# 1. A torsor of liftings

**1.1.** Suppose that we have a diagram of morphism of schemes

$$\begin{tikzcd} & & X \arrow[d, "f"] \\ Y_0 \arrow[r, "i"] \arrow[urr, bend left, "u_0"] & Y \arrow[r] \arrow[ur, shift left, "u_1"] \arrow[ur, shift right, "u_2"'] & S \end{tikzcd} \tag{1}$$

where $u_1 \circ i = u_2 \circ i = u_0$ and $i : Y_0 \to Y$ is a closed embedding whose ideal of definition $\mathscr{I} \subseteq \mathscr{O}_Y$ is square-zero.

**1.2. Remark.** Note that $i : Y_0 \to Y$ is (or can be viewed as[1]) the *identity* on the underlying topological spaces because $\mathscr{I}$ is square-zero. It may be more convenient, as in almost all literatures, to omit the inverse image functor $i^{-1}$ and the direct image functor $i_*$ from notations, by just identifying the underlying topological spaces of $Y$ and $Y_0$. Hence it may be better to write $\mathscr{O}_Y$ (resp. $\mathscr{I}$) for the sheaf of rings $i^{-1}\mathscr{O}_Y$ (resp. sheaf of ideals $i^{-1}\mathscr{I} \subseteq i^{-1}\mathscr{O}_Y$ as well as the sheaf of $\mathscr{O}_{Y_0}$-modules $i^*\mathscr{I} = (i^{-1}\mathscr{I}) \otimes_{i^{-1}\mathscr{O}_Y} \mathscr{O}_{Y_0} = i^{-1}\mathscr{I} \otimes_{i^{-1}\mathscr{O}_Y} i^{-1}(\mathscr{I}/\mathscr{I}^2) = i^{-1}\mathscr{I}$) on $Y_0$. However, it may (or may not) lead to some confusion. In the present notes, I will try to write down explicitly all these notations, which may *look* more formal and functorial, and may better remind us where sheaves live and which modules they are. Of course, we do not really get anything new and we have to pay some price in writing in this style.

**1.3.** First of all, each $u_j$, $j = 1, 2$ induces a morphism of *sheaves of rings*

$$u_j^* := i^{-1}(u_j^\natural) : u_0^{-1}\mathscr{O}_X = i^{-1}(u_j^{-1}\mathscr{O}_X) \longrightarrow i^{-1}\mathscr{O}_Y. \tag{2}$$

On checks that the difference of these two maps

$$u_2^* - u_1^* : u_0^{-1}\mathscr{O}_X \longrightarrow i^{-1}\mathscr{I} \subseteq i^{-1}\mathscr{O}_Y \tag{3}$$

is an *S-Derivation* of $(u_0^{-1}\mathscr{O}_X)$-modules, i.e.,

1. it is $(u_0^{-1}f^{-1}\mathscr{O}_S)$-linear, and
2. it satisfies the Leibniz rule,

where the $(u_0^{-1}\mathscr{O}_X)$-module structure of $i^{-1}\mathscr{I}$ is given by restricting scalars from $i^{-1}\mathscr{O}_Y$ to $u_0^{-1}\mathscr{O}_X$ via (2). This follows from the fact that $u_0 = u_1 \circ i = u_2 \circ i$ (exercise).

**1.4.** Secondly, the above commutative diagram induces a commutative diagram

$$\begin{tikzcd} Y_0 \arrow[r, "i"] \arrow[d, "u_0"] & Y \arrow[d, "{(u_1,u_2)}"] \\ X \arrow[r, "\Delta=\Delta_f"] & X \times_S X. \end{tikzcd}$$

This commutative diagram induces a $u_0$-morphism of the corresponding *conormal sheaf* $\Omega^1_{X/S}$ of $X$ in $X \times_S X$ and $i^{-1}\mathscr{I}$ of $Y_0$ in $Y$:

$$\psi : u_0^{-1}\Omega^1_{X/S} \longrightarrow i^{-1}\mathscr{I} \subseteq i^{-1}\mathscr{O}_Y. \tag{4}$$

[1]According to the very definition, $i$ is just a homeomorphism on the underlying topological spaces. But we can always "alter" $i$ so that it becomes the *canonical* one, which is identity on the underlying space — this is very boring as I am too scrupulous here.

Actually, this is the morphism that comes from the universal property of the relative differential $\Omega^1_{X/S}$, that is, it is the unique morphism of $(u_0^{-1}\mathscr{O}_X)$-module morphism making the diagram

$$\begin{tikzcd} u_0^{-1}\mathscr{O}_X \arrow[r, "u_0^{-1}\mathrm{d}_{X/S}"] \arrow[d, "u_2^*-u_1^*"'] & u_0^{-1}\Omega^1_{X/S} \arrow[dl, "\psi"] \\ i^{-1}\mathscr{I} \subseteq i^{-1}\mathscr{O}_Y & \end{tikzcd} \tag{5}$$

**1.5.** By adjointness, the $u_0^{-1}\mathscr{O}_X$-morphism (4) is equivalent to an $i^{-1}\mathscr{O}_Y$-morphism

$$u_0^{-1}\Omega^1_{X/S} \otimes_{u_0^{-1}\mathscr{O}_X} i^{-1}\mathscr{O}_Y \longrightarrow i^{-1}\mathscr{I},$$

which, applied by $- \otimes_{i^{-1}\mathscr{O}_Y} \mathscr{O}_{Y_0}$, becomes an $\mathscr{O}_{Y_0}$-morphism

$$u_0^*\Omega^1_{X/S} \longrightarrow i^*\mathscr{I}.$$

Note that, as a sheaf of sets, $i^*\mathscr{I} = i^{-1}\mathscr{I}$. (If you prefer to view $\mathscr{I}$ as a sheaf on $Y_0$, then $i^{-1}\mathscr{I} = i^{-1}\mathscr{I} = \mathscr{I}$.)

**1.6.** Moreover, given $u_1 : Y \to X$ with $u_1 \circ i = u_0$ and *any* $\psi$ in $\mathrm{Hom}_{\mathscr{O}_{Y_0}}(u_0^*\Omega^1_{X/s}, i^*\mathscr{I})$, one can obtain another $u_2$ with $u_2 \circ i = u_0$ by reversing the above process. Actually, $u_2$ should has the same map on the underlying topological space, and on the level of structure sheaves, one can recover $u_2^\natural$ from $u_2^* = u_1^* + \psi \circ (u_0^{-1}\mathrm{d}_{X/S})$ by (5). To conclude we get [EGA IV 16.5.17].

**1.7. Theorem.** Given a diagram

$$\begin{tikzcd} & & X \arrow[d, "f"] \\ Y_0 \arrow[r, "i"] \arrow[urr, bend left, "u_0"] & Y \arrow[r] & S \end{tikzcd}$$

where $i$ is a closed immersion defined by a square-zero ideal $\mathscr{I} \subset \mathscr{O}_Y$. Then the assignment

$$U \longmapsto \{v : U \to X : v \circ (i|_{i^{-1}(U)}) = (u_0)|_{i^{-1}(U)} : i^{-1}(U) \to X\}$$

for each Zariski open subset $U \subseteq Y$ is a *pseudo-torsor* under the sheaf of groups $\mathscr{H}om_{\mathscr{O}_{Y_0}}(u_0^*\Omega^1_{X/S}, i^*\mathscr{I})$ on $Y$.

**1.8. Remark.** Actually this is a torsor, not just a pseudo-torsor.

## 2.

**2.1.** Obviously, we have a short exact sequence of

$$0 \to p\mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p \to 0$$

$\mathbb{Z}/p^2\mathbb{Z}$-modules, and the multiplication-by-$p$ map $\times p : \mathbb{F}_p \to p\mathbb{Z}/p^2\mathbb{Z}$ is an isomorphism of $\mathbb{Z}/p^2\mathbb{Z}$-modules. We can translate this as the multiplication-by-$p$ map

$$\times p : \iota_*\mathscr{O}_{\operatorname{Spec}\mathbb{F}_p} \to p\mathscr{O}_{\operatorname{Spec}\mathbb{Z}/p^2\mathbb{Z}} = \mathrm{Ker}(\iota^\sharp : \mathscr{O}_{\operatorname{Spec}\mathbb{Z}/p^2\mathbb{Z}} \twoheadrightarrow \iota_*\mathscr{O}_{\operatorname{Spec}\mathbb{F}_p}) \tag{6}$$

is an isomorphism of $\mathscr{O}_{\operatorname{Spec}\mathbb{Z}/p^2\mathbb{Z}}$-modules, where $\iota : \operatorname{Spec}\mathbb{F}_p \to \operatorname{Spec}\mathbb{Z}/p^2\mathbb{Z}$ is the closed embedding corresponding to the quotient $\mathbb{Z}/p^2\mathbb{Z} \to \mathbb{F}_p$.

**2.2.** Now for any Cartesian square

$$\begin{tikzcd}
S_0 \arrow[r, hook, "i"] \arrow[d, "\pi_0"] & S \arrow[d, "\pi"', "\text{flat}"] \\
\operatorname{Spec} \mathbb{F}_p \arrow[r, hook, "\iota"] & \operatorname{Spec} \mathbb{Z}/p^2\mathbb{Z}
\end{tikzcd}$$

we have, by pulling back (6) along $\pi$, we get

$$\begin{tikzcd}
\pi^*\iota_*\mathscr{O}_{\operatorname{Spec}\mathbb{F}_p} \arrow[rr, "\sim"] \arrow[d, equal, "\iota \text{ affine}"] & & \pi^*(p\mathscr{O}_{\operatorname{Spec}\mathbb{Z}/p^2\mathbb{Z}}) \arrow[dd, equal, "\pi \text{ flat}"] \\
i_*(\pi_0)^*\mathscr{O}_{\operatorname{Spec}\mathbb{F}_p} \arrow[d, equal] & & \\
i_*\mathscr{O}_{S_0} \arrow[r, "\sim", "\times p"'] & p\mathscr{O}_S = \operatorname{Ker}(i^\sharp : \mathscr{O}_S \to i_*\mathscr{O}_{S_0}) &
\end{tikzcd} \tag{7}$$

**2.3. The Gerbe of liftings..** According to Theorem 1.7 and (7), there is a $\Theta_X$-gerbe.

STATE THIS LATER.

