

# Foundations of Software Fall 2022

Week 3

# Review (and more details)

## Recall: Simple Arithmetic Expressions

The set  $\mathcal{T}$  of terms is defined by the following abstract grammar:

$t ::=$

`true`

`false`

`if t then t else t`

`0`

`succ t`

`pred t`

`iszero t`

*terms*

*constant true*

*constant false*

*conditional*

*constant zero*

*successor*

*predecessor*

*zero test*

## Recall: Inference Rule Notation

More explicitly: The set  $\mathcal{T}$  is the *smallest* set *closed* under the following rules.

$$\begin{array}{c} \text{true} \in \mathcal{T} \\ \hline \text{succ } t_1 \in \mathcal{T} \end{array} \quad \begin{array}{c} \text{false} \in \mathcal{T} \\ \hline \text{pred } t_1 \in \mathcal{T} \end{array} \quad \begin{array}{c} 0 \in \mathcal{T} \\ \hline \text{iszero } t_1 \in \mathcal{T} \end{array}$$
$$\frac{t_1 \in \mathcal{T} \quad t_2 \in \mathcal{T} \quad t_3 \in \mathcal{T}}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \in \mathcal{T}}$$

# Generating Functions

Each of these rules can be thought of as a *generating function* that, given some elements from  $\mathcal{T}$ , generates some other element of  $\mathcal{T}$ . Saying that  $\mathcal{T}$  is closed under these rules means that  $\mathcal{T}$  cannot be made any bigger using these generating functions — it already contains everything “justified by its members.”

$$\begin{array}{ccc} \text{true} \in \mathcal{T} & \text{false} \in \mathcal{T} & 0 \in \mathcal{T} \\[1em] \frac{t_1 \in \mathcal{T}}{\text{succ } t_1 \in \mathcal{T}} & \frac{t_1 \in \mathcal{T}}{\text{pred } t_1 \in \mathcal{T}} & \frac{t_1 \in \mathcal{T}}{\text{iszero } t_1 \in \mathcal{T}} \\[1em] \frac{t_1 \in \mathcal{T} \quad t_2 \in \mathcal{T} \quad t_3 \in \mathcal{T}}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \in \mathcal{T}} \end{array}$$

Let's write these generating functions explicitly.

$$F_1(U) = \{\text{true}\}$$

$$F_2(U) = \{\text{false}\}$$

$$F_3(U) = \{0\}$$

$$F_4(U) = \{\text{succ } t_1 \mid t_1 \in U\}$$

$$F_5(U) = \{\text{pred } t_1 \mid t_1 \in U\}$$

$$F_6(U) = \{\text{iszero } t_1 \mid t_1 \in U\}$$

$$F_7(U) = \{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \mid t_1, t_2, t_3 \in U\}$$

Each one takes a set of terms  $U$  as input and produces a set of “terms justified by  $U$ ” as output.

If we now define a generating function for the whole set of inference rules (by combining the generating functions for the individual rules),

$$F(U) = F_1(U) \cup F_2(U) \cup F_3(U) \cup F_4(U) \cup F_5(U) \cup F_6(U) \cup F_7(U)$$

then we can restate the previous definition of the set of terms  $\mathcal{T}$  like this:

**Definition:**

- ▶ A set  $U$  is said to be “closed under  $F$ ” (or “ $F$ -closed”) if  $F(U) \subseteq U$ .
- ▶ The set of terms  $\mathcal{T}$  is the smallest  $F$ -closed set.  
(I.e., if  $\mathcal{O}$  is another set such that  $F(\mathcal{O}) \subseteq \mathcal{O}$ , then  $\mathcal{T} \subseteq \mathcal{O}$ .)

Our alternate definition of the set of terms can also be stated using the generating function  $F$ :

$$\begin{aligned}\mathcal{S}_0 &= \emptyset \\ \mathcal{S}_{i+1} &= F(\mathcal{S}_i) \\ \mathcal{S} &= \bigcup_i \mathcal{S}_i\end{aligned}$$

Compare this definition of  $\mathcal{S}$  with the one we saw last time:

$$\begin{aligned}\mathcal{S}_0 &= \emptyset \\ \mathcal{S}_{i+1} &= \{\text{true, false, 0}\} \\ &\quad \cup \{\text{succ } t_1, \text{pred } t_1, \text{iszero } t_1 \mid t_1 \in \mathcal{S}_i\} \\ &\quad \cup \{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \mid t_1, t_2, t_3 \in \mathcal{S}_i\}\end{aligned}$$

$$\mathcal{S} = \bigcup_i \mathcal{S}_i$$

We have “pulled out”  $F$  and given it a name.



Note that our two definitions of terms characterize the same set from different directions:

- ▶ “from above,” as the intersection of all  $F$ -closed sets;
- ▶ “from below,” as the limit (union) of a series of sets that start from  $\emptyset$  and get “closer and closer to being  $F$ -closed.”

