

Foundations of Software Fall 2022

Week 11

Different Kinds of Maps

What is missing?

$$\begin{array}{ll} \textit{Term} & \rightarrow \textit{Term} \quad (\lambda x.t) \\ \textit{Type} & \rightarrow \textit{Term} \quad (\Lambda X.t) \end{array}$$

Different Kinds of Maps

What is missing?

<i>Term</i>	\rightarrow	<i>Term</i>	$(\lambda x.t)$
<i>Type</i>	\rightarrow	<i>Term</i>	$(\Lambda X.t)$
<i>Type</i>	\rightarrow	<i>Type</i>	???
<i>Term</i>	\rightarrow	<i>Type</i>	???

Agenda today:

- ▶ Type operators
- ▶ Dependent types

Type Operators and System F_ω

Type Operators

Example. Type operators in Scala:

```
type MkFun[T] = T => T
val f: MkFun[Int] = (x: Int) => x
```

Type Operators

Example. Type operators in Scala:

```
type MkFun[T] = T => T
val f: MkFun[Int] = (x: Int) => x
```

Type operators are functions at type-level.

$\lambda X :: K.T$

Type Operators

Example. Type operators in Scala:

```
type MkFun[T] = T => T  
val f: MkFun[Int] = (x: Int) => x
```

Type operators are functions at type-level.

$\lambda X :: K. T$

Two Problems:

- ▶ Type checking of type operators
- ▶ Equivalence of types

Kinding

Problem: avoid meaningless types, like $MkFun[Int, String]$.

Kinding

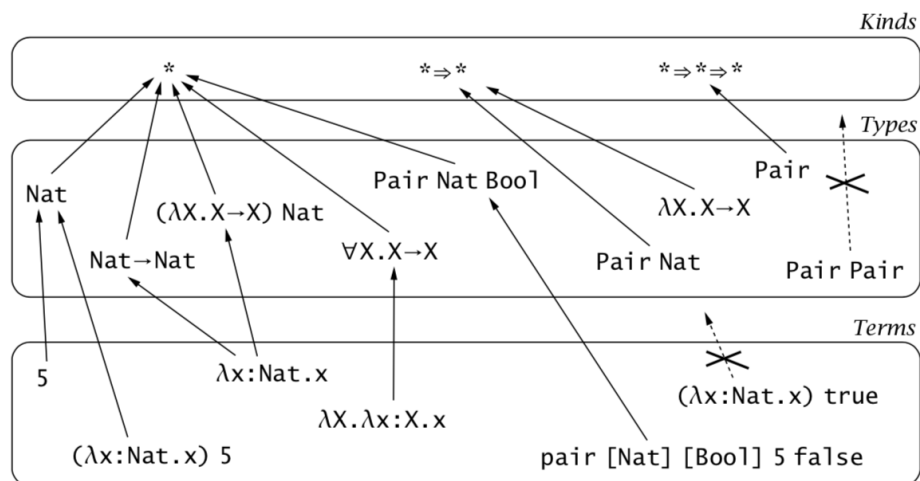
Problem: avoid meaningless types, like *MkFun[Int, String]*.

- $*$ proper types, e.g. *Bool*, $Int \rightarrow Int$
- $* \Rightarrow *$ type operators: map proper type to proper type
- $* \Rightarrow * \Rightarrow *$ two-argument operators
- $(* \Rightarrow *) \Rightarrow *$ type operators: map type operators to proper types

Kinding

Problem: avoid meaningless types, like *MkFun[Int, String]*.

- $*$ proper types, e.g. *Bool*, $Int \rightarrow Int$
- $* \Rightarrow *$ type operators: map proper type to proper type
- $* \Rightarrow * \Rightarrow *$ two-argument operators
- $(* \Rightarrow *) \Rightarrow *$ type operators: map type operators to proper types



Equivalence of Types

Problem: all the types below are equivalent

$$\begin{array}{lll} \text{Nat} \rightarrow \text{Bool} & \text{Nat} \rightarrow \text{Id Bool} & \text{Id Nat} \rightarrow \text{Id Bool} \\ \text{Id Nat} \rightarrow \text{Bool} & \text{Id (Nat} \rightarrow \text{Bool)} & \text{Id (Id (Id Nat} \rightarrow \text{Bool))} \end{array}$$

We need to introduce *definitional equivalence* relation on types, written $S \equiv T$. The most important rule is:

$$(\lambda X :: K. S) T \equiv [X \mapsto T]S \quad (\text{Q-APPABS})$$

And we need one typing rule:

$$\frac{\Gamma \vdash t : S \quad S \equiv T}{\Gamma \vdash t : T} \quad (\text{T-EQ})$$

First-class Type Operators

Scala supports passing type operators as argument:

```
def makeInt[F[_]](f: () => F[Int]): F[Int] = f()
```

```
makeInt[List]() => List[Int](3)
```

```
makeInt[Option]() => None
```

First-class type operators supports *polymorphism* for type operators, which enables more patterns in type-safe functional programming.

