Seminar on Crystalline Cohomology

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1 Divided Power Algebra—October 31, 2016

1.1 Divided Power Structure

Definition 1 ([Stacks, Tag 07GL]) Let A be a ring. Let I be an ideal of A. A collection of maps $\gamma_n : I \to I$, $n > 0$ is called a *divided power structure* (or *PD-structure*, after the French *puissances divisées*) on I if for all $n \geq 0$, $m > 0$, $x, y \in I$, and $a \in A$ we have

1) $\gamma_1(x) = x$, we also set $\gamma_0(x) = 1$,
2) $\gamma_n(x)\gamma_m(x) = \binom{m+n}{m}\gamma_{n+m}(x)$,
3) $\gamma_n(ax) = a^n\gamma_n(x)$,
4) $\gamma_n(x+y) = \sum_{i=0}^{n}\gamma_i(x)\gamma_{n-i}(y)$,
5) $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x)$. □

Remark 1 1) Note that $\frac{(nm)!}{n!(m!)^n}$ is an integer. In fact, it's easy to see that

$$\frac{(nm)!}{n!(m!)^n} = \frac{((n-1)m)!}{(n-1)!(m!)^{n-1}}\binom{(n-1)m}{m-1},$$

So by induction, we have[1]

$$\frac{(nm)!}{n!(m!)^n} = \binom{2m-1}{m-1}\binom{3m-1}{m-1}\cdots\binom{mn-1}{m-1}.$$

2) It follows from 3. that $\gamma_n(0) = 0$ for all $n \geq 1$.
3) Sometimes when γ is clear from context, we write $x^{[n]} := \gamma_n(x)$ for $x \in I$. □

Lemma 1 ([Stacks, Tag 07GM]) *Let A be a ring. Let I be an ideal of A.*

1) If γ is a divided power structure on I, then $n!\gamma_n(x) = x^n$ for $n \geq 1$, $x \in I$.

Assume A is torsion free as a $\mathbb{Z}$-module.

2) A divided power structure on I, if it exists, is unique.
3) If $\gamma_n : I \to I$ are maps then γ is a divided power structure if and only if $n!\gamma_n(x) = x^n$ for all $x \in I$ and $n \geq 1$.
4) The ideal I has a divided power structure if and only if there exists a set of generators x_i of I as an ideal such that for all $n \geq 1$ we have $x_i^n \in (n!)I$.

Proof The first two assertions are easy to check.

To show 3), note that if A is torsion free as a $\mathbb{Z}$-module. Then we have $A \subseteq A \otimes_{\mathbb{Z}} \mathbb{Q}$. So we can assume that A is a $\mathbb{Q}$-algebra. Then everything is easy to check.

That " $\Longrightarrow$ " in 4) is clear. To show " $\Longleftarrow$ ", assume that there exists such a set of generators x_i. We claim that every $x \in I$ satisfies that $x^n \in (n!)I$ for all $n \geq 1$, so that we could define $\gamma_n(x) := x^n/n!$. The key points are $(ax_i)^n = a^n x_i^n \in (n!)I$ and

$$(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k} \in \sum_k \binom{n}{k}(k!)I \cdot (n-k)!I \subseteq (n!)I$$

if the statement holds for x and y. So we can show this by induction. ■

Definition 2 ([Stacks, Tag 07GU]) A *divided power ring* is a triple (A, I, γ) where A is a ring, $I \subset A$ is an ideal, and $\gamma = (\gamma_n)_{n\geq 1}$ is a divided power structure on I. A *homomorphism of divided power rings* $f : (A, I, \gamma) \to (B, J, \delta)$ is a ring homomorphism $f : A \to B$ such that $f(I) \subset J$ and such that $\delta_n(f(x)) = f(\gamma_n(x))$ for all $x \in I$ and $n \geq 1$. □

Remark 2 If A and B are both torsion-free as $\mathbb{Z}$-module, then the requirement that $\delta_n(f(x)) = f(\gamma_n(x))$ is unnecessary. This follows from Lemma 1 ($n!\gamma_n(x) = x^n$). So this condition is interesting only in positive characteristic case. □

Example 1 1) The first trivial example is $(A, (0), 0)$.
2) $\mathbb{Q}$-algebra A with any ideal $I \subseteq A$. We have seen this in Lemma 1.
3) [Stacks, Stacks Project, 07GN] Let p be a prime number. Let A be a ring such that every integer n not divisible by p is invertible, i.e., A is a $\mathbf{Z}_{(p)}$-algebra. Then $I = pA$ has a canonical divided power structure. Namely, given $x = pa \in I$ we set

$$\gamma_n(x) = \frac{p^n}{n!}a^n.$$

The reader verifies immediately that $p^n/n! \in p\mathbf{Z}_{(p)}$ for $n \geq 1$ (**Exercise!**[2]). So the definition makes sense and gives us a sequence of maps $\gamma_n : I \to I$. It is a straightforward exercise to verify that conditions (1) – (5) of Definition 1 are satisfied. Alternatively, it is clear that the definition works for $A_0 = \mathbf{Z}_{(p)}$ and then the result follows from Proposition 3.

[1]This is the number of partitions of a set with mn elements into n subsets with q each (cf. [BO78, 3.1]).

[2]for instance, this can be derived from the fact that the exponent of p in the prime factorization of $n!$ is $\lfloor p/n \rfloor + \lfloor p/n^2 \rfloor + \lfloor p/n^3 \rfloor + \ldots$.

4) [BO78, 3.2, 3)] Let $(A, \mathfrak{m})$ be discrete valuation ring of mixed characteristics, i.e., its fraction field $\mathrm{Frac}(A)$ has characteristic 0 and its residue field $A/\mathfrak{m}$ is of characteristic $p > 0$. In this case, A is an $\mathbb{Z}_{(p)}$-algebra. Suppose $\mathfrak{m} = (\pi)$, i.e., π is a local parameter (or uniformizer). Then $p = u\pi^e$ for some integer $e \geq 1$ and u a unit in A. (The integer e is called the *absolute ramification index*.) Then $\mathfrak{m} = (\pi)$ has a (unique, e.g., by Lemma 1) divided power structure if and only if $e \leq p - 1$. (**Exercise!**[3])
5) [BO78, 3.2, 4)] Let A be a ring with $mA = 0$ for some m and I a divided power ideal. Then (by Lemma 1) for all $x \in I$, $x^n = n!\gamma_n(x) = 0$ whenever $n \geq m$. That is to say I is a *nil ideal*. If moreover I is finitely generated, we know $I^n = 0$ for n large enough. In the latter case, I is a *nilpotent ideal*.
   If $(m-1)!$ is a unit in A and $I^m = 0$, then I has a (not necessarily unique) divided power structure given by
$$\gamma_n(x) = \begin{cases} x^n/n!, & \text{if } n < m, \\ 0, & \text{if } n \geq m. \end{cases}$$
   All together, in characteristic $p > 0$, every divided power ideal satisfies $I^{(p)} := \{x^p : x \in I\} = 0$ and every ideal with I^p has a divided power structure.
   However, $I^{(p)} = 0$ is not sufficient for I to admit a divided power structure. For example, let k be a ring or characteristic $p > 0$ and set
$$A := \frac{k[x_1, \ldots, x_6]}{(x_1^p, \ldots, x_6^p, x_2x_2 + x_3x_4 + x_5x_6)}$$
   and $I := (x_1, \ldots, x_6)$. Then I do not admit any divided power structure. (**Exercise!**) $\square$

Proposition 1 ([Stacks, Tags 07GV, 07GX, and 07GY]) *The category of divided power rings has all limits and colimits. Moreover, the forgetful functor $(A, I, \gamma) \mapsto A$ commutes with limits but not colimits.*

Proof See [Stacks, Tags 07GV and 07GX] for the proof. And [Stacks, Tag 07GY] is a counter-example showing that the forgetful functor does not commute with colimits. ■

An special case of colimit is the following proposition.

Proposition 2 ([BO78, 3.7]) *Suppose A is a ring and B and C are two A-algebras. Let $J \subseteq B$ and $K \subseteq C$ are augmentation ideals (i.e., the natural map $B \to B/J$ has a section $B/J \to B$, such that the composition $B/J \to B \to B/J$ is $\mathrm{id}_{B/J}$ and similar for $C \to C/K$) with divided power structures δ and ϵ respectively. Then the kernel $I := B \otimes_A C \to (B/J) \otimes_A (C/J)$ admits a unique divided power structure γ such that*
$$(B, J, \delta) \to (B \otimes_A C, I, \gamma), \quad \textit{and} \quad (C, K, \epsilon) \to (B \otimes_A C, I, \gamma)$$
are both divided power morphisms.

Proof The statement could also be found at [Ber74, I, 1.7.1] and the proof is in [Rob65]. ■

Definition 3 Let (A, I, γ) be a divided power ring. An ideal $J \subseteq I$ is called a *divided power sub-ideal* if γ restricts to a divided power structure on J. $\square$

1.2 Extension of Divided Power Structure

Definition 4 ([Stacks, Tag 07H0]) Given a divided power ring (A, I, γ) and a ring map $A \to B$ we say γ *extends* to B if there exists a divided power structure $\bar{\gamma}$ on IB such that $(A, I, \gamma) \to (B, IB, \bar{\gamma})$ is a homomorphism of divided power rings. $\square$

[3] See for example [Ber74, I, 3.2.3]

Lemma 2 ([BO78, 3.5]) *Assume that (A, I, γ) is a divided power ring and $J \subseteq A$ an ideal of A. Then γ extends (necessarily uniquely) to A/J if and only if $I \cap J$ is a divided power sub-ideal, i.e., $\gamma_i(x) \in I \cap J$ for all $x \in I \cap J \subseteq I$.*

Proof See [BO78, 3.5] or [Ber74, I, 1.6.2]. Related results see [Stacks, Tag 07H2]. ■

Proposition 3 ([Stacks, Tag 07H1]) *Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. If γ extends to B then it extends uniquely. Assume (at least) one of the following conditions holds*

1) $IB = 0$,
2) I is principal, or
3) $A \to B$ is flat (see also Corollary 6).

Then γ extends to B.

Proof See [Stacks, Tag 07H1]. The proof of the second case could also be found in [BO78, 3.15]. The last statement appear again in Corollary 6. But [Stacks, Tag 07H1] gives a direct proof (without using divided power envelop). ■

1.3 Compatible Divided Power structure

Lemma 3 ([Stacks, Tag 07GQ]) *Let A be a ring with two ideals $I, J \subset A$. Let γ be a divided power structure on I and let δ be a divided power structure on J. Then*

1) γ and δ agree on IJ,
2) if γ and δ agree on $I \cap J$ then they are the restriction of a unique divided power structure ϵ on $I + J$. □

Proof For 1), note that

$$\gamma_n(xy) = y^n \gamma(x) = n! \delta_n(y) \gamma_n(x) = \delta_n(y) x^n = \delta_n(xy).$$

Then use the formula for $\gamma_n(x + y)$ to show this holds for all elements in IJ. For the second statement, see [BO78, 3.12] and [Ber74, I, 1.6.4]. ■

Lemma 4 ([BO78, 3.16]) *Let (A, I, γ) and (B, J, δ) be divided power rings. Let $A \to B$ be a ring homomorphism. Then the followings are equivalent.*

1) γ extends to $\bar{\gamma}$ on IB such that $\bar{\gamma}$ and δ coincide on $IB \cap J$.
2) there exists a (necessarily unique) divided power structure $\bar{\gamma}$ on $J + IB$ such that

$$(A, I, \gamma) \to (B, J + IB, \bar{\gamma}) \quad \text{and} \quad (B, J, \delta) \to (B, J + IB, \bar{\gamma})$$

are homomorphisms of divided power rings.

