Chapter 9 Analysis of Control Systems in State Space

9.1 Introduction

9.2 State-Space and State-Equation

9.3 Establishing State-Space of Linear System

9.4 Solving the Linear Time-Invariant State Equation

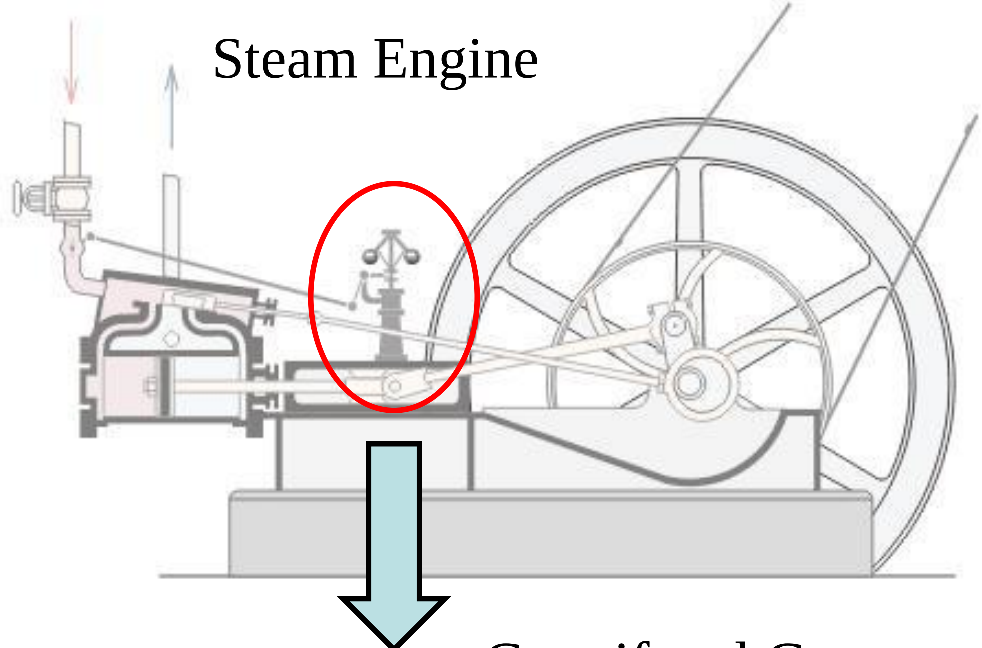
9.5 Controllability and Observability

9.6 Feedback Structure and State-Observers

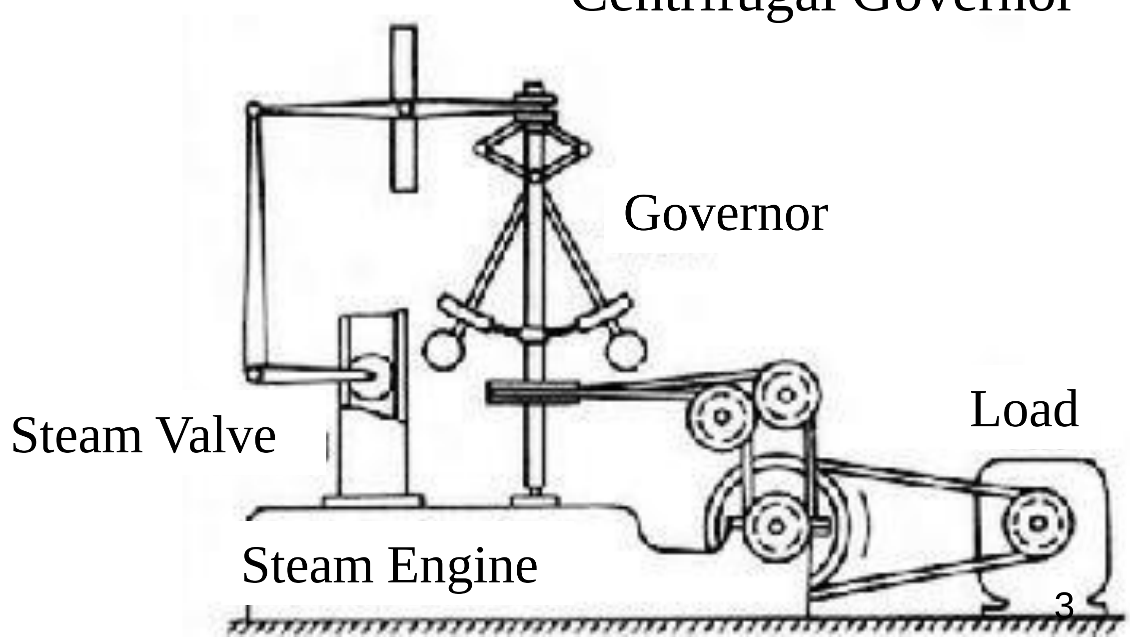
9.1 Introduction

Mr. J. Watt

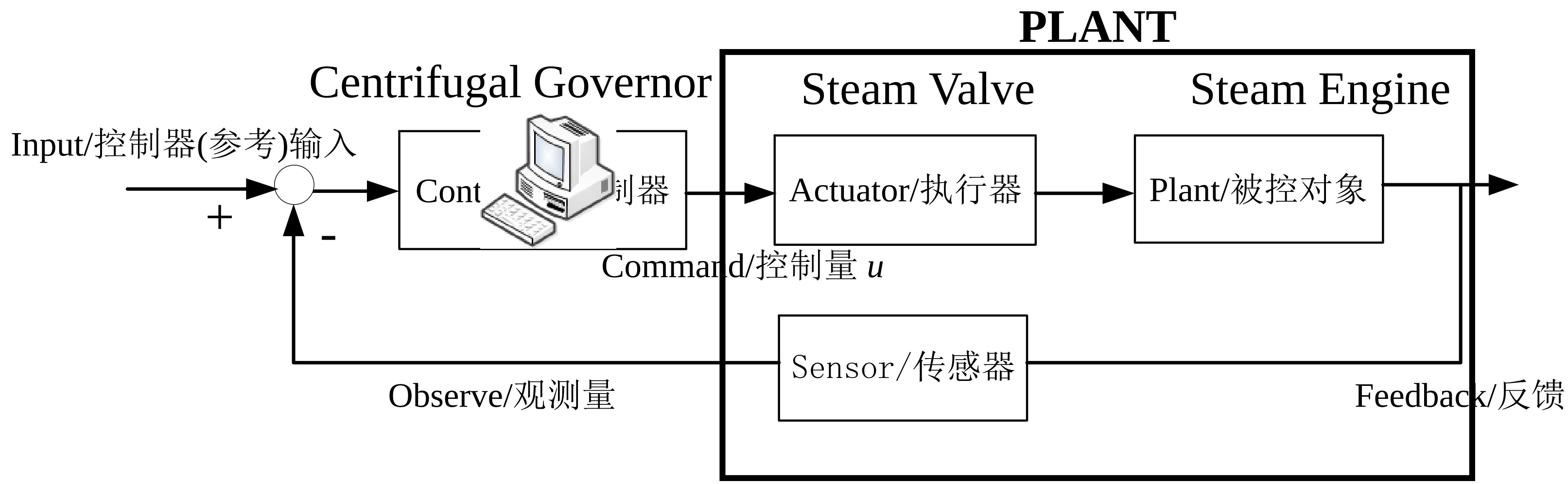
Steam Engine

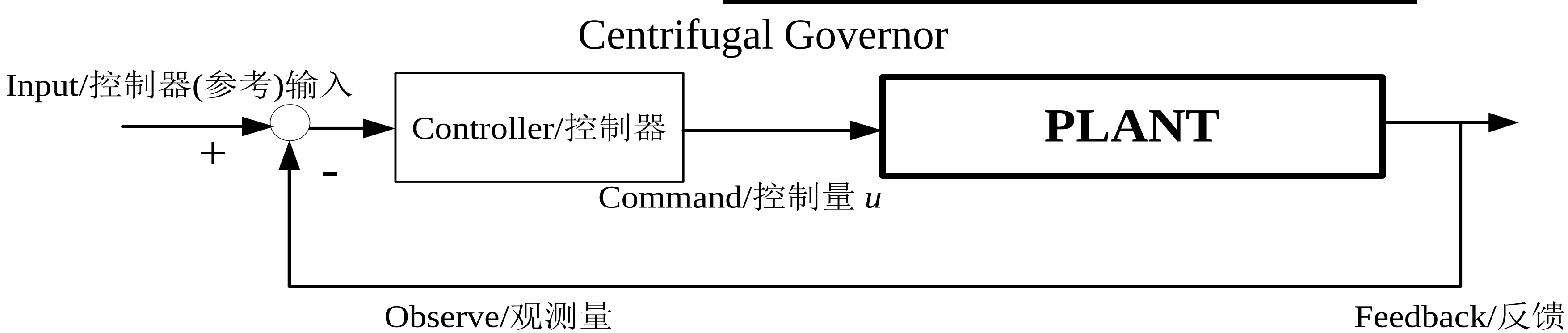


Centrifugal Governor



• Review：Control System





$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Classic Control Theory

- 1784, Watt, Speed Controller of Steam Engine, Feedback Implement
- 1868, Maxwell, “On Governors” (调节器)
- 1877, Routh stability analysis
- 1895, Hurwitz stability analysis
- 1945, Bode frequent method
- 1948, Evans’ Root Locus
- … …

Classic Control Theory

- ✓ Target：SISO, Time-Invariant system
- ✓ Description：Differential Equation, Frequency Character, Transfer Function
- ✓ Method ：Time Domain, Frequency Domain, Root-Locus

