

Foundations of Software Fall 2023

Week 3

Review (and more details)

Recall: Simple Arithmetic Expressions

The set \mathcal{T} of terms is defined by the following abstract grammar:

$t ::=$

`true`

`false`

`if t then t else t`

`0`

`succ t`

`pred t`

`iszero t`

terms

constant true

constant false

conditional

constant zero

successor

predecessor

zero test

Recall: Inference Rule Notation

More explicitly: The set \mathcal{T} is the *smallest* set *closed* under the following rules.

$$\begin{array}{ccc} \text{true} \in \mathcal{T} & \text{false} \in \mathcal{T} & 0 \in \mathcal{T} \\[1em] \frac{t_1 \in \mathcal{T}}{\text{succ } t_1 \in \mathcal{T}} & \frac{t_1 \in \mathcal{T}}{\text{pred } t_1 \in \mathcal{T}} & \frac{t_1 \in \mathcal{T}}{\text{iszero } t_1 \in \mathcal{T}} \\[1em] \frac{t_1 \in \mathcal{T} \quad t_2 \in \mathcal{T} \quad t_3 \in \mathcal{T}}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \in \mathcal{T}} \end{array}$$

Generating Functions

Each of these rules can be thought of as a *generating function* that, given some elements from \mathcal{T} , generates some other element of \mathcal{T} . Saying that \mathcal{T} is closed under these rules means that \mathcal{T} cannot be made any bigger using these generating functions — it already contains everything “justified by its members.”

$$\begin{array}{ccc} \text{true} \in \mathcal{T} & \text{false} \in \mathcal{T} & 0 \in \mathcal{T} \\[1em] \frac{t_1 \in \mathcal{T}}{\text{succ } t_1 \in \mathcal{T}} & \frac{t_1 \in \mathcal{T}}{\text{pred } t_1 \in \mathcal{T}} & \frac{t_1 \in \mathcal{T}}{\text{iszero } t_1 \in \mathcal{T}} \\[1em] \frac{t_1 \in \mathcal{T} \quad t_2 \in \mathcal{T} \quad t_3 \in \mathcal{T}}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \in \mathcal{T}} \end{array}$$

Let's write these generating functions explicitly.

$$F_1(U) = \{\text{true}\}$$

$$F_2(U) = \{\text{false}\}$$

$$F_3(U) = \{0\}$$

$$F_4(U) = \{\text{succ } t_1 \mid t_1 \in U\}$$

$$F_5(U) = \{\text{pred } t_1 \mid t_1 \in U\}$$

$$F_6(U) = \{\text{iszero } t_1 \mid t_1 \in U\}$$

$$F_7(U) = \{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \mid t_1, t_2, t_3 \in U\}$$

Each one takes a set of terms U as input and produces a set of “terms justified by U ” as output.

If we now define a generating function for the whole set of inference rules (by combining the generating functions for the individual rules),

$$F(U) = F_1(U) \cup F_2(U) \cup F_3(U) \cup F_4(U) \cup F_5(U) \cup F_6(U) \cup F_7(U)$$

then we can restate the previous definition of the set of terms \mathcal{T} like this:

Definition:

- ▶ A set U is said to be “closed under F ” (or “ F -closed”) if $F(U) \subseteq U$.
- ▶ The set of terms \mathcal{T} is the smallest F -closed set.
(I.e., if \mathcal{O} is another set such that $F(\mathcal{O}) \subseteq \mathcal{O}$, then $\mathcal{T} \subseteq \mathcal{O}$.)

Our alternate definition of the set of terms can also be stated using the generating function F :

$$\begin{aligned}\mathcal{S}_0 &= \emptyset \\ \mathcal{S}_{i+1} &= F(\mathcal{S}_i) \\ \mathcal{S} &= \bigcup_i \mathcal{S}_i\end{aligned}$$

Compare this definition of \mathcal{S} with the one we saw last time:

$$\begin{aligned}\mathcal{S}_0 &= \emptyset \\ \mathcal{S}_{i+1} &= \{\text{true, false, 0}\} \\ &\quad \cup \{\text{succ } t_1, \text{pred } t_1, \text{iszero } t_1 \mid t_1 \in \mathcal{S}_i\} \\ &\quad \cup \{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \mid t_1, t_2, t_3 \in \mathcal{S}_i\}\end{aligned}$$

$$\mathcal{S} = \bigcup_i \mathcal{S}_i$$

We have “pulled out” F and given it a name.

Note that our two definitions of terms characterize the same set from different directions:

- ▶ “from above,” as the intersection of all F -closed sets;
- ▶ “from below,” as the limit (union) of a series of sets that start from \emptyset and get “closer and closer to being F -closed.”

Proposition 3.2.6 in the book shows that these two definitions actually define the same set.

Warning: Hard hats on for the next slide!

Structural Induction

The principle of structural induction on terms can also be re-stated using generating functions:

Suppose T is the smallest F -closed set.

If, for each set U ,

from the assumption “ $P(u)$ holds for every $u \in U$ ”

we can show “ $P(v)$ holds for any $v \in F(U)$,”

then $P(t)$ holds for all $t \in T$.

Structural Induction

The principle of structural induction on terms can also be re-stated using generating functions:

Suppose T is the smallest F -closed set.

