

9.5 Controllability and Observability of Linear Time-invariant System

- Background
- Concept of Controllability and Observability of Linear Continual System
- Criteria of Controllability and Observability of Linear Continual System
- Duality Principle(对偶原理)

The background of Controllability and Observability

In 1960s, Kalman ... from state space description.

In modern control theory, we consider about a issue, that whether all the states of the system are *affected* and *reflected* by the *input* and *output* in state equation and output equation description of the system, which is the Controllability and Observability Issue.

Controllability: system input affect all the state of the system to achieve the control.

Observability: system output can reflect all the state of the system to achieve the observation.

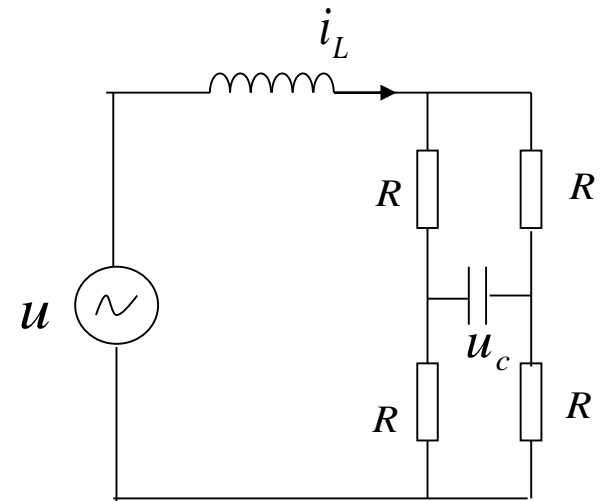
Ex. In a bridge circuit, choose the current of the inductance i_L and the voltage of the capacitance u_c as state variables, the input is u and the output is $y=u_c$

Analysis:

Assume $x_1 = i_L$, $x_2 = u_c$

If $x_2(t_0) = 0$

and $t \geq t_0$, $x_2(t) \equiv 0$



Thus, input u cannot control the state x_2 , x_2 is uncontrollable.

Furthermore, because $y=u_c \equiv 0$, output y cannot reflect the changes of variable x_1 , thus x_1 is unobservable.

9.5.1 the Controllability of the linear time-invariable continual system

1. Definition of Controllability

State-equations: $\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in T_t$

$x(t) \in R^n, \quad u(t) \in R^r, \quad A(t) \in R^{n \times n}, \quad B(t) \in R^{n \times r}, \quad T_t: \text{time space}$

State controllability

It is called the state x_0 is controllable at t_0 , if for a non-zero initial state $x(t_0)=x_0$ with the initial time $t_0 \in T$, exists certain time $t_1 \in T_t$, $t_1 > t_0$ and an unrestricted control $u(t)$, which makes the state transfer from $x(t_0)=x_0$ to $x(t_1)=0$.

System Controllability

It is called the system is controllable at time t_0 , if at time $t_0 \in T$, the non-zero initial states in the state space are all controllable.

Incomplete Controllable

It is called the system is incomplete controllable, if there are one or some non-zero state variables uncontrollable in the state space.

The controllability of linear time-invariable system has no relation to the initial time t_0 .

2. The State Controllability algebraic criteria of time-invariable system

Assume the final state is the origin of the state space, and the initial time $t_0=0$.

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

From the definition of controllability of the state,

We have:
$$x(t_1) = 0 = e^{At_1}x(0) + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau$$

Or
$$x(0) = -\int_0^{t_1} e^{-A\tau}Bu(\tau)d\tau$$

We can rewrite the $e^{-A\tau}$ to the polynomial of matrix A:

$$e^{-A\tau} = \sum_{k=0}^{n-1} \alpha_k(\tau)A^k$$

$$\int_0^{t_1} a_k(\tau)u(\tau)d\tau = \beta_k$$

$$x(0) = -\sum_{k=0}^{n-1} A^k B \int_0^{t_1} a_k(\tau)u(\tau)d\tau = -[B \ AB \ \cdots \ A^{n-1}B] \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix}$$

Thus if system is controllable, the following equation about the initial state $x(0)$ has unique solution.

$$x(0) = -[B \ AB \ \cdots \ A^{n-1}B] \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix}$$

In which β_k ($k = 0, 1, 2, \dots, n-1$) is the scalar quantity set.

Which is equivalent to the situation that the nxn matrix Q:

$$Q = [B \ AB \ \cdots \ A^{n-1}B]$$

Satisfy $\text{rank}(Q)=n$

ALL SUM UP, we have **n.s.** conditions of controllability for linear time-invariable system:

The algebraic criteria of State Controllability :

Iff (if and only if) the $n \times n$ matrix Q is full rank:

$$\text{rank } Q = \text{rank}[B \ AB \ \cdots \ A^{n-1}B] = n$$

the system is controllable.

Extensive result:

For r dimension control vector u , if the system state equation is:

$$\dot{x} = Ax + Bu$$

$$x(t) \in R^n, u(t) \in R^r, A \in R^{n \times n}, B \in R^{n \times r}$$

Controllable condition is for $n \times nr$ matrix Q :

$$Q = [B \ AB \ \cdots \ A^{n-1}B]$$



Controllability Matrix

$\text{Rank}(Q)=n$, which is equivalent that n column vectors in Q are linearly independent.

