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A new family of rational surfaces in $\mathbb{P}^4$ ☆

Hans-Christian Graf v. Bothmer*, Cord Erdenberger, Katharina Ludwig

Institiut für Mathematik, Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany

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Abstract

We describe a new method of constructing rational surfaces with given invariants in $\mathbb{P}^4$ and present a family of degree 11 rational surfaces of sectional genus 11 with 2 six-secants that we found with this method.

Keywords: Rational surfaces; Construction; Computer algebra; Finite fields

1. Introduction

In 1989 Ellingsrud and Peskine showed that the degree of non-general type surfaces in $\mathbb{P}^4$ is bounded (Ellingsrud and Peskine, 1989). Since then their degree bound has been sharpened by various authors, most recently by Decker and Schreyer (2000) to 52. On the other hand one has tried to construct and classify non-general type surfaces in $\mathbb{P}^4$. Decker et al. (1993) lists the 51 families of such surfaces known at that time of which 18 are rational. Since then Schreyer (1996) has found 4 more families, 3 of them parametrizing rational surfaces. Five more families of non-rational, non-general type surfaces were found by Aure et al. (1997), Abo et al. (1998) and Abo and Ranestad (2002). Recently Abo

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* Corresponding author. Tel.: +49 511 762 2315; fax: +49 511 762 5803.
E-mail address: bothmer@math.uni-hannover.de (H.-C. Graf v. Bothmer).

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announced the existence of a family of degree 12 rational surfaces in $\mathbb{P}^4$. Non-general type surfaces are classified up to degree 10 (see Decker and Schreyer (2000) for an overview and references); the largest known degree of a non-general type surface in $\mathbb{P}^4$ is 15. Rational surfaces are only known up to degree 12.

In this paper we describe a new method for finding rational surfaces in $\mathbb{P}^4$ and present a new family of rational degree 11 surfaces in $\mathbb{P}^4$ which we found with this method.

Our method is partly based on an idea of Schreyer (1996) who explicitly constructed surfaces in $\mathbb{P}^4$ over small fields using computer algebra programs and provided a method of lifting these surfaces to characteristic 0. There he used the observation that modules with special syzygies are much more common over small fields than over characteristic zero to find those modules by a random search. Here we use a random search over $\mathbb{F}_2$ to find linear systems with special configurations of basepoints on $\mathbb{P}^2$.

We construct one of our new surfaces over $\mathbb{F}_2$ in Section 2. In Section 3 we prove that this surface lies in a family that is also defined in characteristic zero. Finally we explain our search algorithm in Section 4.

Part of our calculations were done at the Gauss Laboratory at the University of Göttingen. We would like to thank Yuri Tschinkel for giving us this opportunity.

2. The surface

Let us first fix some notation. We work over the fields $\mathbb{F}_2, \mathbb{F}_{2^{14}}$ and $\mathbb{F}_{2^5}$ which we realize as

$$\mathbb{F}_{2^{14}} = \mathbb{F}_2[t]/(t^{14} + t^{13} + t^{11} + t^{10} + t^8 + t^6 + t^4 + t + 1)$$

and $\mathbb{F}_{2^5} = \mathbb{F}_2[t]/(t^5 + t^3 + t^2 + t + 1)$.

Over these fields we consider the points

$$\begin{aligned} P &= (0 : 0 : 1) \in \mathbb{P}^2(\mathbb{F}_2) \\ Q &= (t^{11898} : t^{137} : 1) \in \mathbb{P}^2(\mathbb{F}_{2^{14}}) \\ R &= (t^6 : t^{15} : 1) \in \mathbb{P}^2(\mathbb{F}_{2^5}). \end{aligned}$$

Lemma 2.1. *The orbits of Q and R under the Frobenius endomorphism are of degree* 14 *and* 5 *respectively. We denote the corresponding points by $Q_1, \ldots, Q_{14}$ and $R_1, \ldots, R_5$.*

Proof. The orbits of Q and R are defined by the kernels of

$$\mathbb{F}_2[x, y, z] \to \mathbb{F}_{2^{14}}[x, y, z]/I_Q$$

and

$$\mathbb{F}_2[x, y, z] \to \mathbb{F}_{2^5}[x, y, z]/I_R$$

where I_Q and I_R are the ideals of Q and R respectively. Using Script A.1 one can calculate these kernels and check that the degrees of their vanishing sets are 14 and 5.

