Connections, Stratifications, and D-Modules

To see the linearity more clearly, we explicitly write $B \otimes_{f,A,g} C$ when B (resp. C) is viewed as an A-module (algebra) via f (resp. g). If the A-module structure is clear from context, then f and/or g are/is omitted from the notation. Similarly, $\underline{\mathrm{Hom}}_{A,f}(M, N)$ indicates maps are f-linear.

Keep the notations as in [Ber74]. Suppose that $X = (A, P_X^n, d_0^n, d_1^n, \pi^n, \delta^{m,n})$ is a formal category. Set $I_n := \mathrm{Ker}(\pi^n : P_X^n \to A)$.

For any A-modules M and N, set

$$\underline{\mathrm{Diff}}^n(M, N) := \underline{\mathrm{Hom}}_{A,d_0^n}(P_X^n \otimes_{d_1^n,A} M, N).$$

Set $(P_X^n)^\vee := \underline{\mathrm{Hom}}_{A,d_0^n}(P_X^n, A) \cong \underline{\mathrm{Diff}}(A, A)$.[1] Then $\underline{\mathrm{Diff}}^n(M, A)$ has a natural P_X^n-module structure hence *two* different A-module structures. Set $\underline{\mathrm{Diff}}(M, N) := \cup_n \underline{\mathrm{Diff}}(M, n)$ and $\mathcal{D} := \varinjlim_n (P_X^n)^\vee = \cup_n (P_X^n)^{\vee}$[2]. There is a (non-commutative) ring structure on $\underline{\mathrm{Diff}}(M, N)$. The product or composition $D_m \circ D_n$ of sections D_m of $\underline{\mathrm{Hom}}_{A,d_0^n}(P_X^m \otimes_{d_0^n,A} M, N)$ and D_n of $\underline{\mathrm{Hom}}_{A,d_0^n}(P_X^n \otimes_{d_0^n,A} L, M)$ is the composition

$$P_X^{m+n} \otimes_{d_0^n,A} L \xrightarrow{(\delta^{m,n}\otimes \mathrm{id}_L)} P_X^m \otimes_{d_1^n,A,d_0^n} P_X^n \otimes_{d_1^n,A} L \xrightarrow{(\mathrm{id}_{P_X^m} \otimes D_n)} P_X^m \otimes_A M \xrightarrow{D_m} N.$$

In particular, if D_m and D_n are sections of $(P_X^n)^\vee$ and $(P_X^n)^\vee$, then $D_m \circ D_n$ is the composition

$$P_X^{m+n} \xrightarrow{\delta^{m,n}} P_X^m \otimes_A P_X^n \xrightarrow{\mathrm{id}_{P_X^m} \otimes D_n} P_X^m \otimes_{d_1^n,A} A \cong P_X^n \xrightarrow{D_m} A.$$

Recall also that any section f of $\underline{\mathrm{Hom}}_A(M, N)$ determines a degree 0 differential operator given by the composition

$$P_X^0 \otimes_{d_0^0,A} M \xrightarrow{\pi^0\otimes \mathrm{id}_M} A \otimes_A M \cong M \xrightarrow{f} N.$$

In particular, a section a of A determines a degree 0 differential operator by

$$P_X^0 \xrightarrow{\pi^0} A \xrightarrow{\cdot a} A.$$

As π^0 is surjective, we have $\underline{\mathrm{Hom}}_A(M, N) \subseteq \underline{\mathrm{Diff}}^0(M, N)$ and $A = \underline{\mathrm{End}}_A(A) \subseteq (P_X^0)^\vee \cong \underline{\mathrm{Diff}}^0(A, A)$.

1 Connections

An *n-connection* on M by definition is a P_X^n-module homomorphism

$$\varepsilon_n : P_X^n \otimes_{d_1^n,A} M \longrightarrow M \otimes_{A,d_0^n} P_X^n$$

that induces id_M modulo I.

[1] Sometimes it's denoted by $\mathcal{D}^{\leq n}$, $\mathcal{D}^{(n)}$ or $\mathcal{D}^n$, whose sections are called differential operators of orders no more than n. Moreover, sometimes we explicitly distinguish a d_0^n-linear map $P_X^n \otimes_{d_1^n,A} A \to A$ its image $P_X^n \to A$ under the isomorphism $\underline{\mathrm{Hom}}_{A,d_0^n}(P_X^n \otimes_{d_1^n,A} A, A) \cong \underline{\mathrm{Hom}}_{A,d_0^n}(P_X^n, A)$.

