldsymbol{b}_k\boldsymbol{t}^k + \cdots$$

$\boldsymbol{x}, \boldsymbol{b}_0, \boldsymbol{b}_1, \cdots \boldsymbol{b}_k \cdots$ are n dimensional vectors.

Calculate the derivative of above equation:

$$\dot{x} = b_1 + 2b_2 t + \cdots + k b_k t^{k-1} + \cdots = A(b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots)$$

$$\dot{x} = b_1 + 2b_2 t + \cdots + k b_k t^{k-1} + \cdots = A(b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots)$$

Assume the coefficients with the same power are uniform.

$$b_1 = Ab_0$$

$$b_2 = \frac{1}{2} A b_1 = \frac{1}{2} A^2 b_0$$

$$b_3 = \frac{1}{3} A b_2 = \frac{1}{3 \times 2} A^3 b_0$$

$$\vdots$$

$$b_k = \frac{1}{k} A b_{k-1} = \frac{1}{k!} A^k b_0$$

$$\vdots$$

$$\boldsymbol{x(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots}$$

$$\because \quad x(0) = b_0$$

$$\therefore \quad x(t) = (I + At + \frac{1}{2} A^2 t^2 + \cdots + \frac{1}{k!} A^k t^k + \cdots) x(0)$$

Define: $e^{At} = I + At + \frac{1}{2} A^2 t^2 + \cdots + \frac{1}{k!} A^k t^k + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$

$$x(t) = e^{At} x(0)$$

e^{At} —— Matrix exponential function, called ***state transition matrix***: **$\Phi(t)$.**

- Laplace transformation for $\dot{x} = Ax$

$$sx(s) - x(0) = Ax(s)$$

$$(Is - A)x(s) = x(0)$$

$$x(s) = (Is - A)^{-1}x(0)$$

Inverse Laplace Transformation:

$$x(t) = L^{-1}[(sI - A)^{-1}]x(0)$$

Compare with the power series method: $x(t) = e^{At}x(0)$

$$\boxed{\Phi(t) = e^{At} = L^{-1}[(sI - A)^{-1}]}$$

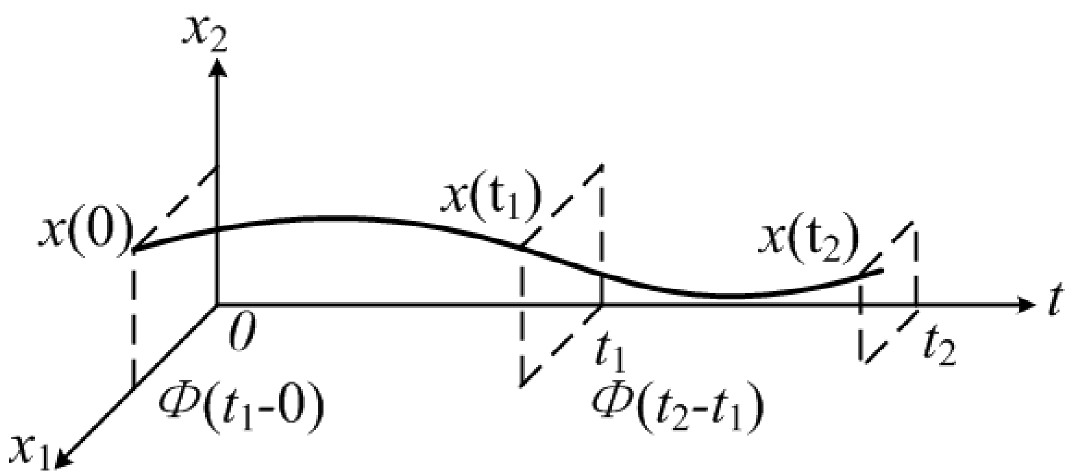
$$\Uparrow$$

The closed form (闭合形式) (analytic form 解析形式) of the state transition matrix, which is convergent (收敛的).

Discussion:

$$\dot{x} = Ax \Rightarrow x(t) = e^{At}x(0) \text{ OR } x(t) = e^{A(t-t_0)}x(t_0)$$

The solution of homogeneous state equation describe a freedom motion (自由运动) of the system without the input $u(t)$, which is the transition of the initial state only based on the state transition matrix $\mathbf{e}^{\mathbf{A(t-t0)}}$.



2. The solution of non-homogeneous state equation

Give the non-homogeneous state equation:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \qquad \boldsymbol{x}(t) \in R^n, \boldsymbol{u}(t) \in R^r, A \in R^{n\times n}, B \in R^{n\times r}$$

- Direct method（Integral method 积分法） $\dot{x}(t) - Ax(t) = Bu(t)$

Left multiply e^{-At} simultaneously: $\underline{\underline{e^{-At}[\dot{x}(t) - Ax(t)]}} = e^{-At}Bu(t)$

$$\boxed{\frac{d}{dt}[e^{-At}x(t)]}$$

$$\frac{d}{dt}[e^{-At}x(t)] = e^{-At}Bu(t)$$

$$e^{-At}x(t) - x(0) = \int_o^t e^{-A\tau}Bu(\tau)d\tau \qquad \boldsymbol{x}(t)\big|_{t=0} = \boldsymbol{x}(0)$$

$$x(t) = e^{At}x(0) + \int_o^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$x(t) = \Phi(t)x(0) + \int_o^t \Phi(t-\tau)Bu(\tau)d\tau$$

Response of initial condition

Response of input u(t)

Discussion:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \Rightarrow x(t) = \Phi(t)x(0) + \int_o^t \Phi(t-\tau)Bu(\tau)d\tau$$

The solution of non-homogeneous state equation is composed by two parts

- The freedom motion of the initial state: $\boldsymbol{\Phi}(t)x(0)$, which is called zero-input response;
- The controlled motion by the input: $\int_0^t \boldsymbol{\Phi}(t\text{-}\tau)\boldsymbol{Bu}(\tau)\mathrm{d}\tau$, which is called zero-state response.

