Mathematical Modeling

Lesson 4 – Discrete Models

Modeling

Process of creating a representation of a system or process.

- Physical Object
  - Ex. Model of a building
- Abstract Concept
  - Ex. Mathematical Equation

Types of Models

- Physical Models
- Mathematical Models
- Computational Models

Physical Model

tangible representations of objects or systems

Mathematical Model

his is an abstract, symbolic representation of a system using mathematical concepts and equations.

Computational Model

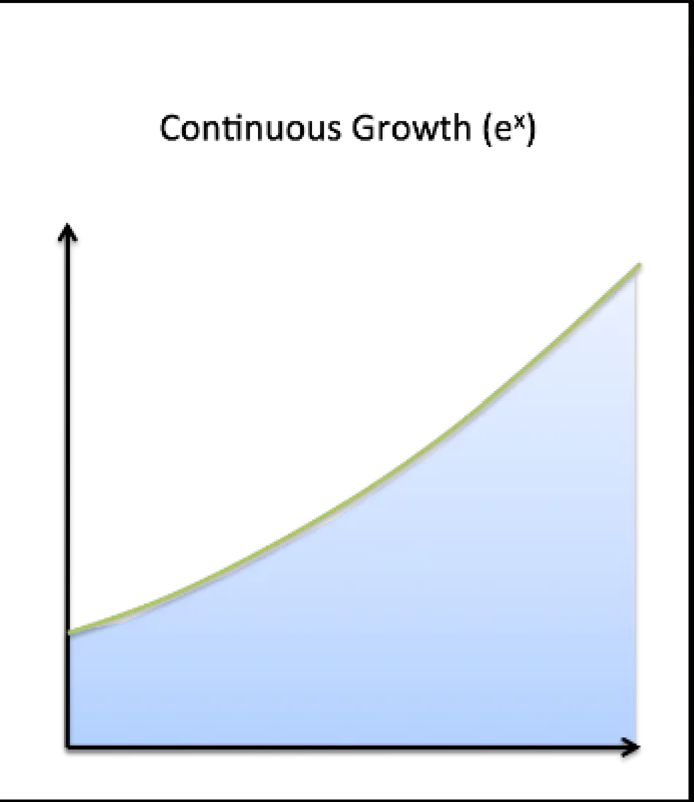
This type of model uses algorithms and numerical methods to simulate complex systems on computers.

Mathematical Model

his is an abstract, symbolic representation of a system using mathematical concepts and equations.

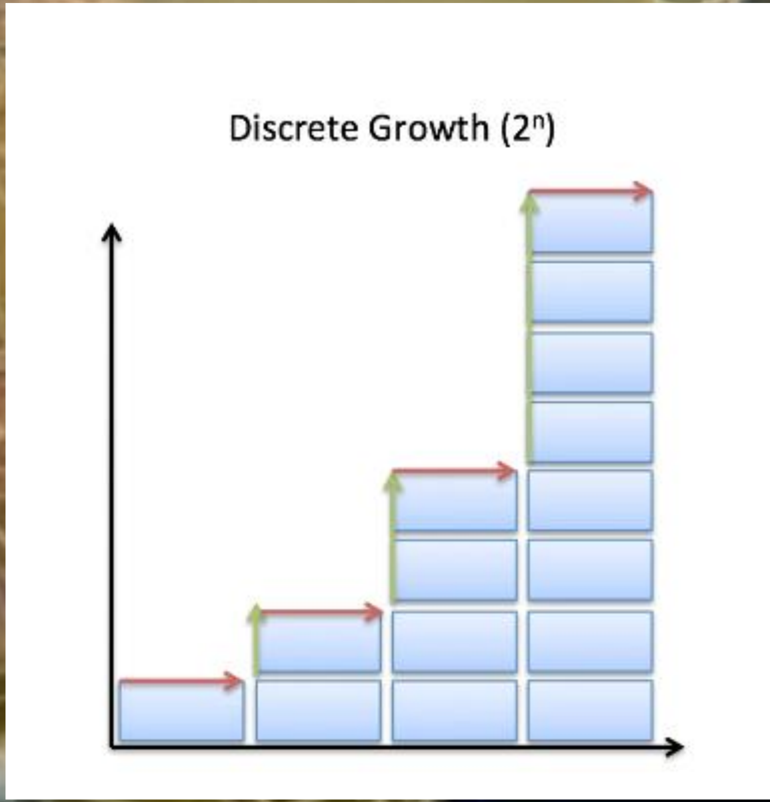
Continuous model

- is one where the variables change smoothly over time, without any jumps or interruptions.
- ideal for representing processes that evolve gradually and can be described using differential equations



Discrete model

- deal with systems where changes occur at specific intervals or steps.
- useful when we're looking at situations where events happen one after the other.