3) there exists a divided power structure ϵ on some $K \supseteq J + IB$ such that

$$(A, I, \gamma) \to (B, K, \bar{\epsilon}) \quad \text{and} \quad (B, J, \delta) \to (B, K, \bar{\epsilon})$$

are homomorphisms of divided power rings.

Proof 1) $\implies$ 2): Lemma 3.
2) $\implies$ 3): obvious.
3) $\implies$ 1): (Sketch) Observes that if $(A, I, \gamma) \to (B, J, \delta)$ is a divided power morphism, so IB is a divided power sub-ideal of J (**Exercise!**[4]). Then the desired result follows from this observation.
See also[Ber74, I, 2.2]. ■

Definition 5 ([BO78, 3.17]) Let (A, I, γ) and (B, J, δ) be divided power rings. Let $A \to B$ be a ring map. We say *δ is compatible with γ* if one (hence all) of the equivalent conditions in Lemma 4 holds. □

Lemma 5 *If (A, I, γ) and (B, J, δ) are compatible, then γ extends to B/J.*

Proof This follows from Lemma 4 and Lemma 2. ■

[4]See fro example [BO78, 3.14]

1.4 Divided Power Algebra Associated to a Module

Detailed proofs of statements in this section could be found in [Rob65; Rob63]. This construction is similar as the construction of the symmetric algebra associated to a module.

Definition 6 ([Ber74, I, 1.1.1]) Let A be a ring. We say a divided power ring (B, J, δ) is a *divided power A-algebra* if B is an A-algebra. Morphisms between two divided power A-algebras are divided power homomorphisms. □

Denote by C the category of divided power A-algebras. Let ω be the forgetful functor

$$\omega : \mathsf{C} \longmapsto \mathsf{Mod}_A, \quad (B, J, \delta) \longmapsto J.$$

Theorem 1 ([Ber74, I, 1.4.1]) *With the notations defined as above, the functor ω admits a left adjoint Γ_A. That is to say*

$$\mathrm{Hom}_{\mathsf{C}}\left(\Gamma_A(M), (B, J, \delta)\right) \cong \mathrm{Hom}_{\mathsf{Mod}_A}(M, J) \tag{1}$$

□

Proposition 4 ([Ber74, I, 1.4.2]) *Let M be an A module. Then $\Gamma_A(M)$ has a natural $\mathbb{Z}_{\geq 0}$-grading, such that*

$$\Gamma_A(M) = \bigoplus_{n \geq 0} \Gamma_A^n(M), \quad \Gamma_A^0(M) = A, \quad \Gamma_A^1(M) = M.$$

For any A-module homomorphism $M \to N$, $\Gamma_A(M \to N)$ is a homomorphism of graded A-algebras. Moreover the divided power ideal of $\Gamma_A(M)$ is

$$\Gamma_A^+(M) := \bigoplus_{n>0} \Gamma_A^n(M).$$

The divided power structure on $\Gamma_A^+(M)$ is usually denoted by $-^{[\ ]}$. And $\Gamma_A^n(M)$ is generated as an A-module by elements of the form

$$x_1^{[n_1]} x_2^{[n_2]} \cdots x_t^{[n_t]}, \text{ with } x_i \in M \text{ and } n_1 + n_2 + \cdots + n_t = n.$$ □

If furthermore M is a free A-module, then $\Gamma_A(M)$ is also free.

Remark 3 Recall that if an adjoint functor exists, then it exists uniquely up to a natural isomorphism.

We could rephrase Theorem 1 as follow: For each A-module M, there exists a divided power A-algebra $(\Gamma_A(M), \Gamma_A^+(M), -^{[\ ]})$, together with an A-module homomorphism $\varphi : M \to \Gamma_A^+(M)$, such that for each A-module J which is a divided power ideal of a divided power A-algebra (B, J, δ), and for any A-module morphism $\alpha :: M \to J$, there exists a unique a divided power morphism $\psi : (\Gamma_A(M), \Gamma_A^+(M), -^{[\ ]}) \to (B, J, \delta)$, such that $\alpha = (\psi|_{\Gamma_A^+(M)}) \circ \varphi$. □

Example 2 ([Ber74, I, 1.5]) Let A be a ring and I an index set. The *divided power polynomial algebra* relative to a family of indeterminants $(x_i)_i$, the divided power algebra associated to the free A-module A^I, and it's denote by $A\langle x_i \rangle_{i \in I}$. □

Example 3 ([Stacks, Tag 07H4, Tag 07H6]) Let A be a ring. Let $t \geq 1$. We will denote $A\langle x_1, \ldots, x_t \rangle$ the following A-algebra: As an A-module we set

$$A\langle x_1, \ldots, x_t \rangle = \bigoplus\nolimits_{n_1, \ldots, n_t \geq 0} A x_1^{[n_1]} \ldots x_t^{[n_t]}$$

with multiplication given by

$$x_i^{[n]} x_i^{[m]} = \frac{(n+m)!}{n!m!} x_i^{[n+m]}.$$

We also set $x_i = x_i^{[1]}$. Note that $1 = x_1^{[0]} \dots x_t^{[0]}$. There is a similar construction which gives the divided power polynomial algebra in infinitely many variables. There is an canonical A-algebra map $A\langle x_1, \dots, x_t\rangle \to A$ sending $x_i^{[n]}$ to zero for $n > 0$. The kernel of this map is denoted $A\langle x_1, \dots, x_t\rangle_+$.

If $t < s$, there is a natural morphism

$$A\langle x_1, \dots, x_t\rangle \to A\langle x_1, \dots, x_s\rangle$$

Then for any index set I, we define

$$A\langle x_i\rangle_{i\in I} := \varinjlim_t A\langle x_1, \dots, x_t\rangle.$$

□

1.5 Divided Power Envelop

Let (A, I, γ) be fixed a divided power ring. Let C_1 be the category of divided power rings over (A, I, γ). That is to say, objects of C_1 are divided power rings (B, J, δ) together with a divided power homomorphism $(A, I, \gamma) \to (B, J, \delta)$; and morphisms of C_1 are divided power homomorphisms over (A, I, γ). Let C_1' be the category of pairs (B, J) with B an A-algebra such that $IB \subseteq J$. Morphisms in C_1' are A-algebra homomorphisms which induces A-linear morphism between the given ideals. We have the forgetful functor

$$\omega_1 : \mathsf{C}_1 \longrightarrow \mathsf{C}_1', \quad (B, J, \delta) \longmapsto (B, J).$$

Theorem 2 ([Ber74, I, 2.3.1]) *With the notations defined as above, the functor ω_1 admits a left adjoint funtor D_γ. That is to say*

$$\operatorname{Hom}_{\mathsf{C}_1}\left(D_\gamma(B, J), (C, K, \epsilon)\right) \cong \operatorname{Hom}_{\mathsf{C}_1'}\left((B, J), (C, K)\right). \tag{2}$$

Proof (sketch) Here I sketch a proof following [BO78, 3.19]. The same proof could also be found in [Ber74, 2.3.1]. Another totally different but interesting (category-theoretic) proof can be found at [Stacks, Tag 07H8].

We will write $(\bar{J}, \bar{\gamma})$ the corresponding divided power ideal. Naturally, we would like to construct $D_\gamma(B, J)$ from the divided power algebra $\Gamma_B(J)$. Our goal is to construct $D_\gamma(B, J)$ as a quotient of $\Gamma_B(A)$ so that $\bar{J}$ is the image of $\Gamma_B^+(J)$ in the quotient. Of course, Lemma 2 is useful. Now suppose we have maps (dotted arrows and $\mathfrak{a}$ are to construct)

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, hook] & \Gamma_B(J) \arrow[r, dotted] & \Gamma_B(J)/\mathfrak{a} \\ I \arrow[r, "f|_I"] & J \arrow[r, "\varphi"] & \Gamma_B^+(J) \arrow[r, dotted] & (\Gamma_B^+(J) + \mathfrak{a})/\mathfrak{a} \end{tikzcd}$$

First all all, we are supposed to have $JD_\gamma(B, J) \subseteq \bar{J}$. But under the canonical B-algebra structure $B \hookrightarrow \Gamma_B(J)$, J is mapped into the degree 0 part. So we want to identify the J in the 0 degree part of $\Gamma_B(J)$ with the J in the positive degree part, i.e., $\varphi(J)$. So one family of relations that we need is

$$x - \varphi(x), \quad x \in J \subseteq \Gamma_B(J). \tag{i}$$

Moreover $D_\gamma(B, J)$ should be in the category C_1, that is to say if $D_\gamma(B, J)$ exists, we should have a divided power homomorphism $(A, I, \gamma) \to (D_\gamma, \bar{J}, \bar{\gamma})$. We are expected to have for all $x \in I$, that $\left(\varphi(f(x))\right)^{[n]} = \varphi(f(\gamma_n(x)))$. To force this to be true, we need another set of relations

$$\left(\varphi(f(x))\right)^{[n]} - \varphi(f(\gamma_n(x))) \in \Gamma_B^+(J), \quad x \in I. \tag{ii}$$

Now let $\mathfrak{a}$ be the ideal of $\Gamma_B(J)$ generated by elements of forms as in eqs. (i) and (ii), and set $D := \Gamma_B(J)/\mathfrak{a}$. To use Lemma 2, we need to show that $\Gamma_B^+(J) \cap \mathfrak{a}$ is a divided power sub-ideal respect to $-^{[\ ]}$.

Note that $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$ where $\mathfrak{a}$ is the ideal generated by elements of eq. (i) and $\mathfrak{a}_2$ of eq. (ii). We have $\mathfrak{a}_2 \subseteq \Gamma_B^+(J)$. Hence $\mathfrak{a} \cap \Gamma_B^+(J) = \mathfrak{a}_1 \cap \Gamma_B^+(J) + \mathfrak{a}_2$. Applying the formula for $(x+y)^{[n]}$, it suffices to show $-^{[\ ]}$ restricts to $\mathfrak{a}_1 \cap \Gamma_B^+(J)$ and $\mathfrak{a}_2$.

Now suppose $x \in \mathfrak{a}_1 \cap \Gamma_B^+(J)$. Write $x = \sum a_i(x_i - \varphi(x_i))$ with $a_i \in \Gamma_B(J)$. Each a_i could be decomposed as $a_i = a_i^0 + a_i^+$ with $a_i^0 \in \Gamma_B^0(J) = B$ and $a_i^+ \in \Gamma_B^+(J)$. So

$$x = \sum a_i^0(x_i - \varphi(x_i)) + \sum a_i^0\varphi(x_i) + \sum a_i^0\varphi(x_i).$$

The last term on the RHS should be zero as $x \in \Gamma_B^+(J)$ (as the last term is the degree 0 part of x). It follows that $0 = \phi(\sum a_i^0\varphi(x_i)) = \sum a_i^0\varphi(x_i)$, i.e., the second term in the above equation is also zero. Hence $x = \sum a_i^0(x_i - \varphi(x_i)) \in \mathfrak{a}_1\Gamma_B^+(J)$. So we have $\mathfrak{a}_1 \cap \Gamma_B^+(J) = \mathfrak{a}_1\Gamma_B^+(J)$. It's straightly forward to check **(Exercise!)** that $-^{[\ ]}$ restricts to $\mathfrak{a}_1 \cap \Gamma_B^+(J)$.

