Torsors (and Gerbes)

some formal nonsense

1 Definition

References are [Stacks, Tags 03AH, 0497, 036Z, etc.], [Vis05] and [Gir71]. In this section, all results and definitions are quite formal. There is no algebraic geometry except when giving examples. Nevertheless one need to note that, in concrete settings in algebraic geometry, algebraic topology, differential geometry, one usually requires more for something to be a torsor.

1.1 Notations

$\mathtt{C} := (\mathtt{C}, \tau)$	a site: category $\mathtt{C}$ + (pre-)topology τ. *assumptions*: 1) $\mathtt{C}$ is *locally small*: $\mathrm{Mor}(X, Y)$ is a set for all $X, Y \in \mathrm{Obj}(\mathtt{C})$ 2) τ is *subcanonical*: $h_X := \mathrm{Mor}(-, X)$ is a sheaf for all $X \in \mathrm{Obj}(\mathtt{C})$.
$\mathtt{Sh}(\mathtt{C})$	a topos: sheaves of sets on $\mathtt{C}$.
$G, \mathcal{G}$	group objects in $\mathtt{C}$, $\mathtt{Sh}(\mathtt{C})$ respectively.
$X, \mathcal{X}$	objects in $\mathtt{C}$, $\mathtt{Sh}(\mathtt{C})$ respectively.
$\mathtt{C}/X$, $\mathtt{Sh}(\mathtt{C})/\mathcal{X}$	the slice of C, $\mathtt{Sh}(\mathtt{C})$ over X, $\mathcal{X}$ respectively, with the induced (pre-)topology.[1]
$\mathcal{F}\vert_X \in \mathrm{Obj}(\mathtt{Sh}(\mathtt{C}/X))$	the restriction of $\mathcal{F} \in \mathrm{Obj}(\mathtt{Sh}(\mathtt{C}))$ to the slice $\mathtt{C}/X$.

The assumptions on $(\mathtt{C}, \tau)$ may not be enough in some sense. Some people, for example in [Vis05, §4.4], further require that $\mathtt{C}$ admits (finite) limits/products and that $\mathtt{C}$ has a final object. These further assumptions will make the presentation much more cleaner. But we will put on these assumptions only when we need.

Recall the following basic definition. Let G be a group. A *G-torsor* is a pair (X, ρ), consisting of a (non-empty) set X and a G-action $\rho : X \times G \to X$, $(x, g) \mapsto \rho(x, g) =: x \cdot g$, such that ρ is *simply transitive*, i.e., for all $x, y \in X$, there is a unique $g \in G$, such that $x \cdot g = y$. A *homomorphism* of two G-torsors X and Y is a G-equivariant map. A homomorhpism is an *isomorphism of G-torsors* if it is bijective on the underlying sets (it follows automatically that its inverse is also G-equivariant). If X is an G-torsor, then any choice of an element $x \in X$ gives an isomorphism of G-torsors

$$G \to X, \qquad g \mapsto x \cdot g,$$

where G is the *trivial G-torsor* with its right translation map. Hence a morphism of G-torsors is necessarily an isomorphism, as they are both (non-canonically) isomorphism to the trivial G-torsor G.

Observe that if X is a G-torsor, then the map $X \times G \to X \times X$, $(x, g) \mapsto (x, x \cdot g)$ is bijective. Conversely, if this map is bijective, we can see that X is a G-torsor, as long as X is not empty.

1.2 Pseudo torsors

Now let us try to generalize the above basic concept to a *global* setting. This should be straight forward if we think a little bit; however, the direct generalization gives us the so-called pseudo torsors, as we will have to deal with "non-emptiness". Anyway, let us define pseudo torsors first.

Fix $\mathtt{C}$, $\mathtt{Sh}(\mathtt{C})$, X, G and $\mathcal{X}, \mathcal{G}$ as above. Consider the following data.

[1] These are called *comma category*, and *comma topology*, for example, in [Vis05, Def. 2.5.8]. However, a slice cateogry is only a special case of the more general notion of comma categories.

1. A pair $(\mathcal{P}, \rho)$ with $\mathcal{P} \in \mathrm{Obj}(\mathtt{Sh}(\mathtt{C}))$ and $\rho \in \mathrm{Hom}_{\mathtt{Sh}(\mathtt{C})}(\mathcal{P} \times \mathcal{G}, \mathcal{P})$ a group action satisfying that for all $T \in \mathrm{Obj}(\mathtt{C})$, whenever $\mathcal{P}(T) \neq \emptyset$,
$$\rho(T) : \mathcal{P}(T) \times \mathcal{G}(T) \to \mathcal{P}(T)$$
is a simply transitive action.
2. A pair $(\mathcal{P}, \rho)$ with $\mathcal{P} \in \mathrm{Obj}(\mathtt{Sh}(\mathtt{C}))$ and $\rho \in \mathrm{Hom}_{\mathtt{Sh}(\mathtt{C})}(\mathcal{P} \times h_G, \mathcal{P})$ a group action satisfying that for all $T \in \mathrm{Obj}(\mathtt{C})$, whenever $\mathcal{P}(T) \neq \emptyset$,
$$\rho(T) : \mathcal{P}(T) \times \mathrm{Hom}_{\mathtt{C}}(T, G) \to \mathcal{P}(T)$$
is a simply transitive action.
3. A pair (P, ρ) with $P \in \mathrm{Obj}(\mathtt{C})$ and $\rho \in \mathrm{Hom}_{\mathtt{Sh}(\mathtt{C})}(h_P \times \mathcal{G}, h_P)$ a group action, satisfying that for all $T \in \mathrm{Obj}(\mathtt{C})$, whenever $\mathrm{Hom}_{\mathtt{C}}(T, P) \neq \emptyset$,
$$\rho(T) : \mathrm{Hom}_{\mathtt{C}}(T, P) \times \mathcal{G}(T) \to \mathrm{Hom}_{\mathtt{C}}(T, P)$$
is a simply transitive action.
4. A pair (P, ρ) with $P \in \mathrm{Obj}(\mathtt{C})$ and $\rho \in \mathrm{Hom}_{\mathtt{C}}(P \times G, P)$ a group action satisfying that for all $T \in \mathrm{Obj}(\mathtt{C})$, whenever $\mathrm{Hom}_{\mathtt{C}}(T, P) \neq \emptyset$,
$$\rho(T) : \mathrm{Hom}_{\mathtt{C}}(T, P) \times \mathrm{Hom}_{\mathtt{C}}(T, G) \to \mathrm{Hom}_{\mathtt{C}}(T, G)$$
is a simply transitive action.

