CONSTRUCTION OF RATIONAL SURFACES OF DEGREE 12 IN PROJECTIVE FOURSPACE

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ABSTRACT. The aim of this paper is to present a construction of smooth rational surfaces in projective fourspace with degree 12 and sectional genus 13. In particular, we establish the existences of five different families of smooth rational surfaces in projective fourspace with the prescribed invariants.

1. INTRODUCTION

Hartshone conjectured that only finitely many components of the Hilbert scheme of surfaces in $\mathbb{P}^4$ contain smooth rational surfaces. In 1989, this conjecture was positively solved by Ellingsrud and Peskine [8]. The exact bound for the degree is, however, still open, and hence the question concerning the exact bound motivates a search for smooth rational surfaces in $\mathbb{P}^4$. The goal of this paper is to construct five different families of smooth rational surfaces in $\mathbb{P}^4$ with degree 12 and sectional genus 13. The rational surfaces in $\mathbb{P}^4$ were previously known up to degree 11. In this paper, we will prove the following theorem:

Theorem 1.1. *There exist five distinct families of smooth rational surfaces in $\mathbb{P}^4$ over $\mathbb{C}$ with $d = 12$ and $\pi = 13$. These surfaces are all isomorphic to $\mathbb{P}^2$ blown up in 21 points, but their embedding linear system are of the following five different kinds:*

(i) $\left|12L - \sum_{i=1}^{2} 4E_i - \sum_{j=3}^{11} 3E_j - \sum_{k=12}^{14} 2E_k - \sum_{l=15}^{21} E_l\right|$,

(ii) $\left|12L - \sum_{i=1}^{3} 4E_i - \sum_{j=4}^{9} 3E_j - \sum_{k=10}^{15} 2E_k - \sum_{l=16}^{21} E_l\right|$,

(iii) $\left|12L - \sum_{i=1}^{4} 4E_i - \sum_{j=5}^{7} 3E_j - \sum_{k=8}^{16} 2E_k - \sum_{l=17}^{21} E_l\right|$,

(iv) $\left|12L - \sum_{i=1}^{5} 4E_i - \sum_{j=6}^{17} 2E_j - \sum_{k=18}^{21} E_k\right|$

(v) $\left|12L - 4E_1 - \sum_{i}^{13} 3E_i - \sum_{j=13}^{21} E_j\right|$,

where L is the pullback from $\mathbb{P}^2$ of a line, while the E_i are the exceptional curves of the blowup.

Family (i) was found by Schreyer and the first author in [3]. The surfaces were found by the construction of a Beilinson monad for their ideal sheaf. Let V be a finite-dimensional vector space over a field K and let W be its dual space. A Beilinson monad represents a given coherent sheaf on $\mathbb{P}(W)$ as a homology of a finite complex whose objects are direct sums of bundles of differentials. The differentials in the monad are given by homogeneous matrices over an exterior algebra $E = \bigwedge V$. To construct a Beilinson monad for a given coherent sheaf, we typically need to determine the type of the Beilinson monad, that is, determine each object, and then find the differentials in the monad.

Let X be a smooth rational surface in $\mathbb{P}^4 = \mathbb{P}(W)$ with degree 12 and sectional genus 13. The type of a Beilinson monad for the (suitably twisted) ideal sheaf of X can be derived from the knowledge of its cohomology groups. Such information is partially determined from general results such as the Riemann-Roch formula and Kodaira vanishing theorem. It is, however, hard to determine the dimensions of all cohomology groups needed to determine the type of the Beilinson monad. For this reason, we assume that the ideal sheaf of X has so-called "minimal cohomology" in some range of twists. From this assumption, the Beilinson monad for the twisted ideal sheaf $\mathcal{I}_X(4)$ of X has the following form:

$$4\Omega^3(3) \xrightarrow{A} 2\Omega^2(2) \oplus 2\Omega^1(1) \xrightarrow{B} 3\mathcal{O}. \tag{1.1}$$

The most difficult part is to find the differentials in (1.1). In [3], Schreyer and the first author provided a computational approach to find such differentials A and B. This approach is based on exterior algebra methods presented by Eisenbud, Fløystad and Schreyer [7], and finite field searches developed by Schreyer [11]

The construction we will present in this paper stems from the following observation on Family (i). Let A_1 and B_1 be the linear parts of the differentials A and B in (1.1) respectively. In the example, the locus C_A in $\mathbb{P}(V)$, where A_1 is not surjective, is a rational normal curve, while the locus S_B in $\mathbb{P}(V)$, where B_1 is not injective, is a rational cubic surface scroll in $\mathbb{P}(V)$. Furthermore, the curve C_A does not intersect the surface S_B. From A_1 and B_1 in the example, one can reconstruct a smooth rational surface in $\mathbb{P}^4$ with the same invariants. Indeed, the condition that the composite of B and A is zero gives rise to the homogeneous system of 120 linear equations with 140 unknowns. In this case, the solution space has dimension 26. Thus, we can choose 26 variables freely to determine A and B. A random choice of values for 26 parameters gives the Beilinson monad of type (1.1), and the homology of this monad is the twisted ideal sheaf of a smooth surface of the desired type. Our approach generalizes this procedure. Let $\mathfrak{F}$ be the family of rational normal curves in $\mathbb{P}(V)$. For a fixed rational cubic surface scroll, and for each $20 \leq N \leq 26$, we find a subfamily $\mathfrak{F}(N)$ of rational normal curves such that the associated homogeneous system of linear equations has the N-dimensional solution space. The subfamily $\mathfrak{F}(N)$ is of codimension $N-20$ in $\mathfrak{F}$ and consists of curves such that $N-20$ of its trisecant planes intersect the cubic surface along a conic section. Performing a random search we can expect to find a point p in $\mathfrak{F}(N)(\mathbb{F}_q)$ from $\mathfrak{F}(\mathbb{F}_q)$ at a rate of $\left(1 : q^{N-20}\right)$. As in the case above we find examples of surfaces over $\mathbb{F}_q$. Finally, a lifting argument introduced by Schreyer (cf. [11]) is applied to show the existence of a smooth rational surface in $\mathbb{P}^4$ over $\mathbb{C}$ with the desired invariants are established, when $N = 22, 23, 24, 25$ and 26.