Proposition 2.2. *Let L be the class of a line in $\mathbb{P}^2$. Then*

$$|9L - 3P - 2Q_1 - \cdots - 2Q_{14} - R_1 - \cdots - R_5| = \mathbb{P}^4$$

and this linear system has only the assigned base points.

Proof. Using Script A.1 we intersect the ideal of $3P$, the ideal of the orbit of $2Q$ and the ideal of the orbit of R. We check that the intersection contains exactly five independent ninetics whose base locus is of degree $1 \cdot 6 + 14 \cdot 3 + 5 \cdot 1 = 53$.

Proposition 2.3. *Let S be the blow-up of $\mathbb{P}^2$ in $P, Q_1, \ldots, Q_{14}, R_1, \ldots, R_5$ and $E_1, \ldots, E_{20}$ be the corresponding exceptional divisors. Then the linear system*

$$\left| 9L - 3E_1 - \sum_{i=2}^{15} 2E_i - \sum_{i=16}^{20} E_i \right|$$

is very ample and embeds $S \subset \mathbb{P}^4$ as a smooth surface of degree 11 *and sectional genus* 11.

Proof. In Script A.1 we define a morphism

$$\mathbb{F}_2[x_0, \ldots, x_4] \to \mathbb{F}_2[x, y, z]$$

using the five independent ninetics found. The kernel of this map is the ideal of $S \subset \mathbb{P}^4$. We then calculate that S has degree 11, and sectional genus 11 and finally check smoothness by the Jacobi criterion.

Proposition 2.4. *$S \subset \mathbb{P}^4$ has two* 6*-secants. In particular S cannot lie in one of the known families.*

Proof. Every 6-secant of S must be contained in all quintics that contain S. The final lines of Script A.1 calculate that the vanishing locus $(I_S)_{\leq 5}$ contains S and two lines, which turn out to be 6-secants. The known families of rational surfaces of degree 11 and sectional genus 11 have zero, one or infinitely many 6-secants (Decker et al., 1993).

3. Lifting to characteristic zero

In this section we denote schemes defined over spec $\mathbb{Z}$ with a subscript $\mathbb{Z}$ and their fibers over points of spec $\mathbb{Z}$ with subscripts $\mathbb{F}_p$ or $\mathbb{Q}$.

Proposition 3.1 (*Schreyer, 1996*). *Consider a smooth projective variety $X_{\mathbb{Z}} \subset \mathbb{P}^N_{\mathbb{Z}}$ over* spec $\mathbb{Z}$*, a map*

$$\phi \colon \mathcal{F} \to \mathcal{G}$$

between vector bundles of rank f and g on $X_{\mathbb{Z}}$, the determinantal subvariety $Y_{\mathbb{Z}} \subset X_{\mathbb{Z}}$ where ϕ has rank k, and a $\mathbb{F}_p$-rational point $y \in Y_{\mathbb{F}_p}$.

If the tangent space $T_{Y_{\mathbb{F}_p},y}$ of $Y_{\mathbb{F}_p}$ in y is a linear subspace of codimension $(f-k)(g-k)$ in $T_{X_{\mathbb{F}_p},y}$ then y lies on an irreducible component $Z_{\mathbb{Z}}$ of $Y_{\mathbb{Z}}$ that has nonempty fibers over an open subscheme of spec $\mathbb{Z}$.

Proof. Since $Y_{\mathbb{F}_p}$ is determinantal, the codimension of $Y_{\mathbb{F}_p}$ in $X_{\mathbb{F}_p}$ is at most $c = (f-k)(g-k)$. The condition on the tangent space ensures that $Y_{\mathbb{F}_p}$ is smooth of this codimension in y, or equivalently $Y_{\mathbb{F}_p}$ is of dimension $d = \dim X_{\mathbb{F}_p} - c$ in y. Let now $Z_{\mathbb{Z}}$ be a component of $Y_{\mathbb{Z}}$ that contains y. Since $Y_{\mathbb{Z}}$ is determinantal in $X_{\mathbb{Z}}$ each component of $Y_{\mathbb{Z}}$ has also at most codimension c (Eisenbud, 1995, Ex 10.9, p. 246). Since $\dim X_{\mathbb{Z}} = \dim X_{\mathbb{F}_p} + 1$ the dimension of $Z_{\mathbb{Z}}$ is at least $d + 1$. $Z_{\mathbb{F}_p}$ contains y and is therefore of dimension at most d. Hence $Z_{\mathbb{Z}}$ cannot be contained in the fiber $Y_{\mathbb{F}_p}$ and has nonempty fibers over an open subscheme of spec $\mathbb{Z}$.