- Laplace transformation method $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$

$$\dot{x}(t) - Ax(t) = Bu(t)$$

$$sX(s)\text{-}x(0)\text{-}AX(s)=Bu(s)$$

$$(sI\text{-}A)X(s)=x(0)+Bu(s)$$

$$X(s)=(sI\text{-}A)^{-1}x(0)+(sI\text{-}A)^{-1}Bu(s)$$

Then $x(t)=L^{-1}[(sI\text{-}A)^{-1}x(0)]+L^{-1}[(sI\text{-}A)^{-1}Bu(s)]$

$$x(t) = \Phi(t)x(0) + \int_o^t \Phi(t-\tau)Bu(\tau)d\tau$$

9.4.2 Properties of state transition matrix

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!}A^kt^k$$

1. Initial state: $\Phi(0) = I$
2. $\dot{\Phi}(t) = A\Phi(t) = \Phi(t)A \qquad \dot{\Phi}(0) = A$
3. Linear relationship: $\Phi(t_1 \pm t_2) = \Phi(t_1)\Phi(\pm t_2) = \Phi(\pm t_2)\Phi(t_1)$
4. Reversibility: $\Phi^{-1}(t) = \Phi(-t), \ \ \Phi^{-1}(-t) = \Phi(t)$
5. $x(t_2) = \Phi(t_2 - t_1)x(t_1)$
6. Segmentation: $\Phi(t_2 - t_0) = \Phi(t_2 - t_1)\Phi(t_1 - t_0)$
7. $[\Phi(t)]^k = \Phi(kt)$
8. if $AB = BA, \quad e^{(A+B)t} = e^{At}e^{Bt} = e^{Bt}e^{At};$

   if $AB \neq BA, \quad e^{(A+B)t} \neq e^{At}e^{Bt} \neq e^{Bt}e^{At}$

9. if $\Phi(t)$ is state transition matrix of $\dot{x}(t)=Ax(t)$, the newly state tranition matrix after non-singular transform $x=P\overline{x}$ is:

$$\overline{\Phi}(t)=P^{-1}e^{At}P$$

10. Two common state transition matrices

If A is n-order Diagonal Matrix,

$$A=\begin{bmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n\end{bmatrix},\qquad \Phi(t)=\begin{bmatrix}e^{\lambda_1 t} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & e^{\lambda_n t}\end{bmatrix}$$

If A is m-order Jordan Matrix,

$$A=\begin{bmatrix}\lambda & 1 & \cdots & 0\\ 0 & \lambda & \ddots & \vdots\\ \vdots & \ddots & \ddots & 1\\ 0 & \cdots & 0 & \lambda\end{bmatrix}_{m\times m},\qquad \Phi(t)=\begin{bmatrix}e^{\lambda t} & te^{\lambda t} & \cdots & \dfrac{t^{m-1}}{(m-1)!}e^{\lambda t}\\ 0 & e^{\lambda t} & \ddots & \vdots\\ \vdots & \ddots & \ddots & te^{\lambda t}\\ 0 & \cdots & 0 & e^{\lambda t}\end{bmatrix}$$

9.4.3 Calculation of matrix transition function e^{At}

- Method One: Direct method (matrix power function)

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

For any constant matrix A and limited t, the above infinite series is convergent.

9.4.3 Calculation of matrix transition function e^{At}

- Method Two: Linear transform method (diagonal form method and Jordan form method)

If the matrix A can be transited to the diagonal form, e^{At} can be given as:

$$e^{At} = Pe^{\Lambda t}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}P^{-1}$$

In the above equation, P is the non-singular linear transform matrix for A.

Similarly, if matrix A can be transformed to Jordan form, e^{At} can be given as:

$$e^{At} = Se^{Jt}S^{-1}$$

9. if $\Phi(t)$ is state transition matrix of $\dot{x}(t) = Ax(t)$, the newly state tranition matrix after non-singular transform $x = P\overline{x}$ is:

$$\overline{\Phi}(t) = P^{-1}e^{At}P$$

➢ Method Three: Laplace transform method

$$e^{At} = L^{-1}[(sI - A)^{-1}]$$

For e^{At}, it is essential to calculate the inverse of (sI-A).

Ex. Consider following system matrix, try to find the proper e^{At} by linear transform method and Laplace transform method.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

Solution:

Linear transform method: the eigenvalues of A is 0 and -2 (λ_1=0, λ_2=-2), thus, the transform matrix P is:

$$P = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

from

$$e^{At} = Pe^{\Lambda t}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}P^{-1}$$

we have,

$$e^{At} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} e^{o} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

Laplace transform method:

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}$$

We have:

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

thus:

$$\mathrm{e}^{\mathrm{At}} = L^{-1}[(sI - A)^{-1}] = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

<u>Ex.</u> **Find the state transition matrix Φ(t) and its inverse of following linear time-invariant system.**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

<u>EX.</u> Find the state transition matrix Φ(t) and its inverse of following linear time-invariant system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution: $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$\Phi(t) = e^{At} = L^{-1}[(sI - A)^{-1}]$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$(sI-A)^{-1}=\begin{bmatrix}\dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)}\\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)}\end{bmatrix}$$

$$\Phi(t)=e^{At}=L^{-1}[(sI-A)^{-1}]$$

$$=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}$$

Then calculate the inverse of state transition matrix $\Phi^{-1}(t)$.

