A mini-course on ∞-categories

notes taken by Yun Hao

September 9, 2017

These notes are based on four lectures given by Michael Gröchenig at FU Berlin in the summer semester 2017. The lectures were almost self-contained. However time was limited to cover all details. So some remarks and boring precisions were added as footnotes and many texts were changed and expanded accordingly. So these notes are not able to reflect the conciseness and elegance of the lectures. These notes are written to ensure I myself understand the materials and could play around with them, but for no other reason. There are quite a lot of pretty written introductory notes available on the Internet. All mistakes are due to me. All footnotes can be ignored.

1 n-categories

There are some different notions of n-categories. What we are going to study here, in modern languages, are called $(n,1)$-*categories*, which are different from (∞, n)-categories or *strict n-categories*.[1]

1.1. Definition. A *groupoïd* (or a $(1,0)$-*category*) $\mathscr{G}$ is a (small)[2] category such that every morphism is invertible (hence an isomorphism). A *homomorphism* of groupoïds is just a *functor* between two groupoïds. A homomorphism $f : \mathscr{G}_1 \to \mathscr{G}$ is an *equivalence* of groupoïds, if there is another homomorhpism $g : \mathscr{G}_2 \to \mathscr{G}_1$ such that $g \circ f \simeq \mathrm{id}_{\mathscr{G}_1}$ (naturally isomorphism) and $f \circ g \simeq \mathrm{id}_{\mathscr{G}_2}$. Tow groupoïds are equivalent if there is an equivalence between them.

For any objects x, y in a groupoïd $\mathscr{G}$, the *hom-set* $\mathrm{Hom}_{\mathscr{G}}(x, y)$, if non-empty, is then a group; and in this case, x and y are isomorphic.

1.2. We have a category $\mathtt{Grpd}$ of groupoïds, whose objects are groupoïds and arrows are homomorphisms of groupoïds.[3] There is a well defined notion of *products* of groupoïds, which is just the product of categories.[4]

---

[1] Actually, there are plenty of variations. See the following nice survey for more stories.

- Tom Leinster, *A Survey of Definitions of n-Category*, Theory Appl. Categ. 10, 1–70

In the lecture, the name *strict n-categories* were used. However, what were defined are different from the strict n-categories that people usually refer to.

[2] Usually people do not put this restriction. Or if one ignores set-theory issues, one just ignores such condition on the size of categories. However, for example, for $\mathrm{Hom}_{\mathscr{G}}(x, y)$ being a group, we need $\mathscr{G}$ to be locally small; and for $\pi_0(\mathscr{G})$ to be a set, it is necessary to require that $\mathscr{G}$ is essentially small. But this is not essential in our discussion.

[3] This is sometimes called the *naive* category of groupoïds. It is naive in the sense that it does not reflect homotopies, or natural isomorphisms between homomorphisms of groupo ids. The "correct" category to consider is the 2-category of groupoïds that will be defined later.

[4] The notion of products gives us a *symmetric monoidal structure* on $\mathtt{Grpd}$. Recall that a *monoidal category* is a category $\mathscr{K}$ with

- a functor $\otimes : \mathscr{K} \times \mathscr{K} \to \mathscr{K}$, $(X, Y) \mapsto X \otimes Y$;
- an *associator*, i.e., a natural isomorphism $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$;
- an object I, called the identity or unit object, with natural isomorphisms
  - the *left unitor* $\lambda_X : I \otimes X \xrightarrow{\sim} X$, and
  - the *right unitor* $\rho_X : X \otimes I \xrightarrow{\sim} X$.

satisfying the following coherence properties (commutative diagrams)

- the *triangle diagram*:

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "\alpha_{X,I,Y}"] \arrow[dr, "\rho_X \otimes \mathrm{id}_Y"'] & & X \otimes (I \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes \lambda_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

1.3. Example. Let G be a group. There is a groupoïd $\mathrm{B}G$, the *delooping* of G, with only one object $\bullet$, and for every $g \in G$, a morphism α_g of $\bullet$ itself. In diagram, this groupoïd looks like

$$\begin{tikzcd} \bullet \arrow[loop above] \arrow[loop right, "\alpha_g"] \arrow[loop below] \end{tikzcd}$$

If G_1 and G_2 are tow isomorphic groups, then the two categories $\mathrm{B}G_1$ and $\mathrm{B}G_2$ are equivalent.

1.4. Example. Let X be a topological space. The *fundamental groupoïd* $\pi_{\leq 1}(X)$ of X consists of objects points in X, and for each $x, y \in X$, the set of morphisms $\mathrm{Hom}_{\pi_{\leq 1}(X)}(x, y)$ is the set of *homotopy classes* of paths connecting x and y.

1.5. Definition. Let $\mathscr{G}$ be a groupoïd. Then

- the *set of connected components* $\pi_0(\mathscr{G})$ of X is the set of isomorphism classes of $\mathscr{G}$;
- for any object x of $\mathscr{G}$, the *fundamental group*[5] of $\mathscr{G}$ at x, denoted by $\pi_1(\mathscr{G}, x)$, is the group $\mathrm{Aut}_{\mathscr{G}}(x) = \mathrm{Hom}_{\mathscr{G}}(x, x)$.

Clearly if $x \simeq y$ are two isomorphic objects of $\mathscr{G}$, then $\pi_1(\mathscr{G}, x) \simeq \pi_1(\mathscr{G}, y)$ given by conjugation. So it make sense to speak of $\pi_1(\mathscr{G}, [x])$ for any $[x] \in \pi_0(\mathscr{G})$. For cleaner notations, we will drop the square brackets, which will not cause any confusion.

1.6. Lemma. If $\mathscr{G}$ is a groupoïd, then

$$\mathscr{G} \simeq \bigsqcup_{x \in \pi_0(\mathscr{G})} \mathrm{B}\pi_1(\mathscr{G}, x).$$

Interpretation: *A groupoïd is like a set (the set of connected components) where elements have automorphism groups.*

1.7. Example. Let V be a set, and G a group acting on V. We define the *action groupoïd* $[V/G]$ to be the category whose

- objects are elements of V, and
- morphisms are $\alpha_{(g,v)} : v \longrightarrow gv$, for each pair $(g, v) \in G \times V$. That is to say, an arrow f in $\mathrm{Hom}_{[V/G]}(v, w)$ is an element $g \in G$, such that $gv = w$.

It follows that $\pi_0([V/G]) = V/G$, and for all $v \in V$, $\pi_1([V/G], v) = \mathrm{Stab}_G(v)$.

- the *pentagon diagram*:

$$\begin{tikzcd} X \otimes (Y \otimes (Z \otimes W)) & & (X \otimes Y) \otimes (Z \otimes W) \arrow[ll, "\alpha_{X,Y,Z\otimes W}"'] & & ((X \otimes Y) \otimes Z) \otimes W \arrow[ll, "\alpha_{X\otimes Y,Z,W}"'] \arrow[dl, "\alpha_{X,Y,Z}\otimes \mathrm{id}_W"] \\ & X \otimes ((Y \otimes Z) \otimes W) \arrow[ul, "\mathrm{id}_X \otimes \alpha_{Y,Z,W}"] & & (X \otimes (Y \otimes Z)) \otimes W \arrow[ll, "\alpha_{X,(Y\otimes Z),W}"'] & \end{tikzcd}$$

expressing the fact that $\otimes$ is associative and has left and right identities. A functor between to monoidal categories that preserves the monoidal structure is called a *monoidal functor*. (Though this sounds trivial to define, but there are some issues about weak or strict commutativity of diagrams.)

A *symmetric monoidal cateogry* is a monoidal category with

- a natural isomorphism $B_{X,Y} : X \otimes Y \to Y \otimes X$, called the braiding,

satisfying

- the *hexagon diagram* (commutative diagram)

$$\begin{tikzcd} & (X \otimes Y) \otimes Z \arrow[r, "\alpha_{X,Y,Z}"] & X \otimes (Y \otimes Z) \arrow[dr, "\mathrm{id}_X \otimes B_{Y,Z}"] & \\ Z \otimes (X \otimes Y) \arrow[ur, "B_{Z,X\otimes Y}"] & & & X \otimes (Z \otimes Y) \\ & (Z \otimes X) \otimes Y \arrow[ul, "\alpha_{Z,X,Y}"] & (X \otimes Z) \otimes Y \arrow[l, "B_{X,Z}\otimes \mathrm{id}_Y"] \arrow[ur, "\alpha_{X,Z,Y}"'] & \end{tikzcd}$$

- $B_{Y,X} \circ B_{X,Y} = \mathrm{id}_{X \otimes Y}$ (strictly equal).

A monoidal functor that preserves the symmetric monodal structure is called a *symmetric monodial functor*.

