Two Talks on Tate's Thesis

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In two talks, I will talk about J. Tate's 1950 Princeton doctoral dissertation [Tat50], which is written under the supervision of E. Artin, and was reprinted as [Tat67a]. He developed the method of expressing ζ-functions or more generally L-functions as adèlic integrals over the idèle groups, and deriving this way their fundamental properties such as their analytic continuation, their functional equation and their Euler product form, which had been previously proven by E. Hecke in [Hec18; Hec20]. Poisson's formula played the role of Hecke's complicated theta formulas, and is the number theoretic analog of the Riemann-Roch theorem.

I mainly follow the more recent lecture notes [Poo05]. There are many other references. The book [RV99] is devoted to this theory explaining all background knowledge. As commented in [Tat67a, A few Commens on Recent Related Lierature], in the book [Lan64], "the global results are renormalized in a way which corresponds more closely to the classical theory and which is more suitable for applications". [Kud03] and [Wei66] emphasize the distribution viewpoint, which is not discussed in my talks. Sections with a star beside the section number are omitted in the talk.

As a side remark, redefining the classical ζ-functions had already been done by Matchett in her 1946 thesis [Mat46], who was also a student of E. Artin. Besides, essentially the method (without an analog of the local theory in Tate's thesis but with more results on the finiteness of the class number and Dirichlet's theorem on units) was independently discovered by Iwasawa (Iwasawa Kenkichi 岩澤 健吉). He announced it in his 1950 ICM talk *A note on L-functions* and his letter [Iwa92] to Dieudonné in 1952. In an interview [Iwa93], he recalled that

> I also applied for a short talk of ten minutes, and explained how Hecke's L-functions can be expressed as integrals on idele groups. Just after my talk, Artin came to me, and told me that one of his students was doing essentially the same thing. It was Tate.

Now the method is sometimes referred as *Iwasawa-Tate theory.*

*私はガラスを食べられます。それは私を傷つけません。他們爲什麼不講中文？Updates see https://haoyun.github.io/files/17SS-FS-Tates-Thesis.pdf.

Local Theory, 27.4.2017

1 Reivew on Measure and Integration Theory

For comprehensive introduction on measure theory on locally compact Hausdorff spaces, see [Bou04a] and [Bou04b].

***1.1 Measure Space.** Let X be a set, and $\mathscr{M}$ be a collection of subsets of X. Then $\mathscr{M}$ is said to be a **σ-algebra** if $\mathscr{M}$ is closed under complement and countable unions (including finite and empty unions). Elements of a σ-algebra are called **measurable sets**. A **measurable space** is a pair $(X, \mathscr{M})$, where X is a set and $\mathscr{M}$ is a σ-algebra. If $(X, \mathscr{M})$ and $(Y, \mathscr{N})$ are two measurable spaces, a **measurable function** is a map $f : X \to Y$ such that $f^{-1}(N) \in \mathscr{M}$ for any $N \in \mathscr{N}$.

A **measure** on $(X, \mathscr{M})$ is a function $\mu : \mathscr{M} \to [0, \infty]$, such that μ is countably additive, that is, for any countable (or finite) collection of disjoint measurable sets (A_i), $\mu(\cup A_i) = \sum \mu(A_i)$. A **measure space** is a triple $(X, \mathscr{M}, \mu)$, with $(X, \mathscr{M})$ a measurable space and μ a measure on $(X, \mathscr{M})$.

***1.2** If X is a topological spaces, denote by $\mathscr{B} := \mathscr{B}(X)$ the σ-algebra generated by open subsets of X, which is called the **Borel σ-algebra**. A measure on $(X, \mathscr{B}(X))$ for a topological space X is called a **Borel measure**.

If not specified, a measure space whose underlying set is a topological space is always endowed with a Borel measure. In particular, $\mathbb{R}^n$ and $\mathbb{C}^n$ are endowed with the usual Lebesgue measure.

***1.3 Integration.** From now on, unless specified, a function is a $\mathbb{C}$-valued function. Fix a measure space $(X, \mathscr{M}, \mu)$. For any $S \in \mathscr{M}$ with $\mu(S) < \infty$, define $\int 1_S := \mu(S)$. A **step function** f is a *finite* $\mathbb{C}$-linear combination of such 1_S's. For step functions, define $\int f$ so that it is linear in f. A function $f : X \to \mathbb{C}$ is **integrable** if outside a **null set**, i.e., a subset that is contained in a measure-0 set, f equals to the pointwise limit of some L^1-Cauchy sequence of step functions (f_i); then define

$$\int_X f \, \mathrm{d}\mu := \lim_{i \to \infty} \int f_i \in \mathbb{C}.$$

A measurable $f : X \to \mathbb{C}$ is integrable if and only if $|f|$ is integrable. Moreover, in this case, $|\int f| \leq \int |f|$. The $\mathbb{C}$-vector space of integrable functions is denoted by $\mathscr{L}^1(X)$.

1.4 Haar Measure. A Hausdorff topological spaces is **locally compact** if every $x \in X$ has a compact neighborhood. An **outer Radon measure** on a *locally compact Hausdorff space* X is a Borel measure $\mu : \mathscr{B} := \mathscr{B}(X) \to [0, \infty]$ that is

- *locally finite*, i.e., every $x \in X$ has an open neighborhood U such that $\mu(U) < \infty$;
- *outer regular*, i.e., every $S \in \mathscr{B}$ satisfies $\mu(S) = \inf_{\text{open } U \supseteq S} \mu(U)$; and
- *inner regular on open sets*, i.e., every open $U \subseteq X$ satisfies $\mu(U) = \sup_{\text{compact } K \subseteq U} \mu(K)$.

***1.4.1 Remark.** By definition, every compact subset K (which lies $\mathscr{B}$ as for Hausdorff spaces, compact subsets are closed) has finite measure. In fact, for each $x \in K$, there is an open neighbourhood U_x of x with $\mu(U_x) < \infty$ by *locally finiteness.* Then the cover $(U_x)_{x \in K}$ of K has a finite sub-cover by compactness. So $\mu(K) < \infty$ by by *outer-regularity.*

1.4.2 A (positive) **Radon integral** on a locally compact Hausdorff space X is a $\mathbb{C}$-linear map

$$I : C_c(X) := \{f : X \to \mathbb{C} : f \text{ is continuous and } \operatorname{Supp} f \text{ is compact}\} \to \mathbb{C}$$

such that $I(f) \geq 0$ whenever $f \geq 0$. Given an outer Radon measure μ, we can define

$$I_\mu : C_c(X) \longrightarrow \mathbb{C}, \quad f \longmapsto \int_X f \, d\mu.$$

1.4.3 Riesz Representation Theorem. (Riesz-Markov-Kakutani Theorem). Let X be a locally compact Hausdorff space. Then

$$\begin{aligned} \{\text{outer Radon measure on } X\} &\longrightarrow \{\text{Radon integrals on } X\} \\ \mu &\longmapsto I_\mu \end{aligned}$$

is a *bijection.*

1.4.4 A Borel measure μ on a topological group G is **left-invariant** if $\mu(gS) = \mu(S)$ for any $g \in G$ and $S \in \mathscr{B}(G)$. A **left Haar measure** on a *locally compact Hausdorff topological group* G is a non-zero left-invariant outer Radon measure on G.

***1.4.5 Remarks.**

(1) By definition, every non-empty open subset has non-zero Haar measure. In fact, if there was an open U with $\mu(U) = 0$. Then we can cover every compact subset $K \in \mathscr{B}$ by $(xU)_{x \in K}$. So by *outer-regularity*, every compact subset has measure zero. Then it follows from *inner-regularity* that every open subset has measure zero. Then using *outer-regularity* again, we know $\mu \equiv 0$. But we assume a Haar measure is not zero.
(2) It then follows that every compact subset that contains a non-empty open subset has strictly positive and finite Haar measure.

1.4.6 Theorem. (Existence and uniqueness of Haar measure) Let G be a locally compact Hausdorff topological group.

(1) There exists a left Haar measure μ on G.
(2) Every other left Haar measure on G is $c\mu$, for some $c \in \mathbb{R}_{>0}$.

1.4.7 Example.

(1) On $\mathbb{R}^n$, the Lebesgue measure is a Haar measure.
(2) On a discrete group, the counting measure assigning measure 1 to each singleton is a Haar measure.

***1.4.8** In case G is compact, the **normalized Haar measure on** G is the unique Haar measure μ such that $\mu(G) = 1$.

2 Pontryagin duality and Fourier transform

2.1 LCA groups. An **LCA** group is a locally compact abelian *Hausdorff* topological group. LCA groups form a category, with continuous homomorphisms as arrows.

2.1.1 Example. $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. It is an LCA group and $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$.

2.1.2 Example. Local fields (Section 3.1) are LCA groups. The Adèdel ring of a global field (Section 6.1) is an LCA group.

2.2 A **character** of G is a *continuous* homomorphism $\chi : G \to \mathbb{C}^\times = \mathrm{GL}(1, \mathbb{C})$. A **unitary character** of G is a *continuous* homomorphism $\chi : G \to \mathbb{T} := \mathrm{U}(1, \mathbb{C})$. In other words, a (unitary) character is a one-dimensional (unitary) *representation* of G.

*__2.2.1 Remark.__ Some authors (e.g., [Kud03] and [Tat67a]) call our characters (resp. unitary characters) **quasi-characters** (resp. characters).

*__2.2.2 Proposition.__ If G is compact, then every character of G is unitary.

Proof Consider the composition $G \to \mathbb{C}^\times \to \mathbb{R}_{>0}$, then it follows from the fact that the only compact subgroup of $\mathbb{R}_{>0}$ is $\{1\}$.

2.2.3 The group of all characters, resp. unitary characters, is denoted by

$$X(G), \quad \text{resp.} \quad \widehat{G}.$$

The group low is defined by point-wise multiplication. The group $\widehat{G}$ endowed with the *compact-open topology*, i.e., the topology generated by sets $\{\chi \in \widehat{G} : \chi(K) \subseteq U\}$ for every compact $K \subseteq G$ and every open $U \subseteq \mathbb{T}$, is called the **Pontryagin dual** of G. It turns out that $\widehat{G}$ is again an LCA group.

2.3 Theorem. (**Pontryagin-van Kampen Duality**). The canonical homomorphism

$$\omega_G : G \longrightarrow \widehat{\widehat{G}}, \quad g \longmapsto \omega_G(g) := \big(\chi \mapsto \chi(g)\big) \tag{1}$$

is an isomorphism of LCA groups.

*__2.3.1 Remark.__ One can show that the map ω_G define in (1) is a group homomorphism for any topological abelian group. It is continuous if G is locally compact.[1] Moreover, taking the Pontryagin dual is an *exact* (see §2.5.2) contravariant functor form the category of LCA groups to itself.

