# Grothendieck Connections

Work in the local case. Suppose that $B$ is an $A$-algebra. Let $I := I_{B/A}$ be the kernel of the canonical surjective ring homomorphism $m : B \otimes_A B \to B$, $b_1 \otimes b_2 \mapsto b_1 b_2$. There are two $B$-algebra structures of $B \otimes_A B$, which are denoted by

$$\begin{aligned} j_1 : B \longrightarrow B \otimes_A B, \quad & b \mapsto b \otimes 1 \\ j_2 : B \longrightarrow B \otimes_A B, \quad & b \mapsto 1 \otimes b \end{aligned}$$

Let $P^1_{B/A} := (B \otimes_A B)/I^2$ be the *first principal part*. Then $j_2$ and $j_2$ induces two $B$-algebra structures on $P^1$.

$$p_i := \bar{j}_i : B \longrightarrow B \otimes_A B \longrightarrow P^1,$$

where the second map is the natural quotient map. One the other hand, $\Omega^1_{B/A} := I/I^2$ has only one $B$-module structure. This is because for any $\sum a_i \otimes b_i \in \Omega^1_{B/A}$, we have $m(\sum a_i \otimes b_i) = \sum a_i b_i = 0$. Hence

$$\sum a_i \otimes b_i = \sum a_i \otimes b_i - \sum a_i b_i \otimes 1 = \sum (a_i \otimes 1) \cdot (1 \otimes b_i - b_i \otimes 1)$$

So $\Omega^1_{B/A}$ is generated as an *left* (i.e., induced by $p_1$) $B$-module by elements of the form $1 \otimes b - b \otimes 1$. Moreover, note that $(1 \otimes a - a \otimes 1) \cdot (1 \otimes b - b \otimes 1) \in I^2$, we know that $(1 \otimes a)(1 \otimes b - b \otimes 1) = (a \otimes 1)(1 \otimes b - b \otimes 1) \in I/I^2$. So the left (induced by $p_1$) and right (induced by $p_2$) $B$-module structures of $\Omega^1_{B/A}$ coincide. Hence those elements also generate $\Omega^1_{B/A}$ as an *right* $B$-module. Set

$$\partial := \partial_{B/A} : B \longrightarrow \Omega^1_{B/A}, \quad b \mapsto (p_2 - p_1)(b) = 1 \otimes b - b \otimes 1.\text{[1]}$$

We have $\partial(b_1 b_2) = b_1 \partial b_2 + b_2 \partial b_1$. It follows that we have an splitting exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \Omega^1_{B/A} \arrow[r] & P^1_{B/A} \arrow[r, "\bar{m}"'] & B \arrow[r] \arrow[l, bend right, "p_i"'] & 0, \end{tikzcd} \tag{1}$$

where $i = 1$ when $P^1_{B/A}$ is viewed as a left $B$-module (induced by $p_1$), and $i = 2$ when $P^1_{B/A}$ is viewed as a right $B$-module (induced by $p_2$). So we have isomorphism[2]

$$\begin{aligned} P^1_{B/A} &\cong \Omega^1_{B/A} \oplus B, \quad b_1 \otimes b_2 \mapsto (\ \ b_1 \partial b_2, b_1 b_2), && \text{as a left } B\text{-module,} \\ P^2_{B/A} &\cong \Omega^1_{B/A} \oplus B, \quad b_1 \otimes b_2 \mapsto (-b_2 \partial b_1, b_1 b_2), && \text{as a right } B\text{-module.} \end{aligned}$$

Given the ring structure $(\omega, b) \cdot (\omega', b') := (b\omega' + b'\omega, bb')$, on the direct sum $\Omega^1_{B/A} \oplus B_2$, then above $B$-module isomorphisms becomes *ring isomorphisms*.

Now suppose that $E$ is an $B$-module, together with an $A$-connection

$$\nabla : E \longrightarrow E \otimes_B \Omega^1_{B/A}.\text{[3]}$$

[1] In EGA $0_{\mathrm{IV}}$, §20, $p_1 - p_2$ is used. But in EGA $\mathrm{IV}_4$, §16, the convention $p_2 - p_1$ is used.

[2] all the following construction are based on the observations from the definition of these two isomorphism. For example, the difference of the first components gives the derivation $\partial(b_1 b_2)$.