[2] Sometimes this is also denoted by $\mathcal{D}^\infty$.

1.1. An n-connection ε induces the following maps

(C1) A d_1^n-linear map $\theta_n : M \to M \otimes_{A,d_0^n} P_X^n$ that induces id_M modulo I.
In fact, θ_n is induced as the composition

$$M \xrightarrow{d_1^n \otimes \mathrm{id}_M} P_X^n \otimes_{d_1^n,A} M \xrightarrow{\varepsilon_n} M \otimes_{A,d_0^n} P_X^n.$$

(C2) A P_X^n-linear map $\nabla_n : \underline{\mathrm{Diff}}^n(E,F) \longrightarrow \underline{\mathrm{Diff}}^n(M \otimes_A E, M \otimes_A F)$ that induces $M \otimes_A \underline{\mathrm{Hom}}_A(E,F) \to \underline{\mathrm{Hom}}_A(M \otimes_A E, M \otimes_A F)$ (recall that $\underline{\mathrm{Hom}}_A(E,F) \subset \underline{\mathrm{Diff}}^0(E,F)$ induced by the surjective ring homomorphism π^0, see [Ber74, II, (2.1.4)]). In particular ∇_n is is d_0^n-linear.
Actually, given any $D : P_X^n \otimes_{d_1^n,A} E \to F$, the map $\nabla_n(D)$ is defined as the composition

$$P_X^n \otimes_{d_1^n,A} M \otimes_{d_1^n,A} E \xrightarrow{\varepsilon_n \otimes \mathrm{id}_E} M \otimes_{A,d_0^n} P_X^n \otimes_{d_1^n,A} E \xrightarrow{\mathrm{id}_M \otimes D} M \otimes_A F.$$

The data in (C2) also implies the following.

(C3) A P_X^n-linear map $\nabla_n' : \underline{\mathrm{Diff}}^n(A,A) \longrightarrow \underline{\mathrm{Diff}}^n(M,M)$ that induces the natural map $A \to \underline{\mathrm{Hom}}_A(M,M)$ viewing any section a as multiplication on M (order 0 differential operator).

1.2. Only under some extra conditions, the above two maps also determine ε_n. Consider the following assumptions.

(A1) X is a formal groupoid: there is a map $\sigma^n : P_X^n \to P_X^n$ such that ... (See [Ber74, II, Définition 1.1.3].)

(A2) I_n is locally *nilpotent*.
(A3) P^n is locally free of finite type.

1.2.1. If (A1) and (A2) is satisfied, then ε_n can be recovered from (C1).
Actually, by universal property for the tensor product $P_X^n \otimes_{d_1^n,A} M$, a d_1^n-linear map θ_n as in (C1) determines a P_X^n-linear map ε_n that induces id_M modulo I. Then the existence of σ^n determines σ^n-isomorphism $\sigma_M^n : M \otimes_{A,d_0^n} P_X^n \to P_X^n \otimes_{d_0,A} M$ such that $(\sigma_M^n \circ \varepsilon_n)^2$ is P_X^n-linear and induces identity on M modulo a nilpotent ideal I. This implies that $(\sigma_M^n \circ \varepsilon_n)^2$ is an isomorphism hence ε.

1.2.2. If (A1), (A2) and (A3) are satisfied, then ε_n can be recovered from (C3) or from (C2).
In fact, consider the composition

$$P_X^n \otimes_{d_1^n,A} M \longrightarrow \underline{\mathrm{Hom}}_{A,d_0^n}\left(\underline{\mathrm{Hom}}_{A,d_0^n}(P_X^n \otimes_{d_1^n,A} M, M), M\right) \longrightarrow \underline{\mathrm{Hom}}_{A,d_0^n}\left(\underline{\mathrm{Hom}}_{A,d_0^n}(P_X^n, A,), M\right),$$

where the first map is the natural map $M \to (M^\vee)^\vee$, and the second one is induced by ∇_n given in (C2). *Note that here we only use the fact that ∇_n is d_0^n-linear.* Moreover, due to (A3), we have natural isomorphisms

$$M \otimes_{A,d_0^n} P_X^n \xrightarrow{\sim} M \otimes_{A,d_0^n} \underline{\mathrm{Hom}}_{A,d_0^n}\left(\underline{\mathrm{Hom}}_{A,d_0^n}(P_X^n, A), A\right) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{A,d_0^n}\left(\underline{\mathrm{Hom}}_{A,d_0^n}(P^n, A), M\right)$$

