

# Foundations of Software Fall 2021

Week 10

## Different Kinds of Maps

What is missing?

$$\begin{array}{lcl} \textit{Term} & \rightarrow & \textit{Term} \ (\lambda x.t) \\ \textit{Type} & \rightarrow & \textit{Term} \ (\Lambda X.t) \end{array}$$

## Different Kinds of Maps

What is missing?

$$\begin{array}{lcl} \textit{Term} & \rightarrow & \textit{Term} \ (\lambda x.t) \\ \textit{Type} & \rightarrow & \textit{Term} \ (\Lambda X.t) \\ \textit{Type} & \rightarrow & \textit{Type} \ ??? \\ \textit{Term} & \rightarrow & \textit{Type} \ ??? \end{array}$$

Agenda today:

- ▶ Type operators
- ▶ Dependent types

## Type Operators and System $F_\omega$

## Type Operators

Example. Type operators in Scala:

```
type MkFun[T] = T => T
val f: MkFun[Int] = (x: Int) => x
```

## Type Operators

Example. Type operators in Scala:

```
type MkFun[T] = T => T
val f: MkFun[Int] = (x: Int) => x
```

*Type operators* are functions at type-level.

$\lambda X :: K. T$

## Type Operators

Example. Type operators in Scala:

```
type MkFun[T] = T => T
val f: MkFun[Int] = (x: Int) => x
```

*Type operators* are functions at type-level.

$\lambda X :: K. T$

Two Problems:

- ▶ Type checking of type operators
- ▶ Equivalence of types

## Kinding

Problem: avoid meaningless types, like `MkFun[Int, String]`.

## Kinding

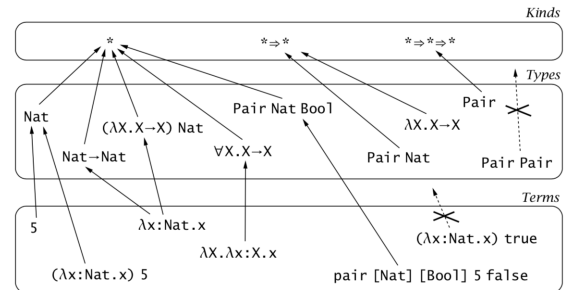
Problem: avoid meaningless types, like `MkFun[Int, String]`.

- `*` proper types, e.g. `Bool`, `Int → Int`
- `* ⇒ *` type operators: map proper type to proper type
- `* ⇒ * ⇒ *` two-argument operators
- `(* ⇒ *) ⇒ *` type operators: map type operators to proper types

## Kinding

Problem: avoid meaningless types, like `MkFun[Int, String]`.

- `*` proper types, e.g. `Bool`, `Int → Int`
- `* ⇒ *` type operators: map proper type to proper type
- `* ⇒ * ⇒ *` two-argument operators
- `(* ⇒ *) ⇒ *` type operators: map type operators to proper types



## Equivalence of Types

Problem: all the types below are equivalent

`Nat → Bool`   `Nat → Id Bool`   `Id Nat → Id Bool`  
`Id Nat → Bool`   `Id (Nat → Bool)`   `Id (Id (Id Nat → Bool))`

We need to introduce *definitional equivalence* relation on types, written  $S \equiv T$ . The most important rule is:

$$(\lambda X :: K. S) T \equiv [X \mapsto T] S \quad (\text{Q-APPABS})$$

And we need one typing rule:

$$\frac{\Gamma \vdash t : S \quad S \equiv T}{\Gamma \vdash t : T} \quad (\text{T-EQ})$$

## First-class Type Operators

Scala supports passing type operators as argument:

```
def makeInt[F[_]](f: () => F[Int]): F[Int] = f()
```

```
makeInt[List]() => List[Int](3)
```

```
makeInt[Option]() => None
```

First-class type operators supports *polymorphism* for type operators, which enables more patterns in type-safe functional programming.

