Divided-power Envelop

This is just a summary of [BO78, Appendix A]. Fix a ring A. Let M be an A-module.

1 Construction

Let $G_A(M) := A[\{(x, n) : x \in M, n \in \mathbb{N}\}]$ be the polynomial A-algebra on the set of indeterminates $\{(x, n) : x \in M, n \in \mathbb{N}\}$. The algebra $G_A(M)$ has a natural grading by letting (x, n) have degree n. Write $G^\bullet_A(M)$ for this graded A-algebra. Let $I_A(M)$ be the ideal of G generated by

1. $(x, 0) - 1$,
2. $(\lambda x, n) - \lambda^n(x, n)$, $\lambda \in A$.
3. $(x, n)(x, m) - \frac{(n+m)!}{n!m!}(x, n + m)$.
4. $(x + y, n) - \sum_{i+j=n}(x, i)(y, j)$.

Then $I_A(M)$ is a *homogeneous ideal*. Then define

$$\Gamma^\bullet_A(M) := G^\bullet_A(M)/I_A(M), \qquad \Gamma^+_A(M) := \bigoplus_{n\geq 1} \Gamma^n_A(M)$$

Write $x^{[n]}$ as the image of (x, n) in $\Gamma^\bullet_A(M)$. We have natural isomorphisms $\Gamma^0_A(M) \cong A$ and $\Gamma^1_A(M) \cong M$.

2 Universal Properties

2.1

For any A-algebra R, set $\exp(R)$[1] to be the subgroup of units of $R[[T]]$, consisting of those $f \in R[[T]]$ that is *of exponential type*, i.e., $f(T_1 + T_2) = f(T_1)f(T_2)$ for indeterminates T_1 and T_2. Then we have natural bijection

$$\begin{aligned} \mathrm{Hom}_{\mathtt{Alg}/A}\left(\Gamma_A(M), R\right) &\longrightarrow \mathrm{Hom}_{\mathtt{Mod}/A}\left(M, \exp(R)\right) \\ \alpha &\longmapsto \left(x \mapsto \sum_{n\geq 0} \alpha(x^{[n]})T^n\right) \end{aligned}$$

As a corollary, we have

[1] One-parameter subgroups of $\hat{\mathbb{G}}_m(R)$.

1. $\Gamma_A(M) \otimes_A A' \cong \Gamma_{A'}(M \otimes_A A')$.
2. $\varinjlim_\lambda \Gamma_A(M_\lambda) \cong \Gamma_A(\varinjlim_\lambda M_\lambda)$.
3. $\Gamma_A(M) \otimes_A \Gamma_A(N) \cong \Gamma_A(M \oplus N)$.

If M is free with basis $(x_i : i \in I)$, then $\Gamma_A(M)$ is free with basis $\left(x_1^{[q_1]} x_2^{[q_2]} \cdots x_r^{[q_r]} : \sum_i q_i = n\right)$.

2.2

If M and N are two A-modules. Let $P(M, N)$ be the set of *polynomial functions*.[2] Namely, elements of $P(M, N)$ are compatible collection of set of maps $\{f_R : M \otimes_A R \to N \otimes_A R\}$. A polynomial function f is said to have weight n, if $f_R(r \cdot m) = r^n \cdot f_R(m)$, for all $r \in R$, and all $m \in M \otimes R$. Denote by the set of polynomial functions of weight n by $P_n(M, N)$. The A-module structure on N gives an A-module structure on $P(M, N)$.

For any $n \in \mathbb{N}$, and any A-algebra R, we have an isomorphsim $\alpha_{n,R} : \Gamma_A^n(M) \otimes_A R \to \Gamma_R^n(M \otimes_A R)$ and natural map $M \otimes_A R \to \Gamma_R(M \otimes_A R)$, $x \mapsto x^{[n]}$. Set

$$\ell_{n,R} : M \otimes_A R \to \Gamma_A(M) \otimes_A R, \quad x \mapsto \alpha_{n,R}^{-1}(x^{[n]})$$

Then $\ell_n := \{\ell_{n,R}\}$ is an element of $P_n(M, \Gamma_A^n(M))$, and it is the universal one in the following sense.

There is a natural bijection

$$\begin{aligned} \mathrm{Hom}_{\mathtt{Mod}/A}\left(\Gamma_n(M), N\right) &\longrightarrow P_n(M, N) \\ f &\longmapsto f \circ \ell_n := \{(f \otimes_A R) \circ \ell_{n,R}\} \end{aligned}$$

As corollaries, we have if $M' \overset{f}{\underset{g}{\rightrightarrows}} M \xrightarrow{h} M'' \to 0$ is an exact sequence, then for each $n \in \mathbb{N}$ and any A-module N, we have exact sequences

1. $0 \longrightarrow P_n(M'', N) \longrightarrow P_n(M, n) \rightrightarrows P_n(M', N)$
2. $\Gamma_A^n(M') \rightrightarrows \Gamma_A^n(M) \longrightarrow \Gamma_A^n(M'') \longrightarrow 0$

Moreover, if $0 \to K \to M \to N \to 0$ is an exact sequence of A-modules, then $\mathrm{Ker}(\Gamma_A^(M) \to \Gamma_A(N))$ is the ideal generated by $\{x^{[n]} : x \in K, n > 0\}$.

2.3

Based on the following two universal properties, one can show that $\Gamma_A^+(M)$ has a unique divided-power structure, which is rather non-trivial. Moreover, for any divided power A-algebra (B, J, δ), we have natural bijection

$$\mathrm{Hom}_{\mathtt{PD\text{-}Alg}/A}\left((\Gamma_A^\bullet(M), \Gamma_A^+(M), -^{[n]}), (B, J, \delta)\right) \longrightarrow \mathrm{Hom}_{\mathtt{Mod}/A}(M, J)$$

[2] Morphisms $\tilde{M} \to \tilde{N}$ of sheaves of sets on the big Zariski set Zar(Spec A).

3 Pairing with symmetric algebra

There is a natural pairing

$$\begin{aligned} &\mathrm{Sym}^n_A\left(\mathrm{Hom}_A(M, A)\right) \times \Gamma^n_A(M) \longrightarrow A \\ &(\phi_1\phi_2\cdots\phi_n, x_1^{[q_1]}x_2^{[q_2]}\cdots x_r^{[q_r]}) \longmapsto \sum_{\alpha}\Big(\prod_{j\in\alpha_i}\phi_j(x_i)\Big) \end{aligned}$$

where $\sum q_i = n$ and $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_r$ is a partition of $\{1, \ldots, n\}$ with $\#\alpha_i = q_i$. The pairing is perfect if M is projective of finite rank.

References

[BO78] Pierre Berthelot and Arthur Ogus. *Notes on crystalline cohomology*. English. Mathematical Notes. Princeton, New Jersey: Princeton University Press. Tokyo: University of Tokyo Press. VI, not consecutively paged. $ 9.50 (1978). 1978.