RECALL THAT THIS GERBE IS ISOMORPHIC TO THE GERBE OF SPLITINGS OF $\tau_{\leq 1}\Omega$ [DI87], ALSO EQUIVALENT TOT THE GERBE OF SPLITTINGS OF THE AZUMAYA ALGEBRA $\mathscr{D}_{X/S}$ [OV07].

**2.4.** Now suppose that $X_0/S_0$ is *smooth* where $S$ is a *flat* lifting of $S_0$ as in the previous subsection. Moreover, suppose that we can lift everything as the following diagram

$$\begin{tikzcd}
& X \arrow[rr, "F"] \arrow[rrrd, bend left=15] & & X' \arrow[dr] & \\
X_0 \arrow[ur, "j"] \arrow[r, "F_0"] \arrow[rrrd, bend right=20] & & X_0' \arrow[ur, "j'"] \arrow[rr] \arrow[dr] & & S \\
& & & S_0 \arrow[ur, "\iota"] &
\end{tikzcd}$$

where $F_0 := F_{X_0/S_0}$ is the *relative Frobenius* morphism, $X$, $X'$, and $F$ are liftings of $X_0$, $X_0'$ and $F_0$ respectively.

**2.5. Remark.** Note that the (absolute and relative) Frobenius morphisms are homeomorphisms/the identity on the underlying topological spaces, so are their liftings. Therefore similar to the remark in §1.2, most authors usually view all sheaves as sheaves on a single topological space: $|X| = |X_0| = |X'| = |X_0'|$. So they do not bother writing inverse image and direct image of sheaves of sets. However, the pullback of sheaves of modules should always be explicitly written down. One possible advantage of this writing style is that, everything follows from functoriality, rather than from computations with local sections of sheaves as in many other references.

**2.6.** By (7), we have an isomorphism

$$\times p : j_*\mathscr{O}_{X_0} \longrightarrow p\mathscr{O}_X \tag{7'}$$

Appying $- \otimes_{\mathscr{O}_X} \Omega^1_{X/S}$, we get an isomorphism of $\mathscr{O}_X$-modules

$$\begin{tikzcd}
j_*\mathscr{O}_{X_0} \otimes_{\mathscr{O}_X} \Omega^1_{X/S} \arrow[r, "\sim"] \arrow[d, equal, "\text{projection formula}"] & p\mathscr{O}_X \otimes_{\mathscr{O}_X} \Omega^1_{X/S} \arrow[d, equal] \\
j_*\Omega^1_{X_0/S_0} \arrow[r, "\sim", "\times p"'] & p\Omega^1_{X/S}
\end{tikzcd} \tag{8}$$

Moreover, the two multiplication map (7′) and (8) are compatible in the sense that the diagram

$$
\begin{tikzcd}
j_*\mathscr{O}_{X_0} \arrow[r, "\mathrm{d}"] \arrow[d, "(7')"] & j_*\Omega^1_{X_0/S_0} \arrow[d, "(8)"] \\
p\mathscr{O}_X \arrow[r, "\mathrm{d}"] & p\Omega^1_{X/S}
\end{tikzcd}
\tag{9}
$$

is commutative.

**2.7.** It is easy to see that the tangent map $T_F : F^{-1}\Omega^1_{X'/S} \to \Omega^1_{X/S}$ has image inside $p\Omega^1_{X/S}$. Then consider its composition with the inverse of (8)

$$F^{-1}\Omega^1_{X'/S} \to p\Omega^1_{X/S} \to j_*\Omega^1_{X_0/S_0},$$

which is $F^{-1}\mathscr{O}_{X'}$-linear. Its adjoint map $j^{-1}F^{-1}\Omega^1_{X'/S} = F_0^{-1}(j')^{-1}\Omega^1_{X'/S} \to \Omega^1_{X_0/S_0}$, which is $F_0^{-1}(j')^{-1}\mathscr{O}_{X'}$-linear, further linearizes (by adjointness) to an $F_0^{-1}\mathscr{O}_{X'}$-linear map

$$\zeta_F : F_0^{-1}(j')^{-1}\Omega^1_{X'/S} \otimes_{F_0^{-1}(j')^{-1}\mathscr{O}_{X'}} F_0^{-1}\mathscr{O}_{X_0} \simeq F_0^{-1}(j')^*\Omega^1_{X'/S} \simeq F_0^{-1}\Omega^1_{X_0'/S_0} \longrightarrow \Omega^1_{X_0/S_0} \tag{10}$$

or even an $\mathscr{O}_{X_0}$-linear map

$$\zeta_F : F_0^{-1}(j')^{-1}\Omega^1_{X'/S} \otimes_{F_0^{-1}(j')^{-1}\mathscr{O}_{X'}} \mathscr{O}_{X_0} \simeq F_0^*(j')^*\Omega^1_{X'/S} \simeq F_0^*\Omega^1_{X_0'/S_0} \longrightarrow \Omega^1_{X_0/S_0} \tag{10′}$$

We can see that $\zeta_F$ is just (the adjoint map of) the tangent map $T_F : F^{-1}\Omega^1_{X'/S} \to \Omega^1_{X/S}$ divided by $p$.

**2.8. Remarks.**

- In the above, we use the same notation $\zeta_F$ two denote different maps. The meaning of it will be clear form context. Actually, the two maps clearly determines each other and thee $\mathscr{O}_{X_0}$-linear map (10′) can be directly derived in the same manner from the $\mathscr{O}_X$-linear map $F^*\Omega^1_{X'/S} \to p\Omega^1_{X/S}$. The reason why we make complicated stuffs here is to emphasize the linearity. Both the $F_0^{-1}\mathscr{O}_{X_0}$-map (10) and the $\mathscr{O}_{X_0}$-linear map (10′) will turn out to be useful later.
- As you may expect, it may be more convenient to express $\zeta_F$ using its adjoint map

$$\zeta_F : \Omega^1_{X_0'/S_0} \to (F_0)_*\Omega^1_{X_0/S_0},$$

which is $\mathscr{O}_{X_0'}$-linear and has two different kinds of adjoint map (10) and (10′). Actually, in most literatures, this pushforward form of $\zeta_F$ is used. But in the current notes, the pullback/inverse image form of $\zeta_F$ will be used in the later part. This is the other reason why we express $\zeta_F$ in this way. Of course, this pushforward form of $\zeta_F$ can be derive directly from the map $\Omega^1_{X'/S} \to F_*(p\Omega^1_{X/S})$ in the same manner as above.
- Alternatively, we can just pullback the $\mathscr{O}_X$-linear (resp. $\mathscr{O}_{X'}$-linear) map $F^*\Omega^1_{X'/S} \to p\Omega^1_{X/S}$ (resp. $\Omega^1_{X'/S} \to F_*(p\Omega^1_{X/S}) = pF_*\Omega^1_{X/S}$) directly along $j$, to derive the $\zeta_F$.

# 3.

**3.1.** Now suppose that there are two liftings of $F_0$ with the same domain and codomain.

$$
\begin{tikzcd}
X \arrow[r, shift left, "F_1"] \arrow[r, shift right, "F_2"'] & X' \\
X_0 \arrow[u, "j"] \arrow[r, "F_0"] & X_0' \arrow[u, "j'"']
\end{tikzcd}
\quad \text{or in the shape of (1)} \quad
\begin{tikzcd}
 & & X' \arrow[d] \\
X_0 \arrow[r, "j"] \arrow[urr, bend left, "F_0 \circ j'"] & X \arrow[ur, shift left, "F_1"] \arrow[ur, shift right, "F_2"'] \arrow[r] & S
\end{tikzcd}
$$

Then we get a commutative diagram as in (5):

$$\begin{tikzcd}
F_0^{-1}(j')^{-1}\mathscr{O}_{X'} \arrow[r, "F_0^{-1}(j')^{-1}\mathrm{d}_{X'/S}"] \arrow[d, "(F_2)^*-(F_1)^*"'] & F_0^{-1}(j')^{-1}\Omega^1_{X'/S} \arrow[dl, dotted, "\exists!"', "\psi_{F_1,F_2}"] \\
j^{-1}(p\mathscr{O}_X) &
\end{tikzcd} \tag{11}$$

where $F_2^* - F_1^*$ is an $S$-derivation and $\psi_{F_1,F_2}$ is $F_0^{-1}(j')^{-1}\mathscr{O}_{X'} = j^{-1}F_r^{-1}\mathscr{O}_{X'}$-linear ($r = 1, 2$). Moreover, we can complete it as a commutative diagram (of $F_0^{-1}(j')^{-1}\mathscr{O}_{X'}$-modules)

$$\begin{tikzcd}
F_0^{-1}(j')^{-1}\mathscr{O}_{X'} \arrow[r, "F_0^{-1}(j')^{-1}\mathrm{d}_{X'/S}"] \arrow[d, "(F_2)^*-(F_1)^*"'] & F_0^{-1}(j')^{-1}\Omega^1_{X'/S} \arrow[dl, dotted, "\psi_{F_1,F_2}"] \arrow[d, "T_{F_1}-T_{F_2}"] \\
j^{-1}(p\mathscr{O}_X) \arrow[r, "j^{-1}\mathrm{d}_{X/S}"'] & j^{-1}(p\Omega^1_{X/S}) \\
\mathscr{O}_{X_0} \arrow[u, "(7')"', "\sim"] \arrow[r, "\mathrm{d}_{X_0/S_0}"] & \Omega^1_{X_0/S_0} \arrow[u, "(8)"', "\sim"]
\end{tikzcd}$$

where $T_{F_j}$ denotes the inverse image under $j$ of the tangent map $F_j^{-1}\Omega^1_{X'/S} \to p\Omega^1_{X/S} \subseteq \Omega^1_{X/S}$. In fact, this diagram without the dotted arrow is commutative by functoriality of $\mathrm{d} : \mathscr{O} \to \Omega^1$ and (9). Moreover, because $\mathrm{d}_{X_0/S_0}$ is $F_0^{-1}\mathscr{O}_{X'_0}$-linear, the compositions

$$\begin{tikzcd}
& F_0^{-1}(j')^{-1}\Omega^1_{X'/S} \arrow[dl, dotted, "\psi_{F_1,F_2}"] \\
j^{-1}(p\mathscr{O}_X) \arrow[d, leftrightarrow, "(7')"', "\sim"] & \\
\mathscr{O}_{X_0} \arrow[r, "\mathrm{d}_{X_0/S_0}"] & \Omega^1_{X_0/S_0}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F_0^{-1}(j')^{-1}\Omega^1_{X'/S} \arrow[d, "T_{F_1}-T_{F_2}"] \\
j^{-1}(p\Omega^1_{X/S}) \arrow[d, leftrightarrow, "(8)"', "\sim"] \\
\Omega^1_{X_0/S_0}
\end{tikzcd}$$

are both $F_0^{-1}(j')^{-1}\mathscr{O}_{X'}$-linear. Observing that $F_0^{-1}(j')^{-1}\Omega^1_{X'/S}$ is generated by the image of $F_0^{-1}(j')^{-1}\mathrm{d}_{X'/s}$, we know that the above two compositions $F_0^{-1}(j')^{-1}\Omega^1_{X'/S} \to \Omega^1_{X_0/S_0}$ agree. So the whole diagram commutes.