If, for each set U ,

from the assumption “ $P(u)$ holds for every $u \in U$ ”

we can show “ $P(v)$ holds for any $v \in F(U)$,”

then $P(t)$ holds for all $t \in T$.

Why?

Structural Induction

Why? Because:

- ▶ We assumed that T was the *smallest* F -closed set, i.e., that $T \subseteq O$ for any other F -closed set O .
- ▶ But showing
 for each set U ,
 given $P(u)$ for all $u \in U$
 we can show $P(v)$ for all $v \in F(U)$

amounts to showing that “the set of all terms satisfying P ”
–call it O – is itself an F -closed set.
(recall: O is F -closed if $F(O) \subseteq O$)

- ▶ Since $T \subseteq O$, every element of T satisfies P .

Structural Induction

Compare this with the structural induction principle for terms from last lecture:

*If, for each term s ,
 given $P(r)$ for all immediate subterms r of s
 we can show $P(s)$,
then $P(t)$ holds for all t .*

Recall, from the definition of \mathcal{S}_i , it is clear that, if a term t is in \mathcal{S}_i , then all of its immediate subterms must be in \mathcal{S}_{i-1} , i.e., they must have strictly smaller depths. Therefore:

*If, for each term s ,
given $P(r)$ for all immediate subterms r of s
we can show $P(s)$,
then $P(t)$ holds for all t .*

Slightly more explicit proof:

- ▶ Assume that for each term s , given $P(r)$ for all immediate subterms of s , we can show $P(s)$.
- ▶ Then show, by induction on i , that $P(t)$ holds for all terms t with depth i .
- ▶ Therefore, $P(t)$ holds for all t .

Operational Semantics and Reasoning

Recall: Abstract Machines

An *abstract machine* consists of:

- ▶ a set of *states*
- ▶ a *transition relation* on states, written \longrightarrow

For the simple languages we are considering at the moment, the term being evaluated is the whole state of the abstract machine.

Recall: Syntax for Booleans

Terms and values

`t ::=`

`true`

`false`

`if t then t else t`

terms

constant true

constant false

conditional

`v ::=`

`true`

`false`

values

true value

false value

Recall: Operational Semantics for Booleans

The evaluation relation $t \longrightarrow t'$ is the smallest relation closed under the following rules:

$\text{if true then } t_2 \text{ else } t_3 \longrightarrow t_2 \quad (\text{E-IFTRUE})$

$\text{if false then } t_2 \text{ else } t_3 \longrightarrow t_3 \quad (\text{E-IFFALSE})$

$$\frac{t_1 \longrightarrow t'_1}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \longrightarrow \text{if } t'_1 \text{ then } t_2 \text{ else } t_3} \quad (\text{E-IF})$$

Derivations

We can record the “justification” for a particular pair of terms that are in the evaluation relation in the form of a tree.

(on the board)

Terminology:

- ▶ These trees are called *derivation trees* (or just *derivations*).
- ▶ The final statement in a derivation is its *conclusion*.
- ▶ We say that the derivation is a *witness* for its conclusion (or a *proof* of its conclusion) — it records all the reasoning steps that justify the conclusion.

Observation

Lemma: Suppose we are given a derivation tree \mathcal{D} witnessing the pair (t, t') in the evaluation relation. Then either

1. the final rule used in \mathcal{D} is E-IFTRUE and we have $t = \text{if true then } t_2 \text{ else } t_3$ and $t' = t_2$, for some t_2 and t_3 , or
2. the final rule used in \mathcal{D} is E-IFFALSE and we have $t = \text{if false then } t_2 \text{ else } t_3$ and $t' = t_3$, for some t_2 and t_3 , or
3. the final rule used in \mathcal{D} is E-IF and we have $t = \text{if } t_1 \text{ then } t_2 \text{ else } t_3$ and $t' = \text{if } t'_1 \text{ then } t_2 \text{ else } t_3$, for some t_1, t'_1, t_2 , and t_3 ; moreover, the immediate subderivation of \mathcal{D} witnesses $(t_1, t'_1) \in \longrightarrow$.

Induction on Derivations

We can now write proofs about evaluation “by induction on derivation trees.”

Given an arbitrary derivation \mathcal{D} with conclusion $t \rightarrow t'$, we assume the desired result for its immediate sub-derivation (if any) and proceed by a case analysis (using the previous lemma) of the final evaluation rule used in constructing the derivation tree.

E.g....

Induction on Derivations — Example

Theorem: If $t \rightarrow t'$, i.e., if $(t, t') \in \rightarrow$, then $\text{size}(t) > \text{size}(t')$.

Proof: By induction on a derivation \mathcal{D} of $t \rightarrow t'$.

1. Suppose the final rule used in \mathcal{D} is E-IFTRUE, with $t = \text{if true then } t_2 \text{ else } t_3$ and $t' = t_2$. Then the result is immediate from the definition of *size*.
2. Suppose the final rule used in \mathcal{D} is E-IFFALSE, with $t = \text{if false then } t_2 \text{ else } t_3$ and $t' = t_3$. Then the result is again immediate from the definition of *size*.
3. Suppose the final rule used in \mathcal{D} is E-IF, with $t = \text{if } t_1 \text{ then } t_2 \text{ else } t_3$ and $t' = \text{if } t'_1 \text{ then } t_2 \text{ else } t_3$, where $(t_1, t'_1) \in \rightarrow$ is witnessed by a derivation \mathcal{D}_1 . By the induction hypothesis, $\text{size}(t_1) > \text{size}(t'_1)$. But then, by the definition of *size*, we have $\text{size}(t) > \text{size}(t')$.