Proposition 3.2.6 in the book shows that these two definitions actually define the same set.

**Warning:** Hard hats on for the next slide!

# Structural Induction

The principle of structural induction on terms can also be re-stated using generating functions:

*Suppose  $T$  is the smallest  $F$ -closed set.*

*If, for each set  $U$ ,*

*from the assumption “ $P(u)$  holds for every  $u \in U$ ”*

*we can show “ $P(v)$  holds for any  $v \in F(U)$ ,”*

*then  $P(t)$  holds for all  $t \in T$ .*

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*then  $P(t)$  holds for all  $t \in T$ .*

Why?

# Structural Induction

Why? Because:

- ▶ We assumed that  $T$  was the *smallest*  $F$ -closed set, i.e., that  $T \subseteq O$  for any other  $F$ -closed set  $O$ .

*for each set  $U$ ,*

*given  $P(u)$  for all  $u \in U$*

*we can show  $P(v)$  for all  $v \in F(U)$*

- ▶ But showing  
amounts to showing that “the set of all terms satisfying  $P$ ”  
–call it  $O$ – is itself an  $F$ -closed set.  
(recall:  $O$  is  $F$ -closed if  $F(O) \subseteq O$ )
- ▶ Since  $T \subseteq O$ , every element of  $T$  satisfies  $P$ .

# Structural Induction

Compare this with the structural induction principle for terms from last lecture:

*If, for each term  $s$ ,  
    given  $P(r)$  for all immediate subterms  $r$  of  $s$   
    we can show  $P(s)$ ,  
then  $P(t)$  holds for all  $t$ .*

Recall, from the definition of  $\mathcal{S}_i$ , it is clear that, if a term  $t$  is in  $\mathcal{S}_i$ , then all of its immediate subterms must be in  $\mathcal{S}_{i-1}$ , i.e., they must have strictly smaller depths. Therefore:

*If, for each term  $s$ ,  
given  $P(r)$  for all immediate subterms  $r$  of  $s$   
we can show  $P(s)$ ,  
then  $P(t)$  holds for all  $t$ .*

### **Slightly more explicit proof:**

- ▶ Assume that for each term  $s$ , given  $P(r)$  for all immediate subterms of  $s$ , we can show  $P(s)$ .
- ▶ Then show, by induction on  $i$ , that  $P(t)$  holds for all terms  $t$  with depth  $i$ .
- ▶ Therefore,  $P(t)$  holds for all  $t$ .

# Operational Semantics and Reasoning



## Recall: Abstract Machines

An *abstract machine* consists of:

- ▶ a set of *states*
- ▶ a *transition relation* on states, written  $\longrightarrow$

For the simple languages we are considering at the moment, the term being evaluated is the whole state of the abstract machine.

# Recall: Syntax for Booleans

## *Terms and values*

`t ::=`

`true`

`false`

`if t then t else t`

*terms*

*constant true*

*constant false*

*conditional*

`v ::=`

`true`

`false`

*values*

*true value*

*false value*

## Recall: Operational Semantics for Booleans

The evaluation relation  $t \longrightarrow t'$  is the smallest relation closed under the following rules:

$\text{if true then } t_2 \text{ else } t_3 \longrightarrow t_2 \quad (\text{E-IFTRUE})$

$\text{if false then } t_2 \text{ else } t_3 \longrightarrow t_3 \quad (\text{E-IFFALSE})$

$$\frac{t_1 \longrightarrow t'_1}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \longrightarrow \text{if } t'_1 \text{ then } t_2 \text{ else } t_3} \quad (\text{E-IF})$$

# Derivations

We can record the “justification” for a particular pair of terms that are in the evaluation relation in the form of a tree.

*(on the board)*

Terminology:

- ▶ These trees are called *derivation trees* (or just *derivations*).
- ▶ The final statement in a derivation is its *conclusion*.
- ▶ We say that the derivation is a *witness* for its conclusion (or a *proof* of its conclusion) — it records all the reasoning steps that justify the conclusion.

## Observation

*Lemma:* Suppose we are given a derivation tree  $\mathcal{D}$  witnessing the pair  $(t, t')$  in the evaluation relation. Then either

1. the final rule used in  $\mathcal{D}$  is E-IFTRUE and we have  $t = \text{if true then } t_2 \text{ else } t_3$  and  $t' = t_2$ , for some  $t_2$  and  $t_3$ , or
2. the final rule used in  $\mathcal{D}$  is E-IFFALSE and we have  $t = \text{if false then } t_2 \text{ else } t_3$  and  $t' = t_3$ , for some  $t_2$  and  $t_3$ , or
3. the final rule used in  $\mathcal{D}$  is E-IF and we have  $t = \text{if } t_1 \text{ then } t_2 \text{ else } t_3$  and  $t' = \text{if } t'_1 \text{ then } t_2 \text{ else } t_3$ , for some  $t_1, t'_1, t_2$ , and  $t_3$ ; moreover, the immediate subderivation of  $\mathcal{D}$  witnesses  $(t_1, t'_1) \in \longrightarrow$ .