We use adjointness (extension of scalars ⊣ restriction of scalars) to linearize them as a commutative diagram

$$\begin{tikzcd}
& F_0^{-1}\Omega^1_{X'_0/S_0} \arrow[dl, "\psi_{F_1,F_2}"'] \arrow[d, "\zeta_{F_2}-\zeta_{F_1}"] \\
\mathscr{O}_{X_0} \arrow[r, "\mathrm{d}_{X_0/S_0}"'] & \Omega^1_{X_0/S_0}
\end{tikzcd} \tag{12}$$

of $(F_0^{-1}\mathscr{O}_{X'_0})$-modules.

**3.2. Remarks.**

- It is important to note that the diagram (12) is a commutative diagram of $F_0^{-1}\mathscr{O}_{X'_0}$-modules. Though $\psi_{F_1,F_2}$ by adjointness corresponds to an $\mathscr{O}_{X_0}$-morphism $F_0^*\mathscr{O}^1_{X_0/S_0} \to \mathscr{O}_{X_0}$, but the $\mathscr{O}_{X_0}$-linear map $\zeta_{F_2} - \zeta_{F_1}$ : $F_0^*\mathscr{O}^1_{X_0/S_0} \to \mathscr{O}_{X_0}$ clearly *cannot* be written as a composition of this $\mathscr{O}_{X_0}$-linear map followed by an $F_0^{-1}\mathscr{O}_{X'_0}$-linear map $\mathrm{d}_{X_0/S_0}$.

Despite of this fact, we will denote the $\mathscr{O}_{X_0}$-linear map $F_0^*\mathscr{O}^1_{X_0/S_0} \to \mathscr{O}_{X_0}$ by $\psi_{F_1,F_2}$ too. The meaning of this notation will be clear from context.

- To avoid such confusion, it may be more convenient to use the adjoint ($F_0^* \dashv (F_0)_*$) map of $\psi_{F_1,F_2}$, so we get a commutative diagram

$$\begin{tikzcd} & \Omega^1_{X_0'/S_0} \arrow[dl, "\psi_{F_1,F_2}"'] \arrow[d, "\zeta_{F_2}-\zeta_{F_1}"] \\ (F_0)_*\mathscr{O}_{X_0} \arrow[r, "\mathrm{d}_{X_0/S_0}"'] & (F_0)_*\Omega^1_{X_0/S_0} \end{tikzcd}$$

of $\mathscr{O}_{X_0'}$-modules.

**3.3.** Now suppose that we have a fixed lift $X'$ of $X_0'$, two *isomorphic* liftings $X_1$ and $X_2$ of $X_0$, and respectively two liftings $F_1$ and $F_2$ of $F_0$. *Choose* an $S$-isomorphism $u : X_1 \to X_2$ lifting identity of $X_0$, then $F_2 \circ u$ and $F_1$ are two lifings of $F_0$ with the same domain and codomain. So we can apply §3.1 to get morphism $\psi_{F_1,F_2\circ u} : F_0^{-1}\Omega_{X_0'/S_0} \to \mathscr{O}_{X_0}$ with $\zeta_{F_2\circ u} - \zeta_{F_1} = \mathrm{d} \circ \psi_{F_1,F_2\circ u} : F_0^{-1}\Omega^1_{X_0'/S_0} \to \mathscr{O}_{X_0} \to \Omega^1_{X_0/S_0}$.

**3.4.** Note that in the discussion above, the map $\psi_{F_1,F_2\circ u}$ does not actually depend on the isomorphism[2] $u : X_1 \to X_2$. In fact, suppose that $u$ and $v$ are two $S$-isomorphism between two given liftings $j_1 : X_0 \to X_1$ and $j_2 : X_0 \to X_2$. Then by definition

$$\begin{aligned}(F_2 \circ u)^* &= j_1^{-1}((F_2 \circ u)^\natural) = j_1^{-1}(u^\natural \circ u^{-1}(F_2^\natural)) \\ &= j_1^{-1}(u^\natural) \circ (u \circ j_1)^{-1}(F_2^\natural) = j_1^{-1}(u^\natural) \circ j_2^{-1}(F_2^\natural) \\ &= u^* \circ F_2^* : (j' \circ F_0)^{-1}\mathscr{O}_{X'} \to j_2^{-1}\mathscr{O}_{X_2} \to j_1^{-1}\mathscr{O}_{X_1}.\end{aligned}$$

So $(F_2 \circ u)^* - (F_2 \circ v)^* = (u^* - v^*) \circ F_2^*$. We know from (5) or (11) that $u^* - v^*$ factors through $\mathrm{d}_{X_2/S} : \mathscr{O}_{X_2} \to \Omega^1_{X_2/S}$, while the image of $F_2^* : F_2^{-1}\mathscr{O}_{X'} \to \mathscr{O}_{X_2}$ is a $p$-th power. Hence $(F_2 \circ u)^* - (F_2 \circ v)^*$ is zero. That is to say, $(F_2 \circ u)^* : (j' \circ F_0)^{-1}\mathscr{O}_{X'} \to (j_1)^{-1}\mathscr{O}_{X_1}$ does not depend on the choice of $u$. So $\psi_{F_1,F_2\circ u}$ does not depend on $u$.

Moreover, it follows from the definition §2.7 of $\zeta$ that $\zeta_{F_2\circ u} = \zeta_{F_2}$, i.e., $\zeta_{F_2\circ u}$ does not depend on $u$ either. In fact, $\zeta_{F_2\circ u}$ (resp. $\zeta_{F_2}$) is the adjoint map of

$$u^{-1}F_2^{-1}\Omega^1_{X'/S} \to p\Omega^1_{X_1/S} \simeq (j_1)_*\Omega^1_{X_0/S_0} \quad (\text{resp. } F_2^{-1}\Omega^1_{X'/S} \to p\Omega^1_{X_2/S} \simeq (j_2)_*\Omega^1_{X_0/S_0})$$

HOW TO SEE THAT THEY ARE THE *SAME* MAP?

So we know in the situation of §3.3, there is a $\psi_{F_1,F_2}$, such that

$$\zeta_{F_2} - \zeta_{F_1} = \mathrm{d} \circ \psi_{F_1,F_2}, \tag{12'}$$

which is $F_0^{-1}\mathscr{O}'_{X_0}$-linear.

**3.5.** In general, any two liftings $X_1$ and $X_2$ of $X_0$ may not be isomorphic as liftings, i.e., there may not be an isomorphism between $u : X_1 \to X_2$ so that $u$ is over $S$ and $u$ lifts $\mathrm{id}_{X_0}$. However, Zariski locally, any two $X_1$ and $X_2$ are isomorphic as liftings EXPLANATION OR REFERENCE. That is to say, there is an open cover $X_0$, so that for each open $U$ in the cover, there is an isomorphism (not necessarily unique) $j_1^{-1}U \to j_2^{-1}U$. These isomorphisms can glue to an isomorphism of $X_1$ and $X_2$ *as schemes*, but NOT a isomorphism of liftings,

[2] If there is an arrow $u : X_1 \to X_2$ over $S$ lifting identity of $X_0$, it must be an isomorphism, in the sense of isomorphism of liftings. This is because once we restrict $u$ to affines, it becomes an isomorphism. EXPLANATION OR REFERENCE.

i.e., NOT a morphism $X_1 \to X_2$ fitting into the following commutative diagram:

$$\begin{tikzcd}
j_2^{-1}(U) \arrow[r] & X_2 \arrow[dr] & \\
 & & S \\
j_1^{-1}(U) \arrow[uu, "\sim", "\exists"'] \arrow[r] & X_1 \arrow[ur] & \\
U \arrow[u] \arrow[uuu, bend left] \arrow[r, hook] & X_0 \arrow[u, "j_1"'] \arrow[uuu, bend right, "j_2"] \arrow[r] & S_0 \arrow[uu]
\end{tikzcd}$$

But the local isomorphisms of liftings are enough to give a global definition of $\psi_{F_1,F_2}$, as the definition $\psi_{F_1,F_2}$ does not depend on the choice of these local isomorphisms, as long as they exist, according to §3.4.

**3.6.** So to conclude, if $X_0'$ lifts to $X_0$, $X_0$ lifts to $X_1$ and $X_2$, and $F_0$ lifts to $F_1$ and $F_2$ respectively, then there is a morphism $\psi_{F_1,F_2} : F_0^*\Omega^1_{X_0'/S_0} \to \mathscr{O}_{X_0}$, such that

$$\zeta_{F_2} - \zeta_{F_1} = \mathrm{d}_{X_0/S_0} \circ \psi_{F_1,F_2} : F_0^{-1}\Omega^1_{X_0'/S_0} \to \mathscr{O}_{X_0} \to \Omega^1_{X_0/S_0}, \tag{12''}$$

which is $F_0^{-1}\mathscr{O}_{X_0'}$-linear.

