

Chapter 7 Analysis and Design of Linear Discrete-Time System (Sampled-data System)

7.1 Introduction

7.2 The Sampling Process and Sampling Theorem

7.3 Signal Recovery and Zero-Order Hold

7.4 Z-Transform and Inverse Z Transform

7.5 Mathematical Models of Discrete-Time Systems

7.6 Performance Analysis of Discrete-Time Systems

7.7 Digital Control Design for Discrete-Time Systems

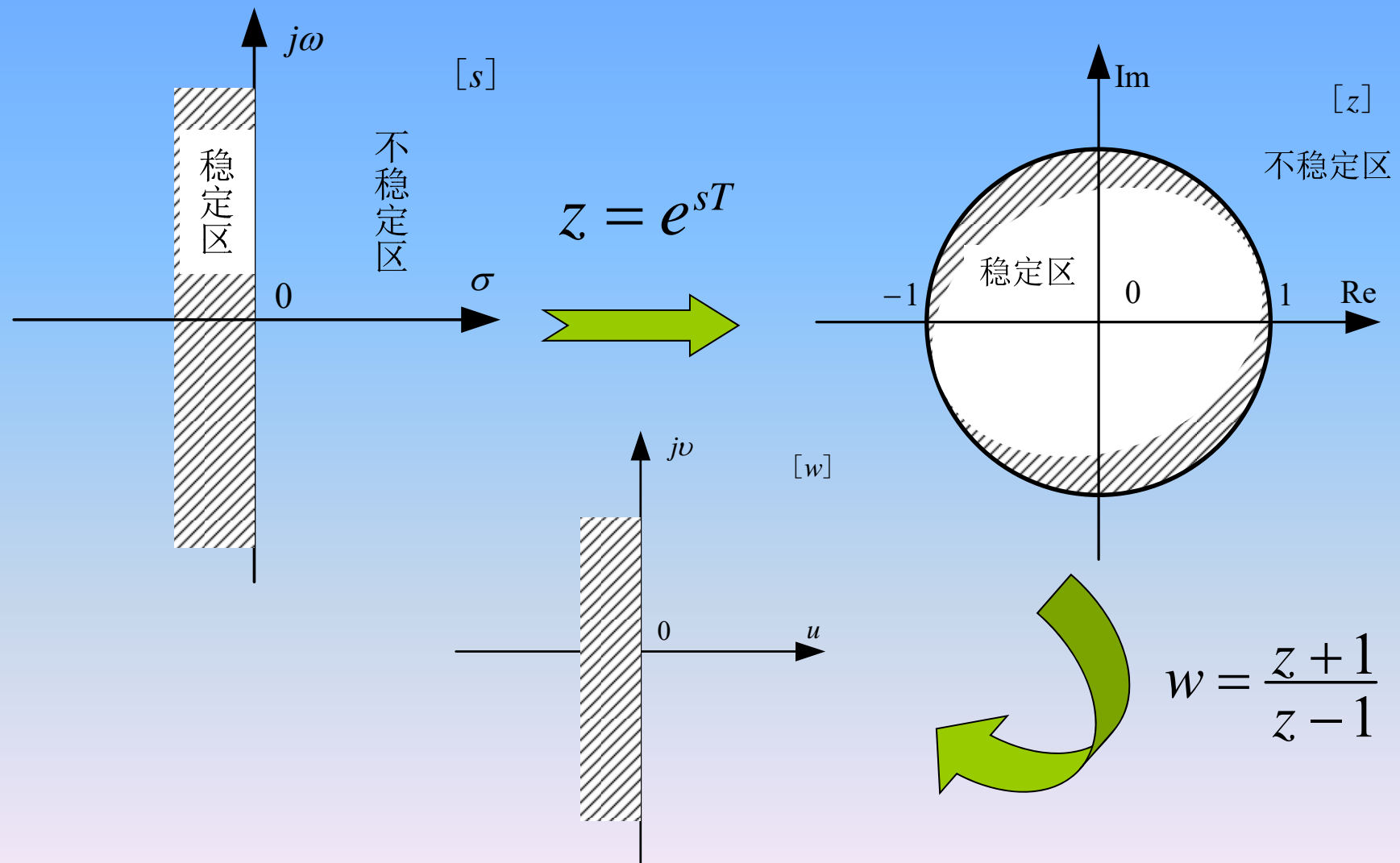
s-Domain to z-Domain Mapping

Necessary and Sufficient Condition for Stability of Linear Discrete-Time Systems

— All poles of $\Phi(z)$ lie in the unit circle of z plane

Routh criterion in w domain (Generalized Routh Criterion)

we've learned three methods to determine the stability of a discrete-time systems.



7.6 Performance Analysis of Discrete-Time Systems

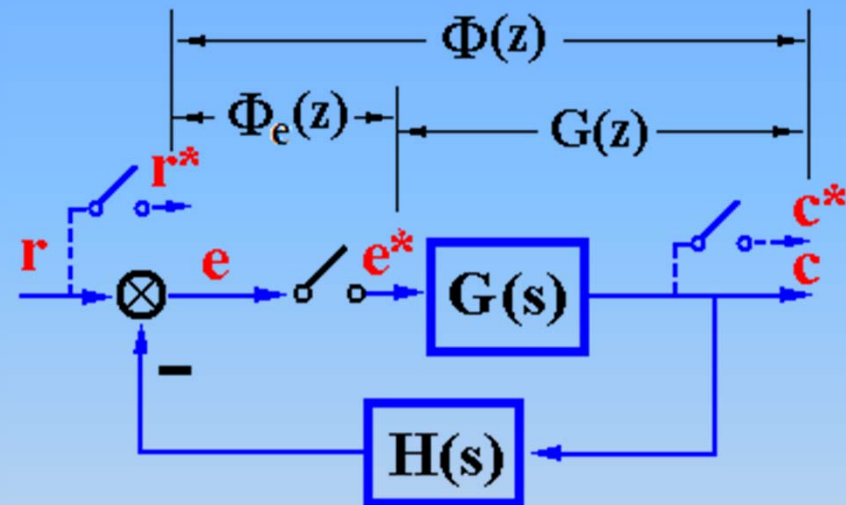
- Stability
- **Dynamic Performance**
- Steady-state Errors

7.6.2 Dynamic Performance Analysis of Discrete-Time Systems

1. General algorithm to obtain the dynamic performance

(1) Obtain the impulse transfer function

$$\text{Let } \begin{cases} GH(z) = Z[G(s)H(s)] \\ \Phi(z) = \frac{G(z)}{1 + GH(z)} = \frac{M(z)}{D(z)} \end{cases}$$

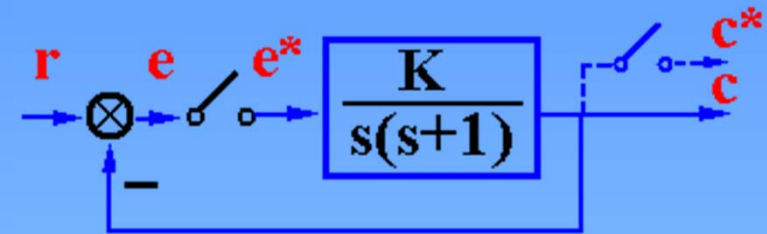


$$\begin{aligned} (2) \text{ Obtain } C(z) &= \Phi(z)R(z) = \frac{M(z)}{D(z)} \cdot \frac{z}{z-1} \\ &= c(0) + c(T)z^{-1} + c(2T)z^{-2} + \dots \end{aligned}$$

$$(3) \quad c^*(t) = c(0)\delta(t) + c(T)\delta(t-T) + c(2T)\delta(t-2T) + \dots$$

(4) Determine the specifications $\sigma\%$, t_s .

