

Foundations of Software Fall 2023

Week 11

1

Different Kinds of Maps

What is missing?

$$\begin{array}{lcl} \textit{Term} & \rightarrow & \textit{Term} \ (\lambda x.t) \\ \textit{Type} & \rightarrow & \textit{Term} \ (\lambda X.t) \end{array}$$

2

Different Kinds of Maps

What is missing?

$$\begin{array}{lcl} \textit{Term} & \rightarrow & \textit{Term} \ (\lambda x.t) \\ \textit{Type} & \rightarrow & \textit{Term} \ (\lambda X.t) \\ \textit{Type} & \rightarrow & \textit{Type} \ ??? \\ \textit{Term} & \rightarrow & \textit{Type} \ ??? \end{array}$$

Agenda today:

- ▶ Type operators
- ▶ Dependent types

2

Type Operators and System F_ω

3

Type Operators

Example. Type operators in Scala:

```
def termIdentity(x: Int): Int = x // similar to  
val termIdentity: Int => Int = (x: Int) => x
```

```
type MkFun[T] = T => T // equiv to  
type MkFun = [T] =>> T => T  
val f: MkFun[Int] = (x: Int) => x
```

4

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Type operators are functions at the type-level.

$\lambda X :: K. T$

4

Type Operators

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type MkFun = [T] =>> T => T  
val f: MkFun[Int] = (x: Int) => x
```

Type operators are functions at the type-level.

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Two Problems:

- ▶ Type checking of type operators
- ▶ Equivalence of types

4

Kinding

Problem: avoid meaningless types, like `MkFun[Int, String]`.

5

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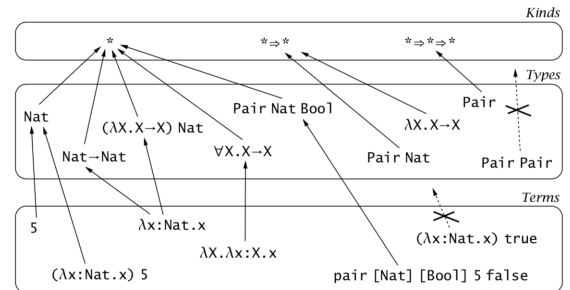
- `*` proper types, e.g. `Bool`, `Int → Int`
- `* ⇒ *` type operators: map proper types to proper types
- `* ⇒ * ⇒ *` two-argument operators
- `(* ⇒ *) ⇒ *` type operators: map type operators to proper types

5

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- `*` proper types, e.g. `Bool`, `Int → Int`
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- `* ⇒ * ⇒ *` two-argument operators
- `(* ⇒ *) ⇒ *` type operators: map type operators to proper types



5

Equivalence of Types

Problem: all the types below are equivalent

`Nat → Bool` `Nat → Id Bool` `Id Nat → Id Bool`
`Id Nat → Bool` `Id (Nat → Bool)` `Id (Id (Id Nat → Bool))`

We need to introduce a *definitional equivalence* relation on types, written $S \equiv T$. The most important rule is:

$$(\lambda X :: K. S) T \equiv [X \mapsto T] S \quad (\text{Q-APPABS})$$

And we need one typing rule:

$$\frac{\Gamma \vdash t : S \quad S \equiv T}{\Gamma \vdash t : T} \quad (\text{T-EQ})$$

6

First-class Type Operators

Scala supports passing type operators as argument:

```
def makeInt[F[_]](f: () => F[Int]): F[Int] = f()
// equiv to
def makeInt[F <: [X] => Any](...): ...
```

