

# Foundations of Software Fall 2021

Week 4

## Programming in the Lambda-Calculus, Continued

## Recall: Church Booleans

```
tru  =  λt. λf. t  
fls  =  λt. λf. f
```

We showed last time that, if  $b$  is a boolean (i.e., it behaves like either `tru` or `fls`), then, for any values  $v$  and  $w$ , either

$$b \ v \ w \longrightarrow^* v$$

(if  $b$  behaves like `tru`) or

$$b \ v \ w \longrightarrow^* w$$

(if  $b$  behaves like `fls`).

## Booleans with “bad” arguments

But what if we apply a boolean to terms that are *not* values?

E.g., what is the result of evaluating

```
tru c0 omega ?
```

## Booleans with “bad” arguments

But what if we apply a boolean to terms that are *not* values?

E.g., what is the result of evaluating

`tru c0 omega ?`

Not what we want!

## A better way

Wrap the branches in an abstraction, and use a dummy “unit value,” to force evaluation of thunks:

`unit = λx. x`

Use a “conditional function”:

`test = λb. λt. λf. b t f unit`

If `tru'` is or behaves like `tru`, `fls'` is or behaves like `fls`, and `s` and `t` are arbitrary terms then

`test tru' (λdummy. s) (λdummy. t) →* s`  
`test fls' (λdummy. s) (λdummy. t) →* t`

## Recall: The z Operator

In the last lecture, we defined an operator `z` that calculates the “fixed point” of a function it is applied to:

```
z =  
  λf. λy. (λx. f (λy. x x y)) (λx. f (λy. x x y)) y
```

That is, if  $z_f = z\ f$  then  $z_f\ v \longrightarrow^* f\ z_f\ v$ .

## Recall: Factorial

As an example, we defined the factorial function as follows:

```
fact =  
  z (λfct.  
    λn.  
      if n=0 then 1  
      else n * (fct (pred n)))
```

For simplicity, we used primitive values from the calculus of numbers and booleans presented in week 2, and even used shortcuts like `1` and `*`.

As mentioned, this can be translated “straightforwardly” into the pure lambda-calculus. Let’s do that.

## Lambda calculus version of Factorial (not!)

Here is the naive translation:

```
badfact =  
  z (λfct.  
    λn.  
      iszro n  
      c1  
      (times n (fct (prd n))))
```

Why is this not what we want?

## Lambda calculus version of Factorial (not!)

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      (times n (fct (prd n))))
```

Why is this not what we want?

(Hint: What happens when we evaluate `badfact c0`?)

## Lambda calculus version of Factorial

A better version:

```
fact =  
  z (λfct.  
    λn.  
      test (iszro n)  
        (λdummy. c1)  
        (λdummy. (times n (fct (prd n))))))
```

## Displaying numbers

```
fact c3 →*
```

## Displaying numbers

```
fact c3  $\longrightarrow^*$  ( $\lambda s. \lambda z.$   
  s (( $\lambda s. \lambda z.$   
    s (( $\lambda s. \lambda z.$   
      s (( $\lambda s. \lambda z.$   
        s (( $\lambda s. \lambda z.$   
          s (( $\lambda s. \lambda z. z$ )  
            s z))  
          s z))  
        s z))  
      s z))  
    s z))  
  s z))  
s z))
```

Ugh!

## Displaying numbers

If we enrich the pure lambda-calculus with “regular numbers,” we can display church numerals by converting them to regular numbers:

```
realnat =  $\lambda n. n (\lambda m. \text{succ } m) 0$ 
```

Now:

```
realnat (times c2 c2)  
   $\longrightarrow^*$   
succ (succ (succ (succ zero))).
```

## Displaying numbers

Alternatively, we can convert a few specific numbers:

```
whack =  
  λn. (equal n c0) c0  
      ((equal n c1) c1  
        ((equal n c2) c2  
          ((equal n c3) c3  
            ((equal n c4) c4  
              ((equal n c5) c5  
                ((equal n c6) c6  
                  n))))))
```

Now:

$$\begin{array}{c} \text{whack (fact } c_3) \\ \xrightarrow{*} \\ \lambda s. \lambda z. s (s (s (s (s (s z)))))) \end{array}$$

## Equivalence of Lambda Terms



## Recall: Church Numerals

We have seen how certain terms in the lambda-calculus can be used to represent natural numbers.

$$\begin{aligned}c_0 &= \lambda s. \lambda z. z \\c_1 &= \lambda s. \lambda z. s \ z \\c_2 &= \lambda s. \lambda z. s \ (s \ z) \\c_3 &= \lambda s. \lambda z. s \ (s \ (s \ z))\end{aligned}$$

Other lambda-terms represent common operations on numbers:

$$scc = \lambda n. \lambda s. \lambda z. s \ (n \ s \ z)$$

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Other lambda-terms represent common operations on numbers:

$$scc = \lambda n. \lambda s. \lambda z. s \ (n \ s \ z)$$

In what sense can we say this representation is “correct”?

In particular, on what basis can we argue that `scc` on church numerals corresponds to ordinary successor on numbers?

## The naive approach

One possibility:

For each  $n$ , the term  $\text{scc } c_n$  evaluates to  $c_{n+1}$ .

## The naive approach... doesn't work

One possibility:

For each  $n$ , the term  $\text{scc } c_n$  evaluates to  $c_{n+1}$ .

Unfortunately, this is false.

E.g.:

$$\begin{aligned} \text{scc } c_2 &= (\lambda n. \lambda s. \lambda z. s (n s z)) (\lambda s. \lambda z. s (s z)) \\ &\longrightarrow \lambda s. \lambda z. s ((\lambda s. \lambda z. s (s z)) s z) \\ &\neq \lambda s. \lambda z. s (s (s z)) \\ &= c_3 \end{aligned}$$

## A better approach

Recall the intuition behind the church numeral representation:

- ▶ a number  $n$  is represented as a term that “does something  $n$  times to something else”
- ▶ `scc` takes a term that “does something  $n$  times to something else” and returns a term that “does something  $n + 1$  times to something else”

I.e., what we really care about is that `scc c2` behaves the same as `c3` when applied to two arguments.

```
scc c2 v w = (λn. λs. λz. s (n s z)) (λs. λz. s (s z)) v w
             → (λs. λz. s ((λs. λz. s (s z)) s z)) v w
             → (λz. v ((λs. λz. s (s z)) v z)) w
             → v ((λs. λz. s (s z)) v w)
             → v ((λz. v (v z)) w)
             → v (v (v w))
```

```
c3 v w      = (λs. λz. s (s (s z))) v w
             → (λz. v (v (v z))) w
             → v (v (v w))
```

## A general question

We have argued that, although  $scc\ c_2$  and  $c_3$  do not evaluate to the same thing, they are nevertheless “behaviorally equivalent.”

What, precisely, does behavioral equivalence mean?

## Intuition

Roughly,

“terms  $s$  and  $t$  are behaviorally equivalent”

should mean:

“there is no ‘test’ that distinguishes  $s$  and  $t$  — i.e., no way to put them in the same context and observe different results.”

## Intuition

Roughly,

“terms  $s$  and  $t$  are behaviorally equivalent”

should mean:

“there is no ‘test’ that distinguishes  $s$  and  $t$  — i.e., no way to put them in the same context and observe different results.”

To make this precise, we need to be clear what we mean by a *testing context* and how we are going to *observe* the results of a test.

## Examples