Example 1 Consider the system shown in the figure, $T=K=1$. Obtain the dynamic specifications. ($\sigma\%$, t_s).



Solution.
$$G(z) = Z \left[\frac{K}{s(s+1)} \right] = \frac{K(1-e^{-T})z}{(z-1)(z-e^{-T})}$$

$$\stackrel{K=T=1}{=} \frac{0.632z}{(z-1)(z-0.368)}$$

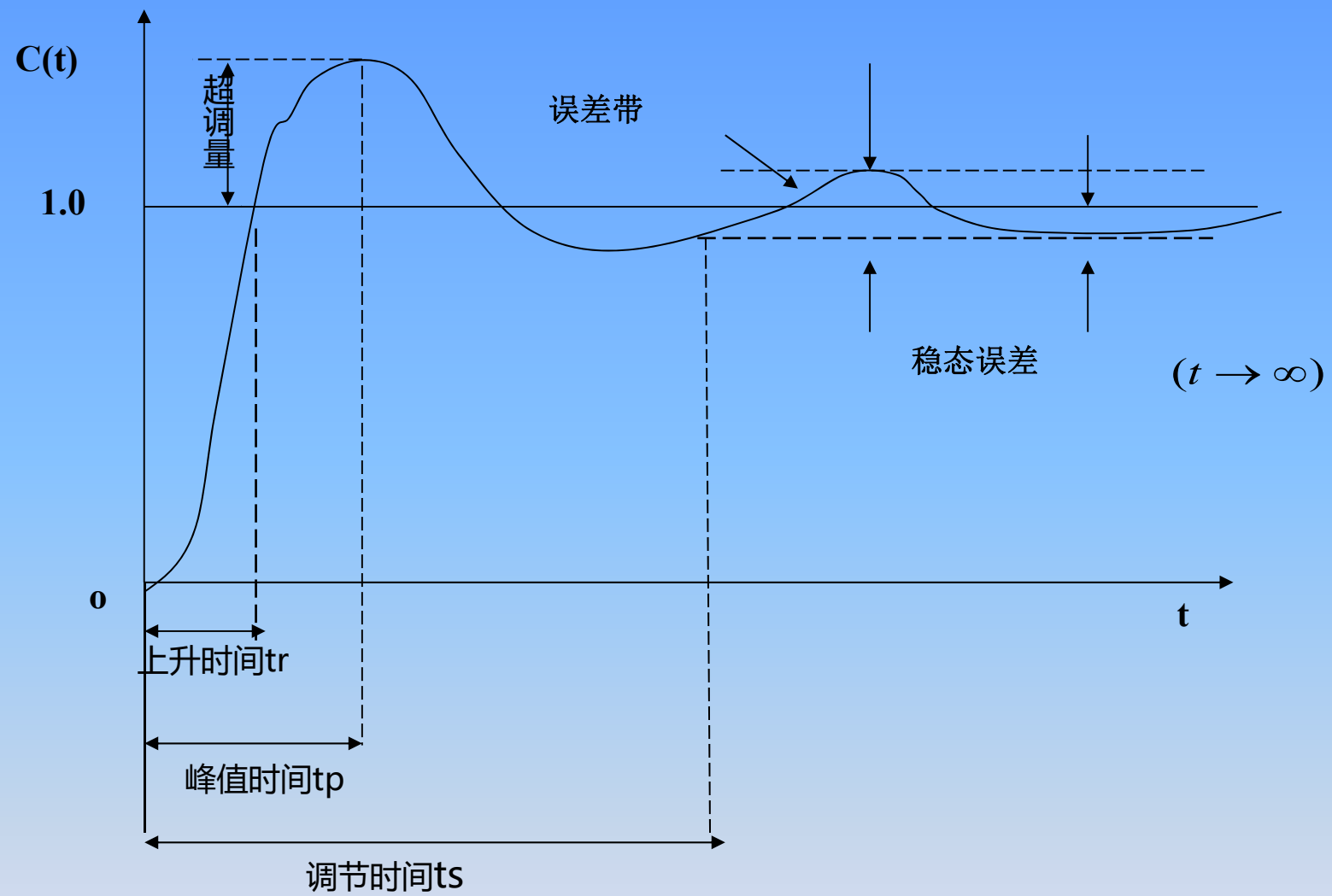
$$\Phi(z) = \frac{G(z)}{1+G(z)} = \frac{0.632z}{z^2 - 0.736z + 0.368}$$

$$c(\infty T) = \lim_{z \rightarrow 1} (z-1) \cdot \Phi(z) \cdot \frac{z}{z-1} = 1$$

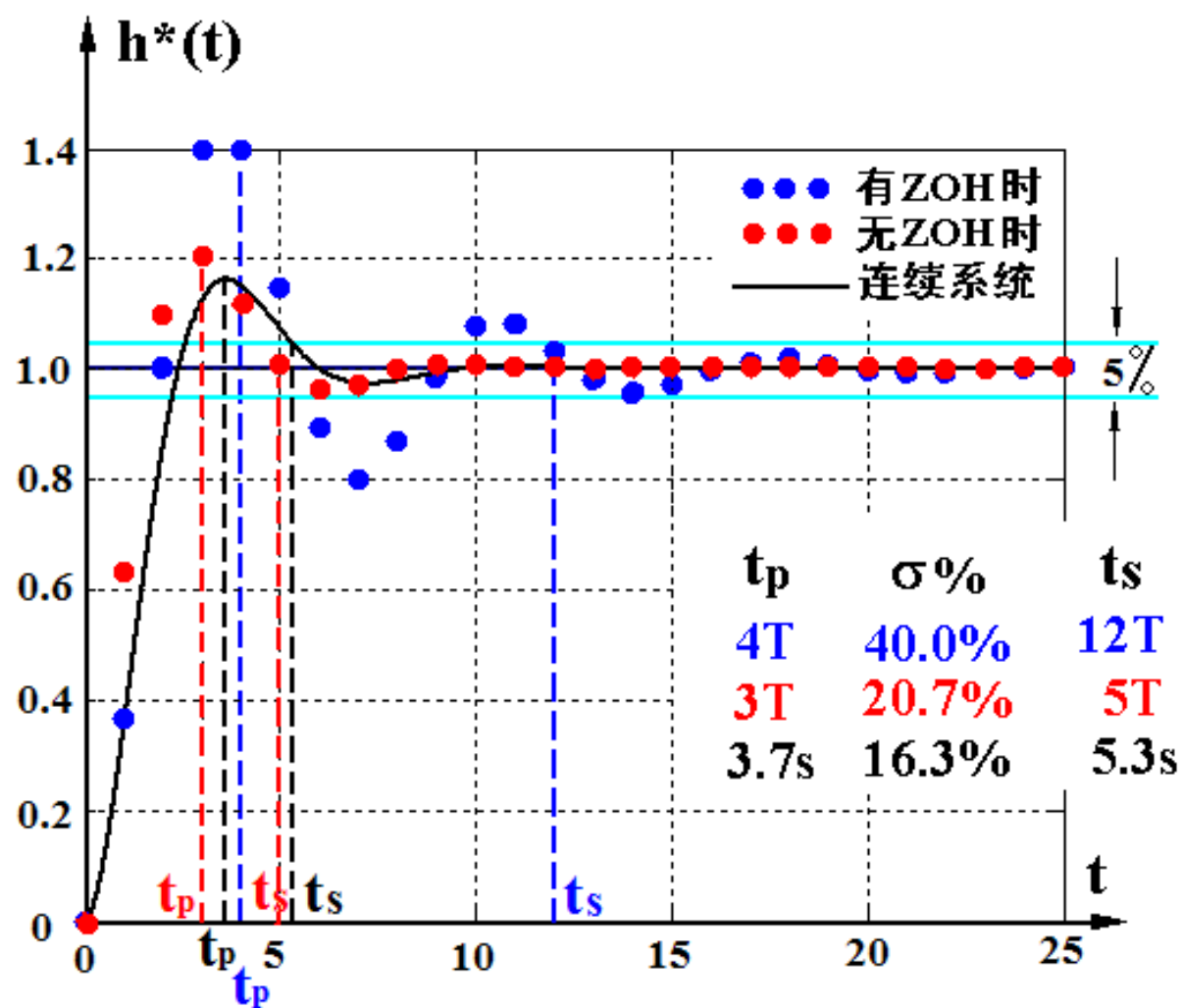
$$C(z) = \Phi(z) \cdot \frac{z}{z-1} = \frac{0.632z^2}{z^3 - 1.736z^2 + 1.104z - 0.368}$$

