7.6.3 Steady-state error of discrete systems

(1) General method: obtain system response

(2) Final value theorem
$$\begin{cases} G(z) \to \Phi_e(z) \\ D(z) \to \text{Stability} \\ e(\infty) = \lim\limits_{z\to 1}(z-1)R(z)\Phi_e(z) \end{cases}$$

(3) Static error constant
$$\begin{cases} G(z) \to v,\ K_p,\ K_v,\ K_a \\ \text{Obtain } e(\infty) \end{cases}$$

Chapter 7 Analysis and Design of Linear Discrete-Time System (Sampled-data System)

7.1 Introduction

7.2 The Sampling Process and Sampling Theorem

7.3 Signal Recovery and Zero-Order Hold

7.4 Z-Transform and Inverse Z Transform

7.5 Mathematical Models of Discrete-Time Systems

7.6 Performance Analysis of Discrete-Time Systems

7.7 Digital Control Design for Discrete-Time Systems

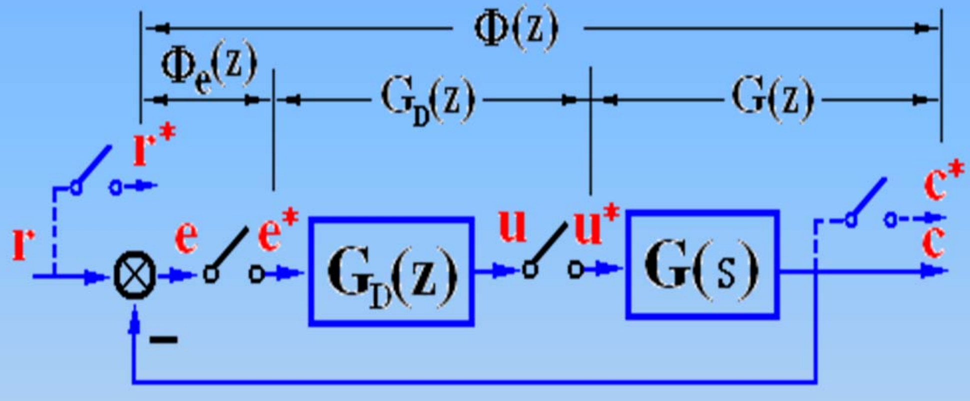
Design for discrete-time systems can be done in s-domain, z-domain and w-domain, respectively.

7.7.1 The Impulse Transfer Function for the Digital Controller

$$\Phi(z) = \frac{G_D(z)\cdot G(z)}{1+G_D(z)\cdot G(z)}$$

$$\Phi_e(z) = \frac{1}{1+G_D(z)\cdot G(z)} = 1-\Phi(z)$$

$$G_D(z)\cdot G(z) = \frac{\Phi(z)}{1-\Phi(z)} = \frac{\Phi(z)}{\Phi_e(z)}$$



$$G_D(z) = \frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}$$

$$\Phi_e(z) = 1-\Phi(z),\ \mathrm{E}(z) = \Phi_e(z)R(z)$$

7.7.2 Deadbeat Control Design 最少拍控制

Deadbeat Control Systems: Matching a particular test input within a number of steps. —— No steady-state error on the sampling point.

(典型输入作用下，能在有限拍内结束响应过程且在采样点上无稳态误差的系统。）

1. A unified description of typical test inputs

$$r(t)=\begin{cases}1(t)\\ t\\ t^2/2\end{cases}\qquad R(z)=\left\{\begin{matrix}\dfrac{z}{z-1}=\dfrac{1}{1-z^{-1}}\\ \dfrac{Tz}{(z-1)^2}=\dfrac{Tz^{-1}}{(1-z^{-1})^2}\\ \dfrac{T^2z(z+1)}{2(z-1)^3}=\dfrac{T^2z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}\end{matrix}\right\}\boxed{\dfrac{A(z)}{(1-z^{-1})^{\nu}}}$$

