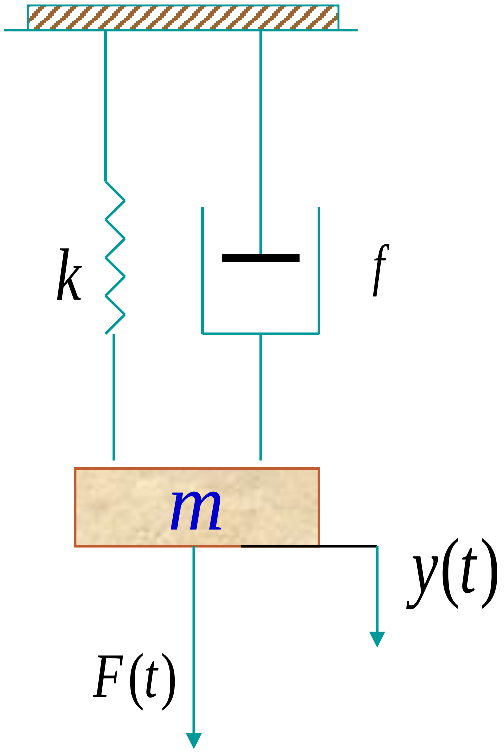
# 9.3 State-space Establishing of Linear System

**General Methodology:**

- From Physics Mechanism of System
- From Differential Equations of System
- From Transfer Functions of System
- From State-variable Diagram of System
- Linear Transformation of State space

# 9.3.1 From Physics Mechanism of System

Ex.9-2 Mechanism system composed by mass, spring(弹簧) and damper(阻尼器) without gravity(重力).




F(t) is Input, y(t) is Output.

Then, if the original displacement and velocity are available, the system's solution of the certain input is available as well.

From **Newton's Law**

$$m\frac{d^2y}{dt^2} = F(t) - f\frac{dy}{dt} - ky$$

Select the displacement and velocity as the state variables

$$x_1 = y, \quad x_2 = v = \dot{y}$$

Input is: $u(t) = F(t)$

$$m\frac{d^2y}{dt^2} = F(t) - f\frac{dy}{dt} - ky$$

State-equations:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{k}{m}x_1 - \dfrac{f}{m}x_2 + \dfrac{1}{m}u \end{cases}$$

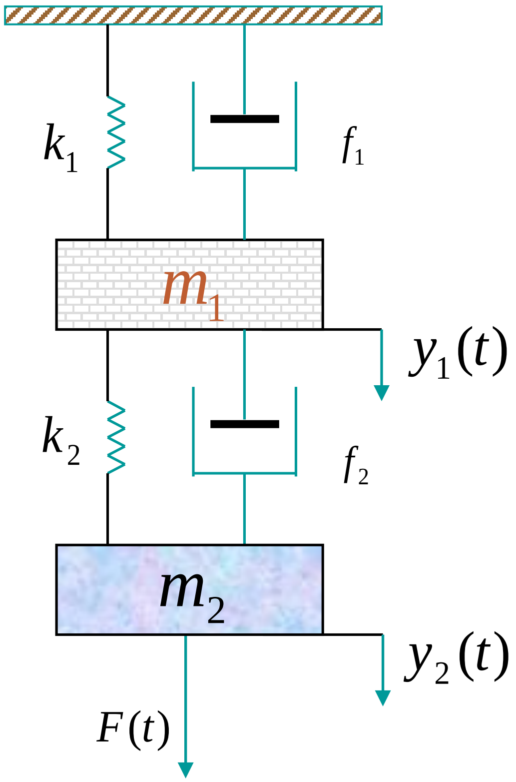
State space representation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{f}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Ex.9-3 The State-space representation of Mechanism without gravity, the input is the pull F, the outputs are the displacement $y_1$ and $y_2$.

From Newton's first Law, we have physics relationship of $m_1$ and $m_2$:




$$m_2\ddot{y}_2 = F(t) - k_2(y_2 - y_1) - f_2(\dot{y}_2 - \dot{y}_1)$$
$$m_1\ddot{y}_1 = k_2(y_2 - y_1) + f_2(\dot{y}_2 - \dot{y}_1) - k_1 y_1 - f_1\dot{y}_1$$

Select 4 independent state variables:

$$x_1 = y_1, \quad x_2 = y_2, \quad x_3 = \dot{y}_1, \quad x_4 = \dot{y}_2$$

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = \dfrac{k_2}{m_1}(x_2 - x_1) + \dfrac{f_2}{m_1}(x_4 - x_3) - \dfrac{k_1}{m_1}x_1 - \dfrac{f_1}{m_1}x_3 \\ \dot{x}_4 = \dfrac{1}{m_2}F(t) - \dfrac{k_2}{m_2}(x_2 - x_1) - \dfrac{f_2}{m_2}(x_4 - x_3) \end{cases}$$

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = \dfrac{k_2}{m_1}(x_2 - x_1) + \dfrac{f_2}{m_1}(x_4 - x_3) - \dfrac{k_1}{m_1}x_1 - \dfrac{f_1}{m_1}x_3 \\ \dot{x}_4 = \dfrac{1}{m_2}F(t) - \dfrac{k_2}{m_2}(x_2 - x_1) - \dfrac{f_2}{m_2}(x_4 - x_3) \end{cases}$$

$x_1 = y_1, \quad x_2 = y_2, \quad x_3 = \dot{y}_1, \quad x_4 = \dot{y}_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1 + k_2}{m_1} & \dfrac{k_2}{m_1} & -\dfrac{f_1 + f_2}{m_1} & \dfrac{f_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2}{m_2} & \dfrac{f_2}{m_2} & -\dfrac{f_2}{m_2} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{m_2} \end{bmatrix}}_{B} F$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

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## 9.3.2 From Differential Equations of System

**Methodology:**

– Establish the differential/difference equations by the physics mechanism of system;

– Establish the state equation focusing on the equations and a group of state-variables;

– Establish the output function based on the relationship between system's outputs and states.

## State-variable Selection

- ✓ Selection of state variable is not unique.
- ✓ Methodology:
  - ☞ Select variable in the initial conditions or related.
  - ☞ Select characteristic variable of independent storage components (energy or information) with certain physical meaning, such as the electric current $i$ of inductance, the voltage $u_c$ of capacitor, and velocity $v$ of mass, etc.

Scenario (1): No derivatives(微分) of input $u$ contained in n-order linear differential equations

Assume the dynamic process of the SISO control system is described as follow:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = bu$$

$y^{(n)}, y^{(n-1)}, \cdots, \dot{y}, y$ —— Derivatives of output

$\boldsymbol{u}$ —— Input

If initial conditions $y(0), y'(0), \ldots, y^{(n-1)}(0)$ of output and the input u(t) of t≥0 are known, the behavior of system at any time can be obtained.

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = bu$$

Select state variables: $$\begin{cases} x_1 = y \\ x_2 = \dot{y} \\ \vdots \\ x_{n-1} = y^{(n-2)} \\ x_n = y^{(n-1)} \end{cases}$$

then: $$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots a_1 x_n + bu \end{cases}$$

State space:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu$$
$$y = C\boldsymbol{x}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_n x_1 - a_{n-1}x_2 - \cdots a_1 x_n + bu \end{cases}$$

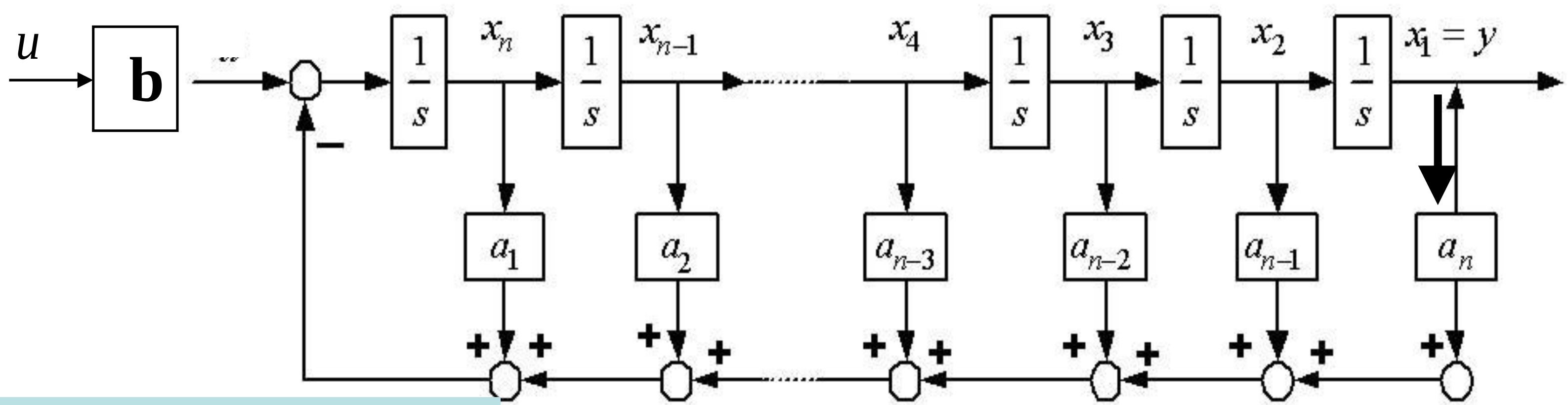
in which, the matrices:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathrm{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ b \end{bmatrix}$$

$$C = [1 \ \ 0 \ \ 0 \ \ \ldots \ \ 0]$$

Then, draw the following block diagram (state variable diagram) among the state variables.

- ✓ The output of each integrator corresponds to each state variable.
- ✓ The state equations are decided by the relationship of I/O.
- ✓ The output equation is on output part.




$$
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = x_3 \\
\vdots \\
\dot{x}_{n-1} = x_n \\
\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots a_1 x_n + bu
\end{cases}
$$

Ex.9-4 Assume the differential equation of the system's dynamic process is $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$ , in which, $u$ and $y$ are input and output.

Try to find the state space representation of the system.

Select state-variables: $x_1 = y, x_2 = \dot{y}, x_3 = \ddot{y},$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -6x_1 - 11x_2 - 6x_3 + 6u \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}}_{\mathbf{B}} u$$

Standard Form:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$$

$$y = \boldsymbol{C}\boldsymbol{x}$$

$$y = \underbrace{[1 \quad 0 \quad 0]}_{\mathbf{C}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

# Practice 9.1 RLC电路

$$\frac{d^2 u_c(t)}{dt^2} + \frac{R}{L}\frac{du_c(t)}{dt} + \frac{1}{LC}u_c(t) = \frac{1}{LC}u_r(t)$$

## Scenario(2): <u>Derivatives of input $u$</u> contained in n-order linear differential equation system

n-order linear differential equation representation:

$$y^{(n)}+a_1\,y^{(n-1)}+\cdots+a_{n-1}\dot{y}+a_n y=b_o\,u^{(n)}+b_1\,u^{(n-1)}+\cdots+b_{n-1}\dot{u}+b_n u$$

Reference scenario(1):

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=x_3\\ \vdots\\ \dot{x}_{n-1}=x_n\\ \dot{x}_n=-a_n y-a_{n-1}\dot{y}-\cdots-a_1 y^{(n-1)}+b_o u^{(n)}+b_1 u^{(n-1)}+\cdots+b_{n-1}\dot{u}+b_n u\\ \qquad=-a_n x_1-a_{n-1}x_2-\cdots-a_1 x_n+b_o u^{(n)}+b_1 u^{(n-1)}+\cdots+b_{n-1}\dot{u}+b_n u\end{cases}$$

$$\begin{cases}x_1=y\\ x_2=\dot{y}\\ \vdots\\ x_{n-1}=y^{(n-2)}\\ x_n=y^{(n-1)}\end{cases}$$

However, the <u>derivative of input $u$</u> is still contained in state equation, which is INCONSEQUENCE(不合理).

Hence, we cannot choose the output y and its derivatives to be state variables of the system. Such group of state variables cannot decide the future state of the system based on the known system input and original state conditions.

**The principle of state variable selection:**

No derivative of the input/operation function could be included in any differential function in the system state equations represented by first-order differential equation sets.

Select the state variables:

The input is contained in the state variables

n-1 state equations

$$\begin{cases} x_1 = y - \beta_0 u \\ x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u \\ x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u \\ \vdots \\ x_n = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \cdots - \beta_{n-2} \dot{u} - \beta_{n-1} u = \dot{x}_{n-1} - \beta_{n-1} u \end{cases}$$

then

$$\begin{cases} \dot{x}_1 = x_2 + \beta_1 u \\ \dot{x}_2 = x_3 + \beta_2 u \\ \vdots \\ \dot{x}_{n-1} = x_n + \beta_{n-1} u \\ \\ \dot{x}_n = ? \quad ? \quad ? \end{cases}$$

How to find the relationship between $x'_n$ and other states: $x_1, x_2, \ldots, x_{n-1}$

? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ?

$$x_n = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \cdots - \beta_{n-2}\dot{u} - \beta_{n-1}u = \dot{x}_{n-1} - \beta_{n-1}u$$

Derivative of $x_n$: $\dot{x}_n = y^{(n)} - \beta_0 u^{(n)} - \beta_1 u^{(n-1)} - \cdots - \beta_{n-2}\ddot{u} - \beta_{n-1}\dot{u}$

Differential equation: $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_o u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$

$$y^{(n)} = -a_1 y^{(n-1)} - \cdots - a_{n-1}\dot{y} - a_n y + b_o u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

Substitute $y^{(n)}$ into $x'_n$:

$$\begin{aligned}
\dot{x}_n &= y^{(n)} - \beta_0 u^{(n)} - \beta_1 u^{(n-1)} - \cdots - \beta_{n-1}\dot{u} \\
&= (-a_1 y^{(n-1)} - a_2 y^{(n-2)} - \cdots - a_{n-1}\dot{y} - a_n y + b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u) \\
&\quad - \beta_0 u^{(n)} - \beta_1 u^{(n-1)} - \cdots - \beta_{n-1}\dot{u} \\
&= -a_1 y^{(n-1)} - a_2 y^{(n-2)} - \cdots - a_{n-1}\dot{y} - a_n y \\
&\quad + (b_0 - \beta_0)u^{(n)} + (b_1 - \beta_1)u^{(n-1)} + \cdots + (b_{n-1} - \beta_{n-1})\dot{u} + b_n u
\end{aligned}$$

state variables: $$\begin{cases} x_1 = y - \beta_0 u \\ x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u \\ x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u \\ \vdots \\ x_n = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \cdots - \beta_{n-2}\dot{u} - \beta_{n-1} u \end{cases}$$

derivatives of y: $$\begin{cases} y = x_1 + \beta_0 u \\ \dot{y} = x_2 + \beta_0 \dot{u} + \beta_1 u \\ \ddot{y} = x_3 + \beta_0 \ddot{u} + \beta_1 \dot{u} + \beta_2 u \\ \vdots \\ y^{(n-1)} = x_n + \beta_0 u^{(n-1)} + \beta_1 u^{(n-2)} + \cdots + \beta_{n-2}\dot{u} + \beta_{n-1} u \end{cases}$$

substitute output $y$ and it derivatives: $y'$,...$y^{(n-1)}$ into $x'_n$:

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_2 x_{n-1} - a_1 x_n$$
$$+(b_0 - \beta_0) u^{(n)} + (b_1 - \beta_1 - a_1 \beta_0) u^{(n-1)} + (b_2 - \beta_2 - a_1 \beta_1 - a_2 \beta_0) u^{(n-2)} + \cdots$$
$$+(b_n - a_1 \beta_{n-1} - a_2 \beta_{n-2} - \cdots - a_{n-1} \beta_1 - a_n \beta_0) u$$

Principle:

No derivative of input $u(t)$ contained in state equations.

