SMOOTH RATIONAL SURFACES IN $\mathbb{P}^4$

FRANK-OLAF SCHREYER

1. Introduction

2. Known Examples

2.1. **Degenerate surfaces.** The plane $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and the cubic surfaces $\mathbb{P}^2(3;1^6)$ are the smooth surfaces which do not span $\mathbb{P}^4$. They have degree 1, 2 and 3 respectively.

2.2. **Non-degenerate surfaces of degree ≤ 7.**

(1) The cubic scroll $X \cong \mathbb{P}^2(2;1) = \{\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \end{pmatrix} < 2\}$
(2) The del Pezzo surface of degree 4, i.e. $\mathbb{P}^2(3;1^{5)} = Q_1 \cap Q_2$, a complete intersection of two quadrics.
(3) The projected Veronese surface of degree 4
(4) The Castelnuovo surface $X \to \mathbb{P}^1$ of degree 5, a conic bundle over $\mathbb{P}^1$
(5) The Bordiga surface $\mathbb{P}^2(4;1^{10})$ of degree 6
(6) The Ionescu-Okonek surface of $\mathbb{P}^2(6;2^6,5^1)$ degree 7 with with Betti table

$$\begin{array}{ccccc} 1 & - & - & - & - \\ - & - & - & - & - \\ - & 1 & - & - & - \\ - & 6 & 10 & 5 & 1 \end{array}$$

(7) The non-special Alexander surface $\mathbb{P}^2(7;2^{10},1)$ of degree 8 and sectional genus $\pi = 5$ with Betti table

$$\begin{array}{cccccc} & 0 & 1 & 2 & 3 & 4 \\ 0: & 1 & . & . & . & . \\ 1: & . & . & . & . & . \\ 2: & . & . & . & . & . \\ 3: & . & 5 & 4 & . & . \\ 4: & . & 4 & 10 & 8 & 2 \end{array}$$

(8) The Okonek surface $\mathbb{P}^2(6;2^4,1^{12})$ of degree 8 and sectional genus $\pi = 7$ with Betti table

$$\begin{array}{cccc} 1 & . & . & . \\ . & . & . & . \\ . & 1 & . & . \\ . & 4 & 5 & 1 \end{array}$$

(9) The non-special Alexander surface $\mathbb{P}^2(13;4^{10})$ of degree 9 and sectional genus $\pi = 6$ with Betti table

$$\begin{array}{ccccc} 1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & 15 & 26 & 15 & 3 \\ . & 1 & 3 & 3 & 1 \end{array}$$

(10) The speciality one Alexander surface $\mathbb{P}^2(9;3^6,2^3,1^6)$ of degree 9 and sectional genus $\pi = 7$ and Betti table

$$\begin{array}{ccccc} 1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & 3 & 1 & . & . \\ . & 6 & 14 & 9 & 2 \end{array}$$

(11) The Ranestad surface $\mathbb{P}^2(14;6,4^9,2,1^2)$ of degree 10 and sectional genus $\pi = 8$ and Betti table

$$\begin{array}{ccccc} 1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & 10 & 13 & 4 & . \\ . & 4 & 11 & 10 & 3 \end{array}$$

(12) The Decker-Ein-Schreyer surface $\mathbb{P}^2(9;3^4,2^7,1^7)$ of degree 10 and sectional genus $\pi = 9$ with Betti table

$$\begin{array}{ccccc} 1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & 1 & . & . & . \\ . & 10 & 18 & 10 & 2 \end{array}$$

(13) The Ranestad surface $\mathbb{P}^2(8; 2^{12}, 1^6)$ of degree 10 and sectional genus $\pi = 9$ and Betti table

	0	1	2	3	4
0 :	1	.	.	.	.
1 :	.	.	.	.	.
2 :	.	.	.	.	.
3 :	.	2	.	.	.
4 :	.	5	9	3	.
5 :	.	1	3	3	1

(14) ≥ 3 families of Schreyer surfaces of degree 11 and sectional genus $\pi = 10$ with Betti table

	0	1	2	3	4
0 :	1	.	.	.	.
1 :	.	.	.	.	.
2 :	.	.	.	.	.
3 :	.	.	.	.	.
4 :	.	5	.	.	.
5 :	.	7	26	20	5

of type $\mathbb{P}^2(18; 6^5, 5^5, 2, 1^4)$, $\mathbb{P}^2(15; 5^5, 4^4, 3^2, 2, 1^3)$. The third family has as last adjoint surface the intersection of $Y = (\mathbb{P}^1 \times \mathbb{P}^2) \cap Q \subset \mathbb{P}^5$ with a quadric Q and $X = Y(H+4R; 4^3, 2^2, 1^3)$ where R denotes the ruling of $\mathbb{P}^1 \times \mathbb{P}^2$. The conic fibration $Y \to \mathbb{P}^1$ has 6 singular fibers.

(15) ≥ 3 families of Schreyer surfaces of degree 11 and sectional genus $\pi = 10$ with Betti table

	0	1	2	3	4
0 :	1	.	.	.	.
1 :	.	.	.	.	.
2 :	.	.	.	.	.
3 :	.	.	.	.	.
4 :	.	5	1	.	.
5 :	.	8	26	20	5

of type $\mathbb{P}^2(13; 4^7, 3^4, 2^2, 1^2)$, $\mathbb{P}^1 \times \mathbb{P}^1((9,9); 4^8, 3, 2^3, 1^2)$ and $\mathbb{P}^2(15; 5^6, 4^2, 3^2, 2^3, 1^2)$.