[5]This group is also called the *vertex group* or *isotropy group* of x in $\mathscr{G}$.

1.8. Definition. A $(2,1)$-*category* (henceforth a 2-*category*), or a *locally groupoidal category*, is a category $\mathscr{C}$ enriched[6] in the category of groupoïds , i.e., for each pair of objects (x,y), there is a *groupoïd* $\mathrm{Hom}_{\mathscr{C}}(x,y)$ and these groupoïds satisfies the composition law $\mathrm{Hom}_{\mathscr{C}}(x,y)\times\mathrm{Hom}_{\mathscr{C}}(y,z)\to\mathrm{Hom}_{\mathscr{C}}(x,z)$ given by the product of groupoïds in `Grpd`, for all objects x, y and z of $\mathscr{C}$.

1.9. Example. There is a 2-category of groupoïds. The category of functors between two groupoïds is naturally a groupoïd itself.

1.10. Now let us consider the category of 2-term chain complexes of *abelian groups*. It has objects chain complexes of the form

$$\mathrm{C}_\bullet = [\ \cdots \longleftarrow 0 \longleftarrow C_0 \xleftarrow{d} C_1 \longleftarrow 0 \longleftarrow \cdots\ ]$$

with $C_i = 0$ for all $i \neq 0, 1$. For simplicity, we will write $C_\bullet = [d : C_1 \to C_0]$. Note that

$$\mathrm{H}_0(\mathrm{C}_\bullet) = \mathrm{Coker}\, d, \quad \text{and} \quad \mathrm{H}_1(\mathrm{C}_\bullet) = \mathrm{Ker}\, d$$

A morphism $f : C_\bullet \to D_\bullet$ of 2-term chain complexes

$$\begin{tikzcd} C_1 \arrow[r, "d^C"] \arrow[d, "f"] & C_0 \arrow[d, "f"] \\ D_1 \arrow[r, "d^D"] & D_0 \end{tikzcd}$$

is a *quasi-isomorphism* if the induced homomorphisms $\mathrm{H}_0(f) : \mathrm{Coker}\, d^C \to \mathrm{Coker}\, d^D$ and $\mathrm{H}_1(f) : \mathrm{Ker}\, d^C \to \mathrm{Ker}\, d^D$ on homology groups are isomorphisms.

1.11. (Strict) Picard Groupoïd. Briefly, a *(strict) Picard groupoïd* is a strictly commutative group object in the category `Grpd` of groupoids. That is to say, a strict Picard groupoïds $(\mathscr{P}, +)$ is a groupoïd $\mathcal{V}$ with a strict abelian group structure $+ : \mathscr{P}\times\mathscr{P}\to\mathscr{P}$, such that the functor $+$ satisfies the axioms of an abelian group on the nose. A *homomorphism* or a functor between two Picard groupoïds is a homomorphism of the underlying groupoïds that preverving the abelian group structure. A homomorphism is an *equivalence* if it's an equivalence as a homomorphism of groupoïds.[7]

1.12. Theorem. (Deligne, SGA4) There is a 1-1 correspondence[8]

$$\left\{\begin{matrix}\text{2-term}\\ \text{chain complexes}\\ \text{of abelian groups}\\ [d : V_1 \to V_0]\end{matrix}\right\}\Big/\{\text{q-isom.}\} \longleftrightarrow \{\text{Strict Picard groupoïds}\}\Big/\{\text{equiv.}\}$$

[6]Let $\mathscr{K}$ be a monoidal category. Then a $\mathscr{K}$*-enriched category* $\mathscr{C}$, or a category enriched in $\mathscr{K}$, is a category $\mathscr{C}$, such that for each pair of objects X and Y in $\mathscr{C}$, we have a hom-object $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ as an object in $\mathscr{K}$, satisfying certain composition laws (associative and unital). For example,

- a category enriched in `Set` is a locally small cateogory;
- a category enriched in (the category of) chain complexes is called a dg-cateogry (differential graded category), as we will see later.
- a category enriched in (the category of) simplicial sets is called a *simplicial category*, as we will see later.

[7]In SGA 4, XVIII. Définition 1.4.2, it's called a *catégorie de Picard strictement commutative*. One needs to take care that *what is an group object* in the category `Grpd` of groupoids. Here we give a slightly different (from SGA4) but essentially the same definition. A *strict Picard groupoids* is a symmetric monoidal category $\mathscr{C}$ such that

- $\mathscr{C}$ is a groupoïd.
- $\pi_0(\mathscr{C})$ is a group, i.e., every object of $\mathscr{C}$ is invertible: for any object x in $\mathscr{C}$, there is an object y such that $x\otimes y\simeq 1$ (not necessarily equal) and $y\otimes x\simeq 1$, where 1 is the identity object. (Some authors require that objects to be strictly invertible, meaning that for every object they put $=$ rather than $\simeq$ in this condition.)
- It is *strictly commutative*, or simply *strict*, in the sense that $B_{x,x}$ is identity for all object x in $\mathscr{C}$.

Take care that $B_{x,x}$ is identity is a strong condition. For example, the category of finite dimensional vector spaces with the usual tensor product, is not strictly commutative. But the category of 1-dimensional vector spaces is.

[8]See SGA IV, XVIII, SS1.4.11–1.1.17. But there a much more general result was proved. The result was credited to Grothendieck.

Idea of Proof. A complex $C_\bullet := [d : V_1 \to V_0]$ gives a natural group action of V_1 on V_0. Set $P(C_\bullet) := [V_0/V_1]$ (Example 1.7). It is a groupoïd by definition. It has a abelian group structure that is induced from that of V_0. One can check that $P(C_\bullet)$ is a strict Picard groupoïd. We have

$$\pi_0(P(C_\bullet) = V_0/V_1 = \operatorname{Coker} d = \mathrm{H}_0(C_\bullet)$$
$$\pi_1(P(C_\bullet), 0) = \operatorname{Stab}_{V_1}(0) = \operatorname{Ker} d = H_1(C_\bullet).$$

Then use the following lemma.

1.13. Lemma. (Whitehead)[9] Let $F : \mathscr{G} \to \mathscr{H}$ be a homomorphism of groupoïds. Then F is an equivalence if and only if $\pi_i(F)$ is an isomorphism for $i = 0, 1$. (Proof is easy, omitted)

1.14. Strict Picard groupoïds form a 2-category, i.e., a category enriched in groupoïds. This 2-category is denoted by $\mathscr{D}_{[0,1]}(\mathbb{Z})$.

1.15. Definition. Let $\mathscr{C}$ be a 2-category. Then the *homotopy category*[10] $\mathrm{Ho}(\mathscr{C})$ of $\mathscr{C}$ is the category with

- objects those of $\mathscr{C}$, and
- the hom-set $\mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(x, y)$ the set $\pi_0\big(\mathrm{Hom}_{\mathscr{C}}(x, y)\big)$ of connected components of the groupoïd $\mathrm{Hom}_{\mathscr{C}}(x, y)$ for each pair of objects x and y.

1.16. We can recover the derived category $D_{[0,1]}(\mathbb{Z}) = \{2 \text{ category of chain cplex } [V_1 \to V_0]\}$ modulo quasi-isomorphism.
$D_{[0,1]} = \mathrm{Ho}(\mathscr{D}_{[0,1]}(\mathbb{Z}))$.

1.17. This 2-category structure is needed to glue complexes of sheaves.

1.18. Example. (Glueing sheaves, details omitted) In order to glue, we need to put the *cocyle conditions*.

1.19. Example. (Glueing 2-term complexes of sheaves)[11] Let X be a topological space and $\mathcal{U} = \{U_i\}$ be a covering of X. For each i, let $\mathcal{F}_\bullet^i = [\mathcal{F}_0^i \to \mathcal{F}_1^i]$ be a 2-term chain complex of sheaves of abelian groups on U_i and for each i, j,

$$\varphi_{ij} : \mathcal{F}_\bullet^j|_{U_{ij}} \to \mathcal{F}_\bullet^i|_{U_{ij}}$$

be a *quasi-isomorphism* satisfying

$$\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik} \tag{1}$$

on U_{ijk} in the *derived category* (in which ϕ_{ij} becomes true isomorphism).

Question. Is there a globally defined 2-term complex $\mathcal{F}_\bullet$ on X, such that we have quasi-isomorphism $\psi : \mathcal{F}_\bullet|_{U_i} \to \mathcal{F}_\bullet^i$. Is there unicity?

Answer. No. However, if we were given more data, then the answer will be Yes. Instead of equalities (1), we have to record "2-morphisms"

$$\alpha_{ijk} : \phi_{ij} \circ \phi_{jk} \xrightarrow{\sim} \phi_{ik}$$

such that these (α_{ijk}) satisfy a cocycle identity of its own:

$$\alpha_{ijk} \circ \alpha_{ijl}^{-1} \circ \alpha_{jkl} \circ \alpha_{jkl}^{-1} = \mathrm{id} \tag{2}$$

In other words, (α_{ijk}) is a Čech 2-cycle.