## Induction on Derivations

We can now write proofs about evaluation “by induction on derivation trees.”

Given an arbitrary derivation  $\mathcal{D}$  with conclusion  $t \rightarrow t'$ , we assume the desired result for its immediate sub-derivation (if any) and proceed by a case analysis (using the previous lemma) of the final evaluation rule used in constructing the derivation tree.

E.g....

## Induction on Derivations — Example

**Theorem:** If  $t \longrightarrow t'$ , i.e., if  $(t, t') \in \longrightarrow$ , then  $\text{size}(t) > \text{size}(t')$ .

**Proof:** By induction on a derivation  $\mathcal{D}$  of  $t \longrightarrow t'$ .

1. Suppose the final rule used in  $\mathcal{D}$  is E-IFTRUE, with  $t = \text{if true then } t_2 \text{ else } t_3$  and  $t' = t_2$ . Then the result is immediate from the definition of *size*.
2. Suppose the final rule used in  $\mathcal{D}$  is E-IFFALSE, with  $t = \text{if false then } t_2 \text{ else } t_3$  and  $t' = t_3$ . Then the result is again immediate from the definition of *size*.
3. Suppose the final rule used in  $\mathcal{D}$  is E-IF, with  $t = \text{if } t_1 \text{ then } t_2 \text{ else } t_3$  and  $t' = \text{if } t'_1 \text{ then } t_2 \text{ else } t_3$ , where  $(t_1, t'_1) \in \longrightarrow$  is witnessed by a derivation  $\mathcal{D}_1$ . By the induction hypothesis,  $\text{size}(t_1) > \text{size}(t'_1)$ . But then, by the definition of *size*, we have  $\text{size}(t) > \text{size}(t')$ .

## Normal forms

A *normal form* is a term that cannot be evaluated any further — i.e., a term  $t$  is a normal form (or “is in normal form”) if there is no  $t'$  such that  $t \longrightarrow t'$ .

A normal form is a state where the abstract machine is halted — i.e., it can be regarded as a “result” of evaluation.



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A normal form is a state where the abstract machine is halted — i.e., it can be regarded as a “result” of evaluation.

Recall that we intended the set of *values* (the boolean constants `true` and `false`) to be exactly the possible “results of evaluation.” Did we get this definition right?

## Values = normal forms

**Theorem:** A term  $t$  is a value iff it is in normal form.

**Proof:**

The  $\implies$  direction is immediate from the definition of the evaluation relation.

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If  $t$  is *not* a value, then it is *not* a normal form.

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For the  $\impliedby$  direction, it is convenient to prove the contrapositive: If  $t$  is *not* a value, then it is *not* a normal form. The argument goes by induction on  $t$ .

Note, first, that  $t$  must have the form `if  $t_1$  then  $t_2$  else  $t_3$`  (otherwise it would be a value). If  $t_1$  is `true` or `false`, then rule E-IFTRUE or E-IFFALSE applies to  $t$ , and we are done.

Otherwise,  $t_1$  is not a value and so, by the induction hypothesis, there is some  $t'_1$  such that  $t_1 \longrightarrow t'_1$ . But then rule E-IF yields

$$\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \longrightarrow \text{if } t'_1 \text{ then } t_2 \text{ else } t_3$$

i.e.,  $t$  is not in normal form.

# Numbers

## *New syntactic forms*

`t ::= ...`  
`0`  
`succ t`  
`pred t`  
`iszero t`

`v ::= ...`  
`nv`

`nv ::=`  
`0`  
`succ nv`

## *terms*

*constant zero*  
*successor*  
*predecessor*  
*zero test*

## *values*

*numeric value*

## *numeric values*

*zero value*  
*successor value*

$$\frac{t_1 \longrightarrow t'_1}{\text{succ } t_1 \longrightarrow \text{succ } t'_1} \quad (\text{E-SUCC})$$

$$\text{pred } 0 \longrightarrow 0 \quad (\text{E-PREDZERO})$$

$$\text{pred } (\text{succ } nv_1) \longrightarrow nv_1 \quad (\text{E-PREDSUCC})$$

$$\frac{t_1 \longrightarrow t'_1}{\text{pred } t_1 \longrightarrow \text{pred } t'_1} \quad (\text{E-PRED})$$

$$\text{iszero } 0 \longrightarrow \text{true} \quad (\text{E-ISZEROZERO})$$

$$\text{iszero } (\text{succ } nv_1) \longrightarrow \text{false} \quad (\text{E-ISZEROSUCC})$$

$$\frac{t_1 \longrightarrow t'_1}{\text{iszero } t_1 \longrightarrow \text{iszero } t'_1} \quad (\text{E-ISZERO})$$

## Values are normal forms

Our observation a few slides ago that all values are in normal form still holds for the extended language.