According to $\Phi^{-1}(t)=\Phi(-t)$, the inverse of state transition matrix is:

$$\Phi^{-1}(t)=e^{-At}=\begin{bmatrix}2e^{t}-e^{2t} & e^{t}-e^{2t}\\ -2e^{t}+2e^{2t} & -e^{t}+2e^{2t}\end{bmatrix}$$

Ex. Try to find the time response relationship of following system, in which, the input $u(t)=\mathbf{1}(t)$, the unit step function at t=0.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Solution:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Phi(t) = e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$ (according to Exercises)

$$\boxed{x(t) = \Phi(t)x(0) + \int_o^t \Phi(t-\tau)Bu(\tau)d\tau}$$

$$x(t) = e^{At}x(0) + \int_o^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1(t)d\tau$$

$$x(t) = e^{At}x(0) + \int_o^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1(t)d\tau$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

If the initial state is zero: **x**(0)=**0**, **x**(t) can be simplified as:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Ex. Assume the dynamic equation is: $\ddot{y}+(a+b)\dot{y}+aby=\dot{u}+cu$

With a, b and c are real constants. Try to find:

(1) The state space equation of the system;

(2) The state transition matrix Φ(t).

Solution:

(1)

$$G(s)=\frac{Y(s)}{U(s)}=\frac{s+c}{s^2+(a+b)s+ab}$$

$$=\frac{s+c}{(s+a)(s+b)}$$

$$=\frac{c-a}{b-a}\cdot\frac{1}{s+a}+\frac{c-b}{a-b}\cdot\frac{1}{s+b}$$

$$\begin{cases}\dot{x}=\begin{bmatrix}-a & 0\\ 0 & -b\end{bmatrix}x+\begin{bmatrix}1\\1\end{bmatrix}u\\ y=\begin{bmatrix}\dfrac{c-a}{b-a} & \dfrac{c-b}{a-b}\end{bmatrix}x\end{cases}$$

(2)

$$\Phi(t)=L^{-1}[(sI-A)^{-1}]$$

$$=L^{-1}\left[\begin{pmatrix}s+a & 0\\ 0 & s+b\end{pmatrix}^{-1}\right]=L^{-1}\begin{bmatrix}\dfrac{1}{s+a} & 0\\ 0 & \dfrac{1}{s+b}\end{bmatrix}=\begin{bmatrix}e^{-at} & 0\\ 0 & e^{-bt}\end{bmatrix}$$

9.4 The solution of linear time-invariant system state equation（小结一）

1. The solution of homogeneous state equation $\dot{x} = Ax$

- Power Series Method (幂级数法) $x(t) = e^{At}x(0)$

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!}A^kt^k$$

- Laplace transformation $x(t) = L^{-1}[(sI - A)^{-1}]x(0)$

$$\Phi(t) = e^{At} = L^{-1}[(sI - A)^{-1}]$$

2. The solution of non-homogeneous state equation $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$

- Direct method（Integral method 积分法）

$$x(t) = e^{At}x(0) + \int_o^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$x(t) = \Phi(t)x(0) + \int_o^t \Phi(t-\tau)Bu(\tau)d\tau$$

- Laplace transformation method $x(t)=L^{-1}[(sI-A)^{-1}x(0)]+L^{-1}[(sI-A)^{-1}Bu(s)]$

3. Properties of state transition matrix

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!}A^kt^k$$

1. Initial state: $\Phi(0) = I$

2. $\dot{\Phi}(t) = A\Phi(t) = \Phi(t)A \qquad \dot{\Phi}(0) = A$

4. Reversibility: $\Phi^{-1}(t) = \Phi(-t), \quad \Phi^{-1}(-t) = \Phi(t)$

7. $[\Phi(t)]^k = \Phi(kt)$

9. if $\Phi(t)$ is state transition matrix of $\dot{x}(t) = Ax(t)$, the newly state tranition matrix after non-singular transform $x = P\bar{x}$ is: $\bar{\Phi}(t) = P^{-1}e^{At}P$

10. Two common state transition matrices

If A is n-order Diagonal Matrix,

$$A = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}, \qquad \Phi(t) = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

If A is m-order Jordan Matrix,

$$A = \begin{bmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda \end{bmatrix}_{m\times m}, \quad \Phi(t) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \cdots & \frac{t^{m-1}}{(m-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & te^{\lambda t} \\ 0 & \cdots & 0 & e^{\lambda t} \end{bmatrix}$$

4. Calculation of matrix transition function e^{At}

- Method One: Direct method (matrix power function)

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

- Method Two: Linear transform method (diagonal form method and Jordan form method)

If the matrix A can be transited to the diagonal form, e^{At} can be given as:

$$e^{At} = Pe^{\Lambda t}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}P^{-1}$$

Similarly, if matrix A can be transformed to Jordan form, e^{At} can be given as:

$$e^{At} = Se^{Jt}S^{-1}$$

- Method Three: Laplace transform method

$$e^{At} = L^{-1}[(sI - A)^{-1}]$$

9.4.4 Establishing and solution of linear discrete system state space representation

1. The state space description of discrete-time linear system:

The discrete-time linear time-variant system:

$$x(k+1) = A(k)x(k) + B(k)u(k)$$
$$y(k) = C(k)x(k) + D(k)u(k)$$

The discrete-time linear time-invariant system:

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

A_{nxn}: system matrix; B_{nxp}: input matrix

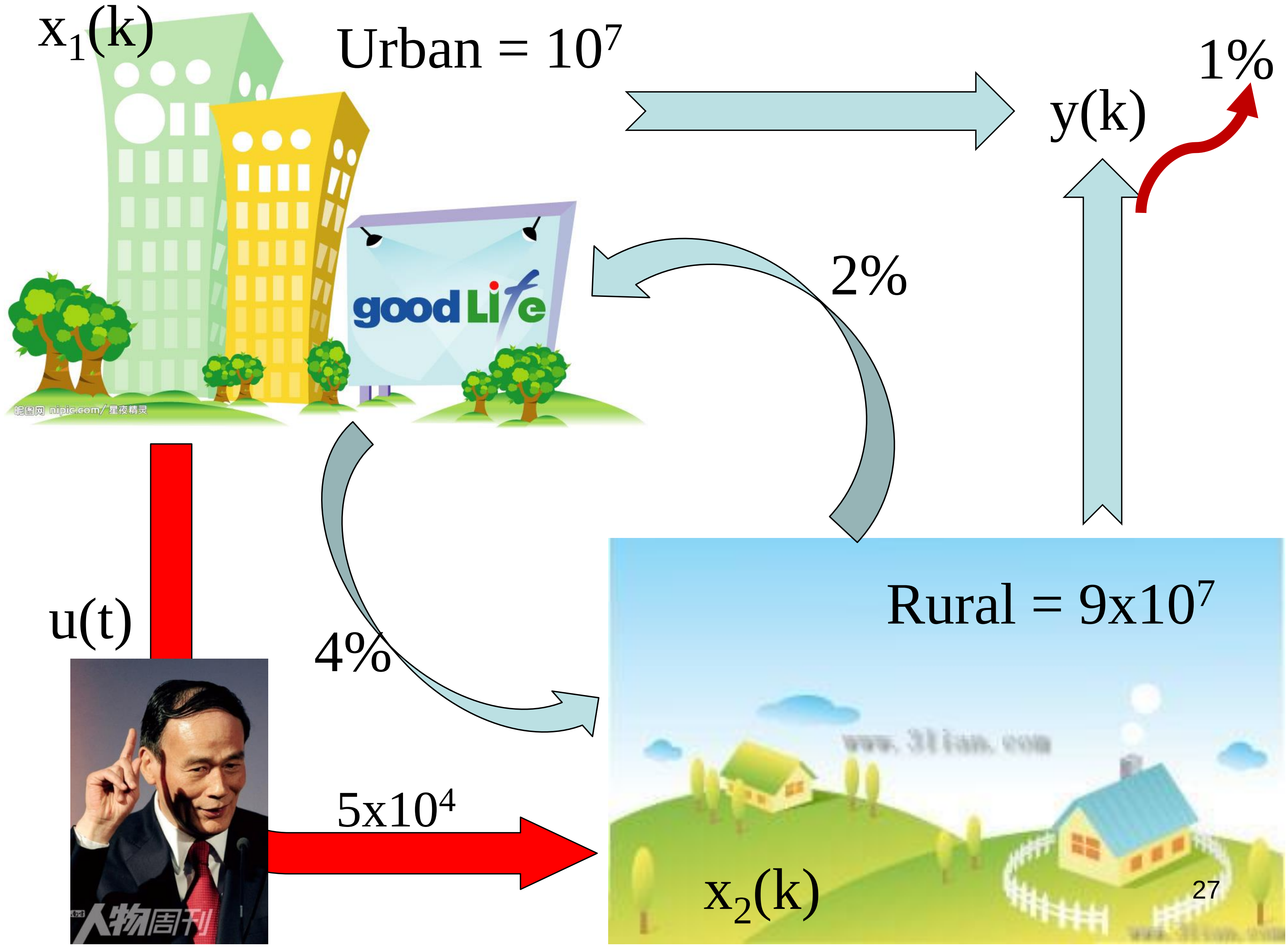
C_{qxn}: output matrix; D_{qxp}: transfer matrix

Population distribution Issue:

Assume the population condition of a country: Urban population is 10^7; Rural population is $9\text{x}10^7$; 4% of urban population transfer to countryside; 2% of rural population transfer to city; increasing rate is 1%.

Assume: (1) k is a discrete-time variable; (2) $x_1(k)$ and $x_2(k)$ are urban and rural population of the k^{th} year; (3) u(k) is population control device of the government: a unit of positive control device can inspirit $5\text{x}10^4$ population move from city to the countryside, and v.v. ; (4) y(k) is the total population of the k^{th} year.

Try to describe such population distribution issue by discrete state space form.


$x_1(k)$
Urban = 10^7
$y(k)$
1%
good Life
2%
Rural = $9x10^7$
u(t)
4%
$5x10^4$
$x_2(k)$

Discrete state equations

$$x_1(k+1) = (1+0.01)\times\left[(1-0.04)x_1(k) + 0.02x_2(k) - 5\times10^4 u(k)\right]$$

$$x_2(k+1) = (1+0.01)\times\left[(1-0.02)x_2(k) + 0.04x_1(k) + 5\times10^4 u(k)\right]$$

Matrix representation:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.9696 & 0.0202 \\ 0.0404 & 0.9898 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} -5.05\times10^4 \\ 5.05\times10^4 \end{bmatrix} u(k)$$

Also standard state space description:

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k) + Du(k)$$

2. Discrete-time state space from differential equations

The difference equation and impulse transfer function are widely used to describe the discrete system in classic control theory. The general form of time-invariant differential equation of SISO system is:

$$
\begin{aligned}
& y(k+n)+a_1 y(k+n-1)+\cdots+a_{n-1} y(k+1)+a_n y(k) \\
& =b_0 u(k+n)+b_1 u(k+n-1)+\cdots+b_{n-1} u(k+1)+b_n u(k)
\end{aligned}
$$

In which, k is time of kT; T is sampling period; u(k) and y(k) are input and output at time of kT; a_i and b_i are constant to describe system performance; consider the Z-transform with zero initial condition:

$$
Z[y(k)]=y(z), \quad Z[(y(k+i)]=z^i y(z)
$$

$$G(z)=\frac{y(z)}{u(z)}=\frac{b_0z^n+b_1z^{n-1}+\cdots+b_{n-1}z+b_n}{z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n}$$

$$=b_0+\frac{\beta_1z^{n-1}+\beta_2z^{n-2}+\cdots+\beta_{n-1}z+\beta_n}{z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n}=b_0+\frac{Y_0(z)}{U_0(z)}$$

Such G(z) is impulse transfer function, which is similar with the form of continual system.