**3.7.** Do we really need the discussion about how the $\psi$ and $\zeta$ depends on the local isomorphism $u$? Here in the application, once the cover $U_\alpha'$ is fixed, and local liftings of Frobenius $F_\alpha$ is chosen, $(F_\alpha)|_{X_{\alpha\beta}}$ and $(F_\beta)_{X_{\alpha\beta}}$ are two liftings from the SAME domain $X_{\alpha\beta}$, i.e., we are in the case of §3.1. So from there we can direct come to this point. I think [DI87] did so.

All the others are used, in some sense, to show how these constructions depend on choices of covering and so on.

In the above, after fixing a lifting $X'$ of $X_0$, we assume the existences of liftings $X$ of $X_0$ and $F$ of $F_0$, which are not generally the case. However, locally this is true. Precisely, there is a covering $\{U_\alpha'\}$ of $X_0'$, such that over each $U_\alpha := F_0^{-1}(U_\alpha)$, there is a lifting $X_\alpha$ of $U_\alpha$ and a lifting $F_\alpha : X_\alpha \to X_\alpha' := X'|_{U_\alpha'}$ of $(F_0)|_{U_\alpha} : U_\alpha \to U_\alpha'$.

$$\begin{tikzcd}
 & X_\alpha \arrow[rr, "F_\alpha"] & & X_\alpha' \arrow[dl] \\
 & & X' & \\
 & U_\alpha \arrow[uu] \arrow[dl] \arrow[rr] & & U_\alpha' \arrow[uu] \arrow[dl] \\
X_0 \arrow[rr, "F_0"] & & X_0' \arrow[uu] &
\end{tikzcd}$$

We use the notions $U_{\alpha\beta} := U_\alpha \cap U_\beta$ for the intersection as usual, similarly for $U_{\alpha\beta\gamma}$ etc..

We write $\zeta_\alpha$ for the map

$$\zeta_{F_\alpha} : (F_0|_{U_\alpha})^*\Omega^1_{U_\alpha'/S_0} \to \Omega^1_{U_\alpha/S_0}$$

defined in (10′).

**3.8. Theorem.** [DI87, Pf. of Thm. 2.1, case c)]/[Ill96, Lemma. 5.4] There is a collection of morphisms

$$\psi_{\alpha\beta} : (F_0)|^*_{U_{\alpha\beta}}\Omega^1_{U'_{\alpha\beta}/S_0} \longrightarrow \mathscr{O}_{U_{\alpha\beta}/S_0},$$

such that

1. over $U_{\alpha\beta}$ (all maps should be restricted to $U_{\alpha\beta}$),

$$\zeta_\beta - \zeta_\alpha = \mathrm{d}_{X_0/S_0} \circ \psi_{\alpha\beta} : F_0^{-1}\Omega^1_{U'_{\alpha\beta}/S_0} \to \mathscr{O}_{U_{\alpha\beta}/S_0} \to \Omega^1_{U_{\alpha\beta}/S_0} \tag{12'''}$$

(In the following, for simplicity, we omit all the "unnecessary" subscripts from our notations. So this relation reads $\zeta_\beta - \zeta_\alpha = \mathrm{d} \circ \psi_{\alpha\beta} : F_0^*\Omega^1_{U'_{\alpha\beta}/S_0} \to \Omega^1_{U_{\alpha\beta}/S_0}$)

2. over $U_{\alpha\beta\gamma}$ (all maps are understood as restricted to $U_{\alpha\beta\gamma}$),

$$\psi_{\alpha\beta} + \psi_{\beta\gamma} + \psi_{\gamma\alpha} = 0 \tag{13}$$

The first relation is just (12″) and the second one is self-evident, or we can say it follows from the uniqueness $\psi$ (Recall (5) and §3.1 that this uniqueness come from the universal property of $\Omega^1$).

# 4. Recall

## 4.1. Cartier Descent.

$$\left\{\begin{array}{ll} \text{Obj.} & (\mathscr{E},\nabla), \text{ with } \mathscr{E} \in \mathrm{Obj}(\mathtt{QCoh}(X)), \\ & \nabla \text{ flat conn, } p\text{-curv. } \psi_\nabla = 0 \\ \text{Arr.} & \text{flat morphisms} \end{array}\right\} \longrightarrow \mathtt{QCoh}(X')$$

$$(\mathscr{E},\nabla) \longmapsto F_*(\mathscr{E}^\nabla)$$

$$(F^*\mathscr{E}', \nabla^{\mathrm{can}}) \longleftarrow \mathscr{E}'$$

For any quasi-coherent sheaf $\mathscr{E}'$ on $X'$, the *canonical connection* is defined as

$$\begin{tikzcd} F^{-1}\mathscr{E}' \otimes_{F^{-1}\mathscr{O}_{X'}} \mathscr{O}_X \arrow[r, "\mathrm{id}_{F^{-1}\mathscr{E}'}\otimes \mathrm{d}"] \arrow[d, equal] & F^{-1}\mathscr{E}' \otimes_{F^{-1}\mathscr{O}_{X'}} \Omega^1_{X/S} \arrow[d, equal] \\ F^*\mathscr{E}' \arrow[r, "\nabla^{\mathrm{can}}"] & F^*\mathscr{E}' \otimes_{\mathscr{O}_X} \Omega^1_{X/S} \end{tikzcd}$$

This makes sense because $\mathrm{d} : \mathscr{O}_X \to \Omega^1_{X/S}$ is $F^{-1}\mathscr{O}_{X'}$-linear.

## 4.2. (Classical) Cartier Transform.

Consider $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ with Frobenius twist[3] $(\mathbb{A}^1_k)' = \operatorname{Spec} k[t]$ and relative Frobenius $F^\sharp : k[t] \to k[x]$, $t \mapsto x^p$.

We have an isomorphism of complexes of $k[t]$-modules.

$$\begin{tikzcd} (F_*\Omega^\bullet_{X/k}, \mathrm{d}) & 0 \arrow[r] & k[x] \arrow[r, "\mathrm{d}"] & k[x]\,\mathrm{d}x \arrow[r] & 0 \\ (\Omega^\bullet_{X'/k}, 0) & 0 \arrow[r] & k[t] \arrow[r, "0"] \arrow[u, "F^\sharp"] & k[t]\,\mathrm{d}t \arrow[r] \arrow[u, "C^{-1}"] & 0 \end{tikzcd}$$

where

$$C^{-1} : f(t)\,\mathrm{d}t \mapsto f(x^p)x^{p-1}\,\mathrm{d}x$$

is the inverse Cartier map.

In general, there is no such a chain morphism that inducing this isomorphism, but we still have isomorphisms *of graded algebras*

$$\Omega^\bullet_{X'/S} \underset{C}{\overset{C^{-1}}{\rightleftarrows}} \mathscr{H}^\bullet(F_*\Omega^\bullet_{X/S}, \mathrm{d}) \tag{14}$$

[3] Some people prefer to use $x^p$ instead of $t$ as the indeterminate, but it may be a little bit confusing when we write $\mathrm{d}(x^p)$, which is not $px^{p-1}\mathrm{d}x$.

### 4.3. Nilpotent Higgs field.

nilpotent of exponent $\leq e$ if for all local sections $\partial_1, \ldots, \partial_e$ of $T_{X/k}$,

$$\theta(\partial_1)\cdots\theta(\partial_n) = 0$$

### 4.4. Nilpotent Connections.

[Kat70, (5.5)].
Nilpotent of exponent $\leq e$ if for all local sections $\partial_1, \ldots, \partial_e$ of $T_{X/k}$,

$$\psi(\partial_1)\cdots\psi(\partial_n) = 0$$

where $\psi$ is the $p$-curvature, in other words, a flat connection is nilpotent if its $p$-curvature, as an $F$-higgs filed, is nilpotent.

- Quasi-nilpotent connections. [Ber74, II, 4.3.5].
- Nilpotent implies quasi-nilpotent (see also [Ber74, II, 4.3.7]).
- Flat quasi-nilpotent connections is equivalent to HPD statifications. [Ber74, II, 4.3.11].

## 5. $p$-curvature

### 5.1. Definition 1 (classical viewpoint).

Briefly recall the definition of $p$-curvature.

- First show $(\nabla(D))^p - \nabla(D^p)$ lies inside $\mathscr{E}nd_{f^{-1}\mathscr{O}_S}(\mathscr{E}) \subseteq \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$, where $f : X \to S$, $\mathscr{E}$ an flat $S$-connection.
- Then show that $(\nabla(D))^p - \nabla(D^p)$ is $p$-linear.
- So obtain that $\mathscr{O}_X$-linear $F_X^*\Theta_{X/S} = F_{X/S}^*\Theta_{X'/S} \to \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$, or equivalently, an $\mathscr{O}_X$-linear map[4]

$$\varphi : \mathscr{E} \to \mathscr{E} \otimes_{\mathscr{O}_X} F_{X/S}^*\Omega^1_{X'/S}.$$

It has property that

- flat/parallel/horizontal with respect to $\nabla$ and $\nabla \otimes \nabla^{\mathrm{can}}$.
- $F$-Higgs.

