Foundations of Software
Fall 2021

Week 10

Different Kinds of Maps

What is missing?

$$\begin{array}{lcll} \mathit{Term} & \rightarrow & \mathit{Term} & (\lambda x.t) \\ \mathit{Type} & \rightarrow & \mathit{Term} & (\Lambda X.t) \end{array}$$

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Agenda today:

- Type operators
- Dependent types

Type Operators and System F_ω

Type Operators

Example. Type operators in Scala:

```
type MkFun[T] = T => T
val f: MkFun[Int] = (x: Int) => x
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Two Problems:

- Type checking of type operators
- Equivalence of types

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$*$	proper types, e.g. $Bool$, $Int \to Int$
$* \Rightarrow *$	type operators: map proper type to proper type
$* \Rightarrow * \Rightarrow *$	two-argument operators
$(* \Rightarrow *) \Rightarrow *$	type operators: map type operators to proper types

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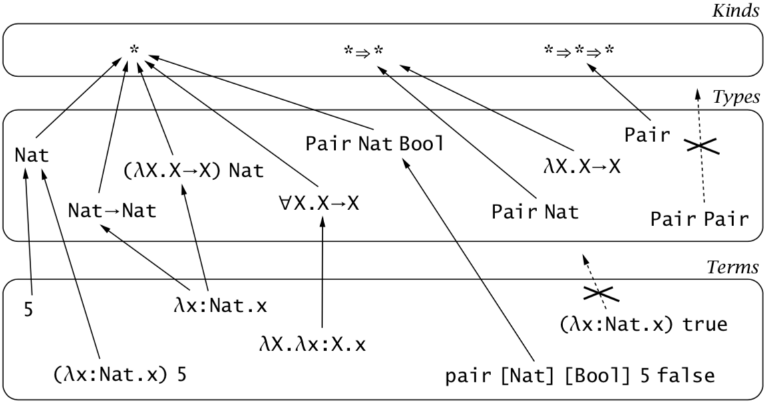

Equivalence of Types

Problem: all the types below are equivalent

$$\begin{array}{lll} Nat \to Bool & Nat \to Id\ Bool & Id\ Nat \to Id\ Bool \\ Id\ Nat \to Bool & Id\ (Nat \to Bool) & Id(Id(Id\ Nat \to Bool) \end{array}$$

We need to introduce *definitional equivalence* relation on types, written $S \equiv T$. The most important rule is:

$$(\lambda X :: K.S)\ T \equiv [X \mapsto T]S \qquad \text{(Q-AppAbs)}$$

And we need one typing rule:

$$\frac{\Gamma \vdash t : S \qquad S \equiv T}{\Gamma \vdash t : T} \qquad \text{(T-Eq)}$$

First-class Type Operators

Scala supports passing type operators as argument:

```
def makeInt[F[_]](f: () => F[Int]): F[Int] = f()