On $\mathbb{P}^2_{\mathbb{Z}}$ we have the map

$$\tau_k : H^0(\mathcal{O}_{\mathbb{P}^2_{\mathbb{Z}}}(a)) \to \mathcal{O}_{\mathbb{P}^2_{\mathbb{Z}}}(a) \oplus 3\mathcal{O}_{\mathbb{P}^2_{\mathbb{Z}}}(a-1) \oplus \cdots \oplus \binom{k+2}{2}\mathcal{O}_{\mathbb{P}^2_{\mathbb{Z}}}(a-k)$$

that associates to each polynomial of degree a the coefficients of its Taylor expansion up to degree k.

Lemma 3.2. *If $a > k$ then the image of τ_k is a vector bundle $\mathcal{F}_k$ of rank $\binom{k+2}{2}$ over* spec $\mathbb{Z}$.

Proof. In each point we consider an affine two-dimensional neighborhood where we can choose the $\binom{k+2}{2}$ coefficients of the affine Taylor expansion independently. This shows that the image has at least this rank everywhere. That this is also the maximal rank follows from the Euler relation.

Remark 3.3. Notice that the morphism $H^0(\mathcal{O}_{\mathbb{P}^2_{\mathbb{Z}}}(a)) \to \binom{k+2}{2}\mathcal{O}_{\mathbb{P}^2_{\mathbb{Z}}}(a-k)$ is not surjective in characteristics that divide a. One really has to consider the whole Taylor expansion.

Set now $X_{\mathbb{Z}} = \mathrm{Hilb}_{1,\mathbb{Z}} \times \mathrm{Hilb}_{14,\mathbb{Z}} \times \mathrm{Hilb}_{5,\mathbb{Z}}$ where $\mathrm{Hilb}_{k,\mathbb{Z}}$ denotes the Hilbert scheme of k points in $\mathbb{P}^2_{\mathbb{Z}}$ over spec $\mathbb{Z}$, and let

$$Y_{\mathbb{Z}} = \{(p, q, r) \mid h^0(9L - 3p - 2q - 1r) \geq 5\} \subset X_{\mathbb{Z}}$$

be the subset where the linear system of ninetics with triple point in p, double points in q and single base points in r is at least of projective dimension 4.

Proposition 3.4. *There exist vector bundles $\mathcal{F}$ and $\mathcal{G}$ of ranks* 55 *and* 53 *respectively on $X_{\mathbb{Z}}$ and a morphism*

$$\phi : \mathcal{F} \otimes \mathcal{O}_{X_{\mathbb{Z}}} \to \mathcal{G}$$

such that the determinantal locus where ϕ has rank 50 *is supported on $Y_{\mathbb{Z}}$.*

Proof. On the Cartesian product

$$\begin{tikzcd} \mathrm{Hilb}_{d,\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathbb{P}^2_{\mathbb{Z}} \\ \mathrm{Hilb}_{d,\mathbb{Z}} & \end{tikzcd}$$

we have the morphisms

$$\pi_2^*\tau_k : H^0(\mathcal{O}_{\mathbb{P}^2_{\mathbb{Z}}}(9)) \otimes \mathcal{O}_{\mathrm{Hilb}_{d,\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}} \to \pi_2^*\mathcal{F}_k.$$

Let now $P_d \subset \mathrm{Hilb}_{d,\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}$ be the universal set of points. Then P_d is a flat family of degree d over $\mathrm{Hilb}_{d,\mathbb{Z}}$ and

$$\mathcal{G}_k := (\pi_1)_*((\pi_2^*\mathcal{F}_k)|_{P_d})$$

is a vector bundle of rank $d\binom{k+2}{2}$ over $\mathrm{Hilb}_{d,\mathbb{Z}}$. On

$$X_{\mathbb{Z}} = \mathrm{Hilb}_{1,\mathbb{Z}} \times \mathrm{Hilb}_{14,\mathbb{Z}} \times \mathrm{Hilb}_{5,\mathbb{Z}}$$

the induced map

$$\phi\colon H^0(\mathcal{O}_{\mathbb{P}^2_{\mathbb{Z}}}(9)) \otimes \mathcal{O}_{X_{\mathbb{Z}}} \xrightarrow{\tau_2 \oplus \tau_1 \oplus \tau_0} \sigma_1^*\mathcal{G}_2 \oplus \sigma_{14}^*\mathcal{G}_1 \oplus \sigma_5^*\mathcal{G}_0$$

has the desired properties, where σ_d denotes the projection to $\mathrm{Hilb}_{d,\mathbb{Z}}$.