*__2.3.2__ For any subset H of G, define $H^\perp := \mathrm{Ann}_{\widehat{G}}(H) := \{\chi \in \widehat{G} : \chi(g) = 1, \forall g \in H\}$, called the **annihilator** of H in $\widehat{G}$. Then

$$X^\perp = \bigcap_{g \in H} \{\chi \in \widehat{G} : \chi(g) = 0\} = \bigcap_{g \in H} \mathrm{Ker}\left(\omega_G(g) : \widehat{G} \to \mathbb{T}\right)$$

is a *closed subgroup* in $\widehat{G}$.

Similarly, for any subset N of $\widehat{G}$, define the annihilator of N in G as $N^\perp := \mathrm{Ann}_G(N) := \{g \in G : \chi(g) = 1, \forall \chi \in N\}$. The notation $N^\perp$ may lead to some confusion because it may also denote the subgroup $\{\gamma \in \widehat{\widehat{G}} : \gamma(\chi) = 1, \forall \chi \in N\}$; but for LCA groups, there will be no confusion due to Corollary 2.3.3 if we identify G and $\widehat{\widehat{G}}$ via ω_G.

[1] This is necessary to ensure the continuity.

***2.3.3 Corollary.** Under the isomorphism (1), there is a bijection

$$\begin{aligned}\{\text{closed subgroups of } G\} &\xleftrightarrow{1:1} \{\text{closed subgroups of } \widehat{G}\}\\ H &\longmapsto H^\perp \\ \omega_G^{-1}(\mathrm{Ann}_{\widehat{\widehat{G}}}(N) \cong \mathrm{Ann}_G(N) &\longleftarrow N\end{aligned}$$

Moreover, for any closed subgroup H of G, one has $H^\perp \cong \widehat{G/H}$ and $\widehat{H} \cong \widehat{G}/H^\perp$.

***2.3.4 Examples.** Let F be a local field (§3.1), e.g., $\mathbb{R}$, $\mathbb{Q}_p$, with the topology induced by the valuation. Let K a global field (§6.1) with the *discrete* topology as a subgroup of $\mathbb{A}_K$ (Proposition 6.2.4). Table 1 gives some examples of Pontryagin duals.

G	$\widehat{G}$ (non-canonical)
F	F (Theorem 3.4.1)
$\mathbb{A}_K$	$\mathbb{A}_K$ (Theorem 6.3.5)
$\mathbb{A}_K/K$	K (Corollary 6.3.6)
$\mathbb{Z}$	$\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$
$\mathbb{Z}_p$	$\mathbb{Q}_p/\mathbb{Z}_p$

G	$\widehat{G}$
discrete	compact
finite	finite
discrete torsion	profinite

Table 1: Example of Pontryagin duals

2.4 Fourier Transform.

Let G be a LCA group and $\mathrm{d}g$ be a Haar measure. If $f \in \mathscr{L}^1(G)$, its **Fourier transform** $\hat{f} : \widehat{G} \to \mathbb{C}$ is defined by

$$\hat{f}(\chi) := \int_G f(g)\overline{\chi(g)}\,\mathrm{d}g.$$

It turns out that $\hat{f}$ is always continuous.

2.5 Fourier Inversion Formula.

Let G be an LCA group. Let $\mathrm{d}g$ be a Haar measure on G. Then there is a *unique* Haar measure $\mathrm{d}\chi$ on $\widehat{G}$, called the **dual measure** or **Plancherel measure**, such that if $f \in \mathscr{L}^1(L)$ and if $\hat{f} \in \mathscr{L}^1(\widehat{G})$, then

$$f(g) = \int_{\widehat{G}} \hat{f}(\chi)\chi(g)\,\mathrm{d}\chi \tag{2}$$

for almost all $g \in G$ (outside a null set). Moreover, if f is continuous, then the above formula holds for all $g \in G$.

2.5.1 Remark. Equation (2) is equivalent to the identity $\hat{\hat{f}}(g) = f(-g)$ (observing that $\chi(-g) = \chi(g)^{-1} = \overline{\chi(g)}$).

***2.5.2** A **short exact sequence** of LCA groups is a sequence

$$1 \to G' \to G \to G'' \to 1$$

which which is a short exact sequence as abstract groups, and G' (resp. G'') has the subspace (resp. quotient space) topology induced from G. This implies that G' is a closed grout G'', otherwise G'' will not be Hausdorff. Conversely, any closed subgroup G' of G gives such a short exact sequence.

Suppose that $1 \to A \to B \to C \to 1$ is a short exact sequence of LCA groups, written multiplicatively. Given Haar measures $\mathrm{d}a$, $\mathrm{d}b$ and $\mathrm{d}c$ on any two of A, B and C, there is a unique Haar measure on the third, which is **compatible** in the sense that

$$\int_B f(b)\,\mathrm{d}b = \int_C \int_A f(ac)\,\mathrm{d}a\mathrm{d}c, \quad \forall f \in C_c(B).$$

3 Local Fields

3.1 A **local field** is a field F that satisfies one (hence all) of the following equivalent conditions (see [**Poo17**, §1.1.2]).

(1) F is a finite extension of $\mathbb{R}$, $\mathbb{Q}_p$, or $\mathbb{F}_p(\!(t)\!)$, for some prime p.
(2) F is isomorphic to one of the following
   - $\mathbb{R}$ or $\mathbb{C}$,
   - a finite extension of $\mathbb{Q}_p$ for some prime p, or
   - $\mathbb{F}_q(\!(u)\!)$ for some prime power q.
(3) F is $\mathbb{R}$ or $\mathbb{C}$, or is the fraction field of a complete discrete valuation ring with finite residue field.
(4) F is a non-discrete locally compact topological field; more precisely, k is locally compact and Hausdorff with respect to some nondiscrete topology for which the field operations are continuous.
(5) F is a completion of a global field (see Section 6.1) with respect to a nontrivial absolute value.

A local field F that is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ is called *archimedean*; other ones are called *nonarchimedean*. In nonarchimedean case, we use the following notations.

$\mathcal{O} = \mathcal{O}_F$	the valuation ring.
$\mathfrak{m} = \mathfrak{m}_F$	the maximal ideal of $\mathcal{O}$.
k	the residue field $\mathcal{O}/\mathfrak{m}$.
π	a uniformizer.
p	characteristic of k.
ord	the valuation on K
q	cardinality of k.

3.2 The additive group $F = F^+$ (with the usual topology or the $\mathfrak{m}$-adic topology) and the subgroup $F^\times$ (with the subspace topology) are both LCA groups. Compare it with Remarks 6.2.1 (2).

3.3 Additive Haar measure and normalized absolute value. Let F be a local field. Then the additive group of F, sometimes denoted by F^+, is an *LCA group*. Let $\mathrm{d}x$ be a Haar measure on F. So we have $\int_F f(x+a)\,\mathrm{d}x = \int_F f(x)\,\mathrm{d}x$ for all $a \in F$ and $f \in C_c(F)$. It turns out that for any $a \in F^\times$ and $b \in F$, we have

$$\int_F f(a(x+y))\,\mathrm{d}x = \int_F f(ax+ab)\,\mathrm{d}x = \int_F f(ax)\mathrm{d}x, \quad \forall f \in C_c(F)$$

Hence $f \mapsto \int_F f(ax)\mathrm{d}x$ is another Haar measure. Hence there is $|a| := |a|_F \in \mathbb{R}_{>0}$, such that

$$\int_F f(x)\,\mathrm{d}x = \int_F f(ax)|a|\,\mathrm{d}x, \quad \forall f \in C_c(F) \tag{3}$$

The number $|a|$ does not depend on the choice of $\mathrm{d}x$ (see [Tat67a, Lemma 2.2.4]). It holds that $|a| \cdot |b| = |a| \cdot |b|$.

The group homomorphism $|\cdot|_F : F^\times \longrightarrow \mathbb{R}_{>0}$, $a \mapsto |a|_F$ (or $F \to \mathbb{R}_{\geq 0}$ by setting $|0| = 0$) is called the **normalized absolute value** of F.

The **normalized** Haar measure on F is the unique Haar measure $\mathrm{d}x$, such that $\mathscr{O}_F$, the maximal compact subring of F, has measure 1, i.e., $\int_{\mathscr{O}} \mathrm{d}x = 1$.

3.3.1 Proposition. ([Tat67a, Lemma 2.2.5]).

- If $F \cong \mathbb{R}$, then $|a|$ is the usual absolute value of a.
- If $F \cong \mathbb{C}$, then $|a|$ is the *square* of the usual absolute value of a. To avoid confusion, sometimes it is written as $\|z\| := z\bar{z} = |z|^2$.
- If F is nonarchimedean, then $|a| = q^{-\operatorname{ord}(a)}$.

Proof Exercise.

3.4 An **additive character** is a nontrivial (i.e., not identically 1) unitary character $\psi : F \to \mathbb{T}$ of the additive group F^+ of F. Given an additive character ψ and $a \in F$, set $\psi_a(x) := \psi(ax)$. Then ψ_a is another unitary character of F.

3.4.1 Theorem. ([Tat67a, Lemma 2.2.1]). Given an additive character ψ, the natural map

$$\Psi : F \longrightarrow \widehat{F}, \quad a \longmapsto \psi_a$$

is an isomorphism of LCA groups.

Proof See [Po005, Theorem 4.4].

***3.4.2** There are standard choices of ψ.

- If $F = \mathbb{R}$, then $\psi(x) := \exp(-2\pi \mathrm{i} x)$.
- If $F = \mathbb{Q}_p$, then ψ is the composition

  $$\mathbb{Q}_p \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Z}[1/p]/\mathbb{Z} \hookrightarrow \mathbb{R}/\mathbb{Z} \cong \mathbb{T}$$

  It is chosen so that $\psi|_{\mathbb{Z}_p} = 1$ and $\psi(1/p^n) = \exp(2\pi\mathrm{i}/p^n)$.
- If $F = \mathbb{F}_p(\!(t)\!)$, $\psi(\sum a_i t^i) := \exp\left(\frac{2\pi \mathrm{i}}{p} a_{-1}\right)$, where a_{-1} is lifted from $\mathbb{F}_p$ to $\mathbb{Z}$.
- If F_0 is one of above and F is a finite separable extension of F_0, let ψ_0 be chosen as above then ψ is chosen to be the composition

  $$F \xrightarrow{\operatorname{Tr}_{F/F_0}} F_0 \xrightarrow{\psi_0} \mathbb{T}$$

  (Recall the trace map $\operatorname{Tr}_{F/F_0}(f)$ is defined as the trace of the multiplication-by-f map on then F_0-vector space F.)

3.4.3 In case that F is nonarchimedean, there exists $r \in \mathbb{Z}$ such that $\psi|_{\mathfrak{m}^r} = 1$ (ψ is continuous and $0 \in \psi^{-1}(U)$ with U a small neighborhood containing $1 \in \mathbb{T}$). The fraction ideal $\mathfrak{m}^r$ with the smallest r such that ψ is trivial on $\mathfrak{m}^r$ is called the **conductor** of ψ. Note $\mathfrak{m}^0 = \mathcal{O}_F$.

3.5 The Schwartz-Bruhat Functions. Recall that a Schwartz function $\mathbb{R}^n \to \mathbb{C}$ is a C^∞-function whose derivatives tend to 0 "rapidly". In case F is nonarchimedean, we cannot take derivatives of a function $F \to \mathbb{C}$.