makeInt[List](() => List[Int](3))
makeInt[Option](() => None)
```

First-class type operators supports *polymorphism* for type operators, which enables more patterns in type-safe functional programming.

System F_ω — Syntax

Formalizing first-class type operators leads to *System* F_ω:

$\mathtt{t}$	::=	...	*terms*
		$\lambda X :: K.t$	*type abstraction*
$\mathtt{T}$	::=		*types*
		$\mathtt{X}$	*type variable*
		$T \to T$	*type of functions*
		$\forall X :: K.T$	*universal type*
		$\lambda X :: K.T$	*operator abstraction*
		$T\ T$	*operator application*
$\mathtt{K}$	::=		*kinds*
		$*$	*kind of proper types*
		$K \Rightarrow K$	*kind of operators*

System F_ω — Semantics

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2} \quad \text{(E-App1)}$$

$$\frac{t_2 \longrightarrow t_2'}{t_1\ t_2 \longrightarrow t_1\ t_2'} \quad \text{(E-App2)}$$

$$(\lambda x{:}T_1.t_1)\ v_2 \longrightarrow [x \mapsto v_2]t_1 \quad \text{(E-AppAbs)}$$

$$\frac{t \longrightarrow t'}{t\ [T] \longrightarrow t'\ [T]} \quad \text{(E-TApp)}$$

$$(\lambda X{::}K.t_1)\ [T] \longrightarrow [X \mapsto T]t_1 \quad \text{(E-TAppTAbs)}$$

System F_ω — Kinding

$$\frac{X :: K \in \Gamma}{\Gamma \vdash X :: K} \qquad \text{(K-TVar)}$$

$$\frac{\Gamma, X{::}K_1 \vdash T_2 : K_2}{\Gamma \vdash \lambda X{::}K_1.T_2 :: K_1 \Rightarrow K_2} \qquad \text{(K-Abs)}$$

$$\frac{\Gamma \vdash T_1 : K_1 \Rightarrow K_2 \qquad \Gamma \vdash T_2 : K_1}{\Gamma \vdash T_1\ T_2 :: K_2} \qquad \text{(K-App)}$$

$$\frac{\Gamma \vdash T_1 : * \qquad \Gamma \vdash T_2 : *}{\Gamma \vdash T_1 \to T_2 :: *} \qquad \text{(K-Arrow)}$$

$$\frac{\Gamma, X{::}K_1 \vdash T_2 :: *}{\Gamma \vdash \forall X{::}K_1.T_2 :: *} \qquad \text{(K-All)}$$

System F_ω — Type Equivalence

$$T \equiv T \qquad \frac{T \equiv S}{S \equiv T} \qquad \frac{S \equiv U \qquad U \equiv T}{S \equiv T}$$

$$\frac{S_1 \equiv T_1 \qquad S_2 \equiv T_2}{S_1 \to S_2 \equiv T_1 \to T_2} \qquad \text{(Q-Arrow)}$$

$$\frac{S_2 \equiv T_2}{\forall X{::}K_1.S_2 \equiv \forall X{::}K_1.T_2} \qquad \text{(K-All)}$$

$$\frac{S_2 \equiv T_2}{\lambda X{::}K_1.S_2 \equiv \lambda X{::}K_1.T_2} \qquad \text{(Q-Abs)}$$

$$\frac{S_1 \equiv T_1 \qquad S_2 \equiv T_2}{S_1\ S_2 \equiv T_1\ T_2} \qquad \text{(Q-App)}$$

$$(\lambda X{::}K.T_1)\ T_2 \equiv [X \mapsto T_2]T_1 \qquad \text{(Q-AppAbs)}$$

System F_ω — Typing

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T} \qquad \text{(T-Var)}$$

$$\frac{\Gamma \vdash T_1 :: * \qquad \Gamma, x{:}T_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda x{:}T_1.t_2 : T_1 \to T_2} \qquad \text{(T-Abs)}$$

$$\frac{\Gamma \vdash t_1 : S \to T \qquad \Gamma \vdash t_2 : S}{\Gamma \vdash t_1\ t_2 : T} \qquad \text{(T-App)}$$

$$\frac{\Gamma, X{::}K_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda X{::}K_1.t_2 : \forall X{::}K_1.T_2} \qquad \text{(T-TAbs)}$$

$$\frac{\Gamma \vdash t : \forall X{::}K.T_2 \qquad \Gamma \vdash T :: K}{\Gamma \vdash t\ [T] : [X \mapsto T]T_2} \qquad \text{(T-TApp)}$$

$$\frac{\Gamma \vdash t : S \qquad S \equiv T \qquad \Gamma \vdash T :: *}{\Gamma \vdash t : T} \qquad \text{(T-Eq)}$$

Example

$$
\begin{aligned}
&\mathit{type}\ \mathit{PairRep}[\mathit{Pair} :: * \Rightarrow * \Rightarrow *] = \{\\
&\qquad \mathit{pair} : \forall X.\forall Y.X \to Y \to (\mathit{Pair}\ X\ Y),\\
&\qquad \mathit{fst} : \forall X.\forall Y.(\mathit{Pair}\ X\ Y) \to X,\\
&\qquad \mathit{snd} : \forall X.\forall Y.(\mathit{Pair}\ X\ Y) \to Y\\
&\}