Obtain the unit step response series $h(k)$ by long division method.

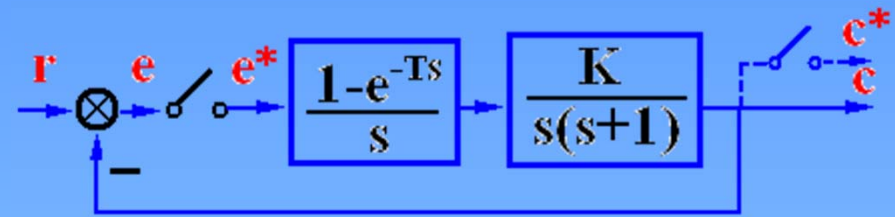
$$\begin{aligned} h(0) &= 0 \\ h(1) &= 0.632 \\ h(2) &= 1.097 \\ h(3) &= 1.207 \\ h(4) &= 1.117 \\ h(5) &= 1.014 \\ h(6) &= 0.964 \\ h(7) &= 0.970 \\ h(8) &= 0.991 \\ h(9) &= 1.004 \\ h(10) &= 1.007 \\ h(11) &= 1.003 \\ h(12) &= 1.000 \\ &\vdots \end{aligned} \quad \left\{ \begin{array}{l} t_p = 3T \\ \sigma\% = 20.7\% \\ t_s = 5T \end{array} \right.$$



控制系统性能指标



Example 1 Consider the system shown in the figure, $T=K=1$. Obtain the dynamic specifications. ($\sigma\%$, t_s).



Solution.
$$G(z) = K \frac{z-1}{z} Z \left[\frac{1}{s^2(s+1)} \right]$$

$$= K \frac{(T-1+e^{-T})z + (1-e^{-T}-Te^{-T})}{(z-1)(z-e^{-T})}$$

$$\stackrel{K=T=1}{=} \frac{0.368z + 0.264}{(z-1)(z-0.368)}$$

$$\Phi(z) = \frac{G(z)}{1+G(z)} = \frac{0.368z + 0.264}{z^2 - z + 0.632}$$

$$c(\infty T) = \lim_{z \rightarrow 1} (z-1) \cdot \Phi(z) \cdot \frac{z}{z-1} = 1$$

$$C(z) = \Phi(z) \cdot \frac{z}{z-1} = \frac{(0.368z + 0.264)z}{z^3 - 2z^2 + 1.632z - 0.632}$$