System F_ω — Syntax

Formalizing first-class type operators leads to System F_ω :

$t ::=$...	<i>terms</i>
	$\lambda X :: K. t$	<i>type abstraction</i>
$T ::=$	X	<i>types</i>
	$T \rightarrow T$	<i>type variable</i>
	$\forall X :: K. T$	<i>type of functions</i>
	$\lambda X :: K. T$	<i>universal type</i>
	$T \ T$	<i>operator abstraction</i>
		<i>operator application</i>
$K ::=$		<i>kinds</i>
	$*$	<i>kind of proper types</i>
	$K \Rightarrow K$	<i>kind of operators</i>

System F_ω — Semantics

$$\frac{t_1 \longrightarrow t'_1}{t_1 \ t_2 \longrightarrow t'_1 \ t_2} \quad (\text{E-APP1})$$

$$\frac{t_2 \longrightarrow t'_2}{t_1 \ t_2 \longrightarrow t_1 \ t'_2} \quad (\text{E-APP2})$$

$$(\lambda x : T_1. t_1) \ v_2 \longrightarrow [x \mapsto v_2] t_1 \quad (\text{E-APPABS})$$

$$\frac{t \longrightarrow t'}{t \ [T] \longrightarrow t' \ [T]} \quad (\text{E-TAPP})$$

$$(\lambda X :: K. t_1) \ [T] \longrightarrow [X \mapsto T] t_1 \quad (\text{E-TAPPTABS})$$

System F_ω — Kinding

$$\begin{array}{c}
 \dfrac{X :: K \in \Gamma}{\Gamma \vdash X :: K} \quad (\text{K-TVAR}) \\
 \\
 \dfrac{\Gamma, X :: K_1 \vdash T_2 : K_2}{\Gamma \vdash \lambda X :: K_1. T_2 :: K_1 \Rightarrow K_2} \quad (\text{K-ABS}) \\
 \\
 \dfrac{\Gamma \vdash T_1 : K_1 \Rightarrow K_2 \quad \Gamma \vdash T_2 : K_1}{\Gamma \vdash T_1 T_2 :: K_2} \quad (\text{K-APP}) \\
 \\
 \dfrac{\Gamma \vdash T_1 : * \quad \Gamma \vdash T_2 : *}{\Gamma \vdash T_1 \rightarrow T_2 :: *} \quad (\text{K-ARROW}) \\
 \\
 \dfrac{\Gamma, X :: K_1 \vdash T_2 :: *}{\Gamma \vdash \forall X :: K_1. T_2 :: *} \quad (\text{K-ALL})
 \end{array}$$

System F_ω — Type Equivalence

$$\begin{array}{c}
 T \equiv T \quad \dfrac{T \equiv S}{S \equiv T} \quad \dfrac{S \equiv U \quad U \equiv T}{S \equiv T} \\
 \\
 \dfrac{S_1 \equiv T_1 \quad S_2 \equiv T_2}{S_1 \rightarrow S_2 \equiv T_1 \rightarrow T_2} \quad (\text{Q-ARROW}) \\
 \\
 \dfrac{S_2 \equiv T_2}{\forall X :: K_1. S_2 \equiv \forall X :: K_1. T_2} \quad (\text{K-ALL}) \\
 \\
 \dfrac{S_2 \equiv T_2}{\lambda X :: K_1. S_2 \equiv \lambda X :: K_1. T_2} \quad (\text{Q-ABS}) \\
 \\
 \dfrac{S_1 \equiv T_1 \quad S_2 \equiv T_2}{S_1 S_2 \equiv T_1 T_2} \quad (\text{Q-APP}) \\
 \\
 (\lambda X :: K. T_1) T_2 \equiv [X \mapsto T_2] T_1 \quad (\text{Q-APPABS})
 \end{array}$$