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{def}\ \mathit{swap}[\mathit{Pair} :: * \Rightarrow * \Rightarrow *, X :: *, Y :: *]\\
&\qquad (\mathit{rep} : \mathit{PairRep}\ \mathit{Pair})\\
&\qquad (\mathit{pair} : \mathit{Pair}\ X\ Y) : \mathit{Pair}\ Y\ X\\
&=\\
&\qquad \mathit{let}\ x = \mathit{rep.fst}\ [X]\ [Y]\ \mathit{pair}\ \mathit{in}\\
&\qquad \mathit{let}\ y = \mathit{rep.snd}\ [X]\ [Y]\ \mathit{pair}\ \mathit{in}\\
&\qquad \mathit{rep.pair}\ [Y]\ [X]\ y\ x
\end{aligned}
$$

The method *swap* works for any representation of pairs.

Properties

Theorem [*Preservation*]: if $\Gamma \vdash t : T$ and $t \longrightarrow t'$, then $\Gamma \vdash t' : T$.

Theorem [*Progress*]: if $\vdash t : T$, then either t is a value or there exists t' with $t \longrightarrow t'$.

Dependent Types

Why Does It Matter?

Example 1. Track length of vectors in types:

$$
\begin{array}{lcl}
\mathit{NVec} & :: & \mathit{Nat} \to * \\
\mathit{first} & : & (n{:}\mathit{Nat}) \to \mathit{NVec}\ (n+1) \to \mathit{Nat}
\end{array}
$$

$(x{:}S) \to T$ is called dependent function type. It is impossible to pass a vector of length 0 to the function first.

Why Does It Matter?

Example 1. Track length of vectors in types:

$$\begin{array}{lll} NVec & :: & Nat \rightarrow * \\ first & : & (n{:}Nat) \rightarrow NVec\ (n+1) \rightarrow Nat \end{array}$$

$(x{:}S) \rightarrow T$ is called dependent function type. It is impossible to pass a vector of length 0 to the function $first$.

Example 2. Safe formatting for $sprintf$:

$$\begin{array}{lll} sprintf & : & (f{:}Format) \rightarrow Data(f) \rightarrow String \\ & & \\ Data([]) & = & Unit \\ Data(''\%d'' :: cs) & = & Nat * Data(cs) \\ Data(''\%s'' :: cs) & = & String * Data(cs) \\ Data(c :: cs) & = & Data(cs) \end{array}$$

Dependent Function Type (a.k.a. Π Types)

A dependent function type is inhabited by *a dependent function*:

$$\lambda x{:}S.t \quad : \quad (x{:}S) \rightarrow T$$

Dependent Function Type (a.k.a. Π Types)

A dependent function type is inhabited by *a dependent function*:

$$\lambda x{:}S.t \quad : \quad (x{:}S) \to T$$

If T does not depend on x, it degenerates to function types:

$$(x{:}S) \to T = S \to T \qquad \textit{where } x \textit{ does not appear free in } T$$

The Calculus of Constructions

The Calculus of Constructions: Syntax

$\texttt{t}$	$::=$		*terms*
		$\texttt{s}$	*sort*
		$\texttt{x}$	*variable*
		$\lambda\texttt{x:t.t}$	*abstraction*
		$\texttt{t t}$	*application*
		$(x{:}t) \to t$	*dependent type*
$\texttt{s}$	$::=$		*sorts*
		$*$	*sort of proper types*
		$\square$	*sort of kinds*
Γ	$::=$		*contexts*
		$\emptyset$	*empty context*
		$\Gamma, x{:}T$	*term variable binding*

The semantics is the usual β-reduction.

The Calculus of Constructions: Typing

$$\vdash * : \Box \;(\textsc{T-Axiom}) \qquad\qquad \frac{x{:}T \in \Gamma}{\Gamma \vdash x : T} \;(\textsc{T-Var})$$

$$\frac{\Gamma \vdash S : s_1 \qquad \Gamma, x{:}S \vdash t : T}{\Gamma \vdash \lambda x{:}S.t : (x{:}S) \to T} \;(\textsc{T-Abs})$$

$$\frac{\Gamma \vdash t_1 : (x{:}S) \to T \qquad \Gamma \vdash t_2 : S}{\Gamma \vdash t_1\ t_2 : [x \mapsto t_2]T} \;(\textsc{T-App})$$

$$\frac{\Gamma \vdash S : s_1 \qquad \Gamma, x{:}S \vdash T : s_2}{\Gamma \vdash (x{:}S) \to T : s_2} \;(\textsc{T-Pi})$$

$$\frac{\Gamma \vdash t : T \qquad T \equiv T' \qquad \Gamma \vdash T' : s}{\Gamma \vdash t : T'} \;(\textsc{T-Conv})$$

The equivalence relation $T \equiv T'$ is based on β-reduction.

Four Kinds of Lambdas

Example	Type
