Comparison Theorem

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This is the note of my talk for the seminar "p-adic Hodge theory" at Freie Universität Berlin in the summer semester 2017.

Warning It is FULL OF TYPOS in [Ols09, §13–14].

WARNING It is FULL OF MISTATES in my note as I did not fully understand these materials.

1 Statement of the Main Theorem

1.1. Notations and conventions.

V, k, K	complete discrete valuation ring V with residue field k and fraction field K.
$W = W(k)$, K_0	ring of Witt vectors of k, $W \subseteq V$, with fraction field K_0.
$\bar{V} \subseteq \bar{K}$	integral closure of V inside a fixed algebraic closure of K.
X/V	smooth proper scheme.
$D \subset X$	divisor with relative normal crossings.
X^0	$X - D$.
(X, M_X)	the log scheme with log-structure M_X defined by D.
$\mathrm{Et}(X)$, $X_{\text{ét}}$	(small) étale site, topos.
$\mathrm{Crys}(X/?)$, X_{crys}	the crystalline(-étale) site,[1] topos.
$\mathrm{Conv}(X/?)$, X_{conv}	convergent site, topos.
$\mathrm{MF}_X^{\nabla}(\Phi)$	filtered F-isocrystals (satisfying Griffiths transersality)[2]

1.2. What is our goal?.

We are in the following situation.

$$\begin{tikzcd}
X_k \arrow[r] \arrow[d] & X \arrow[d] & X_K \arrow[l] \arrow[d] & X_{\bar{K}} \arrow[l] \arrow[d] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} V & \operatorname{Spec} K \arrow[l] & \operatorname{Spec} \bar{K} \arrow[l]
\end{tikzcd}$$

The geometry we really want to understand is that X_K/K (or the open variety X_K^0). In our seminar, we focus on the case X_K has a good reduction. So we just start with a model X/V of X_K/K. The ultimate goal is to compare the algebraic de Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(X_K)$ of X_K/K and the etale cohomology $\mathrm{H}^*_{\text{ét}}(X_{\bar{K}})$ of $X_{\bar{K}}/\bar{K}$. We already have a comparison[3] between $\mathrm{H}^*_{\mathrm{dR}}(X_K/K)$ and the crystalline cohomology $\mathrm{H}^*_{\mathrm{crys}}(X_k/W)$ of the special fibre X_k/k. Now we are trying to relate $\mathrm{H}^*_{\mathrm{crys}}(X_k/k)$ and $\mathrm{H}^*_{\text{ét}}(X_{\bar{K}}/K)$.

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[*]Updates see `https://haoyun.github.io/files/17SS-SS-Comparison.pdf`

[1]This is just called the crystalline site in [Ols09, §14.1] and [Kat89, §5.2], which étale $U \to X$. But it is different form the usual crystalline site, which uses open $U \to X$. It is, in some sense, more reasonable to call it the *crystalline-étale site* (see e.g., [Kat90, Rmk 2.2.4.2]), compared with the so-called *lisse-étale* site. Some people also call it the etale-crystalline site, e.g., [Ols07, §1.3.2] and [BD, §7.10.18]. There are people call this topology the *étale topology on the crystalline site*, but this is not a good name, as a *site* is already equipped with a (pre-)topology by definition. Similarly, the convergent site should better be called the convergent-étale site, e.g., in [Shi02, p. 2.1.3], (a log version of) this is called the convergent site with respect to the étale topology.

[2]See [Fal89, §II] and [Tsu, §1] for more details. Here MF stands for "'modules filtrés", ∇ is a flat connection. Φ is a Frobenius lifting. See also [Ols09, §13.11].

[3]We have seen this last semester's seminar.

1.3. Roughly speaking, an isocrystal is a *crystal up to isogeney*, i.e., a (coherent) crystal in $\mathscr{K} := \mathscr{O} \otimes \mathbb{Q}$-modules, where $\mathscr{O}$ is the structure sheaf of the crystalline (resp. convergent) site,[4] sending (U, T) to $\Gamma(T, \mathscr{O}_T)$. A F-crystal $\mathscr{F}$ (resp. F-isocrystal) is an crystal (resp. isocrystal) with an isomorphism $\phi_{\mathscr{F}} : \sigma^*\mathscr{F} \to \mathscr{F}$, where σ is the Frobenius. On the convergent site, we will use the term *convergent* F-isocrystal.[5]

Let X_k be the special fibre of X. It has an induced log structure M_{X_k}. Denote by $((X, M_{X_k})/W)_{\mathrm{conv}}$ and $((X, M_{X_k})/W)_{\mathrm{crys}}$ be the corresponding crystalline topos. Sometimes, we suppress in our nations the log structure.