```
tru =  $\lambda t. \lambda f. t$   
tru' =  $\lambda t. \lambda f. (\lambda x. x) t$   
fls =  $\lambda t. \lambda f. f$   
omega =  $(\lambda x. x x) (\lambda x. x x)$   
poisonpill =  $\lambda x. \text{omega}$   
placebo =  $\lambda x. \text{tru}$   
 $Y_f = (\lambda x. f (x x)) (\lambda x. f (x x))$ 
```

Which of these are behaviorally equivalent?

## Observational equivalence

As a first step toward defining behavioral equivalence, we can use the notion of *normalizability* to define a simple notion of *test*.

Two terms  $s$  and  $t$  are said to be *observationally equivalent* if either both are normalizable (i.e., they reach a normal form after a finite number of evaluation steps) or both diverge.

I.e., we “observe” a term’s behavior simply by running it and seeing if it halts.

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Aside:

- ▶ Is observational equivalence a decidable property?

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I.e., we “observe” a term’s behavior simply by running it and seeing if it halts.

Aside:

- ▶ Is observational equivalence a decidable property?
- ▶ Does this mean the definition is ill-formed?

## Examples

- ▶ `omega` and `tru` are *not* observationally equivalent

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- ▶ `omega` and `tru` are *not* observationally equivalent
- ▶ `tru` and `fls` are observationally equivalent

## Behavioral Equivalence

This primitive notion of observation now gives us a way of “testing” terms for behavioral equivalence

Terms `s` and `t` are said to be *behaviorally equivalent* if, for every finite sequence of values `v1`, `v2`, ..., `vn`, the applications

`s v1 v2 ... vn`

and

`t v1 v2 ... vn`

are observationally equivalent.



## Examples

These terms are behaviorally equivalent:

```
tru =  $\lambda t. \lambda f. t$   
tru' =  $\lambda t. \lambda f. (\lambda x. x) t$ 
```

So are these:

```
omega =  $(\lambda x. x x) (\lambda x. x x)$   
 $Y_f = (\lambda x. f (x x)) (\lambda x. f (x x))$ 
```

These are not behaviorally equivalent (to each other, or to any of the terms above):