The calculations were done with the computer algebra system `MACAULAY2` developed by Grayson and Stillman [9]. All the `MACAULAY2` scripts needed to construct surfaces are available at [1].

2. Preliminaries

Our main goal is to construct families of smooth rational surfaces X in $\mathbb{P}^4$ with degree $d = 12$, sectional genus $\pi = 13$. The construction discussed in this paper takes the following three steps:

(1) Find a smooth surface X with the prescribed invariants over a finite field of a small characteristic.

(2) Determine the type of the linear system, which embeds X into $\mathbb{P}^4$ to justify that the surface X found in the previous step is rational.
(3) Establish the existence of a lift to characteristic 0.

In this section, we recall some basic facts needed for the construction.

2.1. **Beilinson monads.** Let V be a $(n+1)$-dimensional vector space over a field K with let W be its dual space. We write E for the exterior algebra $\bigoplus_{i=0}^{n+1} \bigwedge^i V$. Step (1) uses the technique of "Beilinson monad". A Beilinson monad represents a given coherent sheaf in terms of a direct sum of (suitably twisted) bundles of differentials and a homomorphism between these bundles, which are given by homogeneous matrices over the exterior algebra E.

Theorem 2.1 ([2]). *For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}(W)$, there is a complex $\mathcal{K}^\cdot$ with*

$$\mathcal{K}^i \simeq \bigoplus_j \mathrm{H}^j(\mathcal{F}(i-j)) \otimes \Omega^{j-i}(j-i)$$

such that

$$\mathrm{H}^i(\mathcal{K}^\cdot) = \begin{cases} \mathcal{F} & i = 0 \\ 0 & i \neq 0. \end{cases}$$

Proof. See [2] for the proof. □

Remark 2.2 ([7]). (i) The differentials of the complex $\mathcal{K}^\cdot$ are given by using the isomorphisms

$$\mathrm{Hom}(\Omega^i(i), \Omega^j(j) \simeq \bigwedge V^{i-j} \simeq \mathrm{Hom}_E(E(i), E(j)).$$

(ii) Let us write

$$\begin{aligned} d_{ij}^{(r)} &\in \mathrm{Hom}(H^j(\mathcal{F}(i-j)) \otimes \Omega^{j-i}(j-i), \mathrm{H}^{j-r+1}(\mathcal{F}(i-j+r)) \otimes \Omega^{j-i-r}(j-i-r) \\ &\simeq \bigwedge^r V \otimes \mathrm{Hom}(\mathrm{H}^j\mathcal{F}(i-j), \mathrm{H}^{j-r+1}\mathcal{F}(i-j+r)) \\ &\simeq \mathrm{Hom}\left(\bigwedge^r W \otimes \mathrm{Hom}(\mathrm{H}^j\mathcal{F}(i-j), \mathrm{H}^{i-j+r}\mathcal{F}(i-j+r)\right) \end{aligned}$$

for the degree r maps actually occurring in $\mathcal{K}^\cdot$. Then the constant maps $d_{ij}^{(0)}$ in $\mathcal{K}^\cdot$ are zero. The linear maps $d_{ij}^{(1)}$ in $\mathcal{K}^\cdot$ correspond to the multiplication maps

$$W \otimes \mathrm{H}^j\mathcal{F}(i-j) \to \mathrm{H}^j\mathcal{F}(i-j+1).$$

So the linear maps induces maps from the set of hyperplanes to the set of linear transformations from $\mathrm{H}^j\mathcal{F}(i-j)$ to $\mathrm{H}^j\mathcal{F}(i-j+1)$.

2.2. **Adjunction theory.** In this subsection, we explain how to spot the surface found in Step (1) within the Enriques-Kodaira classification and determine the type of the linear system that embeds X into $\mathbb{P}^4$. First of all, we recall a result of Sommese and Van de Ven for a surface over $\mathbb{C}$:

Theorem 2.3 ([10]). *Let X be a smooth surface in $\mathbb{P}^n$ over $\mathbb{C}$ with degree d, sectional genus π, geometric genus p_g and irregularity q, let H be its hyperplane class, let K be its canonical divisor and let $N = \pi - 1 + p_g - 1$. Then the adjoint linear system $|H + K|$ defines a birational morphism*

$$\Phi = \Phi_{|H+K|} : X \to \mathbb{P}^{N-1}$$

onto a smooth surface X_1, which blows down precisely all (-1)-curves on X, unless

(i) *X is a plane, or Veronese surface of degree 4, or X is ruled by lines;*
(ii) *X is a Del Pezzo surface or a conic bundle;*
(iii) *X belongs to one of the following four families:*
   (a) *$X = \mathbb{P}^2(p_1, \dots, p_7)$ embedded by $H \equiv 6L - \sum_{i=0}^{7} 2E_i$;*
   (b) *$X = \mathbb{P}^2(p_1, \dots, p_8)$ embedded by $H \equiv 6L - \sum_{i=0}^{7} 2E_i - E_8$;*
   (c) *$X = \mathbb{P}^2(p_1, \dots, p_8)$ embedded by $H \equiv 9L - \sum_{i=0}^{8} 3E_i$;*
   (d) *$X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is an indecomposable rank 2 bundle over an elliptic curve and $H \equiv B$, where B is a section $B^2 = 1$ on X.*

Proof. See [10] for the proof. □

Setting $X = X_1$ and performing the same operation repeatedly, we obtain a sequence

$$X \to X_1 \to X_2 \to \cdots \to X_k.$$

This process will be terminated if $N - 1 \leq 0$. For a surface with nonnegative Kodaira dimension, one obtains the minimal model at the end of the adjunction process. If the Kodaira dimension equals $-\infty$, we end up with a ruled surface, a conic bundle, a Del Pezzo surface, $\mathbb{P}^2$, or one of the few exceptions of Sommese and Van de Ven.