There is a natural morphism of sites[6] $\mathrm{Conv}(X_k/V) \to \mathrm{Conv}(X_k/W)$ hence a morphism of topoï

$$\pi : (X_k/V)_{\mathrm{conv}} \to (X_k/W)_{\mathrm{conv}}$$

For any isocrystal $\mathscr{F} \in (X_k/W)_{\mathrm{conv}}$, the induced morphism on cohomology

$$\mathrm{H}^*\big((X_k/W)_{\mathrm{conv}}, \mathscr{F}\big) \to \mathrm{H}^*\big((X_k/V)_{\mathrm{conv}}, \pi^*\mathscr{F}\big)$$

is an isomorphism.[7]

1.4. Theorem. Let $\mathscr{L}$ be a *crystalline sheaf* associated to filtered module $(\mathscr{F}, \phi_{\mathscr{F}}, \mathrm{Fil}_{\mathscr{F}}) \in \mathrm{Obj}(\mathrm{MF}_X^{\nabla}(\Phi))$. [8] For any decomposition $D = E \cup F$, there is a map[9]

$$\underbrace{\boxed{\mathrm{H}^*\big(((X_k, M_{X_k})/W)_{\mathrm{conv}}, \mathscr{F} \otimes \mathscr{J}_E\big)}}_{\substack{\text{conv. coh. of the special fibre}\\ \text{w/ cpt supp. along } E}} \otimes_{K_0} \tilde{B}_{\mathrm{crys}}(\bar{V}) \xrightarrow{\alpha} \underbrace{\boxed{\mathrm{H}^*_{E,F}(X^0_{\bar{K},\mathrm{\acute{e}t}}, \mathscr{L})}}_{\substack{\text{étale coh. of the geom. generic fibre}\\ \text{w/ partial cpt supp. along } E}} \otimes_{\mathbb{Q}_p} \tilde{B}_{\mathrm{crys}}(\bar{V}) \tag{1}$$

is an isomorphism compatible with

- Frobenius action,
- Galois action,
- cup product,
- Chern classes of vector bundles on X,
- filtration (strictly compatible).

2 Proof 0. Construction, Frobenius, Galois action, cup-product

(These (should) have been discussed in Tanya's talk. For a reference, see [Ols09, §13.17–13.20].)

As mention in the introduction part of [Ols09], the subtle part of the theory to construct a map (in either direction) between the two cohomology theories $\mathrm{H}^*(X_{\bar{K}})$ and $\mathrm{H}^*_{\mathrm{crys}}(X_k)$. Once this is done, the other part is essentially formal.

3 Proof I. Compatible with Chern Classes

[Ols09, §14.1–14.5]

- The crystalline sheaf associated to the *trivial F-isocrystal* $\mathscr{K}_{X_k/W}$ is the $\mathbb{Q}_p$ on $X^0_{K,\mathrm{\acute{e}t}}$.[10]
- Take $D = D \cup \emptyset$, i.e., $E = D$ and $F = \emptyset$. So (1) simplifies to

$$\alpha : \boxed{\mathrm{H}^*\big((X_k/W)_{\mathrm{conv}}, \mathscr{K}_{X_k/W}\big)} \otimes_K \tilde{B}_{\mathrm{crys}}(\bar{V}) \longrightarrow \boxed{\mathrm{H}^*(X^0_{\bar{K},\mathrm{\acute{e}t}}, \mathbb{Q}_p)} \otimes_{\mathbb{Q}_p} \tilde{B}_{\mathrm{crys}}(\bar{V})$$

[4] Note that [Shi02, Prop 2.1.21] states that the categories of isocrystals are equivalent no matter whether we use the étale or Zariski topology, cf. footnote 1.

[5] The category of convergent isocrystals is a full subcategory of that of isocrystals, see [Ogu84, Thm. 0.7.2].

[6] Think about why is $\mathrm{Spf}\, V \to \mathrm{Spf}\, W$ is flat, so that the forgetful functor is well defined.

[7] In [Ols09, §13.1], the author refers to [Ogu84, Corollary 3.2]. But i did not find it relevant. A seemingly related result is [Ogu84, §2.21]. But that seems not enough to get this isomorphism. In Olsson's notes, sometimes $(X_k/V)_{\mathrm{conv}}$ is used, and sometimes $(X_k/W)_{\mathrm{conv}}$ is used. To be consistent throughout the note, I use $(X_k/W)_{\mathrm{conv}}$ only. I think because of this isomorphism, nothing changes if we change to $(X_k/V)_{\mathrm{conv}}$.

[8] So $\mathscr{L}$ is a smooth (lisse) $\mathbb{Q}_p$-sheaf on X_K, and $\mathscr{F}$ is an F-isocrystal on X_k. Moreover, the sheaf $\mathscr{J}_E$ is defined in [Ols09, §13.3].

[9] Recall that $\mathrm{H}^i_{E,F}(X, \mathscr{F}) := \mathrm{H}^i((X-F), j_!\mathscr{F})$, see [Ols09, §8.12].

[10] This should follows from definition. But I did not think about it.

