Let X/S be smooth connected of relative dimension d.

At some point, we will assume $S = \operatorname{Spec} k$, with $k = \bar{k}$ of characteristic p, and X/S smooth proper connected group scheme of (relative) dimension d, i.e., abelian variety. Essential assumptions will be point out where it is used.

$$\begin{tikzcd}
 & \mathrm{T}^*_{X^{(1)}/S} \arrow[d, "\pi"] & \mathscr{D}_{X/S} := \widetilde{F_{X/S,*}D_{X/S}} \\
\begin{array}{c}\text{sheaf of PD-differential} \\ \text{operators (of level 0)} \\ D_{X/S} = D^{(0)}_{X/S}\end{array} \quad X \arrow[r, "F_{X/S}"] & X^{(1)} & F_{X/S,*}D_{X/S}
\end{tikzcd}$$

$F_{X/S}$ is flat locally free of rank p^d.

$\mathscr{D}_{X/S}$ is an Azumaya algebra over $\mathrm{T}^*_{X/S}$ of rank p^{2d}.

For quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$,

$$\mathbf{V}(\mathcal{E}) := \operatorname{Spec} \operatorname{Sym}^\bullet \mathcal{E}^\vee \to S$$

1 Morita

$D \cong \operatorname{End}_R(P)$, then

$$\begin{aligned}
\mathtt{Mod}_R &\longleftrightarrow \mathtt{Mod}_D \\
F &\longmapsto P \otimes_R F \\
\operatorname{Hom}_A(P, G) &\longleftarrow\!\shortmid\; G
\end{aligned}$$

2 Spectral Cover

Fix an $\theta : \mathcal{O}_X$-linear map $E \to E \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$, with $\operatorname{rk} E = n$.

Tautological section

$$\lambda : \mathcal{O}_{\mathrm{T}^*_{X/S}} \to \pi^*\Omega^1_{X/S},$$

tensored with π^*E:

$$\lambda : \pi^*E \to (\pi^*E) \otimes_{\mathcal{O}_{\mathrm{T}^*_{X/S}}} (\pi^*\Omega^1_{X/S})$$

Pullback θ to $\mathrm{T}^*_{X/S}$:

$$\theta : \pi^*E \to (\pi^*E) \otimes_{\mathcal{O}_{\mathrm{T}^*_{X/S}}} (\pi^*\Omega^1_{X/S})$$

Then

$$\textstyle\bigwedge^n E \xrightarrow{\bigwedge^n} \bigwedge^n(\pi^*E \otimes \pi^*\Omega^1_{X/S}) \longrightarrow \bigwedge^n(\pi^*E) \otimes \operatorname{Sym}^n(\pi^*\Omega^n_{X/S}).$$

Thus get

$$\begin{aligned}\chi(\theta) \in \Gamma(\mathrm{T}^*_{X/S}, \mathrm{Sym}^n(\pi^*\Omega^1_{X/S})) &= \Gamma(\mathrm{T}^*_{X/S}, \pi^* \mathrm{Sym}^n \Omega^1_{X/S}) \\ &= \Gamma(X, \mathrm{Sym}^n \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathrm{Sym}^\bullet(\Omega^1_{X/S})^\vee)\end{aligned}$$

Note that $\pi^* \mathrm{Sym}^n \Omega^1_{X/S}$ is locally free of rank $\binom{d+n-1}{n}$ over $\mathrm{T}^*_{X/S}$. So the global section $\chi(\theta)$ defines a closed subscheme

$$i : Y_\chi \hookrightarrow \mathrm{T}^*_{X/S}$$

i.e., the vanishing locus of $\chi(\theta)$.