It is not known whether the adjunction theory holds over a finite field. However, we have the following proposition:

Proposition 2.4 ([6]). *Let X be a smooth surface over a field of arbitrary characteristic. Suppose that the adjoint linear system $|H + K|$ is base point free. If the image X_1 in $\mathbb{P}^N$ under the adjunction map $\Phi_{|H+K|}$ is a surface of the expected degree $(H + K)^2$, the expected sectional genus $\frac{1}{2}(H + K)(H + 2K) + 1$ and with $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X_1})$, then X_1 is smooth and $\Phi : X \to X_1$ is a simultaneous blow down of the $K_1^2 - K^2$ many exceptional lines on X.*

Proof. See Proposition 8.3 in [6] for a proof. □

Remark 2.5. The exceptional divisors contracted in each step are defined over the base field.

Remark 2.6. In [4] and [6], it is described how to compute the adjunction process for a smooth surface given by explicit equations. See, for example, [6] for the computational details.

3. Construction over a small field

Let K be a field, let W be a five-dimensional vector space over K with basis $\{x_i\}_{0 \leq i \leq 4}$, and let V be its dual space with dual basis $\{e_i\}_{0 \leq i \leq 4}$. Let X be a smooth surface in $\mathbb{P}^4 = \mathbb{P}(W)$ with the invariants given above. The first step is to determine the type of the Beilinson monad for the twisted ideal sheaf of X, which is derived from the partial knowledge of its cohomology groups. Such information can be determined from general results such as the Riemann-Roch formula and Kodaira vanishing theorem (see [4] for more detail). Here we assume that X has minimal cohomology in the range $0 \leq j \leq 4$ of twists:

i					
4					
3					
2		4	2		
1				2	3
0					
j	0	1	2	3	4

$h^i \mathcal{I}_X(j)$

Here a zero is represented by the empty box. From Beilinson's Theorem, it follows, therefore, that the corresponding Beilinson monad for $\mathcal{I}_X(4)$ is of the following type:

$$0 \to 4\Omega^3(3) \xrightarrow{A} 2\Omega^2(2) \oplus 2\Omega^1(1) \xrightarrow{B} 3\mathcal{O} \to 0. \tag{3.1}$$

The next step is to describe what maps A and B could be the differentials of the monad (3.1). Let E be the exterior algebra $\bigwedge V$ on V. Then the identifications

$$\mathrm{Hom}(\Omega^i(i), \Omega^j(j)) \simeq \mathrm{Hom}_E(E(i), E(j)) \simeq \bigwedge^{i-j} V$$

allow us to think of the maps A and B as homomorphisms between E-free modules. Since the composite of B and A is zero, each column of A can be written as an E-linear combination of columns of Syz (B). In fact, a reasonable assumption when (3.1) is the Beilinson monad for an ideal sheaf, is that the first graded betti numbers of the minimal free resolution of Coker(B) are:

$$\begin{array}{r|cccc} 0 & 3 & 2 & . & . \\ \text{-1} & . & 2 & 4 & a_2 \\ \text{-2} & . & . & a_1 & * \end{array} \tag{3.2}$$

This assumption is aposteriori is supported in examples. Assume that there exists such a map B. Then this B uniquely determines $A_B : 4E(3) \to 2E(2) \oplus 2E(1)$, which could be the first map in (3.1), up to automorphism of $4E(3)$. The pair (A_B, B) defines a complex

$$4\Omega^3(3) \xrightarrow{A_B} 2\Omega^2(2) \oplus 2\Omega^1(1) \xrightarrow{B} 3\mathcal{O}.$$

Thus we can compute the homology $\ker(B)/\operatorname{im}(A_B)$ and check that this homology corresponds to the twisted ideal sheaf $\mathcal{I}_X(4)$ of X. So the key step is to find a B satisfying Condition (3.2). Our approach to finding such a B takes the following four steps (1), (2), (3) and (4):

(1) Let B_1 be the linear part of B. Suppose that B_1 is the general member of $\mathrm{Hom}_E(2E(1), 3E)$. To ease our calculation, we define B_1 by the matrix

$$B_1 = \begin{pmatrix} e_0 & e_1 \\ e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}.$$

From Remark 2.2 it follows that the hyperplane classes, for which the corresponding linear transformation is not injective, form a rational cubic surface scroll S_B in

$\mathbb{P}(V)$. Let A_1 be an element of $\mathrm{Hom}_E(4E(3), 2E(2))$ and let C_A be the locus of points, where A_1 is not surjective. Now, for a fixed B_1, we find an A_1 such that

(c_1) C_A is smooth and
(c_2) C_A does not intersect S_B.

Note that if C_A is smooth, then C_A is a rational quartic normal curve.

(2) Suppose that there are an $A_2 \in \mathrm{Hom}(4\Omega^3(3), 2\Omega^2(2))$ and a $B_2 \in \mathrm{Hom}(2\Omega^2(2), 3\mathcal{O})$ such that the corresponding sequence

$$4\Omega^3(3) \xrightarrow{A} 2\Omega^2(2) \oplus 2\Omega^1(1) \xrightarrow{B} 3\mathcal{O} \tag{3.3}$$

is a monad for $\mathcal{I}_X(4)$, where $A = {}^t(A_1, A_2)$ and $B = (B_2, B_1)$. Since the sequence (3.3) should be a complex, A_2 and B_2 must satisfy the following matrix equation:

$$B_2 \circ A_1 + B_1 \circ A_2 = 0. \tag{3.4}$$

Let A_2 be the 2×4 matrix whose (k, l) entry is $\sum_{i<j} a_{ij}^{kl} e_i \wedge e_j$ and let B_2 be the 3×2 matrix whose (k, l) entry is $\sum_{i<j} b_{ij}^{kl} e_i \wedge e_j$. Given the fixed matrix B_1 and a linear matrix A_1, condition (3.4) gives rise to a homogeneous system of 120 linear equations in the 140 coefficients of the entries in A_2 and B_2. The rank N of this linear system depends on A_1, and the solution space has dimension $140 - N$.