Normal forms

A *normal form* is a term that cannot be evaluated any further — i.e., a term t is a normal form (or “is in normal form”) if there is no t' such that $t \longrightarrow t'$.

A normal form is a state where the abstract machine is halted — i.e., it can be regarded as a “result” of evaluation.

Normal forms

A *normal form* is a term that cannot be evaluated any further — i.e., a term t is a normal form (or “is in normal form”) if there is no t' such that $t \rightarrow t'$.

A normal form is a state where the abstract machine is halted — i.e., it can be regarded as a “result” of evaluation.

Recall that we intended the set of *values* (the boolean constants `true` and `false`) to be exactly the possible “results of evaluation.” Did we get this definition right?

Values = normal forms

Theorem: A term t is a value iff it is in normal form.

Proof:

The \implies direction is immediate from the definition of the evaluation relation.

Values = normal forms

Theorem: A term t is a value iff it is in normal form.

Proof:

The \Rightarrow direction is immediate from the definition of the evaluation relation.

For the \Leftarrow direction,

Values = normal forms

Theorem: A term t is a value iff it is in normal form.

Proof:

The \Rightarrow direction is immediate from the definition of the evaluation relation.

For the \Leftarrow direction, it is convenient to prove the contrapositive:
If t is *not* a value, then it is *not* a normal form.

Values = normal forms

Theorem: A term t is a value iff it is in normal form.

Proof:

The \Rightarrow direction is immediate from the definition of the evaluation relation.

For the \Leftarrow direction, it is convenient to prove the contrapositive: If t is *not* a value, then it is *not* a normal form. The argument goes by induction on t .

Note, first, that t must have the form `if t_1 then t_2 else t_3` (otherwise it would be a value). If t_1 is `true` or `false`, then rule E-IFTRUE or E-IFFALSE applies to t , and we are done.

Otherwise, t_1 is not a value and so, by the induction hypothesis, there is some t'_1 such that $t_1 \rightarrow t'_1$. But then rule E-IF yields

$$\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \rightarrow \text{if } t'_1 \text{ then } t_2 \text{ else } t_3$$

i.e., t is not in normal form.

Numbers

New syntactic forms

`t ::= ...`
`0`
`succ t`
`pred t`
`iszero t`

`v ::= ...`
`nv`

`nv ::=`
`0`
`succ nv`

terms

constant zero
successor
predecessor
zero test

values

numeric value

numeric values

zero value
successor value

$$\frac{t_1 \longrightarrow t'_1}{\text{succ } t_1 \longrightarrow \text{succ } t'_1} \quad (\text{E-SUCC})$$

$$\text{pred } 0 \longrightarrow 0 \quad (\text{E-PREDZERO})$$

$$\text{pred } (\text{succ } nv_1) \longrightarrow nv_1 \quad (\text{E-PREDSUCC})$$

$$\frac{t_1 \longrightarrow t'_1}{\text{pred } t_1 \longrightarrow \text{pred } t'_1} \quad (\text{E-PRED})$$

$$\text{iszero } 0 \longrightarrow \text{true} \quad (\text{E-ISZEROZERO})$$

$$\text{iszero } (\text{succ } nv_1) \longrightarrow \text{false} \quad (\text{E-ISZEROSUCC})$$

$$\frac{t_1 \longrightarrow t'_1}{\text{iszero } t_1 \longrightarrow \text{iszero } t'_1} \quad (\text{E-ISZERO})$$

Are all normal forms still values?

Clicker question: Propose a term that is in normal form but is *not* a value.

URL: ttpoll.eu

Session ID: [cs452](#)

Stuck terms

Formally, a stuck term is one that is a normal form but not a value.

Stuck terms model run-time errors.

Multi-step evaluation.

The *multi-step evaluation* relation, \longrightarrow^* , is the reflexive, transitive closure of single-step evaluation.

I.e., it is the smallest relation closed under the following rules:

$$\frac{t \longrightarrow t'}{t \longrightarrow^* t'}$$

$$t \longrightarrow^* t$$

$$\frac{t \longrightarrow^* t' \quad t' \longrightarrow^* t''}{t \longrightarrow^* t''}$$

Termination of evaluation

Theorem: For every t there is some normal form t' such that $t \longrightarrow^* t'$.

Proof:

Termination of evaluation

Theorem: For every t there is some normal form t' such that $t \longrightarrow^* t'$.

Proof:

- ▶ First, recall that single-step evaluation strictly reduces the size of the term:

if $t \longrightarrow t'$, then $\text{size}(t) > \text{size}(t')$

- ▶ Now, assume (for a contradiction) that

$t_0, t_1, t_2, t_3, t_4, \dots$

is an infinite-length sequence such that

$t_0 \longrightarrow t_1 \longrightarrow t_2 \longrightarrow t_3 \longrightarrow t_4 \longrightarrow \dots$

- ▶ Then

$\text{size}(t_0) > \text{size}(t_1) > \text{size}(t_2) > \text{size}(t_3) > \dots$

- ▶ But such a sequence cannot exist — contradiction!