[3] Though it makes no difference, many people write $\Omega^1_{B/A} \otimes_B E$, but here it's more convenient to use $E \otimes_B \Omega^1_{B/A}$ instead, because we set $\partial = p_2 - p_1$, rather than $p_1 - p_2$. Otherwise, at some places, an extra minus sign will appear.

We have a $B$-module isomorphisms[4]

$$E \otimes_B P^1_{B/A} = (E \otimes_B \Omega^1_{B/A}) \oplus E, \quad \text{and} \quad P^1_{B/A} \otimes_B E = (\Omega^1_{B/A} \otimes_B E) \oplus E, \tag{2}$$

where $P^1_{B/A}$ on the right of $\otimes$ means that it's tensored using its left $B$-module structure and $P^1_{B/A}$ on the left means that it's tensored using its right $B$-module structure. The induced (right)[5] $P^1_{B/A}$-module structure on $E \otimes_B P^1_{B/A} = (E \otimes_B \Omega^1_{B/A}) \oplus E$ is given by

$$\begin{aligned}(e \otimes \omega, e') \cdot (b_1 \otimes b_2) &= (e \otimes \omega, e') \cdot (b_1 \partial b_2, b_1 b_2) \\ &= (e' \otimes b_1 \partial b_2 + b_1 b_2 e \otimes \omega, b_1 b_2 e')\end{aligned} \tag{3}$$

and the induced (left) $P^1_{B/A}$-module structure on $P^1_{B/A} \otimes_B E = (\Omega^1_{B/A} \otimes_B E) \oplus E$ is given by

$$\begin{aligned}(b_1 \otimes b_2) \cdot (\omega \otimes e, e') &= (-b_2 \partial b_1, b_1 b_2) \cdot (\omega \otimes e, e') \\ &= (-b_2 \partial b_1 \otimes e' + b_1 b_2 \omega \otimes e, b_1 b_2 e')\end{aligned} \tag{4}$$

Based on observations on (4) and (3), define

$$\begin{aligned}\epsilon : (\Omega^1_{B/A} \otimes_B E) \oplus E &\longrightarrow (E \otimes_B \Omega^1_{B/A}) \oplus E \\ (\omega \otimes e, e') &\longmapsto (e \otimes \omega + \nabla(e'), e')\end{aligned} \tag{5}$$

It's straightly forward to check that

$$\begin{aligned}\epsilon(\omega \otimes e, e') \cdot (b_1 \otimes b_2) &= (e \otimes \omega + \nabla(e'), e') \cdot (b_1 \otimes b_2) \\ &= (e' \otimes b_1 \partial b_2 + b_1 b_2 e \otimes \omega + b_1 b_2 \nabla(e'), b_1 b_2 e') \\ \epsilon\big((b_1 \otimes b_2) \cdot (\omega \otimes e, e')\big) &= \epsilon\big(-b_2 \partial b_1 \otimes e' + b_1 b_2 \omega \otimes e, b_1 b_2 e'\big) \\ &= (-e' \otimes b_2 \partial b_1 + b_1 b_2 e \otimes \omega + \nabla(b_1 b_2 e'), b_1 b_2 e') \\ &= (e' \otimes b_1 \partial b_2 + b_1 b_2 e \otimes \omega + b_1 b_2 \nabla(e'), b_1 b_2 e')\end{aligned}$$

where we use the Leibniz rule that $\nabla(b_1 b_2 e') = e' \otimes b_1 \partial b_2 + e' \otimes b_2 \partial b_1 + b_1 b_2 \nabla(e')$. Therefore $\epsilon$ is a $P^1_{B/A}$-module homomorphism. Moreover, it's clear that $\epsilon$ is injective and surjective, hence an isomorphism of $P^1_{B/A}$-modules. So it induce a commutative diagram of isomorphisms $P^1_{B/A}$-modules

$$\begin{tikzcd} P^1_{B/A} \otimes_B E \arrow[r, "\cong"] \arrow[d, dotted, "\varepsilon"', "\cong"] & (\Omega^1_{B/A} \otimes_B E) \otimes E \arrow[d, "\epsilon"] \\ E \otimes_B P^1_{B/A} \arrow[r, "\cong"] & (E \otimes_B \Omega^1_{B/A}) \oplus E. \end{tikzcd}$$