ν	$A(z)$
1	1
2	Tz^{-1}
3	$\dfrac{T^2z^{-1}(1+z^{-1})}{2}$

$$G_D(z) = \frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}$$

Design Idea: **Obtain $G_D(z)$ by constructing $\Phi(z)$ so that the output can match the typical test signal within the minimum steps.**

No $\begin{cases}\text{Zeros}\\ \text{Poles}\end{cases}$ of G(z) on or beyond the unit circle, except for (1, j0)

$$\mathrm{R}(z) = \frac{A(z)}{(1-z^{-1})^{\nu}}$$

$$\mathrm{E}(z) = \Phi_e(z)R(z), \quad \Phi_e(z) = 1-\Phi(z)$$

$$e(\infty) = \lim_{z\to 1}(z-1)\,\Phi_e(z)\,R(z) \quad \Rightarrow \Phi_e(z) = (1-z^{-1})^{\nu}F(z^{-1})$$

To make the D(z) simplest and of the lowest-order, we can choose $F(z^{-1})$ as 1.

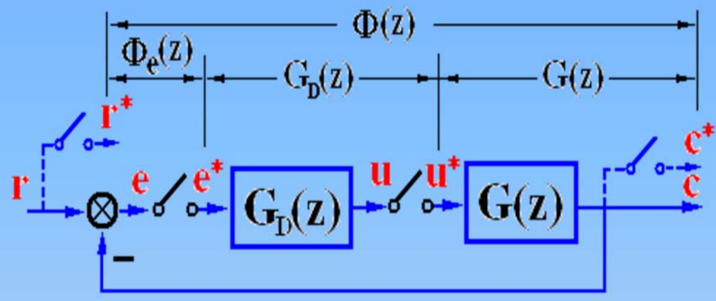
$$\Phi(z) = 1-\Phi_e(z) = 1-(1-z^{-1})^{\nu}$$

From the design idea, we know that $\boldsymbol{e(\infty T)=0}$

$$E(z)=\Phi_e(z)\cdot R(z)=\frac{A(z)}{(1-z^{-1})^{\nu}}\Phi_e(z)$$

$$e(\infty T)=\lim_{z\to 1}(1-z^{-1})\frac{A(z)}{(1-z^{-1})^{\nu}}\Phi_e(z)=0$$

$$\Phi_e(z)=(1-z^{-1})^{\nu}F(z)\overset{F(z)=1}{=}(1-z^{-1})^{\nu}$$



Hence:

$$\Phi(z)=1-\Phi_e(z)=1-(1-z^{-1})^{\nu}=b_1z^{-1}+b_2z^{-2}+\cdots+b_{\nu}z^{-\nu}$$

$$=\frac{b_1z^{\nu-1}+b_2z^{\nu-2}+\cdots+b_{\nu}}{z^{\nu}}$$

$$\boldsymbol{G_D(z)=\frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}}$$

The rule to construct Φ(z): All poles of Φ(z) are located on the origin of z-plane.