Termination Proofs

Most termination proofs have the same basic form:

Theorem: *The relation $R \subseteq X \times X$ is terminating — i.e., there are no infinite sequences x_0, x_1, x_2 , etc. such that $(x_i, x_{i+1}) \in R$ for each i .*

Proof:

1. Choose

- ▶ a well-founded set $(W, <)$ — i.e., a set W with a partial order $<$ such that there are no infinite descending chains $w_0 > w_1 > w_2 > \dots$ in W
- ▶ a function f from X to W

2. Show $f(x) > f(y)$ for all $(x, y) \in R$

3. Conclude that there are no infinite sequences x_0, x_1, x_2 , etc. such that $(x_i, x_{i+1}) \in R$ for each i , since, if there were, we could construct an infinite descending chain in W .

Reading for next week

- ▶ Chapter 4 – Untyped Lambda Calculus
(20 pages!)

The Lambda Calculus

The lambda-calculus

- ▶ If our previous language of arithmetic expressions was the simplest nontrivial programming language, then the lambda-calculus is the simplest *interesting* programming language...
 - ▶ Turing complete
 - ▶ higher order (functions as data)
- ▶ Indeed, in the lambda-calculus, *all* computation happens by means of function abstraction and application.
- ▶ The *e. coli* of programming language research
- ▶ The foundation of many real-world programming language designs (including ML, Haskell, Scheme, Lisp, ...)

Intuitions

Suppose we want to describe a function that adds three to any number we pass it. We might write

$$\text{plus3 } x \quad = \quad \text{succ } (\text{succ } (\text{succ } x))$$

That is, “`plus3 x` is `succ (succ (succ x))`.”

Intuitions

Suppose we want to describe a function that adds three to any number we pass it. We might write

$$\text{plus3 } x = \text{succ } (\text{succ } (\text{succ } x))$$

That is, “`plus3 x` is `succ (succ (succ x))`.”

Q: What is `plus3` itself?

Intuitions

Suppose we want to describe a function that adds three to any number we pass it. We might write

$$\text{plus3 } x \quad = \quad \text{succ } (\text{succ } (\text{succ } x))$$

That is, “`plus3 x` is `succ (succ (succ x))`.”

Q: What is `plus3` itself?

A: `plus3` is the function that, given `x`, yields `succ (succ (succ x))`.

Intuitions

Suppose we want to describe a function that adds three to any number we pass it. We might write

$$\text{plus3 } x = \text{succ } (\text{succ } (\text{succ } x))$$

That is, “`plus3 x` is `succ (succ (succ x))`.”

Q: What is `plus3` itself?

A: `plus3` is the function that, given `x`, yields `succ (succ (succ x))`.

$$\text{plus3} = \lambda x. \text{succ } (\text{succ } (\text{succ } x))$$

This function exists independent of the name `plus3`.

`λx. t` is written “`fun x → t`” in OCaml and “`x ⇒ t`” in Scala.

So `plus3 (succ 0)` is just a convenient shorthand for “the function that, given `x`, yields `succ (succ (succ x))`, applied to `succ 0`.”

$$\begin{aligned} &\text{plus3 (succ 0)} \\ &= \\ &(\lambda x. \text{succ (succ (succ x))}) (\text{succ 0}) \end{aligned}$$

Abstractions over Functions

Consider the λ -abstraction

$$g = \lambda f. f (f (\text{succ } 0))$$

Note that the parameter variable f is used in the *function* position in the body of g . Terms like g are called *higher-order* functions. If we apply g to an argument like plus3 , the “substitution rule” yields a nontrivial computation:

$$\begin{aligned} g \text{ plus3} &= (\lambda f. f (f (\text{succ } 0))) (\lambda x. \text{succ } (\text{succ } (\text{succ } x))) \\ &\text{i.e. } (\lambda x. \text{succ } (\text{succ } (\text{succ } x))) \\ &\quad ((\lambda x. \text{succ } (\text{succ } (\text{succ } x))) (\text{succ } 0)) \\ &\text{i.e. } (\lambda x. \text{succ } (\text{succ } (\text{succ } x))) \\ &\quad (\text{succ } (\text{succ } (\text{succ } (\text{succ } 0)))) \\ &\text{i.e. } \text{succ } (\text{succ } (\text{succ } (\text{succ } (\text{succ } (\text{succ } (\text{succ } 0)))))) \end{aligned}$$

Abstractions Returning Functions

Consider the following variant of `g`:

$$\text{double} = \lambda f. \lambda y. f (f y)$$

I.e., `double` is the function that, when applied to a function `f`, yields a *function* that, when applied to an argument `y`, yields `f (f y)`.

Example

```
double plus3 0
=  (λf. λy. f (f y))
    (λx. succ (succ (succ x)))
    0
i.e. (λy. (λx. succ (succ (succ x)))
        ((λx. succ (succ (succ x))) y))
    0
i.e. (λx. succ (succ (succ x)))
        ((λx. succ (succ (succ x))) 0)
i.e. (λx. succ (succ (succ x)))
        (succ (succ (succ 0)))
i.e. succ (succ (succ (succ (succ (succ 0)))))
```


The Pure Lambda-Calculus

As the preceding examples suggest, once we have λ -abstraction and application, we can throw away all the other language primitives and still have left a rich and powerful programming language.