## System $F_\omega$ — Syntax

Formalizing first-class type operators leads to System  $F_\omega$ :

$t ::= \dots$	<i>terms</i>
$\lambda X :: K. t$	<i>type abstraction</i>
$T ::=$	<i>types</i>
$X$	<i>type variable</i>
$T \rightarrow T$	<i>type of functions</i>
$\forall X :: K. T$	<i>universal type</i>
$\lambda X :: K. T$	<i>operator abstraction</i>
$T \ T$	<i>operator application</i>
$K ::=$	<i>kinds</i>
$*$	<i>kind of proper types</i>
$K \Rightarrow K$	<i>kind of operators</i>

## System $F_\omega$ — Semantics

$$\frac{t_1 \longrightarrow t'_1}{t_1 \ t_2 \longrightarrow t'_1 \ t_2} \quad (\text{E-APP1})$$

$$\frac{t_2 \longrightarrow t'_2}{t_1 \ t_2 \longrightarrow t_1 \ t'_2} \quad (\text{E-APP2})$$

$$(\lambda x :: T_1. t_1) \ v_2 \longrightarrow [x \mapsto v_2] t_1 \quad (\text{E-APPABS})$$

$$\frac{t \longrightarrow t'}{t \ [T] \longrightarrow t' \ [T]} \quad (\text{E-TAPP})$$

$$(\lambda X :: K. t_1) \ [T] \longrightarrow [X \mapsto T] t_1 \quad (\text{E-TAPPTABS})$$

## System $F_\omega$ — Kinding

$\frac{X :: K \in \Gamma}{\Gamma \vdash X :: K}$	(K-TVAR)
$\frac{\Gamma, X :: K_1 \vdash T_2 :: K_2}{\Gamma \vdash \lambda X :: K_1. T_2 :: K_1 \Rightarrow K_2}$	(K-ABS)
$\frac{\Gamma \vdash T_1 :: K_1 \Rightarrow K_2 \quad \Gamma \vdash T_2 :: K_1}{\Gamma \vdash T_1 \ T_2 :: K_2}$	(K-APP)
$\frac{\Gamma \vdash T_1 :: * \quad \Gamma \vdash T_2 :: *}{\Gamma \vdash T_1 \rightarrow T_2 :: *}$	(K-ARROW)
$\frac{\Gamma, X :: K_1 \vdash T_2 :: *}{\Gamma \vdash \forall X :: K_1. T_2 :: *}$	(K-ALL)

## System $F_\omega$ — Type Equivalence

$T \equiv T$	$\frac{T \equiv S}{S \equiv T}$	$\frac{S \equiv U \quad U \equiv T}{S \equiv T}$
$\frac{S_1 \equiv T_1 \quad S_2 \equiv T_2}{S_1 \rightarrow S_2 \equiv T_1 \rightarrow T_2}$	(Q-ARROW)	
$\frac{S_2 \equiv T_2}{\forall X :: K_1. S_2 \equiv \forall X :: K_1. T_2}$	(K-ALL)	
$\frac{S_2 \equiv T_2}{\lambda X :: K_1. S_2 \equiv \lambda X :: K_1. T_2}$	(Q-ABS)	
$\frac{S_1 \equiv T_1 \quad S_2 \equiv T_2}{S_1 \ S_2 \equiv T_1 \ T_2}$	(Q-APP)	
$(\lambda X :: K. T_1) \ T_2 \equiv [X \mapsto T_2] T_1$	(Q-APPABS)	

## System $F_\omega$ — Typing

$$\begin{array}{c}
 \frac{x : T \in \Gamma}{\Gamma \vdash x : T} \quad (\text{T-VAR}) \\
 \\
 \frac{\Gamma \vdash T_1 :: * \quad \Gamma, x : T_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda x : T_1. t_2 : T_1 \rightarrow T_2} \quad (\text{T-ABS}) \\
 \\
 \frac{\Gamma \vdash t_1 : S \rightarrow T \quad \Gamma \vdash t_2 : S}{\Gamma \vdash t_1 t_2 : T} \quad (\text{T-APP}) \\
 \\
 \frac{\Gamma, X :: K_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda X :: K_1. t_2 : \forall X :: K_1. T_2} \quad (\text{T-TABS}) \\
 \\
 \frac{\Gamma \vdash t : \forall X :: K. T_2 \quad \Gamma \vdash T :: K}{\Gamma \vdash t [T] : [X \mapsto T] T_2} \quad (\text{T-TAPP}) \\
 \\
 \frac{\Gamma \vdash t : S \quad S \equiv T \quad \Gamma \vdash T :: *}{\Gamma \vdash t : T} \quad (\text{T-EQ})
 \end{array}$$