- For a vector bunlde $\mathscr{E}$ over X, $\mathscr{E}$ pulls back to vector bundles on X_k and X_K, still denoted by $\mathscr{E}$. The i-th Chern class lives in $\mathrm{H}^{2i}(-)$. More precisely,
  - Crystalline: $\mathrm{H}^{2i}(X_k/W) \rightsquigarrow \boxed{\mathrm{H}^{2i}\big((X_k/W)_{\mathrm{conv}}, \mathscr{K}_{X_k/W}\big)}$.
  - Étale: $\boxed{\mathrm{H}^{2i}\big(X^0_{\bar{K},\text{ét}}, \mathbb{Q}_p(i)\big)}$.
- We first consider the first Chern Class of a line bundle.
  - Crystalline.
    We have a short exact sequence
    $$0 \to \mathscr{I} \to \mathscr{O}_{X_k/W} \to \mathscr{O}_{X_k} \to 0$$
    of sheaves on $\mathrm{Crys}(X_k/W)$, where
    $$\begin{aligned} \mathscr{O}_{X_k/W} &: (U,T) \to \Gamma(T) \\ \mathscr{O}_{X_k} &: (U,T) \to \Gamma(U) \\ \mathscr{I} &: (U,T) \to \mathrm{Ker}(\Gamma(T) \to \Gamma(U)) \end{aligned}$$
    This induces maps
    $$0 \to 1 + \mathscr{I} \to \mathscr{O}^*_{X_k/W} \to \mathscr{O}^*_{X_k} \to 0$$
    $\mathscr{I}$ is a divided power ideal hence there is a logarithm map (recalling that $n!x^{[n]} = x^n$)
    $$\log : 1 + \mathscr{I} \to \mathscr{O}_{X/W}, \qquad 1 + t \mapsto \sum_{m \geq 1} (-1)^{m-1}(m-1)!t^{[m]}$$
    Then we have
    $$\begin{aligned} \mathrm{H}^1(X, \mathscr{O}^*_X) &\to \mathrm{H}^1((X_k/W)_{\mathrm{crys}}, \mathscr{O}^*_{X_k}) \\ &\to \mathrm{H}^2((X_k/W)_{\mathrm{crys}}, 1 + \mathscr{I}) \\ &\to \mathrm{H}^2((X_k/W)_{\mathrm{crys}}, \mathscr{O}_{X_k/W}) \otimes \mathbb{Q} \\ &\simeq \mathrm{H}^2((X_k/W)_{\mathrm{conv}}, \mathscr{K}_{X_k/W}) \end{aligned}$$
    We denote this map by c_1^{cr}.
  - Étale.
    We have the Kummer exact sequence of sheaves on $\mathrm{Et}(X_K)$
    $$0 \to \mu_{p^s} \to \mathbb{G}_m \xrightarrow{-p^s} \mathbb{G}_m \to 0.$$
    This induces
    $$\mathrm{H}^1(X_K, \mathbb{G}_m) \to \mathrm{H}^2(X_K, \mathbb{Z}_p(1)) \otimes \mathbb{Q}_p \simeq \mathrm{H}^2(X_K, \mathbb{Q}_p(1)).$$
    We denote this map by $c_1^{\text{ét}}$.
- Higher Chern classes can be expressed in terms of first Chern classes as follows.
  Suppose that $\mathscr{E}$ is locally free sheaf of rank r over X. Consider the projective bundle[11] $\big(\mathbb{P}(\mathscr{E}), \mathscr{O}(-1)\big) \to X$ associated to $\mathscr{E}$, i.e., $\mathbb{P}(\mathscr{E}) = \mathscr{P}roj(\mathrm{Sym}^\bullet \mathscr{E}^\vee)$. Let $\xi \in \mathrm{H}^2(\mathbb{P}(\mathscr{E}))$ be the first Chern class of the tautological bundle. The cohomology ring $\mathrm{H}^*(\mathbb{P}(\mathscr{E}))$ is a free $\mathrm{H}^*(X)$-module with basis $1, \xi, \xi^2, \ldots, \xi^{r-1}$. Hence there is a relation
  $$\xi^r + c_1\xi^{r-1} + \cdots + c_r = 0 \in \mathrm{H}^{2r}(\mathbb{P}(\mathscr{E}))$$
  Then the i-th Chern class $c_i(\mathscr{E})$ of $\mathscr{E}$ is the coefficient $c_i \in \mathrm{H}^{2i}(X)$.
- So we are to prove that
$$\begin{tikzcd} \mathrm{H}^1(X, \mathscr{O}^*_X) \arrow[r, "c_1^{\text{ét}} \otimes \tilde{B}_{\mathrm{crys}}"] \arrow[d, "c_1^{\mathrm{cr}} \otimes \tilde{B}_{\mathrm{crys}}"] & \mathrm{H}^2\big(X^0_{\bar{K},\text{ét}}, \mathbb{Q}_p(1)\big) \otimes_{\mathbb{Q}_p} \tilde{B}_{\mathrm{crys}}(\bar{V}) \arrow[d, "\beta"] \\ \mathrm{H}^2\big((X_k/W)_{\mathrm{conv}}, \mathscr{K}_{X_k/W}\big) \otimes_{K_0} \tilde{B}_{\mathrm{crys}}(\bar{V}) \arrow[r, "\alpha"] & \mathrm{H}^2(X^0_{\bar{K},\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \tilde{B}_{\mathrm{crys}}(\bar{V}) \end{tikzcd} \tag{2}$$

[11] The terminology differs in literatures. Some call it the projective bundle associate to $\mathscr{E}^\vee$.