Why Use Continuous Simulation?

- use continuous simulation as it depends on differential equation of various parameters associated with the system and their estimated results known to us.

Application Areas

- **Civil engineering** for the construction of dam embankment and tunnel constructions
- **Military applications** for simulation of missile trajectory, simulation of fighter aircraft training, and designing & testing of intelligent controller for underwater vehicles.
- **Logistics** for designing of toll plaza, passenger flow analysis at the airport terminal, and proactive flight schedule evaluation.
- **Business development** for product development planning, staff management planning, and market study analysis

Formulate a continuous model for the rate at which a drug is metabolized in the bloodstream.

Step 1: Understand the Biological Process

Step 2: Assumptions

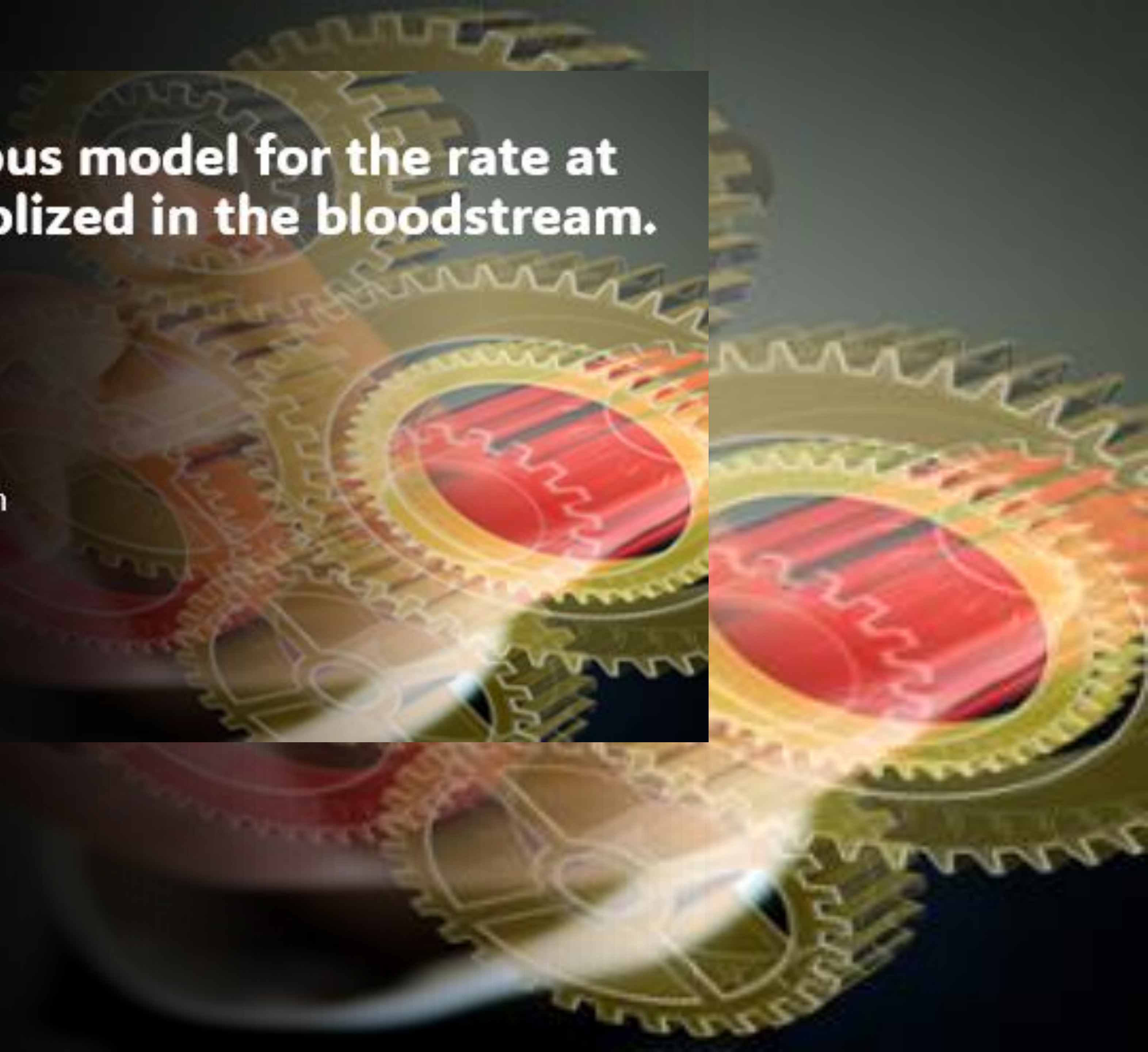
Step 3: Define the Variables

Step 4: Formulate the Differential Equation

Step 5: Solve the Differential Equation

Step 6: Interpretation of the Model

Step 7: Real-World Application



Assignment
Formulating a Continuous Model

Scenario:

A population of bacteria grows in a controlled laboratory environment. The population grows exponentially over time and can be modeled using a continuous function. However, due to limited resources, the growth rate decreases over time as the population approaches the environment's carrying capacity.