## Example

```

type PairRep[Pair :: * ⇒ * ⇒ *] = {
  pair : ∀X.∀Y.X → Y → (Pair X Y),
  fst  : ∀X.∀Y.(Pair X Y) → X,
  snd  : ∀X.∀Y.(Pair X Y) → Y
}

def swap[Pair :: * ⇒ * ⇒ *, X :: *, Y :: *]
  (rep : PairRep Pair)
  (pair : Pair X Y) : Pair Y X
=
  let x = rep.fst [X] [Y] pair in
  let y = rep.snd [X] [Y] pair in
  rep.pair [Y] [X] y x

```

The method *swap* works for any representation of pairs.

## Properties

*Theorem [Preservation]*: if  $\Gamma \vdash t : T$  and  $t \longrightarrow t'$ , then  $\Gamma \vdash t' : T$ .

*Theorem [Progress]*: if  $\vdash t : T$ , then either  $t$  is a value or there exists  $t'$  with  $t \longrightarrow t'$ .

## Dependent Types

## Why Does It Matter?

Example 1. Track length of vectors in types:

```
NVec  :: Nat → *  
first  : (n:Nat) → NVec (n + 1) → Nat
```

$(x:S) \rightarrow T$  is called **dependent function type**. It is impossible to pass a vector of length 0 to the function *first*.

## Why Does It Matter?

Example 1. Track length of vectors in types:

```
NVec  :: Nat → *  
first  : (n:Nat) → NVec (n + 1) → Nat
```

$(x:S) \rightarrow T$  is called **dependent function type**. It is impossible to pass a vector of length 0 to the function *first*.

Example 2. Safe formatting for *sprintf*:

```
sprintf      : (f:Format) → Data(f) → String  
  
Data([])      = Unit  
Data("%d" :: cs) = Nat * Data(cs)  
Data("%s" :: cs) = String * Data(cs)  
Data(c :: cs)  = Data(cs)
```

## Dependent Function Type (a.k.a. $\Pi$ Types)

A dependent function type is inhabited by a *dependent function*:

```
λx:S.t  : (x:S) → T
```

## Dependent Function Type (a.k.a. $\Pi$ Types)

A dependent function type is inhabited by a *dependent function*:

```
λx:S.t  : (x:S) → T
```

If  $T$  does not depend on  $x$ , it degenerates to function types:

```
(x:S) → T = S → T    where x does not appear free in T
```

# The Calculus of Constructions

## The Calculus of Constructions: Syntax

$t ::=$	<i>terms</i>
$s$	<i>sort</i>
$x$	<i>variable</i>
$\lambda x:t. t$	<i>abstraction</i>
$t \ t$	<i>application</i>
$(x:t) \rightarrow t$	<i>dependent type</i>
$s ::=$	<i>sorts</i>
$*$	<i>sort of proper types</i>
$\square$	<i>sort of kinds</i>
$\Gamma ::=$	<i>contexts</i>
$\emptyset$	<i>empty context</i>
$\Gamma, x:T$	<i>term variable binding</i>

The semantics is the usual  $\beta$ -reduction.

## The Calculus of Constructions: Typing

$\vdash * : \square$ (T-AXIOM)	$\frac{x:T \in \Gamma}{\Gamma \vdash x : T}$ (T-VAR)
$\frac{\Gamma \vdash S : s_1 \quad \Gamma, x:S \vdash t : T}{\Gamma \vdash \lambda x:S. t : (x:S) \rightarrow T}$	(T-ABS)
$\frac{\Gamma \vdash t_1 : (x:S) \rightarrow T \quad \Gamma \vdash t_2 : S}{\Gamma \vdash t_1 \ t_2 : [x \mapsto t_2] T}$	(T-APP)
$\frac{\Gamma \vdash S : s_1 \quad \Gamma, x:S \vdash T : s_2}{\Gamma \vdash (x:S) \rightarrow T : s_2}$	(T-PI)
$\frac{\Gamma \vdash t : T \quad T \equiv T' \quad \Gamma \vdash T' : s}{\Gamma \vdash t : T'}$	(T-CONV)

The equivalence relation  $T \equiv T'$  is based on  $\beta$ -reduction.