Recall that, for each i, $\beta^{\otimes i}$ is the *isomorphism* $\beta^{\otimes i} : B_{\text{crys}}(\bar{V})(i) \to B_{\text{crys}}(\bar{V})$, or more precisely, the induced isomorphism on the further localization

$$\beta^{\otimes i} : \tilde{B}_{\text{crys}}(\bar{V})(i) \to \tilde{B}_{\text{crys}}(\bar{V}).$$

(This has been discussed in Drimiti's talk. For a reference, see [Ols09, (11.1.10)].)

3.1. Proposition. The diagram (2) commutes.

Proof The proof is based on computation. Since we didn't define it explicitly, so we omit the proof.

3.2. Corollary. For any vector bundle $\mathscr{E}$ on X, we have

$$\alpha\left(c_i^{\text{cr}(\mathscr{E})}\right) = \beta^{\otimes i}\left(c_i^{\text{ét}}(\mathscr{E})\right)$$

4 Proof II. Behavior under Push-forward

[Ols09, §14.6–14.10]

In this section, we consider the behavior of α under pushforward.[12]

(1) For reasonable cohomology $\mathrm{H}^*(X, -)$, and a (reasonable) sheaf $\mathscr{F}$ on X, a morphism $f : Y \to X$ induces a morphism $\mathrm{H}^i(X, \mathscr{F}) \to \mathrm{H}^i(Y, f^*\mathscr{F})$. Then by some type of Poincare-Verdier duality, we get

$$f_* : \mathrm{H}^i(Y, f^*\mathscr{F}) \to \mathrm{H}^{2\delta+i}(X, \mathscr{F}),$$

where the RHS may have some twist.

(2) We only consider a special case: Let $f : Y \hookrightarrow X$ be a closed embedding of smooth proper V-schemes of of relative codimension $\delta := \dim X - \dim Y$. (Assume that Y meets D transversally.)[13] Moreover, assume that the log structure on Y is induced by that of X. Set $E_Y := E \cap Y$ and $F_Y := Y \cap F$.

(3) In our current setting, we have[14]

$$f_*^{\text{cr}} : \mathrm{H}^i\left((Y_k/W)_{\text{conv}}, f^*\mathscr{F} \otimes \mathscr{J}_{E_Y}\right) \to \mathrm{H}^{2\delta+i}\left((X_k/W)_{\text{conv}}, \mathscr{F} \otimes \mathscr{J}_E\right)$$

for the convergent cohomology and

$$f_*^{\text{ét}} : \mathrm{H}^i_{E_Y,F_Y}(Y^0_{\bar{K},\text{ét}}, f^*\mathscr{L}) \to \mathrm{H}^{2\delta+i}_{E,F}(X^0_{\bar{K},\text{ét}}, \mathscr{L})(\delta)$$

for the étale cohomogoly.

(We have seen these in Marco's talk. For a reference, see [Ols09, §6.17, §8.20].)

(4) We expect α behaves well under push forward. That is, we are expecting a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^i\left((Y_k/W)_{\text{conv}}, f^*\mathscr{F} \otimes \mathscr{J}_{E_Y}\right) \otimes_{K_0} \tilde{B}_{\text{crys}}(\bar{V}) & \xrightarrow{\alpha} & \mathrm{H}^i_{E_Y,F_Y}(Y^0_{\bar{K},\text{ét}}, \mathscr{L}) \otimes_{\mathbb{Q}_p} \tilde{B}_{\text{crys}}(\bar{V}) \\ \downarrow{\scriptstyle f_*^{\text{cr}}} & & \downarrow{\scriptstyle \beta^{\delta} \circ f_*^{\text{ét}}} \\ \mathrm{H}^{2\delta+i}\left((X_k/W)_{\text{conv}}, \mathscr{F} \otimes \mathscr{J}_E\right) \otimes_{K_0} \tilde{B}_{\text{crys}}(\bar{V}) & \xrightarrow{\alpha} & \mathrm{H}^{2\delta+i}_{E,F}(X^0_{\bar{K},\text{ét}}, \mathscr{L}) \otimes_{\mathbb{Q}_p} \tilde{B}_{\text{crys}}(\bar{V}) \end{array} \tag{3}$$

4.1. Theorem. The diagram (3) commutes under above assumptions, i.e.,

$$\alpha \circ f_*^{\text{cr}} = \beta^{\otimes\delta} \circ f_*^{\text{ét}} \circ \alpha$$

[12] The map α respect pullbacks (under mild assumptions) by definition. See also footnote 26.