Then we obtain a P_X^n-linear map $\varepsilon_n : P_X^n \otimes_{d_1^n,A} M \to M \otimes_{A,d_0^n} P_X^n$, which reduces to id_M modulo I. Then use the same argument as the previous proof, we can conclude that ε_n is a P_X^n-isomorphism using (A1) and (A2).

1.3. As under conditions (A1) and (A2), an n-connection is equivalent to a d_1^n-linear map $\theta_n : M \to M \otimes_{A,d_0^n} P_X^n$. Now we can see how this datum determine the map ∇_n as in (C2).

Given a d_1^n-linear map θ_n as in (C1) and a section D of $\underline{\mathrm{Hom}}_{A,d_0^n}(P_X^n \otimes_{d_1^n,A} E, F)$. Consider the composition

$$M \otimes_A E \xrightarrow[d_1^n\text{-linear}]{\theta \otimes \mathrm{id}_E} M \otimes_{A,d_0^n} P_X^n \otimes_{d_1^n,A} E \xrightarrow[d_0^n\text{-linear}]{\mathrm{id}_M \otimes D} M \otimes_A F.$$

This composition is neither d_0^n-linear nor d_1^n-linear. But this map determines a d_0^n-linear map ∇_n as it factors as follows due to the previous equivalence.

$$\begin{tikzcd}
M \otimes_A E \arrow[r, "\theta \otimes \mathrm{id}_E"] \arrow[d, "\text{scaler extension by } d_1^n"'] & M \otimes_{A,d_0^n} P_X^n \otimes_{d_1^n,A} E \arrow[r, "\mathrm{id}_M \otimes D", "d_0^n\text{-linear}"'] & M \otimes_A F \\
P_X^n \otimes_{d_1^n,A} M \otimes_A E \arrow[ur, dotted, "P_X^n\text{-linear}"] \arrow[urr, dotted, "d_0^n\text{-linear}"'] & &
\end{tikzcd}$$

2 (Pseudo-)Stratifications

A *pseudo-stratification* is a collection of compatible n-connections $\varepsilon_n : P^n \otimes_{d_1^n,A} A \to A \otimes_{A,d_0^n} P_X^n$.

$$\begin{tikzcd}
P_X^n \otimes_{d_1^n,A} M \arrow[r, "\varepsilon_n"] \arrow[d] & M \otimes_{A,d_0^n} P_X^n \arrow[d] \\
P_X^m \otimes_{d_1^n,A} M \arrow[r, "\varepsilon_m"] & M \otimes_{A,d_0^n} P_X^m
\end{tikzcd}$$

A *stratification* is a pseudo-stratification that satisfies the cocycle condition.

More precisely, consider the following data.

(S1) (in terms of n-connection) a pseudo-stratification

$$\begin{tikzcd}
P_X^m \otimes_{d_1^n,A,d_0^n} P^n \otimes_{A,d_1^n} M \arrow[rr, "(\delta^{m,n})^*(\varepsilon_{m+n})"] \arrow[dr, "\mathrm{id}_{P_X^m} \otimes \varepsilon_n"'] & & M \otimes_{A,d_0^n} P_X^m \otimes_{d_1^n,A,d_0^n} P_X^n \arrow[dl, "\varepsilon_m \otimes \mathrm{id}_{P_X^n}"] \\
& P_X^m \otimes_{d_1^n,A} M \otimes_{A,d_0^n} P_X^n &
\end{tikzcd}$$

where $(\delta^{m,n})^*(\varepsilon_{m+n})$ is the pullback of the P_X^{m+n}-module morphism $\varepsilon_{m+n} : P_X^{m+n} \otimes_{d_1^n,A} M \to M \otimes_{A,d_0^n} P_X^{m+n}$ along $\delta^{m,n} : P_X^{m+n} \to P_X^m \otimes_{d_1^n,A,d_0^n} P_X^n$.