The same way in continual system can be used in discrete situation: Such as intermediate variable method:

Import intermediate variable Q(z) :

$$\frac{Y_0(z)}{U_0(z)}=\frac{\beta_1z^{n-1}+\beta_2z^{n-2}+\cdots+\beta_{n-1}z+\beta_n}{z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n}\cdot\frac{Q(z)}{Q(z)}$$

$$\frac{Y_0(z)}{U_0(z)} = \frac{\beta_1 z^{n-1} + \beta_2 z^{n-2} + \cdots + \beta_{n-1} z + \beta_n}{z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n} \cdot \frac{Q(z)}{Q(z)}$$

$$u_0(z) = z^n Q(z) + a_1 z^{n-1} Q(z) + \cdots + a_{n-1} z Q(z) + a_n Q(z)$$

$$y_0(z) = \beta_1 z^{n-1} Q(z) + \cdots + \beta_{n-1} z Q(z) + \beta_n Q(z)$$

$Z[y(k)] = y(z), \quad Z[(y(k+i)] = z^i y(z)$

Define state variables:

$$\begin{cases} x_1(z) = Q(z) \\ x_2(z) = zQ(z) = zx_1(z) \\ \quad \vdots \\ x_n(z) = z^{n-1} Q(z) = zx_{n-1}(z) \end{cases}$$

$$u_0(z) = z^n Q(z) + a_1 x_n(z) + \cdots + a_{n-1} x_2(z) + a_n x_1(z)$$

$$z^n Q(z) = -a_n x_1(z) - a_{n-1} x_2(z) \cdots - a_1 x_n(z) + u_0(z)$$

$$y_0(z) = \beta_n x_1(z) + \beta_{n-1} x_2(z) + \cdots + \beta_1 x_n(z)$$

Define state variables: $\begin{cases} x_1(z) = Q(z) \\ x_2(z) = zQ(z) = zx_1(z) \\ \quad \vdots \\ x_n(z) = z^{n-1}Q(z) = zx_{n-1}(z) \end{cases}$

$$z^n Q(z) = -a_n x_1(z) - a_{n-1} x_2(z) \cdots - a_1 x_n(z) + u_0(z)$$

$$y_0(z) = \beta_n x_1(z) + \beta_{n-1} x_2(z) + \cdots + \beta_1 x_n(z)$$

Z inverse transform $\quad Z^{-1}[x_i(z)] = x_i(k), \qquad Z^{-1}[zx_i(z)] = x_i(k+1)$

$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = x_3(k) \\ \quad \vdots \\ x_{n-1}(k+1) = x_n(k) \\ x_n(k+1) = -a_n x_1(k) - a_{n-1} x_2(k) \cdots - a_1 x_n(k) + u_0(k) \end{cases}$$

$$y_0(k) = \beta_n x_1(k) + \beta_{n-1} x_2(k) + \cdots + \beta_1 x_n(k)$$

$$
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(k)
$$

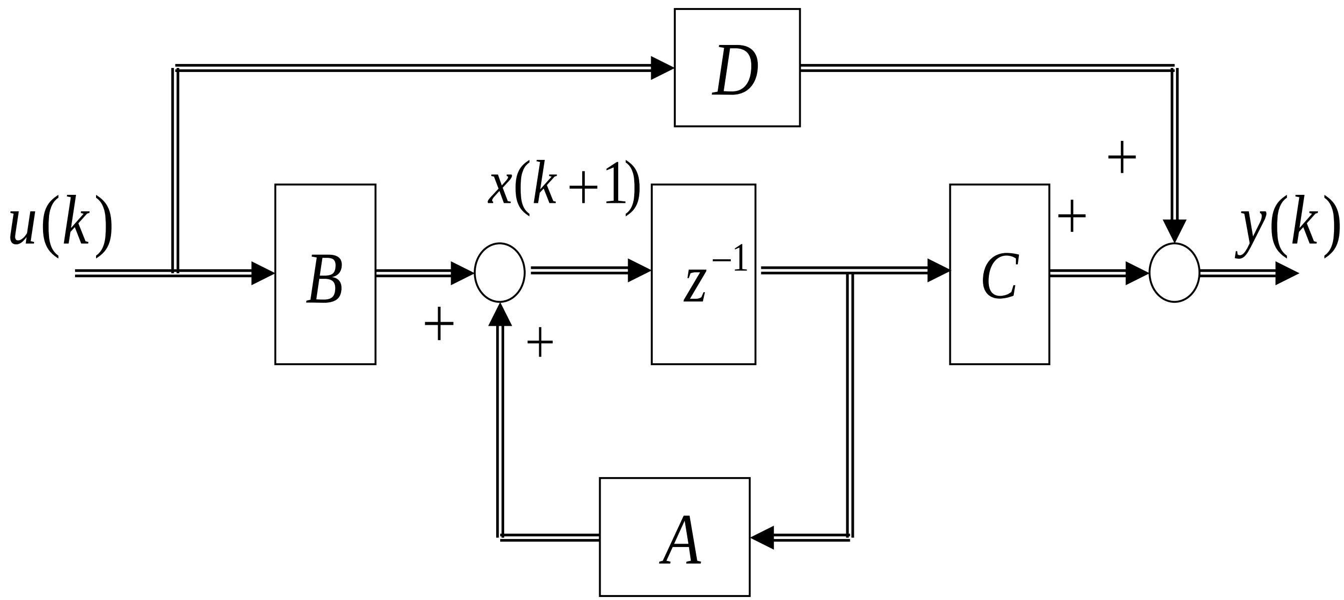
$$
y(k) = [\beta_n \quad \beta_{n-1} \quad \cdots \quad \beta_1] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + Du(k)
$$

Discrete system state equation describes the relationship between the state of the system at (k+1)T, and the state at time kT and input of the system.

Output equation describes the relationship between the output of the system at kT, and the state at kT and input of the system.