[13] This assumption was in [Ols09, Thm. 8.21] and [Fal89, p.63, V.b)].

[14] The twist is missing in [Ols09, (14.6.1) and §8.20] for convergent cohomology. Why? Besides, the superscript cr is a little bit misleading.

4.2. Proof. **Strategy: Deformation to the normal cone.** This should be standard, and is explained in [Ful98, Chapter V].

To simplify notations, write

$$
\begin{aligned}
\mathrm{H}^*_{\mathrm{crys}}(X) &:= \mathrm{H}^i\big((X_k/W)_{\mathrm{conv}}, \mathcal{F} \otimes \mathcal{J}_E\big)\\
\mathrm{H}^*_{\mathrm{crys}}(Y) &:= \mathrm{H}^i\big((Y_k/W)_{\mathrm{conv}}, f^*\mathcal{F} \otimes \mathcal{J}_E\big)\\
\mathrm{H}^*_{\text{ét}}(X) &:= \mathrm{H}^i_{E,F}\big(X^0_{\bar{K},\text{ét}}, \mathcal{L}\big)\\
\mathrm{H}^*_{\text{ét}}(Y) &:= \mathrm{H}^i_{E_Y,F_Y}\big(Y^0_{\bar{K},\text{ét}}, f^*\mathcal{L}\big)
\end{aligned}
$$

and so on.

Consider the blowing up of $X \times \mathbb{P}^1$ with centre $Y \times \{\infty\}$:

Strict transform = blow up

Exceptional divisor Z'_∞

Blow-up Z

$$
\begin{tikzcd}
 & & \overline{\pi^{-1}(Y \times \mathbb{P}^1 - \{\infty\})} \simeq \mathrm{Bl}_{Y\times\{\infty\}}(Y \times \mathbb{P}^1) \arrow[dl] \arrow[d, hook] \arrow[dd, bend left, "\simeq"] \\
\mathrm{E}_{Y\times\{\infty\}}(X \times \mathbb{P}^1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Bl}_{Y\times\{\infty\}}(X \times \mathbb{P}^1) \arrow[d, "\pi"] & \pi^{-1}(Y \times \mathbb{P}^1) \arrow[l, hook'] \arrow[d] \\
Y \times \{\infty\} \arrow[r, hook] & X \times \mathbb{P}^1 & Y \times \mathbb{P}^1 \arrow[l, hook'] \arrow[ul, dotted, "\tilde{f}"]
\end{tikzcd}
$$

For each $t \in \mathbb{P}^1$, denote by Z_t the fibre of the projection $Z \to \mathbb{P}^1$ at $t \in \mathbb{P}^1$. Let $f_t : Y \to Z_t$ be the inclusion induced[15] by $\tilde{f}$. Note that we have

$$
\begin{aligned}
Z_\infty &= \pi^{-1}(X \times \{\infty\})\\
&= \pi^{-1}\big(Y \times \{\infty\} \cup (X - Y) \times \{\infty\}\big)\\
&= \pi^{-1}\big(Y \times \{\infty\}\big) \cup \overline{\pi^{-1}\big((X - Y) \times \{\infty\}\big)}\\
&= \boxed{\mathrm{E}_{Y\times\{\infty\}}(X \times \mathbb{P}^1)} \cup \boxed{\big(\mathrm{Bl}_Y X \times \{\infty\}\big)}\\
&=: Z'_\infty \cup Z''_\infty,
\end{aligned}
$$

and the following facts:[16]

- Z'_∞ and Z''_∞ are irreducible components of Z_∞.
- $Z'_\infty \cap Z''_\infty = (\mathrm{E}_Y X) \times \{\infty\}$, where $\mathrm{E}_Y X$ is the exceptional divisor of the blow up of X with centre Y.
- The image of Y under $f_\infty : Y \to Z_\infty$ lies inside $Z'_\infty - Z''_\infty$.
- The two exceptional divisors $Z'_\infty := \mathrm{E}_{Y\times\{\infty\}}(X \times \mathbb{P}^1)$ and $\mathrm{E}_Y X$ and are projective bundles (projectivization of the normal bundle) over Y, as $Y \hookrightarrow X$ is a regular embedding.