A function $f : F \to \mathbb{C}$ is a **Schwartz-Bruhat function** if it is

- a Schwartz function, if $F = \mathbb{R}$ or $F = \mathbb{C} \cong \mathbb{R}^2$.
- a locally constant function of compact support, if F is nonarchimedean.

Denote by $\mathscr{S} := \mathscr{S}(F)$ the $\mathbb{C}$-vector space of Schwartz-Bruhat functions. If F is nonarchimedean and $f \in \mathscr{S}(F)$, then f is a finite $\mathbb{C}$-linear combination of characteristic function 1_{D_i} for some open disks D_i.

3.6 Fourier Transform for Local Fields. Let F be a local field. Fix a Haar measure $\mathrm{d}x$ on the additive group F, and a nontrivial additive (unitary) character ψ . Recall Theorem 3.4.1 that ψ defines an isomorphism $F \to \hat{F}$, $y \mapsto \hat{\psi}_y$. For any $f \in \mathscr{S}(F)$, define the **Fourier Transform** $\hat{f}$ by[2]

$$\hat{f}(y) := \int_F f(x)\psi(xy)\,\mathrm{d}x = \int_F f(x)\psi_y(x)\mathrm{d}x.$$

It turns out that $\hat{f} \in \mathscr{S}(F)$. In general, $\mathrm{d}x$ is not *self dual*, nevertheless, we have $\hat{\hat{f}}(x) = rf(-x)$ for some $r \in \mathbb{R}_{>0}$. By scaling $\mathrm{d}x$ properly, the *Fourier inversion formula*

$$f(x) = \int_F \hat{f}(y)\overline{\psi(xy)}\,\mathrm{d}y \tag{4}$$

holds for all x.

***3.7** Relative to the standard additive character ψ as in §3.4.2, the unique Haar measure that is **self-dual**, i.e., making (4) hold, can be described as follows.

- If $F = \mathbb{R}$, then $\mathrm{d}x$ is the Lebesgue measure.
- If $F = \mathbb{C}$, then $\mathrm{d}x$ is twice the Lebesgue measure.
- If F is nonarchimedean, $\mathrm{d}x$ is the one such that $\int_{\mathcal{O}_F} \mathrm{d}x = (\#(\mathcal{O}_F/\mathcal{D}_{F/F_0}))^{-1/2}$, where $\mathcal{D}_{F/F_0}$ is the *different* of F/F_0.

3.8 Multiplicative Haar measure. Note that $f \mapsto \int_F f(x)/|x|\,\mathrm{d}x$, $f \in C_c(F^\times)$, where $|\cdot|$ is the normalized absolute value on F, defines a Radon integral. In fact, or any $a \in F^\times$ and any $f \in C_\mathrm{c}(F^\times)$, recalling eq. (3), we obtain that

$$\int_F \frac{f(x)}{|x|}\,\mathrm{d}x = \int_F \frac{f(ax)}{|ax|}|a|\,\mathrm{d}x = \int_F \frac{f(ax)}{|x|}\,\mathrm{d}x.$$

Hence it defines a Haar measure on $F^\times$. So any Haar measure $\mathrm{d}^\times x$ on the multiplicative group $F^\times$ satisfies $\mathrm{d}^\times x = (c/|x|)\mathrm{d}x$ for some $c \in \mathbb{R}_{>0}$, in the sense that for any,

$$\int_{F^\times} f(x)\,\mathrm{d}^\times x = \int_F f(x)\frac{c}{|x|}\,\mathrm{d}x, \quad \forall f \in C_\mathrm{c}(F^\times) \tag{5}$$

[2] In Tate's thesis, a complex conjugate was taken

3.9 Multiplicative Characters. By an **multiplicative character**, we mean a character $\chi \in X(F^\times)$ of the multiplicative group $F^\times$. If $\chi \in X(F^\times)$, its **twisted dual** is the character $\hat{\chi} := \chi^{-1}|\cdot|_F$.

We have a short exact sequence of LCA groups

$$1 \longrightarrow U \longrightarrow F^\times \xrightarrow{|\cdot|_F} |F^\times| \longrightarrow 1$$

where

$$U := U_F := \{x \in F^\times : |x|_F = 1\} = \begin{cases} \{\pm 1\}, & \text{if } F = \mathbb{R}, \\ \mathbb{T}, & \text{if } F = \mathbb{C}, \\ \mathcal{O}^\times & \text{if } F \text{ is nonarchimedean.} \end{cases}$$

and the value group

$$|F^\times| := \{|x| : x \in F^\times\} = \begin{cases} \mathbb{R}_{>0}, & \text{if } F = \mathbb{R} \text{ or } \mathbb{C}, \\ q^{\mathbb{Z}} & \text{if } F \text{ is nonarchimedean.} \end{cases}$$

A character $\chi \in X(F^\times)$ is **unramified** if $\chi|_U = 1$.

3.9.1 Remark. Suppose that F is nonarchimedean. A character η of $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ factors as $\mathrm{Gal}(F^{\mathrm{ab}}/F) \twoheadrightarrow \mathrm{Gal}(L/F) \hookrightarrow \mathbb{C}^\times$ for some finite Galois extension L/F. Let $I = I_F = \mathrm{Gal}(F^{\mathrm{ab}}/F^{\mathrm{ur}})$ be the *inertia subgroup* of $\mathrm{Gal}(F^{\mathrm{ab}}/F)$, which maps onto the inertia subgroup $I_{L/F}$ of $\mathrm{Gal}(L/F)$. Recall the local Artin map $\theta_F : F^\times \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$, which maps U isomorphically to I (see for example, [Ser67, §3.8]). Now consider the composition $\chi : \eta \circ \theta_F$ and the commutative diagram

$$\begin{tikzcd}
F^\times \arrow[r, "\theta_F"] & \mathrm{Gal}(F^{\mathrm{ab}}/F) \arrow[rr, "\eta"] \arrow[dr, two heads] & & \mathbb{C}^\times \\
U \arrow[u, hook] \arrow[r, "\cong"] & I \arrow[u, hook] \arrow[dr] & \mathrm{Gal}(L/F) \arrow[ur, hook] & \\
 & & I_{L/F} \arrow[u, hook] &
\end{tikzcd}$$

We know that the followings are equivalent.

(1) L/F is *unramified.*
(2) $I_{L/F} = \{1\}$ is trivial.
(3) $\eta|_I = 1$ is trivial.
(4) $\chi|_U = 1$ is trivial.

3.9.2 Proposition. For any $\chi \in X(F^\times)$, the followings are equivalent.

(1) χ is unramified,
(2) χ factors through $|F^\times|$.
(3) $\chi = |\cdot|_F^s$ for some $s \in \mathbb{C}$ ([Tat67a, Lemma 2.3.1]).

Proof The only part that need to be proved is (2) implies (3). For this, it suffices to show that every character of $|F^\times|$ is $x \mapsto x^s$ for some $s \in \mathbb{C}$.

- If F is nonarchimedean, choose $s \in \mathbb{C}$ such that $\chi(\pi^{-1}) = q^s$.
- If F is archimedean, note that $(\mathbb{R}_{>0}, \cdot)$ is topologically isomorphic to the group $(\mathbb{R}, +)$ via log with inverse exp. As $(\mathbb{R}, +)$ is simply connected hence every character $(\mathbb{R}, +) \to (\mathbb{C}^\times, \cdot)$ factors through the universal cover $\exp : (\mathbb{C}, +) \twoheadrightarrow (\mathbb{C}^\times, \cdot)$. Now it suffices to consider continuous homomorphisms $(\mathbb{R}, +) \to (\mathbb{C}, +)$. All such maps are of the form $x \mapsto s \cdot x$ for some $s \in \mathbb{C}$, where s is determine by the value of the map at $1 \in \mathbb{R}$.

$$\begin{tikzcd} (\mathbb{R}_{>0}, \cdot) \arrow[r] \arrow[d, shift left, "\log"] & (\mathbb{C}^\times, \cdot) \\ \text{simply connected } (\mathbb{R}, +) \arrow[r, dotted] \arrow[u, shift left, "\exp"] & (\mathbb{C}, +) \arrow[u, "\exp", "\text{universal cover}"'] \end{tikzcd}$$

3.9.3 It then follows that unramified charaters of $F^\times$ form a subgroup of $X(F^\times)$ and this group is isomorphic to

- $\mathbb{C}$, if F is archimedean, or to
- $\mathbb{C}/(2\pi i\mathbb{Z}/\log(q))$, if F is nonarchimedean.

By viewing $X(F^\times)$ as disjoint union of these cosets makes $X(F^\times)$ a Reimann surface with infinitely many connected components.

3.9.4 Corollary. ([Tat67a, Theorem 2.3.1]). Every character $\chi \in X(F^\times)$ is of the form $\eta|\cdot|^s$ for some unitary character η and $s \in \mathbb{C}$

Proof Write χ as $(\chi/|\chi|) \cdot |\chi|$. It's clear that $\chi/|\chi|$ is a unitary character. Meanwhile $|\chi|$ is trivial when restricted to U, hence is unramified and hence is of the form $|\cdot|^s$ for some $s \in \mathbb{C}$ by Proposition 3.9.2.

3.9.5 Corollary. For and $\chi \in X(F^\times)$, one has $|\chi| = |\cdot|^\sigma$ for some $\sigma \in \mathbb{R}$.

Proof Write $\chi = \eta|\cdot|^s$, then $|\chi| = |\cdot|^s$ with $\sigma := \operatorname{Re} s$.

The real number σ is called the **exponent** of χ. Hence a character is unitary if and only if it has exponent 0.

3.9.6 Suppose that F is nonarchimedean. Then there is some $r \in \mathbb{N}$, such that $\chi|_{1+\mathfrak{m}^r} = 1$ $(1 + \mathfrak{m}^0 := \mathcal{O}^\times)$. Then the ideal $\mathfrak{m}^r$ with the smallest such r is called the **conductor** of χ. Hence χ is unramified if and only if the conductor of χ is $\mathcal{O}^\times$. So in some sense, the conductor measure the extent to which χ is ramified.

4 Local zeta integrals and meromorphic continuation

4.1 ([Tat67a, Definition 2.4.1]) Fix a Haar measure $d^\times x$ on $F^\times$. For any $f \in \mathscr{S}(F)$ and $\chi \in X(F^\times)$, define the **local zeta integral**

$$Z(f, \chi) := \int_{F^\times} f(x)\chi(x)\, d^\times x.$$

4.2 Theorem. (Meromorphic continuation) For any $f \in \mathscr{S}(F)$,

(1) For every $f \in \mathscr{S}(F)$, the integral $Z(f,\chi)$ converges for χ of exponent $\sigma > 0$ (see [Tat67a, Lemma 2.4.1]).

(2) For every $f \in \mathscr{S}(F)$, the function $Z(f,\chi)$ extends to a meromorphic function on $X(F^\times)$ (recall the Riemann surface structure of $X(F^\times)$, §3.9.3).

