

# Foundations of Software Fall 2023

## Week 7

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### Plan

PREVIOUSLY: unit, sequencing, let, pairs, tuples

TODAY:

1. options, variants
2. recursion
3. state

NEXT: exceptions?

NEXT: polymorphic (not so simple) typing

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### Records

$t ::= \dots$	<i>terms</i>
$\{l_i = t_i \mid i \in 1..n\}$	<i>record</i>
$t.l$	<i>projection</i>
$v ::= \dots$	<i>values</i>
$\{l_i = v_i \mid i \in 1..n\}$	<i>record value</i>
$T ::= \dots$	<i>types</i>
$\{l_i : T_i \mid i \in 1..n\}$	<i>type of records</i>

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### Evaluation rules for records

$$\{l_i = v_i \mid i \in 1..n\}.l_j \longrightarrow v_j \quad (\text{E-PROJRCD})$$

$$\frac{t_1 \longrightarrow t'_1}{t_1.l \longrightarrow t'_1.l} \quad (\text{E-PROJ})$$

$$\frac{t_j \longrightarrow t'_j}{\{l_i = v_i \mid i \in 1..j-1, l_j = t_j, l_k = t_k \mid k \in j+1..n\} \longrightarrow \{l_i = v_i \mid i \in 1..j-1, l_j = t'_j, l_k = t_k \mid k \in j+1..n\}} \quad (\text{E-RCD})$$

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## Typing rules for records

$$\frac{\text{for each } i \quad \Gamma \vdash t_i : T_i}{\Gamma \vdash \{l_i = t_i \mid i \in 1..n\} : \{l_i : T_i \mid i \in 1..n\}} \quad (\text{T-RCD})$$

$$\frac{\Gamma \vdash t_1 : \{l_i : T_i \mid i \in 1..n\}}{\Gamma \vdash t_1.l_j : T_j} \quad (\text{T-PROJ})$$

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## Sums and variants

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## Sums – motivating example

```
PhysicalAddr = {firstlast:String, addr:String}
VirtualAddr  = {name:String, email:String}
Addr         = PhysicalAddr + VirtualAddr
inl  : "PhysicalAddr → PhysicalAddr+VirtualAddr"
inr  : "VirtualAddr  → PhysicalAddr+VirtualAddr"
```

```
getName = λa:Addr.
  case a of
    inl x ⇒ x.firstlast
  | inr y ⇒ y.name;
```

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## New syntactic forms

$t ::= \dots$	<i>terms</i>
inl t	<i>tagging (left)</i>
inr t	<i>tagging (right)</i>
case t of inl x⇒t   inr x⇒t	<i>case</i>
$v ::= \dots$	<i>values</i>
inl v	<i>tagged value (left)</i>
inr v	<i>tagged value (right)</i>
$T ::= \dots$	<i>types</i>
T+T	<i>sum type</i>

$T_1+T_2$  is a *disjoint union* of  $T_1$  and  $T_2$  (the tags `inl` and `inr` ensure disjointness)

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### New evaluation rules

$t \rightarrow t'$

$\text{case } (\text{inl } v_0) \text{ of } \text{inl } x_1 \Rightarrow t_1 \mid \text{inr } x_2 \Rightarrow t_2 \rightarrow [x_1 \mapsto v_0]t_1$  (E-CASEINL)

$\text{case } (\text{inr } v_0) \text{ of } \text{inl } x_1 \Rightarrow t_1 \mid \text{inr } x_2 \Rightarrow t_2 \rightarrow [x_2 \mapsto v_0]t_2$  (E-CASEINR)

$\frac{t_0 \rightarrow t'_0}{\text{case } t_0 \text{ of } \text{inl } x_1 \Rightarrow t_1 \mid \text{inr } x_2 \Rightarrow t_2 \rightarrow \text{case } t'_0 \text{ of } \text{inl } x_1 \Rightarrow t_1 \mid \text{inr } x_2 \Rightarrow t_2}$  (E-CASE)

$\frac{t_1 \rightarrow t'_1}{\text{inl } t_1 \rightarrow \text{inl } t'_1}$  (E-INL)

$\frac{t_1 \rightarrow t'_1}{\text{inr } t_1 \rightarrow \text{inr } t'_1}$  (E-INR)

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### New typing rules

$\Gamma \vdash t : T$

$\frac{\Gamma \vdash t_1 : T_1}{\Gamma \vdash \text{inl } t_1 : T_1 + T_2}$  (T-INL)

$\frac{\Gamma \vdash t_1 : T_2}{\Gamma \vdash \text{inr } t_1 : T_1 + T_2}$  (T-INR)

$\frac{\Gamma \vdash t_0 : T_1 + T_2 \quad \Gamma, x_1 : T_1 \vdash t_1 : T \quad \Gamma, x_2 : T_2 \vdash t_2 : T}{\Gamma \vdash \text{case } t_0 \text{ of } \text{inl } x_1 \Rightarrow t_1 \mid \text{inr } x_2 \Rightarrow t_2 : T}$  (T-CASE)

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## Sums and Uniqueness of Types

Problem:

If  $t$  has type  $T$ , then  $\text{inl } t$  has type  $T+U$  for every  $U$ .

I.e., we've lost uniqueness of types.

Possible solutions:

- "Infer"  $U$  as needed during typechecking
- Give constructors different names and only allow each name to appear in one sum type (requires generalization to "variants," which we'll see next) — OCaml's solution
- Annotate each  $\text{inl}$  and  $\text{inr}$  with the intended sum type.

For simplicity, let's choose the third.