1.20. The shape of things to come are

1-term	2-term	$\cdots$	n-term
V	$[V_1 \to V_0]$		$[V_n \to \cdots V_0]$
category	2-category		n-category

[9] The classical Whitehead theorem asserts that every weak homotopy equivalence between CW-complexes is a homotopy equivalence. This result is saying that weak homotopy equivalences between groupoïds are equivalences.
[10] The concept *homotopy category* is defined more generally for any *category with weak equivalences*, for example, for a *model category*.
[11] I did not really understand this example.

1.21. Definition. For each integer $n \geq 1$, an $(n, 1)$-*category* (hereinafter an n-category) is defined via iterated enrichment as follows.

- A 1-category is just a (small) category.
- A 1-groupoïd is a synonyms for a groupoïd.
- An n-*category* is a category enriched in $(n-1)$-groupoïds.[12] For an n-category, the *homotopy category* $\mathrm{Ho}(\mathscr{C})$ is the 1-category that has objects those of $\mathscr{C}$ and hom-sets $\mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(x, y) := \pi_0(\mathrm{Hom}_{\mathscr{C}}(x, y))$ for any two objects x and y of $\mathscr{C}$.
- An n-*groupoïd* is a n-category $\mathscr{C}$ such that its homotopy category $\mathrm{Ho}(\mathscr{C})$ is a 1-groupoïd. For an n-groupoïd $\mathscr{C}$, we set $\pi_0(\mathscr{C}) := \pi_0(\mathrm{Ho}(\mathscr{C}))$.

1.22. The above is not a suitable concept. Joyal's quasi-categories are probably the best model of ∞-categories. In particular, two theorems can be proved.

1.23. Theorem. (Conjecture by Grothendieck)[13] ∞-groupoïds modulo equivalence is in one to one correspondence with homotopy types of CW-complexes.

1.24. Theorem. (Dold-Kan, reformulated) Strict ∞-Picard groupoïds modulo equivalence are in one to one correspondence with chain complexes $[\cdots \to V_i \to V_{i-1} \to \cdots \to V_0]$ up to quasi-isomorphisms (This category is denoted by $D_{[0,\infty)}(\mathbb{Z})$).

2 ∞-categories

In this section, we introduce Joyal's quasi-category.[14]

2.1. Definition. Let $\mathbf{\Delta}$ be the category with

- objects: finite nonempty totally ordered sets, and
- morphisms: order-preserving maps.

This category is called the *simplex category*[15] or the *simplicial indexing category*. Denote by $[n]$ the set $\{0, 1, \ldots, n\}$ with the usual (increasing) order. Then $\mathbf{\Delta}$ is *equivalent* to its full-subcategory consisting of objects of the form $[n]$. For convenience, we will refer to $\mathbf{\Delta}$ as this full subcategory.

Note that the *ordered set* $[n]$ can be naturally viewed as a *category*[16] with

- $(n+1)$ objects $0, 1, \ldots, n$, and
- for each $i \leq j$, a unique morphism $i \to j$.

One sometimes write it as

$$[n] = \{0 \to 1 \to 2 \to \cdots \to n\}.$$

[12]By general theory of enriched categories, such a category has a monoidal category structure that induced from that of the ordinary category of $(n-1)$-groupoïds. So that it make sense to speak of categories over them.

[13]This is known as the *Grothendieck's Homotopy Hypothesis*. This is a guiding principle for higher category theory.

It has long been known that CW-complexes with $\pi_i(X, x) = 0$ for all $i > 1$ and all $x \in X$, are described at the level of homotopy by groupoïds. Grothendieck expected and conjectured, that there should be a higher-dimensional analogue: for each $1 \leq n \leq \infty$, the *n-truncated homotopy types* (CW-complexes with $\pi_i(X, x) = 0$ for all $i > n$) are equivalent to the (yet not defined) *n-groupoïds*. This appeared in the unpublished manuscript, *Pursuing Stacks*, in a letter to Daniel Quillen. It is now freely available online at `https://thescrivener.github.io/PursuingStacks/`. For more details or stories, see for examples,

- G. Maltsiniotis, *Infini groupoïdes d'apres Grothendieck*.
- G. Maltsiniotis, *Grothendieck ∞-groupoids, ans still another definition of ∞-categories*.
- M. M. Kapranov, and V. A. Voevodsky, *∞-groupoids and homotopy types*. Unfortunately, it contains an unsalvageable mistake. See also the story *The Origins and Motivations of Univalent Foundations* told by the author himself.
- *Conjectures in Grothendieck's "Pursuing stacks"*, `https://mathoverflow.net/q/115549/19222`.
- *Current status of Grothendieck's homotopy hypothesis and Whitehead's algebraic homotopy programme*, `https://mathoverflow.net/q/266738/19222`.

[14]Jacob Lurie has a short article, *What is an ∞-Category?*, which clearly explained the same ideals as of this section.

[15]Sometimes, it's also called a *simplicial category*, which may lead to some ambiguity, as it may be refer to a $\mathbf{\Delta}$-enriched category or a simplicial object in `Cat`, i.e., a functor $\mathbf{\Delta}^{\mathrm{opp}} \to$ `Cat`.

[16]For this reason, the simplex category can be also viewed as the category of such categories, i.e., viewing each totally ordered set as a category in the obvious way.

2.2. Definition. A *simplicial set*[17] is a functor $\mathscr{S} : \mathbf{\Delta}^{\mathrm{opp}} \to \mathtt{Set}$, i.e., a contra-variant functor from the simplex category $\mathbf{\Delta}$ to the category of sets. Or in other words, simplicial sets are presheaves on $\mathbf{\Delta}$. One usually write a simplicial set $\mathscr{S}$ as $S_\bullet$ and denote by S_n for the set $\mathscr{S}[n]$. A *map of simplicial sets* or a *simplicial map* is a natural transformation of functors. Thus we obtain a category $\mathtt{SSet}$ of simplicial sets and simplicial maps.

The simplicial set

$$\Delta^n := \Delta[n] := \mathrm{Hom}_{\mathbf{\Delta}}(-,[n]) := \mathbf{\Delta}^{\mathrm{opp}} \to \mathtt{Set},$$

represented by $[n]$, is called the *(standard simplicial) n-simplex*.[18] Yoneda lemma, applying to $\mathbf{\Delta} \hookrightarrow \mathtt{SSet}$, implies that we have a natural identification of the set of simplicial maps $\Delta^n \to \mathscr{S}$ and the set $S_n = \mathscr{S}[n]$. An element in this set is called an *n-simplex in $\mathscr{S}$*.

2.3. Proposition-Definition. Let $\mathscr{S}$ and $\mathscr{T}$ be two simplicial sets, then we have a simplicial set

$$\mathscr{S} \times \mathscr{T} : \mathbf{\Delta}^{\mathrm{opp}} \to \mathtt{Set},$$

assigning each $[n]$ the set $\mathscr{S}[n] \times \mathscr{T}[n]$, where the latter product is just the product of sets. Moreover, for each such a pair, we have a simplicial set[19]

$$\mathrm{Fun}(\mathscr{S},\mathscr{T}) := \mathrm{Hom}_{\mathtt{SSet}}(\mathscr{S} \times \Delta^{-},\mathscr{T}) : \mathbf{\Delta}^{\mathrm{opp}} \to \mathtt{Set},$$

sending $[n]$ to the set $\mathrm{Hom}_{\mathtt{SSet}}(\mathscr{S} \times \Delta^n,\mathscr{T})$ of simplicial maps. Clearly, we know the set $\mathrm{Fun}(\mathscr{S},\mathscr{T})[0]$ of 0-simplices in $\mathrm{Fun}(\mathscr{S},\mathscr{T})$ is just the Hom-set of the pair $(\mathscr{S},\mathscr{T})$ in the category $\mathtt{SSet}$. This makes $\mathtt{SSet}$ a category enriched in itself.[20]

2.4. Visualisation. For each $n \in \mathbb{N}$ and each $0 \le i \le n$, we have a map (only for $n \ge 1$) $\delta^i := \delta^i_n : [n-1] \to [n]$, called the *coface map*, which is the unique inject map without taking value i, and a map $\sigma^i := \sigma^i_n : [n+1] \to [n]$, called the *codegeneracy map*, which is the unique surjective map taking the value i twice. Formally,

$$\delta^i(k) = \begin{cases} k, & k < i. \\ k+1, & k \ge i. \end{cases}, \qquad \sigma^i(k) = \begin{cases} k, & k \le i, \\ k-1, & k > i. \end{cases}$$

