Freie Universität Berlin

Intersection of a Correspondence with a Graph of Frobenius

Master Thesis

by

Yun Hao

Under the Supervision of:

Prof. Dr. Dr. h. c. mult. Hélène Esnault

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YUN HAO

Contents

0. Introduction

This is an expository note of Vasharvsky's geometric proof [Var14] of a theorem of Hrushovski [Hru04, Corollary 1.2], which asserts that the intersection of a correspondence and the graph of a sufficiently large power of Frobenius is nonempty.

This note follows Vasharvsky's original paper [Var14] closely. Essentially there is no original work by myself.

The whole structure of this note is basically the same as the original paper, except Section 2. Two appendices are added. Some tools are developed in Sections 3, 4 and 6. The final proof is given in Section 7.

In Section 2, some basic notions and results are given. Most results are stated in a more general setting than that is enough to be applied to the proof of our main theorem. Readers who are familiar with these topics or who do not want to bother with these general discussions would of course skip most part of Section 2 and refer to it only when needed. At the end of this note, there are two appendices: one is on intersection theory and the other is on ℓ-adic cohomology. The two appendices are rather brief.

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1. Statement of the Theorem and Corollaries

1.1. Let $\mathbb{F}_q$ be a finite field with q elements where q is a power of a prime number p. Let $\mathbb{F}$ be a fixed algebraic closure of $\mathbb{F}_q$. Suppose that X^0 is a finite type $\mathbb{F}$-scheme defined over $\mathbb{F}_q$ (§ 2.2). Then we consider the Frobenius endomorphism $\phi_q : X^0 \to X^0$ and let $(\phi_q)^n = \phi_{q^n}$ be its n^{th} iteration (§ 2.18.3). Let $\Gamma^0_{q^n}$ be the *graph* of ϕ_{q^n}, which is a locally closed subscheme of $X^0 \times_{\mathbb{F}} X^0$ (§ 2.13).

1.2. **Theorem.** Assume that C^0 and X^0 are two *irreducible* finite type $\mathbb{F}$-schemes and X^0 is defined over $\mathbb{F}_q$. Let $c^0 := (c^0_1, c^0_2) : C^0 \to X^0 \times_{\mathbb{F}} X^0$ be a morphism such that both c^0_1 and c^0_2 are *dominant*. Then there exists $N \in \mathbb{N}$, such that for all $n \geq N$, the preimage $(c^0)^{-1}(\Gamma^0_{q^n})$ is nonempty.

1.3. **Corollary.** With the assumptions of Theorem 1.2, the union $\cup_n (c^0)^{-1}(\Gamma_{q^n})$ is Zariski dense in C^0.

Proof. Let Z be the Zariski closure in C^0 of $\cup_n (c^0)^{-1}(\Gamma_{q^n})$. Suppose $Z \neq C^0$, then $C' := C^0 \setminus Z$ is Zariski dense in C^0 since C^0 is *irreducible*. Then we can apply Theorem 1.2 to the restriction $c^0|_{C'} : C' \to X^0 \times_{\mathbb{F}} X^0$. So for sufficiently large n, we have $(c^0|_{C'})^{-1}(\Gamma^0_{q^n}) = (c^0)^{-1}(\Gamma^0_{q^n}) \cap C' \neq \emptyset$, which is a contradiction. □

1.4. Let $f : X^0 \to X^0$ be a morphism. An $\mathbb{F}$-point (§ 2.10) $x \in X^0(\mathbb{F})$ is said to be *f-quasi-fixed* if $f(x) = \phi_{q^n}(x) = (\phi_q)^n(x)$ for some $n \in \mathbb{N}$. And it is called *f-periodic* if $f^n(x) = x$ for some $n \in \mathbb{N}$.

1.5. **Corollary.** Let $f : X^0 \to X^0$ be a *dominant* morphism where X^0 is an irreducible finite type $\mathbb{F}$-scheme that is defined over $\mathbb{F}_q$. Then the set of f-quasi-fixed points is Zariski dense in X^0.

Proof. Set $c^0 := (\mathrm{id}_{X^0}, f) : X^0 \to X^0 \times_{\mathbb{F}} X^0$. Then an $\mathbb{F}$-point $x \in (c^0)^{-1}(\Gamma^0_{q^n})$ if and only if $f(x) = (\phi_q)^n(x)$. So the desired result follows immediately from Corollary 1.3. □

1.6. **Corollary.** Let X^0 be a scheme of finite type over $\mathbb{F}$ and $f : X^0 \to X^0$ be a dominant morphism. Then the set of f-periodic points is Zariski dense in X^0.

Proof. As f is dominant, and X^0 is Noetherian thus has finitely many irreducible components, we know there is some $m \in \mathbb{N}$, such that $f^m := f \circ f \circ \cdots \circ f$ restricts to an endomorphism on each component of X^0. Moreover, any f^m-periodic point is clearly an f-periodic point. So it is enough to show the case where X^0 is irreducible. Since X^0 is of finite type over $\mathbb{F}$, there is some $r \in \mathbb{N}$, such that both X^0 and f are defined over $\mathbb{F}_{q^r}$. By replacing $\mathbb{F}_q$ by $\mathbb{F}_{q^r}$, we may assume $r = 1$, i.e., X^0 and f are defined over $\mathbb{F}_q$. Assume $X^0 = \underline{X}^0 \times_{\mathbb{F}_q} \mathbb{F}$, where $\underline{X}^0$ is of finite type over $\mathbb{F}_q$. Note that if $x \in X^0(\mathbb{F}) = \cup_r \underline{X}^0(\mathbb{F}_{q^r})$ (§ 2.10) is a f-quasi-fixed point, then $f(x) = (\phi_q)^n(x)$ for some $n \in \mathbb{N}$. Consequently $f^r(x) = (\phi_q)^{mr}(x) = x$ if $x \in \underline{X}^0(\mathbb{F}_{q^r})$. That is to say, x is f-periodic. So by Corollary 1.5, the set of f-periodic points is Zariski dense. □

1.7. **Outline of Proof.**

1.7.1. Reduce to the case where X^0 is $\mathbb{F}$-quasi-projective, $\dim C^0 = \dim X^0$ and c^0 is a closed embedding. Then we can choose a compactification X of X^0 defined over $\mathbb{F}_q$, and a closed embedding $c : C \to X \times X$, which restricts to $c^0 : C^0 \to X^0 \times X^0$. This will be done in § 7.2.

