Let X/S be smooth connected of relative dimension d.

At some point, we will assume $S = \operatorname{Spec} k$, with $k = \bar{k}$ of characteristic p, and X/S smooth proper connected group scheme of (relative) dimension d, i.e., abelian variety. Essential assumptions will be point out where it is used.

$$T_{X^{(1)}/S}^* \qquad \mathscr{D}_{X/S} := \widetilde{F_{X/S,*}D_{X/S}}$$
 sheaf of PD-differential operators (of level 0)
$$X \xrightarrow{F_{X/S}} X^{(1)} \qquad F_{X/S,*}D_{X/S}$$

$$P_{X/S} = D_{X/S}^{(0)}$$

 $F_{X/S}$ is flat locally free of rank p^d . $\mathcal{D}_{X/S}$ is an Azumaya algebra over $T_{X/S}^*$ of rank p^{2d} . For quasi-coherent \mathcal{O}_S -module \mathcal{E} ,

$$\mathbf{V}(\mathcal{E}) := \operatorname{Spec} \operatorname{Sym}^{\bullet} \mathcal{E}^{\vee} \to S$$

1 Morita

 $D \cong \operatorname{End}_R(P)$, then

$$\operatorname{Mod}_R \longleftrightarrow \operatorname{Mod}_D$$

$$F \longmapsto P \otimes_R F$$
 $\operatorname{Hom}_A(P,G) \longleftrightarrow G$

2 Spectral Cover

Fix an θ : \mathcal{O}_X -linear map $E \to E \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$, with rk E = n.

Tautological section

$$\lambda: \mathcal{O}_{\mathsf{T}^*_{X/S}} \to \pi^*\Omega^1_{X/S},$$

tensored with π^*E :

$$\lambda: \pi^*E \to (\pi^*E) \otimes_{\mathcal{O}_{\mathcal{T}^*_{X/S}}} (\pi^*\Omega^1_{X/S})$$

Pullback θ to $T_{X/S}^*$:

$$\theta:\pi^*E\to (\pi^*E)\otimes_{\mathcal{O}_{\mathcal{T}_{X/S}^*}}(\pi^*\Omega^1_{X/S})$$

Then

$$\wedge^n E \xrightarrow{ \ \ \, \wedge^n} \ \, \wedge^n (\pi^* E \otimes \pi^* \Omega^1_{X/S}) \xrightarrow{ \ \ \, } \ \, \wedge^n (\pi^* E) \otimes \operatorname{Sym}^n (\pi^* \Omega^n_{X/S}).$$

Thus get

$$\begin{split} \chi(\theta) \in \Gamma(\mathbf{T}^*_{X/S}, \operatorname{Sym}^n(\pi^*\Omega^1_{X/S})) &= \Gamma(\mathbf{T}^*_{X/S}, \pi^*\operatorname{Sym}^n\Omega^1_{X/S}) \\ &= \Gamma(X, \operatorname{Sym}^n\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \operatorname{Sym}^{\bullet}(\Omega^1_{X/S})^{\vee}) \end{split}$$

Note that $\pi^*\operatorname{Sym}^n\Omega^1_{X/S}$ is locally free of rank $\binom{d+n-1}{n}$ over $\operatorname{T}^*_{X/S}$. So the global section $\chi(\theta)$ defines a closed subscheme

$$i: Y_{\chi} \hookrightarrow T^*_{X/S}$$

i.e., the vanishing locus of $\chi(\theta)$.

2.1 More details

 $\chi(\theta)$ can be written as

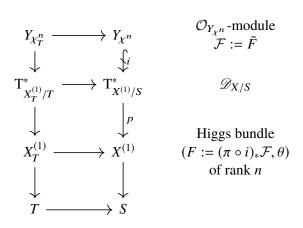
$$\lambda^n - a_1 \lambda^{n-1} + \dots + (-1)^n a_n$$

with $a_i \in \Gamma(X, \operatorname{Sym}^i \Omega^1_{X/S}) \to \Gamma(\operatorname{T}^*_{X/S}, \pi^* \operatorname{Sym}^i \Omega^1_{X/S})$ corresponds to the *trace* of the following map