```
makeInt[List]() => List[Int](3)
makeInt[Option]() => None
makeInt[[T] => (T, T)]() => (3, 4)
```

First-class type operators supports *polymorphism* for type operators, which enables more patterns in type-safe functional programming.

7

System F_ω — Syntax

Formalizing first-class type operators leads to System F_ω :

$t ::= \dots$	<i>terms</i>
$\lambda X :: K. t$	<i>type abstraction</i>
$T ::=$	<i>types</i>
X	<i>type variable</i>
$T \rightarrow T$	<i>type of functions</i>
$\forall X :: K. T$	<i>universal type</i>
$\lambda X :: K. T$	<i>operator abstraction</i>
$T \ T$	<i>operator application</i>
$K ::=$	<i>kinds</i>
$*$	<i>kind of proper types</i>
$K \Rightarrow K$	<i>kind of operators</i>

8

System F_ω — Semantics

$\frac{t_1 \rightarrow t'_1}{t_1 \ t_2 \rightarrow t'_1 \ t_2}$	(E-APP1)
$\frac{t_2 \rightarrow t'_2}{t_1 \ t_2 \rightarrow t_1 \ t'_2}$	(E-APP2)
$(\lambda x :: T_1. t_1) \ v_2 \rightarrow [x \mapsto v_2] t_1$	(E-APPABS)
$\frac{t \rightarrow t'}{t \ [T] \rightarrow t' \ [T]}$	(E-TAPP)
$(\lambda X :: K. t_1) \ [T] \rightarrow [X \mapsto T] t_1$	(E-TAPPTABS)

9

System F_ω — Kinding

$\frac{X :: K \in \Gamma}{\Gamma \vdash X :: K}$	(K-TVAR)
$\frac{\Gamma, X :: K_1 \vdash T_2 :: K_2}{\Gamma \vdash \lambda X :: K_1. T_2 :: K_1 \Rightarrow K_2}$	(K-ABS)
$\frac{\Gamma \vdash T_1 :: K_1 \Rightarrow K_2 \quad \Gamma \vdash T_2 :: K_1}{\Gamma \vdash T_1 \ T_2 :: K_2}$	(K-APP)
$\frac{\Gamma \vdash T_1 :: * \quad \Gamma \vdash T_2 :: *}{\Gamma \vdash T_1 \rightarrow T_2 :: *}$	(K-ARROW)
$\frac{\Gamma, X :: K_1 \vdash T_2 :: *}{\Gamma \vdash \forall X :: K_1. T_2 :: *}$	(K-ALL)

10

System F_ω — Type Equivalence

$T \equiv T$	$\frac{T \equiv S}{S \equiv T}$	$\frac{S \equiv U \quad U \equiv T}{S \equiv T}$
$\frac{S_1 \equiv T_1 \quad S_2 \equiv T_2}{S_1 \rightarrow S_2 \equiv T_1 \rightarrow T_2}$	(Q-ARROW)	
$\frac{S_2 \equiv T_2}{\forall X :: K_1. S_2 \equiv \forall X :: K_1. T_2}$	(K-ALL)	
$\frac{S_2 \equiv T_2}{\lambda X :: K_1. S_2 \equiv \lambda X :: K_1. T_2}$	(Q-ABS)	
$\frac{S_1 \equiv T_1 \quad S_2 \equiv T_2}{S_1 \ S_2 \equiv T_1 \ T_2}$	(Q-APP)	
$(\lambda X :: K. T_1) \ T_2 \equiv [X \mapsto T_2] T_1$	(Q-APPABS)	

11

System F_ω — Typing

$$\begin{array}{c}
 \frac{x : T \in \Gamma}{\Gamma \vdash x : T} \quad (\text{T-VAR}) \\
 \\
 \frac{\Gamma \vdash T_1 :: * \quad \Gamma, x : T_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda x : T_1. t_2 : T_1 \rightarrow T_2} \quad (\text{T-ABS}) \\
 \\
 \frac{\Gamma \vdash t_1 : S \rightarrow T \quad \Gamma \vdash t_2 : S}{\Gamma \vdash t_1 t_2 : T} \quad (\text{T-APP}) \\
 \\
 \frac{\Gamma, X :: K_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda X :: K_1. t_2 : \forall X :: K_1. T_2} \quad (\text{T-TABS}) \\
 \\
 \frac{\Gamma \vdash t : \forall X :: K. T_2 \quad \Gamma \vdash T :: K}{\Gamma \vdash t [T] : [X \mapsto T] T_2} \quad (\text{T-TAPP}) \\
 \\
 \frac{\Gamma \vdash t : S \quad S \equiv T \quad \Gamma \vdash T :: *}{\Gamma \vdash t : T} \quad (\text{T-EQ})
 \end{array}$$

12

Kinding question

Clicker question: What are the kinds of $\lambda X :: *. X \rightarrow X$ and $\forall X :: *. X \rightarrow X$, respectively?

- A. $*$ and $*$
- B. $*$ and $* \Rightarrow *$
- C. $* \Rightarrow *$ and $*$
- D. $* \Rightarrow *$ and $* \Rightarrow *$

URL: ttopoll.eu
Session ID: cs452

13

Example