System F_ω — Typing

$\frac{x : T \in \Gamma}{\Gamma \vdash x : T}$	(T-VAR)
$\frac{\Gamma \vdash T_1 :: * \quad \Gamma, x : T_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda x : T_1. t_2 : T_1 \rightarrow T_2}$	(T-ABS)
$\frac{\Gamma \vdash t_1 : S \rightarrow T \quad \Gamma \vdash t_2 : S}{\Gamma \vdash t_1 t_2 : T}$	(T-APP)
$\frac{\Gamma, X :: K_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda X :: K_1. t_2 : \forall X :: K_1. T_2}$	(T-TABS)
$\frac{\Gamma \vdash t : \forall X :: K. T_2 \quad \Gamma \vdash T :: K}{\Gamma \vdash t [T] : [X \mapsto T] T_2}$	(T-TAPP)
$\frac{\Gamma \vdash t : S \quad S \equiv T \quad \Gamma \vdash T :: *}{\Gamma \vdash t : T}$	(T-EQ)

Example

```
type PairRep[Pair :: * ⇒ * ⇒ *] = {  
  pair : ∀X.∀Y.X → Y → (Pair X Y),  
  fst  : ∀X.∀Y.(Pair X Y) → X,  
  snd  : ∀X.∀Y.(Pair X Y) → Y  
}
```

```
def swap[Pair :: * ⇒ * ⇒ *, X :: *, Y :: *]  
  (rep : PairRep Pair)  
  (pair : Pair X Y) : Pair Y X  
=  
  let x = rep.fst [X] [Y] pair in  
  let y = rep.snd [X] [Y] pair in  
  rep.pair [Y] [X] y x
```

The method `swap` works for any representation of pairs.

Properties

Theorem [Preservation]: if $\Gamma \vdash t : T$ and $t \longrightarrow t'$, then $\Gamma \vdash t' : T$.

Theorem [Progress]: if $\vdash t : T$, then either t is a value or there exists t' with $t \longrightarrow t'$.

Dependent Types

Why Does It Matter?

Example 1. Track length of vectors in types:

$$\begin{aligned} NVec &:: Nat \rightarrow * \\ first &: (n:Nat) \rightarrow NVec (n + 1) \rightarrow Nat \end{aligned}$$

$(x:S) \rightarrow T$ is called **dependent function type**. It is impossible to pass a vector of length 0 to the function *first*.

Why Does It Matter?

Example 1. Track length of vectors in types:

$$\begin{aligned} NVec &:: Nat \rightarrow * \\ first &: (n:Nat) \rightarrow NVec (n + 1) \rightarrow Nat \end{aligned}$$

$(x:S) \rightarrow T$ is called **dependent function type**. It is impossible to pass a vector of length 0 to the function *first*.

Example 2. Safe formatting for *sprintf*:

$$sprintf : (f:Format) \rightarrow Data(f) \rightarrow String$$
$$\begin{aligned} Data([]) &= Unit \\ Data("%d" :: cs) &= Nat * Data(cs) \\ Data("%s" :: cs) &= String * Data(cs) \\ Data(c :: cs) &= Data(cs) \end{aligned}$$

Dependent Function Type (a.k.a. Π Types)

A dependent function type is inhabited by *a dependent function*:

$$\lambda x:S.t \quad : \quad (x:S) \rightarrow T$$

Dependent Function Type (a.k.a. Π Types)

A dependent function type is inhabited by *a dependent function*:

$$\lambda x:S.t \quad : \quad (x:S) \rightarrow T$$

If T does not depend on x , it degenerates to function types:

$$(x:S) \rightarrow T = S \rightarrow T \quad \text{where } x \text{ does not appear free in } T$$

The Calculus of Constructions

The Calculus of Constructions: Syntax

$t ::=$	<i>terms</i>
s	<i>sort</i>
x	<i>variable</i>
$\lambda x:t.t$	<i>abstraction</i>
$t\ t$	<i>application</i>
$(x:t) \rightarrow t$	<i>dependent type</i>
$s ::=$	<i>sorts</i>
$*$	<i>sort of proper types</i>
\square	<i>sort of kinds</i>
$\Gamma ::=$	<i>contexts</i>
\emptyset	<i>empty context</i>
$\Gamma, x:T$	<i>term variable binding</i>

The semantics is the usual β -reduction.