Now take any $x \in \mathfrak{a}_2$. Using the formula of $(x+y)^{[m]}$, it suffices **(Exercise!)** to show that $\left((\varphi(f(x)))^{[n]} - \varphi(f(\gamma_n(x)))\right)^{[m]} \in \mathfrak{a}_2$ for any $m, n \geq 1$. Now write $\psi = \phi \circ f$ to simplify notations[5].

$$\begin{aligned}
\left(\left(\psi(x)\right)^{[n]} - \psi(\gamma_n(x))\right)^{[m]} &= \sum_{r+s=m} \left((\psi(x))^{[n]}\right)^{[r]}(-1)^s(\psi(\gamma_n(x)))^{[s]} \\
&= \sum_{r+s=m} C_{n,r}(\psi(x))^{[nr]}(-1)^s\left(\psi(\gamma_n)\right)^{[s]} \\
&\equiv \sum_{r+s=m} C_{n,r}\psi(\gamma_{nr}(x))(-1)^s\psi(\gamma_s(\gamma_n(x))) \bmod \mathfrak{a}_2 \\
&\equiv \psi\left(\sum_{r+s=m} C_{n,r}\gamma_{nr}(x)(-1)^s\gamma_s(\gamma_n(x))\right) \\
&\equiv \psi\left(\sum_{r+s=m} \gamma_r(\gamma_n(x))(-1)^s\gamma_s(\gamma_n(x))\right) \\
&\equiv \psi\left(\gamma_m(\gamma_n(x) - \gamma(x))\right) \\
&\equiv 0 \bmod \mathfrak{a}_2
\end{aligned}$$

Therefore $-^{[\ ]}$ restricts to $\mathfrak{a} \cap \Gamma_B^+(J)$. So applying lemma 2, we get a divided power ring

$$D_\gamma(B,J) := (D_\gamma(B,J), \bar{J}, \bar{\gamma}) := \left(\Gamma_B(J)/\mathfrak{a}, (\Gamma_B^+(J) + \mathfrak{a})/\mathfrak{a}, -^{[\ ]}\right)$$

The adjoint property then follows from the adjoint property eq. (1) and the very construction of $D_\gamma(B,J)$. **(Exercise!)** ∎

Remark 4 Similar to Remark 3, we could rephrase Theorem 2 as follows: For each pair (B,J), there exists a divided power ring $(D_\gamma(B,J), \bar{J}, \bar{\gamma})$ together with a morphism $(B,J) \to (D_\gamma(B,J), \bar{J})$ in C_1', such that for each divided power ring (C,K,ϵ) in C_1 and each $(B,J) \to (C,K)$ in C_1', there exists a unique morphism $(D_\gamma(B,J), \bar{J}, \bar{\gamma}) \to (C,K,\epsilon)$ in C_1, making the diagram in C_1'

$$\begin{tikzcd}
(D_\gamma(B,J), \bar{J}) \arrow[dr, dotted] & \\
(B,J) \arrow[u] \arrow[r] & (C,K)
\end{tikzcd}$$

commute. □

In fact, with the concept of compatible divided power structures (Definition 5), we could generalize the above result a lit bit. Now let C_2 be the category of divided power A-algebras that are compatible with γ. Morphism in C_2 are just divided power homomorphisms. Let C_2'

[5] we could also omit ϕ using eq. (i), but actually, this part of proof does not rely on the those relations

be the category of pairs (B, J) of an A-algebra B and an *arbitrary* ideal $J \subseteq B$. Morphisms in C_2' are A-algebra homomorphism which induces A-linear morphism between the given ideals. And we still have the forgetful functor

$$\omega_2 : \mathrm{C}_2 \longrightarrow \mathrm{C}_2', \quad (B, J, \delta) \longmapsto (B, J).$$

As we can expected, the conclusion of Theorem 2 still hold.

Theorem 3 ([Ber74, I, 2.4.1]) *With the notations defined as above, the functor ω_2 admits a left adjoint funtor D_γ. That is to say*

$$\mathrm{Hom}_{\mathrm{C}_2}\left(D_\gamma(B, J), (C, K, \epsilon)\right) \cong \mathrm{Hom}_{\mathrm{C}_2'}\left((B, J), (C, K)\right). \tag{3}$$

□

Proof (sketch) See [BO78, 3.19] or [Ber74, I, 2.4.1].

The difference between Theorem 2 is that we do not have $IB \subseteq J$. But we have $(B, J + IB)$ is an object in C_1 Hence we have $D_\gamma(B, J + IB) = (D_\gamma(B, J + IB), \overline{B + IB}, \bar{\gamma})$ as on object in C_1 and an A-algebra homomorphism $B \to D_\gamma(B, J + IB)$. Let $\bar{J}$ be the divided power sub-ideal generated by $JD_\Gamma(B, J + IB)$ inside $\overline{J + IB}$. Now Set

$$D_\gamma(B, J) := \left(D_\gamma(B, J + IB), \bar{J}, \bar{\gamma}\right).$$

One verifies that $D_\gamma(B, J)$ is an object in C_2, i.e., $\bar{\gamma}$ is compatible with γ. The adjoint property then follows from the adjoint property eq. (2) and the construction of $D_\gamma(B, J)$. **(Exercise!)** ■

Remark 5 Of course, there is a more concrete rephrase of Theorem 3 as described in Remark 4. □

Definition 7 Let (A, J, I) be a fixed divided power ring. Let B be an A-algebra and $J \subseteq B$ an ideal. The divided power algebra $D_\gamma(B, J)$ in Theorem 2 and Theorem 3 is called the *divided power envelope of J in B relative (A, I, γ)* □

We will also use the following notations:

$$D_{B,\gamma}(J) := D_B(J) := D_\gamma(B, J),$$

to emphasize different parts that $D_\gamma(B, J)$ depends on.

Corollary 1 ([BO78, 3.20, 4)]) *We have a canonical morphism*

$$B/J \longrightarrow D_\gamma(B, J)/\bar{J}. \tag{4}$$

It is an isomorphism if and only if γ extends to B/J.

Proof (Sketch) See [Ber74, I, 2.3.2 iii) and 2.4.3 iii)].

The existence of this map comes from the universal property.

Note that in the situation of Theorem 2, the isomorphism always an isomorphism by construction. The condition that γ extends to B/J automatically holds (recall Lemma 2).

More generally in the situation of Theorem 3, if it is an isomorphism, then it follows from Lemma 5 that γ extends to B/J. On the other hand, if γ extends to B/J, then $(B/J, 0, 0)$ is an object in C_2, then the adjoint property eq. (i) gives a morphism $(D_\gamma(B, J), \bar{J}) \to (B/J, 0)$, which give an inverse to the canonical map $B/J \to D_\gamma(B, J)/\bar{J}$. **(Exercise!)** ■

Proposition 5 ([BO78, 3.20, 6)]) *Suppose γ extends to B/J and $B \to B/J$ admits a section, then there is a canonical divided power isomorphism*

$$D_0(B, J) \xrightarrow{\cong} D_\gamma(B, J).$$

Proof (Sketch) See [Ber74, I, 2.6.1].

The existence of the natural map comes from the universal property of $B_0(B,J)$. It's surjective (**Exercise!**).

Denote by $\bar{J}$ the divided power ideal of $D_0(B,J)$. We know that γ extends to $D_0(B,J)/\bar{J} \cong B/J$ (by Corollary 1). Besides, The natural surjection $D_0(B,J) \to B/J$ has a section, given by the composition $B/J \to B$ and $B \to D_0(B,J)$. Then one can show that γ is compatible with the divided power structure $\bar{J} \subseteq D_0(B,J)$ because of the existence of the section (see [Ber74, I, 2.2.4]). Then by the universal property of $D_\gamma(B,J)$, we get an inverse of the above natural map (**Exercise!**). ■

Proposition 6 ([BO78, 3.20, 7)]) *Assume (A,I,γ) is a divided power ring and B an A-algebra. Assume also $J,K \subseteq B$ are two ideals of B such that $KD_\gamma(B,J)=0$, i.e., $K \subseteq \mathrm{Ker}(B \to D_\gamma(B,J))$. Then there is a canonical divided power isomorphism*

$$D_\gamma(B,J) \xrightarrow{\cong} D_\gamma(B/K,(J+K)/K)$$

Proof See [Ber74, I, 2.6.2]. ■

Corollary 2 ([BO78, 3.20, 7)]) *Assume (A,I,γ) is a divided power ring and B an A-algebra. Suppose there is an $m \geq 1$ such that $mB=0$ and J is finitely generated, then there is an integer N, such that for all $n>N$, the canonical divided power homomorphism*

$$D_\gamma(B,J) \longrightarrow D_\gamma(B/J^n, J/J^n)$$

is an isomorphism.

Proof It follows from Example 1 5) that J is a nilpotent ideal. So we can apply Proposition 6 to the case $K := J^n$ for large enough n. See [Ber74, I, 2.6.3].

Proposition 7 ([BO78, 3.20, 5)]) *Let M be an A-module. Let $\mathrm{Sym}^\bullet_A(M)$ be the symmetric algebra associated to A and $\mathrm{Sym}^+_A(M)$ be the irrelevant ideal, i.e., the ideal generated by homogeneous elements of positive degrees. Then there is a canonical divided power isomorphism*

$$D_0(\mathrm{Sym}^\bullet_A(M), \mathrm{Sym}^+_A(M))) \xrightarrow{\cong} \Gamma_A(M).$$

Proof See [Ber74, I, 2.5.2]. ■

Corollary 3 ([Ber74, I, 2.5.3])

1.6 Flat Extension of Scalars

Lemma 6 ([Stacks, Tag 07HD]) *Let (A,I,γ) be a divided power ring. Let $B \to B'$ be a homomorphism of A-algebras. Assume that*

1) $B/IB \to B'/IB'$ is flat, and

2) $\mathrm{Tor}_1^B(B', B/IB)=0$.

Then for any ideal $IB \subset J \subset B$ the canonical map

$$D_\gamma(B,J) \otimes_B B' \longrightarrow D_\gamma(B',JB') \tag{5}$$

is an isomorphism.

Proof See [Stacks, Tag 07HD].

The natural map comes from 1) functoriality of D_γ: $D_\gamma(B,J) \to D_\gamma(B',JB')$; 2) universal property of $D_\gamma(B',JB')$; and 3) universal property of tensor product. ∎

Corollary 4 *If $B \to B'$ is flat at all primes of $V(IB') \subset \operatorname{Spec}(B')$, then eq. (5) is an isomorphism. It in particular says that taking the divided power envelope commutes with localization.*

Proof In case $B \to B'$ is flat everywhere, see [BO78, 3.21] for a direct proof. In general case, see [Stacks, Tag 07HD and Tag 051C]. ∎

In characteristic 0, for any ring A and ideal $I \subseteq A$, we always have $A \cong D_0(A,I)$ (**Exercise!**[6]). But this is not generally true.

Corollary 5 ([BO78, 3.23]) *Let A be a ring and $I \subseteq A$ an ideal. Then the canonical morphism $A \to D_0(A,I)$ is an isomorphism modulo torsion.*

Proof See [BO78, 3.23] and [Ber74, I, 2.7.2].