2. Φ(z) for typical test inputs

(1) for $r(t) = 1(t)$

- **The C.L.impulse transfer function:**

$$\nu = 1 \qquad \Phi(z) = z^{-1}$$

$$E(z) = 1$$

The system can track the input by 1 step only。

(2) for $r(t) = t \cdot 1(t)$

- **The C.L.impulse transfer function:**

$$\nu = 2 \quad \Phi(z) = 2z^{-1} - z^{-2}$$

$$E(z) = Tz^{-1}$$

The system can track the input by 2 step。

(3) for $r(t)=\frac{1}{2}t^2\cdot 1(t)$

- **The C.L.impulse transfer function:**

$$\nu = 3 \qquad \Phi(z)=3z^{-1}-3z^{-2}+z^{-3}$$

$$E(z)=\frac{1}{2}T^2z^{-1}+\frac{1}{2}T^2z^{-2}$$

The system can track the input by 3 step。

$$G_D(z) = \frac{\Phi(z)}{\Phi_e(z) \cdot G(z)}$$

Deadbeat Control Design Table

$r(t)$	$R(z)$	$\Phi_e(z) = (1 - z^{-1})^v$	$\Phi(z) = 1 - \Phi_e(z)$	$G_D(z)$	t_s
$1(t)$	$\frac{1}{1-z^{-1}}$	$1 - z^{-1}$	z^{-1}	$\frac{z^{-1}}{(1-z^{-1}) \cdot G(z)}$	T
t	$\frac{Tz^{-1}}{(1-z^{-1})^2}$	$(1-z^{-1})^2$	$2z^{-1} - z^{-2}$	$\frac{z^{-1}(2-z^{-1})}{(1-z^{-1})^2 G(z)}$	$2T$
$\frac{t^2}{2}$	$\frac{T^2 z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}$	$(1-z^{-1})^3$	$3z^{-1} - 3z^{-2} + z^{-3}$	$\frac{z^{-1}(3-3z^{-1}+z^{-2})}{(1-z^{-1})^3 G(z)}$	$3T$

3. Algorithm for Deadbeat Control Design

① **Obtain G(z) — Suppose there are no poles and zeros of G(z) on or beyond the unit circle.**

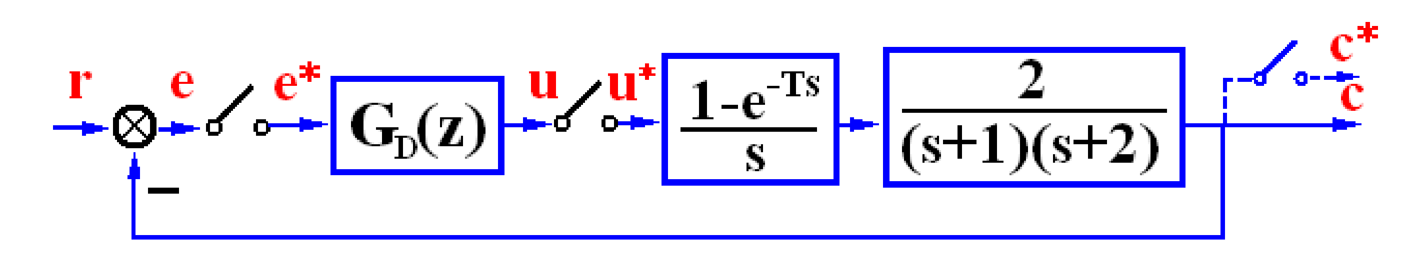
② **Determine $\Phi_e(z)$ for the particular test input**

$$\boldsymbol{r(t)} \quad \Rightarrow \quad R(z) = \frac{A(z)}{(1-z^{-1})^{\nu}} \quad \Rightarrow \quad \Phi_e(z) = (1-z^{-1})^{\nu}$$

$$\Phi_e(z) = (1-z^{-1})^{\nu}\,\mathrm{F}(z)$$

③ **Obtain** $\boldsymbol{\Phi(z) = 1 - \Phi_e(z)}$

④ **Achieve** $\boldsymbol{G_D(z) = \dfrac{\Phi(z)}{\Phi_e(z)\cdot G(z)}}$



Example 1. Consider the system shown in the above figure (T=1). Design deadbeat controllers $G_D(z)$ for $r(t)=1(t)$、t .

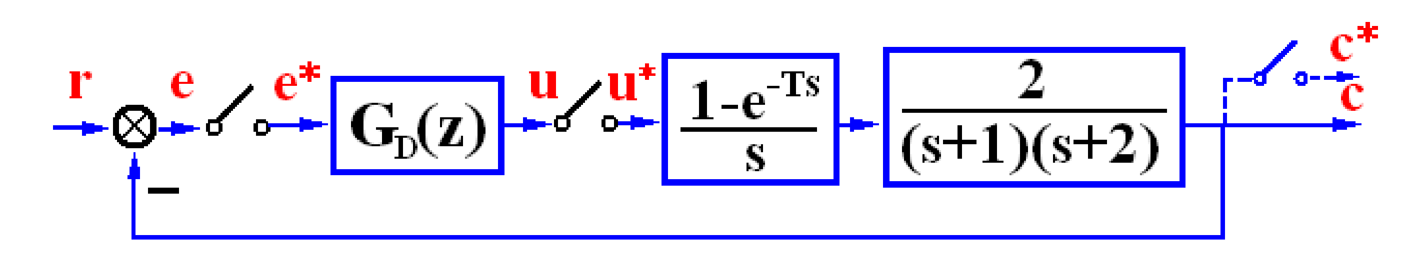
Solution. $$G(z)=Z\left[\frac{1-e^{-Ts}}{s}\cdot\frac{2}{(s+1)(s+2)}\right]=2(1-z^{-1})\cdot Z\left[\frac{C_0}{s}-\frac{C_1}{s+1}+\frac{C_2}{s+2}\right]$$