1.7.2. Reduce to the case where $\partial X := X \setminus X^0$ is *locally $C^{(n)}$-invariant* (Definition 3.2) for all $n \in \mathbb{N}$, where $c^{(n)} := \big((\phi_q)^n \circ c_1, c_2\big) : C \to X \times X$ is the *Frobenius twist* (§ 2.20.5) of c. The main technique will be discussed in Section 4. Then use de Jong's theorem (§ 2.21) on alterations, we can further assume that X is *smooth* and ∂X is a *normal crossing divisor* (§ 2.17) with irreducible components X_i which are *smooth* and defined over $\mathbb{F}_q$. This will be done in § 7.3.

1.7.3. Following a construction in [Pin92], consider a blow-up π of $X \times X$ with center $\cup X_i \times X_i$. Let $\Gamma_{q^n} \subseteq X \times X$ be the graph of the Frobenius endomorphism on X and denote by $\widetilde{C}$ and $\widetilde{\Gamma}_{q^n}$ the proper transform of C and Γ_{q^n}. We will (in Section 5) show that by twisting c with large enough power of Frobenius and by replacing c with $c^{(n)}$, we may reduce to the situation $\widetilde{C} \cap \widetilde{\Gamma}_{q^n} = (c^0)(\Gamma^0_{q^n})$. So it is enough to show that $\widetilde{C} \cap \widetilde{\Gamma}_{q^n} \neq \emptyset$. Now everything is smooth, so it suffices to show the intersection number $[\widetilde{C}] \cdot [\widetilde{\Gamma}_{q^n}] \neq 0$. To show this, we will first show an analogue of the Lefschetz trace formula (Lemma 6.2), then we can complete the proof with the help of Deligne's theorem (§ 2.22), that is the Weil conjecture. This will be done in § 7.4.

2. Notations and Conventions

In this section we go through some basic notions and results that will be used in the proof of Theorem 1.2. Proofs of the statements in this section are omitted but references are given. Most notations and terminologies are standard, while the main references are [Vak15], [Stacks] and [Poo15].

2.1. Throughout this note, p is a fixed prime number. Let q be a power of p. Let $\mathbb{F}_q$ be a finite filed with q elements and $\mathbb{F}$ a fixed algebraic closure of $\mathbb{F}_q$. For any natural number $r \geq 1$, let $\mathbb{F}_{q^r}$ be the subfield of $\mathbb{F}$ with q^r elements. For an arbitrary field k, we denote by $\bar{k}$ a fixed algebraic closure of k.

For results in this section, we do not restrict to schemes over $\mathbb{F}$ or $\mathbb{F}_q$. We state results for general schemes, e.g., S-schemes for an arbitrary scheme S or sometimes k-schemes for an arbitrary field k. We will restrict ourselves to $\mathbb{F}$ or $\mathbb{F}_q$ only when the result will be directly used for our final proof. Fiber products, if the base is not explicitly stated, are always taken over the base we are working on, which will be clear from the context. By abuse of notation, we will sometimes write k to mean $\operatorname{Spec} k$.

If l/k is a field extension and X is a k-scheme, then we write X_l for $X \times_k l$ if k is clear from the context.

2.2. We say a (finite type) $\mathbb{F}$-scheme is *defined over* $\mathbb{F}_{q^r}$ if $X \cong \underline{X} \times_{\mathbb{F}_{q^r}} \mathbb{F}$ for some (finite type) $\mathbb{F}_{q^r}$-scheme $\underline{X}$. A (finite type) morphism $f : X_1 \to X_2$ is *defined over* $\mathbb{F}_{q^r}$ if there is a (finite type) morphism $\underline{f} : \underline{X}_1 \to \underline{X}_2$ such that $f = \underline{f} \times \mathrm{id}_{\mathbb{F}}$ and $X_i = \underline{X}_i \times_{\mathbb{F}_{q^r}} \mathbb{F}$ for each i.

2.3. Let X be a k scheme. Then X is *geometrically irreducible* if $X \times_k \bar{k}$ is irreducible. This is equivalent to say that $X \times_k l$ is irreducible for all field extensions l/k. ([Vak15, § 9.5.2])

In particular, if an $\mathbb{F}$-scheme X is irreducible, and if $X = \underline{X} \times_{\mathbb{F}_{q^r}} \mathbb{F}$, i.e., X is defined over $\mathbb{F}_{q^r}$, then $\underline{X}$ is also irreducible.

All these discussions apply to "*geometrically reduced*" hence also to "*geometrically integral*".

2.4. In this note, we adopt the convention that a morphism $f : X \to S$ is *projective*, and X is a *projective* S-scheme, if there is an isomorphism

$$\begin{tikzcd} X \arrow[rr, "\cong"] \arrow[dr, "f"'] & & \mathcal{P}roj\,\mathcal{I}_\bullet \arrow[dl] \\ & S & \end{tikzcd}$$

for a quasicoherent sheaf of algebras $\mathcal{I}_\bullet$ on S, with the properties that $\mathcal{I}_\bullet$ is generated in degree 1 and $\mathcal{I}_1$ is of finite type ([Vak15, § 17.3.1]).