(3) For every $f \in \mathscr{S}(F)$, the meromorphic function $Z(f,\chi)/L(\chi)$ on $X(F^\times)$ is *holomorphic.* where

$$L(\chi) := \begin{cases} \Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2) & F = \mathbb{R} \\ \Gamma_{\mathbb{C}}(s) := \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s) & F = \mathbb{C} \\ \begin{cases} (1-\chi(\pi))^{-1} & \chi \text{ unramified} \\ 1 & \text{otherwise} \end{cases} & F \text{ nonarchimedean} \end{cases}$$

is called the **local L-factor,**[3] which is a meromorphic function of $\chi \in X(F^\times)$ without zeros in call cases.

(4) For each component X_0 of $X(F^\times)$, there is $f \in \mathscr{S}(F^\times)$, such that the holomorphic function $Z(f,\chi)/L(\chi)$ is non-vanishing on X_0. Moreover, if F is nonarchimedean, X_0 is the subgroup of unramified characters, i.e., $X_0 = \{|\cdot|^s : s \in \mathbb{C}\}$, and if $f = 1_{\mathcal{O}}$, then $Z(f,\chi)/L(\chi)$ is 1 on X_0.

(5) Fix choices of ψ and $\mathrm{d}x$ (to define Fourier transforms). There exists a non-vanishing holomorphic function $\epsilon(\chi,\psi,\mathrm{d}x)$ of $\chi \in X(F^\times)$, called the **local ϵ-factor**, such that

$$\frac{Z(\hat{f},\hat{\chi})}{L(\hat{\chi})} = \epsilon(\chi,\psi,\mathrm{d}x)\frac{Z(f,\chi)}{L(\chi)} \tag{6}$$

for all $f \in \mathscr{S}(F^\times)$. On each component $X_0 = \{\eta|\cdot| : s \in \mathbb{C}\}$ of $X(F^\times)$ with a fixed η, the local ϵ-factor has the form $A\exp(Bs)$ for some $A,B \in \mathbb{C}$. Moreover, if F is nonarchimedean, ψ has conductor $\mathfrak{m}^0$, and if $\int_{\mathcal{O}_F} \mathrm{d}x = 1$, then $\epsilon(|\cdot|^s,\psi,\mathrm{d}x) = 1$ for all $s \in \mathbb{C}$. (cf. [Tat67a, Theorem 2.4.1]).

Proof (partial) • First step is to show that $\zeta(f,\chi)$ converges for $\sigma > 0$. Actually, it absolutely converges. So the question is whether

$$\int_{F^\times} |f(x)||x|^\sigma \mathrm{d}^\times x < \infty.$$

We decompose the integral into two parts, i.e., over $|x| > 1$ and over $0 < |x| \le 1$. Over the former region, the integral is finite as the integrand decays rapidly.

Over $0 < |x| \le 1$, $|f|$ is bounded. So we reduce to the question if $\int_{0<|x|\le 1} |x|^\sigma \mathrm{d}x$ is finite. Then use the translation invariance of $\mathrm{d}^\times$, dividing this region to a union of family of annulus, then the problem reduces to the convergence of $\sum r^\sigma$ with $0 < r < 1$. Done.

• The second step is to show that $Z(f,\chi)$ and $Z(f,\chi)/L(\chi)$ are holomorphic for $\sigma > 0$. We can write $\chi = \eta|\cdot|^s$ with η unitary and $s \in \mathbb{C}$. Fix η. As the integral converges absolutely for $\operatorname{Re} s = \sigma > 0$, We can differentiate under the integral sign. This implies that $L(f,\chi)$ is holomorphic for each component of $X(F^\times)$ for $\sigma > 0$. Since $L(\chi)$ has no zero, $L(f,\chi)/L(\chi)$ is holomorphic on this region too.

[3] In case $F = \mathbb{R}$, This was the archimedean factor used to complete the Riemann zeta function; In case of $F = \mathbb{C}$, the definition can be explained in terms of induced representations when the definition is extended to higher-dimensional representations of Weil groups [Poo05, §4.7].

- First we show the functional equation for a specifically chosen f. More precisely, we show that for each η, there is $f \in \mathscr{S}(F)$, such that for $\chi = \eta|\cdot|^s$,
  - the holomorphic function $L(f,\chi)/L(\chi)$ is non-vanishing on the right half plane $\operatorname{Re} s > 0$;
  - the holomorphic function $L(\hat{f},\hat{\chi})/L(\hat{\chi})$ is non-vannishing on the half plane $\operatorname{Re} s < 1$;
  - there is a non-vanishing holomorphic function $\chi(s) = \epsilon(\chi,\psi,\mathrm{d}x)$ of the form $A\exp(Bs)$ such that the functional equation holds on the strip $0 < \operatorname{Re} s < 1$.

Note that changing ψ or $\mathrm{d}x$ changes $Z(\hat{f},\hat{\chi})$ by a factor. So it suffice to prove this for a choice of ψ and $\mathrm{d}x$. We can also choose $\mathrm{d}^\times x$.

If $F = \mathbb{R}$, take $\psi := \exp(-2\pi\mathrm{i}x)$, $\mathrm{d}x$ to be the Lebesgue measure, and $\mathrm{d}^\times x = \mathrm{d}x/|x|$; If $F = \mathbb{C}$, take $\psi := \exp(-2\pi\mathrm{i}(z+\bar{z}))$, $\mathrm{d}x$ to be *twice* the Lebesgue measure, and $\mathrm{d}^x = \mathrm{d}x/\|x\|$. Then everything follows from direct computation from classical Fourier analysis.

Now suppose that F is nonarchimedean. Choose $\mathrm{d}x$ so that $\int_{\mathcal{O}_F}\mathrm{d}x = 1$ and $\mathrm{d}^\times x = \mathrm{d}x/|x|$. It then follows that

$$\begin{gathered}\int_{\mathfrak{m}^r}\mathrm{d}x = |\pi^r|\int_{\mathcal{O}_F}\mathrm{d}x = p^{-r}, \quad \int_{\mathcal{O}_F^\times}\mathrm{d}^\times x = \int_{\mathcal{O}_F\setminus\mathfrak{m}}\mathrm{d}x = 1-q^{-1}, \\ \int_{1+\mathfrak{m}^r}\mathrm{d}^\times x = \int_{1+\mathfrak{m}^r}\mathrm{d}x = \int_{\mathfrak{m}^r}\mathrm{d}x = q^{-r}, \quad \forall r \geq 1.\end{gathered} \tag{7}$$

[Choose $\psi : F \to \mathbb{T}$ which has conductor $\mathfrak{m}^0 = \mathcal{O}$. According to the conductor of η, we choose appropriate $f = 1_{1+\mathfrak{m}^r}$, then compute $Z(f,\chi)$ and $Z(\hat{f},\hat{\chi})$. *Gauß sums* is used in the computation. Omitted.]

- Then use *Fubini's theorem*[4] to show that

$$Z(f,\chi)Z(\hat{g},\hat{\chi}) = Z(g,\chi)Z(\hat{f},\hat{\chi}) \tag{8}$$

In fact,

$$\begin{aligned}Z(f,\chi)Z(\hat{g},\hat{\chi}) &= \int_{F^\times} f(x)\chi(x)\,\mathrm{d}^\times x\int_{F^\times}\underbrace{\left(\int_F g(z)\psi(yz)\,\mathrm{d}z\right)}_{\hat{g}(y)}\underbrace{\chi(y)^{-1}|y|}_{\hat{\chi}(y)}\,\mathrm{d}^\times y \\ &= \int_{F^\times}\int_{F^\times}\left(f(x)\chi(xy^{-1})|y|\int_{F^\times} g(z)\psi(yz)|z|\,\mathrm{d}^\times z\right)\mathrm{d}^\times x\mathrm{d}^\times y \\ &= \int_{(F^\times)^3} f(x)\chi(xy^{-1})g(z)\psi(yz)|yz|\,\mathrm{d}^\times x\mathrm{d}^\times y\mathrm{d}^\times z && \text{(Fubini)} \\ &= \int_{(F^\times)^3} f(x)g(z)\psi(txz)|txz|\chi(t^{-1})\,\mathrm{d}^x\mathrm{d}^z\mathrm{d}^\times t, && (\text{set } y := xt)\end{aligned}$$

which is symmetric in f and g. Hence eq. (8) holds.

- We now know for our chosen f, it holds that

$$\frac{Z(\hat{f},\hat{\chi})}{L(\hat{\chi})} = \epsilon(\chi,\psi,\mathrm{d}x)\frac{Z(f,\chi)}{L(\chi)}, \qquad 0 < \sigma < 1.$$

[4] Under certain conditions, $\int_X\left(\int_Y f(x,y)\,\mathrm{d}y\right)\mathrm{d}x = \int_Y\left(\int_X f(x,y)\,\mathrm{d}x\right)\mathrm{d}y = \int_{X\times Y} f(x,y)\,\mathrm{d}x\mathrm{d}y$.

Multiplying this by eq. (8) and cancelling out $Z(f,\chi)Z(\hat{f},\hat{\chi})$, we obtain that

$$\frac{Z(\hat{g},\hat{\chi})}{L(\hat{\chi})} = \epsilon(\chi,\psi,\mathrm{d}x)\frac{Z(g,\chi)}{L(\chi)}, \qquad 0<\sigma<1.$$

Meanwhile, RHS is holomorphic for $\sigma > 0$ and LHS holomoprhic $\sigma < 1$. So they glue to a holomorphic extension for RHS. Then it gives a holomorphic extension for $Z(g,\chi)/L(\chi)$.

4.2.1 Remark. With χ fixed, Theorem 4.2 (3) and (4) are saying that $L(\chi)$ is the *greatest common divisor* of the functions $Z(f,\chi)$.

4.2.2 Remark. ([**Poo05**, Remark 4.19]) In Tate's thesis [**Tat67a**, Theorem 2.4.1], as well as in [**Tat67a**], a single function

$$\gamma(\chi,\psi,\mathrm{d}x) := \epsilon(\chi,\psi,\mathrm{d}x)\frac{L(\chi^\vee)}{L(\chi)},$$

sometimes called the **local γ-factor**, to measure the multiplicative error in the functional equation relating $Z(f,\chi)$ and $Z(\hat{f},\hat{\chi})$. Only later, in [**Del73**, §3], did Deligne separate this function into L-factors and an ϵ-factor.

*5 Local zeta integrals as Eigendistributions

The reference to this section is [**Kud03**], which was inspired by the approach of [**Wei66**] in his Bourbaki talk.

Global Theory, 4.5.2017

6 Global Fields, Adèles and Idèles

6.1 A **global field** K is a field that satisfies one of the following conditions (see [Poo17, §1.1.3]).

(1) K is isomorphic to one of the following:

- a finite extension of $\mathbb{Q}$, which is called a **number field**, or
- a finite extension of $\mathbb{F}_p(t)$ for some prime p, or equivalently, the function field of a geometrically integral curve over a finite field $\mathbb{F}_q$ with q a power of p, which is called a **global function filed**.

(2) K is the fraction field of a finitely generated $\mathbb{Z}$-algebra that is an integral domain of Krull dimension 1.