Let $\bar{D}$ be the strict transform of $D \times \mathbb{P}^1$, i.e., $\bar{D} := \overline{\pi^{-1}\big(D \times \mathbb{P}^1 - Y \times \{\infty\}\big)}$.[17] Consider

$$
\begin{tikzcd}
Z - (\bar{D} \cup Z_\infty) \arrow[r, hook, "j"] \arrow[d] & Z - \bar{D} \arrow[d] & Z_\infty - (\bar{D} \cap Z_\infty) \arrow[l, hook', "i"'] \arrow[d] \\
Z - Z_\infty \arrow[r] & Z & Z_\infty \arrow[l]
\end{tikzcd}
$$

Then for any $\mathcal{L}$ on $Z_{\text{ét}}$,[18] there is an exact sequence (see [Mil13, Prop. 8.15])

$$
0 \to j_! j^* \mathcal{L} \to \mathcal{L} \to i_* i^* \mathcal{L} \to 0,
$$

[15] Take the fibre of the $\mathbb{P}^1$-morphism $\tilde{f} : Y \times \mathbb{P}^1 \to Z$ at $t \in \mathbb{P}^1$

[16] I didn't check these facts.

[17] Should it be the strict transform of $D \times \{\infty\}$ or $D \times \mathbb{P}^1$?

[18] Should it be on $Z \setminus \bar{D}$?

hence[19] an exact sequence

$$\overset{\text{coh. w/ cpt supp. along } \bar{D} \cup Z_\infty}{\mathrm{H}^*_{\text{ét}}(Z, Z_\infty)} \xrightarrow{\quad p \quad} \overset{\text{coh. w/ partal cpt supp. along } \bar{D}}{\mathrm{H}^*_{\text{ét}}(Z)} \xrightarrow{\quad q \quad} \mathrm{H}^*_{\text{ét}}(Z_\infty)$$

4.3. Lemma. The composition

$$\mathrm{H}^*_{\text{ét}}(Z, Z_\infty) \xrightarrow{p} \mathrm{H}^*_{\text{ét}}(Z) \xrightarrow{j_0^*} \mathrm{H}^*_{\text{ét}}(Z_0)$$

is zero, where j_0^* is the pull-back map induced by $j_0 : Z_0 \hookrightarrow Z$.

Proof. Note that[20]

$$Z - (\bar{D} \cup Z_\infty) \cong (X - D) \times (\mathbb{P}^1 - \{\infty\}) \cong (X - D) \times \mathbb{A}^1$$

It follows from Künneth formula[21] that

$$\mathrm{H}^*_{\text{ét}}(Z, Z_\infty) \cong \mathrm{H}^*_{\text{ét}}(X - D) \times \mathrm{H}^2_{\text{ét},c}(\mathbb{A}^1) \cong \mathrm{H}^{*-2}_{\text{ét}}(X - D)(1)$$

With this identification, the, the map p is given by the pushforward map

$$(j_0)_* : \mathrm{H}^{*-2}_{\text{ét}}(X - D)(1) \to \mathrm{H}^*_{\text{ét}}(Z)$$

So we reduce the problem to show that

$$j_0^*(j_0)_* : \mathrm{H}^{*-2}_{\text{ét}}(X - D)(1) \to \mathrm{H}^*_{\text{ét}}(X - D)$$

is zero. This follows from the self-intersection formula [SGA5, VII, Thm. 4.1], which shows that this map is given by the cup-product by the first Chern classes of the conormal bundle of Z_0 in Z, which is trivial.[22]

This finishes the proof of the claim.

4.4. Lemma. The Mayer-Vietoris sequence (see [Mil13, Thm. 10.8])[23] induces an exact sequence

$$\mathrm{H}^*_{\text{ét}}(Z, Z_\infty) \xrightarrow{p} \mathrm{H}^*_{\text{ét}}(Z) \longrightarrow \mathrm{H}^*_{\text{ét}}(Z'_\infty) \oplus \mathrm{H}^*_{\text{ét}}(Z''_\infty)$$

Proof Recall that Z'_∞ and $Z'_\infty \cap Z''_\infty$ are both projective bundles over Y. So the map

$$\mathrm{H}^*_{\text{ét}}(Z'_\infty) \to \mathrm{H}^*_{\text{ét}}(Z'_\infty \cap Z''_\infty)$$

is surjective as both are generated over $\mathrm{H}^*_{\text{ét}}(Y)$ by the first Chern class of the tautological line bundle. This implies that

$$\mathrm{H}^*_{\text{ét}}(Z_\infty) \to \mathrm{H}^*_{\text{ét}}(Z'_\infty) \oplus \mathrm{H}^*_{\text{ét}}(Z''_\infty)$$

is injective. Hence the desired exact sequence.

4.5. We have the following commutative diagram:

$$\begin{tikzcd}
X \cong Z_0 \arrow[rr, "j_0"] \arrow[dd] & & Z \arrow[dd, "\pi"] & \\
& Y \cong Y \times \{0\} \arrow[ul, "f"'] \arrow[dl] \arrow[rr, shift right] & & Y \times \mathbb{P}^1 \arrow[ul, "\tilde{f}"'] \arrow[ll, shift right, "\mathrm{pr}_1"'] \arrow[dl] \\
X \times \{0\} \arrow[rr] & & X \times \mathbb{P}^1 &
\end{tikzcd}$$

[19]I didn't quite understand what are these cohomology groups. If there is no D, then everything is easier.