Ex. Determine the controllability of following system:

System 1

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\det Q = \det[B \ AB] = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

Q is a singular matrix, so system is uncontrollable.

System 2

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$Q = [B \ AB \ A^2B] = \begin{bmatrix} 2 & 1 & 3 & 2 & 5 & 4 \\ 1 & 1 & 2 & 2 & 4 & 4 \\ -1 & -1 & -2 & -2 & -4 & -4 \end{bmatrix}$$

The 2nd and 3rd line are linearly dependent

$\text{rank}(Q) = 2 < 3$ So the system is uncontrollable.

- Ex. a bridge circuit Solution:**

Dynamic differential equations:

$$i_L = i_1 + i_2 = i_3 + i_4$$

$$R_3 i_3 - R_4 i_4 = u_c$$

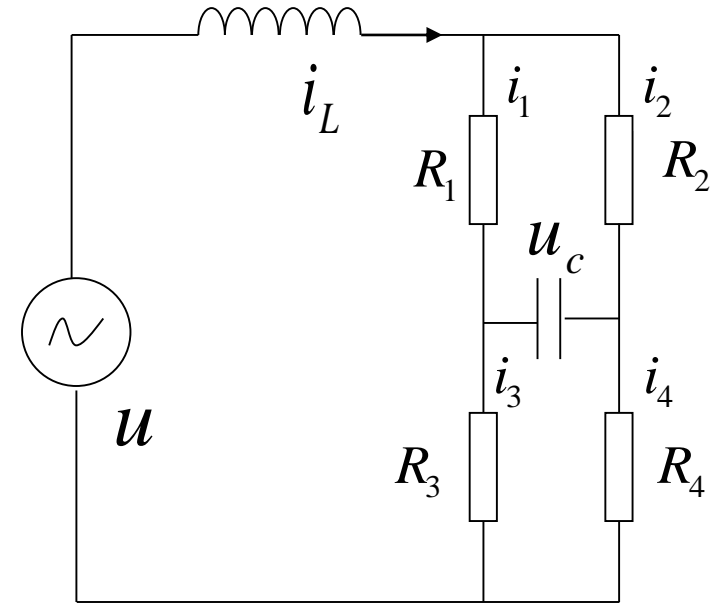
$$R_2 i_2 - R_1 i_1 = u_c$$

$$L \frac{di_L}{dt} + R_1 i_1 + R_3 i_3 = u$$

$$u_c = \frac{1}{C} \int (i_1 - i_3) dt$$

Select state variables: $x_1 = i_L$ $x_2 = u_c$

Eliminate the intermediate variables: i_1, i_2, i_3, i_4



$$\begin{cases} \dot{x}_1 = -\frac{1}{L} \left(\frac{R_1 R_2}{R_1 + R_2} + \frac{R_3 R_4}{R_3 + R_4} \right) x_1 + \frac{1}{L} \left(\frac{R_1}{R_1 + R_2} - \frac{R_3}{R_3 + R_4} \right) x_2 + \frac{1}{L} u \\ \dot{x}_2 = \frac{1}{C} \left(\frac{R_2}{R_1 + R_2} - \frac{R_4}{R_3 + R_4} \right) x_1 - \frac{1}{C} \left(\frac{1}{R_1 + R_2} + \frac{1}{R_3 + R_4} \right) x_2 \end{cases}$$

$$\begin{cases} \dot{x}_1 = -\frac{1}{L} \left(\frac{R_1 R_2}{R_1 + R_2} + \frac{R_3 R_4}{R_3 + R_4} \right) x_1 + \frac{1}{L} \left(\frac{R_1}{R_1 + R_2} - \frac{R_3}{R_3 + R_4} \right) x_2 + \frac{1}{L} u \\ \dot{x}_2 = \frac{1}{C} \left(\frac{R_2}{R_1 + R_2} - \frac{R_4}{R_3 + R_4} \right) x_1 - \frac{1}{C} \left(\frac{1}{R_1 + R_2} + \frac{1}{R_3 + R_4} \right) x_2 \end{cases}$$

- Controllable matrix S

$$\text{rank} S = \text{rank}[b \quad Ab] = \text{rank} \begin{bmatrix} \frac{1}{L} & -\frac{1}{L^2} \left(\frac{R_1 R_2}{R_1 + R_2} + \frac{R_3 R_4}{R_3 + R_4} \right) \\ 0 & -\frac{1}{LC} \left(\frac{R_4}{R_3 + R_4} - \frac{R_2}{R_1 + R_2} \right) \end{bmatrix}$$

- if $\frac{R_4}{R_3 + R_4} \neq \frac{R_2}{R_1 + R_2}$, $\text{rank} S = 2 = n$, system is controllable
- If $\frac{R_2}{R_1 + R_2} = \frac{R_4}{R_3 + R_4}$, $\text{rank} S = 1 < n$, System is uncontrollable

3. Output Controllability

Consider the following state space for a linear time-invariable system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$\mathbf{x} \in R^n, \mathbf{u} \in R^r, \mathbf{y} \in R^m, \mathbf{A} \in R^{n \times n}, \mathbf{B} \in R^{n \times r}, \mathbf{C} \in R^{m \times n}, \mathbf{D} \in R^{m \times r}$$

If we can find an unrestricted control $u(t)$ that in limited period $t_0 \leq t \leq t_1$, which could make any initial output $y(t_0)$ transfer to any final output $y(t_1)$, the system is called output controllable.