Recall that a **place** or a **prime** of a field K is an equivalence[5] class of valuations on K.

6.2 The Adèle Ring and Idéle Group. Let K be a global field. As usual, we introduce the following notations.

K_v	the completion of K at v, which is a local field.
$\mathcal{O}_v$	the valuation ring of K_v, set $\mathcal{O}_v = K_v$ in case v is archimedean.
$\mathfrak{m}_v$	the maximal ideal of $\mathcal{O}_v$.
π_v	a uniformizer
k_v	the residue field of $\mathcal{O}_v$.
v or ord_v	the (multiplicative) valuation.
$\lvert\cdot\rvert_v$	the normalized absolute value on K_v.
q_v	the cardinality of the residue field k_v.

Define the **adèle ring**[6] $\mathbb{A}_K$ and the **idèle group** $\mathbb{I}_K$ as:

$$\mathbb{A} := \mathbb{A}_K := \prod_v{}'(K_v, \mathcal{O}_v) =: \prod_v{}' K_v, \qquad \mathbb{I} := \mathbb{I}_K := \mathbb{A}_K^\times := \prod_v{}'(K_v^\times, \mathcal{O}_v^\times) =: \prod_v{}' K_v^\times.$$

As a *set*, $\mathbb{A}_K$ (resp. $\mathbb{A}_K^\times$) consists of elements of the form $x = (\dots, x_v, \dots)$ with $x_v \in K_v$ (resp. $x_v \in K_v^\times$) for all places v, and for almost all[7] v, $x_v \in \mathcal{O}_v$ (resp. $x_v \in \mathcal{O}_v^\times$). The *ring (resp. group) structure* on $\mathbb{A}_K$ (resp. $\mathbb{A}_K^\times$) is defined component-wise. A basis for the *topology* on $\mathbb{A}$ (resp. $\mathbb{A}_K^\times$) consists of the subsets $\prod_v U_v$, where U_v is open in K_v (resp. $K_v^\times$), and $U_v = \mathcal{O}_v$ (resp. $U_v = \mathcal{O}_v^\times$) for almost all v.

6.2.1 Remarks.

(1) The product $\prod_v'(K_v, \mathcal{O}_v)$ is called the **restricted product** of K_v''s with respect to $\mathcal{O}_v$'s. We will not recall the general definition here, which applies to a collection of (locally compact) topological spaces. The topology on restricted products is finer than the subspace topology induced from the product space with the product topology. See [Cas67, §13] and [Tat67a, §3] for more details.

[5] Two (additive) valuations $\lvert\cdot\rvert_1$ and $\lvert\cdot\rvert_2$ are equivalent if they induce the same topology on K, or equivalently, if $\lvert\cdot\rvert_1 = \lvert\cdot\rvert_2^s$ for some s. See [Cas67, §1 and §4] or [Neu92, II, §3].

[6] We will mainly concentrate on its additive group structure.

[7] It means "for all but finitely many".

(2) The topology defined above on the Idèle group $\mathbb{I}_K$ is finer than the subspace topology as a subspace of $\mathbb{A}_K$.Actually, given a topological ring R, the subgroup $R^\times$ of invertible elements is not in general a topological group if it is endowed with the subset topology because inversion need not be continuous. The "canonical" topology is that the subspace topology of $R^\times$ in $R\times R$ via the map $x \mapsto (x, x^{-1})$. See [**Cas67**, §16].

(3) The name *idèle* dates back to Chevalley, which means *ideal element*, (or in French, *élément idèle*). The name *adèle* was introduced by A. Weil, which means *additive idèle*(or in French, *idèle additif*). An adèle was also called a **repartition**[8] by Chevalley and a **valuation vector** in E. Artin's school (hence in Tate's thesis).

6.2.2 Proposition. The additive group $\mathbb{A}_K$ and multiplicative group $\mathbb{A}_K^\times$ are LCA groups.

Proof This follows from the fact that each K_v is locally compact, and the general theory on restricted products of locally compact topological groups.

6.2.3 For any $a \in K$, and for almost all v, we have $a \in \mathcal{O}_v^\times$. Then the diagonal map $a \mapsto (\cdots, a, a, a, \cdots)$ defines natural ring/group homomorphisms

$$K \hookrightarrow \mathbb{A}, \quad \text{and} \quad K^\times \hookrightarrow \mathbb{A}^\times,$$

which are also *injective*. Identify K (resp. $K^\times$) as a subring (resp. subgroup) of $\mathbb{A}_K$ (resp. $\mathbb{A}_K^\times$). Then elements in K and $K^\times$ are called **principal alèles** and **principal idèles** respectively.

6.2.4 Proposition. K is discrete and cocompact in $\mathbb{A}_K$, i.e., $\mathbb{A}_K/K$ is compact.

Proof See [**Cas67**, §14].

6.2.5 Remark. Compare Proposition 6.2.4 with Proposition 6.5.3.

6.3 Additive Theory.

6.3.1 Additive Characters. As in the local theory, an **additive character** is an character χ_K for the additive group $\mathbb{A}$ that unitary and non-trivial, i.e., a continuous group homomorphism $\chi : \mathbb{A} \to \mathbb{T}$ that is not identically 1.

6.3.2 Remark. As in the local case (§3.4.2), there are standard choices of additive characters on $\mathbb{A}_K$.

- (Number Field Case). Choose the standard ψ_v on each K_v, and set $\prod_v \psi_v$ as the standard ψ. Note that the product is well defined, as for almost all v, K is absolutely unramified hence at that these places, $\psi_v|_{\mathcal{O}_v} = 1$.
- (Function Field Case). Suppose that K is the function field of a (regular integral) curve X over $\mathbb{F}_q$. Let $\Omega_X = \Omega^1_{X/\mathbb{F}_q}$ be the sheaf of differentials of X over $\mathbb{F}_q$. Denote by Ω_K its generic fibre $\Omega^1_{X/\mathbb{F}_q,\eta}$. Hence Ω_K is a 1-dimensional K-vector space, the space of "meromorphic" 1-forms on X. For any closed point v (equivalently, place of K), define

$$\Omega_v := \Omega_K \otimes_K K_v \cong k_v(\!(\pi_v)\!)\, \mathrm{d}u,$$

[8]Also in [**Har77**, p. 248].

where π_v is a uniformizer (local parameter) at the closed point v. And define **residue map**

$$\mathrm{Res}_v : \Omega_v \longrightarrow k_v, \qquad \sum_{i \gg -\infty} a_i u^i \, \mathrm{d}u \longmapsto a_{-1}. \tag{9}$$

(This map does not depend on the choice of the uniformizer π_v, for a proof, see [**Con09**]). Then any choice of a meromorphic 1-form Ω_K gives rise to

$$\psi_v(x) := \exp\left(\frac{2\pi \mathrm{i}}{p} \mathrm{Tr}_{k_v/\mathbb{F}_p} \mathrm{Res}_v(x\omega)\right)$$

for each place v, where we take any lift of $\mathrm{Tr}_{k_v/\mathbb{F}_p} \mathrm{Res}_v(x\omega) \in \mathbb{F}_p$ to $\mathbb{Z}$. The standard ψ is defined as $\prod_v \psi_v$ for a choice of a meromorphic 1-form ω, This is well-defined because the fact that for almost all v, ω generates Ω_v as an $\mathcal{O}_{X,v}$-module, i.e., ω is non-vanishing and regular at v. At these places, $\psi_v|_{\mathcal{O}_v} = 1$ (in other words, ψ_v has conductor $\mathfrak{m}_v^0 = \mathcal{O}_v$). So $\prod_v \psi_v$ is actually a finite product.

6.3.3 Propsotion. For the "standard" choice of ψ as in Remark 6.3.2, we have $\psi|_K = 1$.

Proof The number theory case follows from computations. The function field case follows from the fact that: for any connected proper regular curve over a field k, and for any $\omega \in \Omega_K$, we have $\sum_v \mathrm{Res}_v(\omega) = 0$ (see [**Tat68**, p. 155]).

6.3.4 Remark. It is well-known that on a compact Riemann surface, given a meromorphic differential, the sum of its residues at its poles is zero. Here the result generalizes to (connected proper regular) curves over a field. To find more about the residue map(9) and more story about the relation with *Serre duality*, see [**Har77**, II, Remark 7.14], which cites [**Ser84**, pp. 24–35, 76–81] and [**Tat68**]. Besides, [**Mil06**, III, Corollary 0.8] may be useful. In [**Har11**, §9.5], the Riemann-Roch theorem was proved using this explicit residue map instead of the abstract Serre duality; the author even mentioned that this approach is essentially already in the work of Dedekind and Weber [**DW82**] in 1882!

6.3.5 Theorem. Given the standard additive character ψ, the homomorphism

$$\Psi : \mathbb{A}_K \longrightarrow \widehat{\mathbb{A}_K}, \qquad a \longmapsto (\psi_a : x \mapsto \psi(ax)) \tag{10}$$

is an isomorphism of LCA groups.

Proof A proof is as follows. As a general result for restricted products, there is an isomorphism of LCA groups

$$\begin{aligned} \widehat{\mathbb{A}}_K = \widehat{\prod_v{}' (K_v, \mathcal{O}_v)} &\longrightarrow \prod_v{}' (\widehat{K}_v, \widehat{K_v/\mathcal{O}_v}) \\ \psi &\longmapsto (\psi|_{K_v}) \\ \prod \psi_v &\longleftarrow (\psi_v) \end{aligned} \tag{11}$$

In other words, to give an additive character ψ of $\mathbb{A}$ is the same as giving a collection $(\psi_v)_v$ such that each ψ_v is an additive character of K_v with $\psi_v|_{\mathcal{O}_v} = 1$ for almost all v. Moreover, *for each* places v, the standard ψ_v gives an isomorphism $\Psi_v : K_v \to \widehat{K}_v$ by Theorem 3.4.1. For *almost all* v, $\psi_v|_{\mathcal{O}_v} = 1$, so Ψ_v identifies $\mathcal{O}_v$ and $\widehat{K_v/\mathcal{O}_v}$ (see Corollary 2.3.3). Therefore the product

$$\prod \Psi_v : \prod_v{}' (K_v, \mathcal{O}_v) \longrightarrow \prod_v{}' (\widehat{K}_v, \widehat{K_v/\mathcal{O}_v}) \tag{12}$$

gives an isomorphism.

6.3.6 Corollary. With the same assumptions and notations in Theorem 6.3.5, the map

$$K \longrightarrow \widehat{\mathbb{A}_K/K}, \quad a \longmapsto \psi_a \tag{13}$$

is an isomorphism.

Proof Exercise (note that $\psi|_K = 1$ and that $K^\perp$ as the Pontryagin dual of compact $\mathbb{A}_K/K$ is discrete (see Corollary 2.3.3 and Examples 2.3.4).).

***6.3.7 Remark.** In the proof of Theorem 6.3.5, the isomorphism (11) is canonical. But the isomorphism (12) depends on a choice of $\psi = (\psi_v)$, e.g., in function field case, a choice of meromorphic 1-form $\omega \in \Omega_K$.