```

type PairRep[Pair :: * ⇒ * ⇒ *] = {
  pair : ∀X.∀Y. X → Y → (Pair X Y),
  fst  : ∀X.∀Y. (Pair X Y) → X,
  snd  : ∀X.∀Y. (Pair X Y) → Y
}

def swap[Pair :: * ⇒ * ⇒ *, X :: *, Y :: *]
  (rep : PairRep Pair)
  (pair : Pair X Y) : Pair Y X
=
  let x = rep.fst [X] [Y] pair in
  let y = rep.snd [X] [Y] pair in
  rep.pair [Y] [X] y x

```

The method `swap` works for any representation of pairs.

14

Properties

Theorem [Preservation]: if $\Gamma \vdash t : T$ and $t \longrightarrow t'$, then $\Gamma \vdash t' : T$.

Theorem [Progress]: if $\vdash t : T$, then either t is a value or there exists t' with $t \longrightarrow t'$.

15

Dependent Types

16

Why Does It Matter?

Example 1. Track length of vectors in types:

```
NVec  :: Nat → *  
first : (n:Nat) → NVec (n + 1) → Nat
```

$(x:S) \rightarrow T$ is called **dependent function type**. It is impossible to pass a vector of length 0 to the function *first*.

17

Why Does It Matter?

Example 1. Track length of vectors in types:

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NVec  :: Nat → *  
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```

$(x:S) \rightarrow T$ is called **dependent function type**. It is impossible to pass a vector of length 0 to the function *first*.

Example 2. Safe formatting for *sprintf*:

```
sprintf      : (f:Format) → Data(f) → String
```

```
Data([])      = Unit  
Data("%" :: 'd' :: cs) = Nat * Data(cs)  
Data("%" :: 's' :: cs) = String * Data(cs)  
Data(c :: cs)    = Data(cs)
```

17

Dependent Function Type (a.k.a. Π Types)

A dependent function type is inhabited by a *dependent function*:

```
 $\lambda x:S.t$   : (x:S) → T
```

18

Dependent Function Type (a.k.a. Π Types)

A dependent function type is inhabited by a *dependent function*:

$$\lambda x:S.t \quad : \quad (x:S) \rightarrow T$$

If T does not depend on x , it degenerates to function types:

$$(x:S) \rightarrow T = S \rightarrow T \quad \text{where } x \text{ does not appear free in } T$$

18

The Calculus of Constructions

19

The Calculus of Constructions: Syntax

$t ::=$		<i>terms</i>
s		sort
x		variable
$\lambda x:t.t$		abstraction
$t \ t$		application
$(x:t) \rightarrow t$		dependent type
$s ::=$		<i>sorts</i>
$*$		sort of proper types
\square		sort of kinds
$\Gamma ::=$		<i>contexts</i>
\emptyset		empty context
$\Gamma, x:T$		term variable binding

The semantics is the usual β -reduction.

20

The Calculus of Constructions: Typing

$\vdash * : \square$ (T-AXIOM)	$\frac{x:T \in \Gamma}{\Gamma \vdash x : T}$ (T-VAR)
$\frac{\Gamma \vdash S : s_1 \quad \Gamma, x:S \vdash t : T}{\Gamma \vdash \lambda x:S.t : (x:S) \rightarrow T}$	(T-ABS)
$\frac{\Gamma \vdash t_1 : (x:S) \rightarrow T \quad \Gamma \vdash t_2 : S}{\Gamma \vdash t_1 \ t_2 : [x \mapsto t_2] T}$	(T-APP)
$\frac{\Gamma \vdash S : s_1 \quad \Gamma, x:S \vdash T : s_2}{\Gamma \vdash (x:S) \rightarrow T : s_2}$	(T-PI)
$\frac{\Gamma \vdash t : T \quad T \equiv T' \quad \Gamma \vdash T' : s}{\Gamma \vdash t : T'}$	(T-CONV)

The equivalence relation $T \equiv T'$ is based on β -reduction.

21

Four Kinds of Lambdas

Example	Type
$\lambda x:\mathbb{N}.x + 1$	$\mathbb{N} \rightarrow \mathbb{N}$
$\lambda f:\mathbb{N} \rightarrow \mathbb{N}.f\ x$	$(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$

22

Four Kinds of Lambdas

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$\lambda x:\mathbb{N}.x + 1$	$\mathbb{N} \rightarrow \mathbb{N}$
$\lambda f:\mathbb{N} \rightarrow \mathbb{N}.f\ x$	$(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$
$\lambda X:*. \lambda x:X. x$	$(X:*) \rightarrow X \rightarrow X$
$\lambda F:* \rightarrow *. \lambda x:F\ \mathbb{N}. x$	$(F:* \rightarrow *) \rightarrow (F\ \mathbb{N}) \rightarrow (F\ \mathbb{N})$

22

Four Kinds of Lambdas

Example	Type
$\lambda x:\mathbb{N}.x + 1$	$\mathbb{N} \rightarrow \mathbb{N}$
$\lambda f:\mathbb{N} \rightarrow \mathbb{N}.f\ x$	$(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$
$\lambda X:*. \lambda x:X. x$	$(X:*) \rightarrow X \rightarrow X$
$\lambda F:* \rightarrow *. \lambda x:F\ \mathbb{N}. x$	$(F:* \rightarrow *) \rightarrow (F\ \mathbb{N}) \rightarrow (F\ \mathbb{N})$
$\lambda X:*. X$	$* \rightarrow *$
$\lambda F:* \rightarrow *. F\ \mathbb{N}$	$(* \rightarrow *) \rightarrow *$

22

Four Kinds of Lambdas

Example	Type
$\lambda x:\mathbb{N}.x + 1$	$\mathbb{N} \rightarrow \mathbb{N}$
$\lambda f:\mathbb{N} \rightarrow \mathbb{N}.f\ x$	$(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$
$\lambda X:*. \lambda x:X. x$	$(X:*) \rightarrow X \rightarrow X$
$\lambda F:* \rightarrow *. \lambda x:F\ \mathbb{N}. x$	$(F:* \rightarrow *) \rightarrow (F\ \mathbb{N}) \rightarrow (F\ \mathbb{N})$
$\lambda X:*. X$	$* \rightarrow *$
$\lambda F:* \rightarrow *. F\ \mathbb{N}$	$(* \rightarrow *) \rightarrow *$
$\lambda n:\mathbb{N}. NVec\ n$	$\mathbb{N} \rightarrow *$
$\lambda f:\mathbb{N} \rightarrow \mathbb{N}. NVec\ (f\ 6)$	$(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow *$

22

Strong Normalization

Given the following β -reduction rules

$$\frac{t_1 \rightarrow t'_1}{\lambda x: T_1. t_1 \rightarrow \lambda x: T_1. t'_1} \quad (\beta\text{-ABS})$$

$$\frac{t_1 \rightarrow t'_1}{t_1 t_2 \rightarrow t'_1 t_2} \quad (\beta\text{-APP1})$$

$$\frac{t_2 \rightarrow t'_2}{t_1 t_2 \rightarrow t_1 t'_2} \quad (\beta\text{-APP2})$$

$$(\lambda x: T_1. t_1) t_2 \rightarrow [x \mapsto t_2] t_1 \quad (\beta\text{-APPABS})$$

Theorem [Strong Normalization]: if $\Gamma \vdash t : T$, then there is no infinite sequence of terms t_i such that $t = t_1$ and $t_i \rightarrow t_{i+1}$.

23

Pure Type Systems

$$\frac{\Gamma \vdash S : s_i \quad \Gamma, x:S \vdash T : s_j}{\Gamma \vdash (x:S) \rightarrow T : s_j} \quad (\text{T-PI})$$

System	(s_i, s_j)			
λ_{\rightarrow}	{	$(*, *)$	}	
λ_P	{	$(*, *)$, $(*, \square)$	}	
F	{	$(*, *)$, $(\square, *)$	}	
F^ω	{	$(*, *)$, $(\square, *)$, (\square, \square)	}	
CC	{	$(*, *)$, $(*, \square)$, $(\square, *)$, (\square, \square)	}	

The Lambda Cube
 $\lambda_{\rightarrow} \rightarrow F \rightarrow F^\omega \rightarrow CC$

24

Dependent Types in Coq

25

Proof Assistants

Dependent type theories are at the foundation of proof assistants, like Coq, Agda, etc.

By *Curry-Howard Correspondence*

- proofs \longleftrightarrow programs
- propositions \longleftrightarrow types

26

Proof Assistants

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By *Curry-Howard Correspondence*

- ▶ proofs \longleftrightarrow programs
- ▶ propositions \longleftrightarrow types

Two impactful projects based on Coq:

- ▶ CompCert: certified C compiler
- ▶ Mechanized proof of 4-color theorem

26

Type Universes in Coq

The rule $\Gamma \vdash \text{Type} : \text{Type}$ is unsound (Girard's paradox).

$$\Gamma \vdash \text{Prop} : \text{Type}_1$$

$$\Gamma \vdash \text{Set} : \text{Type}_1$$

$$\Gamma \vdash \text{Type}_i : \text{Type}_{i+1}$$

$$\frac{\Gamma, x:A \vdash B : \text{Prop} \quad \Gamma \vdash A : s}{\Gamma \vdash (x : A) \rightarrow B : \text{Prop}}$$

$$\frac{\Gamma, x:A \vdash B : \text{Set} \quad \Gamma \vdash A : s \quad s \in \{\text{Prop}, \text{Set}\}}{\Gamma \vdash (x : A) \rightarrow B : \text{Set}}$$

$$\frac{\Gamma, x:A \vdash B : \text{Type}_i \quad \Gamma \vdash A : \text{Type}_i}{\Gamma \vdash (x : A) \rightarrow B : \text{Type}_i}$$

27

Coq 101 - inductive definitions and recursion