$$h(0)=0$$

$$h(1)=0.3679$$

$$h(2)=1.0000$$

$$h(3)=1.3996$$

$$h(4)=1.3996$$

$$h(5)=1.1470$$

$$h(6)=0.8944$$

$$h(7)=0.8015$$

$$h(8)=0.8682$$

$$h(9)=0.9937$$

$$h(10)=1.0770$$

$$h(11)=1.0810$$

$$h(12)=1.0323$$

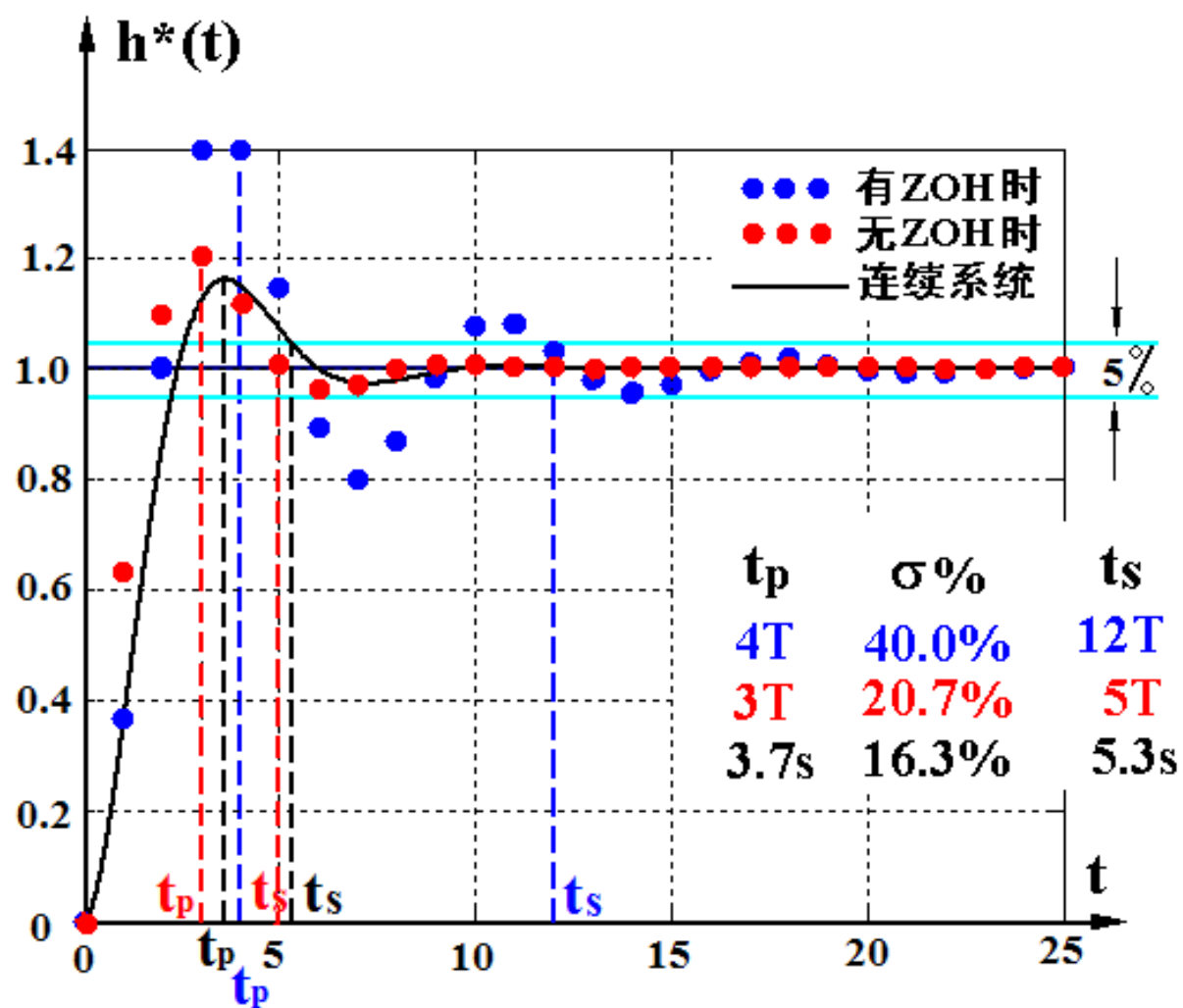
$$h(13)=0.9811$$

$$h(14)=0.9607$$

\vdots

$$\left\{ \begin{array}{l} t_p = 4T \\ \sigma\% = 40\% \end{array} \right.$$

$$t_s = 12T$$



2. Relationship between dynamic response and closed-loop poles

$$\Phi(z) = \frac{M(z)}{D(z)} = \frac{b_m \prod_{i=1}^m (z - z_i)}{a_n \prod_{k=1}^n (z - p_k)} \quad m \leq n$$

$$\begin{aligned} C(z) &= \Phi(z)R(z) = \frac{M(z)}{D(z)} \cdot \frac{z}{z-1} \\ &= \frac{M(1)}{D(1)} \cdot \frac{z}{z-1} + \sum_{k=1}^n \frac{c_k z}{z - p_k} \end{aligned}$$

(1) Single closed-loop poles on the real axis

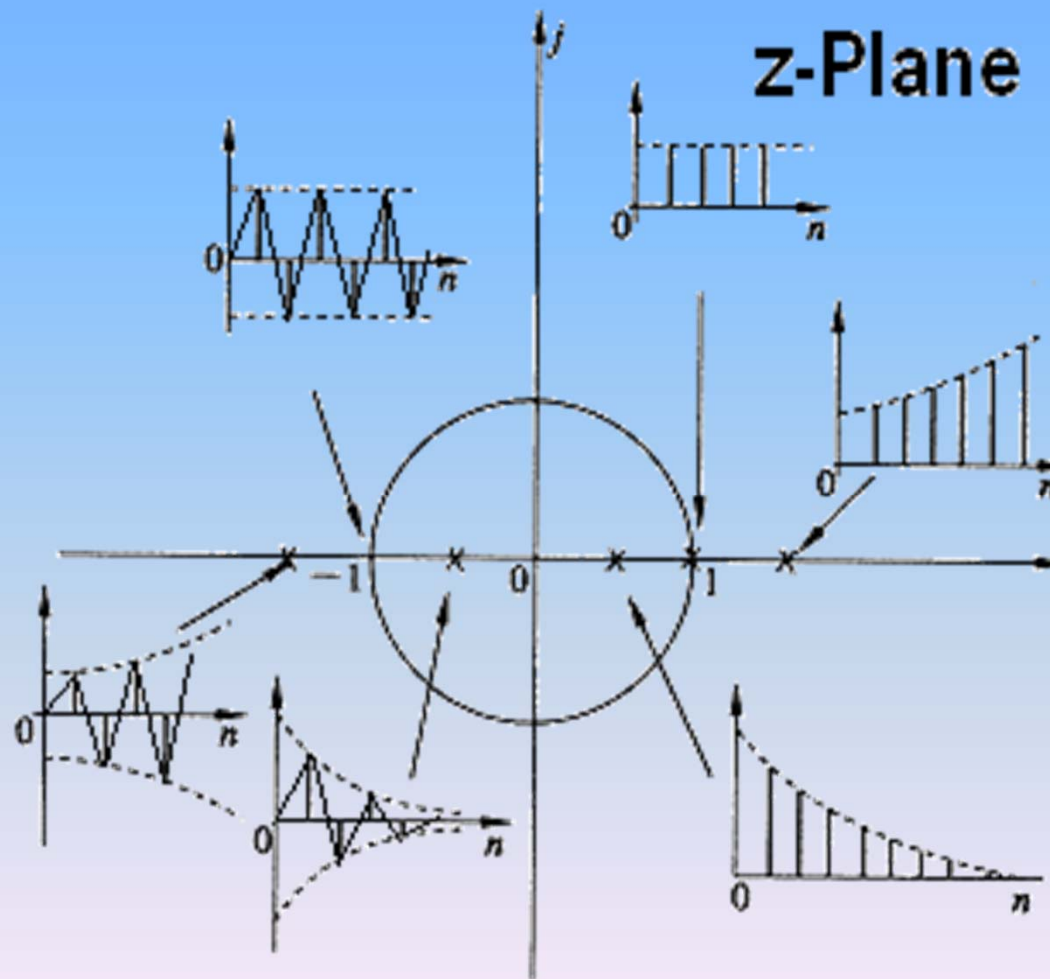
$$c_k^*(t) = Z^{-1} \left[\frac{c_k z}{z - p_k} \right] \quad k = 1, 2, \dots, n$$

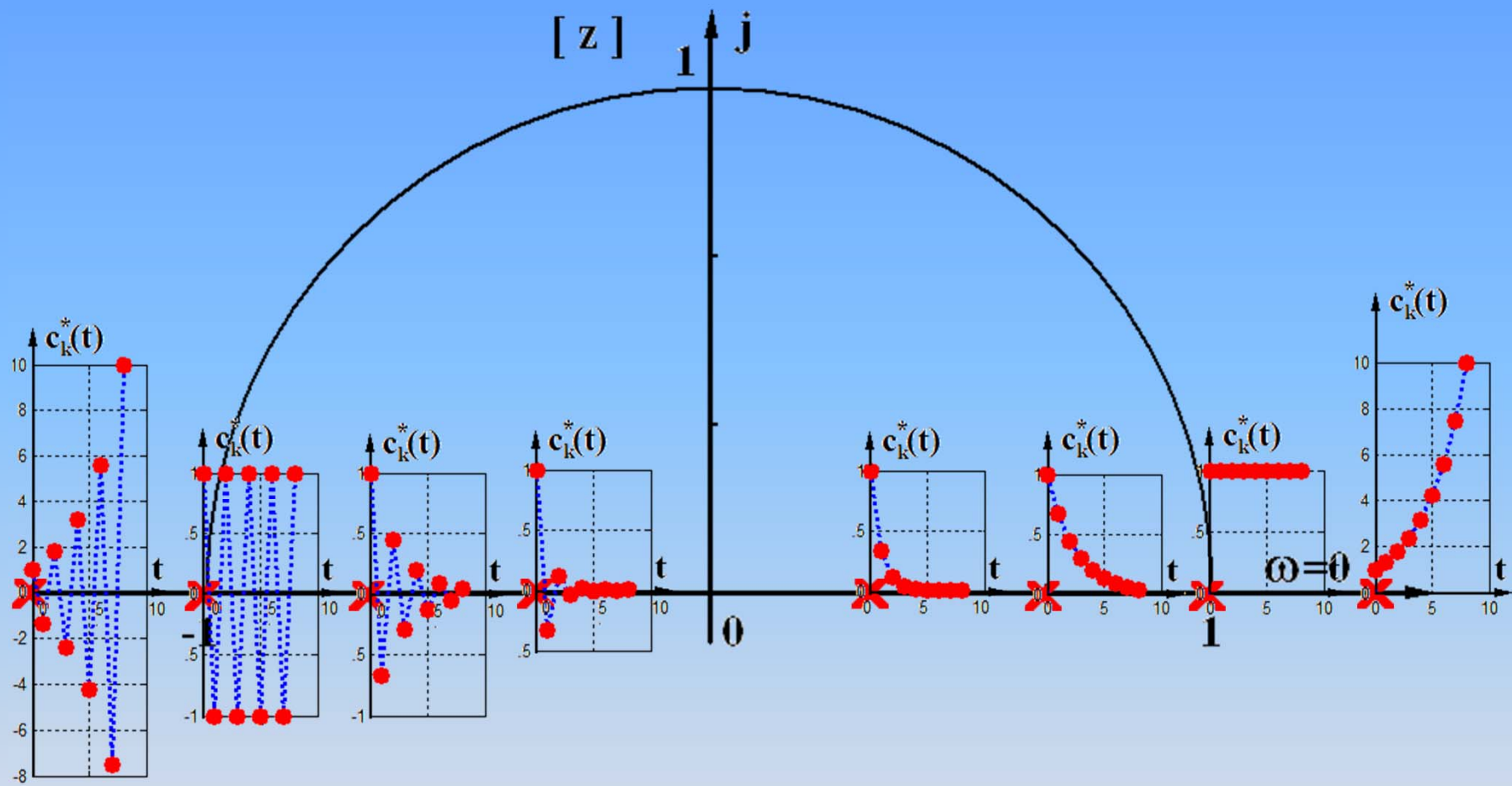
$$c_k(nT) = c_k p_k^n \quad k = 1, 2, \dots, n$$