State space representation of linear time-invariant MIMO discrete system

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{u}(k)$$
$$\boldsymbol{y}(k) = C\boldsymbol{x}(k) + D\boldsymbol{u}(k)$$



3. Discretization of continual system state space expression

The solution of the time-invariant continual state equation under the input u(t) $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \Phi(t-t_0)x(t_0) + \int_{t_0}^t \Phi(t-\tau)Bu(\tau)d\tau$$

Assume $t_0 = kT, \quad x(t_0) = x(kT) = x(k)$

$$t = (k+1)T, \quad x(t) = x[(k+1)T] = x(k+1)$$

at $t \in [k, k+1]$, u(k)=u(k+1) is constant

$$\boldsymbol{x}(k+1) = \Phi[(k+1)T - kT]\boldsymbol{x}(k) + \boxed{\left(\int_{kT}^{(k+1)T} \Phi[(k+1)T - \tau]Bd\tau\right)}\boldsymbol{u}(k)$$

$$G(T) = \int_{kT}^{(k+1)T} \Phi[((k+1)T - \tau)]Bd\tau$$

Variable replacement $(k+1)T - \tau = \tau'$

then

$$G(T) = \int_0^T \Phi(\tau)Bd\tau$$

$$\boldsymbol{x}(k+1) = \Phi[(k+1)T - kT]\boldsymbol{x}(k) + \left(\int_{kT}^{(k+1)T} \Phi[(k+1)T-\tau]Bd\tau\right)\boldsymbol{u}(k)$$

$$\boxed{\boldsymbol{x}(k+1) = \Phi(T)\boldsymbol{x}(k) + \left(\int_0^T \Phi(\tau)Bd\tau\right)\boldsymbol{u}(k)}$$

State equation of discrete system is: $G(T) = \int_0^T \Phi(\tau)Bd\tau$

$$\boldsymbol{x}(k+1) = \Phi(T)\boldsymbol{x}(k) + G(T)\boldsymbol{u}(k)$$

The relationship between $\Phi(T)$ and state transition matrix $\Phi(t)$ of continual system:

$$\Phi(T) = \Phi(t)\big|_{t=T}$$

The output equation of discrete system is:

$$y(k) = C\boldsymbol{x}(k) + D\mathrm{u}(k)$$

Ex. Find the discrete state equation with T=1s from following continual system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Solution: From Ex., the state transition matrix Φ(t) of above continual system is:

$$\Phi(t) = e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\Phi(T) = \Phi(t)\Big|_{t=T=1} = \begin{bmatrix} 0.6004 & 0.2325 \\ -0.4651 & -.0972 \end{bmatrix}$$

$$G(T) = \int_0^T \Phi(\tau) B d\tau = \int_0^T \begin{pmatrix} e^{-\tau} - e^{-2\tau} \\ -e^{-\tau} + 2e^{-2\tau} \end{pmatrix} d\tau = \begin{bmatrix} 1/2 - e^{-T} + 1/2e^{-2T} \\ e^{-T} - e^{-2T} \end{bmatrix}$$

$$G(T)\Big|_{T=1} = \begin{bmatrix} 0.1998 \\ 0.2325 \end{bmatrix}$$

$$\boldsymbol{x}(k+1) = \Phi(T)\boldsymbol{x}(k) + G(T)\boldsymbol{u}(k)$$

4. Solution of time-invariant discrete system dynamic equation

➢ Recurrence method (递推法)

$$\boldsymbol{x}(k+1)=\Phi(T)\boldsymbol{x}(k)+G(T)\boldsymbol{u}(k) \qquad \boldsymbol{k=0,1,\cdots,k-1,}$$

The states at time of T, 2T, … ,kT time:

$$k=0 \qquad \boldsymbol{x}(1)=\Phi(T)\boldsymbol{x}(0)+G(T)\boldsymbol{u}(0)$$

$$k=1 \qquad \begin{aligned}\boldsymbol{x}(2)&=\Phi(T)\boldsymbol{x}(1)+G(T)\boldsymbol{u}(1)\\ &=\Phi^2(T)\boldsymbol{x}(0)+\Phi(T)G(T)u(0)+G(T)\boldsymbol{u}(1)\end{aligned}$$

$$k=2 \qquad \begin{aligned}\boldsymbol{x}(3)&=\Phi(T)\boldsymbol{x}(2)+G(T)\boldsymbol{u}(2)\\ &=\Phi^3(T)\boldsymbol{x}(0)+\Phi^2(T)G(T)u(0)+\Phi(T)G(T)u(1)+G(T)\boldsymbol{u}(2)\end{aligned}$$

$$\vdots \qquad \vdots$$

$$k=k-1 \qquad \begin{aligned}\boldsymbol{x}(k)&=\Phi(T)\boldsymbol{x}(k-1)+G(T)\boldsymbol{u}(k-1)\\ &=\Phi^k(T)\boldsymbol{x}(0)+\Phi^{k-1}(T)G(T)u(0)+\Phi^{k-2}(T)G(T)u(1)+\cdots\\ &\quad+\Phi(T)G(T)\boldsymbol{u}(k-2)+G(T)\boldsymbol{u}(k-1)\\ &=\Phi^k(T)\boldsymbol{x}(0)+\sum_{i=0}^{k-1}\Phi^{k-i-1}(T)G(T)\boldsymbol{u}(i)\end{aligned}$$

$$G(T)=\int_0^T \Phi(\tau)Bd\tau$$

It is the solution of discrete state equation, which is Discrete State Transition Equation.