Besides, we know from the proofs, any character ψ or equivalently by (11), any collection (ψ_v), with $\psi_v|_{\mathcal{O}_v} = 1$ for almost all v, can determine a non-canonical isomorphism (10). If moreover $\psi|_K = 1$, ψ also determine a non-canonical isomorphism (13). These facts give us some freedom to choose "arbitrary" additive characters for later use. But in practice, it's more convenient to just take the "standard" one(s).

6.3.8 Additive Measure. Recall that $\mathbb{A}_K$ is an LCA group hence admits a unique Haar measure up to scaling. If we have a measure $\mathrm{d}x_v$ for each place v, such that for almost all v, $\mathrm{d}x_v$ is normalized, i.e., $\mathcal{O}_v$ gets measure 1, then it make sense to define $\mathrm{d}x := \prod_v \mathrm{d}x_v$, in the sense that for each open set $\prod U_v \subseteq \mathbb{A}_K$, we have

$$\int_{\prod U_v} \mathrm{d}x = \prod \int_{U_v} \mathrm{d}x_v.$$

This defines a Haar measure on $\mathbb{A}_K$.

For general discussion on Haar measures on restricted products, see [**Cas67**, §13] and [**Tat67a**, §3.3].

6.3.9 There is a "standard" choice of Haar measure on $\mathbb{A}_K$. For each v, let $\mathrm{d}x_v$ be the *self-dual* measure on K_v with respect to the standard choice of the additive character ψ_v (§3.4.2 and §3.7). Note that for almost all v, K is absolutely unramified, hence for almost all v, the standard $\mathrm{d}x_v$ is normalized. So we can define define the standard Haar measure $\mathrm{d}x$ on $\mathbb{A}_K$ as the product of these self-dual measure. Once we define the Fourier transform (see Fourier Transform for $\mathbb{A}_K$ 7.2), we can see this measure is also *self-dual*.

This measure is also called the **Tamagawa measure** (after Tamagawa Tsuneo, 玉河 恒夫) on $\mathbb{A}_K$. It is a special case of the more general Tamagawa measure on *adelic algebraic groups*.

6.3.10 Absolute Value. Recall also that for any $a \in K_v^\times$, the multiplication-by-a map defines an automorphism of K_v^+. And this leads to the notion of normalized absolute value. In the global case, we have similar result.

First observe that the map $\mathbb{A}_K \to \mathbb{A}_K$, $x \mapsto ax$ is an isomorphism if and only if a is an idèle ([**Tat67a**, Lemma 4.1.1]). If $\mathrm{d}x$ is any Haar measure on $\mathbb{A}_K$, then for any $a \in \mathbb{A}_K^\times$, there is an $|a|_K \in \mathbb{R}_{>0}^\times$, such that

$$\int_{\mathbb{A}_K} f(x)\,\mathrm{d}x = \int_{\mathbb{A}_K} f(ax)|a|_K\,\mathrm{d}x, \quad \forall f \in C_c(\mathbb{A}_K), \tag{14}$$

which we just write $\mathrm{d}(ax) = |a|_K \mathrm{d}x$. It turns out that we obtain a morphism of LCA groups $|\cdot|_K : \mathbb{A}_K^\times \to \mathbb{R}_{>0}^\times$. This maps is usually called the **norm**, **absolute value** or **content** of idèles. It is independent of the choice of Haar measures on $\mathbb{A}_K$.

6.3.11 Propsotion. The absolute value on $\mathbb{A}_K^\times$ can be explicitly described as

$$|\cdot| := |\cdot|_K : \mathbb{A}_K^\times \longrightarrow \mathbb{R}_{>0}, \quad a \longmapsto \prod_v |a_v|_v,$$

where $|\cdot|_v$ is the *normalized absolute value* on K_v. Moreover, it restricts to constant map 1 on $K^\times$, i.e., for any $a \in K^\times$, we have $|a|_K = 1$.

Proof For the proof the explicit description, see [**Tat67a**, Lemma 4.1.2] or [**Cas67**, §16].
The second result, $|a|_K = 1$, is a classical result, and is called the **product formula**. It can be proven directly, see for example [**Neu92**, III, (1.3)] and [**Cas67**, §12]). Another proof will be given in §6.3.14.

6.3.12 The *counting measure* on K (see Proposition 6.2.4 and Example 1.4.7) and a Haar measure $\mathrm{d}x$ on $\mathbb{A}_K$ induces a Haar measure on $\mathbb{A}_K/K$ *compatible* (see §2.5.2) with respect to the short exact sequence

$$0 \to K \to \mathbb{A}_K \to \mathbb{A}_K/K \to 0.$$

The quotient group $\mathbb{A}/K$ has finite positive volume. In fact, $\mathbb{A}_K/K$ is compact and contains a non-empty open subset (K is discrete in $\mathbb{A}_K$) by Proposition 6.2.4, so it has finite positive measure by Remarks 1.4.5.
A Haar measure on $\mathbb{A}_K$ is **normalized**, if $\mathbb{A}_K/K$ gets volume 1 with the induced measure defined above.

6.3.13 Proposition. The standard choice of $\mathrm{d}x$ on $\mathbb{A}_K$ is normalized.

Proof The volume with respect to the standard measure, can be computed directly. See [**Tat67a**, Theorem 4.1.3] and [**Poo05**]. Another computation-free proof see Remark 7.5.1.

6.3.14 We can give another proof of the product formula: $|a|_K = 1$ for all $a \in K^\times$. Any $a \in K^\times$ defines an isomorphism $\mathbb{A}_K \to \mathbb{A}_K$, $x \mapsto ax$ and it induces isomorphism $\mathbb{A}_K/K \to \mathbb{A}_K/K$. It then follows that $\mathrm{Vol}(\mathbb{A}_K/K) = |a|\,\mathrm{Vol}(\mathbb{A}_K/K)$. So $|a| = 1$ as $0 < \mathrm{Vol}(\mathbb{A}_K/K) < \infty$.

6.4 Multiplicative Theory.

6.4.1 Multiplicative Haar Measure. As $\mathbb{A}_K^\times$ is an LCA group, it admits a unique Haar measure up to scaling. As for $\mathbb{A}_K$, given a collection of Haar measures $\mathrm{d}^\times x_v$ on K_v for each v, such that for almost all v, $\mathcal{O}_v^\times$ get measure 1, we can then define a Haar measure $\mathrm{d}^\times x := \prod \mathrm{d}^\times x_v$ on $\mathbb{A}_K^\times$.

Recall (5) that on each K_v, a multiplicative measure $\mathrm{d}^\times x_v$ and an additive measure $\mathrm{d}x_v$ are related to each other by the relation $\mathrm{d}^\times x_v = (c_v/|x_v|_v)\mathrm{d}x_v$ for some $c_v > 0$. If we have chosen $\mathrm{d}x_v$ to be normalized for almost all v, we should take care of the constant c_v to ensure that $\mathrm{d}^\times x_v$ is also normalized. Recall (7), for a nonarchimedean place v, that if we choose $\mathrm{d}x_v$ such that $\int_{\mathcal{O}_v} \mathrm{d}x_v = 1$ and $\mathrm{d}^\times x_v = \mathrm{d}x_v/|x_v|_v$, then it follows that

$$\int_{\mathcal{O}_v^\times} \mathrm{d}^\times x_v = 1 - q_v^{-1}.$$

To to define a "compatible" measure on $\mathbb{A}_K$, we should set for almost all (nonarchimedean) v,

$$\mathrm{d}^\times x_v = (1 - q_v)^{-1} \frac{\mathrm{d}x_v}{|x_v|_v}.$$

Then for almost all v, $\int_{\mathcal{O}_v^\times} \mathrm{d}^\times x_v = 1$. So it makes sense to define $\mathrm{d}^\times x := \prod \mathrm{d}^\times x_v = \prod_v ((1 - q_v)|x_v|_v)^{-1}\mathrm{d}x_v$.

***6.5** As we have the standard choice of the measure on K_v (§3.7). Accordingly, the we have standard choices of measures on $K_v^\times$, which is defined as

$$\mathrm{d}^\times x_v := \begin{cases} \mathrm{d}x_v/|x_v|_v, & \text{if } v \text{ is archimedean,} \\ (1 - q_v^{-1})^{-1}\mathrm{d}x_v/|x_v|_v, & \text{if } v \text{ is nonarchimedean,} \end{cases}$$

where $\mathrm{d}x_v$ is the standard self dual additive measure. It's not hard to see for all non-archimedean v, we have ([**Tat67a**, Lemma 2.3.3])

$$\int_{\mathcal{O}_v^\times} \mathrm{d}^\times x_v = \int_{\mathcal{O}_v} \mathrm{d}x_v = (\#(\mathcal{O}_F/\mathcal{D}_{F/F_0}))^{-1/2}.$$

For almost all v, this value is 1, because for almost all v, K is absolutely unramified.

6.5.1 As we did in the local case, consider the following short exact sequence

$$1 \longrightarrow \mathbb{A}_{K,1}^\times \longrightarrow \mathbb{A}_K^\times \xrightarrow{|\cdot|_K} |\mathbb{A}_K^\times| \longrightarrow 1$$

where $\mathbb{I}_K^1 := \mathbb{A}_{K,1}^\times$ is the kernel of $|\cdot|_K$ and $|\mathbb{A}^\times|_K$ is the value group. By product formula ($|a|_K = 1$ for all $a \in K^\times$), we identify $K^\times$ as a subgroup of $\mathbb{A}_{K,1}^\times$.

***6.5.2** In case K is a number field, $|\mathbb{A}_K^\times| = \mathbb{R}_{>0}$; equip it with the standard measure $\mathrm{d}t/t = \mathrm{d}(\log t)$. In case K is a function field, $|\mathbb{A}_K^\times| = q^{\mathbb{Z}}$; equip it with the counting measure multiplied by $\log q$. The latter measure is also denoted by $\mathrm{d}t/t$. Denote by $\mathrm{d}^* x$ the induced measure on $\mathbb{A}_{K,1}^\times$ (see §2.5.2). Define for each $t > 0$,

$$\mathbb{A}_{K,t}^\times := \{x \in \mathbb{A}_K^\times : |x|_K = t\}.$$

Any $a \in \mathbb{A}_{K,t}^\times$ gives a homeomorphism $\mathbb{A}_{K,1}^\times \to \mathbb{A}_{K,t}^\times$, $x \mapsto ax$. Hence the push-forward of $\mathrm{d}^* x$ gives a measure on $\mathbb{A}_{K,t}^\times$. This measure is independent of the choice of a because of translation invariance.

6.5.3 Proposition. The subgroup $K^\times$ is discrete in $\mathbb{A}_K^\times$ and is cocompact in $\mathbb{A}_{K,1}^\times$.

Proof See [Cas67, §16].