Let $A' := A \otimes_{\mathbb{Z}} \mathbb{Q}$ (hence $\mathbb{Z}$-torsion is then missing). Then $A \to A'$ is flat (**Exercise!**). Applying $- \otimes_{\mathbb{Z}} \mathbb{Q}$ to the natural map $A \to D_0(A,I)$, we have

$$A' \to D_0(A,I) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow[\text{eq. (5)}]{\cong} D_0(A', IA'),$$

This can be identified with the natural map $A' \to D_0(A',IA')$ and we know it's an isomorphism because A' is of characteristic 0. ∎

Corollary 6 ([BO78, 3.22]) *Let (A,I,γ) be a divided power ring and B a flat A-algebra. Then γ extends to B.*

Proof See [BO78, 3.22] and [Ber74, I, 2.7.3]. ∎

Lemma 7 ([Stacks, Tag 07HE]) *Let $(B,I,\gamma) \to (B',I',\gamma')$ be a homomorphism of divided power rings. Let $I \subset J \subset B$ and $I' \subset J' \subset B'$ be ideals. Assume*

1) $B/I \to B'/I'$ is flat, and

2) $J' = JB' + I'$.

Then the canonical map

$$D_\gamma(B,J) \otimes_B B' \longrightarrow D_{\gamma'}(B',J') \tag{6}$$

is an isomorphism.

Proof See [Stacks, Tag 07HE], which is a generalization of [Ber74, I, 2.8.2]. ∎

1.7 Divided Power Nilpotent Ideals

Let (A,I,γ) be a divided power ring. Following [BO78, 3.24] and [Stacks, Tag 07HQ], define $I_\gamma^{[n]} := I^{[n]}$ to be the ideal of A generated by

$$\gamma_{e_1}(x_1) \cdots \gamma_{e_t}(x_t) \text{ with } \sum e_j \geq n \text{ and } x_j \in I.$$

Remark 6 1) we have $I^n \subset I^{[n]}$. Note that $I^{[1]} = I$. Sometimes we also set $I^{[0]} = A$.

2) [Ber74, I, 3.1.2] We can see that $I^{[n]}$ is a divided power sub-ideal for each $n \geq 1$.

3) Take care that $I^{[n]}$ is not generated by such elements with $\sum e_j = n$, but $\sum e_j \geq n$.

4) [Ber74, I, 3.2.1] In characteristic 0, we have $I^{[n]} = I^n$. □

[6] See [Ber74, 2.5.1] for example

Definition 8 ([BO78, 3.27]) A divided power ideal I is *divided power nilpotent* if $I^{[n]} = 0$ for some $n \geq 1$. It is *divided power quasi-nilpotent* if $m \cdot I^{[n]} = 0$ for some $0 \neq m \in \mathbb{N}$ and $n \geq 1$. □

Fix $0 \neq m \in \mathbb{N}$ and $n \geq 1$, and let $\mathsf{C}_2(m,n)$ be the full subcategory of C_c with objects divided power rings (B, J, δ) such that $m \cdot J^{[n]} = 0$. We have a inclusion functor and a forgetful functor

$$\iota : \mathsf{C}_2(m,n) \to \mathsf{C}_2, \quad \omega_2^{m,n} : \mathsf{C}_2(m,n) \to \mathsf{C}_2'.$$

Theorem 4 ([Ber74, I, 3.3.1]) *The functor ι admits a left adjoint. Hence the functor $\omega_2^{m,n}$ also admits a left adjoint $D_\gamma^{m,n}$.*

Proof Let left adjoint of ι is given by $(B, J, \delta) \mapsto (B/m \cdot J^{[n+1]}, J/m \cdot J^{[n+1]}, \bar{\delta})$, i.e., extending δ to the quotient $B/m \cdot J^{[n+1]}$ (**Exercise**: check this is well-defined). Then using the adjoint property eq. (ii), we get the left adjoint of $\omega_2^{m,n}$ ■

So according to the proof, we see that

$$D_\gamma^{m,n}(B, J) = D_\gamma(B, J)/m \cdot \bar{J}^{[n+1]}. \tag{7}$$

As before, we sometimes also use notations

$$D_{B,\gamma}^{m,n}(J) := D_B^{m,n}(J) := D_\gamma^{m,n}(B, J), \quad \text{and} \quad D_{B,\gamma}^{n}(J) := D_B^{n}(J) := D_\gamma^{n}(B, J) := D_\gamma^{1,n}(B, J).$$

Proposition 8 ([Ber74, I, 3.3.2])

$$D_\gamma^{m,n}(B, J) \xrightarrow{\cong} D_\gamma^{m,n}(B/K, J + K/K)$$

Proof This follows from Proposition 6 and eq. (7). ■

Corollary 7 ([Ber74, I, 3.3.3])

$$D_\gamma^{n}(B, J) \xrightarrow{\cong} D_\gamma^{n}(B/J^{n+1}, J/J^{n+1})$$

Proof This follows from 7 and Proposition 8 and it is a generalization of Corollary 2. ■

Proposition 9

$$A \xrightarrow{\cong} D_0^{m-1}(A, I)$$

Proof This follows from Corollary 5 and eq. (7). ■

2 Calculus with Divided Powers — November 7, 2016

For this part, I mainly follow [Ber74, I, §4, and II].

2.1 Divided Power Schemes

Definition 9 ([Stacks, Tag 07I2]) Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. A *divided power structure* γ on $\mathcal{I}$ is a sequence of maps $\gamma_n : \mathcal{I} \to \mathcal{I}$, $n \geq 1$ such that for any object U of $\mathcal{C}$ the triple

$$(\mathcal{O}(U), \mathcal{I}(U), \gamma)$$

is a divided power ring. A triple $(\mathcal{C}, \mathcal{I}, \gamma)$ as in the definition above is sometimes called a *divided power topos*. Given a second $(\mathcal{C}', \mathcal{I}', \gamma')$ and given a morphism of ringed topoi $(f, f^\sharp) : (\mathrm{Sh}(\mathcal{C}), \mathcal{O}) \to (\mathrm{Sh}(\mathcal{C}'), \mathcal{O}')$ we say that $(f, f^\sharp)$ induces a *morphism of divided power topoi* if $f^\sharp(f^{-1}\mathcal{I}') \subset \mathcal{I}$ and the diagrams

$$\begin{array}{ccc} f^{-1}\mathcal{I}' & \xrightarrow{f^\sharp} & \mathcal{I} \\ \downarrow{\scriptstyle f^{-1}\gamma'_n} & & \downarrow{\scriptstyle \gamma_n} \\ f^{-1}\mathcal{I}' & \xrightarrow{f^\sharp} & \mathcal{I} \end{array}$$

are commutative for all $n \geq 1$. If f comes from a morphism of sites induced by a functor $u : \mathcal{C}' \to \mathcal{C}$ then this just means that

$$(\mathcal{O}'(U'), \mathcal{I}'(U'), \gamma') \longrightarrow (\mathcal{O}(u(U')), \mathcal{I}(u(U')), \gamma)$$

is a homomorphism of divided power rings for all objects U' of $\mathcal{C}'$. □

Definition 10 ([Stacks, Tag 07II]) A *divided power scheme* is a triple $(S, \mathcal{I}, \gamma)$ where S is a scheme, $\mathcal{I}$ is a quasi-coherent sheaf of ideals, and γ is a divided power structure on $\mathcal{I}$. A *morphism of divided power schemes* $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ is a morphism of schemes $f : S \to S'$ such that $(f^{-1}\mathcal{I}')\mathcal{O}_S \subset \mathcal{I}$ and such that

$$(\mathcal{O}_{S'}(U'), \mathcal{I}'(U'), \gamma') \longrightarrow (\mathcal{O}_S(f^{-1}U'), \mathcal{I}(f^{-1}U'), \gamma)$$

is a homomorphism of divided power rings for all $U' \subset S'$ open. □

Remark 7
1) Given a divided power scheme $(T, \mathcal{J}, \gamma)$ we get a canonical closed immersion $U \to T$ defined by $\mathcal{J}$. Conversely, given a closed immersion $U \to T$ and a divided power structure γ on the sheaf of ideals $\mathcal{J}$ associated to $U \to T$ we obtain a divided power scheme $(T, \mathcal{J}, \gamma)$.
2) One can easily define the direct image (or push-forward) functor f_* and inverse image functor f^{-1}. See [Ber74, I, 1.9.2] for details. □

Proposition 10 ([Ber74, I, 4.1.1]) *Let $(S, \mathcal{I}, \gamma)$ be a divided power scheme. Let X be an S-scheme with $f : X \to S$. Suppose $\mathcal{B}$ is a quasi-coherent $\mathcal{O}_X$-algebra and $\mathcal{J} \subseteq \mathcal{B}$ a quasi-coherent ideal. Then $\mathcal{B}$ is an $f^{-1}\mathcal{O}_X$-algebra via the natural maps $f^{-1}\mathcal{O}_S \to \mathcal{O}_X \to \mathcal{B}$. Then $\mathcal{D}_\gamma(\mathcal{B}, \mathcal{J}) := \mathcal{D}_{f^{-1}\gamma}(\mathcal{B}, \mathcal{J})$ is a quasi-coherent $\mathcal{O}_X$-algebra.*

Proof See [Ber74, 4.1.1]. ■

We will denote by $\overline{\mathcal{J}}$ for the divided power ideal of the envelope $\mathcal{D}_\gamma(\mathcal{B}, \mathcal{J})$.

Corollary 8 ([Ber74, I, 4.1.2]) *Let (S, I, γ) be a divided power scheme. We also have that $\mathcal{D}^{m,n}_\gamma(\mathcal{B}, \mathcal{J}) := \mathcal{D}^{m,n}_{f^{-1}\gamma}(\mathcal{B}, \mathcal{J})$ is a quasi-coherent $\mathcal{O}_X$-algebra.* □

2.2 Infinitesimal Divided Power Neighbourhood

Suppose that $i : Y \hookrightarrow X$ is a closed embedding over S which corresponds to the exact sequence

$$0 \to \mathcal{J} \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0.$$

It follows Proposition 10 and Corollary 8 that we could define for $m, n \in \mathbb{N}$ with $m \neq 0$, that

$$\begin{aligned} D_Y(X) &:= D_{Y,\gamma}(X) := \mathcal{S}pec\, \mathcal{D}_\gamma(\mathcal{O}_X, \mathcal{J}), \\ D_Y^{m,n}(X) &:= D_{Y,\gamma}^{m,n}(X) := D_Y^{m,n}(X) := \mathcal{S}pec\, \mathcal{D}_\gamma^{m,n}(\mathcal{O}_X, \mathcal{J}), \\ D_Y^{n}(X) &:= D_{Y,\gamma}^{n}(X) := \mathcal{S}pec\, \mathcal{D}_\gamma^{n}(\mathcal{O}_X, \mathcal{J}). \end{aligned}$$

All the above schemes are affine over X by definition and $D_Y^{m,n}(X)$ is the closed sub-scheme defined by $m \cdot \overline{\mathcal{J}}^{[n+1]}$.

Remark 8 It makes sense to speak about γ extends to Y.

Recall Corollary 1 that, if γ extends to Y, then we have isomorphism

$$\mathcal{D}_\gamma^0(\mathcal{O}_X, \mathcal{J}) = \mathcal{D}_\gamma(\mathcal{O}_X, \mathcal{J})/\mathcal{I} \cong \mathcal{O}_X/\mathcal{J} \cong i_* \mathcal{O}_Y.$$

It follows that we have natural isomorphism

$$Y \xrightarrow{\cong} \mathcal{S}pec(i_* \mathcal{O}_Y) \xrightarrow{\cong} D_Y^0(X).$$

Moreover, note that $J/J^{[n+1]}$ is a nilpotent ideal, as $J^{n+1} \subseteq J^{[n+1]}$ (Remark 6 1)). It follows that all $D_Y^i(X)$ has the same underlying topological space as Y.