For example, if $n = 3$, the coface maps can be represented by diagrams

$$\begin{tikzcd}[column sep=small] 1 \arrow[dr] \arrow[ddr] & \\ & 3 \\ 2 \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] 0 \arrow[rr] \arrow[dr] & & 3 \\ & 2 \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & 1 \arrow[ddr] \\ 0 \arrow[ur] \arrow[drr] & & \\ & & 2 \end{tikzcd}$$

$$\delta^0_3 \qquad\qquad \delta^1_3 \qquad\qquad \delta^2_3 \qquad\qquad \delta^3_3$$

If $n = 1$, the codegeneracy maps $\sigma^i_n : [n+1] \to [n]$ can be represented by diagrams

$$\begin{tikzcd}[column sep=small] & 0 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 1 \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 1 \end{tikzcd}$$

$$\sigma^0_1 \qquad\qquad \sigma^1_1$$

For any simplicial set $S_\bullet : \mathbf{\Delta}^{\mathrm{opp}} \to \mathtt{Set}$, maps $\delta^i_n : [n-1] \to [n]$ and $\sigma^i_n : [n+1] \to [n]$ induce maps $d_i := d^n_i : S_n \to S_{n-1}$ and $s_i := s^n_i : S_n \to S_{n+1}$ which are called the *face maps* and *degeneracy maps* respectively.[21]

[17] More generally, for any category $\mathscr{C}$, a *simplicial object* in $\mathscr{C}$ is a functor $\mathbf{\Delta}^{\mathrm{opp}} \to \mathscr{C}$. So a simplicial set is a simplicial object in $\mathtt{Set}$.

[18] In the lectures, the notation $\mathbf{\Delta}[n]$ was used for both the category $[n]$ and the simplicial set represented by $[n]$. And the latter was also denoted by $\mathrm{N}\mathbf{\Delta}[n]$. Observe also that the set $\mathrm{Hom}_{\mathbf{\Delta}}([m],[n])$ has $\binom{m+n+1}{n}$ elements.

[19] In literatures, it's also denoted by $\mathtt{SSet}(\mathscr{S},\mathscr{T})$, $\mathscr{T}^{\mathscr{S}}$ or $[\mathscr{S},\mathscr{T}]$. It's more often to use the notation $\mathrm{Fun}(\mathscr{S},\mathscr{T})$ if $\mathscr{T}$ is an ∞-category.

[20] The product $(- \times -)$ gives a *symmetric* monoidal structure on the category $\mathtt{SSet}$. This is actually a special case of the monoidal structure on the category of presheaves. The bifunctor $\mathrm{Fun}(-,-)$ is just an *internal*-Hom in this monoidal category.

[21] These maps satisfy the *simplicial identities*:

$$d_i d_j = d_{j-1} d_i, \quad i < j$$

$$s_i s_j = s_{j+1} s_i, \quad i \le j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i, & i < j, \\ 1 & i = j, j+1, \\ s_j d_{i-1} & \text{otherwise.} \end{cases}$$

Now take for example $n = 3$. Then an 3-simplex, which is an element in S_3, or equivalently, a functor $\Delta^3 \to S_\bullet$, can be understand as a graph

$$\begin{tikzcd} & x_1 \arrow[dr] \arrow[dd] & \\ x_0 \arrow[ur] \arrow[rr] \arrow[dr] & & x_3 \\ & x_2 \arrow[ur] & \end{tikzcd}$$

Then under the face map d^3_1, for example, this n-simplex goes to the 2-simplex represented by the graph

$$\begin{tikzcd} x_0 \arrow[rr] \arrow[dr] & & x_3 \\ & x_2 \arrow[ur] & \end{tikzcd}$$

which is exactly the "face" that is opposite to x_2. The vertices of this graphs are elements in S_0, i.e., 0-simplices.

Now it may be an *easy exercise* to understand the face and degeneracy maps for $\mathscr{S} \times \mathscr{T}$ and $\mathrm{Fun}(\mathscr{S}, \mathscr{T})$.

Not only an n-simplex $\Delta^n \to \mathscr{S}$, but more generally any simplicial map $\mathscr{I} \to \mathscr{S}$, or equivalently, any 0-simplex in $\mathrm{Fun}(\mathscr{I}, \mathscr{S})$, can be regarded as an *$\mathscr{S}$-indexed diagram*, or *$\mathscr{S}$-shaped diagram*, in $\mathscr{S}$. For example, a $\Delta^1 \times \Delta^1$ diagram is a square. All these intuition may be better understood if $\mathscr{S}$ is the *nerve* of a category.

2.5. Definition. Let $\mathscr{C}$ be a (small)[22] category and $n \geq 0$ an integer. An *$[n]$-commutative diagram*[23] in $\mathscr{C}$ is a functor $[n] \to \mathscr{C}$. Such a functor is described by its image in $\mathscr{C}$, which is in the usual sense a *commutative* diagram. For example, one usually depict them as

- $n = 0$, C_0
- $n = 1$, $C_0 \longrightarrow C_1$
- $n = 2$, $\begin{tikzcd} & C_1 \arrow[dr] & \\ C_0 \arrow[ur] \arrow[rr] & & C_2 \end{tikzcd}$ or $C_0 \to C_1 \to C_2$.

We denote the *set* of $[n]$-commutative diagrams by[24] $\mathrm{N}_n\mathscr{C}$, and call it the *n-nerve* of $\mathscr{C}$.

For any morphism $[m] \to [n]$ in Δ, we have a induced map $\mathrm{N}_n\mathscr{C} \to \mathrm{N}_m\mathscr{C}$ of sets, given by pre-composing any $\big([n] \to \mathscr{C}\big) \in \mathrm{N}_n\mathscr{C}$ with $[n] \to [m]$ so to get a map $\big([m] \to \mathscr{C}\big) \in \mathrm{N}_m\mathscr{C}$. In this way we obtain a simplicial set

$$\mathrm{N}\mathscr{C} := \mathrm{N}_\bullet\mathscr{C} : \Delta^{\mathrm{opp}} \to \mathtt{Set}, \quad [n] \to \mathrm{N}_n\mathscr{C}.$$

This is called the *(simplicial) nerve* of $\mathscr{C}$. Once again, Yoneda lemma implies that the set $\mathrm{N}_n\mathscr{C}$ is naturally identified with the set of maps $\Delta^n \to \mathrm{N}_\bullet\mathscr{C}$ of simplicial sets. In other words, $\mathrm{N}_n\mathscr{C}$ is the set of n-simplices in $\mathrm{N}\mathscr{C}$. Taking the nerve of a category is a functor

$$\mathrm{N} : \mathtt{SmallCat} \to \mathtt{SSet}.$$

2.6. Example. Let $n \in \mathbb{N}$ be a natural number. The nerve $\mathrm{N}[n]$ of the category $[n]$ is exactly the simplical set Δ^n represented by $[n]$.

2.7. Theorem. The category $\mathscr{C}$ ban be "reconstructed" (up to *isomorphism*) from the simplicial set $\mathrm{N}\mathscr{C}$.[25]

Proof. This is clear from definition of the nerve $\mathrm{N}\mathscr{C}$ of $\mathscr{C}$: $\mathrm{N}_0\mathscr{C}$ is the objects of $\mathscr{C}$. $\mathrm{N}_1\mathscr{C}$ is the morphism of C. The composition law in $\mathscr{C}$ can be recovered from $\mathrm{N}_2\mathscr{C}$.

2.8. Conversely, we can also describe exactly which simplicial sets come from the nerve of a category.

A simplicial set can be defined as a collection of sets S_n together with maps d_i and s_i satisfying the above relations.

[22] To get the *sets* $\mathrm{N}_n\mathscr{C}$, we need to require the category to be small, i.e., to have a *set* of objects.

[23] More generally, for any category $\mathscr{I}$, usually taken to be small or with finite or countable many objects, an *$\mathscr{I}$-shaped diagram in $\mathscr{C}$* is a functor $\mathscr{I} \to \mathscr{C}$. However, some authors use that convention that a $\mathscr{I}$-shaped diagram is a functor $\mathscr{I}^{\mathrm{opp}} \to \mathscr{C}$. This divergence of conventions may sometimes lead to confusions.

[24] In literatures, it is also denoted by $\mathrm{N}\mathscr{C}_n$.

[25] Actually, the nerve functor $\mathrm{N} : \mathtt{SmallCat} \to \mathtt{SSet}$ is fully faithful.