Is the converse true? I.e., is every normal form a value?



## Values are normal forms, but we have stuck terms

Our observation a few slides ago that all values are in normal form still holds for the extended language.

Is the converse true? I.e., is every normal form a value?

No: some terms are *stuck*.

Formally, a stuck term is one that is a normal form but not a value.  
What are some examples?

Stuck terms model run-time errors.

## Multi-step evaluation.

The *multi-step evaluation* relation,  $\longrightarrow^*$ , is the reflexive, transitive closure of single-step evaluation.

I.e., it is the smallest relation closed under the following rules:

$$\frac{t \longrightarrow t'}{t \longrightarrow^* t'}$$

$$t \longrightarrow^* t$$

$$\frac{t \longrightarrow^* t' \quad t' \longrightarrow^* t''}{t \longrightarrow^* t''}$$

## Termination of evaluation

**Theorem:** For every  $t$  there is some normal form  $t'$  such that  $t \longrightarrow^* t'$ .

**Proof:**

# Termination of evaluation

**Theorem:** For every  $t$  there is some normal form  $t'$  such that  $t \longrightarrow^* t'$ .

**Proof:**

- ▶ First, recall that single-step evaluation strictly reduces the size of the term:

*if  $t \longrightarrow t'$ , then  $\text{size}(t) > \text{size}(t')$*

- ▶ Now, assume (for a contradiction) that

$t_0, t_1, t_2, t_3, t_4, \dots$

is an infinite-length sequence such that

$t_0 \longrightarrow t_1 \longrightarrow t_2 \longrightarrow t_3 \longrightarrow t_4 \longrightarrow \dots$

- ▶ Then

$\text{size}(t_0) > \text{size}(t_1) > \text{size}(t_2) > \text{size}(t_3) > \dots$

- ▶ But such a sequence cannot exist — contradiction!

# Termination Proofs

Most termination proofs have the same basic form:

**Theorem:** *The relation  $R \subseteq X \times X$  is terminating — i.e., there are no infinite sequences  $x_0, x_1, x_2$ , etc. such that  $(x_i, x_{i+1}) \in R$  for each  $i$ .*

**Proof:**

1. Choose

- ▶ a well-founded set  $(W, <)$  — i.e., a set  $W$  with a partial order  $<$  such that there are no infinite descending chains  $w_0 > w_1 > w_2 > \dots$  in  $W$
- ▶ a function  $f$  from  $X$  to  $W$

2. Show  $f(x) > f(y)$  for all  $(x, y) \in R$

3. Conclude that there are no infinite sequences  $x_0, x_1, x_2$ , etc. such that  $(x_i, x_{i+1}) \in R$  for each  $i$ , since, if there were, we could construct an infinite descending chain in  $W$ .

# The Lambda Calculus

# The lambda-calculus

- ▶ If our previous language of arithmetic expressions was the simplest nontrivial programming language, then the lambda-calculus is the simplest *interesting* programming language...
  - ▶ Turing complete
  - ▶ higher order (functions as data)
- ▶ Indeed, in the lambda-calculus, *all* computation happens by means of function abstraction and application.
- ▶ The *e. coli* of programming language research
- ▶ The foundation of many real-world programming language designs (including ML, Haskell, Scheme, Lisp, ...)

## Intuitions

Suppose we want to describe a function that adds three to any number we pass it. We might write

$$\text{plus3 } x \quad = \quad \text{succ } (\text{succ } (\text{succ } x))$$

That is, “`plus3 x` is `succ (succ (succ x))`.”



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That is, “ $\text{plus3 } x$  is  $\text{succ } (\text{succ } (\text{succ } x))$ .”

Q: What is  $\text{plus3}$  itself?

A:  $\text{plus3}$  is the function that, given  $x$ , yields  $\text{succ } (\text{succ } (\text{succ } x))$ .

$$\text{plus3} = \lambda x. \text{succ } (\text{succ } (\text{succ } x))$$

This function exists independent of the name  $\text{plus3}$ .

$\lambda x. t$  is written “ $\text{fun } x \rightarrow t$ ” in OCaml and “ $x \Rightarrow t$ ” in Scala.

So `plus3 (succ 0)` is just a convenient shorthand for “the function that, given `x`, yields `succ (succ (succ x))`, applied to `succ 0`.”

$$\begin{aligned} &\text{plus3 (succ 0)} \\ &= \\ &(\lambda x. \text{succ (succ (succ x))}) (\text{succ 0}) \end{aligned}$$