Impactful for Linear, Time-Invariant System Analysis and Design

Classic Control Theory

- ✓ Target： SISO, Time-Invariant system
- ✓ Description： Differential Equation, Frequency Character, Transfer Function
- ✓ Method： Time Domain, Frequency Domain, Root-Locus

Impactful for Linear, Time-Invariant System Analysis and Design

Limitative for Nonlinear, Time-Variant, Multi-variable System Analysis and Design

Input / Output relationship, ignore the internal relation of the system, such as coupling(耦合) issue.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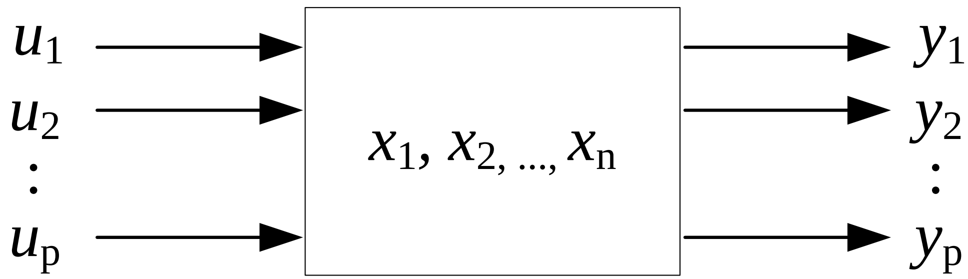

u_1
u_2
u_p
$x_1, x_2, \ldots, x_n$
y_1
y_2
y_p


United States


王永卓 王东明编制新华社发
模拟图

1950s, “Disordered Time” for Control Theory Development

- ✓ Bellman Dynamic Programming / 动态规划
- ✓ Pontryagin’s Maximum Principle / 极大值原理
- ✓ Kalman Controllablility & Observability Theory
- ✓ Pole Placement / 极点配置
- ✓ Observer / 观测器

1960s ~ 1970s, “Modern Control Theory”（State-Space Method）

Areas:

Aerospace Science; Robotics; Modern Industry…

	经典控制理论 (50年代前)	现代控制理论 (50年代后)