2.9. Definition. For an integer $n \geq 0$, and any $0 \leq i \leq n$, the (n,i)-*horn* Λ^n_i is the union of all *faces* of $\mathbf{\Delta}^n$ except the i-th one. Precisely, Λ^n_i is the *simplicial subset* (subfunctor)

$$\Lambda^n_i := \Lambda_i[n] : \mathbf{\Delta}^{\mathrm{opp}} \longrightarrow \mathtt{Set}$$

of $\mathbf{\Delta}^n$ sending $[m]$ to the subset of $\mathrm{Hom}_{\mathbf{\Delta}}([m],[n])$, consisting functors $f^k : [m] \to [n]$ that factors as

$$\begin{tikzcd} {[m]} \arrow[r, "f^k"] \arrow[d, dotted] & {[n]} \\ {[n-1]} \arrow[ur, "\delta^k"'] & \end{tikzcd}$$

for some $k \neq i$, where $\delta^k : [n-1] \to [n]$ is the coface map. An (n,i)-horn in a simplicial set $S_\bullet$ is a simplicial map $\Lambda^n_i \to S_\bullet$.

2.10. Visualization. Formally, an (n,i)-horn in $S_\bullet$ can be defined as a sequence of $(n-1)$-simplices $(e_j)_{0\leq j\leq n, j\neq i}$ of elements of S_{n-1}, which are "compatible" in the following sense:

$$d_j e_k = d_{k-1} e_j \in S_{n-2}, \qquad \forall\, 0 \leq j < k \leq n,\ j,k \neq i. \tag{3}$$

We have a restriction map

$$\begin{aligned} S_n \simeq \mathrm{Hom}_{SSet}(\mathbf{\Delta}^n, S_\bullet) &\longrightarrow \mathrm{Hom}_{SSet}(\Lambda^n_i, S_\bullet) \\ \tau &\longmapsto (d_j\tau)_{0\leq j\leq n, j\neq i} \end{aligned}$$

To build up intuition, let us take $n = 2$ as an example. A 2-simplex $s_2 : \mathbf{\Delta}^2 \to S_\bullet$ can be represented by a graph

$$s_2 : \begin{tikzcd} & x_1 \arrow[dr, "e_0"] & \\ x_0 \arrow[ur, "e_2"] \arrow[rr, "e_1"] & & x_2 \end{tikzcd}$$

And its three faces are represented by

$$\begin{tikzcd} x_1 \arrow[dr, "e_0"] & \\ & x_2 \end{tikzcd} \qquad \begin{tikzcd} x_0 \arrow[r, "e_1"] & x_2 \end{tikzcd} \qquad \begin{tikzcd} & x_1 \\ x_0 \arrow[ur, "e_2"] & \end{tikzcd}$$

$$d^2_0(s_2) \qquad\qquad d^2_1(s_2) \qquad\qquad d^2_2(s_2)$$

The $(2,i)$-horns, $i = 0,1,2$ can be represented as

$$\begin{tikzcd} & x_1 & \\ x_0 \arrow[ur, "e_2"] \arrow[rr, "e_1"] & & x_2 \end{tikzcd} \qquad \begin{tikzcd} & x_1 \arrow[dr, "e_0"] & \\ x_0 \arrow[ur, "e_2"] & & x_2 \end{tikzcd} \qquad \begin{tikzcd} & x_1 \arrow[dr, "e_0"] & \\ x_0 \arrow[rr, "e_1"] & & x_2 \end{tikzcd}$$

$$\begin{array}{ccc} \Lambda^2_0 \to S_\bullet & \Lambda^2_1 \to S_\bullet & \Lambda^2_1 \to S_\bullet \\ d_1e_2 = d_1e_1 & d_0e_2 = d_1e_0 & d_0e_1 = d_0e_0 \end{array}$$

2.11. Theorem. A simpicial set $S_\bullet$ is equal to the nerve of a category $\mathscr{C}$, if and only if every *inner horn* can be filled *uniquely* to a simplex, precisely, if and only if for all $n \in \mathbb{N}$ and $0 < i < n$, any simplicial map $\Lambda^n_i \to S_\bullet$ factors uniquely through $\Lambda^n_i \hookrightarrow \mathbf{\Delta}^n$:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & S_\bullet \\ \mathbf{\Delta}^n \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

Proof. Let $\mathscr{C}$ be the small category with *objects* elements of S_0 and *morphisms* objects on S_1. This is well defined.

For each morphism $f \in S_1$, $d^1_0(f) \in S_0$ and $d^1_1(f)$ are the *source* and *target* of f respectively. For each pair of morphisms $f, g \in S_1$ with $d^1_1(f) = s^1_0(g)$, we get a $(1,1)$-horn (recall the description (3))

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] & & x_2 \end{tikzcd}$$

in $S_\bullet$, hence there is a unique element $\sigma_{f,g} \in S_2$ restricting to this horn. Then the 1-st face $d^2_1(\sigma_{f,g}) \in S_1$ of $\sigma_{f,g}$ is the composition $g \circ f$ of f and g. The associativity of the composition law can be checked using the filling property for $\Lambda^3_i \to S_\bullet$.

2.12. Definition. A *quasi-category*, a *weak Kan complex*[26] or an $(\infty, 1)$-*cateogry* (henceforth an ∞-*category*) is a simplicial set $\mathscr{C}$ such that any *inner horn* can be filled (not necessarily uniquely) to a simplex:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \mathscr{C} \\ \Delta^n \arrow[ur, dotted, "\exists"'] & \end{tikzcd} \qquad \forall n \in \mathbb{N}, 0 < i < n.$$

A *functor* between ∞-categories is just a simplicial map, i.e., a natural transformation of functors.

2.13. Remark. One can think of an ∞-category $\mathscr{C} = C_\bullet$ as follows. Regard C_0 as a set of "objects" and C_1 as a set of "arrows" between the objects. Then given $f, g \in C_1$ such that $d^1_1(f) = d^1_0(g)$, i.e., a $(1,1)$-horn:

$$\begin{tikzcd} & y \arrow[dr] & \\ x \arrow[ur, "f"] & & z \end{tikzcd} \tag{4}$$

Then there exists a (non-unique) 2-simplex σ,

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\sigma"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & {} & z \end{tikzcd} \tag{5}$$

regarded as a "commutative diagram", filling the $(1,1)$-horn. Then we obtain a third arrow $h := d_1(\sigma) \in C_1$, of which we think it as a *candidate* of the "composition" $g \circ f$. But such an h is not unique. In other words, the "composition" of morphisms in an ∞-category is not unique. In contrast, in a *category* (in the usual sense), the composition $h = f \circ g$ is uniquely defined.

With this intuition, one usually refer to 0-simplices (resp. 1-simplices) as *objects* or *vertices* (resp. *arrows* or *morphisms*) of an ∞-category.

2.14. Proposition. Let $\mathscr{S}$ and $\mathscr{C}$ be two simplical sets. Then [27]

- The simplicial set $\mathscr{S} \times \mathscr{C}$ is an ∞-category if $\mathscr{S}$ and $\mathscr{C}$ are both ∞-cateogries.
- The simplicial set $\mathrm{Fun}(\mathscr{S}, \mathscr{C})$ is an ∞-category if $\mathscr{C}$ is an ∞-category.

[26] In contrast, a *Kan complex* is a simplicial set $\mathscr{C}$ such that *every* (n,i)-horn can be filled to an n-simplex in $\mathscr{C}$. As an example, the nerve $\mathrm{N}\mathscr{C}$ of a category $\mathscr{C}$ is a Kan complex if and only if $\mathscr{C}$ is a groupoïd.

[27] Besides these, there are still many other useful constructions on simplicial sets, that produce ∞-categories out of ∞-categories. For example, if $\mathscr{S}$ and $\mathscr{T}$ are two simpliciial sets, then

- the *join* $\mathscr{S} \star \mathscr{T}$, ... (Note that the join is associative but not symmetric.)
- the *left cone* (or *cone*) $\mathscr{S}^{\triangleleft} := \Delta^0 \star \mathscr{S}$, and
- the *right cone* (or *cocone*) $\mathscr{S}^{\triangleright} := \mathscr{S} \star \Delta^0$,

are all ∞-categories if $\mathscr{S}$ and $\mathscr{T}$ are. Actually, these constructions are generlisations of that for ordinary categories. Finally, to see some examples, we have $\Delta^m \star \Delta^n \simeq \Delta^{m+1+n}$, and in particular, $\Delta^{n+1} \simeq (\Delta^n)^{\triangleright} \simeq (\Delta^n)^{\triangleleft}$ as simplicial sets.

We also have more generally comma category (2-limit) and cocomma category (2-colimt). recall also `http://www.math.harvard.edu/~amathew/notesHTT.pdf` page 10.