Moreover, We also have some variations.[2]

$1'$. A pair $(\mathcal{P}, \rho)$ with $\mathcal{P} \in \mathrm{Obj}(\mathtt{Sh}(\mathtt{C}))$ and $\rho : \mathcal{P} \times \mathcal{G} \to \mathcal{P}$ a group action such that the map
$$(\mathrm{pr}_1, \rho) : \mathcal{P} \times \mathcal{G} \to \mathcal{P} \times \mathcal{P}$$
is an isormosphimin $\mathtt{Sh}(\mathtt{C})$.

$4'$. A pair (P, ρ) with $P \in \mathrm{Obj}(\mathtt{C})$ and $\rho : P \times G \to P$ a group action such that the map
$$(\mathrm{pr}_1, \rho) : P \times G \to P \times P$$
is an isomorphism in $\mathtt{C}$.

We know easily 1. $\Leftrightarrow$ $1'$. and 4. $\Leftrightarrow$ $4'$. If $\mathcal{G}$ is representable by G, then 1. $\Leftrightarrow$ 2. and 3. $\Leftrightarrow$ 4.. If $\mathcal{P}$ is representable by P, then 1. $\Leftrightarrow$ 3. and 2. $\Leftrightarrow$ 4..

Definition 1 (pseudo torsors as sheaves) Given a site $\mathtt{C}$ and a group object $\mathcal{G}$ in $\mathtt{Sh}(\mathtt{C})$ (resp. a group object G in $\mathtt{C}$), a *pseudo $\mathcal{G}$-torsor* (resp. *pseudo G-torsor*) is a pair $(\mathcal{P}, \rho)$ as in 1. or 1'. (resp. 2.).

Sometimes, people use the term (pseudo) G-torsor in a more restrictive sense, requiring a (pseudo) torsor to be an object P in $\mathtt{C}$ rather than a sheaf $\mathcal{P}$ on $\mathtt{C}$.

Definition 2 (pseudo torsors as objects) Given a site $\mathtt{C}$ and a group object $\mathcal{G}$ in $\mathtt{Sh}(\mathtt{C})$ (resp. a group object G in $\mathtt{C}$), a *psuedo $\mathcal{G}$-torsor* (resp. *pseudo G-torsor*) is a pair (P, ρ) as in 3. (resp. 4. or 4'.).

Note that in general, the sheaf $\mathcal{P}$ defined in Definition 1 may not be representable. Hence Definitions 1 and 2 are not equivalent. People use different definitions in different contexts. However no matter whether we use Definition 1 or Definition 2, a pseudo G-torsor is always the same thing as a pseudo (h_G)-torsor. So sometimes we will only talk about torsors for a sheaf of groups $\mathcal{G}$. For simplicity, one usually say $\mathcal{P}$ is a $\mathcal{G}$-torsor (over X) without specifying the group action ρ. The problem whether or when a (pseudo) torsor $\mathcal{P}$ is representable will be discussed later in a more concrete setting.

One may note that, the above definition(s) may not be practical enough or may not be the definition one was familiar with if he or she has seen something about $\mathcal{G}$-torosors. For example if we take $\mathtt{C} = (\mathtt{Sch}/k)_{\text{ét}}$ to be the big étale site of k-schemes and G an algebraic group over k, we usually would like to be able to talk about a G-torsor *over a k-scheme X*. This can be defined with a slight modification.

[2] Of course we have similar variations for the data 2. and 3.. But they are not very common to see in literatures.

Definition 3 (pseudo torsors over an object) Let X be an object in $\mathtt{C}$. Then a *pseudo $\mathcal{G}$-torsor (resp. pseudo G-torsor) over X*, depending on whether you would like to define a pseudo torsor to be a sheaf or an object, is defined as a pseudo $(\mathcal{G}|_X)$-torsor (resp. $(h_G)|_X$-torsor) for the site $\mathtt{C}/X$. In more detail, it can be one of the followings.