(3) Solving those equations for b_{ij}^{kl}'s, we obtain $60 - (140 - N) = N - 80$ relations among b_{ij}^{kl}'s, and hence $140 - N$ variables in b_{ij}^{kl}'s can be chosen freely to determine B_2. Then check that B_1 and the B_2 given by the random choices of values for $140 - N$ parameters define a homomorphism from $2E(2) \oplus 2E(1)$ to $3E$ satisfying condition (3.2).

(4) If B_1 and B_2 satisfy condition (3.2), then $B = (B_2, B_1)$ determines a homomorphism $A_B : 4E(3) \to 2E(2) \oplus 2E(1)$ uniquely up to automorphisms of $4E(3)$ and the pair (A_B, B) defines a complex. Compute the homology $\ker(B)/\operatorname{im}(A_B)$ of the complex. If the homology corresponds to the twisted ideal sheaf of a surface in $\mathbb{P}^4$ with the expected invariants, then check smoothness of this surface by using the Jacobian criterion.

In this section, using the construction described above, we establish the existence of the family of smooth rational surfaces in $\mathbb{P}^4$ over a finite field for each $114 \leq N \leq 117$ and show that there exists a lift to characteristic zero for each family. We also establish the existence of a smooth rational surface for $N = 113$. In this case, the intersection of C_A and S_B is no longer empty.

Remark 3.1. Let $A_1 \in \mathrm{Hom}_E(4E(3), 2E(2))$ and let U be the solution space to the homogeneous system of linear equations associated with A_1. Let U_B be the vector space formed by the pairs of two E-linear combinations of columns of B_1. Then U_B can be regarded as a subspace of $\mathrm{Hom}_E(2E(2), 3E)$. Since U contains U_B as a twenty-dimensional subspace, both columns of B_2 can be written as E-linear combinations of columns of B_1 if and only if $N = 120$, so we may exclude this case.

To find A_1 with $N < 120$, we define two further varieties associated to the two linear matrices. By the general assumption on these matrices, every column of A_1 and every row of B_1 have rank 2, i.e., they define elements in $G = G(2, V)$. If V_A and V_B are the column space of A_1 and the row space of B_1 respectively, then the corresponding maps of $\mathbb{P}(V_A)$ and $\mathbb{P}(V_B)$ into $G \subset \mathbb{P}\left(\bigwedge^2 V\right)$ are the double embeddings. We let Z_A and Z_B be the images of $\mathbb{P}(V_A)$ and $\mathbb{P}(V_B)$ under these double embeddings respectively, i.e., Z_A is a Veronese 3-fold; while Z_B is a Veronese surface.

Lemma 3.2. *Let A_1 be an element of $\mathrm{Hom}_E(4E(3), 2E(2))$ satisfying (c_1) and (c_2), and let $\Gamma = Z_A \cap Z_B$. Suppose that Γ is finite. Then Γ consists of at most six points.*

Proof. First we prove that Γ either consists of at most six points or four points in Γ lie on a line in both $\mathbb{P}(V_A)$ and $\mathbb{P}(V_B)$.

Notice first that Z_A itself spans $\mathbb{P}^9$ while Z_B spans $\mathbb{P}^5$. This $\mathbb{P}^5$ intersects G in the union of Z_B and a plane P. Thus Γ thought of as a subscheme of $\mathbb{P}^3 = \mathbb{P}(V_A)$ is contained in the four quadrics defined by restricting the linear forms that vanish on Z_B to Z_A. Assume now that Γ consists of at least seven distinct points. If five of them are in a plane, then the conic through these five is a fixed curve in all quadrics through Γ, which means that the intersection of $\mathbb{P}^5$ with Z_A contains a curve C. But $\mathbb{P}^5$ intersects G in the union of Z_B and a plane, so the curve C must be contained in this plane, i.e., it must be a conic, the image of a line in $\mathbb{P}^3$. The intersection of the plane and Z_B is also a conic, so Γ would contain four collinear points.

On the other hand, if at most three points are in a plane, there is a unique twisted cubic through the six points. If four points lie in a plane, this curve degenerates into a conic and a line or three lines. In either case this possibly reducible twisted cubic lies in three quadrics, and the six points are defined by four quadrics, a contradiction.

Assume that Z_B and Z_A intersect in finite number of points and four of them lie on a line both in $\mathbb{P}(V_B)$ and $\mathbb{P}(V_A)$. The image of these two lines are two conics in $\mathbb{P}^5$ that clearly lie in the plane P. On the one hand the four planes in $\mathbb{P}(W)$ corresponding to the four points each intersects the rational cubic scroll S_B in a conic and the rational normal curve C_A in three points. Let U be the union of these four planes.

The conic, which is obtained as the intersection of the plane P and Z_B, corresponds to the line whose underlying vector space spanned by two rows of B_1. The determinant of the 2×2 matrix generated by these two rows defines a quadric hypersurface Q containing the union U and S_B. This quadric may have rank 3 or 4. We will prove that Q also contains C_A and that therefore S_B and C_A must intersect.