$\lambda x{:}\mathbb{N}.x + 1$	$\mathbb{N} \to \mathbb{N}$
$\lambda f{:}\mathbb{N} \to \mathbb{N}.f\ x$	$(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$

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$\lambda f{:}\mathbb{N} \to \mathbb{N}.f\ x$	$(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$
$\lambda X{:}{*}.\lambda x{:}X.\,x$	$(X{:}{*}) \to X \to X$
$\lambda F{:}{*} \to {*}.\lambda x{:}F\ \mathbb{N}.\,x$	$(F{:}{*} \to {*}) \to (F\ \mathbb{N}) \to (F\ \mathbb{N})$

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$\lambda F{:}{*} \to {*}.\lambda x{:}F\ \mathbb{N}.\,x$	$(F{:}{*} \to {*}) \to (F\ \mathbb{N}) \to (F\ \mathbb{N})$
$\lambda X{:}{*}.X$	${*} \to {*}$
$\lambda F{:}{*} \to {*}.F\ \mathbb{N}$	$({*} \to {*}) \to {*}$

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$\lambda x{:}\mathbb{N}.x + 1$	$\mathbb{N} \to \mathbb{N}$
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$\lambda X{:}{*}.X$	${*} \to {*}$
$\lambda F{:}{*} \to {*}.F\ \mathbb{N}$	$({*} \to {*}) \to {*}$
$\lambda n{:}\mathbb{N}.NVec\ n$	$\mathbb{N} \to {*}$
$\lambda f{:}\mathbb{N} \to \mathbb{N}.NVec\ (f\ 6)$	$(\mathbb{N} \to \mathbb{N}) \to {*}$

Strong Normalization

Given the following β-reduction rules

$$\frac{t_1 \longrightarrow t_1'}{\lambda x{:}T_1.t_1 \longrightarrow \lambda x{:}T_1.t_1'} \qquad (\beta\text{-}\textsc{Abs})$$

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2} \qquad (\beta\text{-}\textsc{App1})$$

$$\frac{t_2 \longrightarrow t_2'}{t_1\ t_2 \longrightarrow t_1\ t_2'} \qquad (\beta\text{-}\textsc{App2})$$

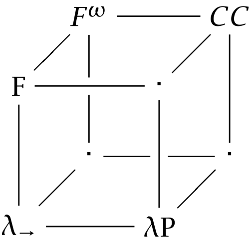
$$(\lambda x{:}T_1.t_1)t_2 \longrightarrow [x \mapsto t_2]t_1 \qquad (\beta\text{-}\textsc{AppAbs})$$

Theorem [*Strong Normalization*]: if $\Gamma \vdash t : T$, then there is no infinite sequence of terms t_i such that $t = t_1$ and $t_i \longrightarrow t_{i+1}$.

Pure Type Systems

$$\frac{\Gamma \vdash S : s_i \qquad \Gamma, x{:}S \vdash T : s_j}{\Gamma \vdash (x{:}S) \to T : s_j} \tag{T-Pi}$$

System	(s_i, s_j)			
$\lambda_{\to}$	$\{ (*, *)$			$\}$
λP	$\{ (*, *),$	$(*, \Box)$		$\}$
F	$\{ (*, *),$		$(\Box, *)$	$\}$
F^{ω}	$\{ (*, *),$		$(\Box, *)$	$(\Box, \Box) \}$
CC	$\{ (*, *),$	$(*, \Box)$	$(\Box, *)$	$(\Box, \Box) \}$



The Lambda Cube

$\lambda_{\to} \longrightarrow F \longrightarrow F^{\omega} \longrightarrow CC$

Dependent Types in Coq

Proof Assistants

Dependent type theories are at the foundation of proof assistants, like Coq, Agda, etc.

By *Curry-Howard Correspondence*

- proofs $\longleftrightarrow$ programs
- propositions $\longleftrightarrow$ types

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Two impactful projects based on Coq:

- CompCert: certified C compiler
- Mechanized proof of 4-color theorem

Type Universes in Coq

The rule $\Gamma \vdash \mathit{Type} : \mathit{Type}$ is unsound (Girard's paradox).

$$\Gamma \vdash \mathit{Prop} : \mathit{Type}_1$$

$$\Gamma \vdash \mathit{Set} : \mathit{Type}_1$$

$$\Gamma \vdash \mathit{Type}_i : \mathit{Type}_{i+1}$$

$$\frac{\Gamma, x{:}A \vdash B : \mathit{Prop} \qquad \Gamma \vdash A : s}{\Gamma \vdash (x : A) \to B : \mathit{Prop}}$$

$$\frac{\Gamma, x{:}A \vdash B : \mathit{Set} \qquad \Gamma \vdash A : s \qquad s \in \{\mathit{Prop}, \mathit{Set}\}}{\Gamma \vdash (x : A) \to B : \mathit{Set}}$$

$$\frac{\Gamma, x{:}A \vdash B : \mathit{Type}_i \qquad \Gamma \vdash A : \mathit{Type}_i}{\Gamma \vdash (x : A) \to B : \mathit{Type}_i}$$

Coq 101 - inductive definitions and recursion