研究对象	单输入单输出的线性定常系统	可以比较复杂
数学模型	传递函数 (输入、输出描述)	状态方程 (可描述内部行为)
数学基础	运算微积、复变函数	线性代数、矩阵理论
设计方法的特点	非唯一性、试凑成份多,经验起很大作用。主要在复数域进行。	设计的解析性，与计算机结合，主要在时间域进行。

- **Classify**
  - **Classic Control Theory**
    - Routh-Hurwitz Stability Criterion, Bode Diagram, Nyquist plot/curve, Root Locus…
  - **Modern Control Theory**
    - State Space, Controllability, Observability …
  - **Postmodern Control Theory**
    - Robust Control, Adaptive Control, Nonlinear Control, Intelligent Control …

9.2 State-Space and State-Equation

预备知识——矩阵微分

1、向量函数对数量函数的导数

$$\vec{X}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

定义：$$\frac{d\vec{X}}{dt}=\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}$$

2、矩阵函数对数量函数的导数

$$A=\begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

定义：$$\frac{dA}{dt}=\begin{bmatrix} \dot{a}_{11}(t) & \dot{a}_{12}(t) & \cdots & \dot{a}_{1n}(t) \\ \dot{a}_{21}(t) & \dot{a}_{22}(t) & \cdots & \dot{a}_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ \dot{a}_{n1}(t) & \dot{a}_{n2}(t) & \cdots & \dot{a}_{nn}(t) \end{bmatrix}$$