## Abstractions over Functions

Consider the  $\lambda$ -abstraction

$$g = \lambda f. f (f (\text{succ } 0))$$

Note that the parameter variable  $f$  is used in the *function* position in the body of  $g$ . Terms like  $g$  are called *higher-order* functions. If we apply  $g$  to an argument like  $\text{plus3}$ , the “substitution rule” yields a nontrivial computation:

$$\begin{aligned} g \text{ plus3} &= (\lambda f. f (f (\text{succ } 0))) (\lambda x. \text{succ } (\text{succ } (\text{succ } x))) \\ &\text{i.e. } (\lambda x. \text{succ } (\text{succ } (\text{succ } x))) \\ &\quad ((\lambda x. \text{succ } (\text{succ } (\text{succ } x))) (\text{succ } 0)) \\ &\text{i.e. } (\lambda x. \text{succ } (\text{succ } (\text{succ } x))) \\ &\quad (\text{succ } (\text{succ } (\text{succ } (\text{succ } 0)))) \\ &\text{i.e. } \text{succ } (\text{succ } (\text{succ } (\text{succ } (\text{succ } (\text{succ } (\text{succ } 0)))))) \end{aligned}$$

## Abstractions Returning Functions

Consider the following variant of `g`:

$$\text{double} = \lambda f. \lambda y. f (f y)$$

I.e., `double` is the function that, when applied to a function `f`, yields a *function* that, when applied to an argument `y`, yields `f (f y)`.

## Example

```
double plus3 0
=  (λf. λy. f (f y))
   (λx. succ (succ (succ x)))
   0
i.e. (λy. (λx. succ (succ (succ x)))
      ((λx. succ (succ (succ x))) y))
      0
i.e. (λx. succ (succ (succ x)))
      ((λx. succ (succ (succ x))) 0)
i.e. (λx. succ (succ (succ x)))
      (succ (succ (succ 0)))
i.e. succ (succ (succ (succ (succ (succ 0)))))
```

# The Pure Lambda-Calculus

As the preceding examples suggest, once we have  $\lambda$ -abstraction and application, we can throw away all the other language primitives and still have left a rich and powerful programming language.

In this language — the “pure lambda-calculus” — *everything* is a function.

- ▶ Variables always denote functions
- ▶ Functions always take other functions as parameters
- ▶ The result of a function is always a function



# Formalities

# Syntax

$t ::=$

$x$

$\lambda x. t$

$t \ t$

*terms*

*variable*

*abstraction*

*application*

*Terminology:*

- ▶ terms in the pure  $\lambda$ -calculus are often called  $\lambda$ -terms
- ▶ terms of the form  $\lambda x. t$  are called  $\lambda$ -abstractions or just *abstractions*

## Syntactic conventions

Since  $\lambda$ -calculus provides only one-argument functions, all multi-argument functions must be written in curried style.

The following conventions make the linear forms of terms easier to read and write:

- ▶ Application associates to the left

*E.g.,  $t\ u\ v$  means  $(t\ u)\ v$ , not  $t\ (u\ v)$*

- ▶ Bodies of  $\lambda$ - abstractions extend as far to the right as possible

*E.g.,  $\lambda x. \lambda y. x\ y$  means  $\lambda x. (\lambda y. x\ y)$ , not  $\lambda x. (\lambda y. x)\ y$*

# Scope

The  $\lambda$ -abstraction term  $\lambda x. t$  *binds* the variable  $x$ .

The *scope* of this binding is the *body*  $t$ .

Occurrences of  $x$  inside  $t$  are said to be *bound* by the abstraction.

Occurrences of  $x$  that are *not* within the scope of an abstraction binding  $x$  are said to be *free*.

Test:

$$\lambda x. \lambda y. x y z$$

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Test:

$$\lambda x. \lambda y. x y z$$
$$\lambda x. (\lambda y. z y) y$$

# Values

$v ::=$

$\lambda x. t$

*values*

*abstraction value*

# Operational Semantics

Computation rule:

$$(\lambda x. t_{12}) \ v_2 \longrightarrow [x \mapsto v_2] t_{12} \quad (\text{E-APPABS})$$

*Notation:  $[x \mapsto v_2] t_{12}$  is “the term that results from substituting free occurrences of  $x$  in  $t_{12}$  with  $v_2$ .”*

# Operational Semantics

Computation rule:

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*Notation:  $[x \mapsto v_2] t_{12}$  is “the term that results from substituting free occurrences of  $x$  in  $t_{12}$  with  $v_2$ .”*

Congruence rules:

$$\frac{t_1 \longrightarrow t'_1}{t_1 \ t_2 \longrightarrow t'_1 \ t_2} \quad (\text{E-APP1})$$

$$\frac{t_2 \longrightarrow t'_2}{v_1 \ t_2 \longrightarrow v_1 \ t'_2} \quad (\text{E-APP2})$$



## Terminology

A term of the form  $(\lambda x. t) v$  — that is, a  $\lambda$ -abstraction applied to a *value* — is called a *redex* (short for “reducible expression”).

## Alternative evaluation strategies

Strictly speaking, the language we have defined is called the *pure, call-by-value lambda-calculus*.

The evaluation strategy we have chosen — *call by value* — reflects standard conventions found in most mainstream languages.