$$=2\cdot\frac{z-1}{z}\cdot Z\left[\frac{1}{2}\cdot\frac{1}{s}-\frac{1}{s+1}+\frac{1}{2}\cdot\frac{1}{s+2}\right]$$

$$=\frac{z-1}{z}\left[\frac{z}{z-1}-\frac{2z}{z-e^{-T}}+\frac{z}{z-e^{-2T}}\right]=1-\frac{2(z-1)}{z-e^{-T}}+\frac{z-1}{z-e^{-2T}}$$

$$=\frac{(1+e^{-2T}-2e^{-T})z+(e^{-3T}+e^{-T}-2e^{-2T})}{(z-e^{-T})(z-e^{-2T})}$$

$$\overset{T=1}{=}\frac{0.4(z+0.365)}{(z-0.368)(z-0.136)}$$



Referring to the result for $r(t) = 1(t)$ in the Design Table

$$R(z) = \frac{z}{z-1} \qquad \text{Choose} \begin{cases} \Phi_e(z) = 1 - z^{-1} \\ \Phi(z) = 1 - \Phi_e(z) = z^{-1} \end{cases}$$

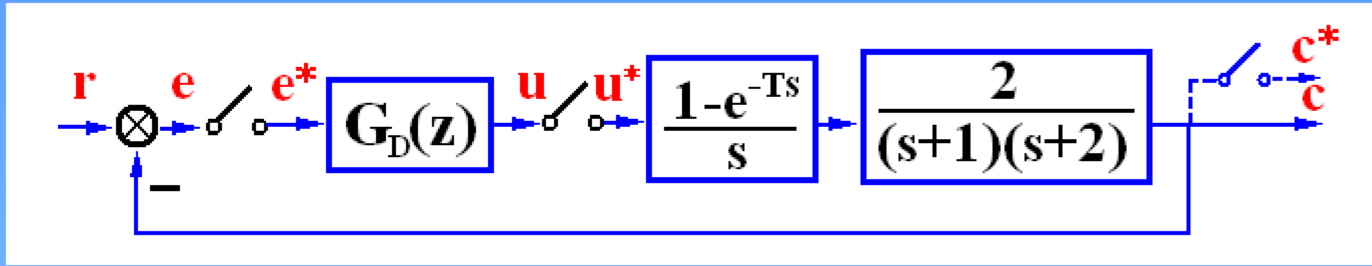
$$G_D(z) = \frac{\Phi(z)}{\Phi_e(z)\cdot G(z)} = \frac{z^{-1}}{1-z^{-1}} \cdot \frac{(z-0.368)(z-0.136)}{0.4(z+0.365)}$$

$$= \frac{2.5(z-0.368)(z-0.136)}{(z-1)(z+0.365)}$$

$$C(z) = \Phi(z)R(z) = z^{-1}\cdot\frac{1}{1-z^{-1}}$$

$$= z^{-1}[1+z^{-1}+z^{-2}+\cdots] = z^{-1}+z^{-2}+z^{-3}+\cdots$$

$$E(z) = \Phi_e(z)R(z) = (1-z^{-1})\cdot\frac{1}{1-z^{-1}} = 1$$



For $r(t)=t$

$$R(z)=\frac{Tz^{-1}}{(1-z^{-1})^2}$$

Choose $\begin{cases}\Phi_e(z)=(1-z^{-1})^2\\ \Phi(z)=1-\Phi_e(z)=2z^{-1}-z^{-2}\end{cases}$