In this language — the “pure lambda-calculus” — *everything* is a function.

- ▶ Variables always denote functions
- ▶ Functions always take other functions as parameters
- ▶ The result of a function is always a function

Formalities

Syntax

$t ::=$

x

$\lambda x. t$

$t \ t$

terms

variable

abstraction

application

Terminology:

- ▶ terms in the pure λ -calculus are often called λ -terms
- ▶ terms of the form $\lambda x. t$ are called λ -abstractions or just *abstractions*

Syntactic conventions

Since λ -calculus provides only one-argument functions, all multi-argument functions must be written in curried style.

The following conventions make the linear forms of terms easier to read and write:

- ▶ Application associates to the left

E.g., $t\ u\ v$ means $(t\ u)\ v$, not $t\ (u\ v)$

- ▶ Bodies of λ -abstractions extend as far to the right as possible

E.g., $\lambda x. \lambda y. x\ y$ means $\lambda x. (\lambda y. x\ y)$, not $\lambda x. (\lambda y. x)\ y$

Scope

The λ -abstraction term $\lambda x. t$ *binds* the variable x .

The *scope* of this binding is the *body* t .

Occurrences of x inside t are said to be *bound* by the abstraction.

Occurrences of x that are *not* within the scope of an abstraction binding x are said to be *free*.

Test:

$$\lambda x. \lambda y. x y z$$

Scope

The λ -abstraction term $\lambda x. t$ *binds* the variable x .

The *scope* of this binding is the *body* t .

Occurrences of x inside t are said to be *bound* by the abstraction.

Occurrences of x that are *not* within the scope of an abstraction binding x are said to be *free*.

Test:

$$\lambda x. \lambda y. x y z$$
$$\lambda x. (\lambda y. z y) y$$

Values

$v ::=$

$\lambda x. t$

values

abstraction value

Operational Semantics

Computation rule:

$$(\lambda x. t_{12}) \ v_2 \longrightarrow [x \mapsto v_2] t_{12} \quad (\text{E-APPABS})$$

Notation: $[x \mapsto v_2] t_{12}$ is “the term that results from substituting free occurrences of x in t_{12} with v_2 .”

Operational Semantics

Computation rule:

$$(\lambda x. t_{12}) \ v_2 \longrightarrow [x \mapsto v_2] t_{12} \quad (\text{E-APPABS})$$

Notation: $[x \mapsto v_2] t_{12}$ is “the term that results from substituting free occurrences of x in t_{12} with v_2 .”

Congruence rules:

$$\frac{t_1 \longrightarrow t'_1}{t_1 \ t_2 \longrightarrow t'_1 \ t_2} \quad (\text{E-APP1})$$

$$\frac{t_2 \longrightarrow t'_2}{v_1 \ t_2 \longrightarrow v_1 \ t'_2} \quad (\text{E-APP2})$$

Terminology

A term of the form $(\lambda x. t) v$ — that is, a λ -abstraction applied to a *value* — is called a *redex* (short for “reducible expression”).

Alternative evaluation strategies

Strictly speaking, the language we have defined is called the *pure, call-by-value lambda-calculus*.

The evaluation strategy we have chosen — *call by value* — reflects standard conventions found in most mainstream languages.

Some other common ones:

- ▶ Call by name (cf. Haskell)
- ▶ Normal order (leftmost/outermost)
- ▶ Full (non-deterministic) beta-reduction

Classical Lambda Calculus

Full beta reduction

The classical lambda calculus allows full beta reduction.

- ▶ The argument of a β -reduction to be an arbitrary term, not just a value.
- ▶ Reduction may appear anywhere in a term.

Full beta reduction

The classical lambda calculus allows full beta reduction.

- ▶ The argument of a β -reduction to be an arbitrary term, not just a value.
- ▶ Reduction may appear anywhere in a term.

Computation rule:

$$(\lambda x. t_{12}) \ t_2 \longrightarrow [x \mapsto t_2] t_{12} \quad (\text{E-APPABS})$$

Full beta reduction

The classical lambda calculus allows full beta reduction.

- ▶ The argument of a β -reduction to be an arbitrary term, not just a value.
- ▶ Reduction may appear anywhere in a term.

Computation rule:

$$(\lambda x. t_{12}) \ t_2 \longrightarrow [x \mapsto t_2] t_{12} \quad (\text{E-APPABS})$$

Congruence rules:

$$\frac{t_1 \longrightarrow t'_1}{t_1 \ t_2 \longrightarrow t'_1 \ t_2} \quad (\text{E-APP1})$$

$$\frac{t_2 \longrightarrow t'_2}{t_1 \ t_2 \longrightarrow t_1 \ t'_2} \quad (\text{E-APP2})$$

$$\frac{t \longrightarrow t'}{\lambda x. t \longrightarrow \lambda x. t'} \quad (\text{E-ABS})$$

Substitution revisited

Remember: $[x \mapsto v_2] t_{12}$ is “the term that results from substituting free occurrences of x in t_{12} with v_2 .”

This is trickier than it looks!

For example:

$$\begin{aligned} & (\lambda x. (\lambda y. x)) y \\ \longrightarrow & [x \mapsto y] \lambda y. x \\ = & ??? \end{aligned}$$

Substitution revisited

Remember: $[x \mapsto v_2]t_{12}$ is “the term that results from substituting free occurrences of x in t_{12} with v_2 .”