```
1 Inductive nat : Type :=  
2   | 0  
3   | S (n : nat).
```

28

Coq 101 - inductive definitions and recursion

```
1 Inductive nat : Type :=  
2   | 0  
3   | S (n : nat).  
  
1 Fixpoint double (n : nat) : nat :=  
2   match n with  
3   | 0 => 0  
4   | S n' => S (S (double n'))  
5   end.
```

Recursion has to be **structural**.

28

Coq 101 - inductive definitions and recursion

```

1 Inductive nat : Type :=
2   | 0
3   | S (n : nat).

1 Fixpoint double (n : nat) : nat :=
2   match n with
3   | 0 => 0
4   | S n' => S (S (double n'))
5   end.

Recursion has to be structural.

1 Inductive even : nat -> Prop :=
2   | even0 : even 0
3   | evenS : forall x:nat, even x -> even (S (S x)).

```

28

Coq 101 - proofs

```

1 Definition even_prop := forall x:nat, even (double x).
2
3 Fixpoint even_proof(x: nat): even (double x) :=
4   match x with
5   | 0      => even0
6   | S n'   => evenS (double n') (even_proof n')
7   end.
8
9 Check even_proof : even_prop.

```

29

Coq 101 - proofs

```

1 Definition even_prop := forall x:nat, even (double x).
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3 Fixpoint even_proof(x: nat): even (double x) :=
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7   end.
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9 Check even_proof : even_prop.

```

The 2nd branch has the type $\text{even } S (S (\text{double } n'))$, and Coq knows by normalizing the types:

$$\text{even } S (S (\text{double } n')) \equiv_{\beta} \text{even } (\text{double } (S n'))$$

29

Recap: Curry-Howard Correspondence

Propositions as types in the context of intuitionistic logic.

Proposition	Term & Type
$A \wedge B$	$t : (A, B)$
$A \vee B$	$t : A + B$
$A \rightarrow B$	$t : A \rightarrow B$
\perp	$t : \text{False}$
$\neg A$	$t : A \rightarrow \text{False}$
$\forall x:A. B$	$t : (x : A) \rightarrow B$
$\exists x:A. B$	$t : (x:A, B)$

30

Curry-Howard correspondence in Coq

```
1 Inductive and (A B:Prop) : Prop :=
2   conj : A -> B -> A /\ B
3 where "A /\ B" := (and A B) : type_scope.
```

31

Curry-Howard correspondence in Coq

```
1 Inductive and (A B:Prop) : Prop :=
2   conj : A -> B -> A /\ B
3 where "A /\ B" := (and A B) : type_scope.
```