$$G_D(z)=\frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}=\frac{2z^{-1}-z^{-2}}{(1-z^{-1})^2}\cdot\frac{(z-0.368)(z-0.136)}{0.4(z+0.365)}$$

$$=\frac{5(z-0.5)(z-0.368)(z-0.136)}{(z-1)^2(z+0.365)}$$

$$E(z)=\Phi_e(z)\cdot R(z)=Tz^{-1}$$

$$C(z)=\Phi(z)R(z)=(2z^{-1}-z^{-2})\cdot\frac{Tz^{-1}}{(1-z^{-1})^2}$$

$$=R(z)-E(z)=2Tz^{-2}+3Tz^{-3}+4Tz^{-4}+\cdots$$

4. G(z) has poles or zeros on or beyond the unit circle

suppose

$$G(z)=\frac{z^{-v}\prod_{i=1}^{L}(1-z_i z^{-1})}{\prod_{i=1}^{n}(1-p_i z^{-1})}$$

where z_i **is the zero of** $G(z)$**;**

p_i **is the pole of** $G(z)$ **.**

Then

$$G_D(z)=\frac{\Phi(z)}{\Phi_e(z)G(z)}=\frac{z^{v}\prod_{i=1}^{n}(1-p_i z^{-1})\Phi(z)}{\prod_{i=1}^{L}(1-z_i z^{-1})\Phi(z)}$$

$$G_D(z)=\frac{\Phi(z)}{\Phi_e(z)G(z)}=\frac{z^{v}\prod_{i=1}^{n}(1-p_i z^{-1})\Phi(z)}{\prod_{i=1}^{L}(1-z_i z^{-1})\Phi_e(z)}$$

①**If there is Z^{v} in $G_D(z)$，** **$G_D(z)$ is un-realizable.**

Thus, we have to ensure that there exists Z^{-v} in $\Phi(z)$，which promises $G_D(z)$ is realizable.

② **If there is z_i on or beyond the unit circle, $G_D(z)$ is unstable.**

Then, those z_i will be designed as the zeros of $\Phi(z)$.

③**Note that**

$$\Phi(z) = G_D(z)G(z)\Phi_e(z)$$

If there are p_i on or beyond the unit circle,

$\Phi(z)$ will be unstable,

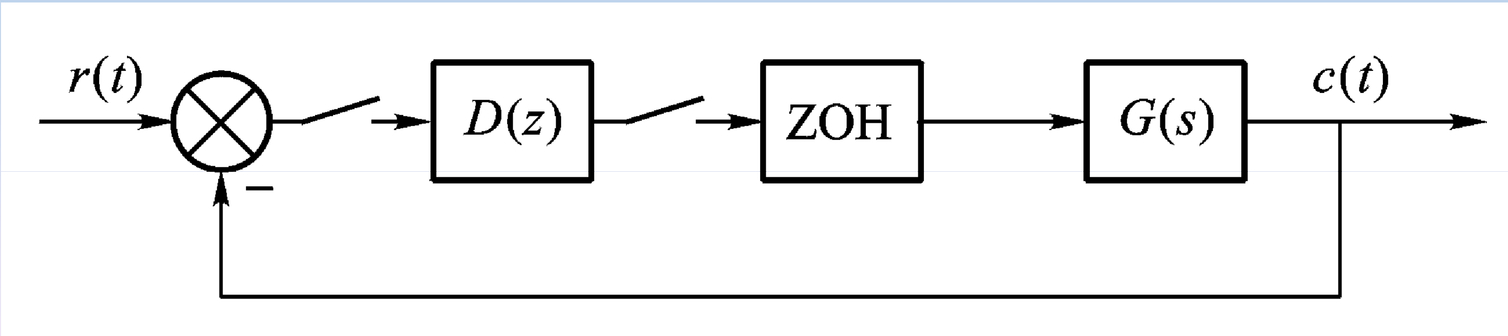
Then **those p_i will be designed as the zeros of $\Phi_e(z)$.**