2.1 More details

$\chi(\theta)$ can be written as

$$\lambda^n - a_1\lambda^{n-1} + \cdots + (-1)^n a_n$$

with $a_i \in \Gamma(X, \mathrm{Sym}^i \Omega^1_{X/S}) \to \Gamma(\mathrm{T}^*_{X/S}, \pi^* \mathrm{Sym}^i \Omega^1_{X/S})$ corresponds to the *trace* of the following map

$$\textstyle\bigwedge^i(E) \xrightarrow{\bigwedge^i} \bigwedge^i(E \otimes_{\mathcal{O}_X} \Omega^1_{X/S}) \longrightarrow \bigwedge^i E \otimes_{\mathcal{O}_X} \mathrm{Sym}^i \Omega^1_{X/S}.$$

3 BNR for Higgs bundles

We have equivalence of categories

$$\begin{tikzcd}
Y_{\chi^n_T} \arrow[r] \arrow[d] & Y_{\chi^n} \arrow[d, hook, "i"] & \begin{array}{c}\mathcal{O}_{Y_{\chi^n}}\text{-module} \\ \mathcal{F} := \tilde{F}\end{array} \\
\mathrm{T}^*_{X^{(1)}_T/T} \arrow[r] \arrow[d] & \mathrm{T}^*_{X^{(1)}/S} \arrow[d, "p"] & \mathscr{D}_{X/S} \\
X^{(1)}_T \arrow[r] \arrow[d] & X^{(1)} \arrow[d] & \begin{array}{c}\text{Higgs bundle} \\ (F := (\pi \circ i)_*\mathcal{F}, \theta) \\ \text{of rank } n\end{array} \\
T \arrow[r] & S &
\end{tikzcd}$$

4 BNR for local systems

$$\begin{tikzcd}
Y_{\chi_T^n} \arrow[rr] \arrow[d, hook, "i_T"] & & Y_{\chi^n} \arrow[d, hook, "i"] & \\
\mathrm{T}^*_{X_T^{(1)}/T} \arrow[rr] \arrow[dd] & & \mathrm{T}^*_{X^{(1)}/S} \arrow[dd, "p"] & \\
& X_T \arrow[rr] \arrow[dl, "F_{X_T/T}"] & & X \arrow[dl, "F_{X/S}"] \\
X_T^{(1)} \arrow[rr] \arrow[d] & & X^{(1)} \arrow[d] & \\
T \arrow[rr] & & S &
\end{tikzcd}$$

$(i^*\mathscr{D}_{X/S})$-module
$\mathcal{E} := \widetilde{F_{X/S,*}E}$

$\mathscr{D}_{X/S}$

Local system
$(E = \widetilde{F_{X/S,*}E}, \nabla)$
of rank n

Higgs bundle
$(F_{X/S,*}E = (\pi \circ i)_*\mathcal{E}, \psi_\nabla)$
of rank $p^d n$

4.1 Two ways to get the degree n characteristic polynomial

1. Consider the F-Higgs bundle $E \to E \otimes_{\mathcal{O}_X} F^*_{X/S}\Omega^1_{X^{(1)}/S}$, whose characteristic polynomial is of order n, but coefficients a priori lie in $\Gamma\big(X, F^*_{X/S}\operatorname{Sym}^n(\Omega^1_{X^{(1)}/S})\big)$. Then use a fact by N. Katz which stating that $[\psi_\nabla(D), \nabla_{D'}] = 0$ for any $D, D' \in (\Omega^1_{X/S})^\vee$. Then Cartier descent shows that the χ is horizontal with respect to the canonical connection ∇^{can} of $F^*_{X/S}\operatorname{Sym}^n(\Omega^1_{X^{(1)}/S})$.

2. Consider the Higgs bundle $(F_{X/S,*}E) \to (F_{X/S,*}E) \otimes_{\mathcal{O}_{X^{(1)}}} \Omega^1_{X^{(1)}/S}$. Its characteristic polynomial has coefficients in $\Gamma(X^{(1)}, \Omega^1_{X^{(1)}})$, but it has degree $p^d n$. One use Morita and BNR for Higgs to show $\chi = (\chi')^{p^d}$ for some χ' with $\deg \chi' = n$.