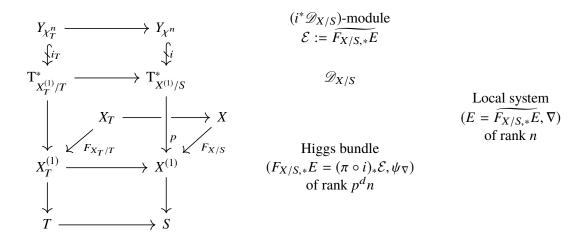
$$\wedge^i(E) \xrightarrow{\wedge^i} \wedge^i(E \otimes_{\mathcal{O}_X} \Omega^1_{X/S}) \longrightarrow \wedge^i E \otimes_{\mathcal{O}_X} \operatorname{Sym}^i \Omega^1_{X/S}.$$

3 BNR for Higgs bundles

We have equivalence of categories



4 BNR for local systems



4.1 Two ways to get the degree n characteristic polynomial

- 1. Consider the F-Higgs bundle $E \to E \otimes_{\mathcal{O}_X} F_{X/S}^* \Omega_{X^{(1)}/S}^1$, whose characteristic polynomial is of order n, but coefficients a priori lie in $\Gamma(X, F_{X/S}^* \operatorname{Sym}^n(\Omega^1_{X^{(1)}/S}))$. Then use a fact by N. Katz which stating that $[\psi_{\nabla}(D), \nabla_{D'}] = 0$ for any $D, D' \in (\Omega^1_{X/S})^{\vee}$. Then Cartier descent shows that the χ is horizontal with respect to the canonical connection $\nabla^{\operatorname{can}}$ of $F_{X/S}^* \operatorname{Sym}^n(\Omega^1_{X^{(1)}/S})$.
- 2. Consider the Higgs bundle $(F_{X/S,*}E) \to (F_{X/S,*}E) \otimes_{\mathcal{O}_{X^{(1)}}} \Omega^1_{X^{(1)}/S}$. Its characteristic polynomial has coefficients in $\Gamma(X^{(1)},\Omega^1_{X^{(1)}})$, but it has degree $p^d n$. One use Morita and BNR for Higgs to show $\chi=(\chi')^{p^d}$ for some χ' with deg $\chi'=n$.

5 Correspondence

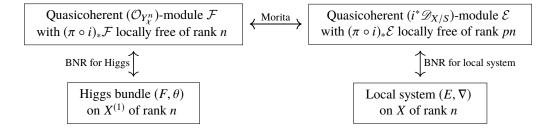
Given

$$\chi^n: S \to \mathbf{V}\left(\bigoplus_{i=1}^n \Gamma(X^{(1)}, \operatorname{Sym}^i \Omega^1_{X^{(1)}/S})\right)$$

It determines $\chi^n \in \Gamma(\mathrm{T}^*_{X/S}, \pi^*\operatorname{Sym}^n\Omega^1_{X^{(1)}/S})$, and $i:Y_{\chi^n} \hookrightarrow \mathrm{T}^*_{X^{(1)}/S}$.

$$i^* \mathcal{D}_{X/S} \cong \mathcal{E}nd_{Y_{\chi^n}}(\mathcal{P}),$$

then



5.1 General case

The above works for any test scheme $T \rightarrow S$ with

$$\chi_T^n: T \to \mathbf{V}\left(\bigoplus_{i=1}^n \Gamma(X^{(1)}, \operatorname{Sym}^i \Omega^1_{X^{(1)}/S})\right)$$

6 Existence of Splittings

Assume X/S abelian variety, $S = \operatorname{Spec} k$.

6.1 Line Bundle Case

 $\chi: S \to \mathbf{V}\big(\bigoplus_{i=1}^n \Gamma(X^{(1)}, \operatorname{Sym}^i \Omega^1_{X^{(1)}/S})\big)$ over S defines a closed subscheme $Y_{\chi} \hookrightarrow T^*_{X^{(1)}/S}$. (This is just the closed subscheme of the cotangent bundle of $X^{(1)}/S$ defined by a one-form.)

Fact 1. $i^* \mathcal{D}_{X/S}$ splits if and only if χ factors through

$$\psi: \operatorname{Pic}_{X/S}^{\natural} \to \mathbf{V}(\Gamma(X^{(1)}, \Omega^1_{X^{(1)}/S})).$$

(This works for general $\chi_T: T \to \mathbf{V}\big(\bigoplus_{i=1}^n \Gamma(X^{(1)}, \operatorname{Sym}^i \Omega^1_{X^{(1)}/S})\big).$)

Fact 2. ψ is smooth surjective.

Conclusion 1. These facts imply that for any *S*-point χ (i.e., k-point), $i^*\mathcal{D}_{X/S}$ splits over the formal neighborhood of Y_{χ} .