预备知识——矩阵微分

3、数量函数对向量的导数

$$f(x_1, x_2, \cdots, x_n), \vec{X} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$$

定义：$$\frac{df}{dx} = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \cdots & \dfrac{\partial f}{\partial x_n} \end{bmatrix}^T$$

4、向量函数对向量的导数

$$A(x) = \begin{bmatrix} a_1(x) & a_2(x) & \cdots & a_n(x) \end{bmatrix}^{\mathrm{T}}$$

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}}$$

定义：$$\frac{dA^{\mathrm{T}}}{dx} = \begin{bmatrix} \dfrac{\partial a_1}{\partial x_1} & \dfrac{\partial a_2}{\partial x_1} & \dots & \dfrac{\partial a_n}{\partial x_1} \\ \dfrac{\partial a_1}{\partial x_2} & \dfrac{\partial a_2}{\partial x_2} & \dots & \dfrac{\partial a_n}{\partial x_2} \\ \vdots & \vdots & \dots & \vdots \\ \dfrac{\partial a_1}{\partial x_n} & \dfrac{\partial a_2}{\partial x_n} & \dots & \dfrac{\partial a_n}{\partial x_n} \end{bmatrix}$$

预备知识——矩阵微分

5、矩阵函数对向量的导数

$$A(x)=\begin{bmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1l}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2l}(x) \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1}(x) & a_{m2}(x) & \cdots & a_{ml}(x) \end{bmatrix}_{m\times l}$$

$$X=\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}_{n\times 1}^T$$

$$\text{定义:}\ \frac{dA}{dx}=\begin{bmatrix} \dfrac{\partial A}{\partial x_1} \\ \dfrac{\partial A}{\partial x_2} \\ \vdots \\ \dfrac{\partial A}{\partial x_n} \end{bmatrix}=\begin{bmatrix} \cdots \end{bmatrix}_{mn\times l}$$

9.2.1 Concepts

1. State: In time domain, a set (集合) of variables to describe System's motion and movement information

2. State-variable: The smallest set of variables that describe the "state" of a system. Intuitively, the state of a system describes enough about the system to determine its future behavior. In an n^{th}-order differential equation there should be n independent state variables.

Attention:

- ✓ State-variable is Sufficient and Necessary to describe dynamic motion of the system.
- ✓ State-variable is not unique.

3. State-vector: n states variables: $x_1(t)$, $x_2(t)$, … , $x_n(t)$ to describe the observed states in State space.

$$X(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T$$

4. State-space: The n dimension space based on the state-variable $x_1(t)$, $x_2(t)$, … , $x_n(t)$.
5. State-locus: At special time t_0, the state X(t_0) will be a point in the state-space; During a period of time t, the state $X(t)$ will be drawn as a trajectory/locus.
6. State-equation: The first-order equation which describes the mathematical relationship between state variables and the input:

$$\dot{x}(t) = f[x(t), u(t)]$$

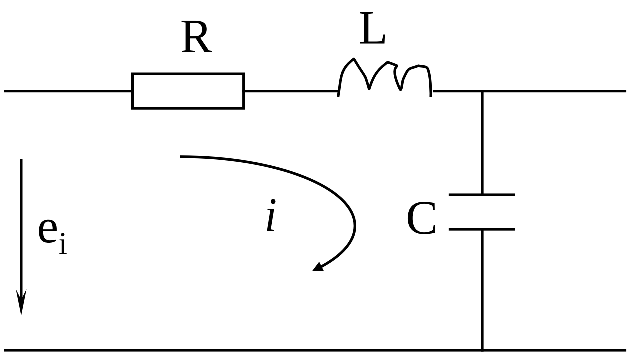
$$x(t_{k+1}) = f[x(t_k), u(t_k)]$$

9.2.2 Examples

Ex. 9-1 RLC electric circuit 's State-space model (figure).