Theorem 3.5. *There exists a family of smooth rational surfaces in $\mathbb{P}^4_{\mathbb{C}}$ with $d = 11$, $\pi = 11$, $K^2 = -11$ and two 6-secants.*

Proof. By determining the infinitesimal deformations of our 20 points $P, Q_1, \ldots, Q_{14}, R_1, \ldots, R_5$ in $\mathbb{P}^2_{\mathbb{F}_2}$ we can check with a Macaulay calculation that

$$y := (P, \{Q_1, \ldots, Q_{14}\}, \{R_1, \ldots, R_5\}) \in Y_{\mathbb{F}_2} \subset X_{\mathbb{F}_2}$$

satisfies the conditions of Proposition 3.1. A script performing these calculations can be obtained from our webpage (Graf v. Bothmer et al., 2004). We therefore have a component $Z_{\mathbb{Z}}$ of $Y_{\mathbb{Z}}$ that contains our configuration of base points. Since the conditions

(i) the points of p, q and r are distinct
(ii) $h^0(9L - 3p - 2q - 1r) = 5$
(iii) the linear system $|9L - 3p - 2q - 1r|$ has no further base points
(iv) the image of the corresponding rational map $\phi\colon \mathbb{P}^2 \to \mathbb{P}^4$ is a smooth surface S
(v) S has two 6-secants

are all open on $Z_{\mathbb{Z}}$ and y is a point on this component that satisfies all conditions, they must hold on a nonempty open subset of $Z_{\mathbb{Z}}$. Since $Z_{\mathbb{Z}}$ is irreducible and has nonzero fibers over the generic point, we obtain smooth surfaces in characteristic zero. The invariants can be calculated from the multiplicities of the base locus using the following proposition.

Proposition 3.6. *Let $S = \mathbb{P}^2_{\mathbb{C}}(p_1, \ldots, p_l)$ be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in l distinct points. We denote by $E_1, \ldots, E_l$ the corresponding exceptional divisors and by L the pullback of a general line in $\mathbb{P}^2_{\mathbb{C}}$ to S. If $|aL - \sum_{i=1}^{l} b_i E_i|$ is a very ample linear system of dimension 4 for suitable a and b_i, then $S \subset \mathbb{P}^4_{\mathbb{C}}$ is a rational surface of degree*

$$d = a^2 - \sum_{i=1}^{l} b_i^2$$

and sectional genus

$$\pi = \binom{a-1}{2} - \sum_{i=1}^{l} \binom{b_i}{2}.$$

The self-intersection of the canonical divisor of S is $K^2 = 9 - l$.

Proof. Set $H = aL - \sum_{i=1}^{l} b_i E_i$. Then

$$d = H^2 = \left(aL - \sum_{i=1}^{l} b_i E_i\right)^2 = a^2 - \sum_{i=1}^{l} b_i^2$$

since $L^2 = 1$, $L.E_i = 0$ and $E_i.E_j = -\delta_{ij}$. The canonical divisor of S is $K = -3L + \sum_{i=1}^{l} E_i$, so $K^2 = 9 - l$. The sectional genus of S can be calculated by adjunction:

$$\pi = \frac{1}{2} H(K+H) + 1 = \binom{a-1}{2} - \sum_{i=1}^{l} \binom{b_i}{2}.$$

4. The search

In this section we will describe our search algorithm. We first need to find suitable linear systems for given invariants. For that we make the following observation which is a direct consequence of Proposition 3.6:

Corollary 4.1. *In the situation of Proposition 3.6 we set $\beta_j = \#\{i \mid b_i = j\}$. The invariants of S are then linear forms in the β_j:*

$$d = a^2 - \sum_j \beta_j j^2$$

$$\pi = \binom{a-1}{2} - \sum_j \beta_j \binom{j}{2}$$

$$K^2 = 9 - \sum_j \beta_j.$$

For given d, π, K^2 and a the linear system above has only finitely many integer solutions. One can find these solutions by integer programming. We have used an algorithm from Cox et al. (1998, Chapter 8).

Example 4.2. For $d = 11$, $\pi = 11$, $K^2 = -11$ and $a = 9$ the only solution is $\beta_3 = 1$, $\beta_2 = 14$ and $\beta_1 = 5$.