The n.s. condition of output controllable:

If the $m \times (n+1)r$ dimension output controllability matrix:

$$\mathbf{Q}' = [\mathbf{CB} : \mathbf{CAB} : \mathbf{CA}^2\mathbf{B} : \cdots : \mathbf{CA}^{n-1}\mathbf{B} : \mathbf{D}]$$

Satisfy $\text{rank}(\mathbf{Q}') = m$, the system is output controllable.

9.5.2 Observability of Linear Continual System

In practical project, sometimes, the states $x(t)$ of the system cannot be measured totally or even are unmeasured. One possible way is to **reflect the states $x(t)$ by the output $y(t)$** , which is the **observability** of the system.

1. Definitions

Completely Observable

For initial time $t_0 \in T_t$, existing a limitary time $t_1 \in T_t$, $t_1 > t_0$, for all $t \in [t_0, t_1]$, the initial value of the states $x(t_0)$ can be determined uniquely by system output $y(t)$, therefore the state is called Observable in $[t_0, t_1]$.

If every state $x(t)$ can be observed by $y(t)$ in the period $t_0 \leq t \leq t_1$, the system is Completely Observable.

Incompletely Observable

For initial time $t_0 \in T_t$, existing a limitary time $t_1 \in T_t$, $t_1 > t_0$, for all $t \in [t_0, t_1]$, if the initial value of all states $x_i(t_0)$, $i=1,2,\dots,n$, cannot be determined by the system output $y(t)$ totally, in other word, at least one state cannot be determined by $y(t)$, therefore the system is called Incompletely Observable in $[t_0, t_1]$, or Unobservable.

For the following state space:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

$$y(t) = Ce^{At} x(0) + C \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau + Du$$

The matrix A , B , C , D and input $u(t)$ are known, therefore the integral part of above equation on the right side is known, which can be removed from measured value $y(t)$.

To discuss the **n.s. condition** of the system's observability, the zero-input system will be considered.

2. The State Observability Algebraic Criteria of Linear time-invariable system

Consider the zero-input state space equation:

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned} \quad x \in R^n, y \in R^m, A \in R^{n \times n}, C \in R^{m \times n}$$

Output vector is: $y(t) = Ce^{At}x(0)$

Rewrite the e^{At} into the polynomial of matrix A :

$$e^{At} = \sum_{k=0}^{n-1} \alpha_k(t) A^k \quad y(t) = \sum_{k=0}^{n-1} \alpha_k(t) CA^k x(0)$$

If system is observable, in the time period $t_0 \leq t \leq t_1$, give a output $y(t)$, then the state $x(0)$ can be determined by the above equation uniquely.

$$\mathbf{y}(t) = \alpha_0(t)C\mathbf{x}(0) + \alpha_1(t)CA\mathbf{x}(0) + \cdots + \alpha_{n-1}(t)CA^{n-1}\mathbf{x}(0)$$

$$= [a_0(t)I_m \quad a_1(t)I_m \quad \cdots \quad a_{n-1}(t)I_m] \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \mathbf{x}(0)$$

Column linear
independence

n.s. condition of System Observability

For the linear time-invariable system:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

If and only if the rank of the $n \times n$ dimension observable matrix R is n , system is observable.

namely

$$R = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad \text{rank} R = n$$

or

$$R^T = [C^T \ A^T C^T \ \dots \ (A^T)^{n-1} C^T] \quad \text{rank} R^T = n$$

Ex. Try to confirm the controllability and observability of following system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

The state controllable matrix: $Q = [B \ AB] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

$\text{rank}Q = 2 = n$ Thus, system is **state controllable**.

The output controllable matrix: $Q' = [CB \ CAB] = \begin{bmatrix} 0 & 1 \end{bmatrix}$

$\text{rank}Q' = 1 = m$ Thus, system is **output controllable**.

The observable matrix: $R^T = [C^T \ A^T C^T] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\text{rank}R^T = 2 = n$ Thus, system is **observable**.

Ex. Analyze the observability of the following system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

In which: $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C = [4 \quad 5 \quad 1]$

Ex. Analyze the observability of the following system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

In which: $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C = [4 \quad 5 \quad 1]$

Solution: The observable matrix:

$$R^T = [C^T \quad A^T C^T \quad (A^T)^2 C^T] = \begin{bmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\begin{vmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{vmatrix} = 0$$

The system is unobservable.

➤ Transfer function description of the State Controllability and Observability Conditions

State Controllability and Observability conditions can be described by transfer function, as well.

