

# Foundations of Software Fall 2023

Week 6

# Plan

PREVIOUSLY:

1. type safety as *progress* and *preservation*
2. typed arithmetic expressions
3. simply typed lambda calculus (STLC)

TODAY:

1. Equivalence of lambda terms
2. Preservation for STLC
3. Extensions to STLC

NEXT: state, exceptions

NEXT: polymorphic (not so simple) typing

# Preservation for STLC

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Which case is the hard one??

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 $\Gamma \vdash t_1 : T_{11} \rightarrow T_{12}$   
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*Subcase:*

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*Subcase:*  $t_1 = \lambda x:T_{11}. t_{12}$   
 $t_2$  a value  $v_2$   
 $t' = [x \mapsto v_2]t_{12}$

Uh oh.

# The “Substitution Lemma”

*Lemma:* Types are preserved under substitution.

That is, if  $\Gamma, x:S \vdash t : T$  and  $\Gamma \vdash s : S$ , then  $\Gamma \vdash [x \mapsto s]t : T$ .

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*Proof:* ...

## Weakening and Permutation

Two other lemmas will be useful.

Weakening tells us that we can *add assumptions* to the context without losing any true typing statements.

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Permutation tells us that the order of assumptions in (the list)  $\Gamma$  does not matter.

*Lemma:* If  $\Gamma \vdash t : T$  and  $\Delta$  is a permutation of  $\Gamma$ , then  $\Delta \vdash t : T$ .

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I.e., “Types are preserved under substitution.”

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*Proof:* By induction on the derivation of  $\Gamma, x:S \vdash t : T$ . Proceed by cases on the final typing rule used in the derivation.



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Case T-APP:  $t = t_1 \ t_2$   
 $\Gamma, x:S \vdash t_1 : T_2 \rightarrow T_1$   
 $\Gamma, x:S \vdash t_2 : T_2$   
 $T = T_1$

By the induction hypothesis,  $\Gamma \vdash [x \mapsto s]t_1 : T_2 \rightarrow T_1$  and  $\Gamma \vdash [x \mapsto s]t_2 : T_2$ . By T-APP,  $\Gamma \vdash [x \mapsto s]t_1 \ [x \mapsto s]t_2 : T$ , i.e.,  $\Gamma \vdash [x \mapsto s](t_1 \ t_2) : T$ .

## The “Substitution Lemma”

*Lemma:* If  $\Gamma, x:S \vdash t : T$  and  $\Gamma \vdash s : S$ , then  $\Gamma \vdash [x \mapsto s]t : T$ .

*Proof:* By induction on the derivation of  $\Gamma, x:S \vdash t : T$ . Proceed by cases on the final typing rule used in the derivation.

Case T-VAR:  $t = z$   
with  $z:T \in (\Gamma, x:S)$

There are two sub-cases to consider, depending on whether  $z$  is  $x$  or another variable. If  $z = x$ , then  $[x \mapsto s]z = s$ . The required result is then  $\Gamma \vdash s : S$ , which is among the assumptions of the lemma. Otherwise,  $[x \mapsto s]z = z$ , and the desired result is immediate.

## The “Substitution Lemma”

*Lemma:* If  $\Gamma, x:S \vdash t : T$  and  $\Gamma \vdash s : S$ , then  $\Gamma \vdash [x \mapsto s]t : T$ .

*Proof:* By induction on the derivation of  $\Gamma, x:S \vdash t : T$ . Proceed by cases on the final typing rule used in the derivation.

Case T-ABS:  $t = \lambda y:T_2. t_1$        $T = T_2 \rightarrow T_1$   
 $\Gamma, x:S, y:T_2 \vdash t_1 : T_1$

By our conventions on choice of bound variable names, we may assume  $x \neq y$  and  $y \notin FV(s)$ . Using *permutation* on the given subderivation, we obtain  $\Gamma, y:T_2, x:S \vdash t_1 : T_1$ . Using *weakening* on the other given derivation ( $\Gamma \vdash s : S$ ), we obtain

$\Gamma, y:T_2 \vdash s : S$ . Now, by the induction hypothesis,

$\Gamma, y:T_2 \vdash [x \mapsto s]t_1 : T_1$ . By T-ABS,

$\Gamma \vdash \lambda y:T_2. [x \mapsto s]t_1 : T_2 \rightarrow T_1$ , i.e. (by the definition of substitution),  $\Gamma \vdash [x \mapsto s]\lambda y:T_2. t_1 : T_2 \rightarrow T_1$ .

## Summary: Preservation

*Theorem:* If  $\Gamma \vdash t : T$  and  $t \longrightarrow t'$ , then  $\Gamma \vdash t' : T$ .