when $\quad u(i)=0, \quad (i=0,1,\cdots,k-1)$

$$x(k)=\Phi^k(T)x(0)+\sum_{i=0}^{k-1}\Phi^{k-i-1}(T)G(T)u(i)$$

$$x(k)=\Phi^k x(0)=\Phi(kT)x(0)=\Phi(k)x(0)$$

$$[\Phi(t)]^k=\Phi(kt)$$

$\Phi(k) \Longrightarrow$ State transition matrix of discrete system

The output equation:

$$G(T)=\int_0^T \Phi(\tau)Bd\tau$$

$$y(k)=Cx(k)+D\mathrm{u}(k)$$

$$=C\Phi^k(T)x(0)+C\sum_{i=0}^{k-1}\Phi^{k-i-1}(T)G(T)u(i)+Du(k)$$

For the following discrete state equation:

$$x(k+1)=Ax(k)+Bu(k)$$

$$x(k+1)=\Phi(T)x(k)+G(T)u(k)$$

$$y(k)=Cx(k)+Du(k)$$

Its solution is:

$$x(k)=A^k x(0)+\sum_{i=0}^{k-1}A^{k-i-1}Bu(i)$$

$$y(k)=CA^k x(0)+C\sum_{i=0}^{k-1}A^{k-i-1}Bu(i)+Du(k)$$

Ex. **The state equation of the linear discrete system**

$$A = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(k+1) = Ax(k) + Bu(k)$$

Find its solution by using the recurrence method with the initial state:

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and the input: $u(k) = 1 \qquad (k \geq 0)$

Solution:

Recurrence method:

when $k = 0$

$$x(1) = Ax(0) + Bu(0)$$

$$= \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \times 1 = \begin{bmatrix} 0 \\ 1.84 \end{bmatrix}$$

when $k=1$

$$x(2)=\begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1.84 \end{bmatrix}+\begin{bmatrix} 1 \\ 1 \end{bmatrix}\times 1=\begin{bmatrix} 2.84 \\ -0.84 \end{bmatrix}$$

when $k=2$

$$x(3)=\begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}\begin{bmatrix} 2.84 \\ -0.84 \end{bmatrix}+\begin{bmatrix} 1 \\ 1 \end{bmatrix}\times 1=\begin{bmatrix} 0.16 \\ 1.39 \end{bmatrix}$$

when $k=3$

$$x(4)=\begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}\begin{bmatrix} 0.16 \\ 1.39 \end{bmatrix}+\begin{bmatrix} 1 \\ 1 \end{bmatrix}\times 1=\begin{bmatrix} 2.39 \\ -0.41 \end{bmatrix}$$

Iterate the operation, and we have $x(k)$ at any sampling time.

We cannot obtain the solutions of the linear discrete equation with closed-form by the recurrence method, rather than use state transition matrix.

➢ Z transform method

$$x(k+1) = \Phi x(k) + Gu(k)$$

$$\Downarrow$$

$$zX(z) - zx(0) = \Phi X(z) + GU(z)$$

$$\Downarrow$$

$$(zI - \Phi)X(z) = zx(0) + GU(z)$$

$$\Downarrow$$

$$X(z) = (zI - \Phi)^{-1}[zx(0) + GU(z)]$$

$$\Downarrow$$

$$x(k) = Z^{-1}[X(z)] = Z^{-1}\{(zI - \Phi)^{-1}[zx(0) + GU(z)]\}$$

$$= Z^{-1}[(zI - \Phi)^{-1} zx(0)] + Z^{-1}[(zI - \Phi)^{-1} GU(z)]$$

Analysis: $x(k)=Z^{-1}[(zI-\Phi)^{-1}zx(0)]+Z^{-1}[(zI-\Phi)^{-1}GU(z)]$

$Z^{-1}[(zI-\Phi)^{-1}zx(0)]$

For scale quantity a: $Z^{-1}\left[\frac{1}{1-az^{-1}}\right]=a^k$

Similarly for matrix Φ: $Z^{-1}\left[(zI-\Phi)^{-1}z\right]=Z^{-1}\left[(1-\Phi z^{-1})^{-1}\right]=\Phi^k$

$Z^{-1}[(zI-\Phi)^{-1}GU(z)]$

For scale function w(k), and relative function W(z)

$$Z^{-1}\{W_1(z)W_2(z)\}=\sum_{i=0}^{k}w_1(k-i)w_2(i)$$

Thus:

$$Z^{-1}[(zI-\Phi)^{-1}GU(z)]=Z^{-1}[(zI-\Phi)^{-1}z\bullet z^{-1}GU(z)]$$

$$=\cdots=\sum_{j=0}^{k-1}\Phi^{k-j-1}Gu(j)$$

Result:

$$x(t)=\Phi^k x(0)+\sum_{j=0}^{k-1}\Phi^{k-j-1}Gu(j)$$

Ex. **the state equation of the linear discrete system**

$$A=\begin{bmatrix}0 & 1\\ -0.16 & -1\end{bmatrix} \qquad B=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$x(k+1)=Ax(k)+Bu(k)$$

Find its solution by using the recurrence method with the initial state: $x(0)=\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}=\begin{bmatrix}1\\ -1\end{bmatrix}$

and the input: $u(k)=1 \qquad (k\geq 0)$

Solution:

Z transform method :

$$X(z)=(zI-\Phi)^{-1}[zx(0)+GU(z)]$$

$$x(k)=Z^{-1}[(zI-\Phi)^{-1}zx(0)]+Z^{-1}[(zI-\Phi)^{-1}GU(z)]$$

Solution:

$$A=\begin{bmatrix}0 & 1\\ -0.16 & -1\end{bmatrix} \qquad B=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$x(k+1)=Ax(k)+Bu(k)$$