1. **Formulate a continuous model that describes the growth of the bacterial population.**
    - Start with the basic exponential growth model and modify it to account for the decreasing growth rate.
    - Define all variables and parameters used in your model.

2. **Solve the differential equation for your model.**
    - Find the general solution for the population at any time t.

3. **Interpret the solution of your model.**
    - Explain what your solution means in terms of the population's behavior over time. Discuss how the population changes as it approaches the carrying capacity.

Reflection

Reflect on the challenges of working with continuous models.

Discuss any difficulties you encountered while formulating, solving, or interpreting continuous models. How did you overcome these challenges? What did you learn from the process?

Mathematical Modeling

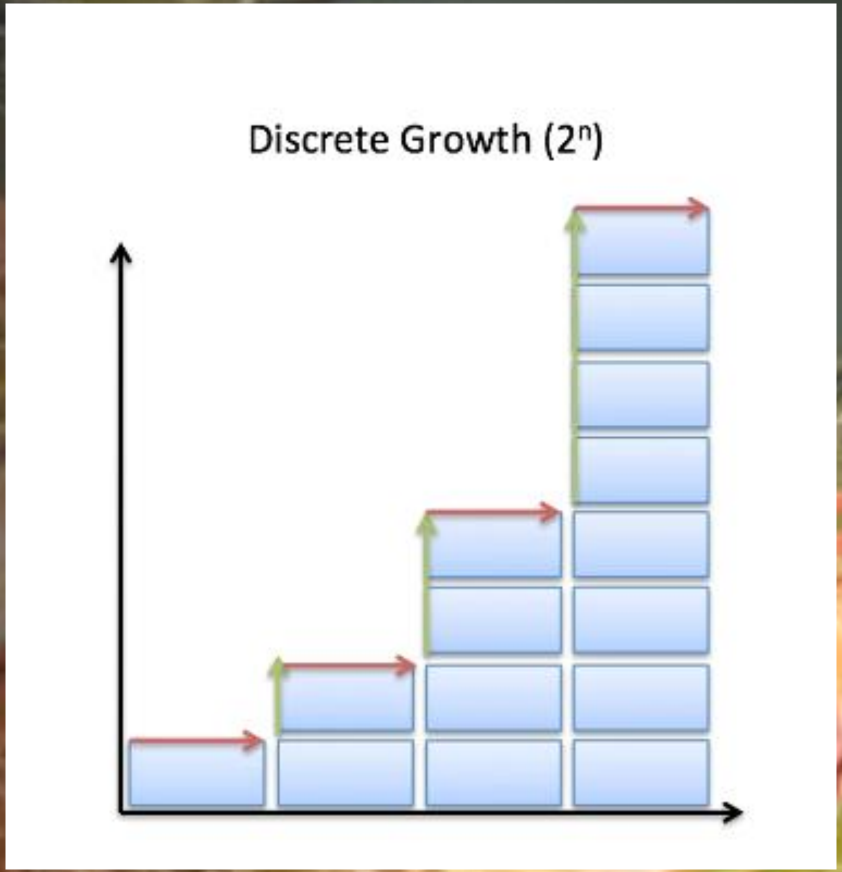
Lesson 4 – Discrete Models

Objectives:

- Understand the concept of discrete models and how they differ from continuous models.
- Define and identify components of a discrete model.
- Apply discrete models to real-world problems, particularly in computer simulations.
- Create simple discrete models using difference equations.

