Foundations of Software
Fall 2022

Week 7

Plan

PREVIOUSLY: unit, sequencing, let, pairs, tuples

TODAY:

1. options, variants
2. recursion
3. state

NEXT: exceptions?
NEXT: polymorphic (not so simple) typing

Records

$\mathtt{t}$	$::=$	$\ldots$	*terms*
		$\{\mathtt{l}_i\mathtt{=}\mathtt{t}_i\ ^{i \in 1..n}\}$	*record*
		$\mathtt{t.l}$	*projection*
$\mathtt{v}$	$::=$	$\ldots$	*values*
		$\{\mathtt{l}_i\mathtt{=}\mathtt{v}_i\ ^{i \in 1..n}\}$	*record value*
$\mathtt{T}$	$::=$	$\ldots$	*types*
		$\{\mathtt{l}_i\mathtt{:}\mathtt{T}_i\ ^{i \in 1..n}\}$	*type of records*

Evaluation rules for records

$$\{\mathtt{l}_i\mathtt{=}\mathtt{v}_i{}^{\,i\in 1..n}\}\mathtt{.l}_j \longrightarrow \mathtt{v}_j \qquad \text{(E-ProjRcd)}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{t}_1\mathtt{.l} \longrightarrow \mathtt{t}_1'\mathtt{.l}} \qquad \text{(E-Proj)}$$

$$\frac{\mathtt{t}_j \longrightarrow \mathtt{t}_j'}{\begin{array}{c}\{\mathtt{l}_i\mathtt{=}\mathtt{v}_i{}^{\,i\in 1..j-1}\mathtt{,}\mathtt{l}_j\mathtt{=}\mathtt{t}_j\mathtt{,}\mathtt{l}_k\mathtt{=}\mathtt{t}_k{}^{\,k\in j+1..n}\}\\ \longrightarrow \{\mathtt{l}_i\mathtt{=}\mathtt{v}_i{}^{\,i\in 1..j-1}\mathtt{,}\mathtt{l}_j\mathtt{=}\mathtt{t}_j'\mathtt{,}\mathtt{l}_k\mathtt{=}\mathtt{t}_k{}^{\,k\in j+1..n}\}\end{array}} \qquad \text{(E-Rcd)}$$

Typing rules for records

$$\frac{\text{for each } i \quad \Gamma \vdash \mathtt{t}_i : \mathtt{T}_i}{\Gamma \vdash \{\mathtt{l}_i\mathtt{=}\mathtt{t}_i\ ^{i \in 1..n}\} : \{\mathtt{l}_i\mathtt{:}\mathtt{T}_i\ ^{i \in 1..n}\}} \qquad \text{(T-Rcd)}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \{\mathtt{l}_i\mathtt{:}\mathtt{T}_i\ ^{i \in 1..n}\}}{\Gamma \vdash \mathtt{t}_1\mathtt{.l}_j : \mathtt{T}_j} \qquad \text{(T-Proj)}$$

Sums and variants

Sums – motivating example

```
PhysicalAddr = {firstlast:String, addr:String}
VirtualAddr  = {name:String, email:String}
Addr         = PhysicalAddr + VirtualAddr
inl   :  "PhysicalAddr → PhysicalAddr+VirtualAddr"
inr   :  "VirtualAddr → PhysicalAddr+VirtualAddr"
```

```
getName = λa:Addr.
  case a of
    inl x ⇒ x.firstlast
  | inr y ⇒ y.name;
```

New syntactic forms

`t`	::=	...	*terms*
		`inl t`	*tagging (left)*
		`inr t`	*tagging (right)*
		`case t of inl x⇒t` | `inr x⇒t`	*case*
`v`	::=	...	*values*
		`inl v`	*tagged value (left)*
		`inr v`	*tagged value (right)*
`T`	::=	...	*types*
		`T+T`	*sum type*

T_1+T_2 is a *disjoint union* of T_1 and T_2 (the tags `inl` and `inr` ensure disjointness)