$$c_k(nT) = c_k p_k^n \quad k = 1, 2, \dots, n$$

$$p_k > 0: \quad p_k > 1 \quad p_k = 1 \quad p_k < 1$$

$$p_k < 0:$$





(2) Closed-loop Complex conjugate poles

$$p_k = |p_k| e^{j\theta_k} \quad \bar{p}_k = |p_k| e^{-j\theta_k}$$

$$\mathbf{c}_{k,k}^*(k) = \mathbf{Z}^{-1} \left[\frac{\mathbf{c}_k z}{z - p_k} + \frac{\bar{\mathbf{c}}_k z}{z - \bar{p}_k} \right]$$

$$\begin{cases} a = \frac{1}{T} \ln |p_k| \\ \omega = \frac{\theta_k}{T} \\ 0 < \theta_k < \pi \end{cases}$$

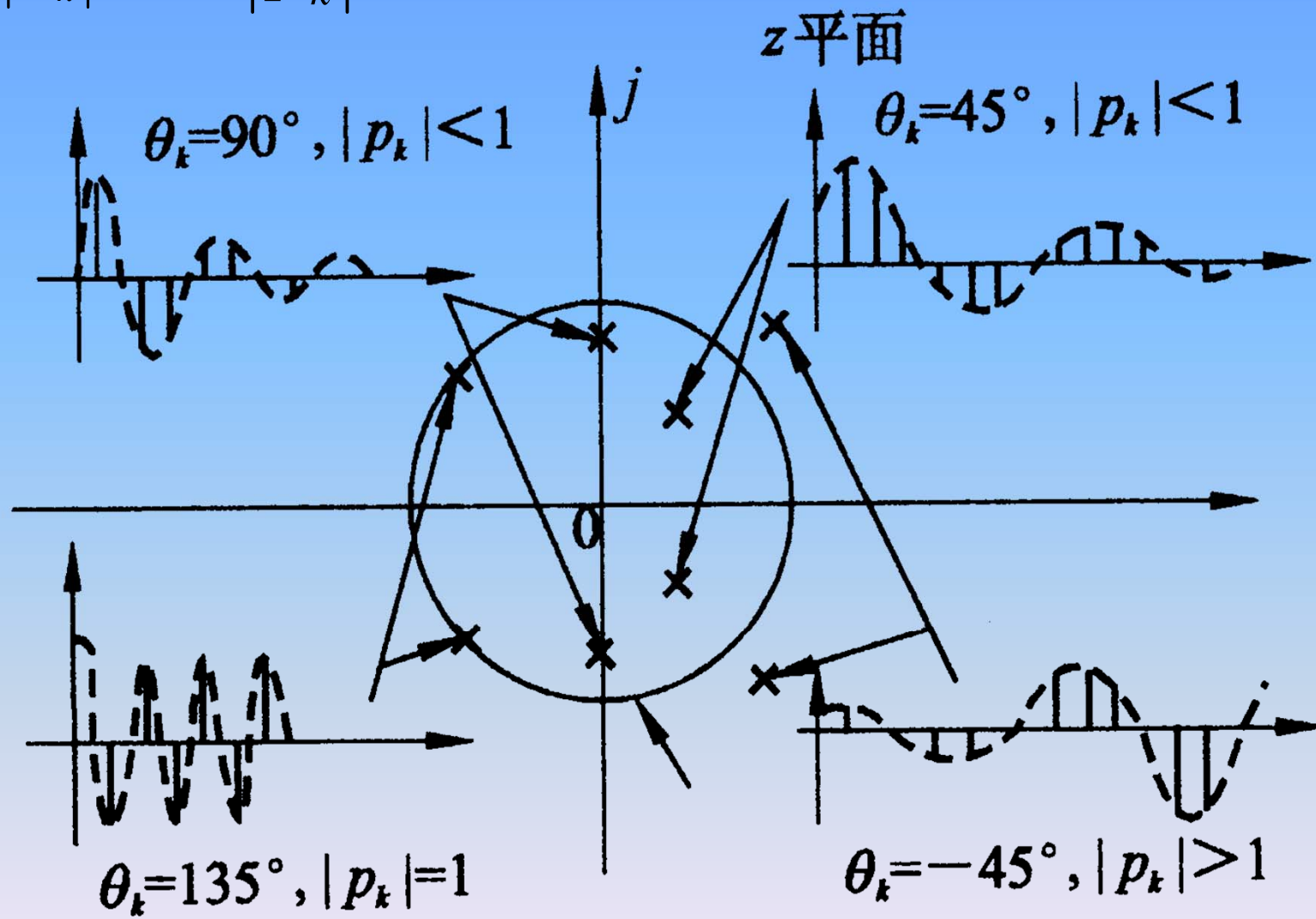
$$\mathbf{c}_{k,k}(nT) = \mathbf{c}_k p_k^n + \bar{\mathbf{c}}_k \bar{p}_k^n$$

$$= \mathbf{c}_k e^{a_k nT} + \bar{\mathbf{c}}_k e^{\bar{a}_k nT}$$

$$= |\mathbf{c}_k| e^{j\varphi_k} e^{(a+j\omega)nT} + |\mathbf{c}_k| e^{-j\varphi_k} e^{(a-j\omega)nT}$$

$$= 2|\mathbf{c}_k| e^{anT} \cos(n\omega T + \varphi_k)$$

$$|p_k| < 1, \quad |p_k| > 1$$



7.6.2 Analysis of discrete-time dynamic performance

(1) General method

$$\begin{cases} G(z) \rightarrow \Phi(z) \longrightarrow C(z) = \sum_{n=0}^{\infty} c(nT)z^{-n} \\ c^*(t) = \sum_{n=0}^{\infty} c(nT)\delta(t - nT) \longrightarrow \text{Obtain } \sigma\%, t_s \text{ by definition} \end{cases}$$