1*. A pair[3] $(\mathcal{P}, \rho)$ with $\mathcal{P} \in \mathrm{Obj}(\mathtt{Sh}(\mathtt{C}/X))$and $\rho \in \mathrm{Hom}_{\mathtt{Sh}(\mathtt{C}/X)}(\mathcal{P} \times \mathcal{G}|_X, \mathcal{P})$ a group action by $\mathcal{G}|_X$ on $\mathcal{P}$, such that (the condition as before, which we omit) ...
2*. A pair $(\mathcal{P}, \rho)$ with $\mathcal{P} \in \mathrm{Obj}(\mathtt{Sh}(\mathtt{C}/X))$ and $\rho \in \mathrm{Hom}_{\mathtt{Sh}(\mathtt{C}/X)}(\mathcal{P} \times h_G|_X, \mathcal{P})$ a group action by $h_G|_X$ on $\mathcal{P}$, such that ....
3*. A pair (P, ρ) with $P \in \mathrm{Obj}(\mathtt{C}/X)$ and $\rho \in \mathrm{Hom}_{\mathtt{Sh}(\mathtt{C}/X)}(h_P \times \mathcal{G}|_X, \mathcal{P})$ a group action by $\mathcal{G}|_X$ on h_p, such that ...
4*. (Suppose $\mathtt{C}$ admits limits/products)
A pair (P, ρ) with $P \in \mathrm{Obj}(\mathtt{C}/X)$ and $\rho \in \mathrm{Hom}_{\mathtt{C}/X}(P \times G \simeq P \times_X (X \times G), P)$[4] a group action (in the category $\mathtt{C}/X$) by $(X \times G)$ on P, or equivalently a group action (in the category $\mathtt{C}$) by G on P that is over X.
1′*. ... (the data as in 1*.) such that the induced map

$$\mathcal{P} \times \mathcal{G}|_X \to \mathcal{P} \times \mathcal{P}$$

is an isomorphism in $\mathtt{Sh}(\mathtt{C}/X)$.
4′*. (Assume $\mathtt{C}$ admits limits/products)
... (the data as in 4*.) such that the induced map

$$P \times G \simeq P \times_X (X \times G) \to P \times_X P$$

is an isomorphism in $\mathtt{C}/X$, where $P \times G$ is viewed as an object in $\mathtt{C}/X$ via the first projection.

On the other hand, if $\mathtt{C}$ admits a final object S, then what Definitions 1 and 2 defined is what Definition 3 defined as a pseudo torsor over S. Even if $\mathtt{C}$ has no final object,[5] we can think of what Definitions 1 and 2 defined as what Definition 3 defined as a pseudo $\mathcal{G}$-torsor *over nothing*, i.e., we just omit all the "$/X$" in Definition 3. Therefore, we can start with either definition to build the theory.

Example 1 The *trivial (pseudo) $\mathcal{G}$-torsor over X* is just $\mathcal{G}|_X$ with its multiplication map $\mathcal{G}|_X \times \mathcal{G}|_X \to \mathcal{G}|_X$. Analogously, assuming $\mathtt{C}$ admits products, the *trivial (pseudo) G-torsor over X* is just $\mathcal{G} \times X$ with the obvious map $(G \times X) \times_X (G \times X) \cong (G \times G) \times X \to G \times X$. These two concepts coincide if $\mathcal{G}$ is representable by G, as in this case $\mathcal{G}|_X \cong (h_G)|_X \cong h_{G \times X} \in \mathrm{Obj}(\mathtt{Sh}(\mathtt{C}/X))$.

Definition 4 Suppose $(\mathcal{P}, \rho)$ and $(\mathcal{P}', \rho')$ are two (pseudo) $\mathcal{G}$-torsors over X. A morphism of torsors $(\mathcal{P}, \rho) \to (\mathcal{P}', \rho')$ is a $\mathcal{G}$-equivariant morphism $\alpha \in \mathrm{Hom}_{\mathtt{Sh}(\mathtt{C})}(\mathcal{P}', \mathcal{P})$, i.e., we have a commutative diagram

$$\begin{tikzcd} \mathcal{P}' \times \mathcal{G}|_X \arrow[r, "\rho'"] \arrow[d, "\alpha \times \mathrm{id}_{\mathcal{G}|_X}"] & \mathcal{P}' \arrow[d, "\alpha"] \\ \mathcal{P} \times \mathcal{G}|_X \arrow[r, "\rho"] & \mathcal{P} \end{tikzcd}$$

of sheaves of sets on the site $\mathtt{C}/X$.

Definition 5 Suppose $\mathcal{P}$ is a pseudo $\mathcal{G}$-torsor over X and $\mathcal{P}'$ is a pseudo $\mathcal{G}$-torsor over X'. A morphism between them is a pair (f, α), consisting of a morphism $f \in \mathrm{Hom}_{\mathtt{C}}(X', X)$ and a $\mathcal{G}$-equivariant morphism $\alpha \in \mathrm{Hom}_{\mathtt{Sh}(\mathtt{C}/X')}(f^{-1}\mathcal{P}, \mathcal{P}')$.

[3]Sometimes it is more convenient to include X into the data the data X so to define it as triple $(X, \mathcal{P}, \rho)$. Free free to do so.
[4] Here "$\times$" without subscript means the direct product in $\mathtt{C}$, while "$\times_X$" stands for the fibre product over X. If $\mathtt{C}$ admits a final object S, then "$\times$" and "$\times_S$" are the same (isomorphic).
[5]For example, the crystalline site does not have a final object.

Question If $\mathcal{X}$ is an object in $\mathtt{Sh}(\mathtt{C})$, what information can we get from the following data in case $\mathcal{X}$ is not representable?

- A pair $(\mathcal{P}, \rho)$ consisting of an object[6] $\mathcal{P} \in \mathtt{Sh}(\mathtt{C})/\mathcal{X}$ and a group action $\rho \in \mathrm{Hom}_{\mathtt{Sh}(\mathtt{C})/\mathcal{X}}(\mathcal{P} \times \mathcal{G}, \mathcal{P})$ such that

  5. for all $\mathcal{T} \in \mathtt{Sh}(\mathtt{C})/\mathcal{X}$, $\mathrm{Hom}_{\mathcal{X}}(\mathcal{T}, \mathcal{G})$ acts on $\mathrm{Hom}_{\mathcal{X}}(\mathcal{T}, \mathcal{P})$ simply transitively, or

  5′. the morphism
$$(\mathrm{pr}_1, \mathcal{P}) : \mathcal{P} \times \mathcal{G} \to \mathcal{P} \times_{\mathcal{X}} \mathcal{P}$$
  in $\mathrm{Hom}_{\mathtt{Sh}(\mathtt{C})/\mathcal{X}}$ is an isomrphism.