There is a different definition from [Har77, II, § 4], the differences and relations between these two definitions could be found in [Vak15, § 17.3] as well as [Stacks, Tag 01W7].

2.5. Let $g : X \to S$ be a morphism. If S is (locally) Noetherian and g (locally) of finite type, then X is (locally) Noetherian. Let $f : X \to Y$ be a morphism of S-schemes and X is locally of finite type over S, then $X \to Y$ is also locally of finite type. ([Stacks, Tag 01T0])

We are interested in the case that $S = \operatorname{Spec} k$ with k a field. In this case, a finite type k-scheme X is Noetherian, so every open subscheme of X is quasi-compact. If $f : X \to Y$ is a morphism between finite type k-schemes X and Y, then f is automatically of finite type. ([Stacks, Tag 01T0])

Any finite type $\bar{k}$-morphism between finite type $\bar{k}$-schemes is defined over some finite extension of k. This applies in particular to the structure morphism of a finite type $\bar{k}$-scheme.

Recall also that Finite morphisms are projective ([Vak15, § 17.3.5]), hence proper ([Vak15, § 10.3.3]).

2.6. Let $f : X \to Y$ be a morphism of schemes. If f is quasi-compact, or X is reduced, then the underlying topological space of the scheme-theoretic image of f is the same as the closure of the set-theoretic image of f ([Vak15, § 8.3.5]).

In particular, suppose $f : X \hookrightarrow Y$ is a locally closed embedding, i.e., $f = j \circ i$ with $i : X \hookrightarrow U$ a closed embedding and $j : U \to Y$ an open embedding. If f is quasi-compact, or X is reduced, then f also factors as an open embedding $X \to \overline{X}$ followed by a closed embedding $\overline{X} \to Y$, where $\overline{X}$ is the scheme-theoretic closure of X in Y ([Vak15, § 8.3.C] and [Stacks, Tag 07RJ]). On the other hand, if $X \to Y$ factors as $X \hookrightarrow Z \hookrightarrow Y$ with $X \hookrightarrow Z$ an open embedding and $Z \to Y$ a closed embedding, then f is always a locally closed embedding ([Vak15, § 8.1.M]).

A special case is that when X is an integral quasi-projective k-scheme, that is, X admits a locally closed embedding into $\mathbb{P}^n_k$, then there is an integral projective k-scheme $\overline{X}$, such that X is open dense in $\overline{X}$.

2.7. Recall that quasi-projective A-schemes are always separated, where A is any ring ([Vak15, § 10.1.14]).

2.8. By a k-variety, we mean a finite type, integral (irreducible and reduced), separated k-scheme. Note that we do not require a variety to be *geometrically integral* as some authors do. Actually, this concept will only be used in § 2.21.

2.9. Suppose $f : X \to Y$ is a *dominant* morphism of integral schemes of finite type over k. Then the set-theoretic image $f(X)$ contains an open dense subset U of Y ([Vak15, § 7.4.L]). Moreover, f maps the generic point of X to the generic point of Y.

Moreover, suppose $\dim X = m$ and $\dim Y = n$. Then $m - n \geq 0$ and the open subset U can be chosen such that for all $y \in U$, the fiber over y has pure dimension $m - n$ ([Vak15, § 11.4.1]). In particular, if $m = n$, then f is *quasifinite* over U.

To be more precise, there are affine open subsets $\operatorname{Spec} A \subseteq X$ and $\operatorname{Spec} B \subseteq Y$, such that $f(\operatorname{Spec} A) \subseteq \operatorname{Spec} B$, and open subset $U \subseteq \operatorname{Spec} B$, such that $f^{-1}(U) \to U$ factors through a finite surjective morphim $f^{-1}(U) \to \mathbb{A}_k^{m-n} \times U$ ([Vak15, § 11.4.1]).

2.10. Suppose X is a finite type $\bar{k}$-scheme that is defined over k, say $X = \underline{X} \times_k \bar{k}$, where $\bar{k}$ is an algebraic closure of k. Then we have the following one-to-one correspondences:

$$\underline{X}(\bar{k}) \longleftrightarrow X(\bar{k}) \longleftrightarrow \{\text{closed points of } X\}.$$

We are particularly interested in the case $k = \mathbb{F}_q$. Besides, we will not distinguish points which corresponds to each other under these correspondences. Another observation is that if $x \in \underline{X}(\bar{k})$, then x factors through $\operatorname{Spec} l \to \operatorname{Spec} k$ for some finite extension l/k (§ 2.5), which gives rise to a point in $\underline{X}(l)$. On the other hand, any $x \in \underline{X}(l)$ with $\bar{k}/l/k$ gives rise to a point in $\underline{X}(\bar{k})$ by composing with $\operatorname{Spec} \bar{k} \to \operatorname{Spec} l$.

2.11. Let $f : X \to Y$ be a morphism of schemes. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are sheaves over X and Y respectively. Then there is a natural adjunction correspondence $\operatorname{Hom}(\mathcal{G}, f_*\mathcal{F}) = \operatorname{Hom}(f^{-1}\mathcal{G}, \mathcal{F})$, where f_* is the push-forward functor and f^{-1} is the inverse image functor. In particular, $\operatorname{Hom}(\mathcal{G}, f_*f^{-1}\mathcal{G}) = \operatorname{Hom}(f^{-1}\mathcal{G}, f^{-1}\mathcal{G})$. Hence there is a unique morphism $f^{-1} : \mathcal{G} \to f_*f^{-1}\mathcal{G}$ of sheaves corresponding to the identity $\mathrm{id} : f^{-1}\mathcal{G} \to f^{-1}\mathcal{G}$. To simplify notations, by a section s of $\mathcal{G}$, we mean a section of $\mathcal{G}$ over some open subset $U \subseteq Y$, i.e., $s \in \mathcal{G}(U)$. Then we just write $f^{-1}(s)$ for $(f_U^{-1})(s) \in f_*f^{-1}\mathcal{G}(U) = (f^{-1}\mathcal{G})(f^{-1}(U))$. In particular we will be interested in the case where s is a *regular function*, i.e., s is a section of $\mathcal{O}_Y$ using our convention. Moreover if $f : X \to Y$ and $g : Y \to Z$ are two morphisms, then we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

2.11.1. *Remark.* In Varshavsky's paper [Var14], $f^{-1}(s)$ is denoted by $f^{\cdot}(s)$ for a section s of $\mathcal{G}$. But we will not use his notation.