For a given set of β_j we have chosen random points in $\mathbb{P}^2$ over $\mathbb{F}_{2^{\beta_j}}$, checked if their orbit under the Frobenius endomorphism had degree β_j; checked whether the corresponding linear system $|aL - \sum_{i=1}^{l} b_i E_i|$ was 4-dimensional; calculated the image of the corresponding map to $\mathbb{P}^4$ and checked whether this image was a smooth rational surface. The script we used is available on our web page (Graf v. Bothmer et al., 2004).

For comparison we also tried to reconstruct the rational surfaces of Decker et al. (1993) using the basepoint multiplicities provided there. Our results are collected in Figs. 1 and 2.

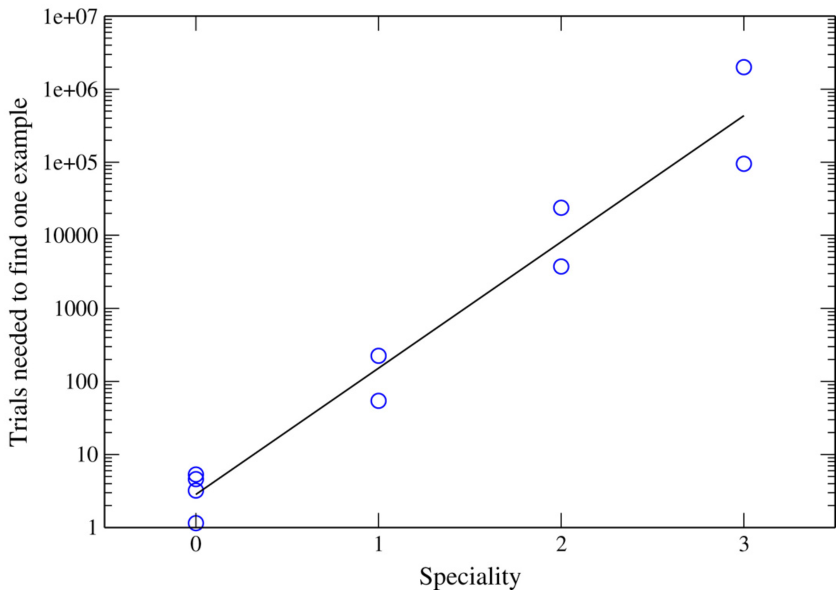

Fig. 1. The difficulty of finding a surface grows exponentially with the speciality.

Example	d	π	Speciality	Trials	Surfaces	Rate (%)	Log rate
B1.7	5	2	0	1000	871	87.1	−0.2
B1.8	6	3	0	1000	311	31.1	−1.7
B1.9	7	4	0	1000	188	18.8	−2.4
B1.10	8	5	0	1000	312	31.2	−1.7
B1.11	8	6	1	10000	184	1.84	−5.8
B1.12	9	6	0	10000	2173	21.73	−2.2
B1.13	9	7	1	100000	446	0.446	−7.8
B1.14	10	8	1	1000000	0	0.0000	$-\infty$
B1.15	10	9	2	1000000	42	0.0042	−14.5
B1.16	10	9	2	1000000	267	0.0267	−11.9
B1.17	11	11	3	10000000	0	0.00000	$-\infty$
B1.18	11	11	3	10000000	5	0.00005	−20.9
B1.19	11	11	3	10000000	0	0.00000	$-\infty$
New	11	11	3	2000000	21	0.00105	−16.5

Fig. 2. Results of random searches using our script. The numbering of the examples is as in Decker et al. (1993).

Notice that the number of trials needed to find an example grows exponentially with the speciality as expected. From this we expect that surfaces of speciality 4 can be found in approximately 500 million trials. Using our program this would take about 500 weeks on a 2 GHz machine.

Remark 4.3. Notice that our approach can only find linear systems where the points in each group of constant multiplicity are in uniform position.

Remark 4.4. Another way of constructing random groups of points is via syzygies and the theorem of Hilbert–Birch. We have also tried this, but found the above approach more effective.

Appendix A

Here we provide a script for the computer algebra program Macaulay 2 (Grayson and Stillman, 2002) that does the calculations needed in Section 2. The script can also be obtained from our webpage (Graf v. Bothmer et al., 2004).

