

# Foundations of Software Fall 2023

## Week 7

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### Plan

PREVIOUSLY: unit, sequencing, let, pairs, tuples

TODAY:

1. options, variants
2. recursion
3. state

NEXT: exceptions?

NEXT: polymorphic (not so simple) typing

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## Records

|                                 |                        |
|---------------------------------|------------------------|
| $t ::= \dots$                   | <i>terms</i>           |
| $\{l_i = t_i \mid i \in 1..n\}$ | <i>record</i>          |
| $t.l$                           | <i>projection</i>      |
| $v ::= \dots$                   | <i>values</i>          |
| $\{l_i = v_i \mid i \in 1..n\}$ | <i>record value</i>    |
| $T ::= \dots$                   | <i>types</i>           |
| $\{l_i : T_i \mid i \in 1..n\}$ | <i>type of records</i> |

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## Evaluation rules for records

$$\{l_i = v_i \mid i \in 1..n\}.l_j \longrightarrow v_j \quad (\text{E-PROJRCD})$$

$$\frac{t_1 \longrightarrow t'_1}{t_1.l \longrightarrow t'_1.l} \quad (\text{E-PROJ})$$

$$\frac{t_j \longrightarrow t'_j}{\begin{array}{l} \{l_i = v_i \mid i \in 1..j-1, l_j = t_j, l_k = t_k \mid k \in j+1..n\} \\ \longrightarrow \{l_i = v_i \mid i \in 1..j-1, l_j = t'_j, l_k = t_k \mid k \in j+1..n\} \end{array}} \quad (\text{E-RCD})$$

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## Typing rules for records

$$\frac{\text{for each } i \quad \Gamma \vdash t_i : T_i}{\Gamma \vdash \{l_i = t_i \mid i \in 1..n\} : \{l_i : T_i \mid i \in 1..n\}} \quad (\text{T-RCD})$$

$$\frac{\Gamma \vdash t_1 : \{l_i : T_i \mid i \in 1..n\}}{\Gamma \vdash t_1.l_j : T_j} \quad (\text{T-PROJ})$$

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## Sums and variants

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## Sums – motivating example

```
PhysicalAddr = {firstlast:String, addr:String}
VirtualAddr  = {name:String, email:String}
Addr         = PhysicalAddr + VirtualAddr
inl  : "PhysicalAddr → PhysicalAddr+VirtualAddr"
inr  : "VirtualAddr → PhysicalAddr+VirtualAddr"
```

```
getName = λa:Addr.
  case a of
    inl x ⇒ x.firstlast
  | inr y ⇒ y.name;
```

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### *New syntactic forms*

|                             |                      |
|-----------------------------|----------------------|
| $t ::= \dots$               | <i>terms</i>         |
| inl t                       | tagging (left)       |
| inr t                       | tagging (right)      |
| case t of inl x⇒t   inr x⇒t | case                 |
| $v ::= \dots$               | <i>values</i>        |
| inl v                       | tagged value (left)  |
| inr v                       | tagged value (right) |
| $T ::= \dots$               | <i>types</i>         |
| T+T                         | sum type             |

$T_1+T_2$  is a *disjoint union* of  $T_1$  and  $T_2$  (the tags `inl` and `inr` ensure disjointness)

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New evaluation rules

$t \longrightarrow t'$

$$\begin{array}{l} \text{case (inl } v_0) \\ \text{of inl } x_1 \Rightarrow t_1 \mid \text{inr } x_2 \Rightarrow t_2 \end{array} \longrightarrow [x_1 \mapsto v_0]t_1 \quad (\text{E-CASEINL})$$

$$\begin{array}{l} \text{case (inr } v_0) \\ \text{of inl } x_1 \Rightarrow t_1 \mid \text{inr } x_2 \Rightarrow t_2 \end{array} \longrightarrow [x_2 \mapsto v_0]t_2 \quad (\text{E-CASEINR})$$