New evaluation rules

$$\boxed{\mathtt{t} \longrightarrow \mathtt{t}'}$$

$$\begin{array}{l}\mathtt{case}\ (\mathtt{inl}\ \mathtt{v}_0) \\ \mathtt{of}\ \mathtt{inl}\ \mathtt{x}_1 \Rightarrow \mathtt{t}_1\ \mid\ \mathtt{inr}\ \mathtt{x}_2 \Rightarrow \mathtt{t}_2\end{array} \longrightarrow [\mathtt{x}_1 \mapsto \mathtt{v}_0]\mathtt{t}_1 \quad \text{(E-CaseInl)}$$

$$\begin{array}{l}\mathtt{case}\ (\mathtt{inr}\ \mathtt{v}_0) \\ \mathtt{of}\ \mathtt{inl}\ \mathtt{x}_1 \Rightarrow \mathtt{t}_1\ \mid\ \mathtt{inr}\ \mathtt{x}_2 \Rightarrow \mathtt{t}_2\end{array} \longrightarrow [\mathtt{x}_2 \mapsto \mathtt{v}_0]\mathtt{t}_2 \quad \text{(E-CaseInr)}$$

$$\frac{\mathtt{t}_0 \longrightarrow \mathtt{t}_0'}{\begin{array}{l}\mathtt{case}\ \mathtt{t}_0\ \mathtt{of}\ \mathtt{inl}\ \mathtt{x}_1 \Rightarrow \mathtt{t}_1\ \mid\ \mathtt{inr}\ \mathtt{x}_2 \Rightarrow \mathtt{t}_2 \\ \longrightarrow \mathtt{case}\ \mathtt{t}_0'\ \mathtt{of}\ \mathtt{inl}\ \mathtt{x}_1 \Rightarrow \mathtt{t}_1\ \mid\ \mathtt{inr}\ \mathtt{x}_2 \Rightarrow \mathtt{t}_2\end{array}} \quad \text{(E-Case)}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{inl}\ \mathtt{t}_1 \longrightarrow \mathtt{inl}\ \mathtt{t}_1'} \quad \text{(E-Inl)}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{inr}\ \mathtt{t}_1 \longrightarrow \mathtt{inr}\ \mathtt{t}_1'} \quad \text{(E-Inr)}$$

New typing rules

$$\boxed{\Gamma \vdash \mathtt{t} : \mathtt{T}}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_1}{\Gamma \vdash \mathtt{inl}\ \mathtt{t}_1 : \mathtt{T}_1\mathtt{+}\mathtt{T}_2} \qquad \text{(T-INL)}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_2}{\Gamma \vdash \mathtt{inr}\ \mathtt{t}_1 : \mathtt{T}_1\mathtt{+}\mathtt{T}_2} \qquad \text{(T-INR)}$$

$$\frac{\begin{array}{c}\Gamma \vdash \mathtt{t}_0 : \mathtt{T}_1\mathtt{+}\mathtt{T}_2 \\ \Gamma, \mathtt{x}_1\mathtt{:}\mathtt{T}_1 \vdash \mathtt{t}_1 : \mathtt{T} \qquad \Gamma, \mathtt{x}_2\mathtt{:}\mathtt{T}_2 \vdash \mathtt{t}_2 : \mathtt{T}\end{array}}{\Gamma \vdash \mathtt{case}\ \mathtt{t}_0\ \mathtt{of}\ \mathtt{inl}\ \mathtt{x}_1 \Rightarrow \mathtt{t}_1\ \mathtt{|}\ \mathtt{inr}\ \mathtt{x}_2 \Rightarrow \mathtt{t}_2 : \mathtt{T}} \qquad \text{(T-CASE)}$$

Sums and Uniqueness of Types

Problem:

> *If `t` has type `T`, then `inl t` has type `T+U` for every `U`.*

I.e., we've lost uniqueness of types.

Possible solutions:

- "Infer" `U` as needed during typechecking
- Give constructors different names and only allow each name to appear in one sum type (requires generalization to "variants," which we'll see next) — OCaml's solution
- Annotate each `inl` and `inr` with the intended sum type.