Some other common ones:

- ▶ Call by name (cf. Haskell)
- ▶ Normal order (leftmost/outermost)
- ▶ Full (non-deterministic) beta-reduction

# Classical Lambda Calculus

## Full beta reduction

The classical lambda calculus allows full beta reduction.

- ▶ The argument of a  $\beta$ -reduction to be an arbitrary term, not just a value.
- ▶ Reduction may appear anywhere in a term.

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Congruence rules:

$$\frac{t_1 \longrightarrow t'_1}{t_1 \ t_2 \longrightarrow t'_1 \ t_2} \quad (\text{E-APP1})$$

$$\frac{t_2 \longrightarrow t'_2}{t_1 \ t_2 \longrightarrow t_1 \ t'_2} \quad (\text{E-APP2})$$

$$\frac{t \longrightarrow t'}{\lambda x. t \longrightarrow \lambda x. t'} \quad (\text{E-ABS})$$

## Substitution revisited

Remember:  $[x \mapsto v_2] t_{12}$  is “the term that results from substituting free occurrences of  $x$  in  $t_{12}$  with  $v_2$ .”

This is trickier than it looks!

For example:

$$\begin{aligned} & (\lambda x. (\lambda y. x)) y \\ \longrightarrow & [x \mapsto y] \lambda y. x \\ = & ??? \end{aligned}$$

## Substitution revisited

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This is trickier than it looks!

For example:

$$\begin{aligned} & (\lambda x. (\lambda y. x)) y \\ \longrightarrow & [x \mapsto y] \lambda y. x \\ = & ??? \end{aligned}$$

Solution:

need to rename bound variables before performing the substitution.

$$\begin{aligned} & (\lambda x. (\lambda y. x)) y \\ = & (\lambda x. (\lambda z. x)) y \\ \longrightarrow & [x \mapsto y] \lambda z. x \\ = & \lambda z. y \end{aligned}$$



## Alpha conversion

Renaming bound variables is formalized as  $\alpha$ -conversion.

Conversion rule:

$$\frac{y \notin \text{fv}(t)}{\lambda x. t =_{\alpha} \lambda y. [x \mapsto y]t} \quad (\alpha)$$

Equivalence rules:

$$\frac{t_1 =_{\alpha} t_2}{t_2 =_{\alpha} t_1} \quad (\alpha\text{-SYMM})$$

$$\frac{t_1 =_{\alpha} t_2 \quad t_2 =_{\alpha} t_3}{t_1 =_{\alpha} t_3} \quad (\alpha\text{-TRANS})$$

Congruence rules: the usual ones.

# Confluence

Full  $\beta$ -reduction makes it possible to have different reduction paths.

Q: Can a term evaluate to more than one normal form?

# Confluence

Full  $\beta$ -reduction makes it possible to have different reduction paths.

Q: Can a term evaluate to more than one normal form?

The answer is no; this is a consequence of the following

**Theorem** [Church-Rosser]

Let  $t$ ,  $t_1$ ,  $t_2$  be terms such that  $t \longrightarrow^* t_1$  and  $t \longrightarrow^* t_2$ . Then there exists a term  $t_3$  such that  $t_1 \longrightarrow^* t_3$  and  $t_2 \longrightarrow^* t_3$ .

# Programming in the Lambda-Calculus

## Multiple arguments

Consider the function `double`, which returns a function as an argument.

$$\text{double} = \lambda f. \lambda y. f (f y)$$

This idiom — a  $\lambda$ -abstraction that does nothing but immediately yield another abstraction — is very common in the  $\lambda$ -calculus.

In general,  $\lambda x. \lambda y. t$  is a function that, given a value  $v$  for  $x$ , yields a function that, given a value  $u$  for  $y$ , yields  $t$  with  $v$  in place of  $x$  and  $u$  in place of  $y$ .

That is,  $\lambda x. \lambda y. t$  is a two-argument function.

(Recall the discussion of *currying* in OCaml.)

# The “Church Booleans”

`tru` =  $\lambda t. \lambda f. t$   
`fls` =  $\lambda t. \lambda f. f$

$\text{tru } v \ w$   
=  $\frac{(\lambda t. \lambda f. t) \ v}{\lambda f. v} \ w$  by definition  
→  $\frac{(\lambda f. v) \ w}{v}$  reducing the underlined redex  
→  $v$  reducing the underlined redex

$\text{fls } v \ w$   
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→  $f$  reducing the underlined redex

## Functions on Booleans

```
not    =  λb. b fls tru
```

That is, `not` is a function that, given a boolean value `v`, returns `fls` if `v` is `tru` and `tru` if `v` is `fls`.

## Functions on Booleans

`and = λb. λc. b c fls`

That is, `and` is a function that, given two boolean values `v` and `w`, returns `w` if `v` is `tru` and `fls` if `v` is `fls`

Thus `and v w` yields `tru` if both `v` and `w` are `tru` and `fls` if either `v` or `w` is `fls`.