This is trickier than it looks!

For example:

$$\begin{aligned} & (\lambda x. (\lambda y. x)) y \\ \longrightarrow & [x \mapsto y] \lambda y. x \\ = & ??? \end{aligned}$$

Solution:

need to rename bound variables before performing the substitution.

$$\begin{aligned} & (\lambda x. (\lambda y. x)) y \\ = & (\lambda x. (\lambda z. x)) y \\ \longrightarrow & [x \mapsto y] \lambda z. x \\ = & \lambda z. y \end{aligned}$$

Alpha conversion

Renaming bound variables is formalized as α -conversion.

Conversion rule:

$$\frac{y \notin \text{fv}(t)}{\lambda x. t =_{\alpha} \lambda y. [x \mapsto y]t} \quad (\alpha)$$

Equivalence rules:

$$\frac{t_1 =_{\alpha} t_2}{t_2 =_{\alpha} t_1} \quad (\alpha\text{-SYMM})$$

$$\frac{t_1 =_{\alpha} t_2 \quad t_2 =_{\alpha} t_3}{t_1 =_{\alpha} t_3} \quad (\alpha\text{-TRANS})$$

Congruence rules: the usual ones.

Confluence

Full β -reduction makes it possible to have different reduction paths.

Q: Can a term evaluate to more than one normal form?

Confluence

Full β -reduction makes it possible to have different reduction paths.

Q: Can a term evaluate to more than one normal form?

The answer is no; this is a consequence of the following

Theorem [Church-Rosser]

Let t , t_1 , t_2 be terms such that $t \longrightarrow^* t_1$ and $t \longrightarrow^* t_2$. Then there exists a term t_3 such that $t_1 \longrightarrow^* t_3$ and $t_2 \longrightarrow^* t_3$.

Programming in the Lambda-Calculus

Multiple arguments

Consider the function `double`, which returns a function as an argument.

$$\text{double} = \lambda f. \lambda y. f (f y)$$

This idiom — a λ -abstraction that does nothing but immediately yield another abstraction — is very common in the λ -calculus.

In general, $\lambda x. \lambda y. t$ is a function that, given a value v for x , yields a function that, given a value u for y , yields t with v in place of x and u in place of y .

That is, $\lambda x. \lambda y. t$ is a two-argument function.

(Recall the discussion of *currying* in OCaml.)

The “Church Booleans”

`tru` = $\lambda t. \lambda f. t$

`fls` = $\lambda t. \lambda f. f$

$$\begin{aligned} & \text{tru } v \ w \\ = & \quad \underline{(\lambda t. \lambda f. t)} \ v \ w && \text{by definition} \\ \longrightarrow & \quad \underline{(\lambda f. v)} \ w && \text{reducing the underlined redex} \\ \longrightarrow & \quad v && \text{reducing the underlined redex} \end{aligned}$$

$$\begin{aligned} & \text{fls } v \ w \\ = & \quad \underline{(\lambda t. \lambda f. f)} \ v \ w && \text{by definition} \\ \longrightarrow & \quad \underline{(\lambda f. f)} \ w && \text{reducing the underlined redex} \\ \longrightarrow & \quad w && \text{reducing the underlined redex} \end{aligned}$$

Functions on Booleans

```
not    =  λb. b fls tru
```

That is, `not` is a function that, given a boolean value `v`, returns `fls` if `v` is `tru` and `tru` if `v` is `fls`.

Functions on Booleans

`and = λb. λc. b c fls`

That is, `and` is a function that, given two boolean values `v` and `w`, returns `w` if `v` is `tru` and `fls` if `v` is `fls`

Thus `and v w` yields `tru` if both `v` and `w` are `tru` and `fls` if either `v` or `w` is `fls`.

Pairs

```
pair = λf.λs.λb. b f s  
fst = λp. p tru  
snd = λp. p fls
```

That is, `pair v w` is a function that, when applied to a boolean value `b`, applies `b` to `v` and `w`.

By the definition of booleans, this application yields `v` if `b` is `tru` and `w` if `b` is `fls`, so the first and second projection functions `fst` and `snd` can be implemented simply by supplying the appropriate boolean.

Example

	$\text{fst } (\text{pair } v \ w)$	
$=$	$\text{fst } ((\lambda f. \lambda s. \lambda b. b \ f \ s) \ v \ w)$	by definition
\longrightarrow	$\text{fst } ((\lambda s. \lambda b. b \ v \ s) \ w)$	reducing
\longrightarrow	$\text{fst } (\lambda b. b \ v \ w)$	reducing
$=$	$(\lambda p. p \ \text{tru}) (\lambda b. b \ v \ w)$	by definition
\longrightarrow	$(\lambda b. b \ v \ w) \ \text{tru}$	reducing
\longrightarrow	$\text{tru } v \ w$	reducing
\longrightarrow^*	v	as before.

Church numerals

Idea: represent the number n by a function that “repeats some action n times.”

$$c_0 = \lambda s. \lambda z. z$$

$$c_1 = \lambda s. \lambda z. s \ z$$

$$c_2 = \lambda s. \lambda z. s \ (s \ z)$$

$$c_3 = \lambda s. \lambda z. s \ (s \ (s \ z))$$

That is, each number n is represented by a term c_n that takes two arguments, s and z (for “successor” and “zero”), and applies s , n times, to z .