Physics relationship:

$$L\frac{di}{dt}+Ri+\frac{1}{c}\int idt=e_i$$



If assume $e_i(t)$ is the input: u(t), $i(t)$ is the output: y(t), and select the proper state-variables $i(t)$ and $\int i(t)$dt:

$$x_1(t)=i(t)$$
$$x_2(t)=\int i(t)dt$$

$$\Longrightarrow$$

$$\frac{dx_1(t)}{dt}=-\frac{R}{L}x_1(t)-\frac{1}{LC}x_2(t)+\frac{1}{L}u(t)$$
$$\frac{dx_2(t)}{dt}=x_1(t)$$

9.2.2 Examples

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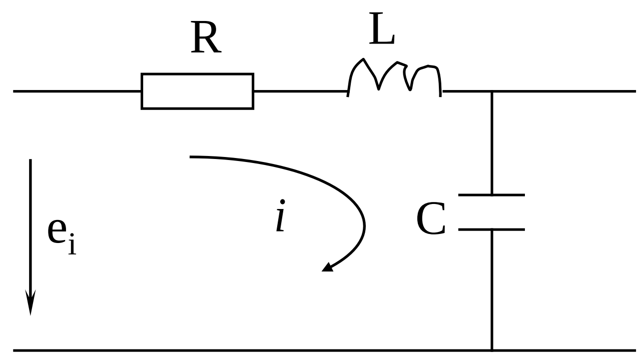

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$$\frac{dx_1(t)}{dt}=-\frac{R}{L}x_1(t)-\frac{1}{LC}x_2(t)+\frac{1}{L}u(t)$$

$$\frac{dx_2(t)}{dt}=x_1(t)$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-\frac{R}{LC} & -\frac{1}{LC}\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}\frac{1}{LC}\\ 0\end{bmatrix}u_r(t)$$

7. Output-equation: The equation which describes the relationship between the output, state variables , and the input.

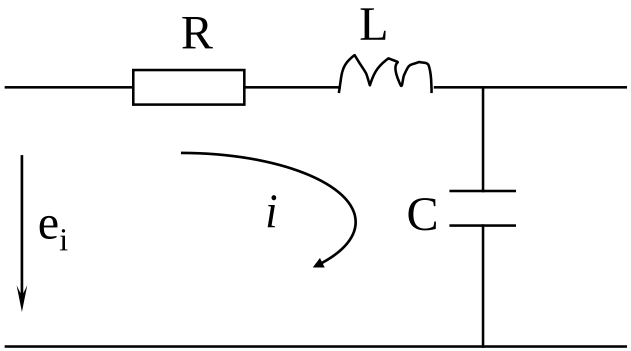
$$y(t) = g[x(t), u(t)]$$

$$y(t_k) = g[x(t_k), u(t_k)]$$

9.2.2 Examples

Ex. 9-1 RLC electric circuit 's State-space model (figure).

Physics relationship:



$$L\frac{di}{dt}+Ri+\frac{1}{c}\int idt=e_i$$

If assume $e_i(t)$ is the input: u(t), $i(t)$ is the output: y(t), and select the proper state-variables $i(t)$ and $\int i(t)$dt:

$$x_1(t)=i(t)$$
$$x_2(t)=\int i(t)dt$$

$$\Longrightarrow \qquad y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

7. Output-equation: Describe the relationship between the Output and State, and between Output and Input.

$$y(t) = g[x(t), u(t)]$$

$$y(t_k) = g[x(t_k), u(t_k)]$$

8. State-space Representation: Represent the system by the State-equations and Output-equations.

$$\begin{cases} \dot{x}(t) = f[x(t), u(t)] \\ y(t) = g[x(t).u(t)] \end{cases}$$

$$\begin{cases} x(t_{k+1}) = f[x(t_k), u(t_k)] \\ y(t_k) = g[x(t_k), u(t_k)] \end{cases}$$

For linear system

State-space Representation
$$\begin{cases} \dot{x}(t) = f[x(t), u(t)] \\ y(t) = g[x(t).u(t)] \end{cases}$$

For linear system, *f* and *g* are linear function.

State-equation: A first-order differential equation.

Output-equation: Algebraic equation(代数方程) for vectors .

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases}$$

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Linear Time-Invariant System

State-equation: For Single-input Linear Time-invariant System

$$\begin{cases} \dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + b_1u(t) \\ \dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_2u(t) \\ \vdots \\ \dot{x}_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + b_nu(t) \end{cases}$$

The constant coefficients $a_{11},\ldots,a_{nn}$; $b_1,\ldots,b_n$ are decided by system characters.