## Pairs

```
pair = λf.λs.λb. b f s  
fst  = λp. p tru  
snd  = λp. p fls
```

That is, `pair v w` is a function that, when applied to a boolean value `b`, applies `b` to `v` and `w`.

By the definition of booleans, this application yields `v` if `b` is `tru` and `w` if `b` is `fls`, so the first and second projection functions `fst` and `snd` can be implemented simply by supplying the appropriate boolean.

## Example

|                     |  |               |
|---------------------|--|---------------|
|                     | $\text{fst } (\text{pair } v \ w)$                                   |               |
| $=$                 | $\text{fst } ((\lambda f. \lambda s. \lambda b. b \ f \ s) \ v \ w)$ | by definition |
| $\longrightarrow$   | $\text{fst } ((\lambda s. \lambda b. b \ v \ s) \ w)$                | reducing      |
| $\longrightarrow$   | $\text{fst } (\lambda b. b \ v \ w)$                                 | reducing      |
| $=$                 | $(\lambda p. p \ \text{tru}) (\lambda b. b \ v \ w)$                 | by definition |
| $\longrightarrow$   | $(\lambda b. b \ v \ w) \ \text{tru}$                                | reducing      |
| $\longrightarrow$   | $\text{tru } v \ w$  | reducing      |
| $\longrightarrow^*$ | $v$  | as before.    |

# Church numerals

Idea: represent the number  $n$  by a function that “repeats some action  $n$  times.”

$$c_0 = \lambda s. \lambda z. z$$

$$c_1 = \lambda s. \lambda z. s \ z$$

$$c_2 = \lambda s. \lambda z. s \ (s \ z)$$

$$c_3 = \lambda s. \lambda z. s \ (s \ (s \ z))$$

That is, each number  $n$  is represented by a term  $c_n$  that takes two arguments,  $s$  and  $z$  (for “successor” and “zero”), and applies  $s$ ,  $n$  times, to  $z$ .

# Functions on Church Numerals

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Multiplication:

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Zero test:

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What about predecessor?

## Predecessor

```
zz = pair c0 c0
```

```
ss = λp. pair (snd p) (scc (snd p))
```

```
prd = λm. fst (m ss zz)
```

# Recursion in the Lambda-Calculus

## Recursion and divergence

Recursion and divergence are intertwined, so we need to consider divergent terms.

$$\text{omega} = (\lambda x. x x) (\lambda x. x x)$$

Note that `omega` evaluates in one step to itself!

So evaluation of `omega` never reaches a normal form: it *diverges*.

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So evaluation of `omega` never reaches a normal form: it *diverges*.

Being able to write a divergent computation does not seem very useful in itself. However, there are variants of `omega` that are *very* useful...



## Recall: Normal forms

- ▶ A *normal form* is a term that cannot take an evaluation step.
- ▶ A *stuck* term is a normal form that is not a value.

Does every term evaluate to a normal form?

No, *omega* is not in normal form.

## Recall: Normal forms

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- ▶ A *stuck* term is a normal form that is not a value.

Does every term evaluate to a normal form?

No,  $\omega$  is not in normal form.

But are there any stuck terms in the pure  $\lambda$ -calculus?

## Towards recursion: Iterated application

Suppose  $f$  is some  $\lambda$ -abstraction, and consider the following variant of  $\omega$ :

$$Y_f = (\lambda x. f (x x)) (\lambda x. f (x x))$$

## Towards recursion: Iterated application

Suppose  $f$  is some  $\lambda$ -abstraction, and consider the following variant of  $\omega$ :

$$Y_f = (\lambda x. f (x x)) (\lambda x. f (x x))$$

Now the “pattern of divergence” becomes more interesting:

$$\begin{aligned} Y_f &= \\ &\quad \underline{(\lambda x. f (x x)) (\lambda x. f (x x))} \\ &\quad \longrightarrow \\ &\quad f \left( \underline{(\lambda x. f (x x)) (\lambda x. f (x x))} \right) \\ &\quad \longrightarrow \\ &\quad f \left( f \left( \underline{(\lambda x. f (x x)) (\lambda x. f (x x))} \right) \right) \\ &\quad \longrightarrow \\ &\quad f \left( f \left( f \left( \underline{(\lambda x. f (x x)) (\lambda x. f (x x))} \right) \right) \right) \\ &\quad \longrightarrow \\ &\quad \dots \end{aligned}$$

$Y_f$  is still not very useful, since (like  $\omega$ ), all it does is diverge.

Is there any way we could “slow it down”?