## Four Kinds of Lambdas

Example	Type
$\lambda x:\mathbb{N}. x + 1$	$\mathbb{N} \rightarrow \mathbb{N}$
$\lambda f:\mathbb{N} \rightarrow \mathbb{N}. f \ x$	$(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$

## Four Kinds of Lambdas

Example	Type
$\lambda x:\mathbb{N}.x + 1$	$\mathbb{N} \rightarrow \mathbb{N}$
$\lambda f:\mathbb{N} \rightarrow \mathbb{N}.f\ x$	$(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$
$\lambda X:*. \lambda x:X. x$	$(X:*) \rightarrow X \rightarrow X$
$\lambda F:*\rightarrow *. \lambda x:F\ \mathbb{N}. x$	$(F:*\rightarrow *) \rightarrow (F\ \mathbb{N}) \rightarrow (F\ \mathbb{N})$

## Four Kinds of Lambdas

Example	Type
$\lambda x:\mathbb{N}.x + 1$	$\mathbb{N} \rightarrow \mathbb{N}$
$\lambda f:\mathbb{N} \rightarrow \mathbb{N}.f\ x$	$(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$
$\lambda X:*. \lambda x:X. x$	$(X:*) \rightarrow X \rightarrow X$
$\lambda F:*\rightarrow *. \lambda x:F\ \mathbb{N}. x$	$(F:*\rightarrow *) \rightarrow (F\ \mathbb{N}) \rightarrow (F\ \mathbb{N})$
$\lambda X:*.X$	$* \rightarrow *$
$\lambda F:*\rightarrow *.F\ \mathbb{N}$	$(*\rightarrow *) \rightarrow *$

## Four Kinds of Lambdas

Example	Type
$\lambda x:\mathbb{N}.x + 1$	$\mathbb{N} \rightarrow \mathbb{N}$
$\lambda f:\mathbb{N} \rightarrow \mathbb{N}.f\ x$	$(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$
$\lambda X:*. \lambda x:X. x$	$(X:*) \rightarrow X \rightarrow X$
$\lambda F:*\rightarrow *. \lambda x:F\ \mathbb{N}. x$	$(F:*\rightarrow *) \rightarrow (F\ \mathbb{N}) \rightarrow (F\ \mathbb{N})$
$\lambda X:*.X$	$* \rightarrow *$
$\lambda F:*\rightarrow *.F\ \mathbb{N}$	$(*\rightarrow *) \rightarrow *$
$\lambda n:\mathbb{N}.NVec\ n$	$\mathbb{N} \rightarrow *$
$\lambda f:\mathbb{N} \rightarrow \mathbb{N}.NVec\ (f\ 6)$	$(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow *$

## Strong Normalization

Given the following  $\beta$ -reduction rules

$$\frac{t_1 \longrightarrow t'_1}{\lambda x: T_1. t_1 \longrightarrow \lambda x: T_1. t'_1} \quad (\beta\text{-ABS})$$

$$\frac{t_1 \longrightarrow t'_1}{t_1\ t_2 \longrightarrow t'_1\ t_2} \quad (\beta\text{-APP1})$$

$$\frac{t_2 \longrightarrow t'_2}{t_1\ t_2 \longrightarrow t_1\ t'_2} \quad (\beta\text{-APP2})$$

$$(\lambda x: T_1. t_1) t_2 \longrightarrow [x \mapsto t_2] t_1 \quad (\beta\text{-APPAbs})$$

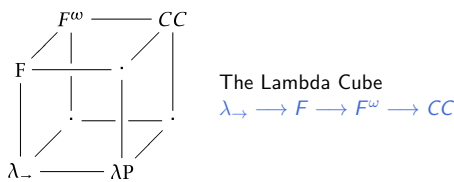
*Theorem [Strong Normalization]:* if  $\Gamma \vdash t : T$ , then there is no infinite sequence of terms  $t_i$  such that  $t = t_1$  and  $t_i \longrightarrow t_{i+1}$ .