```
fls =  $\lambda t. \lambda f. f$   
poisonpill =  $\lambda x. \text{omega}$   
placebo =  $\lambda x. \text{tru}$ 
```

## Proving behavioral equivalence

Given terms  $s$  and  $t$ , how do we *prove* that they are (or are not) behaviorally equivalent?

## Proving behavioral inequivalence

To prove that `s` and `t` are *not* behaviorally equivalent, it suffices to find a sequence of values `v1 ... vn` such that one of

`s v1 v2 ... vn`

and

`t v1 v2 ... vn`

diverges, while the other reaches a normal form.

## Proving behavioral inequivalence

Example:

- ▶ the single argument `unit` demonstrates that `fls` is not behaviorally equivalent to `poisonpill`:

`fls unit`  
`= (λt. λf. f) unit`  
`→* λf. f`

`poisonpill unit`  
`diverges`

## Proving behavioral inequivalence

Example:

- ▶ the argument sequence  $(\lambda x. x) \text{ poisonpill } (\lambda x. x)$  demonstrate that **tru** is not behaviorally equivalent to **fls**:

$$\begin{aligned} & \text{tru } (\lambda x. x) \text{ poisonpill } (\lambda x. x) \\ & \quad \longrightarrow^* (\lambda x. x) (\lambda x. x) \\ & \quad \longrightarrow^* \lambda x. x \\ \\ & \text{fls } (\lambda x. x) \text{ poisonpill } (\lambda x. x) \\ & \longrightarrow^* \text{poisonpill } (\lambda x. x), \text{ which diverges} \end{aligned}$$

## Proving behavioral equivalence

To prove that **s** and **t** are behaviorally equivalent, we have to work harder: we must show that, for every sequence of values  $v_1 \dots v_n$ , either both

$$\mathbf{s} \ v_1 \ v_2 \ \dots \ v_n$$

and

$$\mathbf{t} \ v_1 \ v_2 \ \dots \ v_n$$

diverge, or else both reach a normal form.

How can we do this?

## Proving behavioral equivalence

In general, such proofs require some additional machinery that we will not have time to get into in this course (so-called *applicative bisimulation*). But, in some cases, we can find simple proofs.

*Theorem:* These terms are behaviorally equivalent:

$$\begin{aligned}\text{tru} &= \lambda t. \lambda f. t \\ \text{tru}' &= \lambda t. \lambda f. (\lambda x. x) t\end{aligned}$$

*Proof:* Consider an arbitrary sequence of values  $v_1 \dots v_n$ .

- ▶ For the case where the sequence has up to one element (i.e.,  $n \leq 1$ ), note that both  $\text{tru} / \text{tru } v_1$  and  $\text{tru}' / \text{tru}' v_1$  reach normal forms after zero / one reduction steps.
- ▶ For the case where the sequence has more than one element (i.e.,  $n > 1$ ), note that both  $\text{tru } v_1 v_2 v_3 \dots v_n$  and  $\text{tru}' v_1 v_2 v_3 \dots v_n$  reduce to  $v_1 v_3 \dots v_n$ . So either both normalize or both diverge.

## Proving behavioral equivalence

*Theorem:* These terms are behaviorally equivalent:

$$\begin{aligned}\text{omega} &= (\lambda x. x x) (\lambda x. x x) \\ Y_f &= (\lambda x. f (x x)) (\lambda x. f (x x))\end{aligned}$$

*Proof:* Both

$$\text{omega } v_1 \dots v_n$$

and

$$Y_f v_1 \dots v_n$$

diverge, for every sequence of arguments  $v_1 \dots v_n$ .

# Inductive Proofs about the Lambda Calculus

## Two induction principles

Like before, we have two ways to prove that properties are true of the untyped lambda calculus.

- ▶ Structural induction on terms
- ▶ Induction on a derivation of  $t \rightarrow t'$ .

Let's look at an example of each.

## Structural induction on terms

To show that a property  $\mathcal{P}$  holds for all lambda-terms  $t$ , it suffices to show that

- ▶  $\mathcal{P}$  holds when  $t$  is a variable;
- ▶  $\mathcal{P}$  holds when  $t$  is a lambda-abstraction  $\lambda x. t_1$ , assuming that  $\mathcal{P}$  holds for the immediate subterm  $t_1$ ; and
- ▶  $\mathcal{P}$  holds when  $t$  is an application  $t_1 t_2$ , assuming that  $\mathcal{P}$  holds for the immediate subterms  $t_1$  and  $t_2$ .

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- ▶  $\mathcal{P}$  holds when  $t$  is an application  $t_1 t_2$ , assuming that  $\mathcal{P}$  holds for the immediate subterms  $t_1$  and  $t_2$ .

N.b.: The variant of this principle where “immediate subterm” is replaced by “arbitrary subterm” is also valid. (Cf. *ordinary induction* vs. *complete induction* on the natural numbers.)