The N.S. condition of state controllable and observable is **No Cancellation Appeared in the Transfer Function.**

If there is cancellation in the transfer function, the system is uncontrollable or unobservable, or even uncontrollable and unobservable simultaneously.

$$G(s) = C(sI - A)^{-1}B = \frac{(s - z_1)(s - z_2) \cdots \cancel{(s - z_i)} \cdots}{(s - p_1)(s - p_2) \cdots \cancel{(s - p_j)} \cdots}$$

$z_i = p_j$

Ex.9-24 Consider following transfer function:

$$\frac{Y(s)}{U(s)} = \frac{s + 2.5}{(s + 2.5)(s - 1)}$$

The reducible factors $(s+2.5)$ are included in the numerator and denominator of the transfer function, therefore the system state is uncontrollable or unobservable.

The same result can be derived if we transform the transfer function to the state function:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 2.5 & -1.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= [2.5 \quad 1]x \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2.5 & 1 \\ 2.5 & 1 \end{bmatrix}$$

Rank of the observable matrix $[C / CA]^T$ is 1, the system is state unobservable.

- if we transform the transfer function to another state function:

$$\dot{x} = \begin{bmatrix} 0 & 2.5 \\ 1 & -1.5 \end{bmatrix} x + \begin{bmatrix} 2.5 \\ 1 \end{bmatrix} u \implies [B : AB] = \begin{bmatrix} 2.5 & 2.5 \\ 1 & 1 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Rank of the controllable matrix $[B / AB]^T$ is 1, the system is state uncontrollable.

- We can also found the state space:
$$\begin{cases} \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -2.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

Thus: if there is **Cancellation Appeared in the Transfer Function**, the state space representing will appear uncontrollable or unobservable if we select different state variables to establish the state space.

9.5.3 Duality Principle (对偶原理) -- R.E.Kalman

Discuss the relationship between Controllability and Observability.

Consider the following system state space S_1 and S_2 :

$$S_1: \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad x \in R^n, u \in R^r, y \in R^m, A \in R^{n \times n}, B \in R^{n \times r}, C \in R^{m \times n}$$

$$S_2: \begin{cases} \dot{z} = A^T z + C^T v \\ n = B^T z \end{cases} \quad z \in R^n, v \in R^m, n \in R^r, A^T \in R^{n \times n}, C^T \in R^{n \times m}, B^T \in R^{r \times n}$$

System S_1 and S_2 are called **Dually System**(对偶系统).

Duality Principle:

If and only if the system S_1 is state observable / state controllable, system S_2 will be state controllable / state observable.

➤ Analysis of Duality Principle:

For system S_1 :

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

1. The n.s. condition of state controllable is the rank of $n \times nr$ dimension controllable matrix:

$$\text{rank}[B \quad AB \quad \dots \quad A^{n-1}B] = n$$

2. The n.s. condition of state observable is the rank of $n \times nm$ dimension observable matrix:

$$\text{rank}[C^T \quad A^T C^T \quad \dots \quad (A^T)^{n-1} C^T] = n$$

$$\dot{z} = A^T z + C^T v$$

For system S_2 :

$$n = B^T z$$

1. The n.s. condition of state controllable is the rank of $n \times nm$ dimension controllable matrix:

$$\text{rank}[C^T \quad A^T C^T \quad \cdots \quad (A^T)^{n-1} C^T] = n$$

2. The n.s. condition of state observable is the rank of $n \times nr$ dimension observable matrix:

$$\text{rank}[B \quad AB \quad \cdots \quad A^{n-1} B] = n$$

Based on the Duality Principle, the observability of a certain system can be determined by the state controllability of its dually system.

In brief, the duality is:

$$A \Rightarrow A^T, \quad B \Rightarrow C^T, \quad C \Rightarrow B^T$$

9.5.4 The criterion of Controllability and Observability for a certain linear continual system

➤ Controllability criterion

Criterion 1: The n.s. condition of linear time-invariable continual system state complete controllable is that the controllable matrix Q_c should be full rank.

$$Q_c = [B \ AB \ A^2B \ \cdots \ A^{n-1}B]$$

$$\text{rank} Q_c = n$$

Ex. The system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

try to analyze the controllability of the system.

Solution: $Q_c = [B \ AB] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$$\text{rank} Q_c = 2 = n$$

System is controllable

Ex. Analyze the controllability of the following 3-order 2-input system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$Q_c = [B \ AB \ A^2B] = \begin{bmatrix} 0 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \end{bmatrix}$$

System is uncontrollable

Criterion 2: If linear time-invariable system has unequal eigenvalues, the n.s. condition of system controllable is: The diagonal canonical equation from nonsingular transfer satisfies that **there is no row of zero in the input matrix $\bar{\mathbf{B}}$**

$$\dot{\bar{x}} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \bar{x} + \bar{\mathbf{B}}u$$

Ex. Determine the controllability of following diagonal canonical system.

$$1) \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} u$$

✓

$$2) \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ 7 \end{bmatrix} u$$

✗

$$3) \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 4 & 0 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

✓

$$4) \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 4 & 0 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

✗

Criterion 3: For the Jordan Canonical form $\dot{\bar{x}} = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & \ddots \\ & & & J_k \end{bmatrix} \bar{x} + \bar{B}u$

The elements in the relative **rows** in \bar{B} related with the **last row** in each Jordan block $J_i (i=1,2,\dots,k)$ are **not completely zero**.