**5.2. Lemma.** Let $X/S$ be a smooth $S$-scheme of characteristic $p$. Write $\mathscr{I} \subseteq \mathscr{O}_{X\times X}$ for the diagonal $\Delta$ and $(\mathscr{P} := \mathscr{P}_{X/S}, \overline{\mathscr{I}})$ for the divided power envelop of $\mathscr{I}$ in $\mathscr{O}_{X\times X}$. Define $X\times_S^{\mathrm{PD}}X$ is the $\mathcal{S}pec_{\mathscr{O}_{X\times_S X}}\mathscr{P}$, and $X^{[p]}$ is the closed subscheme of $X\times_S^{\mathrm{PD}}X$ defined by $\overline{\mathscr{I}}^{p+1}$, i.e., $\mathcal{S}pec_{\mathscr{O}_{X\times_S X}}(\mathscr{P}/\overline{\mathscr{I}}^{[p+1]})$. Write its structure sheaf by $\mathscr{P}^{[p]}$. We know from the general theory that $X^{[p]}$ has the same underlying topological space as $X$.

$$\begin{tikzcd}
 & X^{[p]} \arrow[r, hook] & X\times_S^{\mathrm{PD}}X \arrow[d, "\pi", "\text{affine}"'] \\
X \arrow[ur] \arrow[r] \arrow[rr, bend right, "\Delta"'] & X^{(1)} \arrow[r] & X\times_S X
\end{tikzcd}$$

Then there is a natural isomorphism of $\mathscr{O}_X$-modules (see [GLQ10, Prop. 3.2 and 3.3]),

$$F_{X/S}^*\Omega^1_{X/S} \xrightarrow{\sim} \frac{\overline{\mathscr{I}}\mathscr{P}^{[p]}}{\mathscr{I}\mathscr{P}^{[p]}} = \frac{\overline{\mathscr{I}}}{(\overline{\mathscr{I}}^{[p+1]} + \mathscr{I}\mathscr{P})} \tag{15}$$

And it extends to an isomorphism of PD $\mathscr{O}_X$-algebras

$$F_X^*\Gamma^\bullet\Omega^1_{X'/S} \longrightarrow \mathscr{P}/\mathscr{I}\mathscr{P} \tag{16}$$

[4]It is common to denote by the $p$-curvature map by $\psi$. But in this note, $\psi$ has another meaning. So we change to $\varphi$.

**5.3. Lemma/Definition 2 (Stratification viewpoint).** Given a flat connection $\nabla : \mathscr{E} \to \mathscr{E} \otimes_{\mathscr{O}_X} \Omega^1_{X/S}$ on $\mathscr{E}$ over $X$, which is equivalent to an *HPD-stratification*[5] (due to Mochizuki, see [OV07, Prop. 1.7])

$$\epsilon : \mathscr{P}_{X/S} \otimes_{\mathscr{O}_X} \mathscr{E} \xrightarrow{\sim} \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{P}_{X/S}$$

the $p$-curvature $\varphi$ is just the map sending local sections $e$ of $\mathscr{E}$ to the image of $\epsilon(1 \otimes e) - (e \otimes 1)$ in[6]

$$\mathscr{E} \otimes_{\mathscr{O}_X} \frac{\overline{\mathscr{I}}}{(\overline{\mathscr{I}}^{[p+1]} + \mathscr{I}\mathscr{P})} \simeq \mathscr{E} \otimes_{\mathscr{O}_X} F^*_{X/S}\Omega^1_{X'/S}.$$

**5.4. Definition 3 ($\mathscr{D}$-module viewpoint).** Recall that a flat connection on $X$ is equivalent to a $\mathscr{D}$-module, where

$$\mathscr{D} := \mathscr{D}_{X/S} := \varinjlim \mathscr{D}^{[m]} := \varinjlim \mathscr{H}om_{\text{left-}\mathscr{O}_X\text{-mod.}}\left(\mathscr{P}^{[m]}_{X/S}, \mathscr{O}_X\right)$$

is the *crystalline differential operator*, or *PD differential operator (of level 0)*, which we have seen in previous talks. Then (16) induces morphism

$$F^*_{X/S}\,\mathrm{Sym}^n\,\Theta_{X'/S} \longrightarrow \mathscr{D}_{X/S} \tag{17}$$

by taking the dual[7] of the composition $\mathscr{P} \to \mathscr{P}/\mathscr{I}\mathscr{P} \to F^*_{X/S}\Gamma^\bullet\Omega^1_{X'/S}$ of (16), via the natural perfect paring $\Gamma^\bullet\Omega^1_{X'/S} \times \mathrm{Sym}^\bullet\,\Theta_{X'/S} \to \mathscr{O}_{X'}$. Have to take care how to take the dual — taking dual at each step, then take limit.

Recall what Michael did in his talk. Using local sections, (15) is given by the map $\partial^p - \partial^{[p]}$.

## 6. Inverse Cartier

**6.1.** Start with the situation and notations in §3.6. Suppose that we are given an $\mathscr{O}_{X'_0}$-module[8] $\mathscr{H}$ with a Higgs field ($\mathscr{O}_{X'_0}$)-linear and[9] $\theta \wedge \theta = 0$)

$$\theta : \mathscr{H} \to \mathscr{H} \otimes_{\mathscr{O}_{X'_0}} \Omega^1_{X'_0/S_0}$$

nilpotent of exponent $\leq p$, i.e., a Higgs module $(\mathscr{H}, \theta)$. We are to define a flat module $C^{-1}(\mathscr{E}, \theta)$ of nilpotent $\leq p$.

When presenting these on board, it may be too complicated to write down all the indices. The strategy is first write everything without the indices, and leave spaces for the indices. Then claim that these are locally defined, then fill the spaces with the appropriate subscripts.

---

[5]See [Ber74, II, 4.3.11]. There should be assumptions on $S_0$ and $X_0 \to S_0$ for this to be true. But here, in our setting, all conditions verify

[6]Recall that, if we replace $\mathscr{P}$ by first principal part of $X$ in $X \times_S X$, then this difference is exactly the $\nabla$ itself.

[7] It's generally true that, if $\mathscr{M}$ is an $\mathscr{R}$-module locally free of finite type (in any topos),

$$\Gamma^\bullet\mathscr{M} \times \mathrm{Sym}^\bullet\,\mathscr{M} \to \mathscr{R}, \qquad (s^{[n]}, \phi_1 \cdots \phi_n) \mapsto \phi_1(s) \cdots \phi_n(s)$$

is a perfect paring, which gives duality at each step (see [BO78, Prop. A.10]).

In characteristic 0, we do not need divided powers. EXPLAIN MORE FOR GRADING LESS THAN $p$ IN CHARACTERISTIC $p$. REFER TO THE PLACE WHERE THIS FACT IS USED.

[8]When we say modules, we mean quasi-coherent modules.

[9]Say some words on this condition.

**6.2. Local definition.** Write

$$\mathscr{H}_\alpha := \mathscr{H}|_{U'_\alpha}, \quad \theta_\alpha := \theta|_{U'_\alpha}$$

for the restriction of the given higgs module to the open set $U'_\alpha$, and we do the similar for $\mathscr{H}_{\alpha\beta}$ etc.. Set

$$\mathscr{E}_\alpha := F_0^*(\mathscr{H}_\alpha) := (F_0|_{U_\alpha})^*(\mathscr{H}_\alpha) \tag{18}$$

which is a module over $U_\alpha$. Moreover We have an $\mathscr{O}_{U_\alpha}$-linear map

$$\mathscr{E}_\alpha \xrightarrow{(F_0|_{U_\alpha})^*(\theta_\alpha)} \mathscr{E}_\alpha \otimes_{\mathscr{O}_{U_\alpha}} F_0^*\Omega^1_{U'_\alpha/S_0} \xrightarrow{\mathrm{id}_{\mathscr{E}_\alpha} \otimes \zeta_\alpha} \mathscr{E}_\alpha \otimes_{\mathscr{O}_{U_\alpha}} \Omega^1_{U_\alpha/S_0},$$

which, for simplicity, will be written as $\mu_\alpha := \zeta_\alpha \circ F_0^*(\theta_\alpha)$. So

$$\nabla_\alpha := \nabla^{\mathrm{can}} + \mu_\alpha := \nabla^{\mathrm{can}} + \zeta_\alpha \circ F_0^*(\theta). \tag{19}$$

is then a connection.

**6.3. Flatness.** We claim that $\nabla_\alpha$ is a *flat connection* on $\mathscr{E}_\alpha$.

This follows from the local computation and definition of $\zeta$ – the tangent map divied by $p$ (see §2.7) Then use $p^2 = 0$ in $\mathscr{O}_X$, since each time "d" will produce one $p$, then two times "d" (as in the definition of curvature) will produces $p^2 = 0$.

REWRITE. ACTUALLY, $p$ HAS BEEN DIVIDED OUT, WHY DOES IT APPEAR?

**6.4.** Before continuing, we make an elementary remark, or, recall some definitions. Assume only that $X/S$ is smooth and $\mathscr{E}$ an quasi-coherent sheaf. Let $\theta : \mathscr{E} \to \mathscr{E} \otimes_{\mathscr{O}_X} \Omega^1_{X/S}$ be a Higgs field, and $\tilde{\theta} : \Theta_{X/S} \to \mathscr{E}nd_{\mathscr{O}_X}(X)$ the map equivalent to $\theta$. Then for any $\mathscr{O}_X$-linear map $D : \Omega^1_{X/S} \to \mathscr{O}_X$, or in other words, $D \in \Gamma(X, \Theta_{X/S})$ is a vector field, the element $\tilde{\theta}(D) \in \mathrm{End}_{\mathscr{O}_X}(\mathscr{E})$ is just the composition

$$\widetilde{D} : \ \mathscr{E} \xrightarrow{\theta} \mathscr{E} \otimes_{\mathscr{O}_X} \Omega^1_{X/S} \xrightarrow{\mathrm{id} \otimes D} \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_X \simeq \mathscr{E}$$

is an $\mathscr{O}_X$-endomorphism of $\mathscr{E}$. Then

1. the integrability condition of higgs bundles says $\widetilde{D}_1 \circ \widetilde{D}_2 = \widetilde{D}_2 \circ \widetilde{D}_1$ for any two vector fields $D_1$ and $D_2$.
2. the nilpotency condition says that $\widetilde{D}_1 \circ \widetilde{D}_2 \circ \cdots \circ \widetilde{D}_n = 0$ for any $n$ vector field $D_n$ with $n \geq e$, where $e$ is the exponent of nilpotency of $\theta$.

Local coordinates interpretation can be found, for example, in the beginning of [LSZ12], or as cited there, refer to [Kob87].