2.15. Proposition-Definition. For any ∞-category $\mathscr{C}$, there is a (small) category $\mathrm{Ho}(\mathscr{C})$, together with a simplicial map $\mathscr{C} \to \mathrm{N}\,\mathrm{Ho}(\mathscr{C})$, which is universal in the sense that for any (small) category $\mathscr{D}$ and any simplicial map $\mathscr{C} \to \mathrm{N}\mathscr{D}$, there is a unique functor $\mathrm{Ho}(\mathscr{C}) \to \mathscr{D}$, making the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[dr] & \mathrm{N}\,\mathrm{Ho}(\mathscr{C}) \arrow[d, dotted, "\exists!"] \\ & \mathrm{N}\mathscr{D} \end{tikzcd}$$

In other words, Ho is the *left adjoint* functor to the nerve functor N:

$$\mathrm{Ho} \dashv \mathrm{N} : \texttt{SmallCat} \to \infty\text{-}\texttt{Cat}.$$

The category $\mathrm{Ho}(\mathscr{C})$ is called the *homotopy category*[28] of $\mathscr{C}$. It exists uniquely up to equivalence of categories.

The category $\mathrm{Ho}(\mathscr{C})$ can be constructed explicitly. Write C_n be the set of n-simplices in $\mathscr{C}$. We say that two morphisms (1-simplices) $f, g \in C_1$ are *homotopic*, written as $f \simeq g$, if

- $d_i(f) = d_i(g) := x_i \in C_0$, $i = 1, 2$, and
- there is a 2-simplex $\sigma \in C_2$, such that $d_0(\sigma) = s_0(x_1) =: \mathrm{id}_{x_1}$, $d_1(\sigma) = f$ and $d_2(\sigma) = g$.[29] We can depict it as

$$\begin{tikzcd} & x_1 \arrow[dr, "\mathrm{id}_{x_1}"] \arrow[d, phantom, "\sigma"] & \\ x_0 \arrow[ur, "g"] \arrow[rr, "f"'] & {} & x_1. \end{tikzcd}$$

For any two morphisms (1-simplices) f and g, their *composition* h as described in (5), though not unique, but is unique *up to homotopy*. In fact, suppose we have two 2-simplices σ and σ' filling the $(1,1)$-horn (4) such that $h = d_1(\sigma)$ and $h' = d_1(\sigma')$ are two candidates for the "composition" $g \circ f$. We consider the $(3,1)$-horn given by $s_1(g)$, σ and σ':

$$\begin{tikzcd} & y \arrow[drr, "g"] \arrow[ddr, "g"] & & \\ x \arrow[ur, "f"] \arrow[rrr, "h"] \arrow[dr, "h'"'] & & & z \\ & & z \arrow[ur, "\mathrm{id}_z"'] & \end{tikzcd}$$

Then there is a 3-simplex τ filling this horn such that its 1-st face $d_1(\tau) \in C_2$:

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "h'"'] & & z \\ & z \arrow[ur, "\mathrm{id}_z"'] & \end{tikzcd}$$

gives the homotopy between h and h'. One can, using the same method, show that the being homotopic is an equivalence relation on the set C_1. So we obtain a category with objects elements of C_0 and morphisms homotopy equivalence classes in C_1. This is the homotopy category $\mathrm{Ho}(\mathscr{C})$.

2.16. Definition. A morphism (1-simplex) $f : x \to y$ in an ∞-cateogry $\mathscr{C}$ is an *equivalence* if becomes an isomorphism in the homotopy category $\mathrm{Ho}(\mathscr{C})$. Two objects (0-simplices) x and y are *equivalent* if there is an equivalence between them, in other words, if they are isomorphic as objects of $\mathrm{Ho}(\mathscr{C})$.

[28] A more general notion of homotopy category of a simplicial set $\mathscr{C}$ exists, which is sometimes denoted by $\tau_1(\mathscr{C})$. Moreover, $\tau_1 \dashv \mathrm{N} : \texttt{SmallCat} \to \texttt{SSet}$ is an adjoint pair.

[29] There is an equivalent definition, saying that $f \simeq g$ if there is a $\sigma' \in C_2$, depicted as

$$\begin{tikzcd} & x_0 \arrow[dr, "g"] \arrow[d, phantom, "\sigma'"] & \\ x_0 \arrow[ur, "\mathrm{id}_{x_0}"] \arrow[rr, "f"'] & {} & x_1 \end{tikzcd}$$

where $\mathrm{id}_{x_0} := s_0(x_0)$.

2.17. Warning. Many properties, e.g., a morphism being an equivalence and a functor being essentially surjective (would be defined below), can be detected at the level of homotopy categories. However, many properties cannot be checked at the level of homotopy categories.

- Commutative diagrams in $\mathrm{Ho}(\mathscr{C})$ do not always lift with exceptions of (triangle, square diagrams).
- One cannot just construct functors between ∞-categories by describing them at the level of objects and morphisms.

2.18. Proposition-Definition. Given an ∞-category $\mathscr{C}$, and a pair of 0-simplices x and y, there is a *topological space* $\mathrm{Map}(x, y) := \mathrm{Map}_{\mathscr{C}}(x, y)$, called the *mapping space*, such that

$$\pi_0(\mathrm{Map}(x, y)) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(x, y).$$

2.19. Remark. A lot of properties of an ∞-category $\mathscr{C}$ are encoded in $\mathrm{Map}(x, y)$.

- $\mathscr{C}$ is the nerve of a category if and only if $\mathrm{Map}(x, y)$ are discrete topological spaces.
- $\mathscr{C}$ is equivalent to $\mathrm{N}\,\mathrm{Ho}(\mathscr{C})$ if and only if $\pi_i(\mathrm{Map}(x, y)) = 0$ for all $i > 0$.

2.20. Definition. A functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ of ∞-categories is

- *essentially surjective*, if $\mathrm{Ho}(F) : \mathrm{Ho}(\mathscr{C}) \longrightarrow \mathrm{Ho}(\mathscr{D})$ is so.
- *fully faithful* if for all pair x, y, $\mathrm{Map}_{\mathscr{C}}(x, y) \longrightarrow \mathrm{Map}_{\mathscr{D}}(F(x), F(y))$ is a weak homotopy equivalence,[30] and
- an *equivalence* if it is fully faithful and essentially surjective.

2.21. Definition. An object (0-simplex) x in an ∞-category $\mathscr{C}$ is

- *final*, if for all objects y in $\mathscr{C}$, the topological space $\mathrm{Map}(y, x)$ is *weakly contractible*, i.e., $\mathrm{Map}(x, y)$ is weakly homotopy equivalent to a single point, or more concretely speaking, if $\mathrm{Map}(x, y)$ is path connected and all homotopy groups are trivial.[31]
- *initial*, if for all objects y in $\mathscr{C}$, the topological space $\mathrm{Map}(x, y)$ is *weakly contractible*.
- an *zero object*, if it is both initial and final. An ∞-category with a zero object is called a *pointed ∞-category*.

2.22. Example. An object x of an ordinary cateogry $\mathscr{C}$ is final (resp. initial) if and only if x is final (resp. initial) in $\mathrm{N}\mathscr{C}$.

2.23. As we can interpret commutative square diagrams in an ordinary category $\mathscr{C}$ as functors $[1] \times [1] \longrightarrow \mathscr{C}$, a square diagram in an ∞-category is a simplicial map $\Delta^1 \times \Delta^1 \longrightarrow \mathscr{C}$, or in other words, a 0-simplex in the ∞-category $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$. Note that Λ^2_2 is a simplicial subset of $\Delta^1 \times \Delta^1$.[32] Intuitively, this is analogous to the fact that

$$\mathscr{J} := \begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array} \quad \text{is a subcategory of} \quad \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array} (\simeq [1] \times [1])$$

Moreover, to give two morphisms $f : x \longrightarrow z$, $g : y \longrightarrow z$ in an ordinary category $\mathscr{C}$, is the same to give a functor $p : \mathscr{J} \longrightarrow \mathscr{C}$. The fibre product of f and g is the *terminal object* in the subcategory of $\mathrm{Fun}([1] \times [1], \mathscr{C})$ consisting of commutative squares that restrict to $p : \mathscr{J} \longrightarrow \mathscr{C}$ via $\mathscr{J} \hookrightarrow [1] \times [1]$. Now, the analogue in ∞-categories of fibre products goes as follows.

---

[30] Recall that a continuous map $f : X \longrightarrow Y$ of topological spaces is

- a *weak homotopy equivalence*, or *weak equivalence* if the induced map $\pi_0(f) : \pi_0(X) \longrightarrow \pi_0(Y)$ between the sets of path connected components is bijective, and for every $x \in X$ and every $1 \leq i \in \mathbb{N}$, the induced map $\pi_i(f) : \pi_i(X, x) \longrightarrow \pi_i(Y, f(x))$ on homotopy groups is bijective.
- a *homotopy equivalence*, there is a continuous map $g : Y \longrightarrow X$, and *homotopies* $(f \circ g) \sim \mathrm{id}_Y$ and $(g \circ f) \sim \mathrm{id}_X$.