5 Correspondence

Given

$$\chi^n : S \to \mathbf{V}\left(\bigoplus_{i=1}^{n} \Gamma(X^{(1)}, \operatorname{Sym}^i \Omega^1_{X^{(1)}/S})\right)$$

It determines $\chi^n \in \Gamma(\mathrm{T}^*_{X/S}, \pi^* \operatorname{Sym}^n \Omega^1_{X^{(1)}/S})$, and $i : Y_{\chi^n} \hookrightarrow \mathrm{T}^*_{X^{(1)}/S}$.

$$i^*\mathscr{D}_{X/S} \cong \mathcal{E}nd_{Y_{\chi^n}}(\mathcal{P}),$$

then

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Quasicoherent } (\mathcal{O}_{Y_\chi^n})\text{-module } \mathcal{F} \\ \text{with } (\pi \circ i)_* \mathcal{F} \text{ locally free of rank } n\end{array}} \arrow[r, leftrightarrow, "\text{Morita}"] \arrow[d, leftrightarrow, "\text{BNR for Higgs}"'] & \boxed{\begin{array}{c}\text{Quasicoherent } (i^* \mathscr{D}_{X/S})\text{-module } \mathcal{E} \\ \text{with } (\pi \circ i)_* \mathcal{E} \text{ locally free of rank } pn\end{array}} \arrow[d, leftrightarrow, "\text{BNR for local system}"] \\
\boxed{\begin{array}{c}\text{Higgs bundle } (F, \theta) \\ \text{on } X^{(1)} \text{ of rank } n\end{array}} & \boxed{\begin{array}{c}\text{Local system } (E, \nabla) \\ \text{on } X \text{ of rank } n\end{array}}
\end{tikzcd}$$

5.1 General case

The above works for any test scheme $T \to S$ with

$$\chi_T^n : T \to \mathbf{V}\left(\bigoplus_{i=1}^{n} \Gamma(X^{(1)}, \operatorname{Sym}^i \Omega^1_{X^{(1)}/S})\right)$$

6 Existence of Splittings

Assume X/S abelian variety, $S = \operatorname{Spec} k$.

6.1 Line Bundle Case

$\chi : S \to \mathbf{V}\big(\bigoplus_{i=1}^{n} \Gamma(X^{(1)}, \operatorname{Sym}^i \Omega^1_{X^{(1)}/S})\big)$ over S defines a closed subscheme $Y_\chi \hookrightarrow \mathrm{T}^*_{X^{(1)}/S}$. (This is just the closed subscheme of the cotangent bundle of $X^{(1)}/S$ defined by a one-form.)

Fact 1. $i^* \mathscr{D}_{X/S}$ splits if and only if χ factors through

$$\psi : \operatorname{Pic}^{\natural}_{X/S} \to \mathbf{V}\big(\Gamma(X^{(1)}, \Omega^1_{X^{(1)}/S})\big).$$

(This works for general $\chi_T : T \to \mathbf{V}\big(\bigoplus_{i=1}^{n} \Gamma(X^{(1)}, \operatorname{Sym}^i \Omega^1_{X^{(1)}/S})\big)$.)

Fact 2. ψ is *smooth surjective*.

Conclusion 1. These facts imply that for any S-point χ (i.e., k-point), $i^* \mathscr{D}_{X/S}$ splits over the formal neighborhood of Y_χ.

6.2 General Case

$\chi : S \to \mathbf{V}\big(\Gamma(X^{(1)}, \operatorname{Sym}^n \Omega^1_{X^{(1)}/S})\big)$ over S defines a closed subscheme $i : Y_\chi \to \mathrm{T}^*_{X^{(1)}/S}$.

Fact 3. The sheaf of relative differentials $\Omega^1_{X/S}$ is trivial as X is group scheme over $S = \operatorname{Spec} k$.[1] Hence an isomorphism $\Omega^1_{X^{(1)}/S} \cong (\mathcal{O}_{X^{(1)}/S})^{\oplus d}$ gives isomorphisms

$$\begin{aligned} \pi^*\Omega^1_{X^{(1)}/S} &\cong \left(\mathcal{O}_{\mathrm{T}^*_{X^{(1)}/S}}\right)^{\oplus d}, & \pi^* \operatorname{Sym}^n \Omega^1_{X^{(1)}/S} &\cong \left(\mathcal{O}_{\mathrm{T}^*_{X^{(1)}/S}}\right)^{\oplus \binom{d+n-1}{n}}, \\ \lambda &\mapsto (\lambda^1, \dots, \lambda^d) & \lambda^n &\mapsto ((\lambda^1)^n, \dots, (\lambda^d)^n, \dots). \end{aligned}$$

Fact 4. The set of global sections $\Gamma(X^{(1)}, \Omega^1_{X^{(1)}/S})$ consists of only constants, as X/S is proper connected.