(2) Closed-loop poles $p_k \longrightarrow$ Response $c_k(nT) = C_k p_k^n$

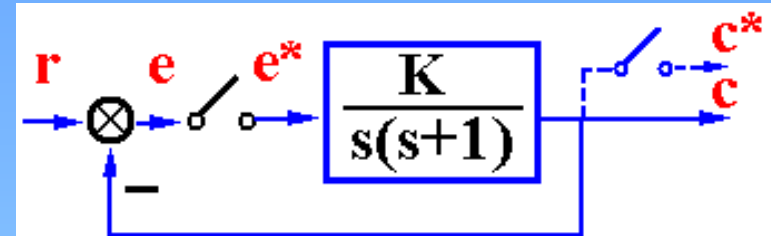
7.6 Performance Analysis of Discrete-Time Systems

- **Stability**
- **Dynamic Performance**
- **Steady-state Errors**

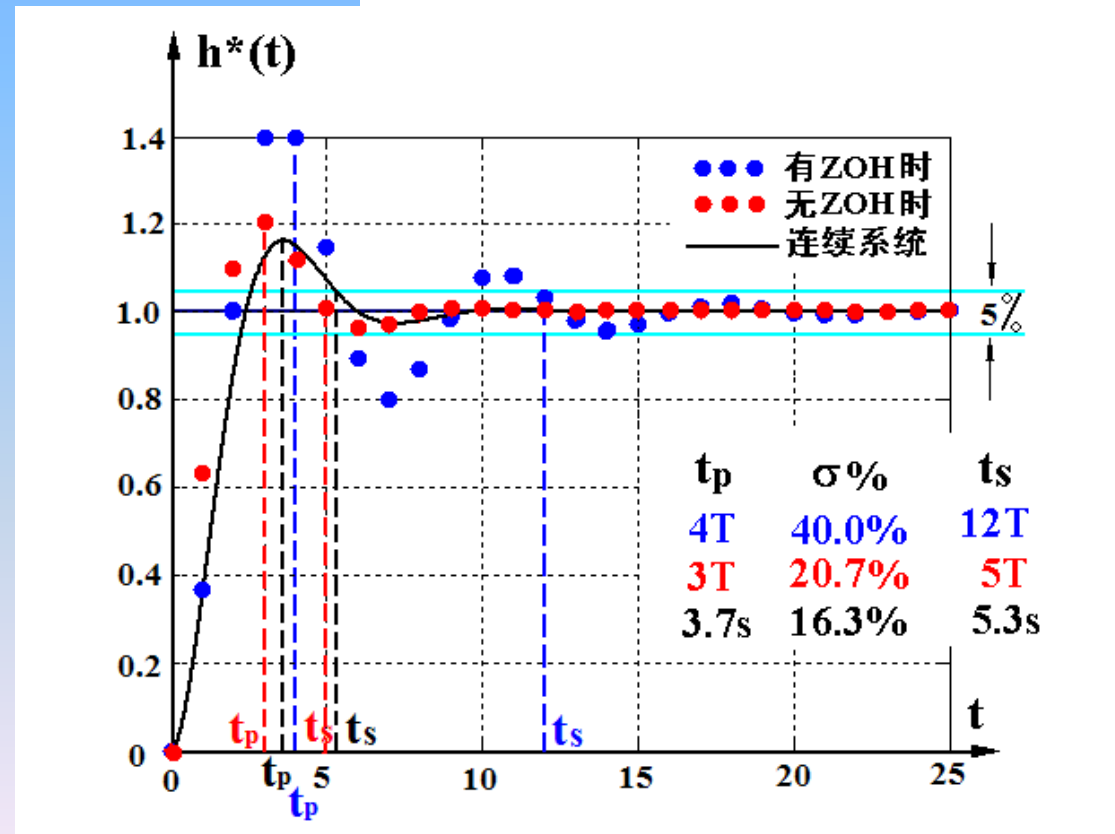
7.6.3 Steady-state error

1. General method to obtain steady-state error

Example Consider the system shown in the figure, $T=K=1$. Obtain the dynamic specifications. ($\sigma\%$, t_s).



$h(0)=0$
 $h(1)=0.632$
 $h(2)=1.097$
 $h(3)=1.207$
 $h(4)=1.117$
 $h(5)=1.014$
 $h(6)=0.964$
 $h(7)=0.970$
 $h(8)=0.991$
 $h(9)=1.004$
 $h(10)=1.007$
 $h(11)=1.003$
 $h(12)=1.000$
 \vdots



2. Using final value theorem to obtain steady-state error

Let $\begin{cases} GH(z) = Z[G(s)H(s)] = \frac{1}{(z-1)^v} GH_0(z) \\ \lim_{z \rightarrow 1} GH_0(z) = K \end{cases}$ **v: System type**

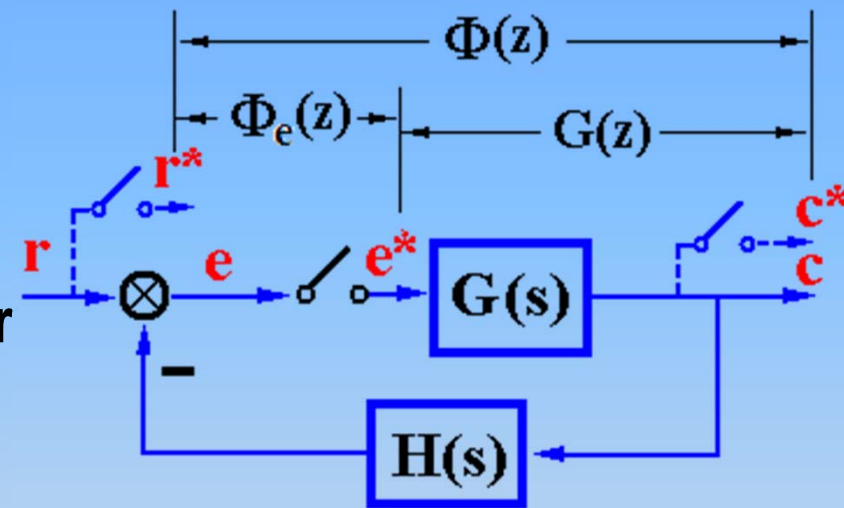
Algorithm:

- (1) Determine the stability
- (2) Obtain the impulse transfer function from $E(z)$ to $C(z)$.

$$\Phi_e(z) = \frac{E(z)}{R(z)} = \frac{1}{1 + GH(z)}$$

- (3) Obtain $e(\infty)$ by the final value theorem

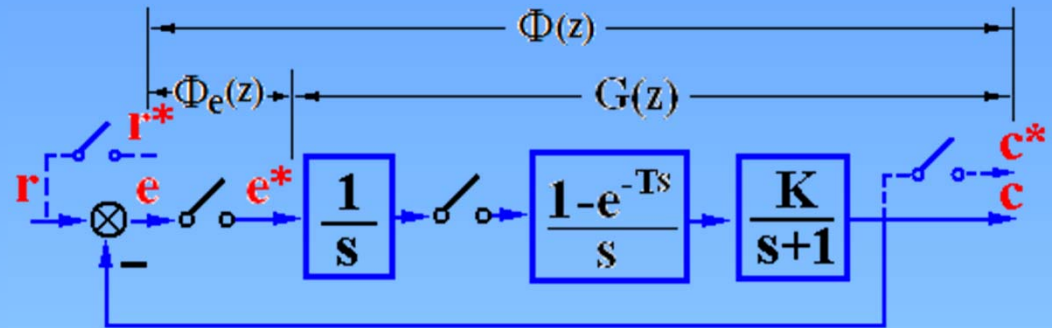
$$e(\infty) = \lim_{z \rightarrow 1} (z-1) \Phi_e(z) R(z) = \lim_{z \rightarrow 1} (z-1) \cdot R(z) \cdot \frac{1}{1 + GH(z)}$$