In case Q has rank 4, Q is a cone with a vertex. Since C_A meets each plane in U in three points, C_A is contained in Q by the Bezout theorem. If Q has rank 3, then any plane defined by a linear combination of the two rows contains the same line, which is the directrix of S_B. Assume that C_A is not contained in Q. Then at least two of three points in each plane are common for all four planes, because otherwise the number of intersection number of Q and C_A is going to be more than 8. These points, however, lie on the directrix of S_B, which is a contradiction. □

Consider again the matrix equation $P_{A,B} = B_1 \circ A_2 + B_2 \circ A_1 = 0$ with exterior multiplication. The entries of $P_{A,B}$ are in $\bigwedge^3 V$, which has rank 10, so the equation $P_{A,B} = 0$ defines a linear system of equations on the coefficients parameterized by $V_A \otimes V_B \otimes \bigwedge^2 V$ of rank $N \leq 120$.

Remark 3.3. A point in $Z_A \cap Z_B$ correspond to a plane in $\mathbb{P}(V)$ that intersect the curve C_A in three points and the surface S_B in a conic section.

Lemma 3.4. *Let A_1 be an element of $\mathrm{Hom}_E(4E(3), 2E(2))$ satisfying (c_1) and (c_2), and let N be the rank of linear system of equations defined by $P_{A,B}$. If Z_A and Z_B intersect in k points, then N is at most $120 - k$.*

Proof. A row R_1 in A_1 and a column L_1 in B_1, the corresponding row R_2 in B_2 and column L_2 in A_2 and a 2-vector $\omega \in \bigwedge^2 V$, define an exterior product $(R_1 \cdot L_2 + R_2 \cdot L_1) \wedge \omega$ which lies in $\bigwedge^5 V \simeq K$. Assume that the subspaces of V generated by L_1 and R_1 coincide, then

$$(R_1 \cdot L_2 + R_2 \cdot L_1) \wedge \left(\bigwedge^2 L_1 \right) = 0$$

independent of L_2 and R_2. Therefore there is one relation among the coefficients of $P_{A,B}$ for each point of intersection of Z_A and Z_B in $\mathbb{P}(\bigwedge^2 V)$. Assume now that there are $r \leq 6$ points of intersection $Z_A \cap Z_B$, and consider their corresponding tensors in $V_A \otimes V_B \otimes \bigwedge^2 V$. Notice that these tensors are all pure, so they have natural projections on each factor. In particular, they are linearly independent if they are linearly independent in one factor. In fact we end our proof by showing that the $r \leq 6$ points in $Z_A \cap Z_B$ are linearly independent in $\mathbb{P}(\bigwedge^2 V)$.

Since Z_A and Z_B are quadratically embedded, no three points on either of them are collinear. If four points are coplanar, the plane of their span meets both Z_A and Z_B in a conic. In the proof of the Lemma 3.2, we saw that this is the case only if C_A and S_B intersect. If five points in $Z_A \cap Z_B$ span only a $\mathbb{P}^3$, this $\mathbb{P}^3$ must intersect the Veronese surface Z_B in a conic and a residual point, i.e., four of the five points are coplanar as in the previous case.

Finally if $Z_A \cap Z_B$ consists of six points that span a $\mathbb{P}^4$, then this $\mathbb{P}^4$ intersects Z_B in rational normal quartic curve, or in two conics. But the span of Z_B intersects G in the union of the plane P and Z_B, so the $\mathbb{P}^4$ intersects G in a curve of degree 4 and a line in P that is bisecant to the curve, or the plane P and a conic that meets P in a point. Since no four of the six points are coplanar and $Z_A \cap Z_B$ is finite, the intersection of $\mathbb{P}^4$ and Z_A cannot contain a curve. The six points in $Z_A \cap Z_B$ considered as points in $\mathbb{P}(V_A)$ therefore lie on five quadrics, which have a finite intersection. If five of them are in a plane, then the conic in the plane through those five points must lie in each quadric, a contradiction. If at most four points are coplanar, there is a unique, possibly reducible twisted cubic curve through the six points. But then the six points lie on only four quadrics, a contradiction completing our proof. □

Corollary 3.5. *Let $\mathfrak{F}$ be the family of rational normal curves in $\mathbb{P}(V)$, and let $\mathfrak{F}(N)$ be the subfamily of $\mathfrak{F}$ formed by rational normal curves C_A satisfying (c_2) such that the rank of linear system of equations defined by $P_{A,B}$ equals N. Then $\mathrm{codim}\,(\mathfrak{F}, \mathfrak{F}(N)) \leq 120 - N$.*

Proof. Each intersection point of Z_A with Z_B imposes an independent condition on C_A as long as the intersection is finite. Therefore, the codimension of the family of rational normal curves C_A such that Z_A intersects Z_B in $k < 7$ points is k, and the corollary follows. □

Remark 3.6. We discuss the cases $N = 118$ and 119, where we could not find any example where the homology of the monad is an ideal sheaf of the desired kind.

If $N = 119$, then for matrices B_2 in $\mathrm{Hom}_E(2E(2), 3E)$ obtained in step (2), the matrix $B = (B_1, B_2)$ has the syzygies of the following type

$$\begin{array}{r|rrrr} 0 & 3 & 2 & . & . \\ -1 & . & 2 & 6 & 10 \\ -2 & . & . & 5 & * \end{array} \tag{3.5}$$

Taking a map from $4E(3)$ to $6E(3)$ at random and compositing the map to the second map in (3.5) give a homomorphism $A : 4E(3) \to 2E(2) \oplus 2E(1)$. Then A and B define a complex of type (3.3). The homology $\ker(B)/\operatorname{im}(A)$, however, has rank three.

Similarly, the map B has syzygies of the following type in the case of $N = 118$:

$$\begin{array}{r|rrrr} 0 & 3 & 2 & . & . \\ -1 & . & 2 & 5 & 3 \\ -2 & . & . & 3 & * \end{array} \tag{3.6}$$

The homology of the associated complex in cohomological degree 0 has rank 1, but the corresponding scheme does not have the desired invariants.