Any homotopy equivalence is also a weak homotopy equivalence. Conversely, the classical *Whitehead theorem* states that, any weak homotopy equivalence between CW complexes is a homotopy equivalence.

[31] In an ordinary category, the any two final objects are uniquely isomorphic to one another. But in the setting of ∞-categories, it's more complicated to state a similar result. See Higher Topos Theory, Proposition 1.2.12.9.

[32] It's a nice example to see that there are different ways to describe the simplicial set $\Delta^1 \times \Delta^1$.

- $\Delta^1 \times \Delta^1 \simeq (\Lambda^2_0)^{\triangleright} \simeq (\Lambda^2_2)^{\triangleleft}$.
- $\Delta^1 \times \Delta^1 \simeq \mathrm{N}([1] \times [1])$
- It's a simplicial subset of Δ^3.

2.24. Definition. To give two morphisms (1-simplices) $f : x \to z$ and $g : y \to z$ in an ∞-category $\mathscr{C}$ is the same as to give a simplicial map $p : \Lambda^2_2 \to \mathscr{C}$, which we depict as

$$\begin{array}{ccc} & & y \\ & & \downarrow{\scriptstyle g} \\ x & \xrightarrow{f} & z. \end{array} \tag{6}$$

We define the *fibre product* of f and g to be the *terminal object* in the ∞-subcategory[33] (simplicial subset) of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$ with objects (0-simplices) square diagrams

$$\begin{array}{ccc} w & \longrightarrow & y \\ \downarrow & & \downarrow{\scriptstyle g} \\ x & \xrightarrow{f} & z \end{array}$$

that restricts via $\Lambda^2_2 \hookrightarrow \Delta^1 \times \Delta^1$ to the map (6). Such a square diagram is called a *pullback square*.

2.25. Remark. Similarly, we can define the notion of *pushout square*, or even more generally, a *limit* and a *colimit* of a *diagram* $p : \mathscr{S} \to \mathscr{C}$ for some simplicial set $\mathscr{C}$.

2.26. Sources of ∞-categories.

- *Localization at weak equivalences.*
  Let $\mathscr{C}$ be a category and W be a subcategoy having all objects of $\mathscr{C}$ (morphisms are *weak equivalences*[34]) to be inverted. There exists an ∞-category $\mathscr{C}[W^{-1}]$ and a functor $F : \mathscr{C} \to \mathscr{C}[W^{-1}]$, such that $F(W)$ is send to equivalences and F is the universal such functor.[35]
  For instance, let $\mathscr{A}$ be an abelian category, we define
  $$\mathscr{D}(\mathscr{A}) := \mathrm{Ch}(\mathscr{A})[\{\text{q-iso.}\}^{-1}], \qquad \mathrm{Ho}(\mathscr{D}(\mathscr{A})) = \mathcal{D}(\mathscr{A}),$$
  where $\mathrm{Ch}(\mathscr{A})$ is the category of chain complexes in $\mathscr{A}$, and $\mathcal{D}(\mathscr{A})$ is the *derived category* of $\mathscr{A}$.
- *Categories enriched in the category* `Top` *of topological spaces.*
  If $\mathscr{C}$ is such a category, then we can associate an ∞-category $\mathrm{N_{top}}\mathscr{C}$, called the *topological nerve* of $\mathscr{C}$, such that $\mathrm{Ho}(\mathrm{N_{top}}\mathscr{C}) = \mathrm{Ho}(\mathscr{C})$. Recall that for a category $\mathscr{C}$ enriched in `Top`, its *homotopy category* $\mathrm{Ho}(\mathscr{C})$ has objects those of $\mathscr{C}$ and $\mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(x, y) = \pi_0\big(\mathrm{Map}_{\mathscr{C}}(x, y)\big)$ for each pair of objects x and y.
- Categories enriched in $\mathrm{Ch}(\mathbb{Z})$.
  Such a category is called a *differential graded category*, or a *dg-category*. To such a category $\mathscr{C}$, we can associated an ∞-category $\mathrm{N_{dg}}(\mathscr{C})$, called the *differential graded nerve* or *dg-nerve* for short.

---

[33]In the lecture, this ∞-category was not precisely defined. This ∞-category is called the *over ∞-category* of $\mathscr{C}$ over p, and its denoted by $\mathscr{C}_{/p}$. It's defined as follows. Suppose $p : \mathscr{K} \to \mathscr{C}$ be any simplicial map into an ∞-category $\mathscr{C}$, then (see also `https://math.stackexchange.com/q/2413228/19690`)

$$\mathscr{C}_{/p}[n] := \mathrm{Hom}_{\mathtt{sSet},p}(\Delta^n \star \mathscr{K}, \mathscr{C}) \subseteq \mathrm{Hom}_{\mathtt{sSet}}(\Delta^n \star \mathscr{K}, \mathscr{C}) \subseteq \mathrm{Hom}_{\mathtt{sSet}}(\Delta^n \times \mathscr{K}^{\triangleleft}, \mathscr{C})$$

where $\mathrm{Hom}_{\mathtt{sSet},p}$ denotes the set of simplicial maps that restrict to p via the natural map $\mathscr{K} \to \Delta^n \star \mathscr{K}$. Similarly, the *under ∞-category* $\mathscr{C}_{p/}$ is defined as $\mathscr{C}_{p/}[n] := \mathrm{Hom}_{\mathtt{sSet},p}(\mathscr{K} \star \Delta^n, \mathscr{C})$.

[34]Morphisms in W are required to satisfy the following conditions.

- all isomorphisms are in W, and
- (*2-out-of-6 property*) for any three composable morphisms f, g and h in $\mathscr{C}$, if $g \circ f$ and $h \circ g$ are in W, then so are f, g, h and $h \circ g \circ f$. This property implies the following one.
- (*2-out-of-3 property*) for any two composable morphisms f and g in $\mathscr{C}$, if two of f, g and $g \circ f$ are in W, then so is the third.

Such a category $\mathscr{C}$ together with the weak equivalences are is called a *homotopical category*. Without the above conditions, such a pair $(\mathscr{C}, W)$ is sometimes called a relative category.

[35]I didn't find a reference to a simple construction of such a localization, except a brief description at `https://ncatlab.org/nlab/show/(infinity,1)-category#homotopical_categories`. See also `https://math.stackexchange.com/q/809608/19690`. One possible reference is

- J. Lurie, *Higher Algebra*, §1.3.4 Inverting Quasi-Isomorphisms.

Naively, one can formally invert all weak equivalence to get an ordinary category such that weak equivalences becomes isomorphisms. But limits and colimits behave poorly in such a naive localization. Then one turn to the notion of a *model category*. But in some sense, choosing a model category structure to study homotopy theory is like to choose a basis to study vector spaces. There is another related concept called the *Dwyer-Kan simplicial localization*. A lovely note is

- Arun Debray, Summer 2016 homotopy theory seminar.

2.27. Remark. Every ∞-category up to equivalence arises from the first two examples, but not the last one.[36]

2.28. Exercise. Show that the ∞-category of spaces, i.e., topological spaces modulo weak equivalence, is not a dg-nerve.

3 Stable infinity categories

> It has long been recognized that for many purposes the derived category is too crude: it identifies homotopic morphisms of chain complexes without remembering *why* they are homotopic. It is possible to correct this defect by viewing the derived category as the homotopy category of an underlying ∞-category $\mathcal{D}(\mathscr{A})$. The ∞-categories which arise in this way have special features that reflect their "additive" origins: they are *stable*.
>
> J. Lurie, *Higher Algebra*, Chapter 1.

3.1. Definition. An ∞-category $\mathscr{C}$ is *stable* if the following axioms are satisfied.

- $\mathscr{C}$ is *pointed*, i.e., it has a *zero object* 0.
- Every morphism $f : Y \to Z$ (resp. $g : X \to Y$) fits into a *pullback* (resp. pushout) diagram[37]

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & Y \arrow[d, "f"] \\ 0 \arrow[r] & Z. \end{tikzcd}$$

- A square diagram is a pullback if and only if it is a pushout.[38]

3.2. Theorem. There is a canonical *triangulated category structure*[39] on the homotopy category $\mathrm{Ho}(\mathscr{C})$ of a *stable* ∞-category $\mathscr{C}$.

[36]There are many other ways to define an ∞-cateogry, such as *simplicially enriched categories*, *Segal categories*, *complete Segal spaces*.

[37]The object X is called the *kernel* of f if the diagram is a pullback diagram. The object Z is called the *cokernel* of g if the diagram is a pushout. So this condition is requiring that *every morphism has kernel and cokernel*.