2.12. When a closed subset Z of the underlying topological space of a scheme X is considered as a closed subscheme, we equip Z with the canonical *reduced* scheme structure. In particular, this convention applies to the irreducible components of a scheme or of a subscheme ([Vak15, § 8.3.9]).

Let $f : X \to Y$ be a morphism of schemes. Let $Z \subseteq Y$ be a closed subscheme of Y corresponding to an ideal sheaf $\mathcal{I}_Z := \mathcal{I}_{Z/Y} \subseteq \mathcal{O}_Y$. Then the scheme-theoretic preimage of Z is the scheme $f^{-1}(Z) := X \times_Y Z$, corresponding to the ideal sheaf $\mathcal{I}_{f^{-1}(Z)} := \mathcal{I}_{f^{-1}(Z)/X} := f^{-1}\mathcal{I}_Z \cdot \mathcal{O}_X$ ([Vak15, § 16.3.9]).

Denote by $\sqrt{\mathcal{I}}$ the *radical* of the ideal sheaf $\mathcal{I}$, i.e., for any *affine* open $U \subseteq Y$, $\sqrt{\mathcal{I}}(U) := \sqrt{\mathcal{I}(U)}$. Then the radical ideal sheaf $\sqrt{\mathcal{I}}$ corresponds to the *reduced* closed subscheme Z_{red} that has the same underlying topological space as Z. Moreover, for any scheme X, we have a natural morphism $X_{\text{red}} \to X$, which is the identity on the underlying topological space ([Vak15, §§ 8.3.9 and 8.3.10]).

2.13. Let $f : X \to Y$ be a morphism of two S-schemes. The *graph morphism* is the morphism $(\mathrm{id}_X, f) : X \to X \times_S Y$. It is a locally closed embedding. We call the corresponding locally closed subscheme of $X \times Y$ the *graph* of f. If Y is separated over S, then the graph morphism (id_X, f) is a closed embedding so in this case the graph is a closed subscheme. ([Vak15, § 10.1.18]).

In particular, if Y is a quasi-projective k-scheme (§ 2.7), then the graph of f is closed.

2.14. **Regularity and Smoothness.** Recall that a locally Noetherian scheme X is *regular at* $x \in X$ if $\mathcal{O}_{X,x}$ is a *regular local ring*, i.e., $\dim_{\kappa(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{X,x}$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$ and $\kappa(x)$ is the residue field of $\mathcal{O}_{X,x}$. A locally Noetherian scheme X is *regular* if it is regular at each $x \in X$.

A morphism $f : X \to Y$ is *smooth at* $x \in X$, if there exist affine open subsets $U \subseteq X$ and $V \subseteq Y$, such that $x \in U$, $f(U) \subseteq V$ and the induced morphism $f|_U : U \to V$ is *standard smooth*, that is, $f|_U$ is isomorphic to

$$\operatorname{Spec} S := \operatorname{Spec} \frac{R[x_1, \ldots, x_n]}{(f_1, \ldots, f_r)} \longrightarrow \operatorname{Spec} R,$$

with $r \leq n$ and $\det\left((\partial f_i/\partial x_j)_{1 \leq i,j \leq r}\right)$ mapping to an invertible element in S. A morphism $f : X \to Y$ is *smooth* if it is smooth at all points $x \in X$. Moreover, $f : X \to Y$ is *smooth of relative dimension* d if f is smooth and the number $d := n - r$ is uniform for all $x \in X$. A morphism is *étale* if it is smooth of relative dimension 0.

There are variants of definitions of smoothness, as provided in [Stacks, Tag 01V9]. A remarkable one says that, a morphism $f : X \to Y$ is smooth of relative dimension d if and only if f is of finite presentation, flat and the sheaf of relative differentials $\Omega_{X/Y}$ is locally free of rank d ([Vak15, § 25.1.5] and [Stacks, Tag 02G2]). Moreover, we know by definition, $\Omega_{X/Y}|_U$ has basis $(\mathrm{d}\, x_i)_{r+1 \leq i \leq n}$ over R. It also follows immediately that the smooth locus is open. As a consequence, smoothness can be checked at closed points.

An important case is when $Y = \operatorname{Spec} k$: a k-scheme is smooth of dimension d over k if its structure morphism $X \to \operatorname{Spec} k$ is smooth of relative dimension d.

2.14.1. As defined above, regularity is an "absolute" notion while smoothness is a "relative" notion. They are related to each other in the following way ([Vak15, § 12.2.10]).

(a) If k is perfect, every regular locally of finite type k-scheme is smooth over k.
(b) Every smooth k-scheme is regular.

2.14.2. From the above discussion, we have the following useful observations.

(a) Smooth k-schemes are geometrically reduced as regular local rings are normal, hence reduced ([Stacks, Tag 056T]).

(b) If l/k is a field extension, then X is smooth over k if and only if X_l is smooth over l ([Vak15, § 12.2.G and §21.3.C]).