Functions on Church Numerals

Successor:

Functions on Church Numerals

Successor:

$$\text{scc} = \lambda n. \lambda s. \lambda z. s \ (n \ s \ z)$$

Functions on Church Numerals

Successor:

$$\text{scc} = \lambda n. \lambda s. \lambda z. s \ (n \ s \ z)$$

Addition:

Functions on Church Numerals

Successor:

$$\text{scc} = \lambda n. \lambda s. \lambda z. s \ (n \ s \ z)$$

Addition:

$$\text{plus} = \lambda m. \lambda n. \lambda s. \lambda z. m \ s \ (n \ s \ z)$$

Functions on Church Numerals

Successor:

$$\text{scc} = \lambda n. \lambda s. \lambda z. s \ (n \ s \ z)$$

Addition:

$$\text{plus} = \lambda m. \lambda n. \lambda s. \lambda z. m \ s \ (n \ s \ z)$$

Multiplication:

Functions on Church Numerals

Successor:

$$\text{succ} = \lambda n. \lambda s. \lambda z. s \ (n \ s \ z)$$

Addition:

$$\text{plus} = \lambda m. \lambda n. \lambda s. \lambda z. m \ s \ (n \ s \ z)$$

Multiplication:

$$\text{times} = \lambda m. \lambda n. m \ (\text{plus } n) \ c_0$$

Functions on Church Numerals

Successor:

$$\text{scc} = \lambda n. \lambda s. \lambda z. s \ (n \ s \ z)$$

Addition:

$$\text{plus} = \lambda m. \lambda n. \lambda s. \lambda z. m \ s \ (n \ s \ z)$$

Multiplication:

$$\text{times} = \lambda m. \lambda n. m \ (\text{plus } n) \ c_0$$

Zero test:

Functions on Church Numerals

Successor:

$$\text{scc} = \lambda n. \lambda s. \lambda z. s \ (n \ s \ z)$$

Addition:

$$\text{plus} = \lambda m. \lambda n. \lambda s. \lambda z. m \ s \ (n \ s \ z)$$

Multiplication:

$$\text{times} = \lambda m. \lambda n. m \ (\text{plus } n) \ c_0$$

Zero test:

$$\text{iszro} = \lambda m. m \ (\lambda x. \text{fls}) \ \text{tru}$$

Functions on Church Numerals

Successor:

$$\text{scc} = \lambda n. \lambda s. \lambda z. s \ (n \ s \ z)$$

Addition:

$$\text{plus} = \lambda m. \lambda n. \lambda s. \lambda z. m \ s \ (n \ s \ z)$$

Multiplication:

$$\text{times} = \lambda m. \lambda n. m \ (\text{plus } n) \ c_0$$

Zero test:

$$\text{iszro} = \lambda m. m \ (\lambda x. \text{fls}) \ \text{tru}$$

What about predecessor?

Predecessor

```
zz = pair c0 c0
```

```
ss =  $\lambda$ p. pair (snd p) (scc (snd p))
```

```
prd =  $\lambda$ m. fst (m ss zz)
```

Recursion in the Lambda-Calculus

Recursion and divergence

Recursion and divergence are intertwined, so we need to consider divergent terms.

$$\text{omega} = (\lambda x. x x) (\lambda x. x x)$$

Note that `omega` evaluates in one step to itself!

So evaluation of `omega` never reaches a normal form: it *diverges*.

Recursion and divergence

Recursion and divergence are intertwined, so we need to consider divergent terms.

$$\text{omega} = (\lambda x. x x) (\lambda x. x x)$$

Note that `omega` evaluates in one step to itself!

So evaluation of `omega` never reaches a normal form: it *diverges*.

Being able to write a divergent computation does not seem very useful in itself. However, there are variants of `omega` that are *very* useful...

Recall: Normal forms

- ▶ A *normal form* is a term that cannot take an evaluation step.
- ▶ A *stuck* term is a normal form that is not a value.

Does every term evaluate to a normal form?

No, `omega` is not in normal form.

Recall: Normal forms

- ▶ A *normal form* is a term that cannot take an evaluation step.
- ▶ A *stuck* term is a normal form that is not a value.

Does every term evaluate to a normal form?

No, `omega` is not in normal form.

But are there any stuck terms in the pure λ -calculus?

Towards recursion: Iterated application

Suppose f is some λ -abstraction, and consider the following variant of ω :

$$Y_f = (\lambda x. f (x x)) (\lambda x. f (x x))$$

Towards recursion: Iterated application

Suppose f is some λ -abstraction, and consider the following variant of ω :

$$Y_f = (\lambda x. f (x x)) (\lambda x. f (x x))$$

Now the “pattern of divergence” becomes more interesting:

$$\begin{aligned} Y_f &= \\ &\quad \frac{(\lambda x. f (x x)) (\lambda x. f (x x))}{\longrightarrow} \\ &\quad f \left(\frac{(\lambda x. f (x x)) (\lambda x. f (x x))}{\longrightarrow} \right) \\ &\quad f \left(f \left(\frac{(\lambda x. f (x x)) (\lambda x. f (x x))}{\longrightarrow} \right) \right) \\ &\quad f \left(f \left(f \left(\frac{(\lambda x. f (x x)) (\lambda x. f (x x))}{\longrightarrow} \right) \right) \right) \\ &\quad \dots \end{aligned}$$

Y_f is still not very useful, since (like ω), all it does is diverge.