## Delaying divergence

`poisonpill = λy. omega`

Note that `poisonpill` is a value — it will only diverge when we actually apply it to an argument. This means that we can safely pass it as an argument to other functions, return it as a result from functions, etc.

$$\begin{array}{c} \hline (\lambda p. \text{fst} (\text{pair } p \text{ fls}) \text{ tru}) \text{ poisonpill} \\ \longrightarrow \\ \text{fst} (\text{pair } \text{poisonpill} \text{ fls}) \text{ tru} \\ \longrightarrow^* \\ \hline \text{poisonpill } \text{tru} \\ \longrightarrow \\ \text{omega} \\ \longrightarrow \\ \dots \end{array}$$

## A delayed variant of omega

Here is a variant of `omega` in which the delay and divergence are a bit more tightly intertwined:

$$\text{omegav} = \lambda y. (\lambda x. (\lambda y. x \ x \ y)) (\lambda x. (\lambda y. x \ x \ y)) \ y$$

Note that `omegav` is a normal form. However, if we apply it to any argument `v`, it diverges:

$$\begin{aligned} & \text{omegav} \ v \\ &= \\ & \frac{(\lambda y. (\lambda x. (\lambda y. x \ x \ y)) (\lambda x. (\lambda y. x \ x \ y)) \ y) \ v}{\longrightarrow} \\ & \frac{(\lambda x. (\lambda y. x \ x \ y)) (\lambda x. (\lambda y. x \ x \ y)) \ v}{\longrightarrow} \\ & (\lambda y. (\lambda x. (\lambda y. x \ x \ y)) (\lambda x. (\lambda y. x \ x \ y)) \ y) \ v \\ &= \\ & \text{omegav} \ v \end{aligned}$$

## Another delayed variant

Suppose  $f$  is a function. Define

$$z_f = \lambda y. (\lambda x. f (\lambda y. x x y)) (\lambda x. f (\lambda y. x x y)) y$$

This term combines the “added  $f$ ” from  $Y_f$  with the “delayed divergence” of  $\omega_{\text{av}}$ .



If we now apply  $z_f$  to an argument  $v$ , something interesting happens:

$$\begin{aligned}
 & z_f \ v \\
 & = \\
 & \frac{(\lambda y. (\lambda x. f (\lambda y. x \ x \ y)) (\lambda x. f (\lambda y. x \ x \ y)) \ y) \ v}{\longrightarrow} \\
 & \quad \frac{(\lambda x. f (\lambda y. x \ x \ y)) (\lambda x. f (\lambda y. x \ x \ y)) \ v}{\longrightarrow} \\
 & f (\lambda y. (\lambda x. f (\lambda y. x \ x \ y)) (\lambda x. f (\lambda y. x \ x \ y)) \ y) \ v \\
 & = \\
 & f \ z_f \ v
 \end{aligned}$$

Since  $z_f$  and  $v$  are both values, the next computation step will be the reduction of  $f \ z_f$  — that is, before we “diverge,”  $f$  gets to do some computation.

Now we are getting somewhere.

# Recursion

Let

```
f  =  λfct.  
      λn.  
        if n=0 then 1  
        else n * (fct (pred n))
```

`f` looks just like the ordinary factorial function, except that, in place of a recursive call in the last line, it calls the function `fct`, which is passed as a parameter.

N.b.: for brevity, this example uses “real” numbers and booleans, infix syntax, etc. It can easily be translated into the pure lambda-calculus (using Church numerals, etc.).

We can use  $z_f$  to “tie the knot” in the definition of  $f$  and obtain a real recursive factorial function:

$$\begin{aligned}
 & z_f \ 3 \\
 & \longrightarrow^* \\
 & f \ z_f \ 3 \\
 & = \\
 & (\lambda fct. \ \lambda n. \ \dots) \ z_f \ 3 \\
 & \longrightarrow \quad \longrightarrow \\
 & \text{if } 3=0 \text{ then } 1 \text{ else } 3 * (z_f \ (\text{pred } 3)) \\
 & \longrightarrow^* \\
 & 3 * (z_f \ (\text{pred } 3)) \\
 & \longrightarrow \\
 & 3 * (z_f \ 2) \\
 & \longrightarrow^* \\
 & 3 * (f \ z_f \ 2) \\
 & \dots
 \end{aligned}$$

## A Generic $z$

If we define

$$z = \lambda f. z_f$$

i.e.,

$$z = \lambda f. \lambda y. (\lambda x. f (\lambda y. x x y)) (\lambda x. f (\lambda y. x x y)) y$$

then we can obtain the behavior of  $z_f$  for any  $f$  we like, simply by applying  $z$  to  $f$ .

$$z f \longrightarrow z_f$$

For example:

```
fact      =      z  ( λfct.  
                      λn.  
                        if n=0 then 1  
                        else n * (fct (pred n)) )
```

## Technical Note

The term `z` here is essentially the same as the `fix` discussed the book.

$$z = \lambda f. \lambda y. (\lambda x. f (\lambda y. x x y)) (\lambda x. f (\lambda y. x x y)) y$$
$$\text{fix} = \lambda f. (\lambda x. f (\lambda y. x x y)) (\lambda x. f (\lambda y. x x y))$$

`z` is hopefully slightly easier to understand, since it has the property that  $z f v \longrightarrow^* f (z f) v$ , which `fix` does not (quite) share.