```
1 Inductive or (A B:Prop) : Prop :=
2   | or_introl : A -> A \/ B
3   | or_intror : B -> A \/ B
4 where "A \/ B" := (or A B) : type_scope.
```

31

Curry-Howard correspondence in Coq

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```
1 Inductive False : Prop :=.
```

31

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```

```
1 Inductive False : Prop :=.
```

```
1 Definition not (A:Prop) := A -> False.
2 Notation "~ x" := (not x) : type_scope.
```

31

Curry-Howard correspondence in Coq - continued

```
1 Notation "A -> B" := (forall (_ : A), B) : type_scope.
2 Definition iff (A B:Prop) := (A -> B) /\ (B -> A).
3 Notation "A <-> B" := (iff A B) : type_scope.
```

32

Curry-Howard correspondence in Coq - continued

```
1 Notation "A -> B" := (forall (_ : A), B) : type_scope.
2 Definition iff (A B:Prop) := (A -> B) /\ (B -> A).
3 Notation "A <-> B" := (iff A B) : type_scope.

1 Inductive ex (A:Type) (P:A -> Prop) : Prop :=
2   ex_intro : forall x:A, P x -> ex (A:=A) P.
3
4 Notation "'exists' x .. y , p" :=
5   (ex (fun x => .. (ex (fun y => p)) ..)) : type_scope.
```

32

Curry-Howard correspondence in Coq - continued

```
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2 Definition iff (A B:Prop) := (A -> B) /\ (B -> A).
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1 Inductive ex (A:Type) (P:A -> Prop) : Prop :=
2   ex_intro : forall x:A, P x -> ex (A:=A) P.
3
4 Notation "'exists' x .. y , p" :=
5   (ex (fun x => .. (ex (fun y => p)) ..)) : type_scope.

1 Inductive eq (A:Type) (x:A) : A -> Prop :=
2   eq_refl : x = x :>A
3
4 Notation "x = y" := (eq x y) : type_scope.
```

32

The equivalence between LEM and DNE

In *intuitionistic logics*, the *law of excluded middle* (LEM) and the *law of double negation* (DNE) are not provable.

- ▶ LEM: $\forall P. P \vee \neg P$
- ▶ DNE: $\forall P. \neg\neg P \rightarrow P$

By curry-howard correspondence, there are no terms that inhabit the types above.

33

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33

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33

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By curry-howard correspondence, there are no terms that inhabit the types above.

However, $\forall P. P \rightarrow \neg\neg P$ can be proved. **How?**

We will prove that LEM is equivalent to DNE:

```
1 Definition LEM: Prop := forall P: Prop, P \/ ~P.
2 Definition DNE: Prop := forall P: Prop, ~~P -> P.
3 Definition LEM_DNE_EQ: Prop := LEM <-> DNE.
```

33

LEM \rightarrow DNE

```
1 Definition LEM_To_DNE :=
2   fun (lem: forall P : Prop, P \/ ~ P) (Q:Prop) (q: ~~Q)
3     =>
4     match lem Q with
5     | or_introl l =>
6       l
7     | or_intror r =>
8       match (q r) with end
9     end.
10
11 Check LEM_To_DNE : LEM -> DNE.
```

34

DNE \rightarrow LEM

```
1 Definition DNE_To_LEM :=
2   fun (dne: forall P : Prop, ~~P -> P) (Q:Prop) =>
3     (dne (Q \ / ~ Q))
4     (fun H: ~(Q \ / ~ Q) =>
5       let nq := (fun q: Q => H (or_intror q))
6       in H (or_intror nq)
7     ).
8
9 Check DNE_To_LEM : DNE -> LEM.
10
11 Definition proof := conj LEM_To_DNE DNE_To_LEM.
12 Check proof : LEM <-> DNE.
```

35

Dependent Types in Programming Languages

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36

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Challenge: the decidability of type checking.

36

Problem with Type Checking

Value constructors:

```
NVec  : ℕ → *
nil   : NVec 0
cons  : ℕ → (ℕ → NVec n → NVec n + 1)
```

Appending vectors:

```
append : (m:ℕ) → (n:ℕ) → NVec m → NVec n → NVec (n + m)
append = λm:ℕ. λn:ℕ. λl:NVec m. λt:NVec n.
  match l with
  | nil => t
  | cons x r y => cons x (r + n) (append r n y t)
```

Question: How does the type checker know $S(r + n) = n + (S r)$?

37