[38]This condition is the same as that *every morphism is the cokernel of its kernel and the kernel of its cokernel.*

[39]Recall that a *triangulated category* (Jean-Louis Verdier) is the following data:

- an *additive* category $\mathscr{C}$, i.e., an $\mathtt{Ab}$-enriched category admitting finite coprodcuts, where $\mathtt{Ab}$ is the category of abelian groups.
- a *shift/translation functor* $\mathscr{C} \to \mathscr{C}$, $X \mapsto X[1]$, which is additive and an equivalence,
- a collection of *distinguished triangles* $X \to Y \to Z \to X[1]$.

satisfying the following axioms:

(T1) Every morphism $f : X \to Y$ in $\mathscr{C}$ can be extended to a distinguished triangle.
(T2) The collection of distinguished triangles is stable under isomorphism.
(T3) For any object X in $\mathscr{C}$,

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$$

is a distinguished triangle.
(T4) A diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle if and only if the induced diagram

$$Y \xrightarrow{g} Z \xrightarrow{h} X \xrightarrow{-f[1]} Y[1]$$

is a distinguished triangle.
(T5) For any commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\alpha"] & Y \arrow[r] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, dotted, "\exists\gamma"] & X[1] \arrow[d, "\alpha[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

such that the two rows are distinguished triangles, there is a dotted arrow making the entire diagram commutative.
(T6) (Octahedral axiom) Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{u} Y/X \longrightarrow X[1],$$
$$Y \xrightarrow{g} Z \xrightarrow{v} Z/Y \longrightarrow Y[1],$$
$$X \xrightarrow{g\circ f} Z \xrightarrow{w} Z/X \longrightarrow X[1],$$

Ideas of Proof. See

- Jacob Lurie, *Higher Algebra*, §1.1.2.
- Alberto García-Raboso, *Stable ∞-categories*, url: http://www.math.toronto.edu/agraboso/files/stableInfCat.pdf.

We need to define a *shift/translation functor* $\mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{C})$, $x \mapsto x[1]$. In general, for any ∞-category (not necessarily stable), we have functors $\Sigma, \Omega : \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{C})$, called the *suspension functor* and *loop functor* respectively, characterized by that for any object x in $\mathscr{C}$, the square diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma x \end{tikzcd} \quad \text{and resp.} \quad \begin{tikzcd} \Omega x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & x \end{tikzcd}$$

is a *pushout* and respectively, a *pullback*. A distinguished triangle $x \to y \to z \to \Sigma x$ is the same thing as a square diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ 0 \arrow[r] & z \end{tikzcd}$$

which is simultaneously a pullback and a pushout. Moreover, for all $i \geq 0$,

$$\pi_i(\mathrm{Map}_{\mathscr{C}}(x, y)) = \mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(\Sigma^i x, y). \tag{7}$$

3.3. Definition. A functor $\mathscr{C} \to \mathscr{D}$ of stable ∞-categories is *stable* if it preserves pullbacks, equivalently, if it preserves pushouts.[40]

3.4. Proposition. Let $F : \mathscr{C} \to \mathscr{D}$ be an *exact* functor between stable ∞-categories. Then

- F is fully faithful if and only if $\mathrm{Ho}(F)$ is fully faithful.
- F is equivalence if and only if $\mathrm{Ho}(F)$ is an equivalence.

Proof. That F being stale is the same as $\mathrm{Ho}(F)(\Sigma x) = \Sigma\,\mathrm{Ho}(F)(x)$. Then for each x, y, the map $\mathrm{Ho}(F)(x, y)$

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(\Sigma^i x, y) \longrightarrow\ & \mathrm{Hom}_{\mathrm{Ho}(\mathscr{D})}\left(\mathrm{Ho}(F)(\Sigma^i x), \mathrm{Ho}(F)(y)\right) \\ \simeq\ & \mathrm{Hom}_{\mathrm{Ho}(\mathscr{D})}\left(\Sigma^i(\mathrm{Ho}(F)(x)), \mathrm{Ho}(F)(y)\right) \end{aligned}$$

is the same as

$$\pi_i(\mathrm{Map}_{\mathscr{C}}(x, y)) \to \pi_i(\mathrm{Map}_{\mathscr{D}}(F(x), F(y)))$$

by eq. (7). The seconed statement follows from the first and §2.20.

3.5. Proposition. Let $\mathscr{C}$ be a stable ∞-category, and $\mathscr{S}$ be any simplicial set. Then the ∞-category $\mathrm{Fun}(\mathscr{S}, \mathscr{C})$ (§2.14) is stable.

---

there exists a fourth distinguished triangle

$$Y/X \to Z/X \to Z/Y \to (Y/X)[1]$$

completing the commutative diagram

$$\begin{tikzcd}[column sep=small] X \arrow[dr] \arrow[rr, bend left] & & Z \arrow[dr] \arrow[rr, bend left] & & Z/Y \arrow[dr] \arrow[rr, bend left] & & (Y/X)[1]. \\ & Y \arrow[ur] \arrow[dr] & & Z/X \arrow[ur] \arrow[dr] & & Y[1] \arrow[ur] & \\ & & Y/X \arrow[ur] \arrow[rr, bend right] & & X[1] \arrow[ur] & & \end{tikzcd}$$

Actually, the above axiom (T5) is redundant, see

- J. P. May, *The axioms for triangulated categories*, url:http://www.math.uchicago.edu/~may/MISC/Triangulate.pdf.

for a proof.

[40] General (left and right) exactness of functors between ∞-categories (not necessarily stable) is a little bit involved to define.

3.6. There is an ∞-category $\mathtt{Cat}_\infty$ of ∞-categories,[41] and an ∞-subcategory $\mathtt{Cat}_\infty^{\mathrm{Ex}}$ of $\mathtt{Cat}_\infty$ that has objects stable ∞-categories and morphisms exact functors.

3.7. Proposition. Stable ∞-categories are closed under *limits* (in $\mathtt{Cat}_\infty^{\mathrm{Ex}}$),[42] but *not* closed under arbitrary *colimits*.

3.8. Definition. Let F be a pre-sheaf (or prestack,). We define

$$\mathscr{D}_{\mathrm{qcoh}}(F) = \lim_{\operatorname{Spec} A \to F} \mathscr{D}_{\mathrm{qcoh}}(\operatorname{Spec} A)$$

where $\mathscr{D}_{\mathrm{qcoh}}(\operatorname{Spec} A) = \mathscr{D}(\mathtt{Mod}_A)$ is the *derived ∞-category*[43] of the abelian category $\mathtt{Mod}_A$. $\mathscr{D}(\mathtt{Mod}_A)$ is stable hence $\mathscr{D}_{\mathrm{qcoh}}(F)$.

3.9. Example. Let X be smooth variety over a field k of characteristic 0. Define a presheaf

$$X_{\mathrm{dR}} : \operatorname{Spec} A \mapsto X(\operatorname{Spec} A^{\mathrm{red}})$$

Then $\operatorname{Ho}(\mathscr{D}_{\mathrm{qcoh}}(X_{\mathrm{dR}})) = \mathcal{D}(\mathtt{Mod}(D_X))$, where D_X is the sheaf of differential operators[44] on X, and $\mathcal{D}$ means the derived category.

---

41 It is defined as a *simplicial nerve* (different from the nerve of an ordinary category) of a simplicial category (see footnote 6) $\mathtt{Cat}_\infty^{\Delta}$ with objects ∞-categories and $\operatorname{Hom}_{\mathtt{Cat}_\infty^{\Delta}}(\mathscr{C}, \mathscr{D})$ the maximal Kan complex (see footnote 26) of $\operatorname{Fun}(\mathscr{C}, \mathscr{D})$. This is the "correct" category to work within, rather than the ordinary category of ∞-categories, which is a subcategory of $\mathtt{SSet}$.

42 This provides a tool for addressing the classical problem of "gluing in the derived category".

43 See Lurie, *Higher Algebra*, §1.3.2 Derived ∞-Categories and §1.3.4 Inverting Quasi-Isomorphisms.

44 Recall that D_X is defined as follows. Let $\Delta : X \to X \times_k X$ be the diagonal morphism and $\mathcal{I}$ be the kernel of $\Delta^{-1}\mathcal{O}_{X\times_k X} \twoheadrightarrow \mathcal{O}_X$. Then the sheaf $\mathcal{P}^n_{X/k} := \Delta^{-1}\mathcal{O}_{X\times_k X}/\mathcal{I}^{n+1}$ has two $\mathcal{O}_X$-module structures induced by the two projections $p_i : X \times_k X \to X$, $i = 1, 2$. Then define $D^{(n)}_{X/k} = \mathcal{H}om\,(\mathcal{P}^n_{X/k}, \mathcal{O}_X)$, and $D_X = D_{X/k} = \bigcup D^{(n)}_{X/k}$.