6.5.4 Remarks.

(1) The subgroup $K^\times$ is not cocompact in $\mathbb{A}_K^\times$. In fact, by the product formula, $|a|_K = 1$ for all $a \in K^*$. But $|\cdot|_K$ is not trivial on $\mathbb{A}_K^\times$. Then $|\cdot|_K$ induces a continuous homomorphism $\mathbb{A}_K^\times/K^\times \twoheadrightarrow \mathbb{R}_{>0}$. This implies that $\mathbb{A}_K^\times/K^\times$ cannot be compact.
(2) Moreover, in function field case, $\mathbb{A}_K^\times/K^\times$ is totally disconnected, see [Cas67, §16].
(3) There are two topologies on $\mathbb{A}_{K,1}^\times$, which are induced form $\mathbb{A}_K$ and $\mathbb{A}_K^\times$ respectively. Luckily, This two topologies coincide. See [Cas67, §16].

6.5.5 An **idèle-class character**, **Hecke character** or **Größencharakter** is a continuous homomorphism $\chi : \mathbb{A}_K^\times \to \mathbb{C}^\times$ such that $\chi|_{K^\times} = 1$. Clearly, idèle-class characters can be identified with characters of the **idèle class group** $\mathbb{A}_K^\times/K^\times$.

6.5.6 Remark. This is not the classical definiton[9] given by Hecke in [Hec20]. Relation between the two notions is sketched in [Kud03, §5] and discussed in details in [Neu92, VII, §6].

6.5.7 Example. (Norm of idèles). By the product formula, we know that the norm of idèles $|\cdot|_K$ is an idèle-clas character.

6.5.8 Example. (Dirichlet character). If $K = \mathbb{Q}$ is the field of rational numbers, then we have

$$\mathbb{A}_\mathbb{Q}^\times = \mathbb{Q}^\times \times \mathbb{R}_{>0}^\times \times \widehat{\mathbb{Z}}^\times.$$

A **Dirichlet character** χ is a homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ for some $N \geq 1$. Thus a Dirichlet character defines an idèle class character

$$\mathbb{A}^\times \twoheadrightarrow \widehat{\mathbb{Z}}^\times \twoheadrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times.$$

One can show that every finite-order idèle-class character for $\mathbb{Q}$ arises in this way.

6.5.9 Remark. Let K^{sep} be a separable closure of K and $K^{\mathrm{ab}} \subseteq K^{\mathrm{sep}}$ be the maximal abelian extension. Let $\theta = \theta_K : \mathbb{A}_K^\times/K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ be the *global Artin map* for K (see e.g., [Tat67b, §5]). Any character $\rho : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathbb{C}^\times$ factors through $\mathrm{Gal}(K^{\mathrm{ab}}/K)$, hence the bottom line of the diagram

$$\begin{tikzcd}
 & & \mathrm{Gal}(K^{\mathrm{sep}}/K) \arrow[d, dotted, two heads] \arrow[dr] & \\
\mathbb{A}_K^\times \arrow[r, two heads] & \mathbb{A}_K^\times/K^\times \arrow[r, "\theta"] & \mathrm{Gal}(K^{ab}/K) \arrow[r] & \mathbb{C}^\times
\end{tikzcd}$$

gives an idèle-class character (cf. Remark 3.9.1).

However not all idèle-class character arise from characters of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. This is because any character of $\mathrm{Gal}(K^{sep}/K)$ has *finite* image as $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ is profinite, but idèle class characters can have infinite image. One can fie this by replacing $\mathrm{Gal}(K^{sep}/K)$ by the **Weil Group** W_K.

[9]See also https://math.stackexchange.com/q/11679/19690.

And in fact, idèle-class characters are in bijection with characters $W_K \to \mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$ of the Weil group. One can generalize the whole theory to continuous representations $W_K \to \mathrm{GL}_n(\mathbb{C})$ for $n \geq 1$. For an introduction, see [**Tat79**].

6.5.10 Consider the exact sequence

$$1 \longrightarrow \mathbb{A}_{K,1}^\times/K^\times \longrightarrow \mathbb{A}_K^\times/K^\times \longrightarrow |\mathbb{A}_K^\times| \longrightarrow 1,$$

where $A_{K,1}^\times/K^\times$ is is compact. Similar to Corollary 3.9.4 and Corollary 3.9.5, It then follows that every idèle-class character χ is of the form

$$\chi = \eta|\cdot|_K^s, \quad \text{with } \eta \text{ unitary and } s \in \mathbb{C}.$$

Let $\sigma = \operatorname{Re} s$ hence $|\chi| = |\cdot|^\sigma$. And as in the local case, σ is called the **exponent** of χ. Define $\hat{\chi} := \chi^{-1}|\cdot|$ and $\chi_v := \chi|_{K_v^\times}$.

The group of all idèle-class characters is denoted by $\mathscr{X} := \mathscr{X}(\mathbb{A}_K^\times)$. It has a structure of Riemann surface as $X(F^\times)$ did in §3.9.3.

7 Poisson summation formula and Riemann-Roch

The statements in this section are not accurate enough, in particular, the convergence issues of integrals and summations are usually neglected.

7.1 Schwartz-Bruhat Functions. If for each place v, we have $f_v \in \mathscr{S}(K_v)$, and $f_v = 1_{\mathcal{O}_v}$ for almost all v. Then $\prod f_v : \mathbb{A}_K \to \mathbb{C}$ is defined.[10] That is $f(x) = \prod_v f_v(x_v)$ for any $x = (x_v) \in \mathbb{A}_K$. A **Schwartz-Bruhat function** $f : \mathbb{A}_K \to \mathbb{C}$ is a finite $\mathbb{C}$-linear combination of such functions. Denote by $\mathscr{S} = \mathscr{S}(\mathbb{A}_K)$ the $\mathbb{C}$-vector space of all Schwartz-Bruhat functions.

7.2 Fourier Transform for $\mathbb{A}_K$. Fix the standard additive character ψ (see Remark 6.3.2) and the standard $\mathrm{d}x$ (see §6.3.9). For any $f \in \mathscr{S}(\mathbb{A}_K)$, define its **Fourier transform**

$$\hat{f}(y) := \int_{\mathbb{A}_K} f(x)\psi(xy)\,\mathrm{d}x.$$

This is again a function in $\mathscr{S}(\mathbb{A}_K)$ (recall the isomorphism (10)). Indeed, it holds that

$$\widehat{\prod f_v} = \prod \hat{f}_v,$$

for the $f = \prod f_v$ defined above. The **Fourier inversion formula** then reads

$$f(x) = \int_{\mathbb{A}_K} \hat{f}(y)\overline{\psi(xy)}\,\mathrm{d}y, \tag{15}$$

equivalently, $\hat{\hat{f}}(x) = f(-x)$.

Here we used the fact that the standard $\mathrm{d}x$ is chosen to be *self-daul* with respect to the standard ψ (see §6.3.9). If we didn't choose so, there would only have $\hat{\hat{f}}(x) = cf(-x)$ for some constant $c > 0$.

[10] Some authors use the notation $\otimes f_v$, e.g., in [**Kud03**].

7.3 Let D the **additive fundamental domain**, so we have

$$\mathbb{A}_K = \coprod_{x\in K}(x+D),$$

whose explicit description can be found in [Tat67a, Definition 4.1.2]. Clear the volume of D (w.r.t. the measure on $\mathbb{A}_K$) is the same of $\mathbb{A}_K/K$ (w.r.t. the measure on $\mathbb{A}_K/K$). We have seen that it has volume 1 under the standard measure.

7.4 **Fourier Transform for $\mathbb{A}_K/K$.** Recall the exact sequence

$$1 \to K \to \mathbb{A}_K \to \mathbb{A}_K/K \to 1$$

and the isomorphism $K \cong \widehat{\mathbb{A}_K/K}$ in Corollary 6.3.6.

Any function f on $\mathbb{A}_K/K$ can be identified as a function on $\mathbb{A}$ that is K-periodic, i.e., $f(x+\kappa) = f(x)$ for all $\kappa \in K$. For any integrable function $f \in \mathscr{L}^1(\mathbb{A}/K)$, define its Fourier transform $\hat{f} : K \to \mathbb{C}$ as

$$\hat{f}(\kappa) := \int_D f(x)\psi(\kappa x)\,\mathrm{d}x.$$

It holds that

$$f(x) = \sum_{\kappa\in K}\hat{f}(\kappa)\overline{\psi(\kappa x)} \tag{16}$$

(the counting measure on K makes an integral to be a sum).

***7.4.1** To obtain the inversion formula (16), we implicitly used the fact that $\mathrm{Vol}(\mathbb{A}_K/K) = 1$, If we do not know this a priori, we should have

$$f(x) = \frac{1}{\mathrm{Vol}(D)}\sum_{\kappa\in K}\hat{f}(\kappa)\overline{\psi(\kappa x)}. \tag{17}$$

7.5 **Theorem.** (Poisson summation formula). If $f \in \mathscr{S}(\mathbb{A}_K)$, then it holds that

$$\sum_{\kappa\in K} f(\kappa) = \sum_{\kappa\in K}\hat{f}(\kappa).$$

Proof The strategy is apply the Fourier inversion formula to the function

$$F(x) := \sum_{\kappa\in K} f(x+\kappa), \quad x \in \mathbb{A}_K/K.$$

We have for each $\kappa \in K$,

$$\begin{aligned}
\hat{F}(\kappa) &= \int_D \sum_{y\in K} f(x+y)\psi(\kappa y)\,\mathrm{d}x = \sum_{y\in K}\int_D f(x+y)\psi(\kappa y)\,\mathrm{d}x && \text{switch } \textstyle\int \text{ and } \sum \\
&= \sum_{y\in K}\int_{D+y} f(z)\psi(\kappa(z-y))\,\mathrm{d}z && (\text{set } z = x+y) \\
&= \int_{\mathbb{A}_K} f(z)\psi(\kappa z)\,\mathrm{d}z && (\mathbb{A}_K = \textstyle\coprod_{y\in K}(y+D)) \\
&= \hat{f}(\kappa).
\end{aligned}$$

Then the Fourier inversion formula gives

$$\sum_{\kappa \in K} f(x+\kappa) =: F(x) = \sum_{\kappa\in K} \hat{F}(\kappa)\overline{\psi(\kappa x)} = \sum_{\kappa \in K} \hat{f}(\kappa)\overline{\psi(\kappa x)}.$$

Evaluate both sides at $x = 0$, and get the desired result.

7.5.1 Remark. ([Tat67a, Thm:4.2.1]). If we do not know $\mathrm{Vol}(D) = 1$ a priori, the above computation using (17), applying to $f \in \mathscr{S}(\mathbb{A})$ and $\hat{f} \in \mathscr{S}(\mathbb{A})$, gives $(\mathrm{Vol}(D))^2 = 1$. Then we deduce that $\mathrm{Vol}(D) = 1$.

7.5.2 Riemann-Roch Theorem. (Arithmetic Form). Let K be a global field and $f \in \mathscr{S}(\mathbb{A}_K)$. For any $a \in \mathbb{A}_K^\times$, we have

$$\frac{1}{|a|}\sum_{\kappa\in K} \hat{f}(\kappa/a) = \sum_{\kappa \in K} f(a\kappa).$$

7.6 In rest of this section, assume that K is the function field of a (proper regular(=smooth) integral) curve X over $\mathbb{F}_q$. Places v of K could be identified with closed points of X.