Proposition 3.7. *There exist five distinct families of smooth rational surfaces in $\mathbb{P}^4$ over $\overline{\mathbb{F}_5}$ with $d = 12$ and $\pi = 13$. The corresponding embedding linear systems are:*

(i) $\left|12L - \sum_{i_1=1}^{2} 4E_{i_1} - \sum_{i_2=3}^{11} 3E_{i_2} - \sum_{i_3=12}^{14} 2E_{i_3} - \sum_{i_4=15}^{21} E_{i_4}\right|$,

(ii) $\left|12L - \sum_{i_1=1}^{3} 4E_{i_1} - \sum_{i_2=4}^{9} 3E_{i_2} - \sum_{i_3=10}^{15} 2E_{i_3} - \sum_{i_4=16}^{21} E_{i_4}\right|$,

(iii) $\left|12L - \sum_{i_1=1}^{4} 4E_{i_1} - \sum_{i_2=5}^{7} 3E_{i_2} - \sum_{i_3=8}^{16} 2E_{i_3} - \sum_{i_4=17}^{21} E_{i_4}\right|$,

(iv) $\left|12L - \sum_{i_1=1}^{5} 4E_{i_1} - \sum_{i_2=6}^{17} 2E_{i_3} - \sum_{i_4=18}^{21} E_{i_4}\right|$.

Proof. By random search over $\mathbb{F}_5$, we can find an $A_1 \in \mathrm{Hom}_E(4E(3), 2E(2))$ satisfying (c_1) and (c_2) for each $114 \leq N \leq 117$:

(i) $\begin{pmatrix} e_0 + 2e_1 + 2e_2 & -e_3 & -2e_1 + e_2 + 2e_3 & e_0 + 2e_1 + 2e_2 - e_3 + 2e_4 \\ e_0 + 2e_1 - e_2 + e_3 & -e_4 & e_1 - 2e_2 & e_3 + 2e_4 \end{pmatrix}$;

(ii) $\begin{pmatrix} 2e_3 & -e_0 + 2e_1 - 2e_2 + 2e_3 - e_4 & 2e_1 + 2e_2 + 2e_4 & e_0 + 2e_2 + e_3 - 2e_4 \\ 2e_4 & e_0 - 2e_1 - 2e_2 - 2e_3 - 2e_4 & e_1 + 2e_2 + e_3 - e_4 & e_2 + 2e_3 - 2e_4 \end{pmatrix}$;

(iii) $\begin{pmatrix} -e_2 - 2e_3 & 2e_0 + e_2 - e_3 - 2e_4 & -e_1 - 2e_2 - e_3 + 2e_4 & e_0 - e_2 + e_3 - e_4 \\ 2e_2 + 2e_3 - e_4 & e_0 + 2e_2 - 2e_3 - e_4 & e_1 + e_2 + e_3 - 2e_4 & e_2 + 2e_3 + e_4 \end{pmatrix}$;

(iv) $\begin{pmatrix} -e_1 + e_4 & -2e_0 + e_1 - e_2 - e_4 & e_1 + e_2 - 2e_3 + 2e_4 & e_0 + 2e_2 - 2e_3 + e_4 \\ -2e_1 - 2e_2 - 2e_3 + 2e_4 & e_0 + 2e_1 - 2e_2 - e_3 & e_1 + e_2 + 2e_3 - 2e_4 & -2e_2 - e_4 \end{pmatrix}$.

A `MACAULAY2` script for finding these A_1's can be obtained from [1]. For each matrix, we can show by (2), (3) and (4) that there is a smooth rational surface in $\mathbb{P}^4$ with the desired invariants. The type of a linear system embedding the surface into $\mathbb{P}^4$ can be determined as in Section 2.2. □

Remark 3.8. (i) To find the matrices given in the proof of Proposition 3.7 more effectively, we take the following extra steps: For two fixed vectors v_1 and v_2 that are contained in V_B, choose a 2×2 matrix A_1' with linear entries randomly to make the augmented matrix $A_1 = (v_1, v_2, A_1')$. Then compute N. In this case, N should be less than or equal to 118 by Corollary 3.5. If $N \leq 117$, proceed with steps (2) to (4) to check whether a smooth surface is found.

(ii) Let $A_1 \in \mathrm{Hom}_E(4E(3), 2E(2))$ with $114 \leq N \leq 117$ satisfying (c_1) and (c_2). By Proposition 3.7, we may assume that there are elements of $\mathrm{Hom}_E(2E(2), 3E)$ obtained by random choices of values for $140 - N$ parameters that give rise to smooth surfaces in $\mathbb{P}^4$ with $d = 12$ and $\pi = 13$. Let B_2 and B_2' be such elements of $\mathrm{Hom}_E(2E(2), 3E)$. Then the corresponding monads are isomorphic if and only if B_2 and B_2' differ only by a constant (modulo U_B). This is equivalent to the random choices for B_2 and B_2' are the same up to constant. It turns out, therefore, that the family of smooth surfaces obtained in this way has dimension $(140 - N - 1) - 20 = 119 - N$.

(iii) For each A_1 given in the proof of Proposition 3.7, we can check that Z_A and Z_B intersect in $120 - N$ points. So, for a general choice, the equality in Lemma 3.4 holds, and hence the codimension of $\mathfrak{F}(N)$ in $\mathfrak{F}$ is expected to be $120 - N$.

The proof of Lemma 3.2 suggests the existence of a 2×4 matrix A_1 with entries from V such that C_A is smooth and Z_A intersects Z_B in more than six points if we allow C_A to intersect S_B.