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_2 x_{n-1} - a_1 x_n$$
$$+(b_0 - \beta_0) u^{(n)} + (b_1 - \beta_1 - a_1 \beta_0) u^{(n-1)} + (b_2 - \beta_2 - a_1 \beta_1 - a_2 \beta_0) u^{(n-2)} + \cdots$$
$$+(b_n - a_1 \beta_{n-1} - a_2 \beta_{n-2} - \cdots - a_{n-1} \beta_1 - a_n \beta_0) u$$

Principle:

No derivative of input $u(t)$ contained in state equations.

thus:
$$\begin{cases} \beta_0 = b_0 \\ \beta_1 = b_1 - a_1 \beta_0 \\ \beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0 \\ \vdots \\ \beta_n = b_n - a_1 \beta_{n-1} - \cdots - a_{n-1} \beta_1 - a_n \beta_0 \end{cases}$$

State-space of the system is:

$$\begin{cases} \dot{x}_1 = x_2 + \beta_1 u \\ \dot{x}_2 = x_3 + \beta_2 u \\ \vdots \\ \dot{x}_{n-1} = x_n + \beta_{n-1} u \\ \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots a_1 x_n + \beta_n u \end{cases}$$

$$\begin{cases} \beta_0 = b_0 \\ \beta_1 = b_1 - a_1\beta_0 \\ \beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 \\ \vdots \\ \beta_n = b_n - a_1\beta_{n-1} - \cdots - a_{n-1}\beta_1 - a_n\beta_0 \end{cases}$$

$$x_1 = y - \beta_0 u$$

Rewrite the system to the matrix representation:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu$$

$$y = C\mathbf{x} + Du$$

In which:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \vdots \\ \beta_n \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \mathbf{D} = \beta_0 = b_0$$

Ex.9-5 Assume the dynamic equation of a control system can be written as the differential equation:

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 2y = 11\dot{u} + 6u$$

try to give its state space description.

**Solution:** compare with the standard differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_o u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

we have:

$$a_1 = 6,\quad a_2 = 11,\quad a_3 = 2,$$
$$b_0 = 0,\quad b_1 = 0,\quad b_2 = 11,\quad b_3 = 6$$

$$x_1 = y - \beta_0 u$$
$$x_2 = \dot{x}_1 - \beta_1 u$$
$$x_3 = \dot{x}_2 - \beta_2 u$$

and the coefficients:

$$\beta_0 = b_0 = 0$$
$$\beta_1 = b_1 - a_1\beta_0 = 0$$
$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = 11$$
$$\beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 = -60$$

$$x_1 = y$$
$$x_2 = \dot{x}_1$$
$$x_3 = \dot{x}_2 - 11u$$

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 2y = 11\dot{u} + 6u$$

$$\boxed{\begin{aligned} \beta_0 &= b_0 = 0 \\ \beta_1 &= b_1 - a_1\beta_0 = 0 \\ \beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 = 11 \\ \beta_3 &= b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 = -60 \end{aligned}}$$

State equations are:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3 + 11u$$

$$\dot{x}_3 = -a_3x_1 - a_2x_2 - a_1x_3 + \beta_3u = -2x_1 - 11x_2 - 6x_3 - 60u$$

State space description of matrix:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 11 \\ -60 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\boxed{\begin{aligned} &\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \vdots \\ \beta_n \end{bmatrix} \\ &\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \mathbf{D} = \beta_0 = b_0 \end{aligned}}$$

The standard form:

$$y^{(n)}(t)+a_1y^{(n-1)}(t)+\cdots+a_{n-1}\dot{y}(t)+a_ny(t)$$
$$=b_0u^{(n)}+b_1u^{(n-1)}+\cdots+b_{n-1}u'+b_nu(t)$$

while if $b_0=0$, we can select another group of state variable as follow:

$$x_n=y$$
$$x_i=\dot{x}_{i+1}+a_{n-i}y-b_{n-i}u \qquad i=1,2,3,\cdots,n-1$$

$$\begin{aligned} x_{n-1}&=\dot{x}_n+a_1y-b_1u \\ x_{n-2}&=\dot{x}_{n-1}+a_2y-b_2u \\ &\vdots \\ x_2&=\dot{x}_3+a_{n-2}y-b_{n-2}u \\ x_1&=\dot{x}_2+a_{n-1}y-b_{n-1}u \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{x}_n&=x_{n-1}-a_1x_n+b_1u \\ \dot{x}_{n-1}&=x_{n-2}-a_2x_n+b_2u \\ &\vdots \\ \dot{x}_3&=x_2-a_{n-2}x_n+b_{n-2}u \\ \dot{x}_2&=x_1-a_{n-1}x_n+b_{n-1}u \end{aligned}$$

$\dot{x}_1$ ???

Output equation: $y = x_n$

Furthermore:

$$x_{n-1} = \dot{x}_n + a_1 y - b_1 u$$
$$= \dot{y} + a_1 y - b_1 u$$
$$x_{n-2} = \dot{x}_{n-1} + a_2 y - b_2 u$$
$$= \ddot{y} + a_1 \dot{y} - b_1 \dot{u} + a_2 y - b_2 u$$
$$\vdots$$
$$x_2 = \dot{x}_3 + a_{n-2} y - b_{n-2} u$$
$$= y^{(n-2)} + a_1 y^{(n-3)} - b_1 u^{(n-3)} + a_2 y^{(n-4)} - b_2 u^{(n-4)} + \cdots + a_{n-2} y - b_{n-2} u$$
$$x_1 = \dot{x}_2 + a_{n-1} y - b_{n-1} u$$
$$= y^{(n-1)} + a_1 y^{(n-2)} - b_1 u^{(n-2)} + a_2 y^{(n-3)} - b_2 u^{(n-3)} + \cdots + a_{n-1} y - b_{n-1} u$$

$$x_1 = \dot{x}_2 + a_{n-1}y - b_{n-1}u$$
$$= y^{(n-1)} + a_1 y^{(n-2)} - b_1 u^{(n-2)} + a_2 y^{(n-3)} - b_2 u^{(n-3)} + \cdots + a_{n-1}y - b_{n-1}u$$

calculate derivate of $x_1$ ⬇

$$\dot{x}_1 = y^{(n)} + a_1 y^{(n-1)} - b_1 u^{(n-1)} + a_2 y^{(n-2)} - b_2 u^{(n-2)} + \cdots + a_{n-1}\dot{y} - b_{n-1}\dot{u}$$

bring $y^{(n)}$ into $x'_1$ according to:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1}\dot{y}(t) + a_n y(t) = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

⬇

$$\dot{x}_1 = -a_n x_n + b_n u$$

Matrix description: $\dot{x} = \mathbf{A}x + \mathbf{b}u$

$$y = \mathbf{C}x + \mathbf{d}u$$

$$x_n = y$$

$$\dot{x}_n = x_{n-1} - a_1 x_n + b_1 u$$

$$\dot{x}_{n-1} = x_{n-2} - a_2 x_n + b_2 u$$

$$\vdots$$

$$\dot{x}_3 = x_2 - a_{n-2} x_n + b_{n-2} u$$

$$\dot{x}_2 = x_1 - a_{n-1} x_n + b_{n-1} u$$

$$\boldsymbol{\dot{x}_1 = -a_n x_n + b_n u}$$

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \qquad \mathbf{d} = \mathbf{0}$$

Ex.9-5(II) Differential equation of control system is

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 2y = 11\dot{u} + 6u$$

try to give its state space description.

Solution: standard differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_o u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

$$a_1 = 6, \quad a_2 = 11, \quad a_3 = 2,$$

$$b_0 = 0, \quad b_1 = 0, \quad b_2 = 11, \quad b_3 = 6$$

select state variable

$$\begin{aligned} x_3 &= y \\ x_2 &= \dot{x}_3 + a_1 y - b_1 u \\ x_1 &= \dot{x}_2 + a_2 y - b_2 u \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_3 &= y \\ x_2 &= \dot{x}_3 - 6y \\ x_1 &= \dot{x}_2 - 11y + 11u \end{aligned}$$

$$\dddot{y}+6\ddot{y}+11\dot{y}+2y=11\dot{u}+6u$$

$$\dot{x}_1=-2x_3-6u$$
$$\dot{x}_2=x_1-11x_3+11u$$
$$\dot{x}_1=x_2-6x_3$$

State space description:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 0 & -2\\ 1 & 0 & -11\\ 0 & 1 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}6\\ 11\\ 0\end{bmatrix}u$$

$$y=\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$A=\begin{bmatrix}0 & 0 & \cdots & 0 & -a_n\\ 1 & 0 & \cdots & 0 & -a_{n-1}\\ 0 & 1 & \cdots & 0 & -a_{n-2}\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & 1 & -a_1\end{bmatrix}\qquad \mathbf{b}=\begin{bmatrix}b_n\\ b_{n-1}\\ \vdots\\ b_1\end{bmatrix}$$

$$\mathbf{c}=\begin{bmatrix}0 & 0 & \cdots & 1\end{bmatrix}\qquad \mathbf{d}=\mathbf{0}$$

Conclusion: for a certain system, selection of state variables is not unique.

Scenario (1): No derivatives(微分) of input $u$ contained in n-order linear differential equations

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = bu$$

State space:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu$$

$$y = C\mathbf{x}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathrm{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ b \end{bmatrix}$$

$$C = [1 \;\; 0 \;\; 0 \;\; \ldots \;\; 0]$$

Scenario(2): Derivatives of input $u$ contained in n-order linear differential equation system

$$y^{(n)}+a_1 y^{(n-1)}+\cdots+a_{n-1}\dot{y}+a_n y=b_o u^{(n)}+b_1 u^{(n-1)}+\cdots+b_{n-1}\dot{u}+b_n u$$

$$\begin{cases}\beta_0=b_0\\ \beta_1=b_1-a_1\beta_0\\ \beta_2=b_2-a_1\beta_1-a_2\beta_0\\ \vdots\\ \beta_n=b_n-a_1\beta_{n-1}-\cdots-a_{n-1}\beta_1-a_n\beta_0\end{cases}$$

$$\dot{\boldsymbol{x}}=A\boldsymbol{x}+Bu$$
$$y=C\boldsymbol{x}+Du$$

$$x=\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}\quad \mathbf{A}=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \vdots\\ \beta_n\end{bmatrix}$$

$$\mathbf{C}=\begin{bmatrix}1 & 0 & 0 & \cdots & 0\end{bmatrix}\qquad \mathbf{D}=\beta_0=b_0$$

$$y^{(n)}(t)+a_1 y^{(n-1)}(t)+\cdots+a_{n-1}\dot{y}(t)+a_n y(t)$$
$$=b_0 u^{(n)}+b_1 u^{(n-1)}+\cdots+b_{n-1}u'+b_n u(t)$$

while if $b_0=0$, we can select another group of state variable as follow:

$$\dot{x}=\mathbf{A}x+\mathbf{b}u$$
$$y=\mathbf{C}x+\mathbf{d}u$$

$$A=\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_1 \end{bmatrix} \qquad \mathbf{b}=\begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_1 \end{bmatrix}$$

$$\mathbf{C}=\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \qquad \mathrm{d}=0$$

例：设系统微分方程为$\dddot{y}(t)+4\ddot{y}(t)+2\dot{y}(t)+y(t)=\ddot{u}(t)+\dot{u}(t)+3u(t)$

用上述两种方法建立状态空间表达式。

$$y^{(n)}+a_1 y^{(n-1)}+\cdots+a_{n-1}\dot{y}+a_n y=b_o u^{(n)}+b_1 u^{(n-1)}+\cdots+b_{n-1}\dot{u}+b_n u$$

**解：方法一，n=3, 而$b_0$=0， $b_1$=1， $b_2$=1， $b_3$=3，$a_1$=4， $a_2$=2， $a_3$=1**

$$\beta_0=b_0=0$$

$$\beta_1=b_1-a_1\beta_0=1$$

$$\beta_2=b_2-a_1\beta_1-a_2\beta_0=1-4\times 1=-3$$

$$\beta_3=b_3-a_1\beta_2-a_2\beta_1-a_3\beta_0=3+4*3-2*1=13$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1\\ -3\\ 13\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1\end{bmatrix}\qquad \mathbf{B}=\begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \vdots\\ \beta_n\end{bmatrix}$$

$$\mathbf{C}=\begin{bmatrix}1 & 0 & 0 & \cdots & 0\end{bmatrix}\qquad \mathbf{D}=\beta_0=b_0$$

$$\dddot{y}(t)+4\ddot{y}(t)+2\dot{y}(t)+y(t)=\ddot{u}+\dot{u}+3u(t)$$

方法二，因为**$b_3$=0**，所以可以用方法二，可直接根据微分方程的系数写出状态空间表达式。

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}0&0&-1\\1&0&-2\\0&1&-4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}3\\1\\1\end{bmatrix}u$$

$$y=\begin{bmatrix}0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$A=\begin{bmatrix}0&0&\cdots&0&-a_n\\1&0&\cdots&0&-a_{n-1}\\0&1&\cdots&0&-a_{n-2}\\\vdots&\vdots&&\vdots&\vdots\\0&0&0&1&-a_1\end{bmatrix}\qquad \mathbf{b}=\begin{bmatrix}b_n\\b_{n-1}\\\vdots\\b_1\end{bmatrix}$$

$$\mathbf{C}=\begin{bmatrix}0&0&\cdots&1\end{bmatrix}\qquad \mathbf{d}=\mathbf{0}$$

Ex.9-6 The equations of a 2input/2output system are:

$$\ddot{y}_1 + a_1\dot{y}_1 + a_2 y_2 = b_1\dot{u}_1 + b_2 u_1 + b_3 u_2$$

$$\dot{y}_2 + a_3 y_2 + a_4 y_1 = b_4 u_2$$

try to give its state space description.