For simplicity, let's choose the third.

New syntactic forms

`t`	::=	...	*terms*
		`inl t as T`	*tagging (left)*
		`inr t as T`	*tagging (right)*
`v`	::=	...	*values*
		`inl v as T`	*tagged value (left)*
		`inr v as T`	*tagged value (right)*

Note that `as T` here is not the ascription operator that we saw before — i.e., not a separate syntactic form: in essence, there is an ascription "built into" every use of `inl` or `inr`.

New typing rules $\boxed{\Gamma \vdash \mathtt{t} : \mathtt{T}}$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_1}{\Gamma \vdash \mathtt{inl}\ \mathtt{t}_1\ \mathtt{as}\ \mathtt{T}_1\mathtt{+}\mathtt{T}_2 : \mathtt{T}_1\mathtt{+}\mathtt{T}_2} \quad \text{(T-Inl)}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_2}{\Gamma \vdash \mathtt{inr}\ \mathtt{t}_1\ \mathtt{as}\ \mathtt{T}_1\mathtt{+}\mathtt{T}_2 : \mathtt{T}_1\mathtt{+}\mathtt{T}_2} \quad \text{(T-Inr)}$$

Evaluation rules ignore annotations: $\boxed{\mathtt{t} \longrightarrow \mathtt{t}'}$

$$\begin{array}{l}\mathtt{case}\ (\mathtt{inl}\ \mathtt{v}_0\ \mathtt{as}\ \mathtt{T}_0)\\ \mathtt{of}\ \mathtt{inl}\ \mathtt{x}_1 \Rightarrow \mathtt{t}_1 \mid \mathtt{inr}\ \mathtt{x}_2 \Rightarrow \mathtt{t}_2\\ \qquad \longrightarrow [\mathtt{x}_1 \mapsto \mathtt{v}_0]\mathtt{t}_1\end{array} \tag{E-CaseInl}$$

$$\begin{array}{l}\mathtt{case}\ (\mathtt{inr}\ \mathtt{v}_0\ \mathtt{as}\ \mathtt{T}_0)\\ \mathtt{of}\ \mathtt{inl}\ \mathtt{x}_1 \Rightarrow \mathtt{t}_1 \mid \mathtt{inr}\ \mathtt{x}_2 \Rightarrow \mathtt{t}_2\\ \qquad \longrightarrow [\mathtt{x}_2 \mapsto \mathtt{v}_0]\mathtt{t}_2\end{array} \tag{E-CaseInr}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{inl}\ \mathtt{t}_1\ \mathtt{as}\ \mathtt{T}_2 \longrightarrow \mathtt{inl}\ \mathtt{t}_1'\ \mathtt{as}\ \mathtt{T}_2} \tag{E-Inl}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{inr}\ \mathtt{t}_1\ \mathtt{as}\ \mathtt{T}_2 \longrightarrow \mathtt{inr}\ \mathtt{t}_1'\ \mathtt{as}\ \mathtt{T}_2} \tag{E-Inr}$$

Variants

Just as we generalized binary products to labeled records, we can generalize binary sums to labeled *variants*.

New syntactic forms

$\mathtt{t}$	$::=$	...	*terms*
		$\mathtt{<l=t>\ as\ T}$	*tagging*
		$\mathtt{case\ t\ of\ <l_i=x_i>\Rightarrow t_i}\ ^{i\in 1..n}$	*case*
$\mathtt{T}$	$::=$	...	*types*
		$\mathtt{<l_i:T_i}\ ^{i\in 1..n}\mathtt{>}$	*type of variants*