Proposition 3.9. *There exists a smooth rational surface in $\mathbb{P}^4$ over $\overline{\mathbb{F}}_3$ with $d = 12$ and $\pi = 13$ embedded by*

$$\left| 12L - 4E_1 - \sum_{i_2=2}^{13} 3E_{i_2} - \sum_{i_4=13}^{21} E_{i_4} \right|.$$

Proof. We can find an $A_1 \in \mathrm{Hom}_E(4E(3), 3E(2))$ such that C_A is smooth and Z_A intersects Z_B in seven points over $\mathbb{F}_3$ by random search:

$$\text{(v)} \quad \begin{pmatrix} -e_4 & -e_2 - e_3 + e_4 & -e_1 & e_0 - e_1 - e_2 + e_3 + e_4 \\ -e_2 - e_3 + e_4 & e_0 + e_1 + e_2 + e_3 - e_4 & e_2 & -e_1 - e_2 + e_3 - e_4 \end{pmatrix}.$$

In this example, $N = 113$. So the codimension of $\mathfrak{F}(N)$ in $\mathfrak{F}$ is expected to be 7. For this A_1, there is a smooth rational surface X in $\mathbb{P}^4$ with the desired invariants. The type of a linear system embedding X into $\mathbb{P}^4$ can be determined by the adjunction theory (see Section 2.2). □

4. Lift to characteristic 0

In the previous section, we constructed smooth rational surfaces in $\mathbb{P}^4$ over a small field. We want, however, to find examples defined over the complex numbers $\mathbb{C}$. In this section, we show the existence of a lift to characteristic 0, using an argument due to Schreyer, as follows: Let $\mathfrak{F}$ and $\mathfrak{F}(N)$ be given as in §3.

Lemma 4.1 ([11]). *Let A_1 be a point of $\mathfrak{F}(N)$, where $\mathfrak{F}(N)$ has codimension $120 - N$. Then there exists a number field $\mathbb{L}$ and a prime $\mathfrak{p}$ in $\mathbb{L}$ such that the residue field $\mathcal{O}_{\mathbb{L},\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathbb{L},\mathfrak{p}}$ is in $\mathbb{F}_p$. Furthermore, if the surface $X/\mathbb{F}_p$ corresponding to A_1 is smooth, then the surface $X/\mathbb{L}$ corresponding to the generic point $\mathrm{Spec}\,\mathbb{L} \subset \mathrm{Spec}\,\mathcal{O}_{\mathbb{L},\mathfrak{p}}$ is also smooth.*

Proof. Let p be a prime number. If this is not the case, $\mathbb{Z}$ has to be replaced by the ring of integers in a number field which is $\mathbb{F}_p$ as the residue field.

Since $\mathfrak{F}(N)(\mathbb{F}_p)$ has pure codimension $120 - N$ in A_1, there are hyperplanes $H_1, \ldots, H_{120-N}$ in $\mathfrak{F}(\mathbb{F}_p)$ such that A_1 is an isolated point of $\mathfrak{F}(N)(\mathbb{F}_p) \cap H_1 \cap \cdots \cap H_{120-N}$. We may assume that $H_1, \ldots, H_{120-N}$ are defined over $\mathrm{Spec}\,\mathbb{Z}$ and that they meet transversally in A_1. This allows us to think that $\mathfrak{F}(N)(\mathbb{F}_p) \cap H_1 \cap$

$\cdots \cap H_{120-N}$ is defined over $\mathbb{Z}$. Let Z be an irreducible component of $\mathfrak{F}(N)(\mathbb{Z})$ containing A_1. Then $\dim Z = 1$.

The residue class field of generic point of Z is a number field $\mathbb{L}$ that is finitely generated over $\mathbb{Q}$, because $\mathfrak{F}(N)(\mathbb{Z})$ is projective. Let $\mathcal{O}_{\mathbb{L}}$ be the ring of integers of $\mathbb{L}$ and let $\mathfrak{p}$ be a prime ideal which lies over $A_1 \in Z$. Then $\operatorname{Spec} \mathcal{O}_{\mathbb{L},\mathfrak{p}} \to Z \subset \mathfrak{F}(N)(\mathbb{Z})$ is an $\mathcal{O}_{\mathbb{L},\mathfrak{p}}$-valued point which lifts A_1. The residue class field $\mathcal{O}_{\mathbb{L},\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathbb{L},\mathfrak{p}}$ is a finite extension of $\mathbb{F}_p$.

Performing the construction of the surface over $\mathcal{O}_{\mathbb{L},\mathfrak{p}}$ gives a flat family $\mathcal{X}$ of surfaces over $\mathcal{O}_{\mathbb{L},\mathfrak{p}}$. Since smoothness is an open property, and since the special fiber $\mathcal{X}_{\mathfrak{p}}$ is smooth, the general fiber $\mathcal{X}_{\mathbb{L}}$ is also smooth. □

Next we argue that the adjunction process of the surface over the number field $\mathbb{L}$ has the same numerical behavior:

Proposition 4.2 ([6]). *Let $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_{\mathbb{L},\mathfrak{p}}$ be a family as in Proposition* 3.7 *or* 3.9*. If the Hilbert polynomial of the first adjoint surface of $X = \mathcal{X} \otimes \mathbb{F}_q$ is as expected, and if $\mathrm{H}^1(X, \mathcal{O}_X(-1)) = 0$, then the adjunction map of the general fiber $\mathcal{X}_{\mathbb{L}}$ blows down the same number of exceptional lines as the adjunction map of the spacial fiber X.*

Proof. See Corollary 8.4 in [6] for a proof. □

Theorem 4.3. *There are at least five different families of smooth rational surfaces in $\mathbb{P}^4$ over $\mathbb{C}$ with $d = 12$ and $\pi = 13$.*