Solution: find derivative of $y_1$, $y_2$ with highest-order

$$\ddot{y}_1 = -a_1\dot{y}_1 + b_1\dot{u}_1 - a_2 y_2 + b_2 u_1 + b_3 u_2$$

$$\dot{y}_2 = -a_3 y_2 - a_4 y_1 + b_4 u_2$$

calculate their integration

$$y_1 = \iint [(-a_1\dot{y}_1 + b_1\dot{u}_1) + (-a_2 y_2 + b_2 u_1 + b_3 u_2)]dt^2$$

$$= \int (-a_1 y_1 + b_1 u_1)dt + \iint (-a_2 y_2 + b_2 u_1 + b_3 u_2)dt^2$$

$$= \int [(-a_1 y_1 + b_1 u_1) + \int (-a_2 y_2 + b_2 u_1 + b_3 u_2)dt]dt$$

$$y_2 = \int [(-a_3 y_2 - a_4 y_1 + b_4 u_2)dt$$

Select state variables: $x_1 = y_1$

$x_2 = y_2$

$$\boxed{\begin{aligned} y_1 &= \int [(-a_1 y_1 + b_1 u_1) + \int (-a_2 y_2 + b_2 u_1 + b_3 u_2) dt] dt \\ y_2 &= \int [(-a_3 y_2 - a_4 y_1 + b_4 u_2) dt \end{aligned}}$$

From the equation of $y_1$:

$$\dot{x}_1 = -a_1 y_1 + b_1 u_1 + \int (-a_2 y_2 + b_2 u_1 + b_3 u_2) dt$$

$$= -a_1 x_1 + b_1 u_1 + \int (-a_2 x_2 + b_2 u_1 + b_3 u_2) dt$$

Select another state variable:

$$x_3 = \int (-a_2 x_2 + b_2 u_1 + b_3 u_2) dt$$

and $\dot{x}_3 = -a_2 x_2 + b_2 u_1 + b_3 u_2$

From the equation of $y_2$:

$$\dot{x}_2 = -a_3 x_2 - a_4 x_1 + b_4 u_2$$

The equation set:
$$\dot{x}_1 = -a x_1 + x_3 + b_1 u_1$$
$$\dot{x}_2 = -a_4 x_1 - a_3 x_2 + b_4 u_2$$
$$\dot{x}_3 = -a_2 x_2 + b_2 u_1 + b_3 u_2$$

$$\dot{x}_1 = -ax_1 + x_3 + b_1u_1$$
$$\dot{x}_2 = -a_4x_1 - a_3x_2 + b_4u_2$$
$$\dot{x}_3 = -a_2x_2 + b_2u_1 + b_3u_2$$

$$x_1 = y_1$$
$$x_2 = y_2$$

Rewrite the equations by the matrixes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 0 & 1 \\ -a_4 & -a_3 & 0 \\ 0 & -a_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 & 0 \\ 0 & b_4 \\ b_2 & b_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The output matrix equation:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

# 9.3 State-space Establishing of Linear System

**General Methodology:**

- From Physics Mechanism of System
- From Differential Equations of System
- From Transfer Functions of System
- From State-variable Diagram of System
- Linear Transformation of State space

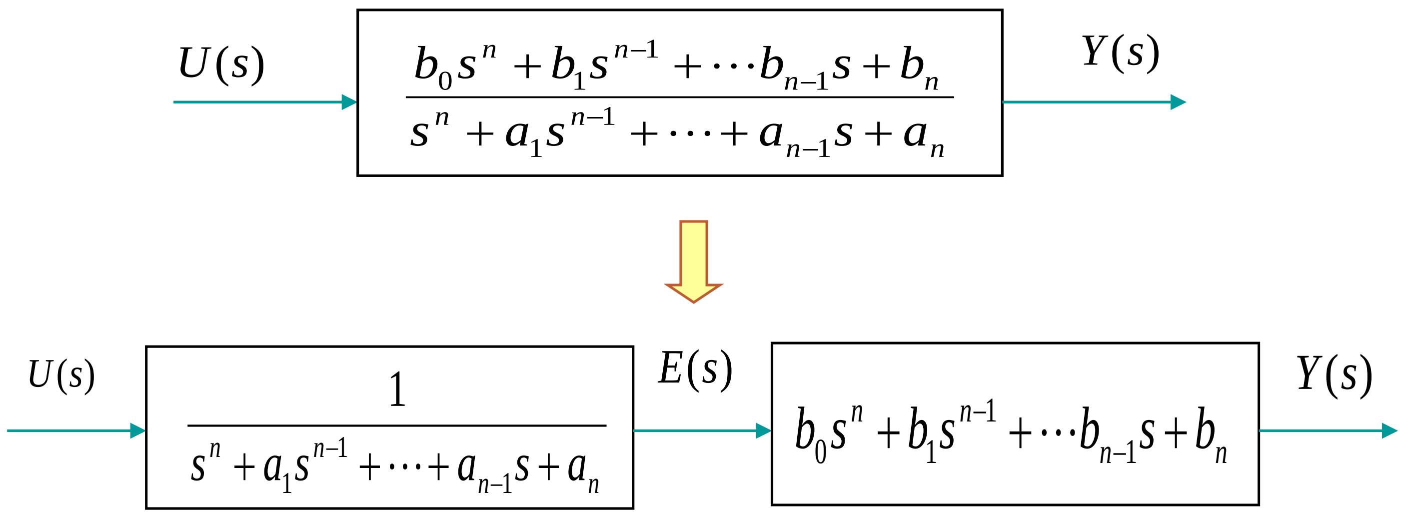
# 9.3.3 From Transfer Functions of System

- **Transfer Functions** ⇔ - **State Space**

## （1）Transfer Functions to State Space

**推导 1：**

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$



$$U(s) \longrightarrow \boxed{\dfrac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}} \xrightarrow{E(s)} \boxed{b_0 s^n + b_1 s^{n-1} + \cdots b_{n-1}s + b_n} \longrightarrow Y(s)$$

$$U(s) = (s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n)E(s)$$

$$Y(s) = (b_0 s^n + b_1 s^{n-1} + \cdots b_{n-1}s + b_n)E(s)$$

Select state variables: $\begin{cases} x_1 = e(t) \\ x_2 = \dot{e}(t) \\ \vdots \\ x_n = e^{(n-1)}(t) \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 = \dot{e}(t) \\ \dot{x}_2 = x_3 = \ddot{e}(t) \\ \vdots \\ \dot{x}_{n-1} = x_n = e^{(n-1)}(t) \\ \dot{x}_n = e^{(n)}(t) \end{cases}$

$$\Downarrow$$

$$u = \dot{x}_n + a_1 x_n + a_2 x_{n-1} + \cdots + a_{n-1}x_2 + a_n x_1$$

$$y = b_0 \dot{x}_n + b_1 x_n + b_2 x_{n-1} + \cdots + b_{n-1}x_2 + b_n x_1$$

$$\frac{Y(s)}{U(s)}=\frac{b_0s^n+b_1s^{n-1}+\cdots b_{n-1}s+b_n}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}$$

$$\begin{cases}\dot{x}_1=x_2=\dot{e}(t)\\ \dot{x}_2=x_3=\ddot{e}(t)\\ \vdots\\ \dot{x}_{n-1}=x_n=e^{(n-1)}(t)\\ \dot{x}_n=e^{(n)}(t)\end{cases}$$

$$u=\dot{x}_n+a_1x_n+a_2x_{n-1}+\cdots+a_{n-1}x_2+a_nx_1$$

$$y=b_0\dot{x}_n+b_1x_n+b_2x_{n-1}+\cdots+b_{n-1}x_2+b_nx_1$$

$$\dot{x}_n=-a_1x_n-a_2x_{n-1}-\cdots-a_{n-1}x_2-a_nx_1+u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}0&1&0&\cdots&0\\ 0&0&1&\cdots&0\\ \vdots&\vdots&\vdots&\cdots&\vdots\\ -a_n&-a_{n-1}&-a_{n-2}&\cdots&-a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \vdots\\ 1\end{bmatrix}u$$

A

B

$$b_0\dot{x}_n=b_0(-a_1x_n-a_2x_{n-1}-\cdots-a_{n-1}x_2-a_nx_1)+b_0u$$

$$y=b_0(-a_n\quad -a_{n-1}\quad\cdots\quad -a_1)\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}+b_0u+(b_n\quad b_{n-1}\cdots b_1)\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}$$

C

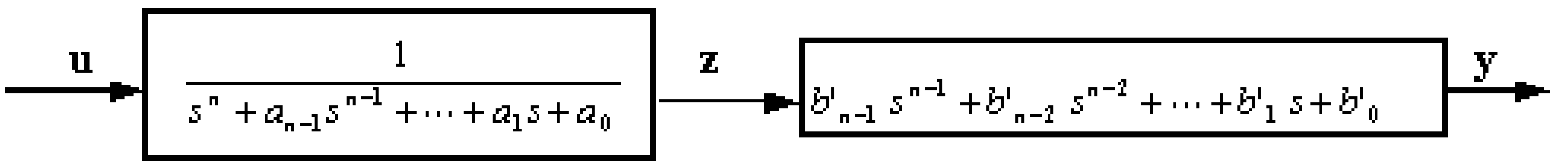
D

If $b_0$=0,the output equation will be simplified.

**推导 2:** $G(s)=\dfrac{Y(s)}{U(s)}=b_n+\dfrac{b'_{n-1}s^{n-1}+\cdots+b'_1 s+b'_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=b_n+G'(s)$

## A：可控标准型

可以将 $G'(s)$ 分解为两部分相串联的形式




$$z^{(n)}+a_{n-1}z^{(n-1)}+\cdots+a_1\dot{z}+a_0 z=u$$

$$y=b'_{n-1}z^{(n-1)}+\cdots+b'_1\dot{z}+b'_0 z$$

选取状态变量为

$$x_1=z,\ x_2=\dot{z},\ x_3=\ddot{z},\cdots,\ x_n=z^{(n-1)}$$

$$\begin{cases}\dot{x}_1=x_2=\dot{e}(t)\\ \dot{x}_2=x_3=\ddot{e}(t)\\ \vdots\\ \dot{x}_{n-1}=x_n=e^{(n-1)}(t)\\ \dot{x}_n=-a_1x_n-a_2x_{n-1}-\cdots-a_{n-1}x_2-a_nx_1+u\end{cases}$$

$$G(s)=\frac{Y(s)}{U(s)}=b_n+\frac{b'_{n-1}s^{n-1}+\cdots+b'_1 s+b'_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=b_n+G'(s)$$

而 $G(s)$ 对应的状态空间表达式为

$\mathbf{A_c}$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \vdots\\ \dot{x}_n(t)\end{bmatrix}=\boxed{\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\ \vdots&\vdots&\vdots&&\vdots\\0&0&0&\vdots&1\\-a_0&-a_1&-a_2&\cdots&-a_{n-1}\end{bmatrix}}\begin{bmatrix}x_1(t)\\x_2(t)\\ \vdots\\x_n(t)\end{bmatrix}+\boxed{\begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix}}u$$

$\mathbf{b_c}$

$$y=\begin{bmatrix}b'_0 & b'_1 & \cdots & b'_{n-1}\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\ \vdots\\x_n(t)\end{bmatrix}+b_n u$$

$$\begin{cases}\dot{x}_1=x_2=\dot{e}(t)\\ \dot{x}_2=x_3=\ddot{e}(t)\\ \vdots\\ \dot{x}_{n-1}=x_n=e^{(n-1)}(t)\\ \dot{x}_n=-a_1x_n-a_2x_{n-1}-\cdots-a_{n-1}x_2-a_nx_1+u\end{cases}$$

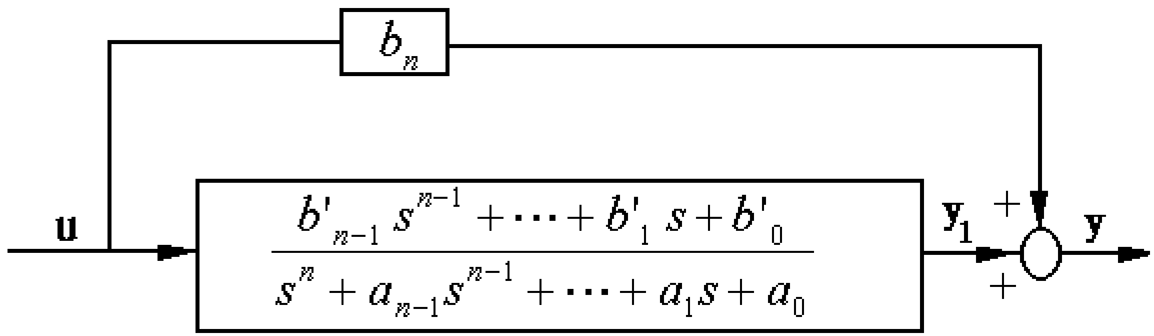
$$y=b'_{n-1}z^{(n-1)}+\cdots+b'_1\dot{z}+b'_0 z$$

**A阵为友矩阵，b向量最后一行为1，其他全为零。**

**可控标准型，用{$\mathbf{A_c}$，$\mathbf{b_c}$}表示**

$$G(s)=\frac{Y(s)}{U(s)}=b_n+\frac{b'_{n-1}s^{n-1}+\cdots+b'_1 s+b'_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=b_n+G'(s)$$

## B：可观测标准型




$$y_1^{(n)}(t)+a_{n-1}y_1^{(n-1)}(t)+\cdots+a_1\dot{y}_1(t)+a_0 y_1(t)$$
$$=b'_{n-1}u^{(n-1)}+\cdots+b'_1\dot{u}+b'_0 u(t)$$

按照微分方程求状态空间表达式的方法二选择状态变量，所不同的是系统状态变量与$y_1$有关。

$$x_n=y_1$$

$$x_i=\dot{x}_{i+1}+a_i y_1-b_i u \qquad i=1,2,3,\cdots,n-1$$

$$y=y_1(t)+b_n u$$

$$\dot{x}_n=x_{n-1}-a_1 x_n+b_1{}'u$$
$$\dot{x}_{n-1}=x_{n-2}-a_2 x_n+b_2{}'u$$
$$\vdots$$
$$\dot{x}_3=x_2-a_{n-2}x_n+b_{n-2}{}'u$$
$$\dot{x}_2=x_1-a_{n-1}x_n+b_{n-1}{}'u$$
$$\dot{x}_1=-a_n x_n+b_n{}'u$$

$$x_n=y$$

$$G(s)=\frac{Y(s)}{U(s)}=b_n+\frac{b'_{n-1}s^{n-1}+\cdots+b'_1s+b'_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}=b_n+G'(s)$$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \vdots\\ \dot{x}_n(t)\end{bmatrix}=\underbrace{\boxed{\begin{bmatrix}0&0&\cdots&0&-a_0\\ 1&0&\cdots&0&-a_1\\ 0&1&\cdots&0&-a_2\\ \vdots&\vdots&&\vdots&\vdots\\ 0&0&0&1&-a_{n-1}\end{bmatrix}}}_{\mathbf{A_o}}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}+\begin{bmatrix}b'_0\\ b'_1\\ \vdots\\ b'_{n-1}\end{bmatrix}u$$

$$y_1=\begin{bmatrix}0&\cdots&0&1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}$$

$$\begin{aligned}\dot{x}_n&=x_{n-1}-a_1x_n+b_1'u\\ \dot{x}_{n-1}&=x_{n-2}-a_2x_n+b_2'u\\ &\vdots\\ \dot{x}_3&=x_2-a_{n-2}x_n+b_{n-2}'u\\ \dot{x}_2&=x_1-a_{n-1}x_n+b_{n-1}'u\\ \dot{x}_1&=-a_nx_n+b_n'u\end{aligned}$$

$$x_n=y$$

对系统**G(s)**而言，所不同的仅仅是输出方程。

$$y=y_1+b_nu=\underbrace{\boxed{\begin{bmatrix}0&\cdots&0&1\end{bmatrix}}}_{\mathbf{c_o}}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}+b_nu$$