6.2 General Case

 $\chi: S \to \mathbf{V}(\Gamma(X^{(1)}, \operatorname{Sym}^n \Omega^1_{X^{(1)}/S}))$ over S defines a closed subscheme $i: Y_\chi \to \mathrm{T}^*_{X^{(1)}/S}$.

Fact 3. The sheaf of relative differentials $\Omega^1_{X/S}$ is trivial as X is group scheme over $S = \operatorname{Spec} k$. Hence an isomorphism $\Omega^1_{X^{(1)}/S} \cong (\mathcal{O}_{X^{(1)}/S})^{\oplus d}$ gives isomorphisms

$$\pi^* \Omega^1_{X^{(1)}/S} \cong \left(\mathcal{O}_{\mathsf{T}^*_{X^{(1)}/S}} \right)^{\oplus d}, \qquad \pi^* \operatorname{Sym}^n \Omega^1_{X^{(1)}/S} \cong \left(\mathcal{O}_{\mathsf{T}^*_{X^{(1)}/S}} \right)^{\oplus \binom{d+n-1}{n}}, \\ \lambda \mapsto (\lambda^1, \dots, \lambda^d) \qquad \qquad \lambda^n \mapsto ((\lambda^1)^n, \dots, (\lambda^d)^n, \dots).$$

Fact 4. The set of global sections $\Gamma(X^{(1)}, \Omega^1_{X^{(1)}/S})$ consists of only constants, as X/S is proper connected.

Conclusion 2. For any *S*-point χ , i.e., a symmetric differential form, the closed subscheme Y_{χ} is a closed subscheme cut out by $\binom{d+n-1}{n}$ monic polynomial equations of degree n with coefficients in $k = \bar{k}$ (Fact 4).

Among these equations, d of them are of the form

$$(\lambda^i)^n - a_1^i (\lambda^i)^{n-1} + \dots + (-1)^n a_n^i, \qquad a_i^i \in k, \quad j = 1, \dots, n, i = 1, \dots, d.$$

where λ^i is the component of the tautological section of $\pi^*\Omega^1_{X^{(1)}/S}$ in the *i*-th part of the decomposition (Fact 3).

These equations can be always written as a product of $(\lambda^i - b^i_j)$'s as k is algebraically closed. These n equations defines a closed subscheme $Y_{\chi'}$ of $T^*_{\chi(1)/S}$ ²:

$$i: Y_{\chi} \hookrightarrow Y_{\chi'} \stackrel{i'}{\hookrightarrow} T^*_{X^{(1)}/S}$$

 $Y_{\chi'}$ is a union of d^n (counting multiplicity) closed subschemes defined by 1-forms. Hence $(i')^* \mathscr{D}_{X/S}$ splits over the formal neighborhood of $Y_{\chi'}$ hence³, $i^* \mathscr{D}_{X/S}$ splits over the formal neighborhood of Y_{χ} .

7 Further discussion

Can consider $\mathcal{D}_{X/S} = \varinjlim D_{X/S}^{(m)}$ -modules instead of local systems, where $\mathcal{D}_{X/S}$ is the differential operators in the sense of Grothendieck and $D_{X/S}^{(m)}$ is the differential operators of level m in the sense of Berthelot.

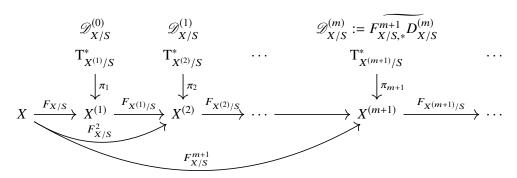
Fact 5. $D_{X/S}^{(0)}$ -modules is equivalent to flat connections. \mathcal{O}_X -coherent $\mathcal{D}_{X/S}$ -modules is equivalent to F-divided bundles/stractfied bundles.

 $^{{}^{1}\}Omega^{1}_{G/S} \cong \pi^{*}e^{*}\Omega^{1}_{G/S}$ for any group $\pi: G \to S$ with identity $e: S \to G$.

²Some computation evidence suggests that $Y_{\chi} = Y_{\chi'}$, i.e., the other equations are superfluous.

³Think about it.

Fact 6. $F_{X/S,*}^{m+1}D_{X/S}^{(m)}$ is an Azumaya algebra over $\mathrm{Sym}^{ullet}(\Omega^1_{X^{(m+1)}/S}).^4$



The so-called p^{m+1} -curvature of E on X gives a Higgs bundle on $X^{(m+1)}$.

⁴ See for example arXiv:0811.1168. See also Berthelot, Notes on