Is there any way we could “slow it down”?

Delaying divergence

`poisonpill = λy. omega`

Note that `poisonpill` is a value — it will only diverge when we actually apply it to an argument. This means that we can safely pass it as an argument to other functions, return it as a result from functions, etc.

$$\begin{array}{c} \hline (\lambda p. \text{fst } (\text{pair } p \text{ fls}) \text{ tru}) \text{ poisonpill} \\ \longrightarrow \\ \text{fst } (\text{pair } \text{poisonpill} \text{ fls}) \text{ tru} \\ \longrightarrow^* \\ \hline \text{poisonpill } \text{tru} \\ \longrightarrow \\ \text{omega} \\ \longrightarrow \\ \dots \end{array}$$

A delayed variant of omega

Here is a variant of `omega` in which the delay and divergence are a bit more tightly intertwined:

$$\text{omegav} = \lambda y. (\lambda x. (\lambda y. x \ x \ y)) (\lambda x. (\lambda y. x \ x \ y)) \ y$$

Note that `omegav` is a normal form. However, if we apply it to any argument `v`, it diverges:

$$\begin{aligned} & \text{omegav} \ v \\ &= \\ & \frac{(\lambda y. (\lambda x. (\lambda y. x \ x \ y)) (\lambda x. (\lambda y. x \ x \ y)) \ y) \ v}{\longrightarrow} \\ & \frac{(\lambda x. (\lambda y. x \ x \ y)) (\lambda x. (\lambda y. x \ x \ y)) \ v}{\longrightarrow} \\ & (\lambda y. (\lambda x. (\lambda y. x \ x \ y)) (\lambda x. (\lambda y. x \ x \ y)) \ y) \ v \\ &= \\ & \text{omegav} \ v \end{aligned}$$

Another delayed variant

Suppose f is a function. Define

$$z_f = \lambda y. (\lambda x. f (\lambda y. x x y)) (\lambda x. f (\lambda y. x x y)) y$$

This term combines the “added f ” from Y_f with the “delayed divergence” of ω_{av} .

If we now apply z_f to an argument v , something interesting happens:

$$\begin{aligned}
 & z_f \ v \\
 & = \\
 & \frac{(\lambda y. (\lambda x. f (\lambda y. x \ x \ y)) (\lambda x. f (\lambda y. x \ x \ y)) \ y) \ v}{\longrightarrow} \\
 & \quad \frac{(\lambda x. f (\lambda y. x \ x \ y)) (\lambda x. f (\lambda y. x \ x \ y)) \ v}{\longrightarrow} \\
 & f (\lambda y. (\lambda x. f (\lambda y. x \ x \ y)) (\lambda x. f (\lambda y. x \ x \ y)) \ y) \ v \\
 & = \\
 & f \ z_f \ v
 \end{aligned}$$

Since z_f and v are both values, the next computation step will be the reduction of $f \ z_f$ — that is, before we “diverge,” f gets to do some computation.

Now we are getting somewhere.

Recursion

Let

```
f  =  λfct.  
      λn.  
        if n=0 then 1  
        else n * (fct (pred n))
```

`f` looks just like the ordinary factorial function, except that, in place of a recursive call in the last line, it calls the function `fct`, which is passed as a parameter.

N.b.: for brevity, this example uses “real” numbers and booleans, infix syntax, etc. It can easily be translated into the pure lambda-calculus (using Church numerals, etc.).

We can use z_f to “tie the knot” in the definition of f and obtain a real recursive factorial function:

$$\begin{aligned}
 & z_f \ 3 \\
 & \longrightarrow^* \\
 & f \ z_f \ 3 \\
 & = \\
 & (\lambda fct. \ \lambda n. \ \dots) \ z_f \ 3 \\
 & \longrightarrow \quad \longrightarrow \\
 & \text{if } 3=0 \text{ then } 1 \text{ else } 3 * (z_f \ (\text{pred } 3)) \\
 & \longrightarrow^* \\
 & 3 * (z_f \ (\text{pred } 3)) \\
 & \longrightarrow \\
 & 3 * (z_f \ 2) \\
 & \longrightarrow^* \\
 & 3 * (f \ z_f \ 2) \\
 & \dots
 \end{aligned}$$

A Generic z

If we define

$$z = \lambda f. z_f$$

i.e.,

$$z = \lambda f. \lambda y. (\lambda x. f (\lambda y. x x y)) (\lambda x. f (\lambda y. x x y)) y$$

then we can obtain the behavior of z_f for any f we like, simply by applying z to f .

$$z f \longrightarrow z_f$$

For example:

```
fact      =      z  ( λfct.  
                      λn.  
                        if n=0 then 1  
                        else n * (fct (pred n)) )
```

Technical Note

The term `z` here is essentially the same as the `fix` discussed the book.

```
z =  
  λf. λy. (λx. f (λy. x x y)) (λx. f (λy. x x y)) y
```

```
fix =  
  λf. (λx. f (λy. x x y)) (λx. f (λy. x x y))
```

`z` is hopefully slightly easier to understand, since it has the property that $z\ f\ v \longrightarrow^* f\ (z\ f)\ v$, which `fix` does not (quite) share.