(If two of Jordan Blocks have the same eigenvalue, the result is not exists.)

Ex. Analysis the controllability of following Jordan canonical system.

$$1) \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} u$$



$$2) \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



➤ Observability Criterion

Criterion 1: The n.s. condition of linear time-invariable continual system state complete observable is that the observable matrix R should be **full rank**.

$$R = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Namely, $\text{rank} R = n$

Ex. From the matrix A and C, analyze system's observability.

$$A = \begin{bmatrix} -4 & 5 \\ 1 & 0 \end{bmatrix} \quad C = [1 \quad -1]$$

Solution:

$$CA = [1 \quad -1] \begin{bmatrix} -4 & 5 \\ 1 & 0 \end{bmatrix} = [-5 \quad 5]$$

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -5 & 5 \end{bmatrix}$$



Ex. Try to determine the observability of the following system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 2 & -1 \\ -2 & 1 \end{bmatrix}$$



Criterion 2: If the linear time-invariable continual system has unequal eigenvalue, the n.s. condition of state observability is that the diagonal canonical form of the system from nonsingular transfer satisfy that **there is no column of zero in the output matrix \bar{C}**

$$\begin{cases} \dot{\bar{x}} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \bar{x} \\ y = \bar{C}\bar{x} \end{cases}$$

Ex. determine the observability of the system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



Criterion 3:

In the Jordan Canonical Form:

$$\begin{cases} \dot{\bar{x}} = \begin{bmatrix} J_1 & & 0 \\ & J_2 & \\ & & \ddots \\ 0 & & & J_k \end{bmatrix} \bar{x} \\ y = \bar{c}\bar{x} \end{cases}$$

The elements in the relative **column** in \bar{c} related with the **first row** in each Jordan block $J_i (i=1,2,\dots,k)$ are **not completely zero**.

(If two of Jordan Blocks have the same eigenvalue, the result is not exists.)

Ex.9-33 try to determine the observability of following system.

$$1) \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 2 & 1 & & 0 \\ 0 & 2 & & \\ & & 3 & 1 \\ 0 & & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$



$$2) \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u \quad y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



Controllability and Observability (小结)

Controllability: system input affect all the state of the system to achieve the control.

Observability: system output can reflect all the state of the system to achieve the observation.

1. State Controllability

The algebraic criteria of State Controllability :

Iff (if and only if) the $n \times n$ matrix Q is full rank, the system is controllable.

$$\text{rank} Q = \text{rank}[B \ AB \ \cdots \ A^{n-1}B] = n$$

2. Output Controllability

The n.s. condition of output controllable:

Iff the $m \times (n+1)r$ output controllability matrix Q is full rank, the system is output controllable.

$$\text{rank} Q' = \text{rank}[CB \vdots CAB \vdots CA^2B \vdots \cdots \vdots CA^{n-1}B \vdots D] = m$$

3. Observability

If and only if the rank of the $n \times n$ dimension observable matrix R is n , system is observable.

$$\text{rank} R = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \quad \text{或} \quad \text{rank} R^T = \text{rank} [C^T \ A^T C^T \ \cdots \ (A^T)^{n-1} C^T] = n$$

4. Transfer function description of the State Controllability and Observability Conditions

The N.S. condition of state controllable and observable is **No Cancellation Appeared in the Transfer Function.**

$$G(s) = C(sI - A)^{-1} B = \frac{(s - z_1)(s - z_2) \cdots \cancel{(s - z_i)} \cdots}{(s - p_1)(s - p_2) \cdots \cancel{(s - p_j)} \cdots} \quad z_i = p_j$$

5. Duality Principle (对偶原理) -- R.E.Kalman

Consider the following system state space S_1 and S_2 :

$$S_1: \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad x \in R^n, u \in R^r, y \in R^m, A \in R^{n \times n}, B \in R^{n \times r}, C \in R^{m \times n}$$

$$S_2: \begin{cases} \dot{z} = A^T z + C^T v \\ n = B^T z \end{cases} \quad z \in R^n, v \in R^m, n \in R^r, A^T \in R^{n \times n}, C^T \in R^{n \times m}, B^T \in R^{r \times n}$$

System S_1 and S_2 are called **Dually System**(对偶系统).

Duality Principle:

if and only if the system S_1 is state observable / state controllable, system S_2 will be state controllable / state observable.

6. The criterion of Controllability and Observability for a certain linear continual system

➤ Controllability criterion

Criterion 1: the n.s. condition of linear time-invariable continual system state complete controllable is that the controllable matrix Q_c should be full rank.

$$Q_c = [B \ AB \ A^2B \ \cdots \ A^{n-1}B]$$

$$\text{rank} Q_c = n$$

Criterion 2: if linear time-invariable system has unequal eigenvalues, the n.s. condition of system controllable is:
the diagonal canonical equation from nonsingular transfer satisfies that **there is no row of zero in the input matrix \bar{B}**

$$\dot{\bar{x}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{bmatrix} \bar{x} + \bar{B}u$$

Criterion 3: For the Jordan Canonical form $\dot{\bar{x}} = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & \ddots \\ & & & J_k \end{bmatrix} \bar{x} + \bar{B}u$

the elements in the relative **rows** in \bar{B} related with the **last row** in each Jordan block $J_i (i=1,2,\dots,k)$ are **not completely zero**.