```
Inductive nat : Type :=
  | O
  | S (n : nat).
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```
Fixpoint double (n : nat) : nat :=
  match n with
    | O => O
    | S n' => S (S (double n'))
  end.
```

Recursion has to be structural.

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Recursion has to be structural.

```
Inductive even : nat -> Prop :=
  | even0 : even O
  | evenS : forall x:nat, even x -> even (S (S x)).
```

Coq 101 - proofs

```
Definition even_prop := forall x:nat, even (double x).

Fixpoint even_proof(x: nat): even (double x) :=
  match x with
  | O      => even0
  | S n'   => evenS (double n') (even_proof n')
  end.

Check even_proof : even_prop.
```

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  end.

Check even_proof : even_prop.
```

The 2nd branch has the type $\mathit{even}\, S\,(S\,(\mathit{double}\ n'))$, and Coq knows by normalizing the types:

$$\mathit{even}\, S\,(S\,(\mathit{double}\ n')) \equiv_\beta \mathit{even}\,(\mathit{double}\,(S\ n'))$$

Recap: Curry-Howard Correspondence

Propositions as types in the context of intuitionistic logic.

Proposition	Term & Type
$A \wedge B$	$t : (A, B)$
$A \vee B$	$t : A + B$
$A \rightarrow B$	$t : A \rightarrow B$
$\bot$	$t : \mathit{False}$
$\neg A$	$t : A \rightarrow \mathit{False}$
$\forall x{:}A.\, B$	$t : (x : A) \rightarrow B$
$\exists x{:}A.\, B$	$t : (x{:}A, B)$

Curry-Howard correspondence in Coq

```
Inductive and (A B:Prop) : Prop :=
  conj : A -> B -> A /\ B
where "A /\ B" := (and A B) : type_scope.
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```
Inductive or (A B:Prop) : Prop :=
  | or_introl : A -> A \/ B
  | or_intror : B -> A \/ B
where "A \/ B" := (or A B) : type_scope.
```