The Calculus of Constructions: Typing

$$\begin{array}{c}
 \vdash * : \square \text{ (T-AXIOM)} \qquad \frac{x:T \in \Gamma}{\Gamma \vdash x : T} \text{ (T-VAR)} \\
 \\
 \frac{\Gamma \vdash S : s_1 \quad \Gamma, x:S \vdash t : T}{\Gamma \vdash \lambda x:S. t : (x:S) \rightarrow T} \text{ (T-ABS)} \\
 \\
 \frac{\Gamma \vdash t_1 : (x:S) \rightarrow T \quad \Gamma \vdash t_2 : S}{\Gamma \vdash t_1 \ t_2 : [x \mapsto t_2] T} \text{ (T-APP)} \\
 \\
 \frac{\Gamma \vdash S : s_1 \quad \Gamma, x:S \vdash T : s_2}{\Gamma \vdash (x:S) \rightarrow T : s_2} \text{ (T-PI)} \\
 \\
 \frac{\Gamma \vdash t : T \quad T \equiv T' \quad \Gamma \vdash T' : s}{\Gamma \vdash t : T'} \text{ (T-CONV)}
 \end{array}$$

The equivalence relation $T \equiv T'$ is based on β -reduction.

Four Kinds of Lambdas

Example	Type
$\lambda x:\mathbb{N}. x + 1$	$\mathbb{N} \rightarrow \mathbb{N}$
$\lambda f:\mathbb{N} \rightarrow \mathbb{N}. f \ x$	$(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$

Four Kinds of Lambdas

Example	Type
$\lambda x:\mathbb{N}.x + 1$	$\mathbb{N} \rightarrow \mathbb{N}$
$\lambda f:\mathbb{N} \rightarrow \mathbb{N}.f\ x$	$(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$
$\lambda X:*. \lambda x:X. x$	$(X:*) \rightarrow X \rightarrow X$
$\lambda F:* \rightarrow *. \lambda x:F\ \mathbb{N}. x$	$(F:* \rightarrow *) \rightarrow (F\ \mathbb{N}) \rightarrow (F\ \mathbb{N})$

Four Kinds of Lambdas

Example	Type
$\lambda x:\mathbb{N}.x + 1$	$\mathbb{N} \rightarrow \mathbb{N}$
$\lambda f:\mathbb{N} \rightarrow \mathbb{N}.f\ x$	$(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$
$\lambda X:*. \lambda x:X. x$	$(X:*) \rightarrow X \rightarrow X$
$\lambda F:* \rightarrow *. \lambda x:F\ \mathbb{N}. x$	$(F:* \rightarrow *) \rightarrow (F\ \mathbb{N}) \rightarrow (F\ \mathbb{N})$
$\lambda X:*. X$	$* \rightarrow *$
$\lambda F:* \rightarrow *. F\ \mathbb{N}$	$(* \rightarrow *) \rightarrow *$

Four Kinds of Lambdas

Example	Type
$\lambda x:\mathbb{N}.x + 1$	$\mathbb{N} \rightarrow \mathbb{N}$
$\lambda f:\mathbb{N} \rightarrow \mathbb{N}.f\ x$	$(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$
$\lambda X:*. \lambda x:X. x$	$(X:*) \rightarrow X \rightarrow X$
$\lambda F:* \rightarrow *. \lambda x:F\ \mathbb{N}. x$	$(F:* \rightarrow *) \rightarrow (F\ \mathbb{N}) \rightarrow (F\ \mathbb{N})$
$\lambda X:*. X$	$* \rightarrow *$
$\lambda F:* \rightarrow *. F\ \mathbb{N}$	$(* \rightarrow *) \rightarrow *$
$\lambda n:\mathbb{N}. NVec\ n$	$\mathbb{N} \rightarrow *$
$\lambda f:\mathbb{N} \rightarrow \mathbb{N}. NVec\ (f\ 6)$	$(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow *$

Strong Normalization

Given the following β -reduction rules

$$\frac{t_1 \longrightarrow t'_1}{\lambda x:T_1. t_1 \longrightarrow \lambda x:T_1. t'_1} \quad (\beta\text{-ABS})$$

$$\frac{t_1 \longrightarrow t'_1}{t_1\ t_2 \longrightarrow t'_1\ t_2} \quad (\beta\text{-APP1})$$

$$\frac{t_2 \longrightarrow t'_2}{t_1\ t_2 \longrightarrow t_1\ t'_2} \quad (\beta\text{-APP2})$$

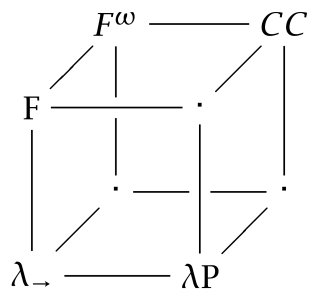
$$(\lambda x:T_1. t_1) t_2 \longrightarrow [x \mapsto t_2] t_1 \quad (\beta\text{-APPABS})$$

Theorem [Strong Normalization]: if $\Gamma \vdash t : T$, then there is no infinite sequence of terms t_i such that $t = t_1$ and $t_i \longrightarrow t_{i+1}$.