Example 1 Consider the discrete system shown in the figure, $K=2$, $T=1$;
Obtain $e(\infty)$ for $r(t)=1(t)$, t , $t^2/2$.

$$G(z) = Z \left[\frac{1}{s} \right] \cdot Z \left[\frac{1 - e^{-Ts}}{s} \cdot \frac{K}{s+1} \right]$$

$$= \frac{K(1 - e^{-T})z}{(z-1)(z - e^{-T})} \quad \nu = 1$$



$$\Phi_e(z) = \frac{1}{1 + \frac{K(1 - e^{-T})z}{(z - 1)(z - e^{-T})}} = \frac{(z - 1)(z - e^{-T})}{(z - 1)(z - e^{-T}) + K(1 - e^{-T})z}$$

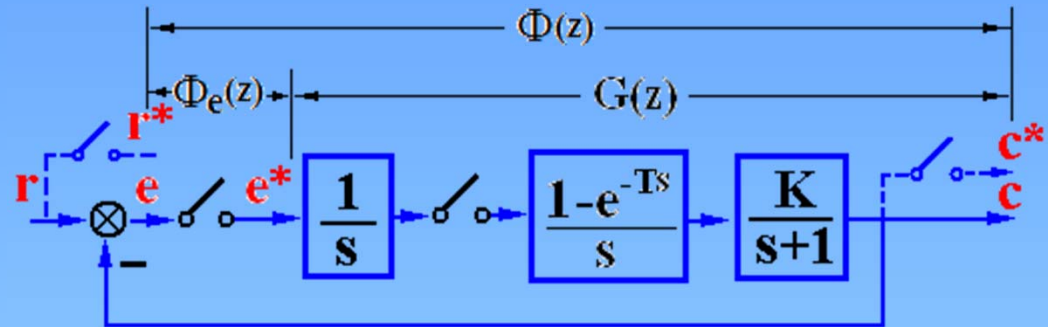
$$D(z) = z^2 + [K(1 - e^{-T}) - (1 + e^{-T})]z + e^{-T} = 0$$

$$0 < K < \frac{2(1 + e^{-T})}{(1 - e^{-T})} \stackrel{T=1}{=} 4.33$$

Example 1 Consider the discrete system shown in the figure, $K=2$, $T=1$;
Obtain $e(\infty)$ for $r(t)=1(t)$, t , $t^2/2$.

$$0 < K < 4.33$$

$$e(\infty) = \lim_{z \rightarrow 1} (z-1)R(z)\Phi_e(z)$$



$$\Phi_e(z) = \frac{(z-1)(z-e^{-T})}{(z-1)(z-e^{-T}) + K(1-e^{-T})z}$$

$$r_1(t) = 1(t) \quad e_1(\infty) = \lim_{z \rightarrow 1} (z-1) \frac{z}{z-1} \cdot \frac{(z-1)(z-e^{-T})}{(z-1)(z-e^{-T}) + K(1-e^{-T})z} = 0$$

$$r_2(t) = t \quad e_2(\infty) = \lim_{z \rightarrow 1} (z-1) \frac{Tz}{(z-1)^2} \cdot \frac{(z-1)(z-e^{-T})}{(z-1)(z-e^{-T}) + K(1-e^{-T})z} = \frac{T}{K}$$

$$r_3(t) = \frac{t^2}{2} \quad e_3(\infty) = \lim_{z \rightarrow 1} (z-1) \frac{Tz(z+1)}{2(z-1)^3} \cdot \frac{(z-1)(z-e^{-T})}{(z-1)(z-e^{-T}) + K(1-e^{-T})z} = \infty$$

3. Static Error Constant Method

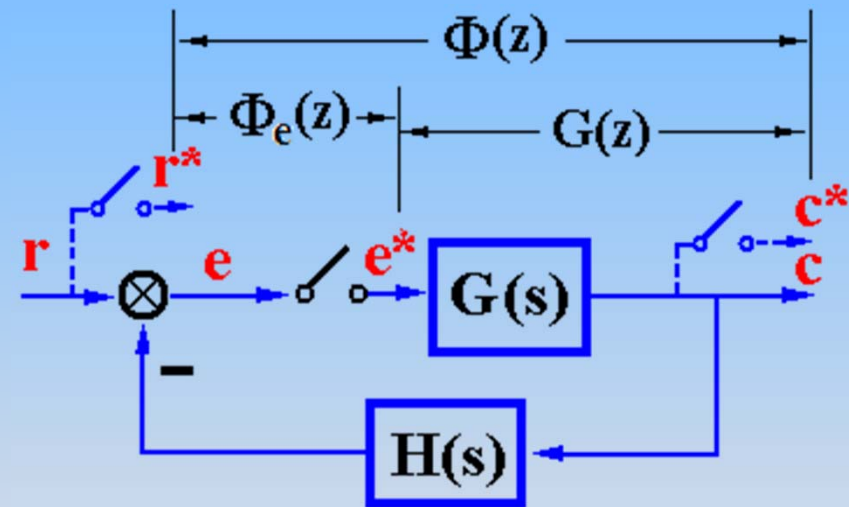
shows how $e(\infty)$ changes with $r(t)$

(For stable linear discrete systems subject to $r(t)$ and sampled at the error signal)

Let $\begin{cases} GH(z) = Z[G(s)H(s)] = \frac{1}{(z-1)^v} GH_0(z) \\ \lim_{z \rightarrow 1} GH_0(z) = K \end{cases}$ **v: System type**

$$\Phi_e(z) = \frac{E(z)}{R(z)} = \frac{1}{1 + GH(z)}$$

$$\begin{aligned} e(\infty) &= \lim_{z \rightarrow 1} (z-1) \Phi_e(z) R(z) \\ &= \lim_{z \rightarrow 1} (z-1) \cdot R(z) \cdot \frac{1}{1 + GH(z)} \end{aligned}$$



$$e(\infty T) = \lim_{z \rightarrow 1} (z - 1) \Phi_e(z) R(z) = \lim_{z \rightarrow 1} (z - 1) \cdot R(z) \cdot \frac{1}{1 + GH(z)}$$

$$r(t) = A \cdot 1(t) \quad e(\infty T) = \lim_{z \rightarrow 1} (z - 1) \cdot \frac{Az}{z - 1} \cdot \frac{1}{1 + GH(z)} = \frac{A}{1 + \lim_{z \rightarrow 1} GH(z)} = \frac{A}{K_p}$$

Static position error constant $K_p = 1 + \lim_{z \rightarrow 1} GH(z)$

$$r(t) = A \cdot t \quad e(\infty T) = \lim_{z \rightarrow 1} (z - 1) \cdot \frac{ATz}{(z - 1)^2} \cdot \frac{1}{1 + GH(z)} = \frac{AT}{\lim_{z \rightarrow 1} (z - 1) GH(z)} = \frac{AT}{K_v}$$