(c) A morphism $f : X \to Y$ is smooth if and only if $X_{\kappa(y)}$ is smooth over $\kappa(y)$ for all $y \in Y$ if and only if $X_{\bar{k}}$ is smooth over $\bar{k}$ for all geometric points $\operatorname{Spec} \bar{k} \to Y$ ([Vak15, § 25.2.2]). In particular, a k-scheme is smooth over k if and only if $X_{\bar{k}}$ is smooth over $\bar{k}$.

(d) A morphism $f : X \to Y$ is smooth if and only if f is locally of finite presentation, flat and has regular geometric fibers. In particular, a locally of finite type k-scheme is smooth if and only if $X_{\bar{k}}$ is regular.

2.14.3. *Étale Coordinate.* ([Stacks, Tag 054L]) Suppose that $f : X \to Y$ is smooth at x. Suppose that $U \subseteq X$ and $V \subseteq Y$ are affine open neighborhoods of x and $f(x)$ respectively, such that $f|_U$ is standard smooth as in § 2.14. Then we get an étale morphism $U \to \mathbb{A}^d_R$, given by the natural map

$$R[x_{r+1}, \ldots, x_n] \to \frac{R[x_1, \ldots, x_n]}{(f_1, \ldots, f_r)},$$

making the diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & U \arrow[l, hook'] \arrow[r, "\text{étale}"] \arrow[d] & \mathbb{A}^d_R \arrow[dl] \\ Y & V \arrow[l, hook'] & \end{tikzcd}$$

commute. A special case is when $Y = \operatorname{Spec} k$ and $f : X \to Y$ is the structure morphism of X. That is roughly speaking, a smooth k-scheme looks étale-locally like an affine space.

2.15. Suppose $f : X \hookrightarrow Y$ is a locally closed embedding. Then f is a *regular embedding of codimension* r if for all $x \in X$, the ideal $\operatorname{Ker}(\mathcal{O}_{Y,f(x)} \twoheadrightarrow \mathcal{O}_{X,x})$ is generated by a regular sequence of length r ([Vak15, § 8.4.7]).

2.15.1. *Proposition.* ([Stacks, Tag 067U] and [Vak15, § 12.2.L]) Let X and Y be smooth S-schemes and $i : X \hookrightarrow Y$ be a locally closed embedding. Then $i : X \to Y$ is a regular embedding.

2.16. Recall that an *effective Cartier divisor* is a closed subscheme D of a scheme X such that its ideal sheaf is locally generated by a regular function that is not a zero-divisor ([Vak15, § 8.4.1] and [Stacks, Tag 01WR]). If X is locally Noetherian, then this is equivalent to require that $D \hookrightarrow X$ is a regular embedding of codimension 1 ([Vak15, § 8.4.H and § 11.3.3]).

2.17. Suppose that X is a Noetherian scheme. A *strict normal crossing divisor* is an effective Cartier divisor such that (a) D is reduced with irreducible components D_i's, in other words, D is a scheme-theoretic union $D = \cup_{i\in I} D_i$, where I is a finite index set; (b) for any $x \in D$, the local ring $\mathcal{O}_{X,x}$ is regular, and (c) for each subset $\emptyset \neq J \subseteq I$, the closed subscheme $D_J := \cap_{j\in J} D_j$ is a regular scheme of codimension $|J|$. A *normal crossing divisor* is a divisor D of X with the property that there is an étale cover $X' \to X$ such that the pull-back $D' := D \times_X X'$ is a strict normal crossing divisor in X' ([Jon96, § 2.4]).

2.17.1. *Remark.* Suppose D is a strict normal crossing divisor on a locally of finite type k-scheme. Then by definition of strict normal crossing divisor as well as § 2.14.1, we know that all D_J's are automatically smooth over k, as long as k is perfect.

2.18. **Frobenius Morphisms.** Let $p > 0$ be a fixed prime number, and q be a power of p. A scheme X is *of characteristic p* if $p\mathcal{O}_X = 0$. For a scheme of characteristic p, there are some different but closely related Frobenius morphisms associated with X. The main reference for this paragraph is [Poo15, § 7.5.5].

2.18.1. *Absolute/Relative Frobenius.* Let X be a scheme of characteristic p, then there is a well-defined morphism

$$\operatorname{Fr}_X := \operatorname{Fr}_{X,q} : X \to X,$$

which is the *identity* on the underlying topological space of X and the q^{th} power map on the structure sheaf $\mathcal{O}_X$ of X, which is well defined because X is of characteristic p. This morphism is called the *absolute q-Frobenius*, *q^{th} power absolute Frobenius* or simply *absolute Frobenius* when q is clear from the context.

Now suppose that X is an S-scheme with structure morphism α and S is of characteristic p. Then X is also of characteristic p. Denote by $X^{(q)}$, or $X^{(q/S)}$, the base change of X along the absolute q-Frobenius $\mathrm{Fr}_S : S \to S$. By the universal property of the fiber product, there is a unique morphism

$$\mathrm{Fr}_{X/S} := (\alpha, \mathrm{Fr}_X) : X \to X^{(q)}$$

making the diagram

$$\begin{tikzcd} X \arrow[drr, bend left, "\mathrm{Fr}_X"] \arrow[dr, dashed, "\mathrm{Fr}_{X/S}"] \arrow[ddr, bend right, "\alpha"'] & & \\ & X^{(q)} \arrow[r] \arrow[d] & X \arrow[d, "\alpha"] \\ & S \arrow[r, "\mathrm{Fr}_S"] & S \end{tikzcd} \tag{2.1}$$

commute, where Fr_X is the absolute q-Frobenius on X. This morphism is called the *relative q-Frobenius* on X. Again, we will usually simply call it relative Frobenius when q is clear from the context. In particular, if Fr_S is an automorphism, e.g., $S = \operatorname{Spec}\mathbb{F}_q$ or $S = \operatorname{Spec}\mathbb{F}_p$, then $X^{(q)} \cong X$ and the relative Frobenius $\mathrm{Fr}_{X/S}$ becomes an endomorphism of X over S.