For any divisor[11] $D = \sum_v d_v v$, where $d_v \in \mathbb{Z}$ and $d_v = 0$ for almost all v, define $\deg D := \sum_v d_v[k_v : \mathbb{F}_q]$. Each $f \in K^\times$ gives rise to a **principal divisor** $\sum_v \mathrm{ord}_v(f) d_v$. Moreover, the chosen $\omega \in \Omega_K$ gives a **canonical divisor** $\mathscr{K} = \sum_v c_v v$, where c_v is the "order of vanishing of ω at v". On the other hand, any $x = (x_v) \in \mathbb{A}_k^\times$ defines a divisor $\sum \mathrm{ord}_v(x_v)v$. If we define the subring of **integral adèles**

$$\mathcal{O}_{\mathbb{A}_K} := \mathbb{O}_K := \prod_v \mathcal{O}_v \subseteq \mathbb{A}_K$$

of $\mathbb{A}_K$, we then observe that the quotient $\mathbb{O}_K^\times \backslash \mathbb{A}_K^\times$ is exactly the divisor group and the double quotient $\mathbb{O}_K^\times \backslash \mathbb{A}_K^\times / K^\times$ is then the *divisor class group*, or the *Picard group*.

Given a divisor $D = \sum_v d_v v$, define[12]

$$\mathcal{O}_{\mathbb{A}_K}(D) := \prod_v \mathfrak{m}_v^{-d_v} \subseteq \mathbb{A}_K.$$

Then the $\mathbb{F}_q$-vector space[13]

$$L(D) := K \cap \mathcal{O}_{\mathbb{A}_K}(D).$$

consists of rational functions f on X having at worst a pole of order d_v at v for each v. Set $\ell(D) := \dim_{\mathbb{F}_q} L(D)$. The genus of X, or K is defined as $g := \ell(\mathscr{K}) \in \mathbb{Z}_{\geq 0}$.

7.7 Riemann-Roch Theorem. (Geometric form). Let K be a function field of a (proper regular integral) curve X over $\mathbb{F}_q$. Then it holds that

$$\ell(D) - \ell(\mathscr{K} - D) = \deg D + 1 - g. \tag{18}$$

for all divisor D on X.

[11]For the notation D, do not get confused with the fundamental domain D.
[12]Here we prefer the notation $\mathcal{O}_{\mathbb{A}_K}$ rather than $\mathbb{O}_K$, because the former one looks more like a structure sheaf.
[13]The letter L stands for Laurent series.

PROOF One can compute (Exercise) that

$$\hat{1}_{\mathfrak{m}_v^{-d_v}} = q^{d_v - c_v/2} \hat{1}_{\mathfrak{m}_v^{d_v - c_v}}.$$

Taking product yields

$$\hat{1}_{\mathbb{O}_{\mathbb{A}_K}(D)} = q^{\deg D - \deg K/2} \hat{1}_{\mathbb{O}_{\mathbb{A}_K}(\mathcal{K}-D)}$$

Then applying Poisson summation formula to $f = 1_{\mathbb{O}_{\mathbb{A}_K}(D)}$ gives

$$\begin{array}{ccl} \sum_{x\in K} 1_{\mathbb{O}_{\mathbb{A}_K}(D)} & = & \sum_{x\in L(D)} 1 = q^{\ell(D)} \\ \Big\| \text{Poisson Summation Formula} & & \\ \sum_{x\in K} \hat{1}_{\mathbb{O}_{\mathbb{A}_K}(D)} & = & q^{\deg D - \deg K/2} \sum_{x\in L(\mathcal{K}-D)} 1 = q^{\deg D - \frac{\deg \mathcal{K}}{2} + \ell(\mathcal{K}-D)}. \end{array}$$

So

$$\ell(D) = \deg D - \frac{\deg \mathcal{K}}{2} + \ell(\mathcal{K} - D).$$

Now consider the case $D = 0$, we can see $L(0) = \mathbb{F}_q$, hence $\ell(0) = 1$ (Exercise). It then follows that $\deg \mathcal{K}/2 = 1 - g$ hence eq. (18).

7.7.1 Remarks.

(1) The correspondence between $\mathbb{O}_K^\times \backslash \mathbb{A}_K^\times / K^\times = \mathrm{GL}_1(\mathbb{O}_K) \backslash \mathrm{GL}_1(\mathbb{A}_K)/\mathrm{GL}_1(K)$ and Picard group, generalizes to a correspondence between $\mathrm{GL}_n(\mathbb{O}_K) \backslash \mathrm{GL}_n(\mathbb{A}_K)/\mathrm{GL}_n(K)$ and the set of isomorphism classes rank n vector bundles over X.

(2) There is a number field version of Riemann-Roch theorem in *Arakelov theory*. See [Neu92, Kapitel III] for details.

8 Global zeta integrals

8.1 Given $f \in \mathscr{S}(\mathbb{A}_K)$, and an idèle-class character χ, define the *global zeta integral* as

$$Z(f, \chi) := \int_{\mathbb{A}_K^\times} f(x)\chi(x) \, \mathrm{d}^\times x.$$

This is a generalization of the *completed* Riemann zeta function $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

8.2 Theorem. (Meromorphic continuation and Functional equation of global zeta integrals). Fix an $f \in \mathscr{S}(\mathbb{A}_K)$.

(1) The integral $Z(f, \chi)$ converges for idèle-class characters χ of exponent $\sigma > 1$.

(2) The function $Z(f, \chi)$ of χ extends to a meromorphic function on $\mathscr{X}$. More precisely, it is holomorphic except for

- a simple pole at $|\cdot|^0$ with residue $-V f(0)$, and
- a simple pole at $|\cdot|^1$ with residue $V \hat{f}(0)$.

where $V = \mathrm{Vol}(\mathbb{A}_{K,1}^\times / K^\times)$.

(3) It holds that

$$Z(f,\chi) = Z(\hat{f},\hat{\chi}),$$

as meromorphic functions of $\chi \in \mathscr{X}$.

PROOF (SKETCH) • Convergence for $\sigma > 1$. It's enough to show it is absolutely convergent. So we may replace χ by $|\cdot|^\sigma$. Now

$$Z(f,\chi) = \prod Z(f_v, |\cdot|_v^\sigma).$$

For almost all v, $f_v = 1_{\mathcal{O}_v}$ and $Z(f_v, |\cdot|_v^\sigma) = (1 - q_v^{-\sigma})^{-1}$. It suffices to prove the convergence of

$$\prod_{\text{non-arch. } v} (1 - q_v^\sigma)^{-1}$$

For number field, this is the **Dedekind zeta function**, which converges for $\sigma > 1$. Details omitted.

• Slice $\mathbb{A}_K^\times$ by norm (recall §6.5.2). One has for $\sigma > 1$,

$$Z(f,\chi) = \int_{|\mathbb{A}_K^\times|} \left(\int_{\mathbb{A}_{K,t}^\times} f(x)\chi(x)\,\mathrm{d}^*x \right) \frac{\mathrm{d}t}{t} := \int_{|\mathbb{A}_K^\times|} Z_t(f,\chi)\frac{\mathrm{d}t}{t}.$$

• Using Poisson summation formula Theorem 7.5, One can show there is a functional equation for $Z_t(f,\chi)$:

$$Z_t(f,\chi) + f(0)\int_{\mathbb{A}_{K,t}^\times} \chi(x)\,\mathrm{d}^*x = Z_{1/t}(\hat{f},\hat{\chi}) + \hat{f}(0)\int_{\mathbb{A}_{K,1/t}^\times} \hat{\chi}(x)\,\mathrm{d}^*x.$$

• Then we write

$$Z(f,\chi) = \int_0^1 Z_t(f,\chi)\frac{\mathrm{d}t}{t} + \int_1^\infty Z_t(f,\chi)\frac{\mathrm{d}t}{t} =: J(f,\chi) + I(f,\chi).$$

The second part $I(f,\chi)$ is easy to handle for $\sigma > 1$. The first part $J(f,\chi)$ can be dealt with the flipping $t \mapsto 1/t$ showed in the previous step.

Then with some observation on the symmetry of the equations obtained, one can finish the proof. Omitted.

8.3 For each idèle-class character χ, define

$$\epsilon(\chi) := \prod_v \epsilon(\chi_v), \qquad L(\chi) := \prod_v L(\chi_v).$$

8.4 Theorem.

(1) The product $\epsilon(\chi)$ converges to a non-vanishing holomorphic function on all of $\mathscr{X}$.

(2) The product $L(\chi)$ converges for $\sigma > 1$, and extends to a meromorphic function on all of $\mathscr{X}$, holomorphic everywhere outside $|\cdot|^0$ and $|\cdot|^1$.

(3) We have $L(\chi) = \epsilon(\chi)L(\hat{\chi})$ as meromorphic functions on $\mathscr{X}$.

PROOF Restrict attention to one component $\mathscr{X}_0$ of $\mathscr{X}$.

In the local case Theorem 4.2, we have shown that i)' $\epsilon_v(\chi_v)$ is non-vanishing holomorphic function on $\mathscr{X}_0$, which equals to 1 for almost all v;

there is $f_v \in \mathscr{X}(K_v)$ with $f_v = 1_{\mathcal{O}_v}$ for almost all v such that

ii)' $Z(f_v, \chi_v)/L(\chi_v)$ is non-vanishing holomorphic function on $\mathscr{X}_0$, equal to 1 for almost all v; and

iii)' the local functional equation

$$\frac{Z(\hat{f}_v, \hat{\chi}_v)}{L(\hat{\chi}_v)} = \epsilon(\chi_v)\frac{Z(f_v, \chi_v)}{L(\chi_v)}$$

holds for all v.

Taking product over all v, and setting $f := \prod f_v \in \mathscr{S}(\mathbb{A}_K)$, we get

i) $\epsilon(\chi)$ is non-vanishing holomorphic on $\mathscr{X}_0$.

ii) $\prod_v(Z(f_v, \chi_v)/L(\chi_v))$ is non-vanishing holomorphic on $\mathscr{X}_0$, so the properties of $L(\chi)$ in (2) follows from those of $Z(f, \chi)$ in Theorem 8.2 (2).

iii) It holds that

$$\frac{Z(\hat{f}, \hat{\chi})}{L(\hat{\chi})} = \epsilon(\chi)\frac{Z(f, \chi)}{L(\chi)}.$$

Together with Theorem 8.2 (3), this implies $L(\chi) = \epsilon(\chi)L(\chi^\vee)$.

***8.5 Hecke L-function.** Fixan idèle-class character χ, define

$$\epsilon(s, \chi) := \epsilon(\chi|\cdot|_K^s), \quad \text{and} \quad L(s, \chi) := L(\chi|\cdot|_K^s), \qquad s \in \mathbb{C}.$$

Functions of the form $L(s, \chi)$ are called **Heck L-functions**. The functional equation reads

$$L(s, \chi) = \epsilon(s, \chi)L(1 - s, \chi^{-1}).$$

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