If X is a torsion scheme ($m\mathcal{O}_X = 0$ for some m), then by Example 1 5), $\bar{\mathcal{J}}$ is a nil ideal (not necessarily a nilpotent ideal), hence $D_Y(X)$ has the same underlying topological space as $D_Y^1(X)$, that is to say, $D_Y(X)$ has the same underlying topological space as Y.

If $Y \hookrightarrow X$ is a locally closed embedding, which could be written as

$$\begin{tikzcd} Y \arrow[rr, hook] \arrow[dr, hook] & & X \\ & U \arrow[ur, hook] & \end{tikzcd} \tag{8}$$

Proposition 11 ([Ber74, I, 4.2.1]) *Suppose $Y \to X$ is a locally closed embedding with a factorization like eq. (8). Then*

1) *The schemes $D_Y^{m,n}(U)$ and $D_Y^n(U)$ are independent of the choice of U.*
2) *If X is a torsion scheme ($m\mathcal{O}_X = 0$ for some $m \in \mathbb{N}$), then $D_Y(U)$ is also independent of the choice of U.* □

Definition 11 ([Ber74, I, 4.1.7]) Suppose $Y \hookrightarrow X$ is a locally closed embedding and suppose that γ extends to Y. Then

1) the schemes $\mathcal{D}_Y^n(X)$ is called the *n-th divided power infinitesimal neighborhood* of Y in X;
2) if in addition X is a torsion scheme, the scheme $D_Y(X)$ is called the *divided power infinitesimal neighborhood* of Y in X. □

To summarize, we have the following commutative diagram.

$$\begin{tikzcd} D_Y^1(X) \arrow[r, hook] & D_Y^2(X) \arrow[r, hook] & \cdots \arrow[r, hook] & D_Y^n(X) \arrow[r, hook] \arrow[drr] & \cdots \arrow[r, hook] & D_Y(X) \arrow[d, "\text{affine}"', "\text{not neces. cl.}"] \\ Y \arrow[u, "\cong"] \arrow[rrrrr, hook, "i"] & & & & & X \end{tikzcd} \tag{9}$$

Remark 9 1) Note that in general $D_Y^n(X)$ is not a subscheme of X, though they are called "neighborhood".
2) universal property. [Ber74, I, 4.1.5]. □

The following proposition is just to summarize we have obtained.

Proposition 12 *Let $i : Y \to X$ be a locally closed embedding. Then*

1) *$D_Y^{m,n}$ and $D_Y(X)$ whenever it is defined are affine schemes over X.*

2) $D_Y^{m,n}(X)$ *has the same underlying topological space as* Y*.*
3) if X *is torsion, then* $D_Y(X)$ *has the same topological space as* Y*.* □

Theorem 5 ([Ber74, I, 4.5.1, 4.5.2]) *Let* S *be a scheme and* $i : Y \to X$ *a locally closed embedding and* X *and* Y *are both smooth over* S*. In this case* i *is a regular embedding ([Stacks, Tag 067T]). Suppose* i *is of codimension* d*. Then*

1) locally on X*, for every* $m \neq 0$ *and every* n*, there exists an isomorphism*

$$\mathcal{S}pec(\mathcal{O}_Y\langle T_1, \dots, T_d\rangle / m \cdot J^{[n+1]}) \xrightarrow{\cong} D_Y^{m,n}(X)$$

of divided power schemes, where J *is the divided power ideal generated by* T_i*'s.*

2) if X *is a torsion scheme, then locally on* X*, there is an isomorphism*

$$\mathcal{S}pec(\mathcal{O}_Y\langle T_1, \dots, T_d\rangle) \xrightarrow{\cong} D_Y(X).$$ □

Now let $f : X \to S$ be an S scheme and $(S, \mathcal{I}, \gamma)$ is a divided power scheme. Let $\Delta_f^k : X \to X_{/S}^{k+1} := \underbrace{X \times_S \cdots \times_S X}_{k \text{ times}}$ be the diagonal morphism, which is a locally closed embedding. If γ extends to X, then by Proposition 5, the construction is independent of the choice of γ. Suppose so and define

$$D_{X/S}^{m,n}(k) := D_X^{m,n}(X_{/S}^{k+1}), \qquad D_{X/S}^{n}(k) := D_X^{n}(X_{/S}^{k+1}).$$

Now suppose moreover (at least) one of the following conditions is satisfied.

1) X/S is separated;
2) X is a torsion scheme,

This allows us to define

$$D_{X/S}(k) := D_X(X_{/S}^{k+1}).$$

This is because $D_X(X_{/S}^{k+1})$ is always well-defined for closed embeddings but only well-defined for locally closed embeddings when X is a torsion scheme (Proposition 11).

Recall Proposition 12 that $D_{X/S}^{m,n}(k)$ (and $D_{X/S}$ if X is torsion) has the same underlying topological space as X, and each projection $\mathrm{pr}_i : X_{/S}^{k+1} \to X$, which is a section of $\Delta_f^k : X \to X_{/S}^{k+1}$, provides the structure sheaf of $D_{X/S}^{m,n}(k)$ (and $D_{X/S}$ if X is torsion) an $\mathcal{O}_X$-module (indeed $\mathcal{O}_X$-algebra) structure. And by a little bit abuse of notations, we define $\mathcal{D}_{X/S}^{m,n}$, $\mathcal{D}_{X/S}^{n}(k)$, and $\mathcal{D}_{X/S}(k)$ to be the structure sheaf of $D_{X/S}^{m,n}(k)$, $D_{X/S}^{n}(k)$, and $D_{X/S}(k)$ respectively.

To simplify notations, when $k = 1$, we omit (k) in the above notations. For example we will write $D_{X/S}^n$ and $\mathcal{D}_{X/S}^n$ instead of $D_{X/S}^n(1)$ $\mathcal{D}_{X/S}^n(1)$.

Proposition 13 ([Ber74, I, 4.4.3]) *Let* $(S, \mathcal{I}, \gamma)$ *be a divided power scheme and* X/S *an* S*-scheme such that* γ *extends to* X*. Then the natural homomorphism*

$$\mathcal{P}_{X/S}^1(k) \to \mathcal{D}_{X/S}^1(k)$$

is an isomorphism, where $\mathcal{P}_{X/S}^1$ *is the sheaf of first principal parts.*

Proof This follows from Proposition 9 and Proposition 5. ∎

Remark 10 For the relations between $\mathcal{P}_{X/S}^n$ and $\mathcal{D}_{X/S}^n$, see for example [Ber74, II, 1.1.5, b)]. □

2.3 Divided Power Differential Operators

Let X/S be an scheme over S. Recall that the two projections $\mathrm{pr}_i : X \times_S X \to X$, $i = 0, 1$ define two $\mathcal{O}_X$-algebra structures $d_i^n : \mathcal{O}_X \to \mathcal{D}^n_{X/S}$. Moreover, there are natural morphisms $\pi^n : \mathcal{D}^n_{X/S} \to \mathcal{O}_X$ by construction (see for example eq. (9)).

Recall Proposition 2 and theorem 4 that there is a natural morphism

$$\delta^{m,n} : \mathcal{D}^{m+n}_{X/S} \to \mathcal{D}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^n_{X/S}. \tag{10}$$

Now the datum we have is $(\mathcal{O}_X, \mathcal{D}^n_{X/S}, d_0^n, d_1^n, \pi^n, \delta^{m,n})$.

Definition 12 ([Ber74, II, 2.1.3, b) & c)]) Let $\mathcal{E}$, and $\mathcal{F}$ be two $\mathcal{O}_X$-modules.

1) A *divided power differential operator of order no more than n (relative to S)* is a morphism $f : \mathcal{D}_{X/S} \otimes \mathcal{E} \to \mathcal{F}$ of $\mathcal{O}_X$-modules.
2) If X is torsion scheme, a *divided power hyper-differential operator (relative to S)* is morphism $f : \mathcal{D}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{F}$ of $\mathcal{O}_X$-modules. □

([Ber74, II, 2.1.2]) Note that any divided power differential operator $f : \mathcal{D}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{F}$ induces a morphism $\mathcal{E} \to \mathcal{F}$ of sheaves of groups (not necessarily of $\mathcal{O}_X$-modules, but only $f^{-1}\mathcal{O}_S$-linear) $f^\flat : \mathcal{E} \to \mathcal{F}$. In fact, $f^\flat$ is the composition

$$\mathcal{E} \cong \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{d_1^n \otimes \mathrm{id}} \mathcal{D}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{f} \mathcal{F}. \tag{11}$$

Be careful that $f \mapsto f^\flat$ is not injective (see Example 4).

([Ber74, II, 2.1.6]) Suppose that $\mathcal{E}, \mathcal{F}, \mathcal{G}$ are $\mathcal{O}_X$ modules and $f : \mathcal{D}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{F}$ and $g : \mathcal{D}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{G}$. We can define the *composition* $g \circ f$ of f and g by the composition of maps

$$\mathcal{D}^{n+m}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\delta^{n,m} \otimes \mathrm{id}} \mathcal{D}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\mathrm{id} \otimes f} \mathcal{D}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{g} \mathcal{G}. \tag{12}$$

Moreover, one checks that

$$(g \circ f)^\flat = g^\flat \circ f^\flat. \tag{13}$$

In fact, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{D}^{n+m}_{X/S} \otimes \mathcal{E} \arrow[rr, "\delta^{n,m} \otimes \mathrm{id}"] & & \mathcal{D}^n_{X/S} \otimes \mathcal{D}^m_{X/S} \otimes \mathcal{E} \arrow[dr, "\mathrm{id} \otimes f"] & & \\
& \mathcal{D}^m_{X/S} \otimes \mathcal{E} \arrow[ur] \arrow[dr, "f"] & & \mathcal{D}^n_{X/S} \otimes \mathcal{F} \arrow[dr, "g"] & \\
\mathcal{E} \arrow[uu] \arrow[ur] \arrow[rr, "f^\flat"] & & \mathcal{F} \arrow[ur] \arrow[rr, "g^\flat"] & & \mathcal{G}
\end{tikzcd}$$

Remark 11 We will denote by PD-Diff$^n_{X/S}(\mathcal{E}, \mathcal{F})$ the ring of divided power differential operators. One can also define the sheaf of divided power (hyper-)differential operators, which is denoted by PD-$\mathcal{D}iff^n_{X/S}(\mathcal{E}, \mathcal{F})$.