➤ Observability Criterion

Criterion 1: the n.s. condition of linear time-invariable continual system state complete observable is that the observable matrix R should be **full rank**.

$$\text{rank} R = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

Criterion 2: if the linear time-invariable continual system has unequal eigenvalue, the n.s. condition of state observability is that the diagonal canonical form of the system from nonsingular transfer satisfy that **there is no column of zero in the output matrix \bar{C}**

$$\begin{cases} \dot{\bar{x}} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \bar{x} \\ y = \bar{C}\bar{x} \end{cases}$$

Criterion 3:

In the Jordan Canonical Form:

$$\begin{cases} \dot{\bar{x}} = \begin{bmatrix} J_1 & & 0 \\ & J_2 & \\ & & \ddots \\ 0 & & & J_k \end{bmatrix} \bar{x} \\ y = \bar{c}\bar{x} \end{cases}$$

the elements in the relative **column** in \bar{c} related with the **first row** in each Jordan block $J_i (i=1,2,\dots,k)$ are **not completely zero**.

(If two of Jordan Blocks have the same eigenvalue, the result is not exists.)

9.5.5 Controllability and Observability of Linear Time-invariant Discrete System

➤ Criterion of Controllability

State equations of discrete system:
$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) \end{cases}$$

$\mathbf{x} \in R^{n \times 1}$, $\mathbf{A} \in R^{n \times n}$ is nonsingular matrix

$\mathbf{B} \in R^{n \times r}$, $\mathbf{C} \in R^{m \times n}$, $\mathbf{D} \in R^{m \times r}$, $\mathbf{y} \in R^{m \times 1}$, $\mathbf{u} \in R^{r \times 1}$

If a sequence of the unrestricted control vector $u(0), u(1), \dots, u(n-1)$ can be obtained to make the system transform from $x(0)$ to $x(n)=0$, the system is controllable.

From the solution existence theorem, the n.s. condition of MIMO linear discrete system controllability is

$$\text{Assume } \text{rank } Q_d = \text{rank } [B \ AB \ \cdots \ A^{n-1}B]$$

$$\text{Then } \text{rank } Q_d = n$$

The criterion of output complete controllability

$$\text{rank } Q_d^o = \text{rank } [CB \ CAB \ \cdots \ CA^{n-1}B : D] = n$$

Ex. state equations of the time-invariable discrete system:

$$x(k+1) = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u(k)$$

Determine the controllability of the system.

Solution: The rank of the controllable matrix:

$$\text{rank}[B \quad AB \quad A^2B] = \text{rank} \begin{bmatrix} 1 & 0 & 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 4 & 2 \end{bmatrix} = 3$$

The system is controllable.

The controllable matrix of multi-input time-invariable discrete system is $n \times nr$ dimension. And the controllable condition is that the rank of the matrix is n . Therefore, the calculation could be enough when rank is n .

➤ Criterion of Observability

Consider the state equation of the system

$$x(k+1) = Ax(k)$$

$$y(k) = Cx(k)$$

If the output $y(k)$ in the finite sample period can determine the initial state vector $x(0)$, the system is observable.

Notice that $y(k)$ is an m dimension vector, thus the above n matrix equations will have nm algebraic equations, which contains $x_1(0), x_2(0), \dots, x_n(0)$.

To obtain unique solution $x_1(0), x_2(0), \dots, x_n(0)$ from the nm equations, the rank of the $nm \times n$ coefficient matrix should be n .

$$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

The ranks for the matrix and its transposition are the same, therefore, the n.s. condition of system observable is:

$$\text{rank } R_d^T = \text{rank} [C^T \quad A^T C^T \quad \dots \quad (A^T)^{n-1} C^T] = n$$

Ex. determine the observability of the following systems

$$S_1: \quad x(k+1) = \begin{bmatrix} 2 & 0 & 3 \\ -1 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix} x(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k)$$

Obtain the observable matrix

$$\text{rank} \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \\ -1 & -2 & 0 \\ 4 & 3 & 12 \\ 0 & 4 & -3 \end{bmatrix} = 3$$

or

$$\text{rank} \begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 2 & -1 & 4 & 0 \\ 0 & 1 & 0 & -2 & 3 & 4 \\ 0 & 0 & 3 & 0 & 12 & 3 \end{bmatrix} = 3$$

System 1 is observable.

$$S_2 \quad x(k+1) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 3 & 0 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} x(k)$$

Obtain the observable matrix

$$\text{rank} \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 3 & 0 & 2 \\ 1 & 0 & -1 \\ 9 & 0 & 1 \\ -2 & 0 & -3 \end{bmatrix} < 3$$

System 2 is unobservable

9.5.6 Structure analysis of continual time linear time-invariable system

➤ Controllable Decomposition

The dynamic equation of an uncontrollable system is:

$$\dot{x} = Ax + Bu \quad y = Cx$$

The rank of the controllable matrix $r < n$, **select r linear independent columns and any $n-r$ columns**, to construct the nonsingular transformation **T^{-1}** and satisfy:

$$x = T^{-1}\bar{x} \quad \bar{x} = \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix}$$