Actually, this is the approach taken in [Gir71], where torsors on a topos $\mathbf{T}$ was defined in III §1.4.1 at first, then the torsors on a site $\mathtt{C}$ were defined as torsors on the associated topos $\mathtt{Sh}(\mathtt{C})$ to this site in III §1.7.1. If we start with this definition, we will make this article shorter, but ....

1.3 Torsors

Let $(\mathcal{P}, \rho)$ be a $\mathcal{G}$-torsor over X. Consider the following conditions.[7]

i. There is a covering $(U := \coprod_{i \in I} U_i \to X)$ in $\mathtt{C}$, such that $\mathcal{P}(U) \neq \emptyset$.
ii. There is a covering $(U := \coprod_{i \in I} U_i \to X)$ in $\mathtt{C}$, such that $\mathrm{Hom}_X(U, P) \neq \emptyset$, equivalently, $P_U := P \times_X U \to U$ has a section $U \to P_U$.

Clearly, if $\mathcal{P}$ is representable by P, then i. and ii. are equivalent.

Definition 6 A *$\mathcal{G}$-torsor over X*, a *torsor under $\mathcal{G}$ over X*, or a *$\mathcal{G}$-principal homogeneous space over X*, is a pseudo $\mathcal{G}$-torsor over X such that condition i. is satisfied. A *G-torsor over X* is just a h_G-torsor over X, equivalently, a pseudo G-torsor that satisfies condition ii. A morphism of torsors is just a morphism of pseudo torsors.

Note that we may as well define a $\mathcal{G}$-torsor witout referring to an object X. If $\mathtt{C}$ has a final object S, then a $\mathcal{G}$-torsor in $\mathtt{C}$ is a just a $\mathcal{G}$-torsor over S as defined in Definition 6. Note that, the above conditions i. and ii. are equivalent to

i′. For all $T \in \mathrm{Obj}(\mathtt{C}/X)$, there is a covering $(U := \coprod_{i \in I} U_i \to T)$, such that $\mathcal{P}(U) \neq \emptyset$.
ii′. For all $T \in \mathrm{Obj}(\mathtt{C}/X)$, there is a covering $(U := \coprod_{i \in I} U_i \to T)$, such that $\mathrm{Hom}_T(U, P) \neq \emptyset$.

respectively. So if $\mathtt{C}$ has no final object, a $\mathcal{G}$-torsor (resp. a G-torsor) in $\mathtt{C}$ is defined as a pseudo $\mathcal{G}$-torsor (resp. pseudo G-torsor) satisfying the condition i″. (resp. ii″.) as below.

i″. For all $T \in \mathrm{Obj}(\mathtt{C})$, there is a covering $(U := \coprod_{i \in I} U_i \to T)$, such that $\mathcal{P}(U) \neq \emptyset$.
ii″. For all $T \in \mathrm{Obj}(\mathtt{C})$, there is a covering $(U := \coprod_{i \in I} U_i \to T)$, such that $\mathrm{Hom}_T(U, P) \neq \emptyset$.

Lemma 1 *A morphism $\mathcal{P}' \to \mathcal{P}$ of two $\mathcal{G}$-torsors $(\mathcal{P}, \rho)$ and $(\mathcal{P}', \rho')$ over X is necessarily an isomorphism. In other words, the category of $\mathcal{G}$-torsors over X is a groupoid. (Hence a morphism of two $\mathcal{G}$-torsors $\mathcal{P}'$ over X' and $\mathcal{P}$ over X is necessarily Cartesian.)*

Proof For any $U \in \mathrm{Obj}(\mathtt{C}/X)$ and any $\mathcal{P} \in \mathrm{Obj}(\mathtt{Sh}(\mathtt{C}/X))$, if $\mathcal{P}(U) \neq \emptyset$, then for all $V \in \mathrm{Obj}(\mathtt{C}/U)$, $(\mathcal{P}|_T)(V) = \mathcal{P}(V) = \mathrm{Im}\big((-|_V) : \mathcal{P}(U) \to \mathcal{P}(V)\big) \neq \emptyset$. So if $\mathcal{P}(U) \neq \emptyset$, a section $s_0 \in \mathcal{P}(U)$ defines a morphism of sheaves
$$\beta : \mathcal{G}|_U \to \mathcal{P}|_U,$$
such that for any $V \in \mathrm{Obj}(\mathtt{C}/U)$
$$\begin{aligned} \beta(V) : (\mathcal{G}|_U)(V) &\longrightarrow (\mathcal{P}|_U)(V) \\ g &\longmapsto \rho(V)\big((s_0)|_V, g\big), \end{aligned}$$

[6] It seems that it is necessary to require that $\mathcal{P} \to \mathcal{X}$ is an epimorphism, as in [Hv15] and [Gir71, III§1.4.1].
[7] Here we assume arbitrary coproduct (disjoint union) exists in $\mathtt{C}$, which may not be the case in practice, for example when we consider the category of *finite type schemes*. But this is not essential: we can just replace statements about U by statements about U_i for all i.

where $\rho(V)$ is the action $\mathcal{P}(V) \times (\mathcal{G}|_X)(V) \to \mathcal{P}(V)$. We know over each V, $\beta(V)$ is an isomorphism as discussed in the beginning of the article. Hence β defines an isomorphism of $\mathcal{P}|_U$ and $\mathcal{G}|_U$ compatible with their $\mathcal{G}|_U$-actions.