[20]Should think about this: $Z - Z_\infty = \pi^{-1}(X \times (\mathbb{P}^1 - \{\infty\})) \cong X \times (\mathbb{P}^1 - \{\infty\})$, as the latter does not meet the centre of the blow up. Then $Z - (Z_\infty \cup \bar{D}) = (Z - Z_\infty) - \pi^{-1}(D - Y) = \cdots$.

[21]Reference?

[22]I didn't check: what does the reference say and why is the Chern class trivial.

[23]It reads $\cdots \to \mathrm{H}^i(X) \to \mathrm{H}^i(U_1) \oplus \mathrm{H}^i(U_2) \to \mathrm{H}^i(U_1 \cap U_2) \to \mathrm{H}^{i+1}(X) \to \cdots$, where $X = U_1 \cup U_2$ with U_i open.

In either theory, the composition

$$\mathrm{H}^*(Y) \xrightarrow{\mathrm{pr}_1^*} \mathrm{H}^*(Y \times \mathbb{P}^1) \xrightarrow{\tilde{f}_*} \mathrm{H}^*(Z) \xrightarrow{j_0^*} \mathrm{H}^*(Z_0) = \mathrm{H}^*(X)$$

equals to f_*.[24]

With our simplified notations, the diagram (3) whose commutative we are going to show, now simplifies as[25]

$$\begin{tikzcd}
\mathrm{H}^*_{\mathrm{crys}}(Y) \arrow[r] \arrow[d, "\alpha"] & \mathrm{H}^*_{\mathrm{crys}}(Y \times \mathbb{P}^1) \arrow[r, "\tilde{f}_*^{\mathrm{cr}}"] & \mathrm{H}^*_{\mathrm{crys}}(Z) \arrow[r, "j_0^*"] \arrow[d, "\alpha"] & \mathrm{H}^*_{\mathrm{crys}}(Z_0) \cong \mathrm{H}^*_{\mathrm{crys}}(X) \arrow[d, "\alpha"] \\
\mathrm{H}^*_{\text{ét}}(Y) \arrow[r] & \mathrm{H}_{\text{ét}}(Y \times \mathbb{P}^1) \arrow[r, "\tilde{f}_*^{\text{ét}}"] & \mathrm{H}^*_{\text{ét}}(Z) \arrow[r, "j_0^*"] \arrow[d] & \mathrm{H}^*_{\text{ét}}(Z_0) \cong \mathrm{H}^*_{\text{ét}}(X) \\
 & & \mathrm{H}^*_{\text{ét}}(Z'_\infty) \oplus \mathrm{H}^*_{\text{ét}}(Z''_\infty) &
\end{tikzcd}$$

We need to show the left square in the above diagram commutes. We identify $a \in \mathrm{H}^*(Y)$ with $\mathrm{pr}_1^*(a) \in \mathrm{H}^*(Y \times \mathbb{P}^1)$ in either theory. So we are to show $\alpha(\tilde{f}_*^{\mathrm{cr}}(a))$ and $f_*^{\text{ét}}(\alpha(a))$ in $\mathrm{H}^*_{\text{ét}}(Z)$ agrees. For this, we only need to show that their images in $\mathrm{H}^*_{\text{ét}}(Z'_\infty)$ and $\mathrm{H}^*_{\text{ét}}(Z''_\infty)$ agree respectively.

Recall that Y does not lie in Z''_∞, so we reduce our problem to the case $f : Y \hookrightarrow Z'_\infty$, with Z'_∞ a projective bundle over Y. In either theory, for any class $\mathrm{H}^*(Y)$, $f_*(\beta) = f_*(f^*p^*\beta) = f_*(1) \cup p^*\beta$, where p is the projection of Z'_∞ to Y.

It suffices[26] to consider the case $f_*(1)$. But in either theory, $f_*(1)$ is given by the Chern class of the conormal bundle. So we finish the proof.

4.6. Corollary. Denote by $\mathrm{Tr}^{\mathrm{cr}}$ and $\mathrm{Tr}^{\text{ét}}$ the trace maps.[27] Then we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{H}^{2d}\big((X_k/W)_{\mathrm{conv}}, \mathcal{K}_{X/V} \otimes \mathcal{J}_D\big) \otimes_{K_0} \tilde{B}_{\mathrm{crys}}(\bar{V}) \arrow[r, "\mathrm{Tr}^{\mathrm{cr}}"] \arrow[d, "\alpha"] & \tilde{B}_{\mathrm{crys}}(\bar{V}) \\
\mathrm{H}^{2d}_c(X^0_{\bar{K},\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \tilde{B}(\bar{V}) \arrow[r, "\mathrm{Tr}^{\text{ét}}"] & \tilde{B}(\bar{V})(d) \arrow[u, "\beta^{\otimes d}"]
\end{tikzcd}$$

In particular, the map

$$\alpha : \mathrm{H}^{2d}((X_k/W)_{\mathrm{conv}}, \mathcal{K} \otimes \mathcal{J}_D) \otimes_{K_0} \tilde{B}_{\mathrm{crys}}(\bar{V}) \to \mathrm{H}^{2d}_c(X_{\bar{K},\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \tilde{B}_{\mathrm{crys}}(\bar{V})$$

is an isomorphism.