Static velocity error constant $K_v = \lim_{z \rightarrow 1} (z - 1) GH(z)$

$$r(t) = \frac{A}{2} t^2 \quad e(\infty T) = \lim_{z \rightarrow 1} (z - 1) \cdot \frac{AT^2 z(z + 1)}{2(z - 1)^3} \cdot \frac{1}{1 + GH(z)} = \frac{AT^2}{\lim_{z \rightarrow 1} (z - 1)^2 GH(z)} = \frac{AT^2}{K_a}$$

Static acceleration error constant $K_a = \lim_{z \rightarrow 1} (z - 1)^2 GH(z)$

Similar to the continuous system, we can divide the discrete-time system as type 0, type I, type II,... according to the numbers of the pole $z=1$ of the impulse transfer function.

$$r(t) = A \cdot 1(t)$$

Static position error constant

$$K_p = 1 + \lim_{z \rightarrow 1} GH(z)$$

Type 0: $K_p = \text{constant}$

Type ≥ 1 : $K_p = \infty$, $e(\infty) = 0$

$$r(t) = A \cdot t$$

Static velocity error constant

$$K_v = \lim_{z \rightarrow 1} (z - 1)GH(z)$$

Type 0: $K_v = 0$, $e(\infty) = \infty$

Type 1: $K_v = \text{constant}$,

Type ≥ 2 : $K_v = \infty$, $e(\infty) = 0$

$r(t) = \frac{A}{2}t^2$ Static acceleration error constant $K_a = \lim_{z \rightarrow 1} (z-1)^2 GH(z)$

Type 0,1: $K_a=0$, $e(\infty)=\infty$

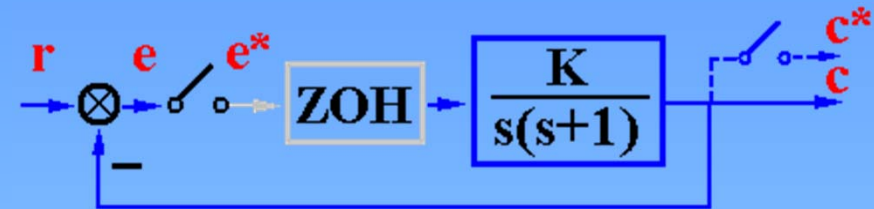
Type =2: $K_a = \text{constant}$,

Type ≥ 3 : $K_a = \infty$, $e(\infty)=0$

$$\begin{cases} GH(z) = \frac{1}{(z-1)^v} GH_0(z) \\ \lim_{z \rightarrow 1} GH_0(z) = K \end{cases}$$

型別	Static Error Constant			Steady-State Error		
V	$K_p = \lim_{z \rightarrow 1} GH(z)$	$K_v = \lim_{z \rightarrow 1} (z-1)GH(z)$	$K_a = \lim_{z \rightarrow 1} (z-1)^2 GH(z)$	$r = A \cdot 1(t)$ $e(\infty) = -\frac{A}{K_p}$	$r = A \cdot t$ $e(\infty) = \frac{AT}{K_v}$	$r = A \cdot t^2/2$ $e(\infty) = \frac{AT^2}{K_a}$
0	K_p	0	0	$\frac{A}{K_p}$	∞	∞
I	∞	K_v	0	0	$\frac{AT}{K_v}$	∞
II	∞	∞	K_a	0	0	$\frac{AT^2}{K_a}$

Example 2 Consider the stable discrete system shown in the figure. When $r(t)=2t$, obtain $e(\infty)$ with/without ZOH.



Solution.

no ZOH

$$\begin{cases} G(z) = Z\left[\frac{K}{s(s+1)}\right] = \frac{K(1-e^{-T})z}{(z-1)(z-e^{-T})} \\ K_v = \lim_{z \rightarrow 1} (z-1)G(z) = \lim_{z \rightarrow 1} \frac{K(1-e^{-T})z}{(z-e^{-T})} = K \end{cases}$$

$$e(\infty) = \frac{AT}{K_v} = \frac{2T}{K}$$

— dependent of T

with ZOH

$$\begin{cases} G(z) = Z\left[\frac{1-e^{-Ts}}{s} \cdot \frac{K}{s(s+1)}\right] = K \frac{z-1}{z} \cdot Z\left[\frac{1}{s^2(s+1)}\right] \\ = K \frac{(T-1+e^{-T})z + (1-e^{-T}-Te^{-T})}{(z-1)(z-e^{-T})} \\ K_v = \lim_{z \rightarrow 1} (z-1)G(z) = \lim_{z \rightarrow 1} \frac{K(T-Te^{-T})}{z-e^{-T}} = KT \end{cases}$$

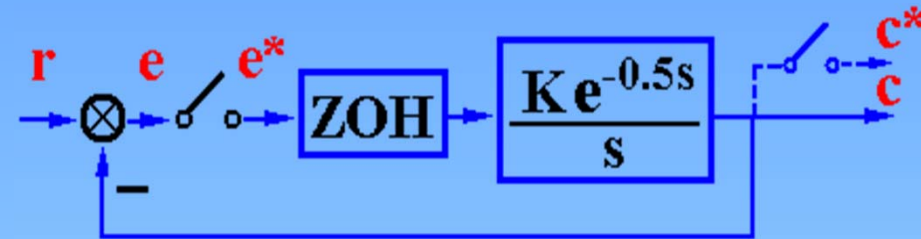
$$e(\infty) = \frac{AT}{K_v} = \frac{A}{K} = \frac{2}{K}$$

— independent of T

Example 3 Consider the system shown in the figure, $T=0.25$. When $r(t)=2 \cdot 1(t)+t$, obtain the range of K for $e(\infty)<0.5$.

Solution. The stable range of K is

$$0 < K < 2.472$$



$$G(z) = Z \left[\frac{1 - e^{-Ts}}{s} \cdot \frac{K e^{-2Ts}}{s} \right]$$

$$= K(1 - z^{-1})z^{-2} Z \left[\frac{1}{s^2} \right] = Kz^{-2} \frac{z-1}{z} \cdot \frac{Tz}{(z-1)^2} = \frac{KT}{z^2(z-1)} \quad \nu = 1$$

$$K_v = \lim_{z \rightarrow 1} (z-1)G(z) = \lim_{z \rightarrow 1} (z-1) \frac{KT}{z^2(z-1)} = KT$$

$$r_1(t) = 2 \cdot 1(t) \quad e_1(\infty) = 0$$

$$r_2(t) = t \quad e_2(\infty) = TA/K_v = 1/K$$

$$2 < K < 2.472$$

$$e(\infty) = e_1(\infty) + e_2(\infty) = 1/K < 0.5 \Rightarrow K > 2$$

7.6.3 Steady-state error of discrete systems

(1) General method: obtain system response

(2) Final value theorem $\left\{ \begin{array}{l} G(z) \rightarrow \Phi_e(z) \\ D(z) \rightarrow \text{Stability} \\ e(\infty) = \lim_{z \rightarrow 1} (z-1)R(z)\Phi_e(z) \end{array} \right.$

(3) Static error constant $\left\{ \begin{array}{l} G(z) \rightarrow v, K_p, K_v, K_a \\ \text{Obtain } e(\infty) \end{array} \right.$

Homework:

P236. 7-10, 7-11

7-10. Consider the system as shown in Fig. 7-65(refer to the textbook), $T=0.5s$,

- (1) determine the stability of the system;
- (2) If $r(t)=1(t)+t$, obtain the steady-state error.

7-11. Consider the system as shown in Fig. 7-66(refer to the textbook), determine the stable range of K , where $T=1s$.