Lemmas to prove:

- ▶ Weakening
- ▶ Permutation
- ▶ Substitution preserves types
- ▶ Reduction preserves types (i.e., preservation)

## Review: Type Systems

To define and verify a type system, you must:

1. Define types
2. Specify typing rules
3. Prove soundness: *progress* and *preservation*

# Two Typing Topics

# Erase

$$\begin{aligned}\text{erase}(x) &= x \\ \text{erase}(\lambda x:T_1. t_2) &= \lambda x. \text{erase}(t_2) \\ \text{erase}(t_1 \ t_2) &= \text{erase}(t_1) \ \text{erase}(t_2)\end{aligned}$$



## Intro vs. elim forms

An *introduction form* for a given type gives us a way of *constructing* elements of this type.

An *elimination form* for a type gives us a way of *using* elements of this type.

# The Curry-Howard Correspondence

In *constructive logics*, a proof of  $P$  must provide *evidence* for  $P$ .

- ▶ “law of the excluded middle” —  $P \vee \neg P$  — not recognized.

A proof of  $P \wedge Q$  is a *pair* of evidence for  $P$  and evidence for  $Q$ .

A proof of  $P \supset Q$  is a *procedure* for transforming evidence for  $P$  into evidence for  $Q$ .

# Propositions as Types

## LOGIC

propositions

proposition  $P \supset Q$

proposition  $P \wedge Q$

proof of proposition  $P$

proposition  $P$  is provable

## PROGRAMMING LANGUAGES

types

type  $P \rightarrow Q$

type  $P \times Q$

term  $t$  of type  $P$

type  $P$  is inhabited (by some term)

# Propositions as Types

## LOGIC

---

propositions

proposition  $P \supset Q$

proposition  $P \wedge Q$

proof of proposition  $P$

proposition  $P$  is provable

proof simplification

(a.k.a. “cut elimination”)

## PROGRAMMING LANGUAGES

---

types

type  $P \rightarrow Q$

type  $P \times Q$

term  $t$  of type  $P$

type  $P$  is inhabited (by some term)

evaluation

# Extensions to STLC

## Base types

Up to now, we've formulated “base types” (e.g. `Nat`) by adding them to the syntax of types, extending the syntax of terms with associated constants (`0`) and operators (`succ`, etc.) and adding appropriate typing and evaluation rules. We can do this for as many base types as we like.

For more theoretical discussions (as opposed to programming) we can often ignore the term-level inhabitants of base types, and just treat these types as uninterpreted constants.

E.g., suppose `B` and `C` are some base types. Then we can ask (without knowing anything more about `B` or `C`) whether there are any types `S` and `T` such that the term

$$(\lambda f:S. \lambda g:T. f \ g) (\lambda x:B. x)$$

is well typed.

# The Unit type

$t ::= \dots$   
 $\text{unit}$

$v ::= \dots$   
 $\text{unit}$

$T ::= \dots$   
 $\text{Unit}$

*terms*  
*constant*  $\text{unit}$

*values*  
*constant*  $\text{unit}$

*types*  
*unit type*

*New typing rules*

$\Gamma \vdash \text{unit} : \text{Unit}$

|                       |
|-----------------------|
| $\Gamma \vdash t : T$ |
|-----------------------|

(T-UNIT)

# Sequencing

$t ::= \dots$   
 $t_1; t_2$

*terms*



# Sequencing

$t ::= \dots$   
 $t_1; t_2$

*terms*

$$\frac{t_1 \longrightarrow t'_1}{t_1; t_2 \longrightarrow t'_1; t_2} \quad (\text{E-SEQ})$$

$$\text{unit}; t_2 \longrightarrow t_2 \quad (\text{E-SEQNEXT})$$

$$\frac{\Gamma \vdash t_1 : \text{Unit} \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash t_1; t_2 : T_2} \quad (\text{T-SEQ})$$

## Derived forms

- ▶ Syntactic sugar
- ▶ Internal language vs. external (surface) language

## Sequencing as a derived form

$$t_1; t_2 \stackrel{\text{def}}{=} (\lambda x:\text{Unit}. t_2) \ t_1$$

where  $x \notin FV(t_2)$

# Equivalence of the two definitions

[board]

# Ascription

*New syntactic forms*

$t ::= \dots$   
 $t \text{ as } T$

*New evaluation rules*

$v_1 \text{ as } T \longrightarrow v_1$

(E-ASCRIIBE)

$$\frac{t_1 \longrightarrow t'_1}{t_1 \text{ as } T \longrightarrow t'_1 \text{ as } T}$$