Example 4 (([Ber74, II, 2.1.7])) Let $f : X \to S$ be an S-scheme and S is of characteristic $p > 0$. Recall that an S-derivation of $\mathcal{O}_X$ determines a differential operator of order no more than 1 (in the sense of [EGA IV$_4$], i.e., a morphism $\mathcal{P}^1 \to \mathcal{O}_X$). Recall Proposition 13 that this also gives a divided power differential operator of order no more than 1. Denote by D the divided power operator and $D^\flat$ the corresponding morphism $\mathcal{O}_X \to \mathcal{O}_X$ defined by eq. (11), i.e., the given derivation. Note that by definition of composition, eq. (12), the p-th power (p-th iterate) of D is a divided power differential operator of order less than p, denoted by D^p. On the other hand, p-th iterate of $D^\flat$ is again a derivation hence corresponds to a (divided power) differential operator of order no more than 1, denoted by $D^{(p)}$. In general, $D^p \neq D^{(p)}$, even though they induces the same endomorphism $(D^\flat)^p$ by eq. (13). As we will see in Proposition 14, D^p will be a divided power differential operator of order p. □

Proposition 14 ([Ber74, II, 4.2.6]) *Suppose that X/S is smooth and $(x_i)_{1\le i\le n}$ is a local coordinates. Set*

$$\xi_i := d_1^k(x_i) - d_0^k(x_i)$$

Recall Theorem 5 that $\xi^{[\mathbf{q}]}$ for $|\mathbf{q}| \le k$ form a basis for $\mathcal{D}^k_{X/S}$. Let $D_{\mathbf{q}}$ be the dual basis of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}^k_{X/S}, \mathcal{O}_X) = \text{PD-}\mathcal{D}iff_{X/S}(\mathcal{O}_X, \mathcal{O}_X)$. Then

$$D_{\mathbf{p}} \circ D_{\mathbf{q}} = D_{\mathbf{p}+\mathbf{q}}.$$

Proof (sketch) [Ber74, II, 4.2.5]. We need to show that

$$(D_{\mathbf{p}} \circ D_{\mathbf{q}})(\xi^{[\mathbf{r}]}) = D_{\mathbf{p}+\mathbf{q}}(\xi^{[\mathbf{r}]}).$$

Note that $(D_{\mathbf{p}} \circ D_{\mathbf{q}})$ is the composition

$$\mathcal{D}^{|\mathbf{p}|+|\mathbf{q}|}_{X/S} \xrightarrow{\delta^{|\mathbf{p}|,|\mathbf{q}|}} \mathcal{D}^{|\mathbf{p}|}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^{|\mathbf{q}|}_{X/S} \xrightarrow{\mathrm{id}\otimes D_{\mathbf{q}}} \mathcal{D}^{|\mathbf{p}|}_{X/S} \xrightarrow{D_{\mathbf{p}}} \mathcal{O}_X.$$

Then one checks readily the desired equality holds (**Exercise!**). ■

2.4 Divided Power Stratification

Now we have three maps from $\mathcal{D}^{m+n}_{X/S} \to \mathcal{D}^m_{X/S} \otimes \mathcal{D}^n_{X/S}$, namely, $\delta^{m,n}$, $q_0^{m,n}$ and $q_1^{m,n}$, where $q_0^{m,n}$ is the composition of natural maps

$$\mathcal{D}^{m+n}_{X/S} \longrightarrow \mathcal{D}^m_{X/S} \longrightarrow \mathcal{D}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^n_{X/S}$$

and $q_0^{m,n}$ is the composition of

$$\mathcal{D}^{m+n}_{X/S} \longrightarrow \mathcal{D}^n_{X/S} \longrightarrow \mathcal{D}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^n_{X/S}$$

We have two maps from $\mathcal{O}_X$ to $\mathcal{D}^{m+n}_{X/S}$, namely, d_0^{m+n} and d_1^{m+n}. Consider their compositions

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, bend left, "d_0^{m+n}"] \arrow[r, bend right, "d_1^{m+n}"'] & \mathcal{D}^{m+n}_{X/S} \arrow[r, bend left, "q_0^{m,n}"] \arrow[r, "\delta^{m,n}"] \arrow[r, bend right, "q_1^{m,n}"'] & \mathcal{D}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^n_{X/S}. \end{tikzcd}$$

which give rise to three maps from $\mathcal{O}_X$ to $\mathcal{D}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^n_{X/S}$ (**Exercise!**[7])

$$\begin{aligned} q_0^{m,n} \circ d_0^{m+n} &= \delta^{m,n} \circ d_0^{m+n} \\ q_0^{m,n} \circ d_1^{m+n} &= q_1^{m,n} \circ d_0^{m+n} \\ q_1^{m,n} \circ d_1^{m+n} &= \delta^{m,n} \circ d_1^{m+n} \end{aligned}$$

For any $\mathcal{O}_X$-module $\mathcal{E}$, write respectively

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^n_{X/S}, \quad \mathcal{D}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}^n_{X/S}, \quad \mathcal{D}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E},$$

for the base change of $\mathcal{E}$ to an $\mathcal{D}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^n_{X/S}$-module via the above three maps.

For any morphism of sheaves of rings $f : \mathcal{A} \to \mathcal{B}$, and any morphism of $\mathcal{A}$-modules $\phi : \mathcal{E} \to \mathcal{F}$, we will write $f^*(\phi)$[8] for the base change of ϕ to a morphism of $\mathcal{B}$-modules.

[7] See for example [Ber74, II, 1.3].

[8] l'étoile étant mise en exposant pour se conformer à l'intuition géométrique. [Ber74, II, 1.3].

Definition 13 (([Ber74, II, 1,2.1, 1.2.2 b), 1.3.1 & 1.3.6])) Let $\mathcal{E}$ be an $\mathcal{O}_X$-module.

1) A *divided power n-connection (relative to S)* is an isomorphism

$$\epsilon_n : \mathcal{D}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}^n_{X/S}$$

of $\mathcal{D}^n_{X/S}$-modules, which reduces to identity modulo the augmentation ideal of $\mathcal{D}_{X/S}$.

2) A *divided power pseudo-stratification* of $\mathcal{E}$ is a collection of n-connections

$$\epsilon_n : \mathcal{D}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}^n_{X/S}$$

such that for all $m \leq n$, the diagram

$$\begin{tikzcd} \mathcal{D}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \arrow[r, "\epsilon_n"] \arrow[d] & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}^n_{X/S} \arrow[d] \\ \mathcal{D}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \arrow[r, "\epsilon_m"] & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}^m_{X/S} \end{tikzcd}$$

is commutative.

3) A *divided power stratification* of $\mathcal{E}$ is divided power pseudo-stratification (ϵ_n), such that for any $0 \leq k \leq n$, the *cocycle condition*

$$(\delta^{k,n-k})^*(\epsilon_n) = (q_0^{k,n-k})^*(\epsilon_n) \circ (q_1^{k,n-k})^*(\epsilon_n).$$

□

Remark 12 There are some equivalent conditions of the cocycle condition. See [Ber74, 1.3.3, 1.4.3 & 1.4.4]. □

Theorem 6 ([BO78, II, 4.8]) *Let X/S be a smooth morphism and $\mathcal{E}$ an $\mathcal{O}_X$-module. TFAE*

1) a divided power stratification.

2) a flat connection Δ on $\mathcal{E}$.

3) a collection of $\mathcal{O}_X$-linear maps

$$\text{PD-}\mathcal{D}\mathit{iff}^n_{X/S}(\mathcal{O}_X, \mathcal{O}_X) \to \text{PD-}\mathcal{D}\mathit{iff}^n_{X/S}(\mathcal{E}, \mathcal{E})$$

which fit together to give a ring homomorphism

$$\varinjlim \text{PD-}\mathcal{D}\mathit{iff}_{X/S}(\mathcal{O}_X, \mathcal{O}_X) \to \varinjlim \text{PD-}\mathcal{D}\mathit{iff}_{X/S}(\mathcal{E}, \mathcal{E}).$$

4) for all $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, maps

$$\text{PD-}\mathcal{D}\mathit{iff}^n_{X/S}(\mathcal{F}, \mathcal{G}) \to \text{PD-}\mathcal{D}\mathit{iff}^n_{X/S}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{G})$$

taking identities to identities and compatible with composition.

Proof Detailed proofs see [Ber74, II, 2,2.5 & 4.2.11]. Sketched proof see [BO78, II, 4.8]. ■

Remark 13 Almost all the above discussion applies to any datum

$$(A, P^n, d_0^n, d_1^n, \pi^n, \delta^{m,n}) \tag{14}$$

that behaves like $(\mathcal{O}_X, \mathcal{D}^n_{X/S}, d_0^n, d_1^n, \pi^n, \delta^{m,n})$. Doing so, we could unify some different theories.

For example we can consider $(\mathcal{O}_X, \mathcal{P}^n_{X/S}, d_0^n, d_1^n, \pi^n, \delta^{m,n})$, where $\mathcal{P}^n$ is the sheaf of principal parts of order n. In fact, Berthelot did so in [Ber74, II] and such a datum like eq. (14) is called a *formal category* ([Ber74, II, 1.1.3]). □

Crystalline Topos/site (Recap)

Let $(S, \mathcal{I}, \gamma)$ be a divided power scheme and $f : X \to S$ be a morphism of schemes. Suppose that γ extends to X as always. We also assume that p is locally nilpotent on all schemes that we consider.

Definition 14 ([Ber74, III, 1.1.1]) The *(small) crystalline site of X (relative to S)*, denoted by $\mathrm{Crys}(X/S, \mathcal{I}, \gamma)$ or $\mathrm{Crys}(X/S)$, is the following data:

1) An object (U, T, δ) of $\mathrm{Crys}(X/S)$ is an open subscheme U of X with a closed embedding $i : U \to T$ over S and a divided power structure δ on $\mathcal{J} := \mathrm{Ker}(\mathcal{O}_T \to i_*\mathcal{O}_U)$ that is compatible with γ. Such an object is called a divided power thickening of U.
2) a morphism $g : (U, T, \delta) \to (U', T', \delta')$ in $\mathrm{Crys}(X/S)$ is a divided power morphism $g : T \to T'$ over S such that the diagram

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, hook] & U' \arrow[d, hook] \\ T \arrow[r, "g"] & T' \end{tikzcd}$$

is commutative, where the map $U \to U'$ is a inclusion.
3) A covering of an object (U, T, δ) is a family of morphisms $(U_i, T_i, \delta_i) \to (U, T, \delta)$ in $\mathrm{Crys}(X/S)$ such that the morphisms $T_i \to T$ are jointly surjective open embeddings onto T.

The *crystalline topos of X (relative to S)*, denoted by $(X/S)_{\mathrm{crys}}$, is the topos associated to the site $\mathrm{Crys}(X/S)$, i.e., the category of sheaves on $\mathrm{Crys}(X/S)$. □

Definition 15 ([Ber74, III, 4.1.1]) *big crystalline site.* □

[SGA IV$_1$, Exercise 4.10.6], [Ber74, III, 1.1.4] Let $\mathcal{F}$ be a sheaf of sets on $\mathrm{Crys}(X/S)$. For any object (U, T, δ) in $\mathrm{Crys}(X/S)$,

$$\mathcal{F}_{(U,T,\delta)}, \quad \text{or} \quad \mathcal{F}_T.$$

For any morphism $g : (U, T, \delta) \to (U', T, \delta')$ in $\mathrm{Crys}(X/S)$,

$$g^*_{\mathcal{F}} : g^{-1}(\mathcal{F}_T) \longrightarrow \mathcal{F}_T.$$

We define the *Structure sheaf* as

$$(\mathcal{O}_{X/S})_T := \mathcal{O}_T, \quad \text{or} \quad \mathcal{O}_{X/S}(T) := \mathcal{O}_T(T).$$

Then $(X/S)_{\mathrm{crys}}$ becomes a ringed topoi $((X/S)_{\mathrm{crys}}, \mathcal{O}_{X/S})$.

Proposition 15 ([Ber74, III, 1.1.5]) *For any object (U, T, δ) in $\mathrm{Crys}(X/S)$, the functor that associate a sheaf $\mathcal{F}$ on $\mathrm{Crys}(X/S)$ to a sheaf $\mathcal{F}_T$ commutes with limit and colimit.* □

Now we fix the following notations. Consider the commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ (S', \mathcal{I}', \gamma') \arrow[r, "u"] & (S, \mathcal{I}, \Gamma) \end{tikzcd}$$

where u is a divided power morphism of divided power schemes.

Definition 16 ([Ber74, III, 2.1.1]) Let (U, T, δ) be an object of $\mathrm{Crys}(X/S)$ and (U', T', δ') an object of $\mathrm{Crys}(X'/S')$. A morphism $h : T' \to T$ is called a *g-divided-power morphism* if the following conditions are satisfied.