Discrete model

- A model where variables change at specific intervals.



graphical representation of its behavior.

Discrete model

- Discrete models use difference equations instead of differential equations.

Example: Consider the growth of a population that increases by a fixed percentage each year. A simple **Discrete Population Growth Model** looks like this:

$$P_{n+1} = P_n + rP_n$$

Where:

- P_n is the population at time step n,
- r is the growth rate (as a decimal, e.g., 0.05 for 5% growth).

Example Problem: Population Growth Model

- Initial Population: 1,000 people
- Growth rate: 10% per year.
- Calculate the population for
- the next 3 years.

$$P_{n+1} = P_n + rP_n$$

First year:

$$P_1 = 1000 + (0.10 \times 1000) = 1000 + 100 = 1100$$

Second year:

$$P_2 = 1100 + (0.10 \times 1100) = 1100 + 110 = 1210$$

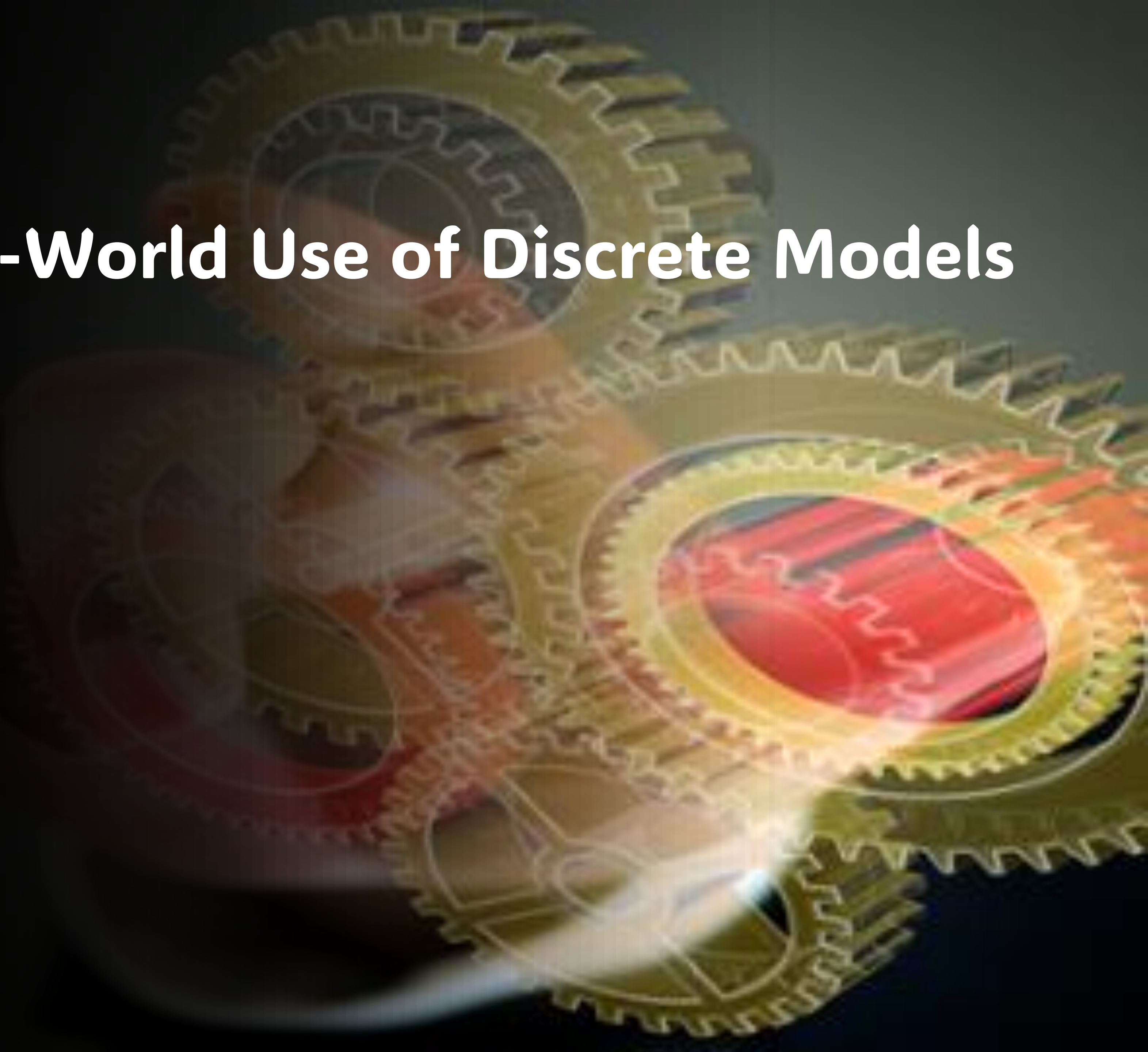
Third year:

$$P_3 = 1210 + (0.10 \times 1210) = 1210 + 121 = 1331$$

- Result: The population after 3 years is 1331.

Application: Real-World Use of Discrete Models

- simulations of network traffic,
- financial modeling,
- algorithms for machine learning



Discrete Model for a Queueing System in a Barber Shop

- In a barber shop, customers arrive at different times, wait in a queue, and are served one by one. We can model this scenario using a discrete model, where time is divided into specific intervals (e.g., minutes or hours), and customer arrivals and services are handled at each interval.

Discrete Model for a Queueing System in a Barber Shop

Assumptions for the Model:

1. **Customer arrival rate:** A customer arrives at the shop every 10 minutes (on average).
2. **Service rate:** The barber takes 15 minutes to serve each customer.
3. **Queue capacity:** The shop can hold a maximum of 5 customers waiting in line.
4. **Working hours:** The barber shop is open for 8 hours per day.
5. **Initial conditions:** At the start of the day, the queue is empty, and the barber is ready to serve.

Discrete Model for a Queueing System in a Barber Shop

Model Components:

1. **Time Steps:** Each time step represents 1 minute.
2. **State Variables:**
   - Q_n: Number of customers in the queue at time step n.
   - S_n: Number of customers being served at time step n (0 or 1).
3. **Arrival and Service Conditions:**
   - Every 10 minutes, a customer arrives if the queue isn't full.
   - Every 15 minutes, the barber completes the service of one customer.
4. **Rules:**
   - If the queue is full (5 customers), new arrivals are turned away.
   - The next customer from the queue starts getting served when the barber finishes with the current one.

Discrete Model for a Queueing System in a Barber Shop

Discrete Model Equations:

1. **Customer Arrival:**

   Let A_n represent the customer arrival at time step n, where $A_n = 1$ if a customer arrives, and $A_n = 0$ otherwise. This depends on whether $n \mod 10 = 0$ (customer arrives every 10 minutes).

$$A_n = \begin{cases} 1 & \text{if } n \mod 10 = 0, \text{ and } Q_n < 5 \\ 0 & \text{otherwise} \end{cases}$$

Discrete Model for a Queueing System in a Barber Shop

2. **Customer Service:**

   Let C_n represent customer service completion at time step n, where $C_n = 1$ if a customer is finished, and $C_n = 0$ otherwise. This happens every 15 minutes ($n \mod 15 = 0$).

$$C_n = \begin{cases} 1 & \text{if } n \mod 15 = 0 \text{ and } S_n = 1 \\ 0 & \text{otherwise} \end{cases}$$

3. **Queue Update:**

   The queue size Q_{n+1} depends on the number of new arrivals and completed services:

$$Q_{n+1} = Q_n + A_n - C_n$$

Discrete Model for a Queueing System in a Barber Shop

4. **Service Update:**

   If the barber finishes serving a customer, the next customer in the queue starts getting served:

$$S_{n+1} = \begin{cases} 1 & \text{if } C_n = 1 \text{ and } Q_n > 0 \\ 0 & \text{if no customers are in the queue} \\ 1 & \text{if still serving a customer} \end{cases}$$

Discrete Model for a Queueing System in a Barber Shop

Simulate the barber shop queue for the first 30 minutes

- Initial Conditions: $Q_0 = 0, S_0 = 0$

Time Step 0:

- No customer in queue, no one is being served.
- $A_0 = 0, C_0 = 0$
- $Q_1 = Q_0 + A_0 - C_0 = 0 + 0 - 0 = 0$

Discrete Model for a Queueing System in a Barber Shop

Simulate the barber shop queue for the first 30 minutes

Time Step 10 (A customer arrives):