Pure Type Systems

$$\frac{\Gamma \vdash S : s_i \quad \Gamma, x:S \vdash T : s_j}{\Gamma \vdash (x:S) \rightarrow T : s_j} \quad (\text{T-PI})$$

System	(s_i, s_j)			
λ_{\rightarrow}	{	$(*, *)$	}	
λP	{	$(*, *)$, $(*, \square)$	}	
F	{	$(*, *)$, $(\square, *)$	}	
F^ω	{	$(*, *)$, $(\square, *)$, (\square, \square)	}	
CC	{	$(*, *)$, $(*, \square)$, $(\square, *)$, (\square, \square)	}	



The Lambda Cube
 $\lambda_{\rightarrow} \longrightarrow F \longrightarrow F^\omega \longrightarrow CC$

Dependent Types in Coq

Proof Assistants

Dependent type theories are at the foundation of proof assistants, like Coq, Agda, etc.

By *Curry-Howard Correspondence*

- ▶ proofs \longleftrightarrow programs
- ▶ propositions \longleftrightarrow types

Proof Assistants

Dependent type theories are at the foundation of proof assistants, like Coq, Agda, etc.

By *Curry-Howard Correspondence*

- ▶ proofs \longleftrightarrow programs
- ▶ propositions \longleftrightarrow types

Two impactful projects based on Coq:

- ▶ CompCert: certified C compiler
- ▶ Mechanized proof of 4-color theorem

Type Universes in Coq

The rule $\Gamma \vdash \text{Type} : \text{Type}$ is unsound (Girard's paradox).

$$\Gamma \vdash \text{Prop} : \text{Type}_1$$

$$\Gamma \vdash \text{Set} : \text{Type}_1$$

$$\Gamma \vdash \text{Type}_i : \text{Type}_{i+1}$$

$$\frac{\Gamma, x:A \vdash B : \text{Prop} \quad \Gamma \vdash A : s}{\Gamma \vdash (x : A) \rightarrow B : \text{Prop}}$$

$$\frac{\Gamma, x:A \vdash B : \text{Set} \quad \Gamma \vdash A : s \quad s \in \{\text{Prop}, \text{Set}\}}{\Gamma \vdash (x : A) \rightarrow B : \text{Set}}$$

$$\frac{\Gamma, x:A \vdash B : \text{Type}_i \quad \Gamma \vdash A : \text{Type}_i}{\Gamma \vdash (x : A) \rightarrow B : \text{Type}_i}$$

Coq 101 - inductive definitions and recursion

```
1 Inductive nat : Type :=  
2   | 0  
3   | S (n : nat).
```

Coq 101 - inductive definitions and recursion

```
1 Inductive nat : Type :=  
2   | 0  
3   | S (n : nat).  
  
1 Fixpoint double (n : nat) : nat :=  
2   match n with  
3     | 0 => 0  
4     | S n' => S (S (double n'))  
5   end.
```

Recursion has to be **structural**.

Coq 101 - inductive definitions and recursion

```
1 Inductive nat : Type :=  
2   | 0  
3   | S (n : nat).  
  
1 Fixpoint double (n : nat) : nat :=  
2   match n with  
3     | 0 => 0  
4     | S n' => S (S (double n'))  
5   end.
```

Recursion has to be **structural**.

```
1 Inductive even : nat -> Prop :=  
2   | even0 : even 0  
3   | evenS : forall x:nat, even x -> even (S (S x)).
```

Coq 101 - proofs

```
1 Definition even_prop := forall x:nat, even (double x).
2
3 Fixpoint even_proof(x: nat): even (double x) :=
4   match x with
5   | 0      => even0
6   | S n'   => evenS (double n') (even_proof n')
7   end.
8
9 Check even_proof : even_prop.
```

Coq 101 - proofs

```
1 Definition even_prop := forall x:nat, even (double x).
2
3 Fixpoint even_proof(x: nat): even (double x) :=
4   match x with
5   | 0      => even0
6   | S n'   => evenS (double n') (even_proof n')
7   end.
8
9 Check even_proof : even_prop.
```

The 2nd branch has the type $\text{even } S (S (\text{double } n'))$, and Coq knows by normalizing the types:

$$\text{even } S (S (\text{double } n')) \equiv_{\beta} \text{even } (\text{double } (S n'))$$

Recap: Curry-Howard Correspondence

Propositions as types in the context of intuitionistic logic.