(16) ≥ 1 families of Schreyer surfaces of degree 11 and sectional genus $\pi = 10$ with Betti table

	0	1	2	3	4
total:	1	14	28	20	5
0 :	1	.	.	.	.
1 :	.	.	.	.	.
2 :	.	.	.	.	.
3 :	.	.	.	.	.
4 :	.	5	2	.	.
5 :	.	9	26	20	5

The adjunction leads to a quadric bundle of class $Y = 2H - R$ in a scroll of type $S(2,1,1)$. The surface X is the blow-up of Y in $3^4, 2^4, 1$ points. $X = Y(2H + 2R; 3^4, 2^4, 1)$.

(17) One family of Schreyer surfaces $\mathbb{P}^2(12; 4^4, 3^5, 2^6)$ of degree 11 and sectional genus $\pi = 10$ with Betti table

	0	1	2	3	4
total:	1	15	29	20	5
0 :	1	.	.	.	.
1 :	.	.	.	.	.
2 :	.	.	.	.	.
3 :	.	.	.	.	.
4 :	.	5	3	.	.
5 :	.	10	26	20	5

(18) The Popsecu surface $\mathbb{P}^2(9; 3^4, 2^7, 1^7)$ of degree 11 and sectional genus $\pi = 11$ with Betti table

1	.	.	.	.
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
.	10	12	3	.
.	.	2	3	1

(19) The Popsecu surface of degree 11 and sectional genus $\pi = 11$ with Betti table

	0	1	2	3	4
0 :	1	.	.	.	.
1 :	.	.	.	.	.
2 :	.	.	.	.	.
3 :	.	.	.	.	.
4 :	.	10	13	4	.
5 :	.	1	3	3	1

(20) The Popsecu surface of degree 11 and sectional genus $\pi = 11$ with Betti table and Tate resolution

	0	1	2	3	4
0 :	1	.	.	.	.
1 :	.	.	.	.	.
2 :	.	.	.	.	.
3 :	.	.	.	.	.
4 :	.	10	14	6	1
5 :	.	2	5	4	1

and

	−1	0	1	2	3	4	5	6	7
−4 :	1	.	.	.	.	.	.	.	.
−3 :	107	64	32	11	.	.	.	.	.
−2 :	.	.	.	.	3	1	.	.	.
−1 :	.	.	.	.	.	2	1	.	.
0 :	.	.	.	.	.	.	10	38	91

respectively.

(21) The vBothmer-Erdenberger-Ludwig surface $\mathbb{P}^2(9;3,2^{14},1^5)$ of degree 11 and sectional genus $\pi = 11$ with Betti table and Tate resolution

	0	1	2	3	4
0 :	1	.	.	.	.
1 :	.	.	.	.	.
2 :	.	.	.	.	.
3 :	.	1	.	.	.
4 :	.	5	4	.	.
5 :	.	2	9	8	2

and

	−1	0	1	2	3	4	5	6	7
−4 :	1	.	.	.	.	.	.	.	.
−3 :	107	64	32	11	.	.	.	.	.
−2 :	.	.	.	.	3	1	.	.	.
−1 :	.	.	.	.	.	2	2	.	.
0 :	.	.	.	.	.	1	10	38	91

respectively.

(22) Six families of Abo-Ranestad surfaces of degree 12 and sectional genus $\pi = 13$ with Betti table

	0	1	2	3	4
0 :	1	.	.	.	.
1 :	.	.	.	.	.
2 :	.	.	.	.	.
3 :	.	.	.	.	.
4 :	.	5	.	.	.
5 :	.	4	18	13	3

of type $\mathbb{P}^2(12;4^5,2^{12},1^4)$, $\mathbb{P}^2(12;4^4,3^3,2^9,1^5)$, $\mathbb{P}^2(12,4^3,3^6,2^6,1^6)$ and $\mathbb{P}^2(12;4^2,3^9,2^3,1^7)$. Further families: The fifth case has as last adjoint surface a quadric bundle of degree 8 which is a divisor Y of class $2H-2R$ on a scroll of degree 5 (and type $S(2,2,1)$?) and $X = Y(H+3R;3^5,2^4,7^1)$. Finally, the sixth case has as last adjoint surface a quadric bundle of degree 9 which is a is a divisor Y of class $2H-3R$ on a scroll of degree 6 (perhaps of type $S(2,2,2)$) and $X = Y(H+3R;3^8,2,8^1)$.

(23) One family of Abo-Ranestad surfaces $\mathbb{P}^2(12; 4^5, 2^{12}, 1^4)$ of degree 12 and sectional genus $\pi = 13$ with Betti table

$$\begin{array}{rccccc}
 & 0 & 1 & 2 & 3 & 4 \\
\text{total:} & 1 & 10 & 19 & 13 & 3 \\
0: & 1 & . & . & . & . \\
1: & . & . & . & . & . \\
2: & . & . & . & . & . \\
3: & . & . & . & . & . \\
4: & . & 5 & 1 & . & . \\
5: & . & 5 & 18 & 13 & 3
\end{array}.$$

This family could be in the closure of the first family of **??**.

So at the moment there are at least $22 + 2 + 2 + 5 = 31$ components of the Hilbert scheme of surfaces in $\mathbb{P}^4$ whose general elements correspond to non-degenerate smooth rational surfaces.

Caveat: Check that all are extendible to characteristic 0 and check whether family **??** lies in the closure of the first family of **??**.

FACHBEREICH MATHEMATIK, UNIVERSITÄT DES SAARLANDES, CAMPUS E2 4, D-66123 SAARBRÜCKEN, GERMANY

Email address: `schreyer@math.uni-sb.de`