$$\frac{t_0 \longrightarrow t'_0}{\begin{array}{l} \text{case } t_0 \text{ of inl } x_1 \Rightarrow t_1 \mid \text{inr } x_2 \Rightarrow t_2 \\ \longrightarrow \text{case } t'_0 \text{ of inl } x_1 \Rightarrow t_1 \mid \text{inr } x_2 \Rightarrow t_2 \end{array}} \quad (\text{E-CASE})$$

$$\frac{t_1 \longrightarrow t'_1}{\text{inl } t_1 \longrightarrow \text{inl } t'_1} \quad (\text{E-INL})$$

$$\frac{t_1 \longrightarrow t'_1}{\text{inr } t_1 \longrightarrow \text{inr } t'_1} \quad (\text{E-INR})$$

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New typing rules

$\Gamma \vdash t : T$

$$\frac{\Gamma \vdash t_1 : T_1}{\Gamma \vdash \text{inl } t_1 : T_1 + T_2} \quad (\text{T-INL})$$

$$\frac{\Gamma \vdash t_1 : T_2}{\Gamma \vdash \text{inr } t_1 : T_1 + T_2} \quad (\text{T-INR})$$

$$\frac{\begin{array}{l} \Gamma \vdash t_0 : T_1 + T_2 \\ \Gamma, x_1 : T_1 \vdash t_1 : T \quad \Gamma, x_2 : T_2 \vdash t_2 : T \end{array}}{\Gamma \vdash \text{case } t_0 \text{ of inl } x_1 \Rightarrow t_1 \mid \text{inr } x_2 \Rightarrow t_2 : T} \quad (\text{T-CASE})$$

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## Sums and Uniqueness of Types

Problem:

*If  $t$  has type  $T$ , then  $\text{inl } t$  has type  $T+U$  for every  $U$ .*

I.e., we've lost uniqueness of types.

Possible solutions:

- ▶ “Infer”  $U$  as needed during typechecking
- ▶ Give constructors different names and only allow each name to appear in one sum type (requires generalization to “variants,” which we’ll see next) — OCaml’s solution
- ▶ Annotate each  $\text{inl}$  and  $\text{inr}$  with the intended sum type.

For simplicity, let's choose the third.

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*New syntactic forms*

```
t ::= ...  
    inl t as T  
    inr t as T
```

*terms*  
tagging (left)  
tagging (right)

```
v ::= ...  
    inl v as T  
    inr v as T
```

*values*  
tagged value (left)  
tagged value (right)

Note that  $\text{as } T$  here is not the ascription operator that we saw before — i.e., not a separate syntactic form: in essence, there is an ascription “built into” every use of  $\text{inl}$  or  $\text{inr}$ .

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*New typing rules*

$\Gamma \vdash t : T$

$$\frac{\Gamma \vdash t_1 : T_1}{\Gamma \vdash \text{inl } t_1 \text{ as } T_1+T_2 : T_1+T_2} \quad (\text{T-INL})$$

$$\frac{\Gamma \vdash t_1 : T_2}{\Gamma \vdash \text{inr } t_1 \text{ as } T_1+T_2 : T_1+T_2} \quad (\text{T-INR})$$

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*Evaluation rules ignore annotations:*

$t \longrightarrow t'$

$$\begin{array}{l} \text{case (inl } v_0 \text{ as } T_0) \\ \text{of inl } x_1 \Rightarrow t_1 \mid \text{inr } x_2 \Rightarrow t_2 \\ \longrightarrow [x_1 \mapsto v_0]t_1 \end{array} \quad (\text{E-CASEINL})$$

$$\begin{array}{l} \text{case (inr } v_0 \text{ as } T_0) \\ \text{of inl } x_1 \Rightarrow t_1 \mid \text{inr } x_2 \Rightarrow t_2 \\ \longrightarrow [x_2 \mapsto v_0]t_2 \end{array} \quad (\text{E-CASEINR})$$

$$\frac{t_1 \longrightarrow t'_1}{\text{inl } t_1 \text{ as } T_2 \longrightarrow \text{inl } t'_1 \text{ as } T_2} \quad (\text{E-INL})$$

$$\frac{t_1 \longrightarrow t'_1}{\text{inr } t_1 \text{ as } T_2 \longrightarrow \text{inr } t'_1 \text{ as } T_2} \quad (\text{E-INR})$$

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## Variants

Just as we generalized binary products to labeled records, we can generalize binary sums to labeled *variants*.