(E-ASCRIIBE1)

*New typing rules*

$\boxed{\Gamma \vdash t : T}$

$$\frac{\Gamma \vdash t_1 : T}{\Gamma \vdash t_1 \text{ as } T : T}$$

(T-ASCRIIBE)

*terms*

*ascription*

$\boxed{t \longrightarrow t'}$

## Ascription as a derived form

$$t \text{ as } T \stackrel{\text{def}}{=} (\lambda x:T. x) \ t$$

# Let-bindings

*New syntactic forms*

$t ::= \dots$   
 $\text{let } x=t \text{ in } t$

*New evaluation rules*

*terms*

*let binding*

$t \longrightarrow t'$

$\text{let } x=v_1 \text{ in } t_2 \longrightarrow [x \mapsto v_1]t_2$  (E-LETV)

$$\frac{t_1 \longrightarrow t'_1}{\text{let } x=t_1 \text{ in } t_2 \longrightarrow \text{let } x=t'_1 \text{ in } t_2}$$
 (E-LET)

*New typing rules*

$\Gamma \vdash t : T$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma, x:T_1 \vdash t_2 : T_2}{\Gamma \vdash \text{let } x=t_1 \text{ in } t_2 : T_2}$$
 (T-LET)

# Pairs

$t ::= \dots$   
 $\{t, t\}$   
 $t.1$   
 $t.2$

*terms*  
*pair*  
*first projection*  
*second projection*

$v ::= \dots$   
 $\{v, v\}$

*values*  
*pair value*

$T ::= \dots$   
 $T_1 \times T_2$

*types*  
*product type*



## Evaluation rules for pairs

$$\{v_1, v_2\}.1 \longrightarrow v_1 \quad (\text{E-PAIRBETA1})$$

$$\{v_1, v_2\}.2 \longrightarrow v_2 \quad (\text{E-PAIRBETA2})$$

$$\frac{t_1 \longrightarrow t'_1}{t_1.1 \longrightarrow t'_1.1} \quad (\text{E-PROJ1})$$

$$\frac{t_1 \longrightarrow t'_1}{t_1.2 \longrightarrow t'_1.2} \quad (\text{E-PROJ2})$$

$$\frac{t_1 \longrightarrow t'_1}{\{t_1, t_2\} \longrightarrow \{t'_1, t_2\}} \quad (\text{E-PAIR1})$$

$$\frac{t_2 \longrightarrow t'_2}{\{v_1, t_2\} \longrightarrow \{v_1, t'_2\}} \quad (\text{E-PAIR2})$$

## Typing rules for pairs

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash \{t_1, t_2\} : T_1 \times T_2} \quad (\text{T-PAIR})$$

$$\frac{\Gamma \vdash t_1 : T_{11} \times T_{12}}{\Gamma \vdash t_{1.1} : T_{11}} \quad (\text{T-PROJ1})$$

$$\frac{\Gamma \vdash t_1 : T_{11} \times T_{12}}{\Gamma \vdash t_{1.2} : T_{12}} \quad (\text{T-PROJ2})$$

# Tuples

$t ::= \dots$   
 $\{t_i \mid i \in 1..n\}$   
 $t.i$

*terms*  
*tuple*  
*projection*

$v ::= \dots$   
 $\{v_i \mid i \in 1..n\}$

*values*  
*tuple value*

$T ::= \dots$   
 $\{T_i \mid i \in 1..n\}$

*types*  
*tuple type*

## Evaluation rules for tuples

$$\{v_i \mid i \in 1..n\}.j \longrightarrow v_j \quad (\text{E-PROJTUPLE})$$

$$\frac{t_1 \longrightarrow t'_1}{t_1.i \longrightarrow t'_1.i} \quad (\text{E-PROJ})$$

$$\frac{t_j \longrightarrow t'_j}{\begin{array}{l} \{v_i \mid i \in 1..j-1, t_j, t_k \mid k \in j+1..n\} \\ \longrightarrow \{v_i \mid i \in 1..j-1, t'_j, t_k \mid k \in j+1..n\} \end{array}} \quad (\text{E-TUPLE})$$

## Typing rules for tuples

$$\frac{\text{for each } i \quad \Gamma \vdash t_i : T_i}{\Gamma \vdash \{t_i\}_{i \in 1..n} : \{T_i\}_{i \in 1..n}} \quad (\text{T-TUPLE})$$

$$\frac{\Gamma \vdash t_1 : \{T_i\}_{i \in 1..n}}{\Gamma \vdash t_1.j : T_j} \quad (\text{T-PROJ})$$