Note that the relative Frobenius $\mathrm{Fr}_{X/S}$ is an S-morphism while the absolute Frobenius Fr_X is not in general.

2.18.2. *Arithmetic/Geometric Frobenius Elements.* Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_q$. It is known that the absolute Galois group is $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q) \cong \widehat{\mathbb{Z}}$. The q^{th} power map $\sigma_q : a \mapsto a^q$, $a \in \mathbb{F}$, and its inverse $\sigma_q^{-1} : a \mapsto a^{1/q}$, $a \in \mathbb{F}$, which are topological generators of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$, are called the *arithmetic* and the *geometric Frobenius element*, respectively.

2.18.3. Now suppose that $\underline{X}$ is an $\mathbb{F}_q$-scheme. Since $\mathrm{Fr}_{\operatorname{Spec}\mathbb{F}_q}$ on $\operatorname{Spec}\mathbb{F}_q$ is the identity, we have $\underline{X}^{(q)} = \underline{X}$ and $\mathrm{Fr}_{\underline{X}} = \mathrm{Fr}_{\underline{X}/\mathbb{F}_q}$. Moreover, if we set $X := \underline{X}_{\mathbb{F}} := \underline{X} \times_{\mathbb{F}_q} \mathbb{F}$ to be the base change of $\underline{X}$ along the natural map $\operatorname{Spec}\mathbb{F} \to \operatorname{Spec}\mathbb{F}_q$, then we get an isomorphism $X^{(q)} \cong X$. Write σ_q and σ_q^{-1} also for the maps $\operatorname{Spec}\mathbb{F} \to \operatorname{Spec}\mathbb{F}$ induced by the arithmetic and the geometric Frobenius elements, respectively. The morphism $\mathrm{id}_{\underline{X}} \times \sigma_q$ is sometimes called the *arithmetic Frobenius morphism*.

Suppose pr_i, $i = 1, 2$, are the projections from $X = \underline{X} \times_{\mathbb{F}_q} \mathbb{F}$ to $\underline{X}$ and $\mathbb{F}$ respectively. Note that $\sigma_q : \operatorname{Spec}\mathbb{F} \to \operatorname{Spec}\mathbb{F}$ is exactly the absolute q-Frobenius on $\operatorname{Spec}\mathbb{F}$. Denote by $\phi_{X,q}$ or ϕ_q the relative Frobenius $\mathrm{Fr}_{X/\mathbb{F}}$. In other words, we specialize the diagram (2.1) to

$$\begin{tikzcd} X \arrow[drr, bend left, "\mathrm{Fr}_X"] \arrow[dr, dashed, "\phi_q"] \arrow[ddr, bend right, "\mathrm{pr}_2"'] & & \\ & X \arrow[r, "\mathrm{id}_{\underline{X}} \times \sigma_q"] \arrow[d, "\mathrm{pr}_2"] & X \arrow[d, "\mathrm{pr}_2"] \\ & \operatorname{Spec}\mathbb{F} \arrow[r, "\sigma_q"] & \operatorname{Spec}\mathbb{F}, \end{tikzcd} \tag{2.2}$$

and it is easy to see that $\phi_q = \mathrm{Fr}_{X/\mathbb{F}}$ is the base extension of the absolute Frobenius $\mathrm{Fr}_{\underline{X}} = \mathrm{Fr}_{\underline{X}/\mathbb{F}_q}$ along the natural map $\operatorname{Spec}\mathbb{F} \to \operatorname{Spec}\mathbb{F}_q$. It is an endomorphism of X over $\operatorname{Spec}\mathbb{F}$. This relative Frobenius ϕ_q is sometimes called the *geometric Frobenius*. We will call it *the* Frobenius in this note. It is *the* Frobenius morphism considered in [SGA 4½] and [Del74].

For any $n \in \mathbb{N}$, it is easy to see that the n^{th} iteration of ϕ_q is exactly the relative q^n-Frobenius ϕ_{q^n}, i.e., $\phi_{q^n} = \phi_q^n := \phi_q \circ \cdots \circ \phi_q$.

2.18.4. If X is a finite type $\mathbb{F}$-scheme that is defined over $\mathbb{F}_q$, say $X = \underline{X} \times_{\mathbb{F}_q} \mathbb{F}$, then these Frobenius morphisms are related to each other as $\phi_q \circ (\mathrm{id}_{\underline{X}} \times \sigma_q) = (\mathrm{id}_{\underline{X}} \times \sigma_q) \circ \phi_q = \mathrm{Fr}_X$.

2.18.5. *Example.* Let $A := \mathbb{F}_q[x_1, \dots, x_n]/(f_1, \dots, f_r)$ be a finite type $\mathbb{F}_q$-algebra and $A \otimes_{\mathbb{F}_q} \mathbb{F} = \mathbb{F}[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Set $\underline{X} := \operatorname{Spec} A$ and $X := \underline{X} \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \mathbb{F} = \operatorname{Spec}(A \otimes_{\mathbb{F}_q} \mathbb{F})$.

- the absolute q-Frobenius $\mathrm{Fr}_X : X \to X$ is induced by the ring homomorphism that raises every polynomial to its q^{th} power, which is equvivalent to raise every coefficient and every indeterminate x_i to their q^{th} power.
- the geometric q-Frobenius $\mathrm{Fr}_{X/\mathbb{F}} : X \to X$ is induced by raising each indeterminate x_i to its q^{th} power.
- the arithmetic Frobenius $(\mathrm{id}_{\underline{X}} \times \sigma_q) : X \to X$ is induced by raising each coefficient to its q-th power.