**6.5. Exponential twists.** Now let us look at overlaps $U_{\alpha\beta}$. We have $\mathscr{O}_{U_{\alpha\beta}}$-linear morphisms

$$\mathscr{E}_{\alpha\beta} \xrightarrow{(F_0|_{U_{\alpha\beta}})^*(\theta_{\alpha\beta})} \mathscr{E}_{\alpha\beta} \otimes_{\mathscr{O}_{U_{\alpha\beta}}} F_0^*\Omega^1_{U'_{\alpha\beta}/S_0} \xrightarrow{\mathrm{id} \otimes \psi_{\alpha\beta}} \mathscr{E}_{\alpha\beta} \otimes_{\mathscr{O}_{U_{\alpha\beta}}} \mathscr{O}_{U_{\alpha\beta}} = \mathscr{E}_{\alpha\beta}$$

which for simplicity, will be written as $\tau_{\alpha\beta} := \psi_{\alpha\beta} \circ (F_0^*\theta_{\alpha\beta}) \in \mathrm{End}_{\mathscr{O}_{U_{\alpha\beta}}}(\mathscr{E}_{\alpha\beta})$.

Define *(truncated) exponential twists*,

$$G_{\alpha\beta} := \exp(\tau_{\alpha\beta}) := \exp\left(\psi_{\alpha\beta} \circ (F_0^*\theta_{\alpha\beta})\right) := \sum_{n=0}^{p-1} \frac{1}{n!}(\tau_{\alpha\beta})^{\circ n} \tag{20}$$

which will be served as transition functions to glue $(\mathscr{E}_\alpha, \nabla_\alpha)$'s. Note that $G_\alpha$ is well-defined as all higher order terms vanishes due to the nilpotent assumption.

**6.6. Sheaves glue.** Over triple overlaps $U_{\alpha\beta\gamma}$ (all maps below should be restricted to $U_{\alpha\beta\gamma}$), we have

$$\tau_{\alpha\beta} \circ \tau_{\beta\gamma} = \tau_{\beta\gamma} \circ \tau_{\alpha\beta} \tag{21}$$

owing to the integrability condition.

Then (21), nilpotent assumption and (13) implies that (all maps should be restricted to $U_{\alpha\beta\gamma}$)

$$\begin{aligned}\exp(\tau_{\alpha\beta}) \circ \exp(\tau_{\beta\gamma}) \circ \exp(\tau_{\gamma\alpha}) &= \exp(\tau_{\alpha\beta} + \tau_{\beta\gamma} + \tau_{\gamma\alpha}) \\ &= \exp\big((\psi_{\alpha\beta} + \psi_{\beta\gamma} + \psi_{\gamma\alpha}) \circ F_0^*(\theta_{\alpha\beta\gamma})\big) \\ &= \mathrm{id}\,.\end{aligned}$$

It also follows that $G_{\alpha\beta} \in \mathrm{Aut}_{\alpha\beta}(\mathscr{E}_{\alpha\beta})$ are automorphisms, which can also be seen directly as $G_{\alpha\beta}^{-1} = G_{\beta\alpha} = \exp(-\tau_\alpha)$.

So we can use these glue functions to glue $\mathscr{E}_\alpha$'s, to a globally defined module $\mathscr{E}$ on $X_0$.

When presenting the following on blackboard, just omit all the indices $\alpha\beta$, and remind the audience that everything should be on $U_{\alpha\beta}$.

**6.7.** For the same reason that $\tau_{\alpha\beta}$'s commutes with each other (21), we have over overlaps $U_{\alpha\beta}$, $\mu_{\alpha\beta}$'s commute with $\mu_\alpha$'s: TO THINK ABOUT IT. SEEMS HAVE TO CHECK THIS USING COORDINATES. BUT FOR THIS IT SEEMS THAT WE HAVE TO ASSUME THAT $\mathscr{E}$ IS COHERENT, NOT JUST QUASI-COHERENT.

$$(\tau_{\alpha\beta} \otimes \mathrm{id}) \circ \mu_\alpha|_{U_{\alpha\beta}} = \mu_\alpha|_{U_{\alpha\beta}} \circ \tau_{\alpha\beta} : \mathscr{E}_{\alpha\beta} \longrightarrow \mathscr{E}_{\alpha\beta} \otimes_{U_{\alpha\beta}} \Omega^1_{U_{\alpha\beta}/S_0}. \tag{22}$$

Hence $G_{\alpha\beta}$'s commute with $\mu_\alpha$'s.

$$(G_{\alpha\beta} \otimes \mathrm{id}) \circ \mu_\alpha|_{U_{\alpha\beta}} = \mu_\alpha|_{U_{\alpha\beta}} \circ G_{\alpha\beta} : \mathscr{E}_{\alpha\beta} \longrightarrow \mathscr{E}_{\alpha\beta} \otimes_{U_{\alpha\beta}} \Omega^1_{U_{\alpha\beta}/S_0}. \tag{23}$$

**6.8. Connections glue.** What is more, $G_{\alpha\beta}$ is flat respect to $\nabla_\alpha$ and $\nabla_\beta$ (all maps should be rested to $U_{\alpha\beta}$), i.e., we have commutative diagram:

$$\begin{tikzcd}
\mathscr{E}_{\alpha\beta} \arrow[r, "\nabla_\alpha"] \arrow[d, "G_{\alpha\beta}"] & \mathscr{E}_{\alpha\beta} \otimes_{\mathscr{O}_{U_{\alpha\beta}}} \Omega^1_{U_{\alpha\beta}} \arrow[d, "G_{\alpha\beta}\otimes\mathrm{id}"] \\
\mathscr{E}_{\alpha\beta} \arrow[r, "\nabla_\beta"] & \mathscr{E}_{\alpha\beta} \otimes_{\mathscr{O}_{U_{\alpha\beta}}} \Omega^1_{U_{\alpha\beta}}
\end{tikzcd}$$

Recall that $\nabla_\alpha = \nabla^{\mathrm{can}} + \mu_\alpha$, where $\mu_\alpha = \zeta_\alpha(F_0^*\theta)$ is $\mathscr{O}_{U_{\alpha\beta}}$-linear. To see this, we need to show

$$(G_{\alpha\beta} \otimes \mathrm{id}) \circ (\nabla^{\mathrm{can}} + \mu_\alpha) = (\nabla^{\mathrm{can}} + \mu_\beta) \circ G_{\alpha\beta},$$

which we can rearrange as

$$\begin{aligned}(G_{\alpha\beta} \otimes \mathrm{id}) \circ \nabla^{\mathrm{can}} - \nabla^{\mathrm{can}} \circ G_{\alpha\beta} &= \mu_\beta \circ G_{\alpha\beta} - (G_{\alpha\beta} \otimes \mathrm{id}) \circ \mu_\alpha \\ &= (\mu_\beta - \mu\alpha) \circ G_{\alpha\beta} && \text{by (23)}\end{aligned}$$

Or equivalent by (12″)

$$(G_{\alpha\beta} \otimes \mathrm{id}) \circ \nabla^{\mathrm{can}} \circ G_{\alpha\beta}^{-1} - \nabla^{\mathrm{can}} = \mu_\beta - \mu_\alpha = \big(1 \otimes (\mathrm{d} \circ \psi_{\alpha\beta})\big) \circ F_0^*(\zeta_{\alpha\beta})$$

Then the this following from the facts

• A change of coordinates formula which follows from the Leibniz rule of $\nabla^{\mathrm{can}}$. In local coordinates,

$$\omega(\mathbf{e}g) = g^{-1}\mathrm{d}g + g^{-1}\omega(\mathrm{e})g,$$

where $\mathrm{e} = (e_\alpha)$ is a local coordinates, $g = (g_\alpha^\beta)$ is an invertible matrix, $\omega = (\omega\alpha^\beta)$ is the *connection form* with respect to e, such that (with Einstein's summation convention)

$$\nabla(e_\alpha) = e_\beta \otimes \omega_\alpha^\beta$$
$$\nabla(e_\alpha\xi^\alpha(\mathrm{e})) = e_\alpha \otimes \mathrm{d}\xi^\alpha(e) + e_\beta \otimes \omega_\alpha^\beta\xi^\alpha(\mathrm{e})$$

$\omega(\mathrm{e}g)$ is the connection form respect to the basis $\mathrm{e}g$. This can be found in, e.g., [Kob87, p. 1.1.7].

• THINK ABOUT WHY CAN WE USE COORDINATES IF E IS NOT LOCALLY FREE.

• In our case, $G$ is the exponential function, we have $g^{-1}\mathrm{d}g = -(\mathrm{d}g^{-1})g$. The LHS of the above equation reduces to $g^{-1}\mathrm{d}g$, and we have $\exp(-x)\mathrm{d}\exp(x) = \exp(-x)\exp(x)\mathrm{d}x$, we got the local expression of the RHS.

So we are allowed to glue not only the $\mathscr{E}_\alpha$'s, but also the local *flat* connections $\nabla$'s. Therefore, we get

$$\boxed{\text{Higgs module } (\mathscr{H}, \theta) \overset{\text{exp. twist}}{\rightsquigarrow} \text{flat modules } \{\mathscr{E}_\alpha, \nabla_\alpha\} \overset{\text{glue}}{\rightsquigarrow} \text{flat module } (\mathscr{E}, \nabla) =: C^{-1}(\mathscr{H}, \theta)}$$

**6.9. Remark.** Note that $\mathscr{E}$ is NOT $F_0^*\mathscr{H}$. We actually cut $F_0^*\mathscr{H}$ into pieces and twist the transition function to glue to a new sheaf $\mathscr{E}$. In other words, we locally pull back $\mathscr{H}$ and glue the pieces in another way.

**6.10. Nilpotency.** By definition/construction, locally over $U_\alpha$, the $p$-curvature of $\nabla'$ is exactly

$$(F_0|_{U_\alpha})^*(\theta_\alpha) : \mathscr{E}_\alpha \longrightarrow \mathscr{E}_\alpha \otimes (F_0|_{U_\alpha})^*\Omega^1_{U'_\alpha/S_0}$$

Recall that the nilpotency of a flat connection is equivalent to/by definition the nilpotency of its $p$-curvature (see §4.4 or [Kat70, (5.5)]).

## 7. Cartier

**7.1.** Start with the situation and notations in §3.6. Suppose that we are given a flat module $(\mathscr{E}, \nabla)$ of nilpotent $\leq p$. We are to define a Higgs module $C(\mathscr{E}, \nabla)$ of nilpotent $\leq p$.