Proof. Let A_1 be one of (i)-(iv) given in the proof of Proposition 3.7 or (v) given in Proposition 3.9. Suppose that $p = 3$ if A_1 is (v) and otherwise $p = 5$. By Lemma 4.1, it suffices to show that for each $113 \le N \le 117$, the subfamily $\mathfrak{F}(N)$ of $\mathfrak{F}$ has the desired codimension at this A_1. Let P be the set of isomorphic classes of monads of type

$$4\Omega^3(3) \to 2\Omega^2(2) \oplus 2\Omega^1(1) \to 3\mathcal{O},$$

let (A_B, B) be the pair obtained from A_1 by (1), (2) and (3) and let $T_{P,(A_B,B)}$ be the Zariski tangent space of P at the point corresponding to (A_B, B). Let

$$\begin{aligned} V_1 &= \operatorname{Hom}(4\Omega^3(3), 2\Omega^2(2) \oplus 2\Omega^1(1)) \simeq 8V \oplus 8 \bigwedge^2 V, \\ V_2 &= \operatorname{Hom}(2\Omega^2(2) \oplus 2\Omega^1(1), 3\mathcal{O}) \simeq 6 \bigwedge^2 V \oplus 6V, \\ V_3 &= \operatorname{Hom}(4\Omega^3(3), 3\mathcal{O}) \simeq 12 \bigwedge^3 V. \end{aligned}$$

Consider the map

$$\phi : V_1 \oplus V_2 \to V_3$$

defined by $(A', B') \mapsto B' \circ A'$. Let $A_B \in V_1$ and $B \in V_2$ be the differentials of the monad for X given in (1). The differential $d\phi : V_1 \oplus V_2 \to V_3$ of the map ϕ at the point (A_B, B) is given by $(A', B') \mapsto B \circ A' + B' \circ A_B$. Consider the subset $\tilde{P}$ of $\phi(0)$ whose elements give monads of type (3.1). This forms an open subset of $\phi(0)$. Let H denote the group

$$\left\{ \begin{pmatrix} C & 0 \\ v & D \end{pmatrix} \middle| C, D \in GL(2, \mathbb{F}_p), v \in GL(2, V) \right\}$$

and let $G = GL(4, \mathbb{F}_p) \times H \times GL(3, \mathbb{F}_p)$. Then $G' = G/\mathbb{F}_p^\times$ acts on $\tilde{P}$ by

$$(A', B')(f, g, h) = \left(g \circ A' \circ f^{-1}, h \circ B' \circ g^{-1}\right).$$

Let P be the set of isomorphic classes of monads of type (3.1) and let $T_{P,(A_B,B)}$ be the Zariski tangent space of P at the point corresponding to (A_B, B). Then $P \simeq \tilde{P}/G'$, and hence $\dim\left(T_{P,(A_B,B)}\right) = \dim\left((d\phi)^{-1}(0)/G'\right)$. For fixed bases of V_1, V_2 and V_3, we can give the matrix that represents the differential $d\phi$ explicitly. This matrix enables us to compute the kernel of $d\phi$. This computation can be done with `MACAULAY2`:

$$\dim\left(T_{P,(A_B,B)}\right) = \dim\left((d\phi)^{-1}(0)\right) - \dim(G') = 90 - (53 - 1) = 38.$$

A `MACAULAY2` script for this computation can be found in [1]. From Remark 3.8 (ii), the dimension of $\mathfrak{F}(N)$ is therefore

$$\dim(\mathfrak{F}(N) = \dim(T_{P,(A_B,B)}) - (18 + (119 - N)) = N - 99.$$

Recall that the parameter space of rational normal curves in $\mathbb{P}(V)$ has dimension 21. Thus

$$\operatorname{codim}(\mathfrak{F}(N), \mathfrak{F}) = 21 - (N - 99) = 120 - N.$$

The type of very ample divisor that embeds the surface into $\mathbb{P}^4$ follows from Proposition 4.2, which completes the proof. $\square$

References

[1] H. Abo, Macaulay 2 scripts for finding rational surfaces in $\mathbb{P}^4$ with degree 12. Available at `http://www.math.colostate.edu/~abo/programs.html`.

[2] Beilinson, A. *Coherent sheaves on $\mathbb{P}^N$ and problems of linear algebra*, Funct. Anal. Appl. **12** (1978), 214–216.

[3] H. Abo and F.-O. Schreyer, *Exterior algebra methods for the construction of rational surfaces in $\mathbb{P}^4$*, in preparation

[4] W. Decker, L. Ein and F.-O. Schreyer, *Construction of surfaces in $\mathbb{P}^4$*, J. Algebr. Geom. **2** (1993) 185–237

[5] W. Decker and D. Eisenbud, *Sheaf algorithms using the exterior algebra*, In: D. Eisenbud, D.R. Grayson, M.E. Stillman, B. Sturmfels (Eds), Computations in Algebraic Geometry with *Macaylay 2*, Algorithms Comput. Math. **8** Springer, Berlin (2002) 215–249

[6] W. Decker and F. -O. Schreyer, *Non-general type surface in $\mathbb{P}^4$: some remarks on bounds and constructions*, J. Symbolic Comput., **25** (2000) 545–582

[7] D. Eisenbud, G. Fløystad and F.-O. Schreyer, *Sheaf cohomology and free resolutions over exterior algebras*, Trans. Amer. Math. Soc. **355** (2003) 4397–4426

[8] G. Ellingsrud and C. Peskine *Sur les surfaces lisse de $\mathbb{P}^4$*, Invent. Math. **95** (1989) 1–12

[9] D. Grayson and M. Stillman, (1991) Macaulay 2, a software system for research in algebraic geometry. Available at `http://www.math.uiuc.edu/Macaulay2`.

[10] A. J. Sommese and A. Van de Ven, *On the adjunction mapping*, Math. Ann. **278** (1987) 593–603.

[11] F.-O. Schreyer, *Small fields in constructive algebraic geometry*, Lecture Notes in Pure and Appl. Math., **179** (1996) 221–228.