## An example of structural induction on terms

Define the set of *free variables* in a lambda-term as follows:

$$\begin{aligned}FV(x) &= \{x\} \\FV(\lambda x. t_1) &= FV(t_1) \setminus \{x\} \\FV(t_1 \ t_2) &= FV(t_1) \cup FV(t_2)\end{aligned}$$

Define the *size* of a lambda-term as follows:

$$\begin{aligned}\text{size}(x) &= 1 \\ \text{size}(\lambda x. t_1) &= \text{size}(t_1) + 1 \\ \text{size}(t_1 \ t_2) &= \text{size}(t_1) + \text{size}(t_2) + 1\end{aligned}$$

*Theorem:*  $|FV(t)| \leq \text{size}(t)$ .

## An example of structural induction on terms

*Theorem:*  $|FV(t)| \leq \text{size}(t)$ .

*Proof:* By induction on the structure of  $t$ .

- ▶ If  $t$  is a variable, then  $|FV(t)| = 1 = \text{size}(t)$ .
- ▶ If  $t$  is an abstraction  $\lambda x. t_1$ , then

$$\begin{aligned}& |FV(t)| \\&= |FV(t_1) \setminus \{x\}| && \text{by defn} \\&\leq |FV(t_1)| && \text{by arithmetic} \\&\leq \text{size}(t_1) && \text{by induction hypothesis} \\&< \text{size}(t_1) + 1 && \text{by arithmetic} \\&= \text{size}(t) && \text{by defn.}\end{aligned}$$

## An example of structural induction on terms

*Theorem:*  $|FV(t)| \leq size(t)$ .

*Proof:* By induction on the structure of  $t$ .

► If  $t$  is an application  $t_1 \ t_2$ , then

$$\begin{aligned} & |FV(t)| \\ = & |FV(t_1) \cup FV(t_2)| && \text{by defn} \\ \leq & |FV(t_1)| + |FV(t_2)| && \text{by arithmetic} \\ \leq & size(t_1) + size(t_2) && \text{by IH and arithmetic} \\ < & size(t_1) + size(t_2) + 1 && \text{by arithmetic} \\ = & size(t) && \text{by defn.} \end{aligned}$$

## Induction on derivations

Recall that the reduction relation is defined as the smallest binary relation on terms satisfying the following rules:

$$(\lambda x. t_1) \ v_2 \longrightarrow [x \mapsto v_2]t_1 \quad (\text{E-APPABS})$$

$$\frac{t_1 \longrightarrow t'_1}{t_1 \ t_2 \longrightarrow t'_1 \ t_2} \quad (\text{E-APP1})$$

$$\frac{t_2 \longrightarrow t'_2}{v_1 \ t_2 \longrightarrow v_1 \ t'_2} \quad (\text{E-APP2})$$



## Induction on derivations

Induction principle for the small-step evaluation relation.

To show that a property  $\mathcal{P}$  holds for all derivations of  $t \longrightarrow t'$ , it suffices to show that

- ▶  $\mathcal{P}$  holds for all derivations that use the rule E-AppAbs;
- ▶  $\mathcal{P}$  holds for all derivations that end with a use of E-App1 assuming that  $\mathcal{P}$  holds for all subderivations; and
- ▶  $\mathcal{P}$  holds for all derivations that end with a use of E-App2 assuming that  $\mathcal{P}$  holds for all subderivations.

## An example of induction on derivations

*Theorem:* if  $t \longrightarrow t'$  then  $FV(t) \supseteq FV(t')$ .

We must prove, for all derivations of  $t \longrightarrow t'$ , that  $FV(t) \supseteq FV(t')$ .

## An example of induction on derivations

*Theorem:* if  $t \longrightarrow t'$  then  $FV(t) \supseteq FV(t')$ .

*Proof:* by induction on the derivation of  $t \longrightarrow t'$ . There are three cases:

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*Theorem:* if  $t \longrightarrow t'$  then  $FV(t) \supseteq FV(t')$ .

*Proof:* by induction on the derivation of  $t \longrightarrow t'$ . There are three cases:

- If the derivation of  $t \longrightarrow t'$  is just a use of E-AppAbs, then  $t$  is  $(\lambda x. t_1)v$  and  $t'$  is  $[x \mapsto v]t_1$ . Reason as follows:

$$\begin{aligned} FV(t) &= FV((\lambda x. t_1)v) \\ &= FV(t_1) \setminus \{x\} \cup FV(v) \\ &\supseteq FV([x \mapsto v]t_1) \\ &= FV(t') \end{aligned}$$

## An example of induction on derivations

*Theorem:* if  $t \longrightarrow t'$  then  $FV(t) \supseteq FV(t')$ .

*Proof:* by induction on the derivation of  $t \longrightarrow t'$ . There are three cases:

- If the derivation ends with a use of E-App1, then  $t$  has the form  $t_1 \ t_2$  and  $t'$  has the form  $t'_1 \ t_2$ , and we have a subderivation of  $t_1 \longrightarrow t'_1$

By the induction hypothesis,  $FV(t_1) \supseteq FV(t'_1)$ . Now calculate:

$$\begin{aligned} FV(t) &= FV(t_1 \ t_2) \\ &= FV(t_1) \cup FV(t_2) \\ &\supseteq FV(t'_1) \cup FV(t_2) \\ &= FV(t'_1 \ t_2) \\ &= FV(t') \end{aligned}$$

- E-App2 is treated similarly.