Proof It suffices to show this after making an extension of V when $X^0 \to \operatorname{Spec} V$ has a section. In this case, the corollary follows from observing that the trace map is characterized by the fact it sends the class of a point in X^0 to 1.

4.7. It follows that there is a unique map

$$\alpha^t : \mathrm{H}^*_{F,E}(X^0_{\bar{K},\text{ét}}, \mathcal{L}) \otimes_{\mathbb{Q}_p} \tilde{B}_{\mathrm{crys}}(\bar{V}) \to \mathrm{H}^*((X_k/W)_{\mathrm{conv}}, \mathcal{F} \otimes \mathcal{J}_F) \otimes_{K_0} \tilde{B}_{\mathrm{crys}}(\bar{V})$$

such that for any

$$\begin{aligned} a &\in \mathrm{H}^*\big((X_k/W)_{\mathrm{conv}}, \mathcal{F} \otimes \mathcal{J}_E\big), \\ b &\in \mathrm{H}^{2\delta-*}_{F,E}(X^0_{\bar{K},\text{ét}}, \mathcal{L}) \end{aligned}$$

it holds that

$$\mathrm{Tr}^{\mathrm{cr}}\left(\alpha^t(b) \cup a\right) = \beta^{\otimes\delta}\Big(\mathrm{Tr}^{\text{ét}}\left(b \cup \alpha(a)\right)\Big).$$

Then using the same trick as in [Olso9, §8.60–8.61], we conclude that α and α^t are both isomorphisms.

[24] I didn't check this.

[25] Take care that $\mathrm{H}^*_{\mathrm{crys}}$ is that of the special fibre and $\mathrm{H}^*_{\text{ét}}$ is that of the geometric generic fibre.

[26] There are more to say here. The map α is compatible with cup product, this is mentioned in the beginning. That α is compatible with pullback is less obvious. But it is true. We have seen the way to deal with it in Marco's talk. See [Olso9, §8.21–§8.54].

[27] In [Olso9, (14.9.1)], the $\mathcal{K}$ in my diagram is M. But I don't know what is M. Maybe it is the $\overline{M}$ introduced in §13.3 or the M in §8.57. As mentioned before, $\mathbb{Q}_p$ is associated to $\mathcal{K}_{X/W}$, so I write $\mathcal{K}_{X/W}$ here. If you know what is M and M is not $\mathcal{K}_{X/W}$, please replace all $\mathcal{K}_{X/W}$ in the following by M.

5 Proof III. Strictly Compatible with Filtration

[Olso9, §14.11–14.12]

(1) Let F be the filtration on $\mathrm{H}^*(X_k/W, \mathscr{F} \otimes \mathscr{J}_E)$ and $\hat{F}$ be that on $\mathrm{H}^*(X_k/W, \mathscr{F}^\vee \otimes \mathscr{J}_F)$, induced by that of $\mathscr{F}$, where $\mathscr{F}^\vee$ is the dual[28] of $\mathscr{F}$. Let G be the filtration on $\mathrm{H}^*_{E,F}(X^0_{\bar{K},\text{ét}}, \mathscr{L}) \otimes_{\mathbb{Q}_p} \tilde{B}_{\text{crys}}(\bar{V})$ induced by that of $\tilde{B}_{\text{crys}}(\bar{V})$.

(2) It suffices to show

$$\mathscr{G}r^s_F\, \mathrm{H}^k(X_k/W, \mathscr{F} \otimes \mathscr{J}_E) \otimes_{K_0} \tilde{B}_{\text{crys}}(\bar{V}) \to \mathscr{G}r^s_G\, \mathrm{H}^k_{E,F}(X_{\bar{K},\text{ét}}) \otimes_{\mathbb{Q}_p} \tilde{B}_{\text{crys}}(\bar{V})$$

is an inclusion.

(3) To show this, it suffices to show that the filtration $F^\bullet$ and its dual $\hat{F}^\bullet$ gives a perfect pairing.

$$\mathscr{G}r^s_F\, \mathrm{H}^k \otimes \mathscr{G}r^{-s}_{\hat{F}}\, \mathrm{H}^{2\delta-k} \to \mathrm{H}^d(X, \Omega^d_X)$$

(4) Go to the associated de Rham complexes and use Poincaré duality to conclude the result.

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[28] What's is the precise definition?