Therefore, given a connection $\nabla$ on $E$, it determines an isomorphism

$$\varepsilon : P^1_{B/A} \otimes_B E \xrightarrow{\cong} E \otimes_B P^1_{B/A}. \tag{6}$$

such that the diagram

$$\begin{tikzcd} P^1_{B/A} \otimes_B E \arrow[r, "\varepsilon"] \arrow[d, "\mathrm{id} \otimes \bar{m}"] & E \otimes_B P^1_{B/A} \arrow[d, "\bar{m} \otimes \mathrm{id}"] \\ B \otimes_B E \cong E \arrow[r] & E \cong E \otimes_B B \end{tikzcd} \tag{7}$$

[4] As the sequence (1) split, we do not need to require $E$ nor $\Omega^1_{B/A}$ to be free. In a geometry setting, this means that we could work with general quasi-coherent sheaves $\mathcal{E}$ over a unnecessarily smooth scheme $X/S$.

[5] Actually, it makes no difference to view it as an left module, which probably will make the computation less confusing.

of $B$-modules commutes, where the bottom arrow is the canonical isomorphism. If we identify the vertical arrows with the projection to $E$ with respect to the direct sum decomposition, we could simply say that $\varepsilon$ induces identity on $E$).

On the other hand, given an isomorphism (6) which induces identity on $E$, it gives rise to an isomorphism (5) that induces identity on $E$. Let $\mathrm{pr}_1 : (E \otimes_B \Omega^1_{B/A}) \oplus E \to E \otimes_B \Omega^1_{B/A}$ be the projection to the first component, and $i_2$ be the inclusion of $E$ into $(\Omega^1_{B/A} \otimes_B E) \oplus E$ as the second component. Then we can recover $\nabla$ by setting $\nabla : E \to E \otimes_B \Omega^1_{B/A}$ as the composition

$$E \xrightarrow{i_2} (\Omega^1_{B/A} \otimes_B E) \oplus E \xrightarrow{\epsilon} (E \otimes_B \Omega^1_{B/A}) \oplus E \xrightarrow{\mathrm{pr}_1} E \otimes_B \Omega^1_{B/A}. \tag{8}$$

And one checks that

$$\begin{aligned}
\nabla(b \cdot e) &= \mathrm{pr}_1 \circ \epsilon \circ i_2(b \cdot e) \\
&= \mathrm{pr}_1 \circ \epsilon\big((0, b \cdot e)\big) \\
&= \mathrm{pr}_1 \circ \epsilon\big((1 \otimes b) \cdot (0, e)\big) && \text{(recall (4))} \\
&= \mathrm{pr}_1 \big(\epsilon(0, e) \cdot (1 \otimes b)\big) && \text{($\epsilon$ is $P^1_{B/A}$-linear)} \\
&= \mathrm{pr}_1 \big((\nabla(e), e) \cdot (1 \otimes b)\big) \\
&= \mathrm{pr}_1 \big((e \otimes \partial b + b\nabla(e), be)\big) && \text{(recall (3))} \\
&= e \otimes \partial b + b\nabla(e)
\end{aligned}$$

That is to say, $\nabla$ satisfies the Leibniz rule.

**Simplification/Summary** The above computations could be simplified as follows. A simple element $(b_1 \otimes b_2) \otimes e \in P^1_{B/A} \otimes_B E$ has image $(-b_2\partial b_1 \otimes e, b_1 b_2 e)$ in $(\Omega^1_{B/A} \otimes_B E) \oplus E$ under the isomorphism in (2). It is mapped by $\epsilon$ to

$$\begin{aligned}
(-e \otimes b_2\partial b_1 + \nabla(b_1 b_2 e), b_1 b_2 e) &= (e \otimes b_1 \partial b_2 + b_1 b_2 \nabla(e), b_1 b_2 e) \\
&= \big(\nabla(e), e\big) \cdot (b_1 \otimes b_2) \in \big(E \otimes_B \Omega^1_{B/A}\big) \oplus E.
\end{aligned}$$