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### *New syntactic forms*

$t ::= \dots$   
     $\langle l=t \rangle \text{ as } T$   
     $\text{case } t \text{ of } \langle l_i=x_i \rangle \Rightarrow t_i \quad i \in 1..n$

*terms*  
    *tagging*  
    *case*

$T ::= \dots$   
     $\langle l_i:T_i \quad i \in 1..n \rangle$

*types*  
    *type of variants*

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New evaluation rules

$t \longrightarrow t'$

$\text{case } (\langle l_j = v_j \rangle \text{ as } T) \text{ of } \langle l_i = x_i \rangle \Rightarrow t_i^{i \in 1..n} \longrightarrow [x_j \mapsto v_j] t_j \quad (\text{E-CASEVARIANT})$

$$\frac{t_0 \longrightarrow t'_0}{\text{case } t_0 \text{ of } \langle l_i = x_i \rangle \Rightarrow t_i^{i \in 1..n} \longrightarrow \text{case } t'_0 \text{ of } \langle l_i = x_i \rangle \Rightarrow t_i^{i \in 1..n}} \quad (\text{E-CASE})$$

$$\frac{t_i \longrightarrow t'_i}{\langle l_i = t_i \rangle \text{ as } T \longrightarrow \langle l_i = t'_i \rangle \text{ as } T} \quad (\text{E-VARIANT})$$

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New typing rules

$\Gamma \vdash t : T$

$$\frac{\Gamma \vdash t_j : T_j}{\Gamma \vdash \langle l_j = t_j \rangle \text{ as } \langle l_i : T_i^{i \in 1..n} \rangle : \langle l_i : T_i^{i \in 1..n} \rangle} \quad (\text{T-VARIANT})$$

$$\frac{\begin{array}{c} \Gamma \vdash t_0 : \langle l_i : T_i^{i \in 1..n} \rangle \\ \text{for each } i \quad \Gamma, x_i : T_i \vdash t_i : T \end{array}}{\Gamma \vdash \text{case } t_0 \text{ of } \langle l_i = x_i \rangle \Rightarrow t_i^{i \in 1..n} : T} \quad (\text{T-CASE})$$

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## Example

```
Addr = <physical:PhysicalAddr, virtual:VirtualAddr>;  
  
a = <physical=pa> as Addr;  
  
getName = λa:Addr.  
  case a of  
    <physical=x> ⇒ x.firstlast  
  | <virtual=y> ⇒ y.name;
```

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## Options

Just like in OCaml...

```
OptionalNat = <none:Unit, some:Nat>;  
  
Table = Nat → OptionalNat;  
  
emptyTable = λn:Nat. <none=unit> as OptionalNat;  
  
extendTable =  
  λt:Table. λm:Nat. λv:Nat.  
    λn:Nat.  
      if equal n m then <some=v> as OptionalNat  
      else t n;  
  
x = case t(5) of  
  <none=u> ⇒ 999  
  | <some=v> ⇒ v;
```

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## Enumerations

```
Weekday = <monday:Unit, tuesday:Unit, wednesday:Unit,  
          thursday:Unit, friday:Unit>;  
  
nextBusinessDay = λw:Weekday.  
  case w of <monday=x>    ⇒ <tuesday=unit> as Weekday  
          | <tuesday=x>    ⇒ <wednesday=unit> as Weekday  
          | <wednesday=x> ⇒ <thursday=unit> as Weekday  
          | <thursday=x>  ⇒ <friday=unit> as Weekday  
          | <friday=x>    ⇒ <monday=unit> as Weekday;
```