## Pure Type Systems

$$\frac{\Gamma \vdash S : s_i \quad \Gamma, x:S \vdash T : s_j}{\Gamma \vdash (x:S) \rightarrow T : s_j} \quad (\text{T-Pr})$$

System	$(s_i, s_j)$
$\lambda_{\rightarrow}$	$\{ (*, *) \}$
$\lambda P$	$\{ (*, *), (*, \square) \}$
$F$	$\{ (*, *), (\square, *) \}$
$F^\omega$	$\{ (*, *), (\square, *) \}$
$CC$	$\{ (*, *), (*, \square), (\square, *) \}$



## Dependent Types in Coq

### Proof Assistants

Dependent type theories are at the foundation of proof assistants, like Coq, Agda, etc.

By *Curry-Howard Correspondence*

- ▶ proofs  $\longleftrightarrow$  programs
- ▶ propositions  $\longleftrightarrow$  types

### Proof Assistants

Dependent type theories are at the foundation of proof assistants, like Coq, Agda, etc.

By *Curry-Howard Correspondence*

- ▶ proofs  $\longleftrightarrow$  programs
- ▶ propositions  $\longleftrightarrow$  types

Two impactful projects based on Coq:

- ▶ CompCert: certified C compiler
- ▶ Mechanized proof of 4-color theorem

## Type Universes in Coq

The rule  $\Gamma \vdash \text{Type} : \text{Type}$  is unsound (Girard's paradox).

$$\Gamma \vdash \text{Prop} : \text{Type}_1$$

$$\Gamma \vdash \text{Set} : \text{Type}_1$$

$$\Gamma \vdash \text{Type}_i : \text{Type}_{i+1}$$

$$\frac{\Gamma, x:A \vdash B : \text{Prop} \quad \Gamma \vdash A : s}{\Gamma \vdash (x : A) \rightarrow B : \text{Prop}}$$

$$\frac{\Gamma, x:A \vdash B : \text{Set} \quad \Gamma \vdash A : s \quad s \in \{\text{Prop}, \text{Set}\}}{\Gamma \vdash (x : A) \rightarrow B : \text{Set}}$$

$$\frac{\Gamma, x:A \vdash B : \text{Type}_i \quad \Gamma \vdash A : \text{Type}_i}{\Gamma \vdash (x : A) \rightarrow B : \text{Type}_i}$$

## Coq 101 - inductive definitions and recursion

```
1 Inductive nat : Type :=  
2   | 0  
3   | S (n : nat).
```

## Coq 101 - inductive definitions and recursion

```
1 Inductive nat : Type :=  
2   | 0  
3   | S (n : nat).
```

```
1 Fixpoint double (n : nat) : nat :=  
2   match n with  
3   | 0 => 0  
4   | S n' => S (S (double n'))  
5   end.
```

Recursion has to be **structural**.

## Coq 101 - inductive definitions and recursion

```
1 Inductive nat : Type :=  
2   | 0  
3   | S (n : nat).
```

```
1 Fixpoint double (n : nat) : nat :=  
2   match n with  
3   | 0 => 0  
4   | S n' => S (S (double n'))  
5   end.
```

Recursion has to be **structural**.

```
1 Inductive even : nat -> Prop :=  
2   | even0 : even 0  
3   | evenS : forall x:nat, even x -> even (S (S x)).
```

## Coq 101 - proofs

```

1 Definition even_prop := forall x:nat, even (double x).
2
3 Fixpoint even_proof(x: nat): even (double x) :=
4   match x with
5   | 0      => even0
6   | S n'   => evenS (double n') (even_proof n')
7   end.
8
9 Check even_proof : even_prop.

```

## Coq 101 - proofs

```

1 Definition even_prop := forall x:nat, even (double x).
2
3 Fixpoint even_proof(x: nat): even (double x) :=
4   match x with
5   | 0      => even0
6   | S n'   => evenS (double n') (even_proof n')
7   end.
8
9 Check even_proof : even_prop.

```

The 2nd branch has the type  $\text{even } S(S(\text{double } n'))$ , and Coq knows by normalizing the types:

$$\text{even } S(S(\text{double } n')) \equiv_{\beta} \text{even } (\text{double } (S n'))$$

## Recap: Curry-Howard Correspondence

Propositions as types in the context of intuitionistic logic.