2.18.6. *Action of Frobenius on Rational Points and Cohomology.* Let X be a smooth proper finite type irreducible $\mathbb{F}$-scheme defined over $\mathbb{F}_q$ with $X = \underline{X} \times_{\mathbb{F}_q} \mathbb{F}$. We know that $\mathrm{H}^i(X, \mathbb{Q}_\ell)$ is finite dimensional $\mathbb{Q}_\ell$-vector space (Appendix B.6). We also know that any morphism $f : X \to X$ induces a $\mathbb{Q}_\ell$-linear endomorphism f^i on $\mathrm{H}^i(X, \mathbb{Q}_\ell)$. It turns out that the absolute Frobenius Fr_X induces the identity, so by § 2.18.4, ϕ_q and $\sigma_q^{-1} \times \mathrm{id}$ induces *the same* action (pull-back) on $\mathrm{H}^i(X, \mathbb{Q}_\ell)$. This is the reason why σ^{-1} is called the geometric Frobenius element ([Poo15, § 7.5.5] and [Del74, § 1.15]).

Moreover, it is clear from the definition that

$$\#\underline{X}(\mathbb{F}_{q^n}) = \#\{x \in X(\mathbb{F}) : \phi_{q^n}(x) = x\} =: \#\{\text{fixed points of } \phi_q^n\}.$$

and the Lefschetz trace formula (Appendix B.10) asserts that

$$\#\{\text{fixed points of } \phi_q^n\} = \operatorname{Tr}\left((\phi_q^n)^* \middle| \mathrm{H}^i(X, \mathbb{Q}_\ell)\right)$$

([Del74, § 1.4] and [Poo15, § 7.5.6]). A very brief summary of ℓ-adic cohomology is given in Appendix B.

2.19. **Blow-Ups.** The main reference of this paragraph is [Vak15, Chapter 22]. Suppose $X \hookrightarrow Y$ is a closed subscheme cut out by a finite type quasicoherent sheaf of ideals $\mathcal{I} := \mathcal{I}_{X/Y}$ of $\mathcal{O}_Y$ (if Y is Noetherian, then the finite type hypothesis on $\mathcal{I}$ automatically holds). The *blow-up* of Y with respect to $\mathcal{I}$ (with *center* X, or along X), is the scheme $\mathrm{Bl}_X Y := \mathrm{Bl}_{\mathcal{I}} Y := \mathcal{P}roj_Y(\mathcal{O}_Y \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \cdots)$ together with the natural morphism

$$\pi : \mathrm{Bl}_X Y \longrightarrow Y.$$

The *exceptional divisor* of the the blow-up is $\mathrm{E}_X Y := \pi^{-1}(X) = X \times_Y \mathrm{Bl}_X$ and we have

$$\mathrm{E}_X Y = \mathcal{P}roj_Y(\mathcal{O}_Y/\mathcal{I} \oplus \mathcal{I}/\mathcal{I}^2 \oplus \mathcal{I}^2/\mathcal{I}^3 \oplus \cdots),$$

which is an effective Cartier divisor (§ 2.16) of $\mathrm{Bl}_X(Y)$ ([Vak15, § 22.3.2]).

We then have a Cartesian diagram

$$\begin{tikzcd} \mathrm{E}_X Y \arrow[r, hook] \arrow[d] & \mathrm{Bl}_X Y \arrow[d, "\pi"] \\ X \arrow[r, hook] & Y. \end{tikzcd} \tag{2.3}$$

This diagram satisfies the universal property that any other Cartesian diagram

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & W \arrow[d] \\ X \arrow[r, hook] & Y, \end{tikzcd}$$

where D is an effective Cartier divisor of W, factors as

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & W \arrow[d] \\ \mathrm{E}_X Y \arrow[r, hook] \arrow[d] & \mathrm{Bl}_X Y \arrow[d, "\pi"] \\ X \arrow[r, hook] & Y. \end{tikzcd}$$

Moreover, the blow-up of Y along X is uniquely (up to unique isomorphism) determined by the above universal property ([Vak15, § 22.2]).

2.19.1. It is easy to see that $\mathrm{Bl}_\emptyset Y \cong Y$, $\mathrm{Bl}_Y Y = \emptyset$ and $\mathrm{Bl}_D Y \cong Y$ if D is an effective Cartier divisor. Moreover, for any open subset $U \subseteq Y$, $\mathrm{Bl}_{U\cap X} U \cong \pi^{-1}(U)$. A consequence is that blow-ups could be computed affine locally. Then it follows that π is an isomorphism away from X, that is, $\pi|_{\mathrm{Bl}_X Y \setminus \mathrm{E}_X Y} : \mathrm{Bl}_X Y \setminus \mathrm{E}_X Y \to Y \setminus X$ is an isomorphism. Obviously by definition, the blow-up morphism π is projective, hence quasicompact, proper, of finite type and separated ([Vak15, § 22.2]).

2.19.2. *Blow-up Closure Lemma.* As in the situation of § 2.19, suppose $f : Z \to Y$ is any morphism and $W := X \times_Y Z$. Then $\mathcal{I}_{W/Z}$ is also a finite type ideal sheaf. Let $\widetilde{Z}$ be the scheme-theoretic closure of $(Z \times_Y \mathrm{Bl}_X Y) \setminus (W \times_Y \mathrm{Bl}_X Y)$ inside $Z \times_Y \mathrm{Bl}_X Y$ and $\mathrm{E}_{\widetilde{Z}} := \widetilde{Z} \times_{\mathrm{Bl}_X Y} \mathrm{E}_X Y$. Then there is a canonical isomorphim $\widetilde{Z} \cong \mathrm{Bl}_W Z$, under which $\mathrm{E}_{\widetilde{Z}} \cong E_W Z$. We have the following diagram