可观测标准型

用{$\mathbf{A_o}$，$\mathbf{c_o}$}表示

**Ex.9-7 Transfer function of a control system is:**

$$\frac{Y(s)}{U(s)}=\frac{s^2+4s+1}{s^3+9s^2+8s}$$

**transform it to the state space representation.**

**Solution:**

$$a_1=9,\ \ a_2=8,\ \ a_3=0,\ \ b_0=0,\ \ b_1=1,\ \ b_2=4,\ \ b_3=1$$

The state equation is:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -8 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$

The output equations is:

$$y=\begin{bmatrix}1 & 4 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

**课堂练习：**已知控制系统传递函数，试求该系统的状态空间描述。

$$G(s)=\frac{3s^5+6s^4-10s^3+6s+2}{2s^5+4s^4-8s^3+2s+2}$$

解：

例 已知控制系统传递函数，试求该系统的状态空间描述。

$$G(s)=\frac{3s^5+6s^4-10s^3+6s+2}{2s^5+4s^4-8s^3+2s+2}$$

解：

$$G(s)=1.5+\frac{s^3+1.5s-0.5}{s^5+2s^4-4s^3+s+1}$$

$$\frac{Y(s)}{U(s)}=1.5+\Delta$$

$$Y(s)=1.5U(s)+\Delta U(s)$$

$$a_0=1, a_1=1, a_2=0, a_3=-4, a_4=2,$$

$$b_0=-0.5, b_1=1.5, b_2=0, b_3=1, b_4=0,$$

$$\therefore \dot{X}=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\-1&-1&0&4&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}+\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}u$$

$$y=\begin{bmatrix}-0.5&1.5&0&1&0\end{bmatrix}X+1.5u$$

**推导 3:** $G(s) = \frac{Num(s)}{Den(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$ $\quad m \le n$

The Denominator: $Den(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$

is the Characteristic Equations of the system.

Assume there are n characteristics roots: $p_i, i = 1, 2, \cdots, n$

If the denominator of G(s) Den(s)=0 has no repeated roots:

G(s) can be decomposed by the summation of n fractions:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \cdots + \frac{c_n}{s - p_n}$$

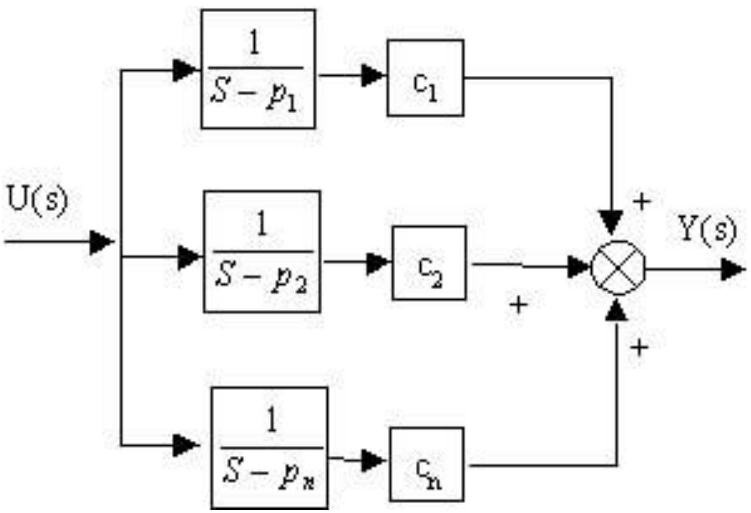
$$= \sum_{i=1}^{n} \frac{c_i}{s - p_i}$$

In which, $c_i = \lim_{s \to p_i} (s - p_i) G(s)$, is called Residue (留数) of pole $p_i$.

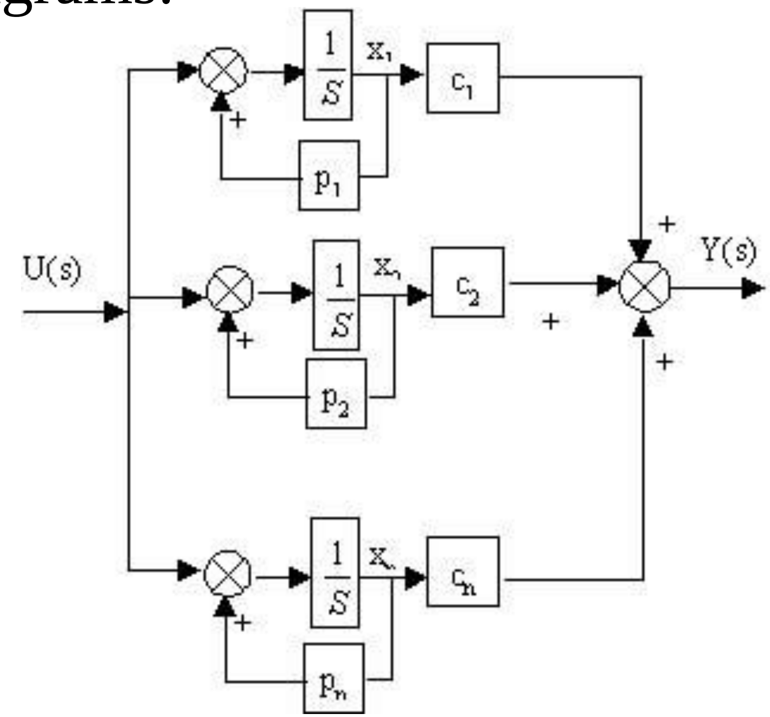
$$Y(s) = \frac{c_1}{s - p_1} U(s) + \frac{c_2}{s - p_2} U(s) + \cdots + \frac{c_n}{s - p_n} U(s)$$

$$= \sum_{i=1}^{n} \frac{c_i}{s - p_i} U(s)$$

The parallel connection diagrams:




(a) Parallel connection




(b) Parallel connection
(No repeated roots)

The state-equations of figure (b):

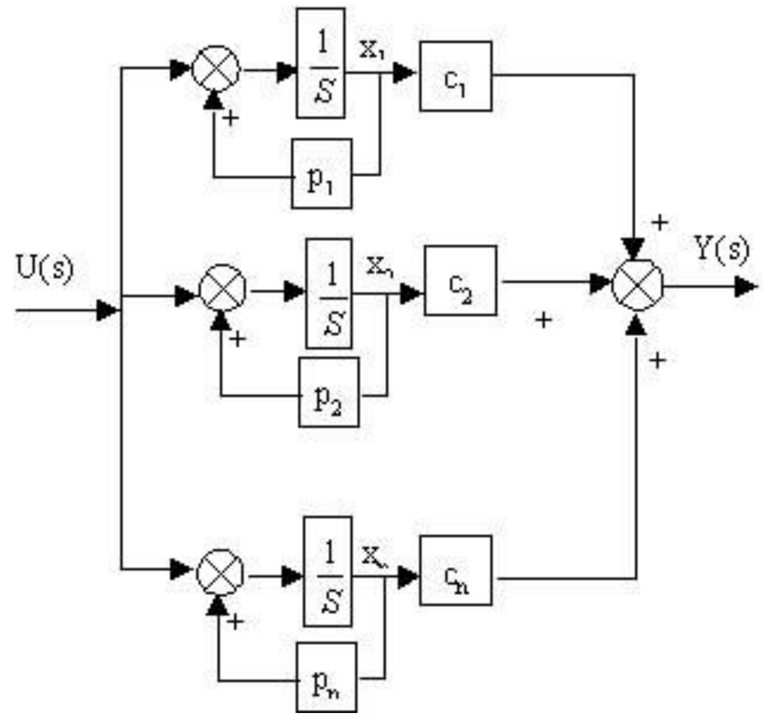
$$\begin{cases} \dot{x}_1 = p_1 x_1 + u \\ \dot{x}_2 = p_2 x_2 + u \\ \vdots \\ \dot{x}_n = p_n x_n + u \end{cases}$$

The output equation:

$$y = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

Matrix representation:

$$\begin{cases} \dot{X} = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & p_n \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \\ Y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} X \end{cases}$$




PS: the system matrix A is a diagonal matrix.

If the denominator of G(s) Den(s)=0 has repeated roots:

$$Den(s) = (s - p_1)^q (s - p_{q+1}) \cdots (s - p_n)$$

$s = p_1$ is the only q times repeated root, we have G(s):

$$G(s) = \frac{Num(s)}{Den(s)}$$

$$= \frac{c_{11}}{s - p_1} + \frac{c_{12}}{(s - p_1)^2} + \cdots + \frac{c_{1q}}{(s - p_1)^q} + \frac{c_{q+1}}{s - p_{q+1}} + \cdots + \frac{c_n}{s - p_n}$$

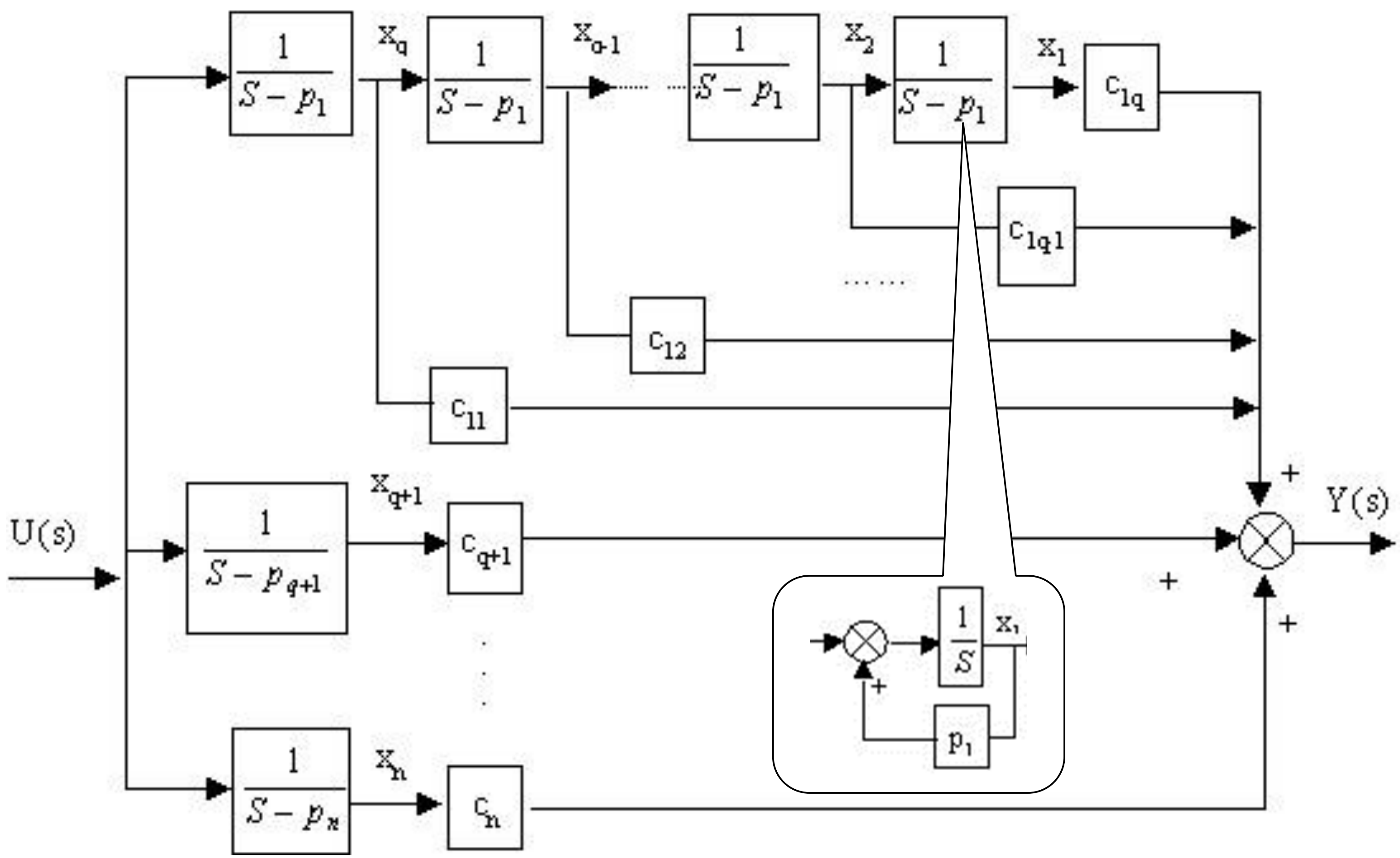
In which:

$$c_{1i} = \frac{1}{(q-i)!} \bullet \lim_{s \to p_1} \frac{d^{q-i}}{ds^{q-i}}[(s - p_1)^q G(s)] \qquad i = 1,2,\ldots,q$$

$$c_j = \lim_{s \to p_j} [(s - p_j) G(s)] \qquad j = q+1, q+2, \ldots, n$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{c_{11}}{s - p_1} + \frac{c_{12}}{(s - p_1)^2} + \cdots + \frac{c_{1q}}{(s - p_1)^q} + \frac{c_{q+1}}{s - p_{q+1}} + \cdots + \frac{c_n}{s - p_n}$$

$$Y(s) = \frac{c_{11}}{s-p_1}U(s) + \frac{c_{12}}{(s-p_1)^2}U(s) + \cdots + \frac{c_{1q}}{(s-p_1)^q}U(s) + \frac{c_{q+1}}{s-p_{q+1}}U(s) + \cdots + \frac{c_n}{s-p_n}U(s)$$