Conclusion 2. For any S-point χ, i.e., a symmetric differential form, the closed subscheme Y_χ is a closed subscheme cut out by $\binom{d+n-1}{n}$ monic polynomial equations of degree n with coefficients in $k = \bar{k}$ (Fact 4).

Among these equations, d of them are of the form

$$(\lambda^i)^n - a^i_1(\lambda^i)^{n-1} + \cdots + (-1)^n a^i_n, \qquad a^i_j \in k, \quad j = 1, \dots, n, i = 1, \dots, d.$$

where λ^i is the component of the tautological section of $\pi^*\Omega^1_{X^{(1)}/S}$ in the i-th part of the decomposition (Fact 3).

These equations can be always written as a product of $(\lambda^i - b^i_j)$'s as k is algebraically closed. These n equations defines a closed subscheme $Y_{\chi'}$ of $\mathrm{T}^*_{X^{(1)}/S}$[2]:

$$i : Y_\chi \lhook\joinrel\longrightarrow Y_{\chi'} \xhookrightarrow{i'} \mathrm{T}^*_{X^{(1)}/S}$$

$Y_{\chi'}$ is a union of d^n (counting multiplicity) closed subschemes defined by 1-forms. Hence $(i')^*\mathscr{D}_{X/S}$ splits over the formal neighborhood of $Y_{\chi'}$ hence[3], $i^*\mathscr{D}_{X/S}$ splits over the formal neighborhood of Y_χ.

7 Further discussion

Can consider $\mathcal{D}_{X/S} = \varinjlim D^{(m)}_{X/S}$-modules instead of local systems, where $\mathcal{D}_{X/S}$ is the differential operators in the sense of Grothendieck and $D^{(m)}_{X/S}$ is the differential operators of level m in the sense of Berthelot.

Fact 5. $D^{(0)}_{X/S}$-modules is equivalent to flat connections.

$\mathcal{O}_X$-coherent $\mathcal{D}_{X/S}$-modules is equivalent to F-divided bundles/stractfied bundles.

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[1] $\Omega^1_{G/S} \cong \pi^* e^* \Omega^1_{G/S}$ for any group $\pi : G \to S$ with identity $e : S \to G$.

[2] Some computation evidence suggests that $Y_\chi = Y_{\chi'}$, i.e., the other equations are superfluous.

[3] Think about it.

Fact 6. $F^{m+1}_{X/S,*}D^{(m)}_{X/S}$ is an Azumaya algebra over $\mathrm{Sym}^\bullet(\Omega^1_{X^{(m+1)}/S})$.[4]

$$
\begin{tikzcd}
 & \mathscr{D}^{(0)}_{X/S} & \mathscr{D}^{(1)}_{X/S} & & \mathscr{D}^{(m)}_{X/S} := \widetilde{F^{m+1}_{X/S,*}D^{(m)}_{X/S}} & \\
 & \mathrm{T}^*_{X^{(1)}/S} \arrow[d, "\pi_1"] & \mathrm{T}^*_{X^{(2)}/S} \arrow[d, "\pi_2"] & \cdots & \mathrm{T}^*_{X^{(m+1)}/S} \arrow[d, "\pi_{m+1}"] & \cdots \\
X \arrow[r, "F_{X/S}"] \arrow[rr, bend right, "F^2_{X/S}"'] \arrow[rrrr, bend right, "F^{m+1}_{X/S}"'] & X^{(1)} \arrow[r, "F_{X^{(1)}/S}"] & X^{(2)} \arrow[r, "F_{X^{(2)}/S}"] & \cdots \arrow[r] & X^{(m+1)} \arrow[r, "F_{X^{(m+1)}/S}"] & \cdots
\end{tikzcd}
$$

The so-called p^{m+1}-curvature of E on X gives a Higgs bundle on $X^{(m+1)}$.

[4] See for example arXiv:0811.1168. See also Berthelot, Notes on . . . .