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$

$$y = \bar{C}\bar{x}$$

In which

$$\bar{A} = TAT^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \quad \bar{B} = TB = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} \quad \bar{C} = CT^{-1} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}$$

after nonsingular transformation, dynamic equations of the system should be:

$$\begin{bmatrix} \dot{x}_C \\ \dot{x}_{\bar{C}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} x_C \\ x_{\bar{C}} \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} \begin{bmatrix} x_C \\ x_{\bar{C}} \end{bmatrix}$$

Thus, the dynamic equation of controllable subsystem is:

$$\dot{x}_C = \bar{A}_{11}x_C + \bar{A}_{12}x_{\bar{C}} + \bar{B}_1u$$
$$y_1 = \bar{C}_1x_C$$

And the dynamic equation of uncontrollable subsystem is:

$$\dot{x}_{\bar{C}} = \bar{A}_{22}x_{\bar{C}}$$
$$y_2 = \bar{C}_2x_{\bar{C}}$$

Ex.9-36 For the system:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad c = [1 \quad -1 \quad 1]$$

Try to decomposed the system by controllability

Solution:

$$\text{rank} \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & -1 & -4 \\ 0 & 0 & 0 \\ 1 & 3 & 8 \end{bmatrix} = 2 < 3$$

System is incompletely controllable,

$$T^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\bar{A} = TAT^{-1} = \left[\begin{array}{cc|c} 0 & -4 & 2 \\ 1 & 4 & -2 \\ \hline 0 & 0 & 1 \end{array} \right] \quad \bar{b} = Tb = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array} \right] \quad \bar{c} = cT^{-1} = [1 \quad 2 \quad | \quad -1]$$

The dynamic equation of the controllable subsystem is:

$$\dot{x}_C = \begin{bmatrix} 0 & -4 \\ 1 & 4 \end{bmatrix} x_C + \begin{bmatrix} 2 \\ -2 \end{bmatrix} x_{\bar{C}} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y_1 = [1 \quad 2] x_C$$

And the dynamic equation of the uncontrollable subsystem is:

$$\dot{x}_{\bar{C}} = x_{\bar{C}}$$

$$y_2 = -x_{\bar{C}}$$

➤ Observable Decomposition

The dynamic equation of the unobservable system is

$$\dot{x} = Ax + Bu \quad y = Cx$$

The rank of the observable matrix is $l < n$, **select l linear independent rows and any $n-l$ rows** to construct a nonsingular transformation **T**

$$x = T^{-1}\bar{x} \quad \bar{x} = \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} \quad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

$$\bar{A} = TAT^{-1} = \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \quad \bar{B} = TB = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} \quad \bar{C} = CT^{-1} = \begin{bmatrix} \bar{C}_1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_o \\ \dot{x}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_1 & 0 \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix}$$

The dynamic of observable subsystem is:

$$\dot{x}_o = \bar{A}_{11}x_o + \bar{B}_1u$$

$$y_1 = \bar{C}_1x_o$$

The dynamic of unobservable subsystem is:

$$\dot{x}_{\bar{o}} = \bar{A}_{21}x_o + \bar{A}_{22}x_{\bar{o}} + \bar{B}_2u$$

$$y_2 = 0$$

- Ex9-38: try to find the observable subsystem of:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} x \end{cases}$$

Solution:

$$\text{rank } Q_o = \text{rank} \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{bmatrix} = 2 < 3$$

System is incomplete observable and the rank of observable part is 2. Choose transform matrix:

$$T = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

- New system after linear transforming:

$$\bar{A} = TAT^{-1} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & -2 & 0 \\ \hline 1 & 0 & -1 \end{array} \right] \quad \bar{B} = TB = \left[\begin{array}{c} 1 \\ -1 \\ 0 \end{array} \right]$$

$$\bar{C} = CT^{-1} = [1 \quad 0 \quad | \quad 0]$$

the state system of observable subsystem is:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

System Canonical decomposition by Controllability and Observability

Assume system (A, B, C) is uncontrollable and unobservable, first of all, decompose system by controllability

$$x = T_c^{-1} \begin{bmatrix} x_C \\ x_{\bar{C}} \end{bmatrix}$$

Then decompose the controllable subsystem and uncontrollable subsystem by observability

$$x_C = T_{O1}^{-1} \begin{bmatrix} x_{CO} \\ x_{C\bar{O}} \end{bmatrix} \quad x_{\bar{C}} = T_{O2}^{-1} \begin{bmatrix} x_{\bar{C}O} \\ x_{\bar{C}\bar{O}} \end{bmatrix}$$

At last we have:

$$x = T_c^{-1} \begin{bmatrix} x_C \\ x_{\bar{C}} \end{bmatrix} = \begin{bmatrix} T_c^{-1} T_{O1}^{-1} & & & \\ & T_c^{-1} T_{O1}^{-1} & & \\ & & T_c^{-1} T_{O2}^{-1} & \\ & & & T_c^{-1} T_{O2}^{-1} \end{bmatrix} \begin{bmatrix} x_{CO} \\ x_{C\bar{O}} \\ x_{\bar{C}O} \\ x_{\bar{C}\bar{O}} \end{bmatrix} = T^{-1} \begin{bmatrix} x_{CO} \\ x_{C\bar{O}} \\ x_{\bar{C}O} \\ x_{\bar{C}\bar{O}} \end{bmatrix}$$