(S2) (in terms of (C1)) a collection of θ_n making the following diagram commutative

$$\begin{tikzcd}
M \arrow[r, "\theta_{m+n}"] \arrow[d, "\theta_n"] & M \otimes_{A,d_0^n} P_X^{m+n} \arrow[d, "\mathrm{id}_M \otimes \delta^{m+n}"] \\
M \otimes_{A,d_0^n} P_X^n \arrow[r, "\theta_m \otimes \mathrm{id}_{P_X^n}"] & M \otimes_{A,d_0^n} P_X^m \otimes_{d_1^n,A,d_0^n} P_X^n.
\end{tikzcd}$$

(S3) (in terms of (C3)) a ring homomorphism $\nabla : \underline{\mathrm{Diff}}(A, A) \to \underline{\mathrm{Diff}}(M, M)$ that induces P_X^n-linear map $\nabla_n : \underline{\mathrm{Diff}}^n(A, A) \to \underline{\mathrm{Diff}}^n(M, M)$.

3 D-Modules

A $\mathcal{D}$*-module* M is an A-module M together with a ring homomorphism

$$\varphi : \mathcal{D} \longrightarrow \underline{\mathrm{End}}(M)$$

extending the natural map $A \hookrightarrow (P^0_X)^\vee \to \underline{\mathrm{End}}(M)$ ([Ber74, II, (2.1.4)]), where $\underline{\mathrm{End}}(M)$ is understood as group endomorphisms. This map is equivalent to a map

$$\varphi' : \mathcal{D} \times M \longrightarrow M$$

such that the usual module axioms are satisfied.

Under some conditions, a $\mathcal{D}$-module structure on an A-module M, that extends the A-module structure of M, is equivalent to a stratification.

3.1. Suppose that we have a ring homomorphism $\varphi : \mathcal{D} \to \underline{\mathrm{End}}(M)$ extending the natural ring homomorphism $A \to \underline{\mathrm{End}}(M)$, $a \mapsto (m \mapsto am)$. For each n, these data determine a group homomorphism $\varphi_n : (P^n_X)^\vee \to \underline{\mathrm{End}}(M)$, that is d^n_0-linear and d^n_1-linear.

To see the linearity, observe that the A-module structure on $(P^n_X)^\vee$ that is induced by d^n_0 (resp. d^n_1) coincides with the left (resp. right) multiplication (composition) of A with $(P^n_X)^\vee$ in the ring $\mathcal{D}$, where A is viewed as a subgroup of $\underline{\mathrm{Hom}}_{A,d^0}(P^0_X, A)$, Take a differential operator D of order no more than n and a section a of A. Observing that for each $m, n \in \mathbb{N}$,

- $(\pi^m \otimes \mathrm{id}_{P^n_X}) \circ \delta^{m,n} : P^{m+n}_X \to P^n$ and $(\mathrm{id}_{P^m_X} \otimes \pi^n) \circ \delta^{m,n} : P^{m+n}_X \to P^m$ are the transition maps. ([Ber74, II, (1.1.10)]),
- $\pi^n \circ d^n_0 = \pi^n \circ d^n_1 = \mathrm{id}_A$ ([Ber74, II, (1.1.8)]),
- π^n is a ring homomorphism, and
- D is d^n_0-linear,

we obtain the following commutative diagrams

$$\begin{tikzcd}
P^n_X \arrow[r, "\delta^{0,n}"] \arrow[ddr, "\mathrm{id}_{P^n_X}"'] & P^0_X \otimes_{d^n_1,A,d^n_0} P^n_X \arrow[r, "\mathrm{id}_{P^n_0} \otimes D"] \arrow[d, "\pi^0 \otimes \mathrm{id}_{P^n_X}"] & P^0_X \otimes_{d^0_1,A} A \arrow[r, "\sim"] & P^0_X \arrow[r, "\pi^0"] & A \arrow[r, "\cdot a"] & A \\
& A \otimes_{A,d^n_0} P^n_X \arrow[d, "\sim"] & & & & \\
& P^n_X \arrow[uurrr, "D"'] & & & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
P^n_X \arrow[r, "\delta^{n,0}"] \arrow[drr, "\mathrm{id}_{P^n_X}"'] & P^n_X \otimes_{d^n_1,A,d^n_0} P^0_X \arrow[r, "\mathrm{id}_{P^n_X} \otimes \pi^0"] & P^n_X \otimes_{d^n_1,A} A \arrow[r, "\mathrm{id}_{P^n_X} \otimes (\cdot a)"] \arrow[d, "\sim"] & P^n_X \otimes_{d^n_1,A} A \arrow[r, "\sim"] & P^n_X \arrow[r, "D"] & A. \\
& & P^n_X \arrow[urr, "\cdot d^n_1(a)"'] & & &
\end{tikzcd}$$