1) $g(U') \subseteq U$

2) h is an S-morphism and the diagram

$$\begin{array}{ccc} U' & \xrightarrow{g|_{U'}} & U \\ \downarrow & & \downarrow \\ T' & \xrightarrow{h} & T \end{array}$$

is commutative.

3) h is a divided power morphism with respect to the divided power structures δ and δ'. □

Suppose that (U, T, δ) is an object of $\mathrm{Crys}(X/S)$. We define a sheaf on $\mathrm{Crys}(X'/S')$, denoted by $g^*(T)$ as follows. For any object (U', T', δ') of $\mathrm{Crys}(X'/S')$, set

$$g^*(T)(U', T', \delta') := \mathrm{Hom}_{g\text{-PD}}(T', T), \tag{15}$$

where $\mathrm{Hom}_{g\text{-PD}}(T', T)$ is the set of g-divided power morphism from from T' to T. This defines a (continuous) functor (cf, [Ber74, III, 2.2.2])

$$g^* : \mathrm{Crys}(X/S) \longrightarrow (X'/S')_{\mathrm{crys}}.$$

Theorem 7 ([Ber74, III, 2.2.3]) *Ther is a unique morphism of topoi*

$$g_{crys} : (X'/S')_{crys} \longrightarrow (X/S)_{crys},$$

such that for any object (U, T, δ) of Crys(X/S), we have

$$g_{crys}^*(\tilde{T}) = g^*(T),$$

where $\tilde{T}$ is the sheaf on crys(X/S) represented by (U, T, δ). □

Corollary 9 ([Ber74, III, 2.2.4]) *The morphisms g_{crys} of topoi as in Theorem 7 is a morphism of ringed topos*

$$g_{crys} : ((X'/S')_{crys}, \mathcal{O}_{X'/S'}) \longrightarrow ((X/S)_{crys}, \mathcal{O}_{X/S}).$$

That is to say, it comes with a natural homomorphism

$$g_{crys}^* \mathcal{O}_{X/S} \longrightarrow \mathcal{O}_{X'/S'},$$

of sheaves of rings. □

3 Crystals — November 28, 2016

We will slightly change our notations: we will denote by f^{-1} to mean the inverse image of sheaves of sets, and f^* to mean the pull-back of sheaves of modules.

3.1 Definition

> Commentaire terminologique: Un cristal possède deux propriétés caractéristiques: la rigidité, et la faculté de croitre, dans un voisinage approprié. Il y a
> soda sulfur
> des cristaux de toute espèce de substance: des cristaux de soude, de souffre, de modules, d'anneaux, de schémas relatifs etc.

— Grothendieck, an excerpt from a letter to Tate. May, 1966.

We fix a base scheme $(S, \mathcal{I}, \gamma)$ once and for all. TODO: p is locally nilpotent.

Definition 17 ([BO78, 6.1] & [Ber74, IV, 1.1.2 i)]) Let $\mathcal{A}$ be a sheaf of rings on $\mathrm{Crys}(X/S)$, and $\mathcal{F}$ be a sheaf of $\mathcal{A}$-modules. Then $\mathcal{F}$ is said to be a *crystal in $\mathcal{A}$-modules*, if for any $g : T' \to T$ in $\mathrm{Crys}(X/S)$, the transition map

$$g^{-1}\mathcal{F}_T \otimes_{g^{-1}A_T} A_{T'} \longrightarrow \mathcal{F}_{T'}$$

is an isomorphism. In case $\mathcal{A} = \mathcal{O}_{X/S}$, a *crystal* in $\mathcal{O}_{X/S}$-modules is simply called a crystal for short. □

Remark 14
1) Clearly, the category of crystals in $\mathcal{A}$-modules have tensor product. cf. [Ber74, IV, 1.1.5].
2) We could also define what is a *crystal in $\mathcal{A}$-algebra* in the same fashion, cf. [Ber74, IV, 1.1.2 ii)].
3) More generally, let $p : \mathcal{C} \to \mathrm{Crys}(X/S)$ be a stack. A *crystal in objects of $\mathcal{C}$ on X relative to S* is a *cartesian section* $\sigma : \mathrm{Crys}(X/S) \to \mathcal{C}$, i.e., a functor σ such that $p \circ \sigma = \mathrm{id}$ and such that $\sigma(f)$ is *strongly cartesian* for all morphisms f of $\mathrm{Crys}(X/S)$. See [Stacks, Tag 07IV] and [Ber74, IV, 1.1.1]. □

Proposition 16 ([Ber74, IV, 1.1.5]) *If $\mathcal{A} \to \mathcal{B}$ is a homomorphism of sheaves of rings on $\mathrm{Crys}(X/S)$, and $\mathcal{B}$ is a crystal in $\mathcal{A}$-algebras. If $\mathcal{F}$ is a crystal in $\mathcal{B}$-modules, then $\mathcal{F}$ become a crystal in $\mathcal{A}$-module by restriction; and if $\mathcal{E}$ is a crystal in $\mathcal{A}$-modules, then $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{B}$ is a crystal in $\mathcal{B}$-modules.*

Proof This follows easily from definition. ■

3.2 Inverse Image (Pullback) of a Crystal

Suppose we have the following commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ (S', \mathcal{I}', \gamma') \arrow[r, "u"] & (S, \mathcal{I}, \Gamma) \end{tikzcd}$$

where u is a divided power homomorphism.

Recall Corollary 9 that we have a natural morphism of ringed topoi

$$g_{\mathrm{crys}} : ((X'/S')_{\mathrm{crys}}, \mathcal{O}_{X'/S'}) \longrightarrow ((X/S)_{\mathrm{crys}}, \mathcal{O}_{X/S}).$$

Proposition 17 ([Ber74, IV, 1.2.4]) *Let (U', T', δ') be an object of $\mathrm{Crys}(X'/S')$ and (U, T, δ) an object of $\mathrm{Crys}(X/S)$. Suppose that $h : T' \to T$ is an g-divided power morphism and $\mathcal{F}$ is a crystal on $\mathrm{Crys}(X/S)$. Then the canonical morphism*

$$h^*(\mathcal{F}_T) \longrightarrow (g^*_{crys}\mathcal{F})_{T'}$$

is an isomorphism and $g^*_{crys}\mathcal{F}$ is a crystal on X'/S'. Here h^* and g^*_{crys} denote the pull-back of modules.

Proof (sketch) This is [BO78, Exercise 6.5]. Detailed discussions could be found in [Ber74, IV, 1.2.2–1.2.4].

First one could see that for any sheaf $\mathcal{F}$ of sets on $\mathrm{Crys}(X/S)$, there is a natural homomorphism (cf. [Ber74, IV, 1.2.2])

$$h^{-1}(\mathcal{F}_T) \longrightarrow (g^{-1}_{\mathrm{crys}}\mathcal{F})_{T'}.$$

Moreover, if $\mathcal{F}$ is a sheaf of $\mathcal{O}_{X/S}$-modules, we have a natural map of $\mathcal{O}_{T'}$-modules

$$h^{-1}(\mathcal{F}_T) \otimes_{h^{-1}\left((\mathcal{O}_{X/S})_T\right)} (g^{-1}_{\mathrm{crys}}\mathcal{O}_{X/S})_{T'} \longrightarrow (g^{-1}_{\mathrm{crys}}\mathcal{F})_{T'}. \tag{16}$$

We could show, by checking at the level of stalks, that if $\mathcal{F}$ is a Crystal, then the above morphism is an isomorphism and $g_{\text{crys}}^{-1}\mathcal{F}$ is a crystal in $g_{\text{crys}}^{-1}\mathcal{O}_{X/S}$-modules (cf. [Ber74, IV, 1.2.3]). Now in our case, we also have a map

$$g_{\text{crys}}^{-1}\mathcal{O}_{X/S} \to \mathcal{O}_{X'/S'}$$

coming form the morphism g_{crys} of ringed topoi. Hence we obtain the natural isomorphism of $\mathcal{O}_{T'}$-modules

$$h^*(\mathcal{F}_T) = h^{-1}(\mathcal{F}_T) \otimes_{h^{-1}\mathcal{O}_T} \mathcal{O}_{T'} \longrightarrow (g_{\text{crys}}^*\mathcal{F})_{T'}$$

by tensoring eq. (16) by $(\mathcal{O}_{X'/S'})_{T'} = \mathcal{O}_{T'}$ over $(g_{\text{crys}}^{-1}\mathcal{O}_{X/S})_{T'}$. That $g_{\text{crys}}^*\mathcal{F}$ is a crystal in $\mathcal{O}_{X'/S'}$-module then follows from Proposition 16. ∎

3.3 Direct Image of a Crystal by a Closed Embedding

Suppose we have a close embedding

$$\begin{tikzcd} Y \arrow[rr, hook, "/"] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

such that γ extends to Y and X.

Lemma 8 ([Ber74, IV, 1.3.1]) *For every object (U, T, δ) of Crys(X/S), the sheaf $i^*(T)$:*

$$i^*(T)(U', T', \delta') = \operatorname{Hom}_{g\text{-}PD}(T', T)$$

as defined in eq. (15) is representable by $(U \cap Y, D_{U\cap Y}(T), \tilde{\gamma})$, where $\tilde{\gamma}$ is the canonical divided power structure on $D_{U\cap Y}(T)$. We have the following commutative diagram.0

$$\begin{tikzcd} D_V(T) \arrow[r, "p_T"] \arrow[dr, phantom, "\circlearrowleft"] & T \\ V := U \cap Y \arrow[u, hook, "/"] \arrow[r, hook, "/"] \arrow[d, hook, "\phi"] \arrow[dr, phantom, "\square"] & U \arrow[u, hook, "/"] \arrow[d, hook, "\phi"] \\ Y \arrow[r, hook, "/"] & X. \end{tikzcd}$$

□

Corollary 10 ([Ber74, IV, 1.3.2]) *The functor $(i_{crys})_*$ is exact and for any sheaf $\mathcal{F}$ on Crys(X/S), there is a canonical isomorphism*

$$\big((i_{crys})_*\mathcal{F}\big)_T \longrightarrow (p_T)_*(\mathcal{F}_{D_V(T)}). \tag{17}$$

Proof Recall Theorem 7 that we have

$$\big((i_{\text{crys}})_*\mathcal{F}\big)(T) = \operatorname{Hom}_{(Y/S)_{\text{crys}}}(i^*(T), \mathcal{F}) = \mathcal{F}(D_V(T)),$$

where we use the adjointness of $(i_{\text{crys}})_*$ and i_{crys}^* and that $i^*(T)$ is representable by Lemma 8. Then the canonical isomorphism follows (**Exercise!**). The exactness of $(i_{\text{crys}})_*$ follows from the fact that p_T is an affine morphism. ∎

Theorem 8 ([Ber74, IV, 1.3.4]) *The direct image $(i_{crys})_*\mathcal{O}_{Y/S}$ is a crystal in $\mathcal{O}_{X/S}$-algebra. Then for every crystal $\mathcal{F}$ on Crys(Y/S), the direct image $(i_{crys})_*\mathcal{F}$ is a crystal on Crys(X/S).* □

Corollary 11 ([Ber74, IV, 1.3.5]) *For every $k \geq 1$ and every choice of the $(k+1)$ $\mathcal{O}_X$-algebra structures of $\mathcal{D}_{X/S}(k)$, there is a canonical isomorphism*

$$\mathcal{D}_Y(X) \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}(k) \longrightarrow \mathcal{D}_Y(X^{k+1}_{/S}),$$

of $\mathcal{O}_X$-modules. In particular, when $k = 1$, we have isomorphisms

$$\mathcal{D}_{X/S}(1) \otimes_{\mathcal{O}_X} \mathcal{D}_Y(X) \xrightarrow{\cong} \mathcal{D}_Y(X \times_S X) \xleftarrow{\cong} \mathcal{D}_Y(X) \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}(1) \tag{18}$$

which give $\mathcal{D}_Y(X)$ a canonical hyper-divided-power stratification.