From T^{-1} transformation, system dynamic is:

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \\ \dot{x}_{\bar{c}o} \\ \dot{x}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{21} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{33} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{44} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_1 & 0 & \bar{C}_3 & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix}$$

the dynamic equation of controllable and observable subsystem is:

$$\dot{x}_{co} = \bar{A}_{11}x_{co} + \bar{A}_{13}x_{\bar{c}o} + \bar{B}_1u$$

$$y_1 = \bar{C}_1x_{co}$$

The dynamic of controllable but unobservable subsystem is:

$$\dot{x}_{c\bar{o}} = \bar{A}_{21}x_{co} + \bar{A}_{22}x_{c\bar{o}} + \bar{A}_{23}x_{\bar{c}o} + \bar{A}_{24}x_{\bar{c}\bar{o}} + \bar{B}_2u$$
$$y_2 = 0$$

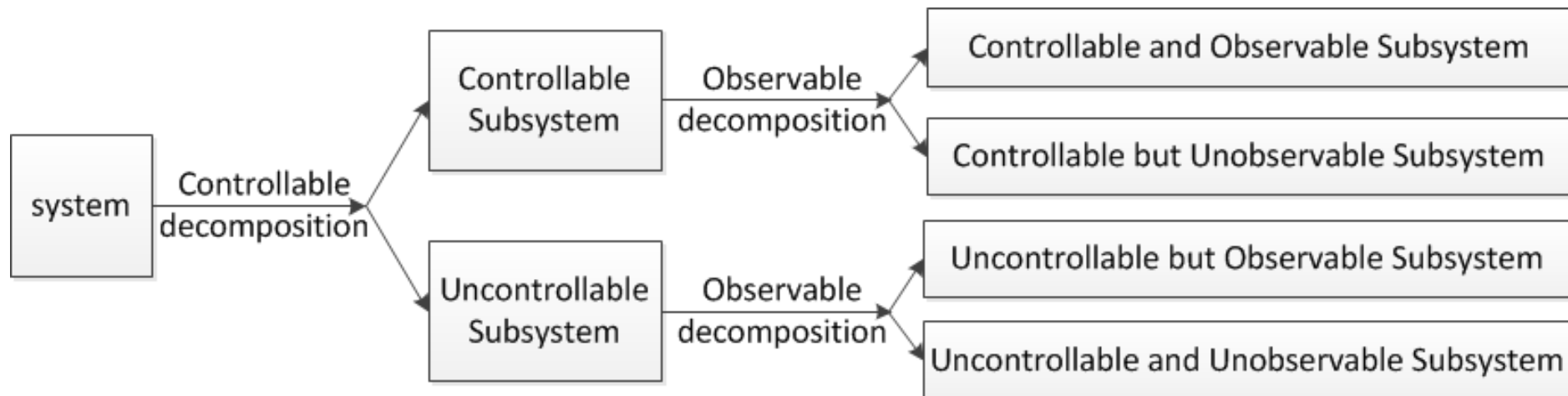
The dynamic of uncontrollable but observable subsystem is:

$$\dot{x}_{\bar{c}o} = \bar{A}_{33}x_{\bar{c}o}$$
$$y_3 = \bar{C}_3x_{\bar{c}o}$$

The dynamic of uncontrollable and unobservable sub system is:

$$\dot{x}_{\bar{c}\bar{o}} = \bar{A}_{43}x_{\bar{c}o} + \bar{A}_{44}x_{\bar{c}\bar{o}}$$
$$y_4 = 0$$

- However, it is complicated to calculate transform matrix T_{co}^{-1} directly.
- The normal method is to decompose the system gradually to controllable, observable and uncontrollable & unobservable subsystem.
- The general decomposition process is:



- Ex9-38: system dynamics:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} x \end{cases}$$

which is incomplete controllable and incomplete observable, try to decompose the system by controllability and observability.

Solution:

(1) Controllability decomposition:

Form the decomposition matrix:

$$T_c^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

- Decompose the system by controllability:

$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & -2 & -2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix}$$

- The uncontrollable subsystem is one dimension system and observable. Thus, it is the uncontrollable but observable subsystem we need.
- (2) Decompose the controllable subsystem by observability.

$$\dot{x}_c = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} x_c + \begin{bmatrix} -1 \\ -2 \end{bmatrix} x_{\bar{c}} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y_1 = \begin{bmatrix} 1 & -1 \end{bmatrix} x_c$$

- Form the observable decomposition matrix:

$$T_{co}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad T_{co} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- Then, the decompose the controllable subsystem by observability:

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} x_{c\bar{o}} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \end{bmatrix}$$

- (3) Combine above transforming, the controllability and observability decomposition equation is:

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \\ \dot{x}_{\bar{co}} \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{co}} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{co}} \end{bmatrix}$$

- The transform matrix:

$$T = T_c \begin{bmatrix} T_{co} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$