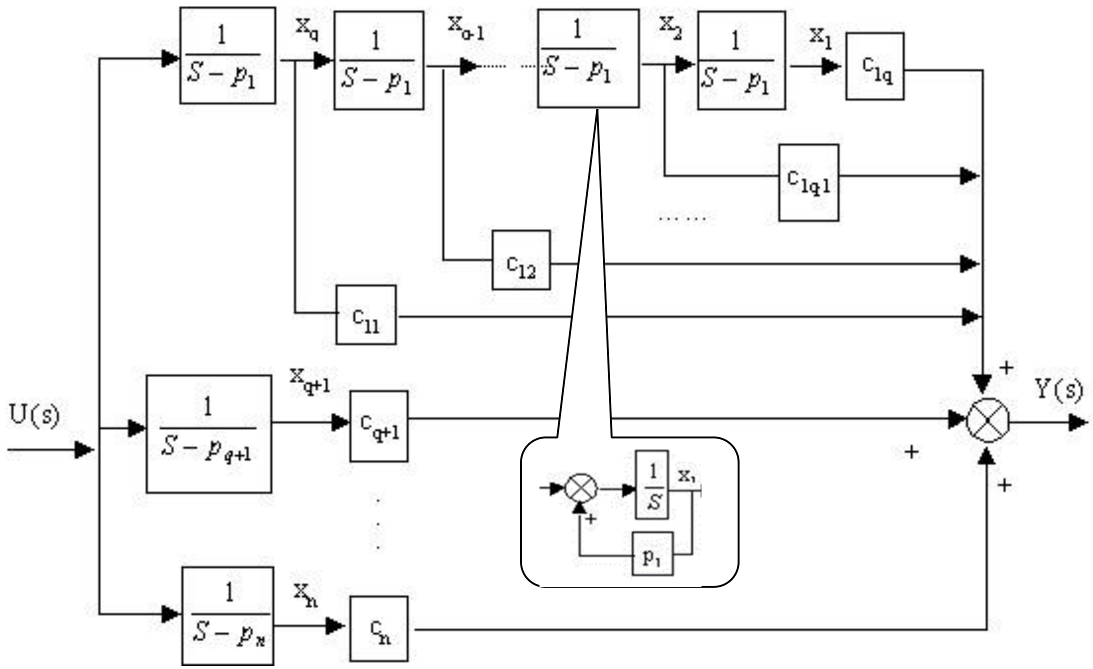



Parallel Connection (Repeated roots)

$$Y(s)=\frac{c_{11}}{s-p_1}U(s)+\frac{c_{12}}{(s-p_1)^2}U(s)+\cdots+\frac{c_{1q}}{(s-p_1)^q}U(s)+\frac{c_{q+1}}{s-p_{q+1}}U(s)+\cdots+\frac{c_n}{s-p_n}U(s)$$

Select the state-variables as the output of the integral items in the diagram:

$$\begin{cases}\dot{x}_1 = p_1 x_1 + x_2\\ \dot{x}_2 = p_1 x_2 + x_3\\ \vdots\\ \dot{x}_{q-1} = p_1 x_{q-1} + x_q\\ \dot{x}_q = p_1 x_q + u\\ \vdots\\ \dot{x}_n = p_n x_n + u\end{cases}$$




$$y = c_{1q}x_1 + c_{1q-1}x_2 + \cdots + c_{11}x_q + c_{q+1}x_{q+1} + \cdots + c_n x_n$$

$$\begin{cases} \dot{X} = \begin{bmatrix} p_1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & p_1 & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & p_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & p_{q+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & p_n \end{bmatrix} X + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} U \\ Y = \begin{bmatrix} c_{1q} & c_{1q-1} & \cdots & c_{11} & c_{q+1} & \cdots & c_n \end{bmatrix} X \end{cases}$$

$$\begin{cases} \dot{x}_1 = p_1 x_1 + x_2 \\ \dot{x}_2 = p_1 x_2 + x_3 \\ \vdots \\ \dot{x}_{q-1} = p_1 x_{q-1} + x_q \\ \dot{x}_q = p_1 x_q + u \\ \vdots \\ \dot{x}_n = p_n x_n + u \end{cases}$$

$$y = c_{1q} x_1 + c_{1q-1} x_2 + \cdots + c_{11} x_q + c_{q+1} x_{q+1} + \cdots + c_n x_n$$

System matrix A is Jordan Standard Form(约当标准型).

注意：

不是所有系统都可以建立对角阵标准型和约当标准型；

全实数单极点　　　　对角阵标准型

全实数有重极点　　　　约当标准型

对角阵标准型和约当标准型仅对系统矩阵**A**而言，

通常用$\Lambda$ 表示对角阵标准型的**A**阵；用**J**表示约当标准型

**Ex.9-8 find the parallel connection of the follow system:**

$$G(s)=\frac{4s^2+10s+5}{s^3+5s^2+8s+4}$$

Solution: the Denominator: $(s+2)^2(s+1)$

$$G(s)=\frac{4s^2+10s+5}{(s+2)^2(s+1)}=\frac{c_{11}}{s+2}+\frac{c_{12}}{(s+2)^2}+\frac{c_3}{s+1}$$

$$c_{12}=\lim_{s\to -2}(s+2)^2G(s)=-1$$

$$c_{11}=\frac{1}{(2-1)!}\lim_{s\to -2}\frac{d^{(2-1)}}{ds^{(2-1)}}[(s+2)^2G(s)]=\lim_{s\to -2}\frac{d}{dS}[\frac{4s^2+10s+5}{(s+1)}]=5$$

$$c_3=\lim_{s\to -1}(s+1)G(s)=\lim_{s\to -1}\frac{4s^2+10s+5}{(s+2)^2}=-1$$

The state equation of the system by parallel connection:

$$\begin{cases}\dot{X}=\begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 0\\ 0 & 0 & -1\end{bmatrix}X+\begin{bmatrix}0\\1\\1\end{bmatrix}u\\ y=\begin{bmatrix}-1 & 5 & -1\end{bmatrix}X\end{cases}$$

## 推导 4： Serialization of the Transfer Function

$$G(s)=\frac{Num(s)}{Den(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0} \qquad m\le n$$

The Numerator is: $Num=b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0$

The Denominator is: $Den=s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0$

If $z_1,z_2,\cdots,z_m$ are m zero points of G(s)，

and $p_1, p_2, \cdots, p_n$ are n pole points of G(s).

then G(s) is：

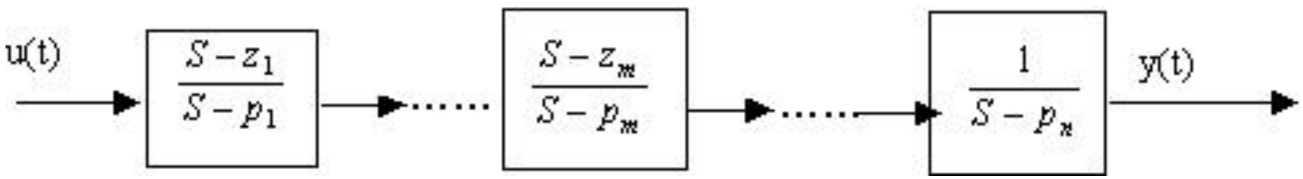
$$G(s)=\frac{b_m(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$$=\frac{s-z_1}{s-p_1}\bullet\frac{s-z_2}{s-p_2}\bullet\cdots\bullet\frac{s-z_m}{s-p_m}\bullet\frac{b_m}{s-p_{m+1}}\bullet\cdots\bullet\frac{1}{s-p_n}$$

Therefore the system is composed serially by n items below:

$$\frac{s-z_1}{s-p_1},\ \frac{s-z_2}{s-p_2},\ \cdots,\ \frac{1}{s-p_n}$$

The structure of the system can be described by figure (a)。

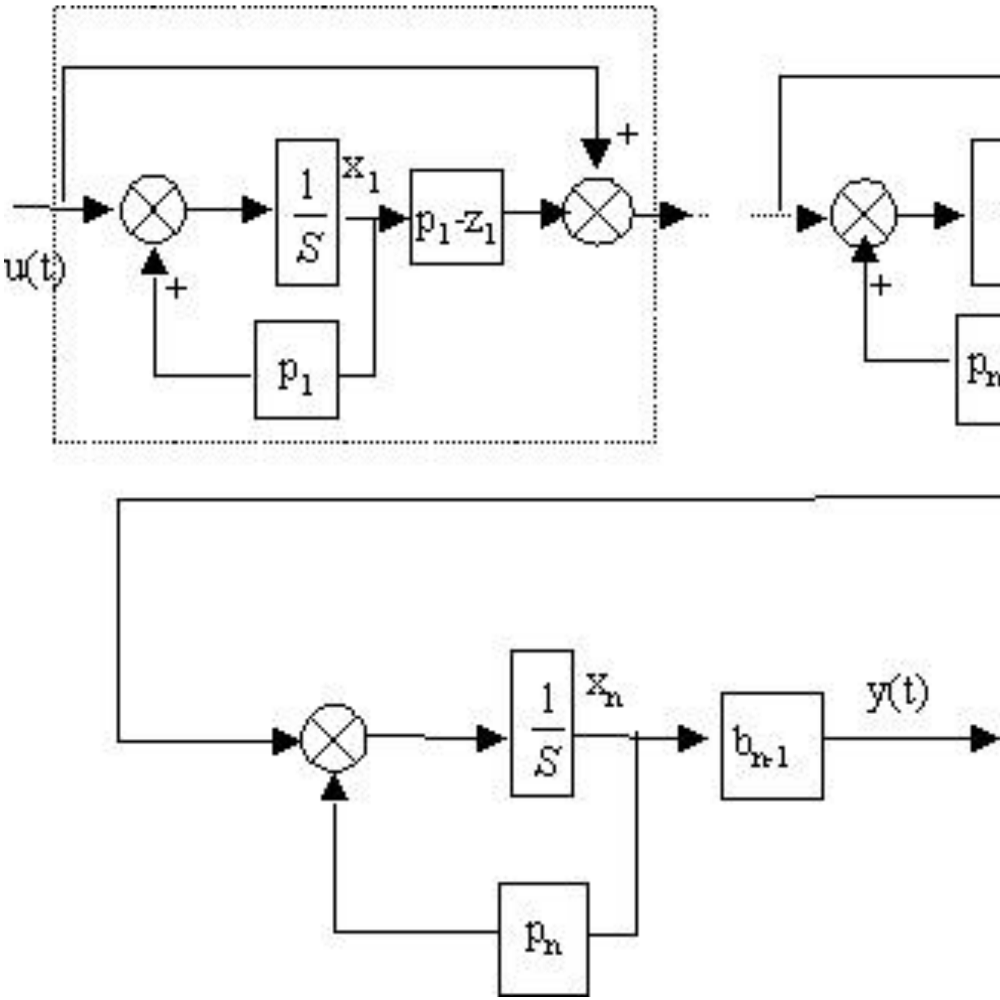



(a)

of which the first block is transformed as follow:

$$\frac{s-z_1}{s-p_1}=1+\frac{p_1-z_1}{s-p_1}=1+(p_1-z_1)\bullet\frac{\frac{1}{s}}{1-p_1\frac{1}{s}}$$

Thus its structure block will be recomposed as in figure (b).




$$1+(p_1-z_1)\bullet\frac{\frac{1}{s}}{1-p_1\frac{1}{s}}$$

m= n-1

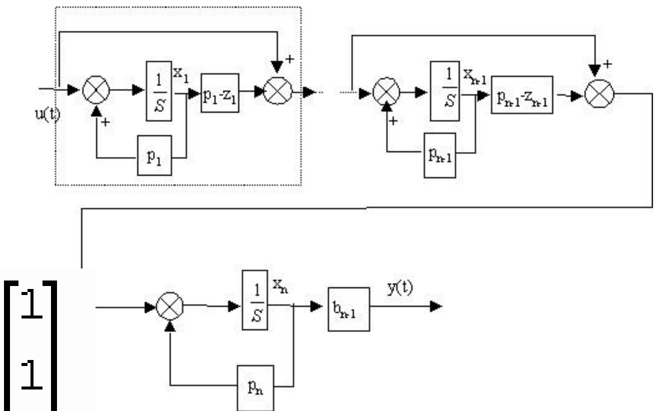
Assume the outputs of integral items are required state variables

The state equations of the system are:

$$G(s) = \frac{Num(s)}{Den(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$\begin{cases} \dot{x}_1 = p_1 x_1 + u \\ \dot{x}_2 = (p_1 - z_1)x_1 + u + p_2 x_2 = (p_1 - z_1)x_1 + p_2 x_2 + u \\ \vdots \\ \dot{x}_n = (p_1 - z_1)x_1 + (p_2 - z_2)x_2 + \cdots + (p_{n-1} - z_{n-1})x_{n-1} + p_n x_n + u \end{cases}$$

$$y = b_m x_n = b_{n-1} x_n, (m = n-1)$$




And the matrix representation:

$$\begin{cases} \dot{X} = \begin{bmatrix} p_1 & 0 & 0 & \cdots & 0 \\ p_1 - z_1 & p_2 & 0 & \cdots & 0 \\ p_1 - z_1 & p_2 - z_2 & p_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ p_1 - z_1 & p_2 - z_2 & p_3 - z_3 & \cdots & p_n \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \\ \\ y = \begin{bmatrix} 0 & 0 & \cdots & 0 & b_{n-1} \end{bmatrix} X \end{cases}$$

# 9.3.3 From Transfer Functions of System

**推导 1：**

$$\frac{Y(s)}{U(s)}=\frac{b_0s^n+b_1s^{n-1}+\cdots b_{n-1}s+b_n}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}$$

$$\dot{x}_n=-a_1x_n-a_2x_{n-1}-\cdots-a_{n-1}x_2-a_nx_1+u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \vdots\\ 1\end{bmatrix}u$$

B

A

C

$$y=b_0(-a_n\quad -a_{n-1}\quad \cdots\quad -a_1)\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}+b_0u+(b_n\quad b_{n-1}\cdots b_1)\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}$$

D

If $b_0$=0,the output equation will be simplified.

**推导 2:** $G(s)=\frac{Y(s)}{U(s)}=b_n+\frac{b'_{n-1}s^{n-1}+\cdots+b'_1 s+b'_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=b_n+G'(s)$

## A：可控标准型

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \vdots\\ \dot{x}_n(t)\end{bmatrix}=\underbrace{\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \vdots & 1\\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}\end{bmatrix}}_{\mathbf{A_c}}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}+\underbrace{\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}}_{\mathbf{b_c}}u$$

$$y=\begin{bmatrix}b'_0 & b'_1 & \cdots & b'_{n-1}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}+b_n u$$

**A阵为友矩阵，b向量最后一行为1，其他全为零。**

可控标准型，用**{$A_c$，$b_c$}**表示

**推导 2:** $$G(s)=\frac{Y(s)}{U(s)}=b_n+\frac{b'_{n-1}s^{n-1}+\cdots+b'_1s+b'_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}=b_n+G'(s)$$

## B：可观标准型

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \vdots\\ \dot{x}_n(t)\end{bmatrix}=\underbrace{\begin{bmatrix}0&0&\cdots&0&-a_0\\ 1&0&\cdots&0&-a_1\\ 0&1&\cdots&0&-a_2\\ \vdots&\vdots&&\vdots&\vdots\\ 0&0&0&1&-a_{n-1}\end{bmatrix}}_{A_o}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}+\begin{bmatrix}b'_0\\ b'_1\\ \vdots\\ b'_{n-1}\end{bmatrix}u$$

$$y=\underbrace{\begin{bmatrix}0&\cdots&0&1\end{bmatrix}}_{c_o}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}+b_nu$$

可观测标准型

用{$A_o$，$c_o$}表示

## 推导 3：

If the denominator of G(s) Den(s)=0 has no repeated roots:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \cdots + \frac{c_n}{s - p_n}$$

$$= \sum_{i=1}^{n} \frac{c_i}{s - p_i}$$

In which, $c_i = \lim_{s \to p_i} (s - p_i) G(s)$, is called Residue (留数) of pole $p_i$.

$$X_i(s) = \frac{1}{s - \lambda_i} U(s)$$

$$X_i(s) = \frac{c_i}{s - \lambda_i} U(s)$$

$$\begin{cases} \dot{X} = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & p_n \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} X \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} X$$

If the denominator of G(s) Den(s)=0 has repeated roots:

$$Den(s) = (s-p_1)^q (s-p_{q+1})\cdots(s-p_n)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{c_{11}}{s-p_1} + \frac{c_{12}}{(s-p_1)^2} + \cdots + \frac{c_{1q}}{(s-p_1)^q} + \frac{c_{q+1}}{s-p_{q+1}} + \cdots + \frac{c_n}{s-p_n}$$

$$c_{1i} = \frac{1}{(q-i)!} \bullet \lim_{s\to p_1} \frac{d^{q-i}}{ds^{q-i}}[(s-p_1)^q G(s)]$$ i =1,2,…,q

$$c_j = \lim_{s\to p_j}[(s-p_j)G(s)]$$ j=q+1,q+2,…,n

$$\begin{cases} \dot{X} = \begin{bmatrix} p_1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & p_1 & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & p_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & p_{q+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & p_n \end{bmatrix} X + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} U \\ Y = \begin{bmatrix} c_{1q} & c_{1q-1} & \cdots & c_{11} & c_{q+1} & \cdots & c_n \end{bmatrix} X \end{cases}$$

System matrix A is Jordan Standard Form(约当标准型).