This two diagrams indicate that for each section p of P^n_X and a of A, we have $(a \circ D)(p) = D(d^n_0(a)p) = a \cdot (D(p))$ and $(D \circ a)(p) = D(d^n_1(a)p)$.

Now assuming that (A3) holds, we can recover θ_n as in 1.2.2:

$$M \longrightarrow \underline{\mathrm{Hom}}_A(\underline{\mathrm{End}}(M), M) \longrightarrow \underline{\mathrm{Hom}}_{A,d^n_0}((P^n_X)^\vee, M) \xleftarrow{\ \sim\ } M \otimes_{A,d^n_0} P^n_X,$$

where the second map is induced by the d^n_0-linear map φ_n and the last isomorphism is due to (A3). This composition is d^1_n-linear as φ_n is.

How is each φ_n related to each other?

3.2. Conversely. Suppose that we have a d_1^n-linear morphism θ_n as in (C1). Then it gives rise to a $(P_X^n)^\vee$-module as follows. For any section D of $(P_X^n)^\vee$, define[3] $\varphi_n(D)$ to be the composition (*not A-linear* but additive)[4]

$$M \xrightarrow{\theta_n} M \otimes_{A,d_0^n} P_X^n \xrightarrow{\mathrm{id}_M \otimes \delta} M \otimes A \cong M$$

This defines a group homomorphism $\varphi_n : (P_X^n)^\vee \to \underline{\mathrm{End}}(M)$, $D \mapsto \varphi_n(D)$.

How to get a ring homomorphism $\mathcal{D} \to \underline{\mathrm{End}}(M)$?

$$\begin{tikzcd}
 & & P_X^m \otimes_{d_1^n,A} M \otimes_{A,d_0^n} P^n \arrow[dr, "\varepsilon_m \otimes \mathrm{id}_{P_X^n}"] & \\
P_X^{m+n} \otimes_{d_1^{m+n},A} M \arrow[r, "\delta^{m,n} \otimes \mathrm{id}_M"] & P_X^m \otimes_{d_1^n,A,d_0^n} P_X^n \otimes_{d_X^1,A} M \arrow[ur, "\mathrm{id}_{P_X^m} \otimes \varepsilon_n"] \arrow[rr, "\delta^{m,n}(\varepsilon_{m+n})"] & & M \otimes_{A,d_0^n} P_X^m \otimes_{d_1^n,A,d_0^n} P_X^m \arrow[d] \\
M & M \otimes_A A \arrow[l] & M \otimes_{A,d_0^n} P_X^m \arrow[l] & M \otimes_{A,d_0^n} P_X^m \otimes_A A \arrow[l, "\sim"']
\end{tikzcd}$$

4 Flat connections

References

[Ber74] Pierre Berthelot. *Cohomologie cristalline des schemas de caractéristique $p > 0$*. French. Lecture Notes in Mathematics. 407. Berlin-Heidelberg-New York: Springer-Verlag. 604 p. 1974.

[BO78] Pierre Berthelot and Arthur Ogus. *Notes on crystalline cohomology*. English. Mathematical Notes. Princeton, New Jersey: Princeton University Press. Tokyo: University of Tokyo Press. VI, not consecutively paged. 1978.

[Kat70] Nicholas M. Katz. "Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin." English. In: *Publ. Math., Inst. Hautes Étud. Sci.* 39 (1970), pp. 175–232.

[3] Recall [Kat70]. The equivalence of a connection $E \to E \otimes \Omega$ and $\mathrm{Der}(X/S) \to \mathrm{End}(E)$ is discussed.

[4] Compare with the map in 1.3. In case $f : X \to S$ and P^n is the n-the principal part, this map can be shown to be $f^{-1}\mathcal{O}_S$-linear.