Matrix expression： $\dot{x}(t) = Ax(t) + bu(t)$

$$x(t) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \dot{x}(t) = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

State-equation: For multi-input Linear Time-Invariant System

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1p}u_p \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2p}u_p \\ \vdots \\ \dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{np}u_p \end{cases}$$

Matrix expression： $\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & a_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}, \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{pmatrix}$$

Output-equation: For Single-input and Single-out Linear Time-Invariant System

$$y(t)=c_1x_1(t)+c_1x_2(t)+\cdots+c_nx_n(t)+du(t)$$

Output is decided by system task.The constant coefficients $c_1,c_2,\ldots,c_n$ and *d* are relative with the system character.

The Matrix representation: $\boxed{y(t)=\mathbf{c}x(t)+du(t)}$

Output-equation: For MIMO system

$$\begin{cases} y_1=c_{11}x_1+c_{12}x_2+\cdots+c_{1n}x_n+d_{11}u_1+d_{12}u_2+\cdots d_{1p}u_p \\ y_2=c_{21}x_1+c_{22}x_2+\cdots+c_{2n}x_n+d_{21}u_1+d_{22}u_2+\cdots d_{2p}u_p \\ \vdots \\ y_q=c_{q1}x_1+c_{q2}x_2+\cdots+c_{qn}x_n+d_{q1}u_1+d_{q2}u_2+\cdots d_{qp}u_p \end{cases}$$

The Matrix representation: $\boxed{y=\mathbf{C}x+\mathbf{D}\boldsymbol{u}}$

State-space equations of Linear Time-Invariant (LTI) System:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

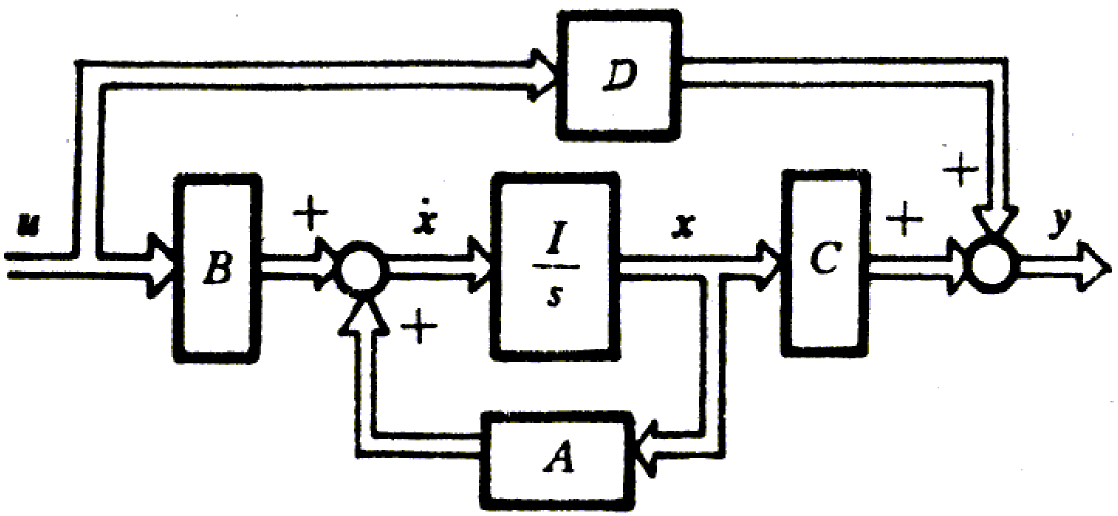
$$y(t) = Cx(t) + Du(t)$$

A—State Matrix (Systems matrix, coefficients matrix)

B—Input Matrix (Control Matrix)

C—Observing Matrix (Output Matrix)

D—Feedforward Matrix (Directly Transfer Matrix)



The Structure of the State-space

Advantage of State-space Analysis Method

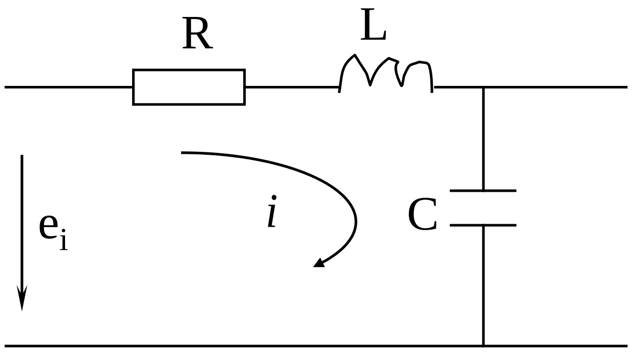
- **Computing:** Using computer to solve the first-order differential equations, easier than higher order equation;
- **Representation:** Using Vector Matrix to simplify the mathematic representation of the differential equations;
- **Field:** MIMO System, Time-Invariant System, Stochastic Process and Sample System, etc.
- **Special:** The use of the state space representation is not limited to systems with linear components and zero initial conditions.