This further corresponds to the element $e \otimes (b_1 \otimes b_2) + \nabla(e) \cdot (b_1 \otimes b_2) \in E \otimes_B P^1_{B/A}$, where $\nabla(e)$ is viewed as its image in $E \otimes_B P^1_{B/A}$ via the natural map $E \otimes_B \Omega^1_{B/A} \to E \otimes_B P^1_{B/A}$, which may *not* be injective in general. So the isomorphism (5) translates to the following. Given a connection $\nabla : E \to E \otimes_B \Omega^1_{B/A}$, we can form the $P^1_{B/A}$-isomorphism

$$\begin{aligned}
\varepsilon : P^1_{B/A} \otimes_B E &\longrightarrow E \otimes_B P^1_{B/A} \\
1 \otimes e &\longmapsto e \otimes 1 + \nabla(e).
\end{aligned} \tag{9}$$

This $\varepsilon$ reduces to identity as (the image of) $\nabla(e)$ lies in the kernel of $\mathrm{id}_E \otimes \tilde{m} : E \otimes_B P^1_{B/A} \to E \otimes_B B \cong E$. Conversely, given a $P^1_{B/A}$-isomorphism as above, we can recover $\nabla$ as the composition

$$\begin{array}{ccccccc}
E & \xrightarrow{i_2} & P^1_{B/A} \otimes_B E & \xrightarrow{\varepsilon} & E \otimes_B P^1_{B/A} & \xrightarrow{\qquad \mathrm{pr}_1 \qquad} & \Omega^1_{B/A} \otimes_B E \\
e & \longmapsto & 1 \otimes e & \longmapsto & \varepsilon(1 \otimes e) & \longmapsto & \varepsilon(1 \otimes e) - (\mathrm{id}_E \circ \tilde{m}) \circ \varepsilon(1 \otimes e).
\end{array} \tag{10}$$

The Leibniz rule follows from direct computations.

**Flat connection** Descent datum.

**Remark** There is a technical point. Usually we set $\partial := p_2 - p_1$, that is $\partial(a) = 1 \otimes a - a \otimes 1$. This makes the isomorphism $P \otimes_B E \to E \otimes P$ more natural. We can rewrite this isomorphism as

1. $\varepsilon : p_2^* E \to p_1^* E$.

On the other hand, we have a natural morphism of *left* $B$-module morphism $E \to P(E) := P \otimes_B E = p_{1*} p_2^* E$.[6] Using adjointness, we get a natural map

2. $\tilde{\varepsilon} : p_1^* E \to p_2^* E$.

Is these two map inverse to each other?

Moreover, we conventionally write a connection as $\nabla : E \to \Omega \otimes E$, rather than $E \to E \otimes \Omega$. Observe (8), if using $p_2^* E \to p_1^* E$, it's more convenient to write a connection as $E \to E \otimes \Omega$.

Atiyah Class. Splitting of the sequence

$$0 \to \Omega \to E \to P \otimes E \to E \to 0$$

# 1 Applications

## 1.1 Pullback of connections

It is very easy to see how a connection pulls back.

Suppose we have a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "u"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

It induces a commutative diagram

$$\begin{tikzcd}
(X')^{(1)} \arrow[r, "u^{(1)}"] \arrow[d, "\Delta_{f'}"] \arrow[dd, bend right=60, "(p')_i^{(1)}"'] & X^{(1)} \arrow[d, "\Delta_f"] \arrow[dd, bend left=60, "p_i^{(1)}"] \\
X' \times_{S'} X' \arrow[r] \arrow[d, "p_i'"] & X \times_S X \arrow[d, "p_i"] \\
X' \arrow[r, "u"] & X
\end{tikzcd}$$

Given an $S$-connection

$$\alpha : (p_2^{(1)})^* \mathcal{E} \longrightarrow (p_1^{(1)})^* \mathcal{E}$$

on a quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$, we can just pullback this map via $u^{(1)}$, and obtain an $S'$ connection

$$(u^{(1)})^* \alpha : (u^{(1)})^* (p_2^{(1)})^* \mathcal{E} \cong ((p')_2^{(1)})^* (u^* \mathcal{E}) \longrightarrow ((p')_1^{(1)})^* (u^* \mathcal{E}) \cong (u^{(1)})^* (p_1^{(1)})^* \mathcal{E}$$

of the $\mathcal{O}_{X'}$-module $u^* \mathcal{E}$.