Now given a covering $(U_i \to X)$ as in condition i., a $\mathcal{G}|_X$-equivariant morphism $\alpha : \mathcal{P}' \to \mathcal{P}$ restricts to a $(G|_{U_i})$-equivariant morphism $(\alpha_i : \mathcal{P}'|_{U_i} \to \mathcal{P}|_{U_i})$ in $\mathtt{Sh}(\mathtt{C}/U_i)$ for all i. Since $\Gamma(U_i, \mathcal{P}|_{U_i}) \neq \emptyset$ and α_i is $(\mathcal{G}|_{U_i})$-equivariant, the above defined isomorphisms $\beta_i : \mathcal{G}|_{U_i} \to \mathcal{P}|_{U_i}$ and $\beta'_i : \mathcal{G}|_{U_i} \to \mathcal{P}'|_{U_i}$ imply that α_i is an isomorphism for all i. These isomorphisms are clearly compatible on $\mathtt{C}/(U_i \times_X U_j)$ for $i, j \in I$ (not necessarily $i \neq j$). Hence α is an isomorphism.

In the above proof, we have actually seen that if a (pseudo) $\mathcal{G}$-torsor $\mathcal{P}$ has a s section over X, i.e., $\mathcal{P}(X) \neq \emptyset$, then we know that $\mathcal{P}$ is then isomorphic to the trivial $\mathcal{G}$-torsor $\mathcal{G}$. So we can say that *a $\mathcal{G}$-torsor is just a locally trivial pseudo $\mathcal{G}$-torsor* — there is a covering $(U_i \to X)$, such that locally on U_i, $\mathcal{P}(U_i) \neq \emptyset$, hence $\mathcal{P}|_{U_i}$ is trivial. Clearly, the meaning of "locally" relies on the choice of (pre)topology τ on $\mathtt{C}$.

1.4 Descent along torsors

1.4.1 General statement

Theorem 1 *Suppose that $P \in \mathrm{Obj}(\mathtt{C}/X)$ is a $\mathcal{G}$-torsor over $X \in \mathrm{Obj}(\mathtt{C})$, and $\mathtt{F} \to \mathtt{C}$ is a* stack*.*

Then there is a canonical equivalence of categories between the fibre $\mathtt{F}(Y)$ of $\mathtt{F}$ over P and the category of $\mathcal{G}$-equivariant objects $\mathtt{F}^{\mathcal{G}}(X)$.

Proof [Vis05, Theorem 4.46].

Question Does the same result hold for $\mathcal{G}$-torsors $\mathcal{P}$.

1.4.2 Special case: Galois descent

1.5 Quotient

2 Interlude: stacks and gerbes

Recall that a category $\mathtt{F}$ with a functor $p : \mathtt{F} \to \mathtt{C}$ is called a *category over* $\mathtt{C}$. An arrow $\phi \in \mathrm{Mor}_{\mathtt{F}}(y, x)$ is said to be *(strongly) Cartesian* if for any arrow $\psi \in \mathrm{Mor}_{\mathtt{F}}(z, x)$, and any arrow $h \in \mathrm{Mor}_{\mathtt{C}}(p(z), p(y))$ with $p(\phi) \circ h = p(\psi)$, there is an arrow $\theta \in \mathrm{Mor}_{\mathtt{F}}(z, y)$, with $\phi \circ \theta = \psi$:

$$\begin{tikzcd}
z \arrow[drr, bend left, "\psi"] \arrow[dr, dotted, "\theta"] \arrow[dd, mapsto] & & \\
& y \arrow[r, "\phi"] \arrow[dd, mapsto] & x \arrow[dd, mapsto] \\
p(z) \arrow[dr, "h"'] \arrow[drr, bend left, "p(\psi)"] & & \\
& p(y) \arrow[r, "p(\phi)"] & p(x)
\end{tikzcd}$$

A category $\mathtt{F}$ over $\mathtt{C}$ is *fibred over* $\mathtt{C}$[8] if for every $x \in \mathrm{Obj}(\mathtt{C})$, any arrow $f \in \mathrm{Mor}_{\mathtt{C}}(V, p(x))$, there is a *(strongly) Cartesian* $\phi \in \mathrm{Mor}_{\mathtt{F}}(y, x)$ lifting f.

$$\begin{tikzcd}
\exists\, y \arrow[r, dotted, "\exists\, \phi"] \arrow[d, mapsto] & x \arrow[d, mapsto] \\
V \arrow[r, "f"] & p(x)
\end{tikzcd}$$

For each object $U \in \mathrm{Obj}(\mathtt{C})$, the *fibre of* $\mathtt{F}$ *over* U, denoted by $\mathtt{F}(U)$ or $\mathtt{F}_U$ is the subcategory of $\mathtt{F}$ with objects in $\mathtt{F}$ that are mapped to U and arrows in $\mathtt{F}$ that are mapped to id_U. A category $\mathtt{F}$ over $\mathtt{C}$ is *fibred in groupoids over* $\mathtt{C}$ if for each $U \in \mathrm{Obj}(\mathtt{C})$, the *fibre* $\mathtt{C}(U)$ over U is a groupoid. Equivalently, a category $\mathtt{F}$ over

[8] Sometimes people say that $\mathtt{F} \to \mathtt{C}$ is a *Grothendieck fibration*.