**注意：**

不是所有系统都可以建立对角阵标准型和约当标准型；

| | |
|---|---|
| 全实数单极点 | 对角阵标准型 |
| 全实数有重极点 | 约当标准型 |

对角阵标准型和约当标准型仅对系统矩阵**A**而言，

通常用**Λ** 表示对角阵标准型的**A**阵；用**J**表示约当标准型

## 2. State Space to Transfer Functions

State Space (Dynamic Equations) VS Transfer Functions

- State Space denote both the input/output relationship and the internal state variables of the system;

  Transfer functions present the input/output relationship only.
- From Transfer Functions to State Space: system realization process, which is complicated and non-unique.

  From State Space to Transfer Functions: simple and unique process.

## ➢ For SISO system: State-space to TF

The State-space representation of a SISO system:

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \\ \boldsymbol{y} = C\boldsymbol{x} + Du \end{cases}$$

$x, \dot{x} \in R^{n\times 1}; A \in R^{n\times n}; B \in R^{n\times 1}; C \in R^{1\times n};$ $D$ is a scalar quantity.

Assume the initial condition zero-input, and use Laplace Transform:

$$sX(s) = AX(s) + BU(s) \qquad X(s) = (sI - A)^{-1}BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

The Transfer Function is:

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Ex.9-8 the state-space of the system is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find its transfer function:

Solution: write the related matrices [A ,B ,C] at first:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = [1 \quad 0]$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 1 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s(s+3)+1} \begin{bmatrix} s+3 & 1 \\ -1 & s \end{bmatrix}$$

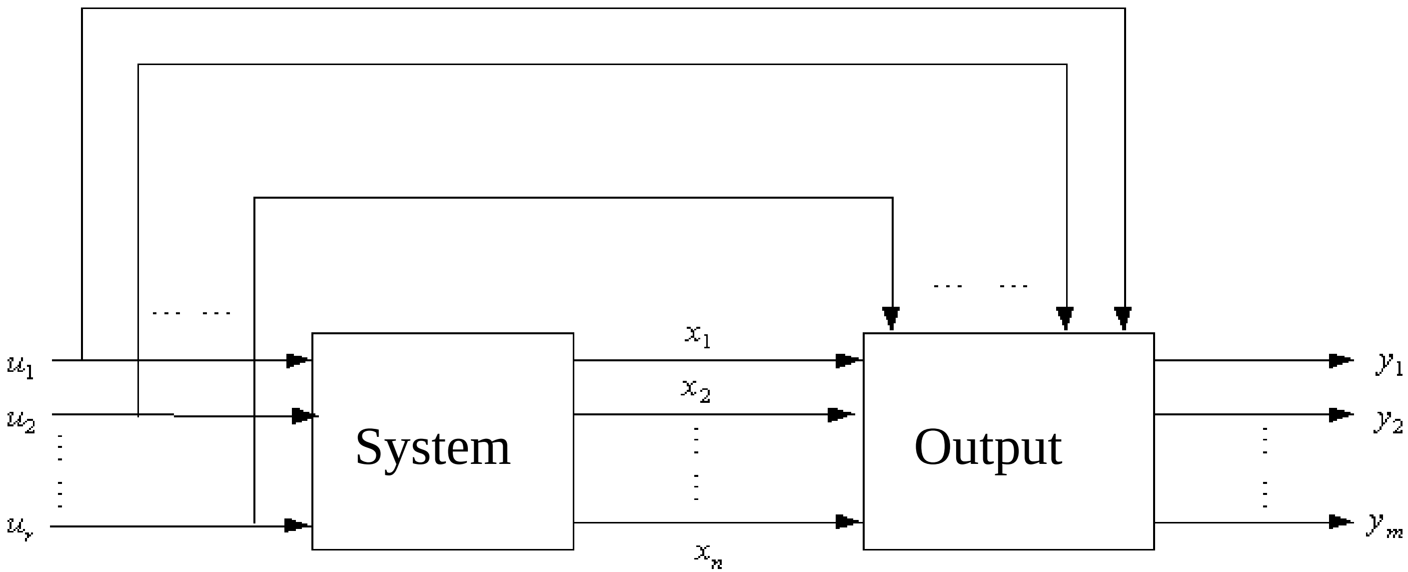
Transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{s+3}{s(s+3)+1} & \dfrac{1}{s(s+3)+1} \\ \dfrac{-1}{s(s+3)+1} & \dfrac{s}{s(s+3)+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s(s+3)+1} = \frac{1}{s^2 + 3s + 1}$$

# ➢ For MIMO system: State-space to TF




$u_1, u_2, \cdots, u_r$ —— input signals of the system

$y_1, y_2, \cdots, y_m$ —— output signals of the system

$x_1, x_2, \cdots, x_n$ —— state variables of the system

The dynamic equations: $\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} = C\boldsymbol{x} + D\boldsymbol{u} \end{cases}$

Equations have the same formation with SISO system, however, the matrices *B*, *C* ,*D* have different dimension.

$$\boldsymbol{x}, \dot{\boldsymbol{x}} \in \boldsymbol{R}^{n\times 1}, \quad \boldsymbol{U} \in \boldsymbol{R}^{r\times 1}, \quad \boldsymbol{Y} \in \boldsymbol{R}^{m\times 1},$$

$$\boldsymbol{A} \in \boldsymbol{R}^{n\times n}, \quad \boldsymbol{B} \in \boldsymbol{R}^{n\times r}, \quad \boldsymbol{C} \in \boldsymbol{R}^{m\times n}, \quad \boldsymbol{D} \in \boldsymbol{R}^{m\times r}$$

Laplace transformation:

$$\boldsymbol{G}(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

$G \in R^{m\times r}$ ———Transfer Function Matrix

**<u>Ex.9-9</u> The dynamic equations are:**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

**Find the Transfer function of the system.**

Solution: $A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $D = 0$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$G(s) = C(sI + A)^{-1} B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

## 课堂练习

可控标准型

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 3u$$

课堂练习

可控标准型

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 3u$$

$$G(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} 1 & 2 & -15 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 6 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 3$$

$$= \frac{3s^3 + 20s + 4}{s^3 + 5s^2 + 6s + 1}$$

可观测标准型

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -6 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 2 & -15\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+3u$$

$$G(s)=C(sI-A)^{-1}B+D=\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}s & 0 & 0\\ -1 & s & 6\\ 0 & -1 & s+5\end{bmatrix}^{-1}\begin{bmatrix}1\\ 2\\ -15\end{bmatrix}+3$$

$$=\frac{3s^3+20s+4}{s^3+5s^2+6s+1}$$

## ➢ the Eigen-equation and Eigenvalue of the system matrix A

The **Eigen-equation** of the system is:

$$\boxed{|\lambda I - A| = 0}$$

The **Eigenvalue** is one of its solutions.

Expand the equation: $|\lambda I - A| = 0$

we have the polynomial: $\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots a_n = 0$

Then solve the equation we have its n eigenvalues.

e.g. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$ then $|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{vmatrix}$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$= (\lambda+1)(\lambda+2)(\lambda+3) = 0$$

3 eigenvalues are －1、－2 and －3.

## ➢ the Eigenvector

The **eigenvectors** satisfy the equations:

$$\boxed{A\mathbf{p}_i = \lambda_i \mathbf{p}_i \ \text{ or } \ (\lambda_i I - A)\mathbf{p}_i = 0}$$

成立，则称***p***为矩阵**A**的关于特征值**λ**的特征向量.

例：求矩阵A的特征值和特征向量。 $A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}$

解： $f(\lambda)=|\lambda I-A|=\begin{vmatrix}\lambda & -1 & 0\\ 0 & \lambda & -1\\ 6 & 11 & \lambda+6\end{vmatrix}$

$$=\lambda^3+6\lambda^2+11\lambda+6=(\lambda+1)(\lambda+2)(\lambda+3)=0$$

-1、-2和-3为A的特征根。求$\lambda_1=-1$的特征向量$P_1$

$$\lambda_1 P_1=AP_1$$

令 $p_1=\begin{bmatrix}p_1\\ p_2\\ p_3\end{bmatrix}=\begin{bmatrix}p_1 & p_2 & p_3\end{bmatrix}^T$ $\begin{bmatrix}\lambda_1 p_1\\ \lambda_1 p_2\\ \lambda_1 p_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}\begin{bmatrix}p_1\\ p_2\\ p_3\end{bmatrix}$

$$\begin{bmatrix} \lambda_1 p_1 \\ \lambda_1 p_2 \\ \lambda_1 p_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$\begin{aligned} \lambda_1 p_1 &= p_2 \\ \lambda_1 p_2 &= p_3 \\ \lambda_1 p_3 &= -6p_1 - 11p_2 - 6p_3 \end{aligned}$$

$$\begin{aligned} p_1 + p_2 &= 0 \\ p_2 + p_3 &= 0 \\ 6p_1 + 11p_2 + 5p_3 &= 0 \end{aligned}$$

秩为2，有无穷组解，有一个自由变量，令

那么

$$\begin{aligned} p_1 &= 1 \\ p_2 &= -p_1 = -1 \\ p_3 &= -p_2 = 1 \end{aligned}$$

$$\mathrm{p}_1 = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

若令

$$\begin{aligned} p_1 &= -1 \\ p_2 &= -p_1 = 1 \\ p_3 &= -p_2 = -1 \end{aligned}$$

$$\mathrm{p}'_1 = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

显然$p_1$和$p'_1$线性相关，表明特征根对应一个独立特征向量。

类似的求$\lambda_2=-2$的特征向量$\boldsymbol{P_2}$

$$p_2=\begin{bmatrix}p_4\\p_5\\p_6\end{bmatrix}\qquad \begin{bmatrix}\lambda_2 p_4\\\lambda_2 p_5\\\lambda_2 p_6\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-6&-11&-6\end{bmatrix}\begin{bmatrix}p_4\\p_5\\p_6\end{bmatrix}\qquad \begin{aligned}2p_4+p_5&=0\\2p_5+p_6&=0\\6p_4+11p_5+4p_6&=0\end{aligned}$$

令 $p_4=1$

$$p_5=-2p_4=-2$$

$$p_6=-2p_5=4$$

$$p_2=\begin{bmatrix}p_4\\p_5\\p_6\end{bmatrix}=\begin{bmatrix}1\\-2\\4\end{bmatrix}$$

类似的求$\lambda_3=-3$的特征向量$\boldsymbol{P_3}$

$$p_3=\begin{bmatrix}p_7\\p_8\\p_9\end{bmatrix}\qquad \begin{bmatrix}\lambda_3 p_7\\\lambda_3 p_8\\\lambda_3 p_9\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-6&-11&-6\end{bmatrix}\begin{bmatrix}p_7\\p_8\\p_9\end{bmatrix}\qquad \begin{aligned}3p_7+p_8&=0\\3p_8+p_9&=0\\6p_7+11p_8+3p_9&=0\end{aligned}$$

令 $p_7=1$

$$p_8=-3p_7=-3$$

$$p_9=-3p_8=9$$

$$p_3=\begin{bmatrix}p_7\\p_8\\p_9\end{bmatrix}=\begin{bmatrix}1\\-3\\9\end{bmatrix}$$

# 小结： State Space to Transfer Functions

（1）The dynamic equations: $\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} = C\boldsymbol{x} + D\boldsymbol{u} \end{cases}$

$$\boldsymbol{G}(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

（2）The **Eigen-equation** of the system is:

$$|\lambda I - A| = 0$$

The **Eigenvalue** is one of its solutions.

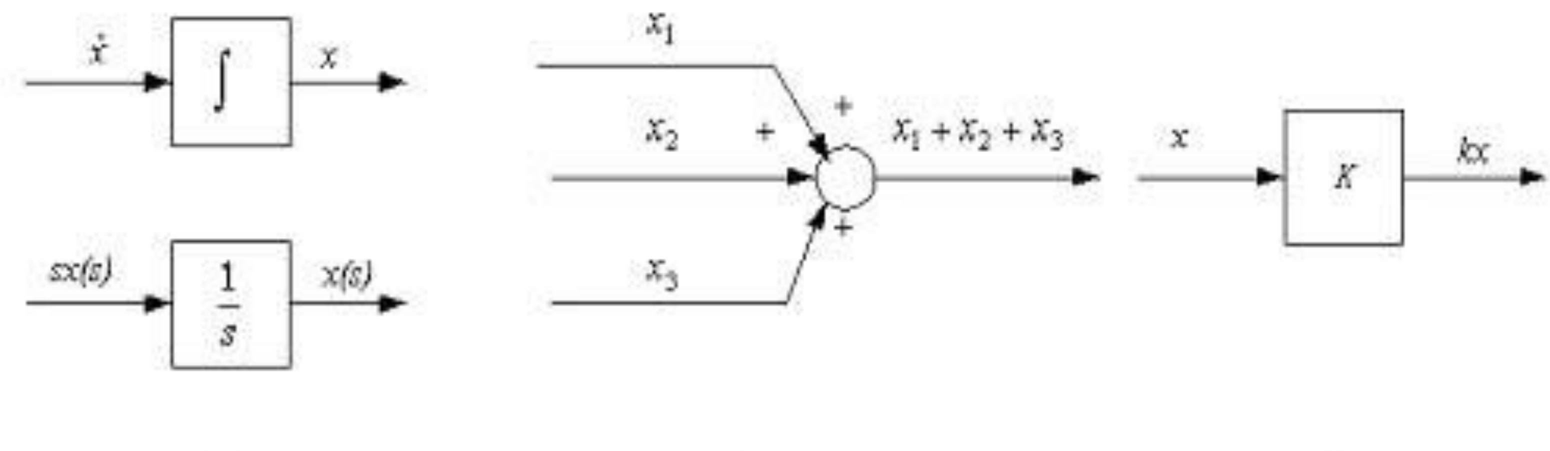
The **eigenvectors** satisfy the equations:

$$A\mathbf{p}_i = \lambda_i \mathbf{p}_i \quad \text{or} \quad (\lambda_i I - A)\mathbf{p}_i = 0$$

## 9.3.4 From State variable diagram of the System

**State variable diagram:** the diagram description of the relationship of state variables, which is composed by the integral items, proportion items and sum symbols.