Proposition	Term & Type
$A \wedge B$	$t : (A, B)$
$A \vee B$	$t : A + B$
$A \rightarrow B$	$t : A \rightarrow B$
\perp	$t : False$
$\neg A$	$t : A \rightarrow False$
$\forall x:A. B$	$t : (x : A) \rightarrow B$
$\exists x:A. B$	$t : (x:A, B)$

Curry-Howard correspondence in Coq

```
1 Inductive and (A B:Prop) : Prop :=
2   conj : A -> B -> A /\ B
3 where "A /\ B" := (and A B) : type_scope.
```

Curry-Howard correspondence in Coq

```
1 Inductive and (A B:Prop) : Prop :=  
2   conj : A -> B -> A /\ B  
3 where "A /\ B" := (and A B) : type_scope.
```

```
1 Inductive or (A B:Prop) : Prop :=  
2   | or_introl : A -> A \/ B  
3   | or_intror : B -> A \/ B  
4 where "A \/ B" := (or A B) : type_scope.
```

Curry-Howard correspondence in Coq

```
1 Inductive and (A B:Prop) : Prop :=  
2   conj : A -> B -> A /\ B  
3 where "A /\ B" := (and A B) : type_scope.
```

```
1 Inductive or (A B:Prop) : Prop :=  
2   | or_introl : A -> A \/ B  
3   | or_intror : B -> A \/ B  
4 where "A \/ B" := (or A B) : type_scope.
```

```
1 Inductive False : Prop :=.
```

Curry-Howard correspondence in Coq

```
1 Inductive and (A B:Prop) : Prop :=  
2   conj : A -> B -> A /\ B  
3 where "A /\ B" := (and A B) : type_scope.
```

```
1 Inductive or (A B:Prop) : Prop :=  
2   | or_introl : A -> A \/ B  
3   | or_intror : B -> A \/ B  
4 where "A \/ B" := (or A B) : type_scope.
```

```
1 Inductive False : Prop :=.
```

```
1 Definition not (A:Prop) := A -> False.  
2 Notation "~ x" := (not x) : type_scope.
```

Curry-Howard correspondence in Coq - continued

```
1 Notation "A -> B" := (forall (_ : A), B) : type_scope.  
2 Definition iff (A B:Prop) := (A -> B) /\ (B -> A).  
3 Notation "A <-> B" := (iff A B) : type_scope.
```


Curry-Howard correspondence in Coq - continued

```
1 Notation "A -> B" := (forall (_ : A), B) : type_scope.
2 Definition iff (A B:Prop) := (A -> B) /\ (B -> A).
3 Notation "A <-> B" := (iff A B) : type_scope.

1 Inductive ex (A:Type) (P:A -> Prop) : Prop :=
2   ex_intro : forall x:A, P x -> ex (A:=A) P.
3
4 Notation "'exists' x .. y , p" :=
5   (ex (fun x => .. (ex (fun y => p)) ..)) : type_scope.
```

Curry-Howard correspondence in Coq - continued

```
1 Notation "A -> B" := (forall (_ : A), B) : type_scope.
2 Definition iff (A B:Prop) := (A -> B) /\ (B -> A).
3 Notation "A <-> B" := (iff A B) : type_scope.

1 Inductive ex (A:Type) (P:A -> Prop) : Prop :=
2   ex_intro : forall x:A, P x -> ex (A:=A) P.
3
4 Notation "'exists' x .. y , p" :=
5   (ex (fun x => .. (ex (fun y => p)) ..)) : type_scope.

1 Inductive eq (A:Type) (x:A) : A -> Prop :=
2   eq_refl : x = x :>A
3
4 Notation "x = y" := (eq x y) : type_scope.
```

The equivalence between LEM and DNE

In **intuitionistic logics**, the *law of excluded middle* (LEM) and the *law of double negation* (DNE) are not provable.