**7.2. Local definition.** Recall the $p$-curvature of $(\mathscr{E}, \nabla)$ is an $\mathscr{O}_{X_0}$-linear map

$$\varphi : \mathscr{E} \longrightarrow \mathscr{E} \otimes_{\mathscr{O}_{X_0}} (F_0)^*\Omega^1_{X'_0/S_0}$$

and even more, it is an $F_0$-higgs field. As before, we use subscripts to indicate the restrictions to $U_\alpha$, e.g., $\mathscr{E}_{\alpha\beta} := \mathscr{E}|_{U_{\alpha\beta}}$, $\varphi_\alpha = \varphi|_{U_\alpha}$, etc..

We have an $\mathscr{O}_{U_\alpha}$-linear morphism

$$\mathscr{E}_\alpha \xrightarrow{\varphi_\alpha} \mathscr{E}_\alpha \otimes_{\mathscr{O}_{U_\alpha}} (F_0|_{U_\alpha})^*\Omega^1_{U'_\alpha/S_0} \xrightarrow{\mathrm{id}\otimes\zeta_\alpha} \mathscr{E}_\alpha \otimes_{\mathscr{O}_{U_\alpha}} \Omega^1_{U_\alpha/S_0}$$

We denote this composition by $\eta_\alpha := \zeta_\alpha \circ \varphi_\alpha := (\mathrm{id}\otimes\zeta_\alpha) \circ (\varphi_\alpha)$, which plays similar role as $\mu_{\alpha\beta}$ in the previous section.

So we can define a new connection

$$\nabla'_\alpha := \nabla_\alpha + \eta_\alpha$$

on $\mathscr{E}_\alpha$. we can check this is a flat connection, for the same reason as §6.3. So we get a collection of flat modules $\{\mathscr{E}_\alpha, \nabla'_\alpha\}$.

Recall that $\varphi_\alpha$ is flat with respect to $\nabla_\alpha$ and $\nabla_\alpha \otimes \nabla^{\mathrm{can}}$ (see §5.1). Moreover, similar to (23), using the integrability condition of the $F_0$-higgs field $\varphi_\alpha$, we can show the commutativity of the following diagram (all maps should be interpreted as their restrictions to $U_\alpha$),

$$\begin{tikzcd}
\mathscr{E}_\alpha \arrow[rr, "\nabla'_\alpha = \nabla_\alpha + \eta_\alpha"] \arrow[d, "\varphi'"] & & \mathscr{E}_\alpha \otimes_{\mathscr{O}_{U_\alpha}} \Omega^1_{U_\alpha/S_0} \arrow[d, "\varphi' \otimes \mathrm{id}"] \\
\mathscr{E}_\alpha \otimes_{\mathscr{O}_{U_\alpha}} F_0^* \Omega^1_{U'_\alpha/S_0} \arrow[rr, "\nabla' \otimes \nabla^{\mathrm{can}}"] & & (\mathscr{E}_\alpha \otimes_{\mathscr{O}_{U_\alpha}} F_0^* \Omega^1_{U'_\alpha/S_0}) \otimes_{\mathscr{O}_{U_\alpha}} \Omega^1_{U_\alpha/S_0}
\end{tikzcd}$$

In other words, the $F_0$-Higgs field $\varphi_\alpha$ is flat with respect to the new connection $\nabla'_\alpha$ and $\nabla' \otimes \nabla^{\mathrm{can}}$.

**7.3. Exponential twists.** Over overlaps $U_{\alpha\beta}$, consider the composition

$$\mathscr{E}_{\alpha\beta} \xrightarrow{\varphi_\alpha} \mathscr{E}_{\alpha\beta} \otimes_{\mathscr{O}_{U_{\alpha\beta}}} (F_0|_{U_{\alpha\beta}})^* \Omega^1_{U'_{\alpha\beta}/S_0} \xrightarrow{\mathrm{id} \otimes \psi_{\alpha\beta}} \mathscr{E}_{\alpha\beta} \otimes_{\mathscr{O}_{U_{\alpha\beta}}} \mathscr{O}_{U_{\alpha\beta}} = \mathscr{E}_{\alpha\beta}$$

We denote this composition by $\rho_{\alpha\beta} = \psi_{\alpha\beta} \circ \varphi_{\alpha\beta} := (\mathrm{id} \otimes \psi_{\alpha\beta}) \circ (\varphi_{\alpha\beta})$, which plays similar role as $\tau_{\alpha\beta}$ does in the previous part.

As before, define transition functions

$$J_{\alpha\beta} := \exp(\rho_{\alpha\beta}) := \exp(\psi_{\alpha\beta} \circ \varphi_{\alpha\beta}) := \sum_{n=0}^{p-1} \frac{1}{n!} (\rho_{\alpha\beta})^{\circ n} \in \mathrm{Aut}_{U_{\alpha\beta}}(\mathscr{E}_{\alpha\beta})$$

**7.4. Local data glue.**

**7.4.1. $\mathscr{E}_\alpha$'s glue.** As we did before, $J_{\alpha\beta}$ glues the local sheaves $\mathscr{E}_\alpha$, which amounts to say that over $U_{\alpha\beta\gamma}$ (all maps in the following should be restricted to $U_{\alpha\beta\gamma}$),

$$J_{\alpha\beta} \circ J_{\beta\gamma} \circ J_{\gamma\alpha} = \mathrm{id}$$

We then can glue $\mathscr{E}_\alpha$'s to a sheaf $\mathscr{E}'$.

**7.4.2. $\nabla_\alpha$'s glue.** Besides, $J_{\alpha\beta}$ is flat with respect to $\nabla'_\alpha$ and $\nabla'_\beta$, i.e., we have a commutative diagram (all maps should be restricted to $U_{\alpha\beta}$)

$$\begin{tikzcd}
\mathscr{E}_{\alpha\beta} \arrow[r, "\nabla'_\alpha"] \arrow[d, "J_{\alpha\beta}"] & \mathscr{E}_{\alpha\beta} \otimes_{\mathscr{O}_{U_{\alpha\beta}}} \Omega^1_{U_{\alpha\beta}} \arrow[d, "J_{\alpha\beta} \otimes \mathrm{id}"] \\
\mathscr{E}_{\alpha\beta} \arrow[r, "\nabla'_\beta"] & \mathscr{E}_{\alpha\beta} \otimes_{\mathscr{O}_{U_{\alpha\beta}}} \Omega^1_{U_{\alpha\beta}}
\end{tikzcd}$$

The reason is the same as (23): $\eta$ commutes with $J$.

We then can glue $\nabla'_\alpha$s to a *flat* connection on $\mathscr{E}'$.

**7.4.3. $\varphi_\alpha$'s maps glue.** For $\varphi_\alpha$'s glue, we need to check the commutativity of the digram

$$\begin{tikzcd}
\mathscr{E}_{\alpha\beta} \arrow[r, "\varphi_\alpha"] \arrow[d, "J_\alpha"] & \mathscr{E}_{\alpha\beta} \otimes_{U_{\alpha\beta}} F_0^* \Omega^1_{U_{\alpha\beta}/S_0} \arrow[d, "J_{\alpha\beta} \otimes \mathrm{id}"] \\
\mathscr{E} \arrow[r, "\varphi_\beta"] & \mathscr{E}_{\alpha\beta} \otimes_{U_{\alpha\beta}} F_0^* \Omega^1_{U_{\alpha\beta}/S_0}
\end{tikzcd}$$

That is (all functions restricted to $U_\alpha\beta$)

$$(J_{\alpha\beta} \otimes \mathrm{id}) \circ \varphi_\alpha = \varphi_\beta \circ J_{\alpha\beta} \tag{24}$$

This can be check by doing some computations!

So we glue $\varphi_\alpha$'s to an $F_0$-Higgs field

$$\varphi' : \mathscr{E}' \longrightarrow \mathscr{E}' \otimes_{\mathscr{O}_{X_0}} F_0^* \Omega^1_{X_0'/S_0}$$

with exponent of nilpontency $\leq p$ on $\mathscr{E}'$.

**7.4.4. Remark.** The glued $\varphi'$ is NOT the $p$-curvature map of the newly obtained flat module.

**7.4.5.** To summarize, we thus obtained

| | | | | |
|---|---|---|---|---|
| flat module $(\mathscr{E}, \nabla)$ with $p$-curv. $\varphi$ | $\xrightarrow{\text{exp. twist}}$ | flat modules $\{(\mathscr{E}_\alpha, \nabla'_\alpha)\}$ with $F_0$-Higgs $\varphi_\alpha$ that is flat w.r.t. $\nabla'_\alpha$. | $\xrightarrow{\text{glue}}$ | flat module with $(\mathscr{E}', \nabla')$ $F_0$-Higgs $\varphi'$ that is flat w.r.t. $\nabla$' |

**7.5. *p*-curvature zero.** This is a tricky part. To check the $p$-curvature $\varphi$ is zero, it suffices to check locally. So we work over $U := U_\alpha$. All maps, if necessary, should be restricted to $U$, and relative differentials are relative to $S_0$. Subscripts $\alpha$ are usually omitted. Keep in mind we are only working locally.

First of all, on $\mathrm{Sym}^{\leq 1}_{\mathscr{O}_U}\left(F_0^*\Omega^1_{U'}\right) = \mathscr{O}_U \oplus F_0^*\Omega^1_{U'} = F_0^*(\mathscr{O}_{U'} \oplus \Omega^1_{U'})$ The map ($f$, $\omega$ are local sections of $\mathscr{O}_U$ and $F_0^*\Omega^1_{U'}$ respectively)

$$\begin{aligned} \nabla_1 := \nabla^{\mathrm{can}} + \big((1,0)\otimes\zeta\big)\circ \mathrm{pr}_2 : \mathscr{O} \oplus F_0^*\Omega^1_U &\longrightarrow (\mathscr{O}_U \oplus F_0^*\Omega_{U'}) \otimes_U \Omega^1_U \\ (f,\omega) &\longmapsto \nabla^{\mathrm{can}}(f,\omega) + (1,0)\otimes\zeta(\omega) \end{aligned}$$

is a *connection* as $\big((1,0)\otimes\zeta\big)\circ\mathrm{pr}_2$ is $\mathscr{O}_U$-linear.