Script A.1.

```
-- construct a surface over F_2 using Frobenius orbits

-- define coordinate ring of P^2 over F_2
F2 = GF(2)
S2 = F2[x,y,z]

-- define coordinate ring of P^2 over F_2^14 and F_2^5
St = F2[x,y,z,t]
use St; I14 = ideal(t^14+t^13+t^11+t^10+t^8+t^6+t^4+t+1);
S14 = St/I14
use St; I5 = ideal(t^5+t^3+t^2+t+1);
S5 = St/I5

-- the points
use S2; P = matrix{{0_S2, 0_S2, 1_S2}}
use S14;Q = matrix{{t^11898, t^137, 1_S14}}
use S5; R = matrix{{t^6, t^15, 1_S5}}

-- their ideals
IP = ideal ((vars S2)*syz P)
IQ = ideal ((vars S14)_{0..2}*syz Q)
IR = ideal ((vars S5)_{0..2}*syz R)

-- their orbits
f14 = map(S14/IQ,S2); Qorbit = ker f14
degree Qorbit -- degree = 14

f5 = map(S5/IR,S2); Rorbit = ker f5
degree Rorbit -- degree = 5

-- ideal of 3P
P3 = IP^3;

-- orbit of 2Q
f14square = map(S14/IQ^2,S2); Q2orbit = ker f14square;

-- ideal of 3P + 2Qorbit + 1Rorbit
I = intersect(P3,Q2orbit,Rorbit);

-- extract 9-tics
H = super basis(9,I)
rank source H -- affine dimension = 5

-- count basepoints (with multiplicities)
degree ideal H -- degree = 53
-- construct map to P^4
```

```
T = F2[x0,x1,x2,x3,x4]
fH = map(S2,T,H);

-- calculate the ideal of the image
Isurface = ker fH;

-- check invariants
betti res coker gens Isurface
codim Isurface -- codim = 2
degree Isurface -- degree = 11
genera Isurface -- genera = {0,11,10}

-- check smoothness
J = jacobian Isurface;
mJ = minors(2,J) + Isurface;
codim mJ -- codim = 5

-- count 6-secants
-- ideal of 1 quartic and 5 quintics
Iquintics = ideal (mingens Isurface)_{0..5};

-- calculate the extra components where these vanish
secants = Iquintics : Isurface;
codim secants -- codim = 3
degree secants -- degree = 2
secantlist = decompose secants -- two components

-- check number of intersections
degree (Isurface+secantlist#0) -- degree = 6
codim (Isurface+secantlist#0) -- codim = 4
degree (Isurface+secantlist#1) -- degree = 6
codim (Isurface+secantlist#1) -- codim = 4
```

References

Abo, H., Decker, W., Sasakura, N., 1998. An elliptic conic bundle in $\mathbb{P}^4$ arising from a stable rank-3 vector bundle. Math. Z. 229 (4), 725–741.

Abo, H., Ranestad, K., 2002. Irregular elliptic surfaces of degree 12 in the projective fourspace. math.AG/0210317.

Aure, A., Decker, W., Hulek, K., Popescu, S., Ranestad, K., 1997. Syzygies of abelian and bielliptic surfaces in $\mathbb{P}^4$. Int. J. Math. 8 (7), 849–919.

Cox, D., Little, J., O'Shea, D., 1998. Using algebraic geometry. Graduate Texts in Mathematics, vol. 185. Springer-Verlag, New York.

Decker, W., Ein, L., Schreyer, F.-O., 1993. Construction of surfaces in $\mathbb{P}^4$. J. Algebraic Geom. 2 (2), 185–237.

Decker, W., Schreyer, F.-O., 2000. Non-general type surfaces in $\mathbb{P}^4$: some remarks on bounds and constructions. J. Symbolic Comput. 29 (4–5), 545–582 (Symbolic computation in algebra, analysis, and geometry, Berkeley, CA, 1998).

Eisenbud, D., 1995. Commutative Algebra with a View Toward Algebraic Geometry. Graduate Texts in Mathematics, vol. 150. Springer-Verlag.

Ellingsrud, G., Peskine, C., 1989. Sur les surfaces lisses de $\mathbb{P}^4$. Invent. Math. 95 (1), 1–11.

Graf v. Bothmer, H.-C., Erdenberger, C., Ludwig, K., 2004. Macaulay 2 scripts for finding rational surfaces in $\mathbb{P}^4$. Available at http://www-ifm.math.uni-hannover.de/~bothmer/surface. See also the LaTeX-file of this article at math.AG/0404492.

Grayson, D.R., Stillman, M.E., 2002. Macaulay 2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2.

Schreyer, F.-O., 1996. Small fields in constructive algebraic geometry. In: Moduli of Vector Bundles. Sanda, 1994, Kyoto, 1994. Lecture Notes in Pure and Appl. Math., vol. 179. Dekker, New York, pp. 221–228.