Proposition	Term & Type
$A \wedge B$	$t : (A, B)$
$A \vee B$	$t : A + B$
$A \rightarrow B$	$t : A \rightarrow B$
$\perp$	$t : \text{False}$
$\neg A$	$t : A \rightarrow \text{False}$
$\forall x:A. B$	$t : (x : A) \rightarrow B$
$\exists x:A. B$	$t : (x:A, B)$

## Curry-Howard correspondence in Coq

```

1 Inductive and (A B:Prop) : Prop :=
2   conj : A -> B -> A /\ B
3   where "A /\ B" := (and A B) : type_scope.

```

## Curry-Howard correspondence in Coq

```
1 Inductive and (A B:Prop) : Prop :=
2   conj : A -> B -> A /\ B
3 where "A /\ B" := (and A B) : type_scope.

1 Inductive or (A B:Prop) : Prop :=
2   | or_introl : A -> A \/ B
3   | or_intror : B -> A \/ B
4 where "A \/ B" := (or A B) : type_scope.
```

## Curry-Howard correspondence in Coq

```
1 Inductive and (A B:Prop) : Prop :=
2   conj : A -> B -> A /\ B
3 where "A /\ B" := (and A B) : type_scope.

1 Inductive or (A B:Prop) : Prop :=
2   | or_introl : A -> A \/ B
3   | or_intror : B -> A \/ B
4 where "A \/ B" := (or A B) : type_scope.

1 Inductive False : Prop :=.
```

## Curry-Howard correspondence in Coq

```
1 Inductive and (A B:Prop) : Prop :=
2   conj : A -> B -> A /\ B
3 where "A /\ B" := (and A B) : type_scope.

1 Inductive or (A B:Prop) : Prop :=
2   | or_introl : A -> A \/ B
3   | or_intror : B -> A \/ B
4 where "A \/ B" := (or A B) : type_scope.

1 Inductive False : Prop :=.

1 Definition not (A:Prop) := A -> False.
2 Notation "~ x" := (not x) : type_scope.
```

## Curry-Howard correspondence in Coq - continued

```
1 Notation "A -> B" := (forall (_ : A), B) : type_scope.
2 Definition iff (A B:Prop) := (A -> B) /\ (B -> A).
3 Notation "A <-> B" := (iff A B) : type_scope.
```

## Curry-Howard correspondence in Coq - continued

```
1 Notation "A -> B" := (forall (_ : A), B) : type_scope.
2 Definition iff (A B:Prop) := (A -> B) /\ (B -> A).
3 Notation "A <-> B" := (iff A B) : type_scope.

1 Inductive ex (A:Type) (P:A -> Prop) : Prop :=
2   ex_intro : forall x:A, P x -> ex (A:=A) P.
3
4 Notation "'exists' x .. y , p" :=
5   (ex (fun x => .. (ex (fun y => p)) ..)) : type_scope.
```

## Curry-Howard correspondence in Coq - continued

```
1 Notation "A -> B" := (forall (_ : A), B) : type_scope.
2 Definition iff (A B:Prop) := (A -> B) /\ (B -> A).
3 Notation "A <-> B" := (iff A B) : type_scope.

1 Inductive ex (A:Type) (P:A -> Prop) : Prop :=
2   ex_intro : forall x:A, P x -> ex (A:=A) P.
3
4 Notation "'exists' x .. y , p" :=
5   (ex (fun x => .. (ex (fun y => p)) ..)) : type_scope.

1 Inductive eq (A:Type) (x:A) : A -> Prop :=
2   eq_refl : x = x :>A
3
4 Notation "x = y" := (eq x y) : type_scope.
```

## The equivalence between LEM and DNE

In *intuitionistic logics*, the *law of excluded middle* (LEM) and the *law of double negation* (DNE) are not provable.

- ▶ LEM:  $\forall P. P \vee \neg P$
- ▶ DNE:  $\forall P. \neg\neg P \rightarrow P$

By curry-howard correspondence, there are no terms that inhabit the types above.

## The equivalence between LEM and DNE

In *intuitionistic logics*, the *law of excluded middle* (LEM) and the *law of double negation* (DNE) are not provable.

- ▶ LEM:  $\forall P. P \vee \neg P$
- ▶ DNE:  $\forall P. \neg\neg P \rightarrow P$

By curry-howard correspondence, there are no terms that inhabit the types above.