$\mathtt{C}$ is a fibred category in groupuoids over $\mathtt{C}$ if every arrow in $\mathtt{C}$ can be lifted to an arrow in $\mathtt{F}$ and every arrow in $\mathtt{F}$ is (strongly) Cartesian. For more details, see [Vis05, §3.3].

A fibred category $\mathtt{F}$ over $\mathtt{C}$ is a *stack* if for each covering $(U_i \to U)$, the functor $\mathtt{F}(U) \to \mathtt{F}(\{U_i \to U\})$ is an equivalence, where $\mathtt{F}(\{U_i \to U\})$ is the category of objects with *descent datum*, in other words, every descent datum is effective. This is discussed in details in [Vis05, §4.1.2]. Some authors use the term stack to mean a *stack in groupoids*, i.e., requiring $\mathtt{F}$ is a fibred in groupoids, see [Stacks, Tag 02ZH].

A *gerbe* is a stack $\mathtt{F}$ in groupoids over $\mathtt{C}$, such that for any object U in $\mathtt{C}$, there is a covering $(U_i \to U)$, such that $\mathtt{F}(U_i) \neq \emptyset$ for all i,[9] and for any object U in $\mathtt{C}$ and any $x, y \in \mathtt{F}(U)$, there is a covering $(U_i \to U)$, such that $x|_{U_i} \simeq y|_{U_i}$ in $\mathtt{F}(U_i)$, i.e., every two objects in the fibre are locally isomorphic. One can say that a gerbe is a stack in groupoids that is *locally non-empty* and *locally connected*. For more details, see [Stacks, Tag 06NY]. For more examples of gerbes, see [MO263832].

3 The gerbe of $\mathcal{G}$-torsors

Let $\mathcal{G}\text{-}\mathtt{torsors}$ be the category of $\mathcal{G}$-torsors,[10] whose objects are triples $(X, \mathcal{P}, \rho)$ with a $\mathcal{G}$-torsor $\mathcal{P} \in \mathrm{Obj}(\mathtt{Sh}(\mathtt{C}/X))$ over $X \in \mathrm{Obj}(\mathtt{C})$, and morphisms are pairs (f, α) as defined in Definition 5. We know from Lemma 1 that α is necessarily an isomorphism. We have a (forgetful) functor

$$p : \mathcal{G}\text{-}\mathtt{torsors} \to \mathtt{C}, \qquad \begin{aligned} (X, \mathcal{P}, \rho) &\mapsto X \\ \big((X', P', \rho') \to (X, P, \rho)\big) &\mapsto (X' \to X) \end{aligned}$$

3.1 $\mathcal{G}\text{-}\mathtt{torsors}$ as a gerbe

Proposition 1 *The functor p makes $\mathcal{G}\text{-}\mathtt{torsors}$ into a* category fibred in groupoids over $\mathtt{C}$.

Proof This just follows from definition and Lemma 1.

Proposition 2 *The functor p makes $\mathcal{G}\text{-}\mathtt{torsors}$ into a* stack in groupoids over $\mathtt{C}$.

Proof It suffices to show that every descent datum is effective. For a reference, see for example [Stacks, Tag 04UK]. The ideas are as follows. Suppose that we have

- a covering $(U_i \to U)$ in $\mathtt{C}$,
- for each i, a $\mathcal{G}|_{U_i}$-torsor $\mathcal{F}_i$ (or if you prefer, in the language of Definition 3, a $\mathcal{G}$-torosr $\mathcal{F}_i$ over U_i).
- for each i, j, an isomorphism
$$\phi_{ij} : \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}},$$
of $\mathcal{G}|_{U_{ij}}$-torsors (meaning ϕ_{ij} is $(\mathcal{G}|_{U_{ij}})$-equivariant and is an isomorphism of sheaves), where $U_{ij} = U_i \times_U U_j$.
- for each i, j, k, the cocycle condition, ...

Then we know sheaves $\mathcal{F}_i$ glue to a sheaf $\mathcal{F}$ on $\mathtt{C}/U$. One then need to check,

- actions of $\mathcal{G}|_{U_i}$ on $\mathcal{F}_i$ glue to a action of $\mathcal{G}|_U$ on $\mathcal{F}$.
- this action makes $\mathcal{F}$ a $\mathcal{G}|_U$-torsor.

Warning Proposition 2 works for torsors as sheaves. If we restricts to torsors that are representable in $\mathtt{C}$, the above proof fails ([Stacks, Tag 04UQ]), as not all torsros are representable and not all descent data for objects in $\mathtt{C}$ are effective if we put no restriction on $\mathtt{C}$. Essentially what we used in the proof is just the descent for sheaves. In practice, in more concrete settings, we will require torsors to be some special kind of sheaves (quasi-coherent sheaf of commutative algebras, or locally free sheaves for example) or special kinds of objects (fppf scheme over X, for example). Then for Proposition 2 to be valid, one need to think about whether such a class of sheaves or objects has the descent property. In most case, there will be no problem for sheaves but one need to take care.

[9] If $\mathtt{C}$ has a final object S, this condition can be stated as there is a covering $(U_i \to S)$, such that $\mathtt{F}(U_i) \neq \emptyset$.

[10] Other common notations are $\mathrm{B}\mathcal{G}$ and $\mathtt{Torsors}(\mathcal{G})$.

Question Is the stack algebraic?[11]

We can proceed one step further.

Theorem 1 *The functor p makes $\mathcal{G}$*`-torsors` *into a* gerbe over $\mathtt{C}$.