✓ The output of each integral item is one of the state variables of the system.




(a) (b) (c)

**Ex.9-10** **The Close-loop TF of the system is:**

$$\frac{Y(s)}{U(s)}=\frac{s^2+3s+2}{s(s^2+7s+12)}$$

**Draw the State variable diagram and find the state space representation of the system.**

**Solution:** rewrite the close-loop TF of the system:

$$\frac{Y(s)}{U(s)}=\frac{s^2+3s+2}{s^3+7s^2+12s}$$

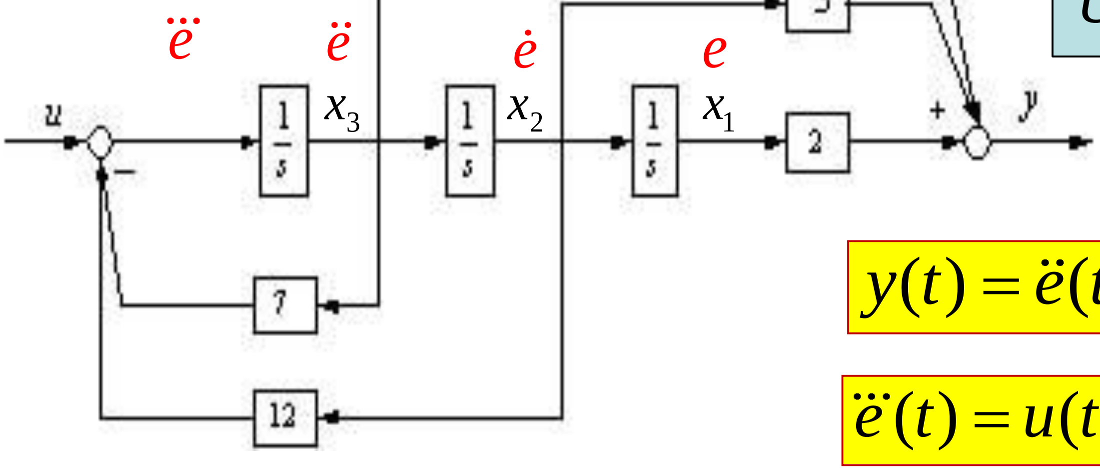
Assume:

$$Y(s)=(s^2+3s+2)E(s) \quad \Rightarrow \quad Y(s)=s^2E(s)+3sE(s)+2E(s)$$

$$\boxed{y(t)=\ddot{e}(t)+3\dot{e}(t)+2e(t)}$$

$$U(s)=(s^3+7s^2+12s)E(s) \quad \Rightarrow \quad S^3E(s)=U(s)-7s^2E(s)-12sE(s)$$

$$\boxed{\dddot{e}(t)=u(t)-7\ddot{e}(t)-12\dot{e}(t)}$$




$$\frac{Y(s)}{U(s)}=\frac{s^2+3s+2}{s^3+7s^2+12s}$$

$$y(t)=\ddot{e}(t)+3\dot{e}(t)+2e(t)$$

$$\dddot{e}(t)=u(t)-7\ddot{e}(t)-12\dot{e}(t)$$

$$\begin{cases}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -12 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u \\ y=[2\ 3\ 1]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\end{cases}$$

## 9.3.5 Linear Transformation of State space

- ✓ State-space equation establishing method review:

  Physics Mechanism / Differential Equations/ Transfer Functions / State-variable Diagram

- ✓ State-variables selection: Non-unique
- ✓ State-space equation: Non-unique
- ✓ The amount of the independent state variables in different state-space equation for a certain physics system: Uniform
- ✓ The connection of different state-space representations: **Linear Transformation**

## • Non-uniqueness of State-space Variables

Assume a State-space Equation:

$$\dot{x} = Ax + Bu$$

Linear Transform the state-variables: $\left(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n\right)$ into another group: $x_1, x_2, \ldots, x_n$

We have

$$x = P\bar{x}$$

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \qquad \begin{cases} x_1 = p_{11}\bar{x}_1 + p_{12}\bar{x}_2 + \cdots + p_{1n} \\ x_2 = p_{21}\bar{x}_1 + p_{22}\bar{x}_2 + \cdots + p_{2n} \\ \cdots\cdots \\ x_n = p_{n1}\bar{x}_1 + p_{n2}\bar{x}_2 + \cdots + p_{nn} \end{cases}$$

If P is a non-singular(非奇异) constant matrix: $|P| \neq 0$

The vector $\overline{x}$ is the state-variable vector of the system: $\dot{x} = Ax + Bu$

Proof: $\dot{\overline{x}} = P^{-1}AP\overline{x} + P^{-1}Bu$

Symbolize: $\overline{A} = P^{-1}AP, \quad \overline{B} = P^{-1}B$

then $\dot{\overline{x}} = \overline{A}\overline{x} + \overline{B}u$

Result: For any control system, selection of state variables is non-unique. Any state variables from linear transformation, which satisfy the **non-singular condition** of the transform matrix P, are the appropriate state variables of the system.

# Invariability(不变性) of linear transformation

System: $\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + Bu \\ \mathbf{y} = C\mathbf{x} + D\mathbf{u} \end{cases}$ If $\mathbf{x} = P\bar{\mathbf{x}}$ $\begin{cases} \dot{\bar{\mathbf{x}}} = P^{-1}AP\bar{\mathbf{x}} + P^{-1}B\mathbf{u} \\ \mathbf{y} = CP\bar{\mathbf{x}} + D\mathbf{u} \end{cases}$

(1) Invariability of eigenequations and eigenvalues

Eigenvalues after transformation:

Analysis:

$$\begin{aligned} \left|\lambda I - P^{-1}AP\right| &= \left|\lambda P^{-1}P - P^{-1}AP\right| = \left|P^{-1}\lambda P - P^{-1}AP\right| \\ &= \left|P^{-1}(\lambda I - A)P\right| = \left|P^{-1}\right|\left|\lambda I - A\right|\left|P\right| \\ &= \left|P^{-1}\right|\left|P\right|\left|\lambda I - A\right| = \left|P^{-1}P\right|\left|\lambda I - A\right| \\ &= \left|I\right|\left|\lambda I - A\right| = \left|\lambda I - A\right| \end{aligned}$$

Obviously, the eigenvalues of the system are same before and after the non-singular linear transformation.

System: $\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + Bu \\ \mathbf{y} = C\mathbf{x} + D\mathbf{u} \end{cases}$ If $\mathbf{x} = P\bar{\mathbf{x}}$ $\begin{cases} \dot{\bar{\mathbf{x}}} = P^{-1}AP\bar{\mathbf{x}} + P^{-1}B\mathbf{u} \\ \mathbf{y} = CP\bar{\mathbf{x}} + D\mathbf{u} \end{cases}$

(2) Invariant of system transfer function matrix

The transfer matrix after transformation:

$$\boldsymbol{G}(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

$$G'(s) = CP(sI - P^{-1}AP)^{-1}P^{-1}B + D$$

$$= CP(P^{-1}sIP - P^{-1}AP)^{-1}P^{-1}B + D$$

$$= CP\left[P^{-1}(sI - A)P\right]^{-1}P^{-1}B + D$$

$$= CPP^{-1}(sI - A)^{-1}PP^{-1}B + D$$

$$= C(sI - A)^{-1}B + D = G(s)$$

The transfer matrix of the system is invariant before and after the non-singular linear transformation.

## ✓Why need Linear Transformation?

Although there are infinite forms of state-space equations according to a certain system, which satisfy the non-singular linear transformation, only several kind of canonical forms are benefit for us:

- **Controllability Canonical Form**
- **Observability Canonical Form**
- **Diagonal Canonical Form**
- **Jordan Canonical Form.**

## ☐ Controllability Canonical Form(可控标准型)

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} c_n & c_{n-1} & \cdots & c_1 \end{bmatrix} x \end{cases}$$

Matrix A is also called companion matrix(友矩阵) .

## ☐ Observability Canonical Form(可观标准型)

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_1 \end{bmatrix} x + \begin{bmatrix} b_n \\ b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} x \end{cases}$$

## Diagonal Canonical Form (I)

Select state variables: $$X_i(s) = \frac{1}{s-\lambda_i}U(s)$$

System Output is: $$Y(s) = \sum_{i=1}^{n} c_i X_i(s)$$

$$\begin{cases} \dot{x} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} x \end{cases}$$

# Diagonal Canonical Form (II)

Select state variables: $$X_i(s) = \frac{c_i}{s-\lambda_i}U(s)$$

System Output is: $$Y(s) = \sum_{i=1}^{n} X_i(s)$$

Inverse Laplace Transformation:

$$\begin{cases} \dot{x} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} x \end{cases}$$

## Jordan Canonical Form

$$\begin{cases} \dot{x} = \left[\begin{array}{ccc|ccc} \lambda_1 & 1 & 0 & & & \\ & \lambda_1 & 1 & & 0 & \\ & & \lambda_1 & & & \\ \hline & & & \lambda_4 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_n \end{array}\right] x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_4 & \cdots & c_n \end{bmatrix} x \end{cases}$$

The Block of repeated poles $\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$ is called Jordan Block.

## (1) Outlines：

$$\left.\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}\right\} \xrightarrow[\text{Equivalent transforming}]{x = P\bar{x}} \left.\begin{aligned}\dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ \bar{y} &= \bar{C}\bar{x} + \bar{D}u = y\end{aligned}\right\}$$

Non-canonical form Canonical form

## (2) Relationship between the coefficient matrices

$$\because \quad x = P\bar{x}, \quad \dot{x} = P\dot{\bar{x}},$$

P is NxN non-singular constant matrix

Substitute into the equations:

$$\left.\begin{aligned}P\dot{\bar{x}} &= AP\bar{x} + Bu \\ y &= CP\bar{x} + Du\end{aligned}\right\} \Rightarrow \left.\begin{aligned}\dot{\bar{x}} &= P^{-1}AP\bar{x} + P^{-1}Bu \\ y &= CP\bar{x} + Du\end{aligned}\right\}$$

$$\Downarrow$$

$$\bar{A} = P^{-1}AP, \quad \bar{B} = P^{-1}B, \quad \bar{C} = CP, \quad \bar{D} = D$$

- The constraint satisfied systems {A,B,C,D} and { $\bar{A},\bar{B},\bar{C},\bar{D}$ } are called Similar Systems;
- The related dynamic equations are called Equivalent Dynamic Equations;
- The linear transformation is called Equivalent Transformation.

The purpose of non-singular transformation is to transform the system matrix $A$ into the canonical form $\bar{A}$

Some Common Linear transformation methods

## (1) Transform *A* to Diagonal Form

(a) Assume a square matrix ***A*** with n different real eigenvalues:

$\lambda_1, \cdots, \lambda_n$ which satisfy the following Eigen-equation.

$$\det(\lambda I - A) = |\lambda I - A| = 0$$

$$\Rightarrow \quad \bar{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

**The non-singular transformation matrix P is composed by real eigenvectors.**

$$\boxed{\mathbf{P} = [\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n]}$$

And the eigenvectors satisfy the equations:

$$\boxed{A\mathbf{p}_i = \lambda_i \mathbf{p}_i \text{ or } (\lambda_i I - A)\mathbf{p}_i = 0}$$

例:将下列状态方程化为对角阵标准型，并求变换后的状态方程。

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

解:

$$f(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{vmatrix} = (\lambda+1)(\lambda+2)(\lambda+3) = 0$$

$\lambda_1 = -1$、$\lambda_2 = -2$、$\lambda_3 = -3$ 为系统的特征根

$$\mathrm{p}_1 = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \mathrm{p}_2 = \begin{bmatrix} p_4 \\ p_5 \\ p_6 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \quad \mathrm{p}_3 = \begin{bmatrix} p_7 \\ p_8 \\ p_9 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \Lambda$$

$$\bar{B} = P^{-1}B = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$$

变换后的状态方程为

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} u$$

注意：求特征向量比较麻烦，对于每个特征根的特征向量不是唯一的，但相互之间是线性相关的，因此变换阵也不唯一。

(b) Assume a companion matrix(友矩阵) $A$ with n different real eigenvalues: $\lambda_1, \cdots, \lambda_n$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

Then the Vander-mode matrix(范德蒙特矩阵) P

$$P = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \\ \lambda_1^{n-1} & & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

Can transform *A* to the diagonal matrix:

$$\Rightarrow \quad \overline{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

例：若状态方程为 $\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-6&-11&-6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}u$

将其化为对角阵，求变换后的状态方程。

解：因为此时A阵为友矩阵，所以系统特征方程为

$$|\lambda I-A|=\lambda^3+6\lambda^2+11\lambda+6=(\lambda+1)(\lambda+2)(\lambda+3)=0$$

有三个特征值为 $\lambda_1=-1,\lambda_2=-2,\lambda_3=-3$

变换阵为

$$P=\begin{bmatrix}1&1&1\\\lambda_1&\lambda_2&\lambda_3\\\lambda_1^2&\lambda_2^2&\lambda_3^2\end{bmatrix}=\begin{bmatrix}1&1&1\\-1&-2&-3\\1&4&9\end{bmatrix}\qquad P^{-1}=\begin{bmatrix}3&2.5&0.5\\-3&-4&-1\\1&1.5&0.5\end{bmatrix}$$

$$\Lambda=P^{-1}AP=\begin{bmatrix}-1&&\\&-2&\\&&-3\end{bmatrix}$$

$$\bar{B}=P^{-1}B=\begin{bmatrix}0.5\\-1\\0.5\end{bmatrix}$$

$$\begin{bmatrix}\dot{\bar{x}}_1\\\dot{\bar{x}}_2\\\dot{\bar{x}}_3\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&-2&0\\0&0&-3\end{bmatrix}\begin{bmatrix}\bar{x}_1\\\bar{x}_2\\\bar{x}_3\end{bmatrix}+\begin{bmatrix}0.5\\-1\\0.5\end{bmatrix}u$$

(c) Assume matrix $A$ has m repeated eigenvalues: $\lambda_1 = \cdots = \lambda_m$ and other (n-m) different eigenvalues. If A still has m independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m$ while solving the equation:

$$A\mathbf{p}_i = \lambda_i \mathbf{p}_i \quad (i = 1 \sim m)$$

the matrix $A$ can be transformed to the diagonal form:

$$\Rightarrow \quad \bar{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & & & & & \mathbf{O} \\ & \ddots & & & & \\ & & \lambda_m & & & \\ & & & \lambda_{m+1} & & \\ & & & & \ddots & \\ \mathbf{O} & & & & & \lambda_n \end{bmatrix}$$

$$P = [\mathbf{p}_1, \cdots, \mathbf{p}_m \vdots \mathbf{p}_{m+1}, \cdots, \mathbf{p}_n]$$

**Ex.9-11: transform the follow state equation to the diagonal form**

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} u$$

**Solution:** characteristic equation

$$\det(\lambda I - A) = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -4 & -5 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 1 \end{vmatrix}$$

$$= (\lambda - 2)(\lambda - 1)^2 = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = \lambda_3 = 1$$

$$(\lambda_1 I - A)\mathbf{p_1} = 0 \quad \begin{bmatrix} 0 & -4 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix} = 0 \quad \mathbf{p_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(\lambda_2 I - A)\mathbf{p}_2 = 0 \quad \begin{bmatrix} -1 & -4 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{21} \\ p_{22} \\ p_{23} \end{bmatrix} = 0 \quad \mathbf{p}_2 = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$$

$$(\lambda_2 I - A)\mathbf{p}_3 = 0 \quad \begin{bmatrix} -1 & -4 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \end{bmatrix} = 0 \quad \mathbf{p}_3 = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore \quad P = [\mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3] = \begin{bmatrix} 1 & 4 & 5 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\overline{A} = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \overline{\mathbf{b}} = P^{-1}\mathbf{b} = \begin{bmatrix} 24 \\ -2 \\ -3 \end{bmatrix}$$

## (2) Transform *A* to Jordan form

(a) Assume matrix A has m repeated real eigenvalues:

$$\lambda_1 = \cdots = \lambda_m$$

others are (n-m) different real eigenvalues.