However,  $\forall P. P \rightarrow \neg\neg P$  can be proved.

## The equivalence between LEM and DNE

In **intuitionistic logics**, the *law of excluded middle* (LEM) and the *law of double negation* (DNE) are not provable.

- ▶ LEM:  $\forall P. P \vee \neg P$
- ▶ DNE:  $\forall P. \neg\neg P \rightarrow P$

By curry-howard correspondence, there are no terms that inhabit the types above.

However,  $\forall P. P \rightarrow \neg\neg P$  can be proved. **How?**

## The equivalence between LEM and DNE

In **intuitionistic logics**, the *law of excluded middle* (LEM) and the *law of double negation* (DNE) are not provable.

- ▶ LEM:  $\forall P. P \vee \neg P$
- ▶ DNE:  $\forall P. \neg\neg P \rightarrow P$

By curry-howard correspondence, there are no terms that inhabit the types above.

However,  $\forall P. P \rightarrow \neg\neg P$  can be proved. **How?**

We will prove that LEM is equivalent to DNE:

- 1 **Definition** LEM: Prop := forall P: Prop, P  $\vee$   $\neg$ P.
- 2 **Definition** DNE: Prop := forall P: Prop,  $\neg\neg$ P  $\rightarrow$  P.
- 3 **Definition** LEM\_DNE\_EQ: Prop := LEM  $\leftrightarrow$  DNE.

## LEM $\rightarrow$ DNE

```
1 Definition LEM_To_DNE :=
2   fun (lem: forall P : Prop, P  $\vee$   $\neg$  P) (Q:Prop) (q:  $\neg\neg$ Q)
3     =>
4     match lem Q with
5     | or_introl l =>
6       l
7     | or_intror r =>
8       match (q r) with end
9     end.
10
11 Check LEM_To_DNE : LEM  $\rightarrow$  DNE.
```

## DNE $\rightarrow$ LEM

```
1 Definition DNE_To_LEM :=
2   fun (dne: forall P : Prop,  $\neg\neg$ P  $\rightarrow$  P) (Q:Prop) =>
3     (dne (Q  $\vee$   $\neg$  Q))
4     (fun H:  $\neg$ (Q  $\vee$   $\neg$ Q) =>
5       let nq := (fun q: Q => H (or_introl q))
6       in H (or_intror nq)
7     ).
8
9 Check DNE_To_LEM : DNE  $\rightarrow$  LEM.
10
11 Definition proof := conj LEM_To_DNE DNE_To_LEM.
12 Check proof : LEM  $\leftrightarrow$  DNE.
```

## Dependent Types in Programming Languages

Despite the huge success in proof assistants, its adoption in programming languages is limited.

- ▶ Scala supports *path-dependent types* and *literal types*.
- ▶ Dependent Haskell is proposed by researchers.

## Dependent Types in Programming Languages

Despite the huge success in proof assistants, its adoption in programming languages is limited.

- ▶ Scala supports *path-dependent types* and *literal types*.
- ▶ Dependent Haskell is proposed by researchers.

**Challenge:** the decidability of type checking.

## Problem with Type Checking

Value constructors:

```

$$\begin{aligned} \text{NVec} &: \mathbb{N} \rightarrow * \\ \text{nil} &: \text{NVec } 0 \\ \text{cons} &: \mathbb{N} \rightarrow (n:\mathbb{N}) \rightarrow \text{NVec } n \rightarrow \text{NVec } n + 1 \end{aligned}$$

```

Appending vectors:

```

$$\begin{aligned} \text{append} &: (m:\mathbb{N}) \rightarrow (n:\mathbb{N}) \rightarrow \text{NVec } m \rightarrow \text{NVec } n \rightarrow \text{NVec } (n + m) \\ \text{append} &= \lambda m:\mathbb{N}. \lambda n:\mathbb{N}. \lambda l:\text{NVec } m. \lambda t:\text{NVec } n. \\ &\quad \text{match } l \text{ with} \\ &\quad | \text{nil} \Rightarrow t \\ &\quad | \text{cons } x \ r \ y \Rightarrow \text{cons } x \ (r + n) \ (\text{append } r \ n \ y \ t) \end{aligned}$$

```

**Question:** How does the type checker know  $S \ (r + n) = n + (S \ r)$ ?