Proof (Sketch) Recall the definitions of $D_Y(X)$ and $D_{X/S}(k)$:

$$\begin{tikzcd}
& D_Y(X) \cong D_Y(D_{X/S}(k)) \arrow[r] \arrow[d] & D_{X/S}(k) \arrow[d] \\
Y \arrow[ur, hook] \arrow[r, hook] & X \arrow[ur, hook] \arrow[r] \arrow[dr, "\mathrm{id}_X"'] & X^{k+1}_{/S} \arrow[d, "\mathrm{pr}_i"] \\
& & X
\end{tikzcd}$$

We can show that the natural morphism $X \to D_{X/S}(k)$ is an object in $\mathrm{Crys}(X/S)$ and the $(k+1)$ projections $D_{X/S}(k) \to X$ defines a morphism $(X, D_{X/S}(k))$ to (X, X) in $\mathrm{Crys}(X/S)$ (**Exercise!**). Recall eq. (17) that we have

$$\big((i_{\mathrm{crys}})_* \mathcal{O}_{Y/S}\big)_{(X,X)} = \mathcal{D}_Y(X), \quad \text{and} \quad \big((i_{\mathrm{crys}})_* \mathcal{O}_{Y/S}\big)_{(X, D_{X/S}(k))} = \mathcal{D}_Y(X^{k+1}_{/S}).$$

Here we use the fact that

$$D_Y\big(D_X(X^{k+1}_{/S})\big) \cong D_Y(X^{k+1}_{/S}).$$

Now as $(i_{\mathrm{crys}})_* \mathcal{O}_{Y/S}$ is a crystal, obtain from the transition map for the chosen projection $D_{X/S}(k) \to X$ that

$$\mathcal{D}_Y(X) \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S} \longrightarrow \mathcal{D}_Y(X^{k+1}_{/S}),$$

is an isomorphism. ■

Lemma 9 ([Ber74, IV, 1.5.5]) *If $f : X \to S$ is a morphism of schemes, then the following statement are equivalent:*

1) *f is smooth;*
2) *f is of finite presentation and there is an open cover $(U_i)_{i \in I}$ of X such that for every closed embedding $Y_0 \to Y$ over S defined by a nil-ideal of $\mathcal{O}_Y$ with Y affine, and for every S-morphism $g : Y_0 \to U_i$, there is an S-morphism $\bar{g} : Y \to U_i$ extending g.*

$$\begin{tikzcd}
Y \arrow[dr, dotted, "\exists \bar{g}"] & & \\
Y_0 \arrow[u, hook, "\mathcal{I}_{Y/Y_0}\ \textit{nil-ideal}"] \arrow[r, "g"] & U_i \arrow[r, hook] & X
\end{tikzcd}$$

□

Remark 15 A morphism $f : X \to S$ satisfies the property in Lemma 9 2) (not necessarily of finite presentation) is said to be *quasi-smooth*. One can show that quasi-smoothness is stable under base change and composition. □

Theorem 9 ([BO78, 6.6]) *Suppose further X/S is smooth, the the following categories are canonically equivalent.*

1) *The category of crystals in $\mathcal{O}_{Y/S}$-modules on $\mathrm{Crys}(Y/S)$.*

2) *The category of $\mathcal{D}_Y(X)$-modules with a hyper-divided-power stratification (as an $\mathcal{O}_X$-modules), which is compatible with the canonical hyper-divided-power stratification given in Corollary 11.*
3) *The category of $\mathcal{D}_Y(X)$-modules with a flat quasi-nilpotent connection (as an $\mathcal{O}_X$-module), which is compatible with the canonical connection on $\mathcal{D}_Y(X)$.*

Proof (Sketch) Detailed discussion see [Ber74, IV, 1.6]. The definition of quasi-nilpotent connection could be found at [Ber74, II, 4.3.6].

Suppose that $\mathcal{F}$ is a crystal on $\mathrm{Crys}(Y/S)$. Consider the following diagram

$$\begin{tikzcd}
 & & D_Y(X\times_S X) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] & D_Y(X) \arrow[d] \\
Y \arrow[r, hook] \arrow[urr] & X \arrow[r] & X\times_S X \arrow[r, shift left] \arrow[r, shift right] & X
\end{tikzcd}$$

The maps $p_i : D_Y(X\times_S X) \to D_Y(X)$ are induced by the projections $X\times_S X \to X$, and one can check that they are arrows in $\mathrm{Crys}(Y/S)$. Since $\mathcal{F}$ is a crystal on $\mathrm{Crys}(Y/S)$, we have natural isomorphisms

$$p_i^*(\mathcal{F}_{D_Y(X)}) \longrightarrow \mathcal{F}_{D_Y(X\times_S X)}, \quad i = 1, 2.$$

These gives an isomorphism ($\mathcal{D}_Y(X\times_S X)$-linear over Y)

$$\mathcal{D}_Y(X\times_S X) \otimes_{\mathcal{D}_Y(X)} \mathcal{F}_{D_Y(X)} \longrightarrow \mathcal{F}_{D_Y(X)} \otimes_{\mathcal{D}_Y(X)} \mathcal{D}_Y(X\times_S X).$$

Recall eq. (18), we get an isomorphism ($\mathcal{D}_{X/S}(1)$-linear over X)

$$\epsilon : \mathcal{D}_{X/S}(1) \otimes_{\mathcal{O}_X} \mathcal{F}_{D_Y(X)} \longrightarrow \mathcal{F}_{D_Y(X)} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}(1).$$

One checks (**Exercise!**) that this defines a hyper-divided-power stratification on $\mathcal{E} := \mathcal{F}_{D_Y(X)}$. Moreover, the compatibility condition is built into the construction.

Conversely, suppose that $\mathcal{E}$ is a $\mathcal{D}_Y(X)$-module with a hyper-divided-power stratification as described in 2). We would like to construct a crystal $\mathcal{F}$ on $\mathrm{Crys}(Y/S)$. As X is smooth, for any sufficiently small object (U, T, δ) in $\mathrm{Crys}(Y/S)$,

$$\begin{tikzcd}
T \arrow[rr, dotted, "h"] & & D_Y(X) \arrow[d] \\
U \arrow[u, hook] \arrow[r, hook] & Y \arrow[ur] \arrow[r, hook] & X
\end{tikzcd}$$

there is a morphism $h : T \to D_Y(X)$ making the above diagram commute (**Exercise!**[9]). Then one define $\mathcal{F}_T := h^*(\mathcal{E})$. If follows from the fact that $\mathcal{E}$ comes with a stratification that $\mathcal{F}_T$ is determined up to a canonical isomorphism. In this way, we define a crystal on $\mathrm{Crys}(X/S)$.

That 2) is equivalent to 3) is omitted, which is similar to the proof of Theorem 6. ■

Proposition 18 ([Ber74, IV, 1.4.1]) *Consider the following Cartesian diagram*

$$\begin{tikzcd}
X_0 \arrow[r, hook, "i"] \arrow[d] & X \arrow[d] \\
S_0 \arrow[r, hook] & (S, \mathcal{I}, \gamma)
\end{tikzcd}$$

where S_0 is defined by a quasi-coherent divided power sub-ideal $\mathcal{I}_0$ of $\mathcal{I}$. Then we have a equivalence of categories

$$(\textit{Crystals on Crys}(X/S)) \underset{(i_{crys})_*}{\overset{i_{crys}^*}{\rightleftarrows}} (\textit{Crystals on Crys}(X_0/S))$$

□

[9] In fact, when U is sufficiently small, the map $U \to Y \to X$ extends to a map $T \to X$ by smoothness of X. Then following from the universal property of divided power envelop, this map factors through $D_Y(X)$.

3.4 Linearization of Hyper-Divided-Power Differential Operators

Suppose that X is an S-scheme and $(S, \mathcal{I}, \gamma)$ is a divided power scheme such that γ extends to X.

Recall that there is a natural morphism ([Ber74, II, (1.1.19)], compared to eq. (10))

$$\delta : \mathcal{D}_{X/S} \longrightarrow \mathcal{D}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}.$$

where we write $\mathcal{D}_{X/S}$ instead of $\mathcal{D}_{X/S}(1)$ to simplify notations.

If $\mathcal{E}$ is an $\mathcal{O}_X$-module, we set

$$L_X(\mathcal{E}) := \mathcal{D}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E},$$

considered as an left $\mathcal{O}_X$-module given by the left $\mathcal{O}_X$-module structure of $\mathcal{D}_{X/S}$ and the tensor product is taken via the right $\mathcal{O}_X$-module structure of $\mathcal{D}_{X/S}$. If $u : \mathcal{D}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{F}$ is a hyper-divided-power differential operator between $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{F}$. We define $L_X(u)$ as the composition

$$\mathcal{D}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\delta \otimes \mathrm{id}_{\mathcal{E}}} \mathcal{D}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\mathrm{id}_{\mathcal{D}_{X/S}} \otimes u} \mathcal{D}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{F}.$$

Lemma 10 ([Ber74, IV, 3.1.2]) *1) The $\mathcal{O}_X$-module $L_X(\mathcal{E})$ comes canonically with a hyper-divided-power stratification relative to S.*

2) The homomorphism $L_X(u)$ is horizontal with respect to the canonical stratifications of $L_X(\mathcal{E})$ and $L_X(\mathcal{F})$.

3) If $v : \mathcal{D}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{G}$ is another hyper-divided-power differential operator, then

$$L_X(v \circ u) = L_X(v) \circ L_X(u).$$

Proof (Sketch) To see that $L_X(\mathcal{E})$ has a hyper-divided-power stratification, we first need to define a canonical isomorphism

$$\mathcal{D}_{X/S} \otimes_{\mathcal{O}_X} L_X(\mathcal{E}) \xrightarrow{\cong} L_X(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}.$$

In fact, this is defined by the composition (write $\mathcal{D}$ instead of $\mathcal{D}_{X/S}$ for short)

$$\mathcal{D} \otimes \mathcal{D} \otimes \mathcal{E} \xrightarrow{\mathrm{id}_{\mathcal{D}} \otimes \delta \otimes \mathrm{id}_{\mathcal{E}}} \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{E} \xrightarrow{(\mathrm{id}_{\mathcal{D}} \cdot \sigma) \otimes \mathrm{id}_{\mathcal{D}} \otimes \mathrm{id}_{\mathcal{D}}} \mathcal{D}{}_{l}\!\otimes \mathcal{D} \otimes \mathcal{E}$$

where $\sigma : \mathcal{D} \to \mathcal{D}$ is the symmetric automorphism, hence $(\mathrm{id}_{\mathcal{D}} \cdot \sigma) : \mathcal{D} \otimes \mathcal{D}, d_1 \otimes d_2 \mapsto d_1 \cdot \sigma(d_2)$, and where the "l" on the left side of tensor means that the module on the left is tensored with its left module structure. One can check this defines a hyper-divided-power stratification. (Exercise!) ∎

So L_X defines a functor

$$\begin{pmatrix} \mathcal{O}_X\text{-modules} \\ \text{HPD differential operators} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{O}_X\text{-modules with a HPD stratification} \\ \text{horizontal homomorphisms} \end{pmatrix}$$

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