Then solve the equation: $A\mathbf{p}_i = \lambda_i \mathbf{p}_i \quad (i = 1 \sim m)$

and receive only one independent eigenvector: $p_1$

Matrix A can be transformed to Jordan Canonical Form only.

$$J = \overline{A} = P^{-1}AP = \left[\begin{array}{cccc|ccc} \lambda_1 & 1 & & & & & \\ & \lambda_1 & \ddots & & & 0 & \\ & & \ddots & 1 & & & \\ & & & \lambda_1 & & & \\ \hline & & & & \lambda_{m+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & \lambda_n \end{array}\right]$$

Jordan Block

$$P = [\mathbf{p}_1, \cdots, \mathbf{p}_m \vdots \mathbf{p}_{m+1}, \cdots, \mathbf{p}_n]$$

Here, $\mathbf{p}_2$, $\mathbf{p}_3$, …, $\mathbf{p}_m$ are generalized eigenvectors (广义特征向量) and satisfy:

$$[\mathbf{p}_1, \cdots, \mathbf{p}_m] \begin{bmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1 \end{bmatrix} = A[\mathbf{p}_1, \cdots, \mathbf{p}_m]$$

$$p_{j-1} + \lambda p_j = A p_j \quad (j=2 \ldots, m)$$

$\mathbf{p}_{m+1}$, …, $\mathbf{p}_n$ are the eigenvectors corresponding to the (n-m) different eigenvalues.

$\boxed{A\mathbf{p}_i = \lambda_i \mathbf{p}_i}$

$\boxed{p_{j-1} + \lambda p_j = A p_j}$

例 已知A阵为 $A = \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}$，求约当阵J。

解：先求 $\lambda_1 = \lambda_2 = 1$ 的特征向量，令特征向量为

$p_1 = \begin{bmatrix} p_{11} & p_{21} & p_{31} \end{bmatrix}^T$，代入 $\lambda_1 p_1 = A p_1$

有
$$\begin{aligned} 6p_{21} - 5p_{31} &= p_{11} \\ p_{11} + 2p_{31} &= p_{21} \\ 3p_{11} + 2p_{21} + 4p_{31} &= p_{31} \end{aligned}$$

化简为：
$$\begin{aligned} 5p_{11} &= -7p_{31} \\ 5p_{21} &= 3p_{31} \end{aligned}$$

取 $p_{11} = 7$ 则有 $p_1 = \begin{bmatrix} 7 & -3 & -5 \end{bmatrix}^T$。只有一个独立的特征向量。

求广义特征向量令 $p_2 = \begin{bmatrix} p_{12} & p_{22} & p_{32} \end{bmatrix}^T$代入下式 $p_1 + \lambda_1 p_2 = A p_2$

有
$$\begin{aligned} -p_{12} + 6p_{22} - 5p_{32} &= 7 \\ p_{12} - p_{22} + 2p_{32} &= -3 \\ 3p_{12} + 2p_{22} + 3p_{32} &= -5 \end{aligned}$$

化简为：
$$\begin{aligned} 5p_{22} - 3p_{32} &= 4 \\ 3p_{12} + 7p_{22} &= -1 \end{aligned}$$

取 $p_{32} = 2$ $\qquad p_2 = \begin{bmatrix} -5 & 2 & 2 \end{bmatrix}^T$

类似的求 $\lambda_3 = 2$, 根据 $\lambda_1 p_1 = Ap_1$ 求出特征向量 $p_3 = \begin{bmatrix} 2 & -1 & -2 \end{bmatrix}^T$

变换阵为 $P = \begin{bmatrix} 7 & -5 & 2 \\ -3 & 2 & -1 \\ -5 & 2 & -2 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 2 & 6 & -1 \\ 1 & 4 & -1 \\ -4 & -11 & 1 \end{bmatrix}$

$$J = P^{-1}AP = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) Assume companion matrix A, which is controllability canonical form matrix, and its m repeated eigenvalues:

$$\lambda_1 = \cdots = \lambda_m$$

There is only one independent real eigenvector:

$$\mathbf{p}_1 = [1\ \lambda_1\ \lambda_1^2 \cdots \lambda_1^{n-1}]^T$$

satisfy:

$$A\mathbf{p}_i = \lambda_i \mathbf{p}_i \quad (i = 1 \sim m)$$

$$P = [\mathbf{p_1}\ \frac{\partial \mathbf{p_1}}{\partial \lambda_1}\ \frac{\partial^2 \mathbf{p_1}}{\partial \lambda_1^2} \cdots \frac{\partial^{m-1} \mathbf{p_1}}{\partial \lambda_1^{m-1}} \vdots \mathbf{p_{m+1}} \cdots \mathbf{p_n}]$$

which can transform A to Jordan form.

(c) Assume a matrix A has 5 repeated eigenvalues:

$$\lambda_1 = \cdots = \lambda_5$$

Satisfy: $A\mathbf{p}_i = \lambda_i \mathbf{p}_i \quad (i = 1 \sim 5)$

With 2 independent real eigenvectors: $p_1$ and $p_2$.

Other (n-5) eigenvalues are different.

The matrix *A* can be transformed to the Jordan form:

$$J = \bar{A} = P^{-1}AP = \left[\begin{array}{ccc|cc|ccc} \lambda_1 & 1 & & & & & & \\ & \lambda_1 & 1 & & & & & \\ & & \lambda_1 & & & & & \\ \hline & & & \lambda_1 & 1 & & & \\ & & & & \lambda_1 & & & \\ \hline & & & & & \lambda_{m+1} & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_n \end{array}\right]$$

$$J = \bar{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & \mathbf{1} & & & & & & \\ & \lambda_1 & \mathbf{1} & & & & & \\ & & \lambda_1 & & & & & \\ & & & \lambda_1 & \mathbf{1} & & & \\ & & & & \lambda_1 & & & \\ & & & & & \lambda_{m+1} & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_n \end{bmatrix}$$

There are 2 upper Jordan blocks in *J*, in which:

$$P = \left[ \begin{array}{ccc:cc:ccc} p_1 & \dfrac{\partial p_1}{\partial \lambda_1} & \dfrac{\partial^2 p_1}{\partial \lambda_1^2} & p_2 & \dfrac{\partial p_2}{\partial \lambda_1} & p_6 & \cdots & p_n \end{array} \right]$$

**课堂练习:**将下列状态方程化为约当阵标准型。

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

课堂练习:将下列状态方程化为约当阵标准型。

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}u$$

解:

$$f(\lambda)=|\lambda I-A|=\begin{vmatrix}\lambda-1&0&0\\0&\lambda-1&0\\0&-1&\lambda-2\end{vmatrix}=(\lambda-1)^2(\lambda-2)=0$$

$\lambda_1=\lambda_2=1$、$\lambda_3=2$ 为系统的特征根.

求 $\lambda_1=\lambda_2=1$ 的特征向量 $p_1=\begin{bmatrix}p_{11} & p_{21} & p_{31}\end{bmatrix}^T$ $\lambda_1 p_1=Ap_1$

$$\begin{bmatrix}\lambda_1 p_{11}\\ \lambda_1 p_{21}\\ \lambda_1 p_{31}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&1&2\end{bmatrix}\begin{bmatrix}p_{11}\\p_{21}\\p_{31}\end{bmatrix}$$

$$p_{11}=p_{11}$$

$$p_{21}=p_{21}$$

$$p_{31}=p_{21}+2p_{31}$$

秩为1，有两个独立的特征向量, 令

$$p_{11}=1, p_{21}=0 \quad p_{31}=0 \qquad p_1=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$$

$$p_{11}=1, p_{21}=1 \quad p_{31}=-1 \qquad p'_1=\begin{bmatrix}1 & 1 & -1\end{bmatrix}^T$$

**2**重根有**2**个特征向量，所以不需要求广义特征向量。

求 $\lambda_3=2$ 的特征向量 $p_3=\begin{bmatrix}p_{13} & p_{23} & p_{33}\end{bmatrix}^T$

$$\begin{bmatrix}\lambda_3 p_{13}\\ \lambda_3 p_{23}\\ \lambda_3 p_{33}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}p_{13}\\ p_{23}\\ p_{33}\end{bmatrix}$$

$$2p_{13}=p_{13} \qquad p_{13}=0$$

$$2p_{23}=p_{23} \qquad p_{23}=0$$

$$2p_{33}=p_{23}+2p_{33} \qquad p_{33}=1\text{(任选的）}$$

$$p_3=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$$

$$P=\begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & -1 & 1\end{bmatrix}\quad P^{-1}=\begin{bmatrix}1 & -1 & 0\\0 & 1 & 0\\0 & 1 & 1\end{bmatrix}\qquad \overline{B}=P^{-1}B=\begin{bmatrix}1 & -1 & 0\\0 & 1 & 0\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$P^{-1}AP=\begin{bmatrix}1 & -1 & 0\\0 & 1 & 0\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{bmatrix}=J=\Lambda$$

有重根的情况下，有时也可能出现对角阵。

## (3) Transform the controllable system to Controllability Canonical Form

The controllability canonical form of a single input linear time-invariant system state equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

Defining the **controllability matrix S** is:

$$\boxed{S = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}}$$

- The corresponding controllability matrix S is a Right Lower Triangular matrix with the main diagonal elements 1：

$$S = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & \times & \times \\ 0 & 1 & -a_{n-1} & \cdots & \times & \times \\ 1 & -a_{n-1} & -a_{n-2} & \cdots & \times & \times \end{bmatrix}$$

- Thus det S $\neq$ 0, the system is controllable, and $A$, $b$ are called **controllability canonical form**.

Any controllable system, if its *A*, b are not controllability canonical form, they can be transformed to the canonical form by appropriate transforming method.

Assume a dynamic system: $\dot{x} = Ax + bu$

Execute the $P^{-1}$ transformation:

$$x = P^{-1} z$$

and we have $\dot{z} = PAP^{-1} z + Pbu$

Satisfy:

$$PAP^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad Pb = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Analyze the transformation matrix: P

$$PAP^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

Assume $P = \begin{bmatrix} p_1^T & p_2^T & \cdots & p_n^T \end{bmatrix}^T$

$$PAP^{-1} = A_c$$

Based on the matrix A, P should satisfy:

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix} A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix}$$

$$\begin{cases} p_1 A = p_2 \\ p_2 A = p_3 \\ \vdots \\ p_{n-1} A = p_n \\ p_n A = -a_0 p_1 - a_1 p_2 - \cdots - a_{n-1} p_n \end{cases}$$

$$\begin{cases} p_1 A = p_2 \\ p_2 A = p_3 \\ \vdots \\ p_{n-1} A = p_n \\ p_n A = -a_0 p_1 - a_1 p_2 - \cdots - a_{n-1} p_n \end{cases}$$

Then:

$$\begin{cases} p_1 A = p_2 \\ p_2 A = p_1 A^2 = p_3 \\ \vdots \\ p_{n-1} A = p_1 A^{n-1} = p_n \end{cases} \Rightarrow$$

Transformation matrix:

$$P = \begin{bmatrix} p_1^T & p_2^T & \cdots & p_n^T \end{bmatrix}^T$$

$$P = \begin{bmatrix} p_1 \\ p_1 A \\ \vdots \\ p_1 A^{n-1} \end{bmatrix}$$

Transformation matrix: $$P = \begin{bmatrix} p_1 \\ p_1 A \\ \vdots \\ p_1 A^{n-1} \end{bmatrix}$$

Then based on vector b, we have:

$$Pb = \begin{bmatrix} p_1 \\ p_1 A \\ \vdots \\ p_1 A^{n-1} \end{bmatrix} b = \begin{bmatrix} p_1 b \\ p_1 Ab \\ \vdots \\ p_1 A^{n-1} b \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$p_1 \begin{bmatrix} b & Ab & \cdots & A^{n-1} b \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$$

Thus $$p_1 = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} b & Ab & \cdots & A^{n-1} b \end{bmatrix}^{-1}$$

It seems that $p_1$ is the last row of the inverse controllability matrix

Therefore, the solution of transformation matrix $P^{-1}$:

(i) Find the controllability matrix $S = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$ ;

(ii) Find the inverse matrix $S^{-1}$, which is:

$$S^{-1} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix}$$

(iii) Take out the last row of $S^{-1}$, (the $n^{th}$ row) and compose the vector:

$$p_1 = \begin{bmatrix} S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix}$$

(iv) Construct matrix P

$$P = \begin{bmatrix} p_1 \\ p_1 A \\ \vdots \\ p_1 A^{n-1} \end{bmatrix}$$

(v) Then, $P^{-1}$ is required transforming matrix from non-canonical form to controllability canonical form.

## (4) Canonical Form of SISO system– From TF

The state space description of the dynamic system:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \qquad x = P\bar{x}$$

- ✓ According to transformation matrix P, we can transform the above system to the canonical forms we need.
- ✓ We can also consider about the transformation using the transfer functions.

The required transfer function is:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

The canonical state space forms of **controllability form**, **observability form** and **diagonal form** (or **Jordan form**) are given as follow.

Example Consider the following transfer function of a certain system:

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$$

Try to find its controllability canonical form, observability canonical form, and diagonal form.

**Solution:**

Its controllability canonical form is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [3 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Observability canonical form is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Diagonal canonical form is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [2 \quad -1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$