Derive $(zI\text{-}A)^{-1}$:

$$|zI-A|=\begin{vmatrix}z & -1\\ 0.16 & z+1\end{vmatrix}=(z+0.2)(z+0.8)$$

$$\left(zI-A\right)^{-1}=\frac{adj(zI-A)}{|zI-A|}=\frac{\begin{bmatrix}z+1 & 1\\ -0.16 & z\end{bmatrix}}{(z+0.2)(z+0.8)}$$

$$=\frac{1}{3}\begin{bmatrix}\dfrac{4}{z+0.2}-\dfrac{1}{z+0.8} & \dfrac{5}{z+0.2}-\dfrac{5}{z+0.8}\\ \dfrac{-0.8}{z+0.2}-\dfrac{0.8}{z+0.8} & \dfrac{-1}{z+0.2}+\dfrac{4}{z+0.8}\end{bmatrix}$$

$$\Phi(k)=Z^{-1}\left[\left(zI-A\right)^{-1}z\right]$$ discrete State transition matrix

$$=\frac{1}{3}\begin{bmatrix}4(-0.2)^k-(-0.8)^k & 5(-0.2)^k-5(-0.8)^k\\ -0.8(-0.2)^k+0.8(-0.8)^k & -(-0.2)^k+4(-0.8)^k\end{bmatrix}$$

$$x(k) = Z^{-1}\{(zI - \Phi)^{-1}[zx(0) + GU(z)]\}$$

$u(k) = 1$ Z-transform $U(z) = \dfrac{z}{z-1}$

$$X(z) = (zI - A)^{-1}\left[zx(0) + BU(z)\right]$$

$$= \begin{bmatrix} \dfrac{(z^2+2)z}{(z+0.2)(z+0.8)(z-1)} \\ \dfrac{(-z^2+1.84z)z}{(z+0.2)(z+0.8)(z-1)} \end{bmatrix} = \frac{1}{18}\begin{bmatrix} \dfrac{-51z}{z+0.2} + \dfrac{44z}{z+0.8} + \dfrac{25z}{z-1} \\ \dfrac{10.2z}{z+0.2} + \dfrac{-35.2z}{z+0.8} + \dfrac{7z}{z-1} \end{bmatrix}$$

$$x(k) = Z^{-1}\left\{X(z)\right\} = \frac{1}{18}\begin{bmatrix} -51(-0.2)^k + 44(-0.8)^k + 25 \\ 10.2(-0.2)^k - 35.2(-0.8)^k + 7 \end{bmatrix}$$

Assume k=0,1,2,3, to double check the results before:

$$x(k) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1.84 \end{bmatrix}, \begin{bmatrix} 2.84 \\ -0.84 \end{bmatrix}, \begin{bmatrix} 0.16 \\ 1.384 \end{bmatrix}$$

5. Discrete-time state space from differential equations

$$y(k+n)+a_1y(k+n-1)+\cdots+a_{n-1}y(k+1)+a_ny(k)$$
$$=b_0u(k+n)+b_1u(k+n-1)+\cdots+b_{n-1}u(k+1)+b_nu(k)$$

$$G(z)=\frac{y(z)}{u(z)}=\frac{b_0z^n+b_1z^{n-1}+\cdots+b_{n-1}z+b_n}{z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n}$$
$$=b_0+\frac{\beta_1z^{n-1}+\beta_2z^{n-2}+\cdots+\beta_{n-1}z+\beta_n}{z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n}=b_0+\frac{N(z)}{D(z)}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}u(k)$$

$$y(k)=[\beta_n \quad \beta_{n-1} \quad \cdots \quad \beta_1]\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix}+Du(k)$$

9.4 The solution of linear time-invariant system state equation（小结二）

6. Discretization of continual system state space expression

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$
$$y = C\boldsymbol{x} + D\boldsymbol{u}$$

$$\boldsymbol{x}(k+1) = \Phi(T)\boldsymbol{x}(k) + \left(\int_0^T \Phi(\tau)Bd\tau\right)\boldsymbol{u}(k) = \Phi(T)\boldsymbol{x}(k) + G(T)\boldsymbol{u}(k)$$

$$G(T) = \int_0^T \Phi(\tau)Bd\tau$$

$$\Phi(T) = \Phi(t)\big|_{t=T}$$

$$y(k) = C\boldsymbol{x}(k) + D\mathrm{u}(k)$$

7. Solution of time-invariant discrete system dynamic equation

- Recurrence method (递推法)

$$\boldsymbol{x}(k+1) = \Phi(T)\boldsymbol{x}(k) + \left(\int_0^T \Phi(\tau)Bd\tau\right)\boldsymbol{u}(k) = \Phi(T)\boldsymbol{x}(k) + G(T)\boldsymbol{u}(k)$$
$$y(k) = C\boldsymbol{x}(k) + D\mathrm{u}(k)$$

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{u}(k)$$
$$\boldsymbol{y}(k) = C\boldsymbol{x}(k) + D\boldsymbol{u}(k)$$

$$\boldsymbol{x}(k) = \Phi^k(T)\boldsymbol{x}(0) + \sum_{i=0}^{k-1}\Phi^{k-i-1}(T)G(T)\boldsymbol{u}(i)$$
$$\boldsymbol{y}(k) = C\boldsymbol{x}(k) + D\mathrm{u}(k)$$
$$= C\Phi^k(T)\boldsymbol{x}(0) + C\sum_{i=0}^{k-1}\Phi^{k-i-1}(T)G(T)\boldsymbol{u}(i) + D\boldsymbol{u}(k)$$

$$\boldsymbol{x}(k) = A^k\boldsymbol{x}(0) + \sum_{i=0}^{k-1}A^{k-i-1}B\boldsymbol{u}(i)$$
$$\boldsymbol{y}(k) = CA^k\boldsymbol{x}(0) + C\sum_{i=0}^{k-1}A^{k-i-1}B\boldsymbol{u}(i) + D\boldsymbol{u}(k)$$

$\Phi(k)$ State transition matrix of discrete system

7. Solution of time-invariant discrete system dynamic equation

- Z transform method

$$x(k+1)=\Phi x(k)+Gu(k)$$

$$x(k)=Z^{-1}[(zI-\Phi)^{-1}zx(0)]+Z^{-1}[(zI-\Phi)^{-1}GU(z)] \xrightarrow{\text{等价}} x(t)=\Phi^{k}x(0)+\sum_{j=0}^{k-1}\Phi^{k-j-1}Gu(j)$$

作业：

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