- $A_{10} = 1$, customer arrives.
- $C_{10} = 0$, no service is complete.
- $Q_{11} = Q_{10} + A_{10} - C_{10} = 0 + 1 - 0 = 1$

$$A_n = \begin{cases} 1 & \text{if } n \mod 10 = 0, \text{ and } Q_n < 5 \\ 0 & \text{otherwise} \end{cases}$$

$$C_n = \begin{cases} 1 & \text{if } n \mod 15 = 0 \text{ and } S_n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$Q_{n+1} = Q_n + A_n - C_n$$

Discrete Model for a Queueing System in a Barber Shop

Simulate the barber shop queue for the first 30 minutes

Time Step 15 (A customer is served):

- $A_{15} = 0$ (no new customer at minute 15).
- $C_{15} = 1$, service is complete.
- $Q_{16} = Q_{15} + A_{15} - C_{15} = 1 + 0 - 1 = 0$
- $S_{16} = 0$ (no customer in queue to serve).

$$A_n = \begin{cases} 1 & \text{if } n \mod 10 = 0, \text{ and } Q_n < 5 \\ 0 & \text{otherwise} \end{cases}$$

$$C_n = \begin{cases} 1 & \text{if } n \mod 15 = 0 \text{ and } S_n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$Q_{n+1} = Q_n + A_n - C_n$$

Discrete Model for a Queueing System in a Barber Shop

Visualization of the Model

Time (min)	Queue Size Q_n	Customer Arrival A_n	Service Completion C_n	Customer Being Served S_n
0	0	0	0	0
10	1	1	0	1
15	0	0	1	0
20	1	1	0	1
30	0	1	1	0

$$A_n = \begin{cases} 1 & \text{if } n \mod 10 = 0, \text{ and } Q_n < 5 \\ 0 & \text{otherwise} \end{cases}$$

$$C_n = \begin{cases} 1 & \text{if } n \mod 15 = 0 \text{ and } S_n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$S_{n+1} = \begin{cases} 1 & \text{if } C_n = 1 \text{ and } Q_n > 0 \\ 0 & \text{if no customers are in the queue} \\ 1 & \text{if still serving a customer} \end{cases}$$

Quiz

1. What is the main characteristic of a discrete model?

a) Changes occur continuously over time

b) Changes occur at specific intervals

c) Changes occur randomly over time

d) Changes do not occur at all

Quiz

2. Which of the following is NOT an example of a discrete model?

a) Inventory restocking every week

b) Population growth measured every year

c) Vehicle speed increasing gradually over time

d) Website visitors tracked every hour

Quiz

3. In a discrete model for population growth, what does the term $P_{n+1} = P_n + rP_n$ represent?

a) Population decreases over time

b) Population stays constant

c) Population at the next time step

d) Population divided into smaller groups

Quiz

4. Which of the following best describes a difference equation in a discrete model?

a) An equation that models continuous change

b) An equation used to solve calculus problems

c) An equation that relates the change between consecutive time steps

d) An equation that models random behavior

Quiz

5. In a queueing system, if customers arrive every 5 minutes and the server takes 10 minutes to serve each customer, how many customers will be in the queue after 30 minutes if no customer leaves?

a) 6

b) 3

c) 12

d) 0

Quiz

6. In a discrete-time model, what does the time step represent?

a) The total number of changes in the system

b) The variable that measures continuous changes

c) The intervals at which the system's state is updated

d) The amount of randomness in the system

Quiz

7. Which of the following is true for a system modeled by a discrete event simulation?

a) The system evolves continuously without interruption

b) The system evolves at specific points when events occur

c) The system's state changes randomly without any specific intervals

d) The system cannot evolve over time

Quiz

8. If a bank account earns 2% interest annually, compounded yearly, what will the balance be after 3 years if the initial balance is $1000? Use the formula $B_{n+1} = B_n + rB_n$, where $r = 0.02$.

a) $1060

b) $1120.80

c) $1061.20

d) $1104.08

Quiz

9. What happens to a discrete model's state variables during each time step?

a) They remain constant

b) They change based on specific rules or equations

c) They change continuously over time

d) They are reset to zero

Quiz

10. Why are discrete models commonly used in computer simulations?

a) Computers can only handle continuous data

b) Computers are unable to model changes at specific intervals

c) Computers are ideal for simulating systems that change in steps

d) Discrete models don't require any equations

Recap of Key Points

- Definition of a Discrete Model
- Difference from Continuous Models
- Key Components: Time Steps, State Variables
- Examples of Discrete Models
- Real-World Applications

Thank You.