[6] The problem may be hidden here. In [EGA IV §16], the functor $E \mapsto P(E)$ is studied. But it seems that $E \to P(E)$ is not a *natural* map. As the tensor product $P \otimes E$ uses the right module structure of $B$, so the natural map, which is the extension of scalers, should be the one $be \mapsto (1 \otimes 1) \otimes be = (1 \otimes b) \otimes e$. This map is not linear if we view $P \otimes E$ as a $B$-module by restriction of scalers via $p_1 : B \to P$.

## 1.2 Canonical connection on Frobenius pullback

**Grothendieck's viewpoint** *This may not be the most direct viewpoint for this problem.*

Denote by $X'/S$ the Frobenius twist of $X/S$ and by $F : X \to X'$ the relative Frobenius morphism. Suppose $\mathcal{E}'$ is a quasi-coherent $\mathcal{O}_{X'}$-module and set $\mathcal{E} := F^*\mathcal{E}'$ to be its Frobnius pullback. Then we know that[7]

$$(p_1^{(1)})^*\mathcal{E} = F^{-1}\mathcal{E}' \otimes_{F^{-1}\mathcal{O}_{X'}} \mathcal{P}^1_{X/S} \quad \text{and} \quad (p_2^{(1)})^*\mathcal{E} = \mathcal{P}^1_{X/S} \otimes_{F^{-1}\mathcal{O}_{X'}} F^{-1}\mathcal{E}'$$

Observe that if we restrict scalars via the natural map $F^{-1}\mathcal{O}_{X'} \to \mathcal{O}_X$, then the two induced $F^{-1}\mathcal{O}_{X'}$-module structures on $\mathcal{P}^1_{X/S}$ coincide.[8] Therefore the natural $F^{-1}\mathcal{O}_{X'}$-module isomorphism

$$\sigma : \mathcal{P}^1_{X/S} \otimes_{F^{-1}\mathcal{O}_{X'}} F^{-1}\mathcal{E}' \longrightarrow F^{-1}\mathcal{E}' \otimes_{F^{-1}\mathcal{O}_{X'}} \mathcal{P}^1_{X/S}, \quad p \otimes e \mapsto e \otimes p$$

actually gives a $P^1_{X/S}$-module isomorphism. It by definition induces to identity on $F^{-1}\mathcal{E}'$.

**Traditional viewpoint.** The key observation is that the derivation map

$$\partial : \mathcal{O}_X \to \Omega^1_{X/S}$$

is an $F^{-1}\mathcal{O}_X$-module homomorphism.[9] Thus the $F^{-1}\mathcal{O}_{X'}$-module homomorphism

$$1 \otimes \partial : F^*\mathcal{E}' = F^{-1}\mathcal{E}' \otimes_{F^{-1}\mathcal{O}_{X'}} \mathcal{O}_X \to F^{-1}\mathcal{E}' \otimes_{F^{-1}\mathcal{O}_{X'}} \Omega^1_{X/S} \cong F^*\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

is well defined. Denote by $\pi$ (resp. $\pi'$) the structure morphism of $X/S$ (resp. $X'/S$). Then the map $1 \otimes \partial$ is automatically $\pi^{-1}\mathcal{O}_S$-linear via the natural homomorphism $F^{-1}(\pi')^\sharp : \pi^{-1}\mathcal{O}_S = F^{-1}(\pi')^{-1}\mathcal{O}_S \to F^{-1}\mathcal{O}_{X'}$. The Leibniz rule is induced from that of $\partial$.

http://math.stackexchange.com/q/1892218/

http://mathoverflow.net/q/13162/

[7] In fact, by identifying the underlying topological spaces of $X$ and $X'$, we can ignore $F^{-1}$ in the following. Feel free to do so.

[8] Basically, this is because in characteristic $p \geq 2$, $t^p \otimes 1 - 1 \otimes t^p = (t \otimes 1 - 1 \otimes t)^p = 0 \in (B \otimes_A B)/I^2 =: P^1_{B/A}$ affine locally. It follows that $(t^p b_1) \otimes b_2 = b_1 \otimes t^p b_2 \in P^1_{B/A}$.

[9] This is because $\partial((at) \cdot b) = \partial(at^p b) = at^p \partial b$, where $at \in B'$ and the $\cdot$ is the action of $B$' on $B$.