Example **Given the discrete system described as in the Following figure, where**

$$G_0(s) = \frac{10}{s(0.1s+1)(0.05s+1)}, \quad G_h(s) = \frac{1-e^{-Ts}}{s}$$

with $\mathrm{T} = 0.2s$

Design a deadbeat controller for $r(t) = 1(t)$



Solution：the O. L. impulse transfer function is

$$G(z)=Z[G_h(z)G_0(z)]=\frac{0.76z^{-1}(1+0.05z^{-1})(1+1.065z^{-1})}{(1-z^{-1})(1-0.135z^{-1})(1-0.0185z^{-1})}$$

For $r(t)=1(t)$ **， we can design**

$$\varphi_e(z)=1-z^{-1} \qquad (1)$$

$$\varphi(z)=z^{-1} \qquad (2)$$

Because there exists *z* = - 1.065 (beyond the unit circle)，

Thus, z should also be the zero of Φ(z)

There exist z^{-1} ***in G(z)* ，**z^{-1} **should be in Φ(z), thus**

$$\varphi(z)= \quad z^{-1}(1+1.065z^{-1}) \qquad (3)$$

Because that

$$\varphi(z) = 1 - \varphi_e(z) \qquad \textbf{(4)}$$

from (3), $\varphi(z)$ **is now a polynomial on** z^{-1} **of order 2,**

To satisfy （4） ， $\varphi_e(z)$ **must be a polynomial on** z^{-1} **of order 2，thus based on (1), we redesign：**

$$\varphi_e(z) = (1 - z^{-1})\ (1 + a_1 z^{-1}) \qquad \textbf{(5)}$$

Where a_1 **is a constant to be chosen later。 (6)**

Thus multiplied by a constant b_1 to be designed later, we get

$$\varphi(z) = b_1 z^{-1}(1 + 1.065 z^{-1})$$

From (5) and (6), we get:

$$a_1 = 0.516 \qquad b_1 = 0.484$$

Thus,

$$\varphi_e(z) = (1 - z^{-1})\,(1 + 0.516z^{-1}) \qquad \textbf{(7)}$$

$$\varphi(z) = 0.484z^{-1}(1 + 1.065z^{-1}) \qquad \textbf{(8)}$$

Then the deadbeat controller is

$$D(z) = \frac{1-\varphi_e(z)}{G(z)\varphi_e(z)}$$

$$= \frac{1-(1-z^{-1})\,(1+0.516z^{-1})}{\dfrac{0.\,76z^{-1}(1+0.05z^{-1})\,(1+0.065z^{-1})}{(1-z^{-1})\,(1-0.135z^{-1})\,(1-0.0185z^{-1})}(1-z^{-1})\,(1+0.516z^{-1})}$$

$$D(z)=\frac{0.637\left(1-0.0185z^{-1}\right)\left(1-0.135z^{-1}\right)}{\left(1+0.05z^{-1}\right)\left(1+0.516z^{-1}\right)}$$

Then the *Z*-transform is

$$C(z)=\varphi(z)R(z)=0.484z^{-1}\left(1+1.085z^{-1}\right)\frac{1}{1-z^{-1}}$$
$$=0.484z^{-1}+z^{-2}+z^{-3}+\cdots+z^{-4}+\cdots$$

System can follow the input at the 2nd step, which is one step later.

Although the deadbeat control system tracks a particular test input accurately within a number of steps, it has the following disadvantages:

(1) It is designed only for a particular input.