This connection extends, by Leibniz rule, to a connection $\nabla_2$ on[10]

$$\mathrm{Sym}^{\leq p-1}_{\mathscr{O}_U}\left(F_0^*\Omega^1_{U'}\right) = F_0^*\left(\mathscr{O}_{U'} \oplus \Omega^1_{U'} \oplus \mathrm{Sym}^2\,\Omega^1_{U'} \oplus \cdots \oplus \mathrm{Sym}^{p-1}\,\Omega_{U'}\right)$$

We then obtain a third connection $\nabla_3 := \nabla_2^\vee$ on the dual[11]

$$\mathrm{Sym}^{\leq p-1}_{\mathscr{O}_U}\left(F_0^*\Theta_{U'}\right)$$

of $\mathrm{Sym}^{\leq p-1}_{\mathscr{O}_U}\left(F_0^*\Theta_{U'}\right)$

Then the idea is follows.

$L := F_0^*(\mathrm{Sym}^\bullet\,\Theta_{X_0'})/S^{\geq p}$ acts on $\mathrm{Sym}^{<p}\,\Theta$ and $\mathscr{E}$ via their $p$-curvatures. They showed/computed that the first action is just a multiplication map and is free of rank one. So there is an isomorphism

$$\lambda : \mathscr{H}om_L(\mathrm{Sym}^{<p}\,\Theta, \mathscr{E}) \xrightarrow{\sim} \mathscr{E}$$

on the LHS, there is a $\nabla_4 := \mathrm{Hom}(\nabla_3, \nabla)$, so it induces another connection on the RHS. They showed this map is nothing but $\nabla'$ we defined. (Remind that we are working locally so $\nabla'$ is $\nabla'_\alpha$)

[10] Here is actually the trick [LSZ15] used. They would like to avoid using divided powers, so rather to extend it the more natural $\Gamma F_0^*\Omega^1_{U'}$, they extend it to $\mathrm{Sym}\,F_0^*\Omega^1_{U'}$. But $\mathrm{Sym}^{\leq p-1} \cong \Gamma^{leqp-1}$ as all $n \leq p-1$ are invertible!

[11] As pointed in the previous remark, it is more NATURAL to have the pairing $\Gamma_A(M) \times \mathrm{Sym}_A(M^\vee) \to A$, rather the $\mathrm{Sym}_A(M) \times \mathrm{Sym}_A(M^\vee) \to A$, while the latter only works in characteristic 0, or when truncated to $\leq p-1$.

**7.6. Descent.** Now we know $(\mathscr{E}', \nabla')$ is a flat connection with zero curvature. So by Cartier descent §4.1, $(\mathscr{E}', \nabla') \simeq (F_0^*\mathscr{H}, \nabla^{\mathrm{can}})$, where $\mathscr{H} := (F_0)_*(\mathscr{E}')^{\nabla'}$. Moreover, the $p$-curvature map

$$\varphi' : \mathscr{E}' \longrightarrow \mathscr{E}' \otimes_{\mathscr{O}_{X_0}} (F_0)^*\Omega^1_{X_0'/S_0}$$

which is flat respect to $\nabla'$ and $\nabla' \otimes \nabla^{\mathrm{can}}$, and is $F_0$-Higgs, descends to a Higgs field

$$\theta : \mathscr{H} \longrightarrow \mathscr{H} \otimes_{\mathscr{O}_{X_0'}} \Omega^1_{X_0'/S_0}$$

To conclude, we have

| | | | | |
|---|---|---|---|---|
| Flat module<br>$(\mathscr{E}, \nabla)$ | $\underset{\text{glue, } p\text{-curv.}}{\overset{\text{exp. twist}}{\rightsquigarrow}}$ | flat module $(\mathscr{E}', \nabla')$<br>with vanishing $p$-curv.<br>and $F_0$-Higgs field $\varphi'$ | $\overset{\text{descent}}{\rightsquigarrow}$ | Higgs module<br>$(\mathscr{H}, \theta) =: C(\mathscr{E}, \nabla)$ |

**7.7. Remark.** Note that $\mathscr{H}$ is not directly obtain from the flat module $(\mathscr{E}, \nabla)$ whose $p$-curvature is not zero. We first twist it to a new flat module $(E', \nabla')$, whose $p$-curvature vanishes. But the Higgs filed is closely related to the $p$-curvature of the origin flat mordule $\mathscr{E}$.

**7.8. Nilpotency.** Because $\theta$ is obtained from the nilpotent $F_0$-Higgs field $\varphi'$, we know $\theta$ is also nilpotent of exponent $\leq p$.

# 8. Inverse to each other

This is not hard.

**8.1. Final Remark.** If there were a lift of the relative Frobenius, we do not need the nilpotency assumption: recall that the only place that we use the nilpotency is the exponential functions.

# A. Some local computations

**A.1. Integrability condition.** Suppose $\theta : \mathscr{E} \to \mathscr{E} \otimes \Omega^1$ is a Higgs field. Take a local basis $(e_i)$ of $\mathscr{E}$. And suppose that

$$\theta(e_i) = e_j \otimes \pi_i^j,$$

where $\pi_i^j$ are local sections of $\Omega$. Then the integrability condition states that $\theta \wedge \theta = 0$, i.e.,

$$e_i \mapsto e_j \otimes \pi_i^j \mapsto e_k \otimes \pi_j^k \otimes \pi_i^j \mapsto e_k \otimes (\pi_j^k \wedge \pi_i^j) = 0$$

so $\pi_j^k \wedge \pi_i^j = 0$ as $(e_i)$ is a basis. Therefore, for any local sections of $\mathscr{H}om_{\mathscr{O}}(\Omega, \mathscr{O})$,

$$0 = (D_1 \wedge D_2)(\pi_j^k \wedge \pi_i^j) = \begin{vmatrix} D_1(\pi_j^k) & D_1(\pi_i^j) \\ D_2(\pi_j^k) & D_2(\pi_i^j) \end{vmatrix} = D_1(\pi_j^k)D_2(\pi_i^j) - D_1(\pi_i^j)D_2(\pi_j^k)$$

Hence

$$\begin{aligned}(\widetilde{D}_1 \circ \widetilde{D}_2)(e_i) &= (\widetilde{D}_1) \circ ((\mathrm{id} \otimes D_2) \circ \theta)(e_i) \\ &= (\widetilde{D}_1)(D_2(\pi_i^j)e_j) \\ &= \cdots = D_1(\pi_j^k)D_2(\pi_i^j)e_k \\ &= (\widetilde{D}_2 \circ \widetilde{D}_2)(e_i)\end{aligned}$$

where $\widetilde{D}$ is the composition

$$\mathscr{E} \xrightarrow{\theta} \mathscr{E} \otimes \Omega \xrightarrow{\mathrm{id} \otimes D} \mathscr{E} \otimes \mathscr{O} \simeq \mathscr{E}$$

and is the image of $D$ in $\mathscr{E}nd_{\mathscr{O}}(\mathscr{E})$ under

$$\tilde{\theta} : \mathscr{H}om_{\mathscr{O}}(\Omega, \mathscr{O}) \longrightarrow \mathscr{E}nd_{\mathscr{O}}(\mathscr{E})$$

So we conclude that for any local sections $D_1$ and $D_2$ in $\mathscr{H}om_{\mathscr{O}}(\Omega, \mathscr{O})$, the endomorphisms that they defined via $\tilde{\theta}$ commute with each other.

**A.2. $\tau$ and $\mu$ commute.** This follows from the integrability condition — applying to $\mathrm{pr}_l \circ \mu$ and $\tau$, where $\mathrm{pr}_l$ is the local projection $\Omega \simeq \mathscr{O}^{\oplus n} \to \mathscr{O}$ to the $l$-the component.

Recall that

$$\begin{aligned} \mu &= (\mathrm{id} \otimes \zeta) \circ F_0^*(\theta) : \mathscr{E} \to \mathscr{E} \otimes F_0^*\mathscr{E} \to \mathscr{E} \otimes \Omega \\ \tau &= (\mathrm{id} \otimes \psi) \circ F_0^*(\theta) : \mathscr{E} \to \mathscr{E} \otimes F_0^*\mathscr{E} \to \mathscr{E} \otimes \mathscr{O} \simeq \mathscr{E} \end{aligned}$$

And the precise statement is to show that

$$(\tau \otimes \mathrm{id}) \circ \mu = \zeta \circ \tau$$

With similar notations in the previous subsection,

$$\begin{aligned} ((\tau \otimes \mathrm{id}) \circ \mu)(e_i) &= (\tau \otimes \mathrm{id})(e_j \otimes \zeta(\pi_i^j)) \\ &= \psi(\pi_j^k)e_k \otimes \zeta(\pi_i^j) \\ (\mu \circ \tau)(e_i) &= \mu(\psi(\pi_i^j)e_j) \\ &= \psi(\pi_i^j)e_k \otimes \zeta(\pi_j^k) \end{aligned}$$

It suffices to see that

$$\psi(\pi_i^j)\begin{pmatrix} \zeta_1(\pi_j^k) \\ \zeta_2(\pi_j^k) \\ \vdots \\ \zeta_n(\pi_j^k) \end{pmatrix} = \psi(\pi_i^j)\zeta(\pi_j^k) = \psi(\pi_j^k)\zeta(\pi_i^j) = \psi(\pi_j^k)\begin{pmatrix} \zeta_1(\pi_i^j) \\ \zeta_2(\pi_i^j) \\ \vdots \\ \zeta_n(\pi_i^j) \end{pmatrix}$$

as local sections of $\Omega$. So it suffices to show that $\psi(\pi_i^j)\zeta_l(\pi_j^k) = \psi(\pi_j^k)\zeta_l(\pi_i^j)$. This follows from the integrability condition.

**A.3. $\tau$ and $\mathrm{d}\tau$ commute.**

**A.4. $\mathrm{d}G = G\mathrm{d}\tau$.** This follows from the previous subsection.

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