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### New syntactic forms

$t ::= \dots$   
 $\text{inl } t \text{ as } T$   
 $\text{inr } t \text{ as } T$

terms  
 tagging (left)  
 tagging (right)

$v ::= \dots$   
 $\text{inl } v \text{ as } T$   
 $\text{inr } v \text{ as } T$

values  
 tagged value (left)  
 tagged value (right)

Note that  $\text{as } T$  here is not the ascription operator that we saw before — i.e., not a separate syntactic form: in essence, there is an ascription "built into" every use of  $\text{inl}$  or  $\text{inr}$ .

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New typing rules

$\boxed{\Gamma \vdash t : T}$

$$\frac{\Gamma \vdash t_1 : T_1}{\Gamma \vdash \text{inl } t_1 \text{ as } T_1+T_2 : T_1+T_2} \quad (\text{T-INL})$$

$$\frac{\Gamma \vdash t_1 : T_2}{\Gamma \vdash \text{inr } t_1 \text{ as } T_1+T_2 : T_1+T_2} \quad (\text{T-INR})$$

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Evaluation rules ignore annotations:

$\boxed{t \longrightarrow t'}$

$$\frac{\text{case } (\text{inl } v_0 \text{ as } T_0) \text{ of } \text{inl } x_1 \Rightarrow t_1 \mid \text{inr } x_2 \Rightarrow t_2}{\longrightarrow [x_1 \mapsto v_0] t_1} \quad (\text{E-CASEINL})$$

$$\frac{\text{case } (\text{inr } v_0 \text{ as } T_0) \text{ of } \text{inl } x_1 \Rightarrow t_1 \mid \text{inr } x_2 \Rightarrow t_2}{\longrightarrow [x_2 \mapsto v_0] t_2} \quad (\text{E-CASEINR})$$

$$\frac{t_1 \longrightarrow t'_1}{\text{inl } t_1 \text{ as } T_2 \longrightarrow \text{inl } t'_1 \text{ as } T_2} \quad (\text{E-INL})$$

$$\frac{t_1 \longrightarrow t'_1}{\text{inr } t_1 \text{ as } T_2 \longrightarrow \text{inr } t'_1 \text{ as } T_2} \quad (\text{E-INR})$$

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## Variants

Just as we generalized binary products to labeled records, we can generalize binary sums to labeled *variants*.

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New syntactic forms

$t ::= \dots$	<i>terms</i>
$\langle l=t \rangle \text{ as } T$	<i>tagging</i>
$\text{case } t \text{ of } \langle l_i=x_i \rangle \Rightarrow t_i \quad i \in 1..n$	<i>case</i>
$T ::= \dots$	<i>types</i>
$\langle l_i:T_i \rangle \quad i \in 1..n$	<i>type of variants</i>

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### New evaluation rules

$$t \longrightarrow t'$$

case  $\langle l_j = v_j \rangle$  as T of  $\langle l_i = x_i \rangle \Rightarrow t_i \quad i \in 1..n$  (E-CASEVARIANT)  
 $\longrightarrow [x_j \mapsto v_j] t_j$

$$\frac{t_0 \longrightarrow t'_0}{\text{case } t_0 \text{ of } \langle l_i = x_i \rangle \Rightarrow t_i \quad i \in 1..n \longrightarrow \text{case } t'_0 \text{ of } \langle l_i = x_i \rangle \Rightarrow t_i \quad i \in 1..n} \quad (\text{E-CASE})$$

$$\frac{t_i \longrightarrow t'_i}{\langle l_j = t_j \rangle \text{ as } T \longrightarrow \langle l_j = t'_j \rangle \text{ as } T} \quad (\text{E-VARIANT})$$

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### New typing rules

$$\Gamma \vdash t : T$$

$$\frac{\Gamma \vdash t_j : T_j}{\Gamma \vdash \langle l_j = t_j \rangle \text{ as } \langle l_i : T_i \quad i \in 1..n \rangle : \langle l_i : T_i \quad i \in 1..n \rangle} \quad (\text{T-VARIANT})$$

$$\frac{\begin{array}{c} \Gamma \vdash t_0 : \langle l_i : T_i \quad i \in 1..n \rangle \\ \text{for each } i \quad \Gamma, x_i : T_i \vdash t_i : T \end{array}}{\Gamma \vdash \text{case } t_0 \text{ of } \langle l_i = x_i \rangle \Rightarrow t_i \quad i \in 1..n : T} \quad (\text{T-CASE})$$

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### Example

```
Addr = <physical:PhysicalAddr, virtual:VirtualAddr>;
a = <physical=pa> as Addr;
getName = λa:Addr.
  case a of
    <physical=x> ⇒ x.firstlast
  | <virtual=y> ⇒ y.name;
```

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### Options

Just like in OCaml...

```
OptionalNat = <none:Unit, some:Nat>;
Table = Nat → OptionalNat;
emptyTable = λn:Nat. <none=unit> as OptionalNat;
extendTable =
  λt:Table. λm:Nat. λv:Nat.
    λn:Nat.
      if equal n m then <some=v> as OptionalNat
      else t n;
x = case t(5) of
  <none=u> ⇒ 999
  | <some=v> ⇒ v;
```

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## Enumerations

```
Weekday = <monday:Unit, tuesday:Unit, wednesday:Unit,  
          thursday:Unit, friday:Unit>;  
  
nextBusinessDay = \w:Weekday.  
  case w of <monday=x>    => <tuesday=unit> as Weekday  
    | <tuesday=x>        => <wednesday=unit> as Weekday  
    | <wednesday=x>      => <thursday=unit> as Weekday  
    | <thursday=x>       => <friday=unit> as Weekday  
    | <friday=x>         => <monday=unit> as Weekday;
```