- ▶ LEM: $\forall P. P \vee \neg P$
- ▶ DNE: $\forall P. \neg\neg P \rightarrow P$

By curry-howard correspondence, there are no terms that inhabit the types above.

The equivalence between LEM and DNE

In **intuitionistic logics**, the *law of excluded middle* (LEM) and the *law of double negation* (DNE) are not provable.

- ▶ LEM: $\forall P. P \vee \neg P$
- ▶ DNE: $\forall P. \neg\neg P \rightarrow P$

By curry-howard correspondence, there are no terms that inhabit the types above.

However, $\forall P. P \rightarrow \neg\neg P$ can be proved.

The equivalence between LEM and DNE

In **intuitionistic logics**, the *law of excluded middle* (LEM) and the *law of double negation* (DNE) are not provable.

- ▶ LEM: $\forall P. P \vee \neg P$
- ▶ DNE: $\forall P. \neg\neg P \rightarrow P$

By curry-howard correspondence, there are no terms that inhabit the types above.

However, $\forall P. P \rightarrow \neg\neg P$ can be proved. **How?**

The equivalence between LEM and DNE

In **intuitionistic logics**, the *law of excluded middle* (LEM) and the *law of double negation* (DNE) are not provable.

- ▶ LEM: $\forall P. P \vee \neg P$
- ▶ DNE: $\forall P. \neg\neg P \rightarrow P$

By curry-howard correspondence, there are no terms that inhabit the types above.

However, $\forall P. P \rightarrow \neg\neg P$ can be proved. **How?**

We will prove that LEM is equivalent to DNE:

- 1 **Definition** LEM: Prop := forall P: Prop, P \vee \neg P.
- 2 **Definition** DNE: Prop := forall P: Prop, $\neg\neg$ P \rightarrow P.
- 3 **Definition** LEM_DNE_EQ: Prop := LEM \leftrightarrow DNE.

LEM \rightarrow DNE

```
1 Definition LEM_To_DNE :=
2   fun (lem: forall P : Prop, P \ / ~ P) (Q:Prop) (q: ~~Q)
3     =>
4     match lem Q with
5     | or_introl l =>
6       l
7     | or_intror r =>
8       match (q r) with end
9     end.
10
11 Check LEM_To_DNE : LEM -> DNE.
```

DNE \rightarrow LEM

```
1 Definition DNE_To_LEM :=
2   fun (dne: forall P : Prop, ~~P -> P) (Q:Prop) =>
3     (dne (Q \ / ~ Q))
4     (fun H: ~(Q \ / ~ Q) =>
5       let nq := (fun q: Q => H (or_introl q))
6       in H (or_intror nq)
7     ).
8
9 Check DNE_To_LEM : DNE -> LEM.
10
11 Definition proof := conj LEM_To_DNE DNE_To_LEM.
12 Check proof : LEM <-> DNE.
```

Dependent Types in Programming Languages

Despite the huge success in proof assistants, its adoption in programming languages is limited.

- ▶ Scala supports *path-dependent types* and *literal types*.
- ▶ Dependent Haskell is proposed by researchers.

Dependent Types in Programming Languages

Despite the huge success in proof assistants, its adoption in programming languages is limited.

- ▶ Scala supports *path-dependent types* and *literal types*.
- ▶ Dependent Haskell is proposed by researchers.

Challenge: the decidability of type checking.

Problem with Type Checking

Value constructors:

$$\begin{aligned} NVec & : \mathbb{N} \rightarrow * \\ nil & : NVec\ 0 \\ cons & : \mathbb{N} \rightarrow (n:\mathbb{N}) \rightarrow NVec\ n \rightarrow NVec\ n + 1 \end{aligned}$$

Appending vectors:

$$\begin{aligned} append & : (m:\mathbb{N}) \rightarrow (n:\mathbb{N}) \rightarrow NVec\ m \rightarrow NVec\ n \rightarrow NVec\ (n + m) \\ append & = \lambda m:\mathbb{N}. \lambda n:\mathbb{N}. \lambda l:NVec\ m. \lambda t:NVec\ n. \\ & \quad \text{match } l \text{ with} \\ & \quad | nil \Rightarrow t \\ & \quad | cons\ x\ r\ y \Rightarrow cons\ x\ (r + n)\ (append\ r\ n\ y\ t) \end{aligned}$$

Question: How does the type checker know $S\ (r + n) = n + (S\ r)$?