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Inductive False : Prop :=.
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```

```
Inductive False : Prop :=.
```

```
Definition not (A:Prop) := A -> False.
Notation "~ x" := (not x) : type_scope.
```

Curry-Howard correspondence in Coq - continued

```
Notation "A -> B" := (forall (_ : A), B) : type_scope.
Definition iff (A B:Prop) := (A -> B) /\ (B -> A).
Notation "A <-> B" := (iff A B) : type_scope.
```

Curry-Howard correspondence in Coq - continued

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Notation "A <-> B" := (iff A B) : type_scope.
```

```
Inductive ex (A:Type) (P:A -> Prop) : Prop :=
  ex_intro : forall x:A, P x -> ex (A:=A) P.

Notation "'exists' x .. y , p" :=
  (ex (fun x => .. (ex (fun y => p)) ..)) : type_scope.
```

Curry-Howard correspondence in Coq - continued

```
Notation "A -> B" := (forall (_ : A), B) : type_scope.
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Notation "'exists' x .. y , p" :=
  (ex (fun x => .. (ex (fun y => p)) ..)) : type_scope.
```

```
Inductive eq (A:Type) (x:A) : A -> Prop :=
  eq_refl : x = x :>A

Notation "x = y" := (eq x y) : type_scope.
```

The equivalence between LEM and DNE

In intuitionistic logics, the *law of excluded middle* (LEM) and the *law of double negation* (DNE) are not provable.

- LEM: $\forall P. P \vee \neg P$
- DNE: $\forall P. \neg\neg P \to P$

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However, $\forall P. P \to \neg\neg P$ can be proved. How?

We will prove that LEM is equivalent to DNE:

```
Definition LEM: Prop := forall P: Prop, P \/~P.
Definition DNE: Prop := forall P: Prop, ~~P -> P.
Definition LEM_DNE_EQ: Prop := LEM <-> DNE.
```

LEM → DNE

```
Definition LEM_To_DNE :=
  fun (lem: forall P : Prop, P \/ ~ P) (Q:Prop) (q: ~~Q)
    =>
    match lem Q with
    | or_introl l =>
      l

    | or_intror r =>
      match (q r) with end
    end.

Check LEM_To_DNE : LEM -> DNE.
```

DNE → LEM

```
Definition DNE_To_LEM :=
  fun (dne: forall P : Prop, ~~P -> P) (Q:Prop) =>
    (dne (Q \/ ~ Q))
      (fun H: ~(Q \/ ~Q) =>
        let nq := (fun q: Q => H (or_introl q))
        in H (or_intror nq)
      ).

Check DNE_To_LEM :  DNE -> LEM.

Definition proof := conj LEM_To_DNE DNE_To_LEM.
Check proof : LEM <-> DNE.
```

Dependent Types in Programming Languages

Despite the huge success in proof assistants, its adoption in programming languages is limited.

- Scala supports *path-dependent types* and *literal types*.
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Challenge: the decidability of type checking.

Problem with Type Checking

Value constructors:

$$
\begin{array}{lcl}
NVec & : & \mathbb{N} \to * \\
nil & : & NVec\ 0 \\
cons & : & \mathbb{N} \to (n{:}\mathbb{N}) \to NVec\ n \to NVec\ n+1
\end{array}
$$

Appending vectors:

$$
\begin{array}{lcl}
append & : & (m{:}\mathbb{N}) \to (n{:}\mathbb{N}) \to NVec\ m \to NVec\ n \to NVec\ (n+m) \\
append & = & \lambda m{:}\mathbb{N}.\, \lambda n{:}\mathbb{N}.\, \lambda l{:}NVec\ m.\, \lambda t{:}NVec\ n. \\
 & & \quad match\ l\ with \\
 & & \quad \mid nil \Rightarrow t \\
 & & \quad \mid cons\ x\ r\ y \Rightarrow cons\ x\ (r+n)\ (append\ r\ n\ y\ t)
\end{array}
$$

Question: How does the type checker know $S\ (r+n) = n + (S\ r)$?