Proof First of all, we have already known that $\mathcal{G}$`-torsors` is a stack in groupoids. For each object U in $\mathtt{C}$, $\mathcal{G}$`-torsors`(U) is non-empty as it at least contains the trivial $\mathcal{G}$-torsor. Moreover, any two $\mathcal{G}$-torsors $\mathcal{P}$ and $\mathcal{P}'$ over U are locally isomorphic to the trivial $\mathcal{G}$-torsor, hence they are locally isomorphic. Therefore, $\mathcal{G}$`-torsors` is a gerbe over $\mathtt{C}$.

It's good to mention that, the gerbe $\mathcal{G}$`-torsors` is very special, in the sense that the functor $\mathcal{G}$`-torsors` $\to \mathtt{C}$ has a section $\mathtt{C} \to \mathcal{G}$`-torsors` that sends every object U to the trivial $\mathcal{G}$-torsor over U, where $\mathtt{C}$ is viewed as the trivial stack. A gerbe $\mathtt{F} \to \mathtt{C}$ that admits a section is called a *neutral gerbe* or *trivial gerbe*. In case $\mathtt{C}$ has a final object S, this condition is equivalent to say that $\mathcal{G}(S) \neq \emptyset$.

3.2 Every neutral gerbe arises as $\mathcal{G}$`-torsors`

The converse of 1 also holds.

Lemma 2 *Suppose that $\mathtt{F}$ is a gerbe over $\mathtt{C}$ and U is an object in $\mathtt{C}$. If $\mathrm{Obj}(\mathtt{F}(U)) \neq \emptyset$, then for any objects $x, y \in \mathtt{F}(U)$, the sheaf $\underline{\mathrm{Isom}}_U(x,y) = \underline{\mathrm{Hom}}_U(x,y)$ on $\mathtt{C}/U$ is an $\underline{\mathrm{Aut}}_U(x)$-torsor for the site $\mathtt{C}/U$.*

Proof It is a pseudo torsor by direct check and it is a torsor by locally connectedness of $\mathtt{F}$. See [Gir71, III §§1.5.3.2 and 2.1.1.1].

Theorem 2 *Let $\mathtt{F}$ be a neutral gerbe. Then $\mathtt{F}$ is isomorphic as gerbes to $\mathcal{G}$*`-torsors` *for some sheaf of groups $\mathcal{G}$ on $\mathtt{C}$.*

Proof Suppose $p : \mathtt{F} \to \mathtt{C}$ is a neutral gerbe with a section $s : \mathtt{C} \to \mathtt{F}$. Define $\underline{\mathrm{Aut}}(s)$ to be the sheaf of groups on $\mathtt{C}$, whose restriction $\underline{\mathrm{Aut}}(s)|_U$ to any object U in $\mathtt{C}$ is the sheaf of groups $\underline{\mathrm{Aut}}_U(s(U))$ assigning each object V in $\mathtt{C}/U$ the group $\mathrm{Aut}_V\left(s(U)|_V\right)$, where $s(U)$ is an object in $F(U)$ and $\mathrm{Aut}_V(s(U)|_V)$ is the *group*[12] of automorphisms in the category $\mathtt{F}(V)$, with group structure defined by composition of automorphisms. This may be a bit confusing. But in case $\mathtt{C}$ has a final object S, and $\sigma = s(S) \in \mathrm{Obj}(\mathtt{F}(S)) \neq \emptyset$ is the object corresponding to the section $s : \mathtt{C} \to \mathtt{F}$, then $\underline{\mathrm{Aut}}(s)$ is exactly the sheaf $\underline{\mathrm{Aut}}_{\mathtt{C}}(\sigma)$ of groups on $\mathtt{C}$ assigning each object U in $\mathtt{C}$ the group $\mathrm{Aut}_U(\sigma|_U)$.

Then by Lemma 2, we obtain a morphism $a : \mathtt{F} \to \mathcal{G}$`-torsors` of neutral gerbes, such that for any object U in $\mathtt{C}$ and any object $y \in \mathtt{F}(U)$, $a(y) := \underline{\mathrm{Isom}}_U(s(U), y)$. Then one checks that this is an isomorphism of gerbes. For a reference, see [Gir71, III §2.2.6].

Theorems 1 and 2 together give an equivalence of neutral gerbes and gerbes of $\mathcal{G}$-torsors (see [Gir71, III §2.2.6.2], a special case of which is [DM82, Example 3.6]).

A gerbe $\mathtt{F} \to \mathtt{C}$ is called an $\mathcal{G}$-gerbe on $\mathtt{C}$ for a sheaf of groups $\mathcal{G}$ on $\mathtt{C}$, if for any object U in $\mathtt{C}$ and any object (if it exists) x in $\mathtt{F}(U)$, we have $\mathcal{G}|_U \cong \underline{\mathrm{Aut}}_U(x)$ as sheaves on $\mathtt{C}/U$. So the gerbe $\mathcal{G}$`-torsors` is a $\mathcal{G}$-gerbe. Besides these "trivial ones", we have another type of $\mathcal{G}$-gerbes.