New evaluation rules

$$\boxed{\mathtt{t} \longrightarrow \mathtt{t}'}$$

$$\begin{array}{l}\mathtt{case}\ (\mathtt{<l}_j\mathtt{=v}_j\mathtt{>}\ \mathtt{as}\ \mathtt{T})\ \mathtt{of}\ \mathtt{<l}_i\mathtt{=x}_i\mathtt{>}\Rightarrow\mathtt{t}_i\ {}^{i\in 1..n} \\ \qquad\qquad \longrightarrow [\mathtt{x}_j \mapsto \mathtt{v}_j]\mathtt{t}_j\end{array} \qquad \text{(E-CaseVariant)}$$

$$\frac{\mathtt{t}_0 \longrightarrow \mathtt{t}_0'}{\begin{array}{c}\mathtt{case}\ \mathtt{t}_0\ \mathtt{of}\ \mathtt{<l}_i\mathtt{=x}_i\mathtt{>}\Rightarrow\mathtt{t}_i\ {}^{i\in 1..n} \\ \longrightarrow \mathtt{case}\ \mathtt{t}_0'\ \mathtt{of}\ \mathtt{<l}_i\mathtt{=x}_i\mathtt{>}\Rightarrow\mathtt{t}_i\ {}^{i\in 1..n}\end{array}} \qquad \text{(E-Case)}$$

$$\frac{\mathtt{t}_i \longrightarrow \mathtt{t}_i'}{\mathtt{<l}_i\mathtt{=t}_i\mathtt{>}\ \mathtt{as}\ \mathtt{T} \longrightarrow \mathtt{<l}_i\mathtt{=t}_i'\mathtt{>}\ \mathtt{as}\ \mathtt{T}} \qquad \text{(E-Variant)}$$

New typing rules $\boxed{\Gamma \vdash \mathtt{t} : \mathtt{T}}$

$$\frac{\Gamma \vdash \mathtt{t}_j : \mathtt{T}_j}{\Gamma \vdash \mathtt{<l}_j\mathtt{=t}_j\mathtt{>}\ \mathtt{as}\ \mathtt{<l}_i\mathtt{:T}_i\ ^{i \in 1..n}\mathtt{>} : \mathtt{<l}_i\mathtt{:T}_i\ ^{i \in 1..n}\mathtt{>}} \quad \text{(T-Variant)}$$

$$\frac{\begin{array}{c}\Gamma \vdash \mathtt{t}_0 : \mathtt{<l}_i\mathtt{:T}_i\ ^{i \in 1..n}\mathtt{>} \\ \text{for each } i \quad \Gamma, \mathtt{x}_i\mathtt{:T}_i \vdash \mathtt{t}_i : \mathtt{T}\end{array}}{\Gamma \vdash \mathtt{case}\ \mathtt{t}_0\ \mathtt{of}\ \mathtt{<l}_i\mathtt{=x}_i\mathtt{>}\Rightarrow\mathtt{t}_i\ ^{i \in 1..n} : \mathtt{T}} \quad \text{(T-Case)}$$

Example

```
Addr = <physical:PhysicalAddr, virtual:VirtualAddr>;

a = <physical=pa> as Addr;

getName = λa:Addr.
  case a of
    <physical=x> ⇒ x.firstlast
  | <virtual=y> ⇒ y.name;
```

Options

Just like in OCaml...

```
OptionalNat = <none:Unit, some:Nat>;

Table = Nat→OptionalNat;

emptyTable = λn:Nat. <none=unit> as OptionalNat;

extendTable =
  λt:Table. λm:Nat. λv:Nat.
    λn:Nat.
      if equal n m then <some=v> as OptionalNat
      else t n;

x = case t(5) of
      <none=u> ⇒ 999
    | <some=v> ⇒ v;
```

Enumerations

```
Weekday = <monday:Unit, tuesday:Unit, wednesday:Unit,
           thursday:Unit, friday:Unit>;

nextBusinessDay = λw:Weekday.
  case w of <monday=x>    ⇒ <tuesday=unit> as Weekday
          | <tuesday=x>   ⇒ <wednesday=unit> as Weekday
          | <wednesday=x> ⇒ <thursday=unit> as Weekday
          | <thursday=x>  ⇒ <friday=unit> as Weekday
          | <friday=x>    ⇒ <monday=unit> as Weekday;
```