9.2.2 Examples

Ex. 9-1 RLC electric circuit 's State-space model (figure).

Physics relationship:

$$L\frac{di}{dt}+Ri+\frac{1}{c}\int idt=e_i$$



If assume $e_i(t)$ is the input: u(t), $i(t)$ is the output: y(t), and select the proper state-variables $i(t)$ and $\int i(t)$dt:

$$x_1(t)=i(t)$$
$$x_2(t)=\int i(t)dt$$

$$\Longrightarrow$$

$$\frac{dx_1(t)}{dt}=-\frac{R}{L}x_1(t)-\frac{1}{LC}x_2(t)+\frac{1}{L}u(t)$$
$$\frac{dx_2(t)}{dt}=x_1(t)$$

$$\frac{dx_1(t)}{dt} = -\frac{R}{L}x_1(t) - \frac{1}{LC}x_2(t) + \frac{1}{L}u(t)$$

$$\frac{dx_2(t)}{dt} = x_1(t)$$

Rewrite to State equations:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{R}{L} & -\frac{1}{LC} \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}}_{B} u(t)$$

$y(t)=i(t)=x_1(t)$, thus we have Output equation:

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

Not the end!!!!

Furthermore!!!!

$$L\frac{di}{dt}+Ri+\frac{1}{c}\int idt=e_i$$

If select some another state variables:

$$x_1=\frac{1}{C}\int idt+Ri,\ x_2=\frac{1}{C}\int idt$$

Then $\quad x_1=x_2+Ri, \qquad L\frac{di}{dt}+x_1=e_i$

We have
$$\begin{cases}\dot{x}_1=\dot{x}_2+R\dfrac{di}{dt}=\dfrac{1}{RC}(x_1-x_2)+\dfrac{R}{L}(-x_1+e_i)\\ \dot{x}_2=\dfrac{1}{C}i=\dfrac{1}{RC}(x_1-x_2)\\ y=\dfrac{1}{R}x_1-\dfrac{1}{R}x_2\end{cases}$$

State-space representation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}\dfrac{1}{RL}-\dfrac{R}{L} & -\dfrac{1}{RC}\\ \dfrac{1}{RC} & -\dfrac{1}{RC}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}\dfrac{R}{L}\\ 0\end{bmatrix}u$$

$$y=\begin{bmatrix}\dfrac{1}{R} & -\dfrac{1}{R}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

1st selection: $x_1(t) = i(t)$

$$x_2(t) = \int i(t)dt$$

2nd selection: $\bar{x}_1 = \frac{1}{c}\int idt + Ri$

$$\bar{x}_2 = \frac{1}{c}\int idt$$

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{LC} \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{1}{RL} - \frac{R}{L} & -\frac{1}{RC} \\ \frac{1}{RC} & -\frac{1}{RC} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{R}{L} \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} \frac{1}{R} & -\frac{1}{R} \end{bmatrix}$$

Conclusion: State-space is non-unique.

- State-variable is not unique.
- State-equation is not unique.

Different state-equations with Different state-variable

1st selection: $x_1(t) = i(t)$

$$x_2(t) = \int i(t)dt$$

2nd selection: $\bar{x}_1 = \frac{1}{c}\int idt + Ri$

$$\bar{x}_2 = \frac{1}{c}\int idt$$

we have

$$x_1 = \frac{1}{R}\bar{x}_1 - \frac{1}{R}\bar{x}_2$$

$$x_2 = c\bar{x}_2$$

matrix representation:

$\boldsymbol{x} = \mathbf{P}\overline{\boldsymbol{x}}$ in which $\mathbf{P} = \begin{bmatrix} \frac{1}{R} & -\frac{1}{R} \\ 0 & c \end{bmatrix}$

Infinite groups of state-variables x are available by any different nonsingular matrix (非奇异阵) **P**.

作业：P348 9-1