Proposition 3 *Suppose that $\mathtt{F}$ is a gerbe, such that for each object U in $\mathtt{C}$ and each object x in $\mathtt{F}(U)$, the sheaf of groups $\underline{\mathrm{Aut}}_U(x) = \underline{\mathrm{Isom}}_U(x,x)$, $V \mapsto \mathrm{Isom}_V(x|_V, x|_V)$ on $\mathtt{C}/U$ is* abelian. *Then for all objects x and x' in $\mathtt{F}(U)$, $\underline{\mathrm{Aut}}_U(x)$ and $\underline{\mathrm{Aut}}_U(x')$ are canonically isomorphic and there is a sheaf of groups $\mathcal{G}$ on $\mathtt{C}$, such that $\mathcal{G}|_U \simeq \underline{\mathrm{Aut}}(x)$ compatible with the isomorphisms between $\underline{\mathrm{Aut}}_U(x)$ and $\underline{\mathrm{Aut}}_U(x')$.*

Proof The ideas is as follows. For any object U in $\mathtt{C}$, $\mathtt{F}(U)$ is non-empty by assumption. Moreover, as $\mathtt{F}$ is a gerbe, $\mathtt{F}(U)$ is *locally connected*. Hence every two object x and y are isomorphic after a change to a covering $(U_i \to U)$, i.e., there is for each i a morphism $\phi : x_i \xrightarrow{\sim} y_i$, where $x_i := x|_{U_i}$ and similarly for y_i. Such an (iso)morphism ϕ_i of objects gives an isomorphism of sheaves $\underline{\mathrm{Aut}}(x_i) \simeq \underline{\mathrm{Aut}}(y_i)$, $\alpha \mapsto \phi_i \circ \alpha \circ \phi_i^{-1}$. As $\underline{\mathrm{Aut}}(x_i)$ by assumption is abelian, this isomorphism is actually independent of ϕ_i. We define $\mathcal{G}|_i$ as $\underline{\mathrm{Aut}}(x_i)$ and can check that these sheaves glue to a sheaf $\mathcal{G}_U$ on $\mathtt{C}/U$. Then for all U, these $\mathcal{G}_U$'s glue to a sheaf of groups $\mathcal{G}$ such that $\mathcal{G}|_U \simeq \mathcal{G}_U$ for all objects U in $\mathtt{C}$. (For a reference, see [Stacks, Tag 0CJY].)

[11] See for example, [Gai09, §3].

[12] We actually ignored the set theory problems here.

4 Contracted Product

5 Cohomology

See [Stacks, Tags 02FN, 03AG and 0CJZ].

6 Torsors in the fppf Topology

6.1 Notations

S	a scheme
$\mathtt{C} = (\mathtt{Sch}/S)_{\mathrm{fppf}}$	the (big) fppf site on S.
G	an fppf group scheme, i.e., the strucutre morphism $G \to S$ is fppf.
$\mathcal{G}$	a sheaf of groups on $(\mathtt{Sch}/S)_{\mathrm{fppf}}$.

6.2 Torsors

In practice, people also consider the étale site.

6.3 Concrete examples

Example 2 Galois extension, [Vis05, §4.45].

Example 3 [Poo17, §5.12.3].
In particular, GL_n-torsors.

Example 4 Link. Example 1. and 2.

To clean up

An useful argument. Link. Let $\phi : G \to S$ be an algebraic group scheme. If G/S is flat, then the existence of the identity section implies that ϕ is surjective, hence faithfully flat. If G is finitely presented over S, then ϕ is fppf. Typically, we will assume G is affine over S, hence $G \to S$ will be fpqc. In particular, we can use descent theory on $G \to S$.

Recall [EGA IV_4, 17.16.2]. If $X \to S$ is fppf, then there is $S' \to S$ fppf, factor through $X \to S$.

Link Important result: If G/S is smooth, then trivial in smooth/etale topology.

In char 0, algebraic groups are smooth.

[MO19339].

[BLR90, Theorem 6.4.1]

Lei suggests that for quasi-projective group-schemes, torsor as a sheaf is the always represented by a scheme., see also [Stacks, Tag 049C].

See also `http://www.math.fu-berlin.de/users/lei/Bun_G%20Berlin`

[Gai09, Proposition 1.9].

References

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models.* English. Berlin etc.: Springer-Verlag, 1990, pp. x + 325 (cit. on p. 8).

[DM82] Pierre Deligne and J.S. Milne. *Tannakian categories.* English. Hodge cycles, motives, and Shimura varieties, Lect. Notes Math. 900, 101-228 (1982). 1982 (cit. on p. 7).

[Gai09] Dennis Gaitsgory. *Seminar notes: G-bundles, stacks and BG.* Sept. 2009. URL: `http://www.math.harvard.edu/~gaitsgde/grad_2009/SeminarNotes/Sept15(stacks).pdf` (cit. on pp. 7, 8).

[Gir71] Jean Giraud. *Cohomologie non abelienne. (Non-Abelian cohomology).* French. Die Grundlehren der mathematischen Wissenschaften. Band 179. Berlin-Heidelberg-New York: Springer-Verlag. IX,467 p. 1971 (cit. on pp. 1, 4, 7).

[Hv15] David Holmes and Raymond van Bommel. *Angelo Vistoli on Gerbes*. Sept. 2015. URL: http://pub.math.leidenuniv.nl/~holmesdst/Vistoli_torsors.pdf (cit. on p. 4).

[MO19339] Chris Schommer-Pries. *Torsors in Algebraic Geometry?* MathOverflow. URL: http://mathoverflow.net/q/19339 (cit. on p. 8).

[MO263832] tttbase. *Phenomena of Gerbes*. MathOverflow. URL: https://mathoverflow.net/q/263832 (cit. on p. 6).

[Poo17] Bjorn Poonen. *Rational Points on Varieties*. Version Jan. 1, 2017. 2017 (cit. on p. 8).

[Stacks] The Stacks Project Authors. *Stacks Project*. http://stacks.math.columbia.edu. 2017 (cit. on pp. 1, 6–8).

[Vis05] Angelo Vistoli. "Grothendieck topologies, fibered categories and descent theory". In: *Fundamental algebraic geometry*. Vol. 123. Math. Surveys Monogr. Amer. Math. Soc., Providence, RI, 2005, pp. 1–104 (cit. on pp. 1, 5, 6, 8).