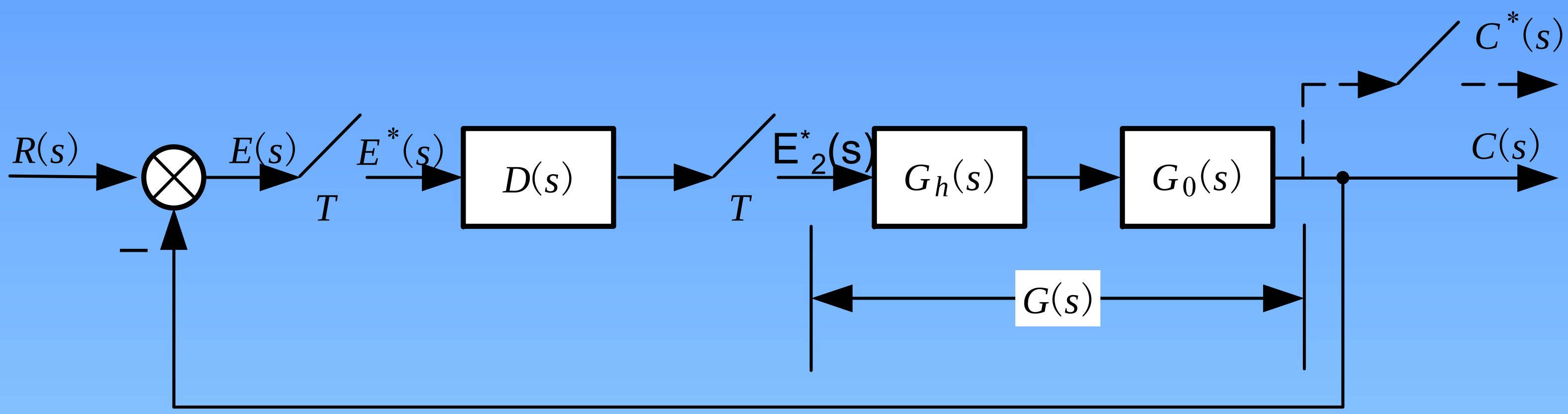
(2) The output has ripples although there are no errors on the sampling points.

(3) The control input changes drastically.

5. Ripple-free deadbeat control design

Ripple： **though the system outputs are stable at the sampling time, they are varying between two sampling time, See p230,Fig.7-50。**

Objective: **Not only tracking the input at the sampling time, the outputs are ripple-free.**



$$E_2(z) = D(z)E(z)$$

solution： ensure $E_2(z)$ being a polynomial on z^{-1} of a finite order.

Condition: $E_2(z)$ is a polynomial on $z^{-1}$of finite order。

$$E_2(z) = D(z)E(z) = D(z)\varphi_e(z)R(z), \quad D(z)\varphi_e(z) = \frac{\varphi(z)}{G(z)}$$

➔ $D(z)\Phi_e(z)$=（*） /z^r, that is the zero of G(z) must be a zero of Φ(z)

Homework:

p238　　7-15, 7-16

7-15. Consider the system as shown in Fig 7-69, $T=1s$, design deadbeat controller $D(z)$ for $r(t)=t$. Draw $r^*(t)$,$e_1^*(t)$,$e_2^*(t)$, $x(t)$, $y(t)$ and $y^*(t)$.

7-16. Furthermore, design a ripple-free deadbeat controller for the system in 7-15.

最少拍设计中， $\Phi(z)$和$\Phi_e(z)$选取时应遵循的原则：

1。D(z)零点的数目不能大于极点的数目；

2。 $\Phi_e(z)$应把G(z)在单位圆上及单位圆外的极点作为自己的零点；

3。 $\Phi(z)$应把G(z)在单位圆上及单位圆外的零点作为自己的零点；

4。当G(z)含有z^{-1}因子时，要求$\Phi(z)$也含有z^{-1}的因子；

5。 因为$\Phi(z)=1-\Phi_e(z)$，他们应该是关于z^{-1}同样阶次的多项式，而且$\Phi_e(z)$还应包含常数项1。

6。当最小拍系统还有无纹波要求时，闭环脉冲传函$\Phi(z)$的零点应抵消G(z)的全部零点（因为最少拍系统设计中G（z）单位圆上及单位圆外的零极点已经被补偿，因此在无纹波的设计中只需抵消G(z)单位圆内的零点）。