$$\begin{tikzcd}
& E_{\widetilde{Z}} \arrow[rr, hook] \arrow[dl, "\cong"'] \arrow[dd, hook] & & \widetilde{Z} \arrow[dl, "\cong"] \arrow[dd, hook] \\
\mathrm{E}_W Z \arrow[rr, hook] \arrow[dd] & & \mathrm{Bl}_W Z \arrow[dd] & \\
& W \times_Y \mathrm{Bl}_X Y \arrow[rr, hook] \arrow[dl] \arrow[dd] & & Z \times_Y \mathrm{Bl}_X Y \arrow[dl] \arrow[dd] \\
W \arrow[rr, hook] \arrow[dd] & & Z \arrow[dd, "f"] & \\
& \mathrm{E}_X Y \arrow[rr, hook] \arrow[dl] & & \mathrm{Bl}_X Y \arrow[dl, "\pi"] \\
X \arrow[rr, hook] & & Y &
\end{tikzcd}$$

such that all squares in this diagram are Cartesian except those involving the two isomorphisms on the top ([Vak15, § 22.2.5]).

2.19.3. *Proper/Strict Transform.* As in the situation of § 2.19.2, the product $Z \times_Y \mathrm{Bl}_X Y$ is called the *total transform* of Z and $\widetilde{Z}$ is called the *proper transform* or *strict transform* of Z. Similarly, the morphism $Z \times_Y \mathrm{Bl}_X Y \to \mathrm{Bl}_X Y$ and $\widetilde{Z} \to \mathrm{Bl}_X Y$ are called the *total transform* and the *proper transform* of the morphism $f : Z \to Y$. If particular, if $f : Z \hookrightarrow Y$ is a closed embedding, then the total transform is $\pi^{-1}(Z)$ and the proper transform is the scheme-theoretic closure $\overline{\pi^{-1}(Z \setminus X)}$. By § 2.19.2, $\widetilde{Z} \to Z$ is proper.

In practice, when X is clear from the context, we usually write $\widetilde{Y}$ for $\mathrm{Bl}_X Y$, $\widetilde{Z}$ and $\widetilde{f}$ for the proper transforms of Z and f.

2.19.4. *Successive Blow-ups.* Let Y be a scheme and $\mathcal{I}, \mathcal{J} \subseteq \mathcal{O}_Y$ are two finite type quasicoherent sheaves of ideals. Let $\pi : \widetilde{Y} := \mathrm{Bl}_{\mathcal{I}} Y \to Y$ be the blow-up of Y with respect to $\mathcal{I}$ and $\tau : \mathrm{Bl}_{\widetilde{\mathcal{J}}} \widetilde{Y} \to \widetilde{Y}$ be the blow-up of $\widetilde{Y}$ with respect to $\widetilde{\mathcal{J}} := \pi^{-1}\mathcal{J} \cdot \mathcal{O}_{\widetilde{Y}}$. Then $\pi \circ \tau$ is canonically isomorphic to the blow-up $\mathrm{Bl}_{\mathcal{I}\cdot\mathcal{J}} Y \to Y$ of Y with respect to $\mathcal{I} \cdot \mathcal{J}$ ([Stacks, Tag 080A]).

From this, we can see that it is more natural to blow up Y with respect to $\mathcal{I} \cdot \mathcal{J}$ rather than $\mathcal{I} \cap \mathcal{J}$, though the latter correspond to the scheme-theoretic union, and $\mathrm{supp}\left(\mathcal{O}_Y/(\mathcal{I} \cap \mathcal{J})\right) = \mathrm{supp}\left(\mathcal{O}_Y/(\mathcal{I} \cdot \mathcal{J})\right) = \mathrm{supp}(\mathcal{O}_Y/\mathcal{I}) \cup \mathrm{supp}(\mathcal{O}_Y/\mathcal{J}) \subseteq Y$.

2.19.5. *Blow-up Commutes with Flat Base Change.* Recall that if $f : Z \to Y$ is a *flat* morphism and D is an effective Cartier divisor on Y, then $f^{-1}(D)$ is an effective Cartier divisor on Z ([Vak15, § 24.2.N]). So from the universal property of blow up, we may conclude that if $Z \to Y$ is flat, then $(\mathrm{Bl}_X Y) \times_Y Z \cong \mathrm{Bl}_{X \times_Y Z} Z \cong \widetilde{Z}$ ([Vak15, § 24.2.P]).

In particular, if l/k is a field extension, we know $\mathrm{Spec}\, l \to \mathrm{Spec}\, k$ is flat so $Y_l \to Y$ is also flat. As a consequence, $(\mathrm{Bl}_X Y)_l \cong \mathrm{Bl}_{X_l}(Y_l)$.

2.19.6. *Blow-up of a Smooth Scheme with Smooth Center is Smooth.* ([Vak15, § 22.3.10] and [Har77, II, Theorem 8.24]) Suppose that $i : X \hookrightarrow Y$ is a closed embedding with X and Y both smooth over k. It follows from § 2.19.5 and § 2.14.2 (c) that we can without loss generality assume $k = \bar{k}$. Recall § 2.15.1 that i is then a regular embedding. It follows that $\mathrm{E}_X Y \cong \mathcal{P}roj\left(\mathrm{Sym}^\bullet(\mathcal{I}/\mathcal{I}^2)\right)$ is a projective bundle over X, hence also smooth, then regular. So $\mathrm{Bl}_X Y$ is also regular at each closed point of $\mathrm{E}_X Y$, as $\mathrm{E}_X Y$ is an effective Cartier divisor ([Vak15, § 12.2.C]). Moreover, $\pi_{Y \setminus X}$ is an isomorphism. So it follows that $\mathrm{Bl}_X Y$ is regular everywhere, hence smooth.

2.19.7. *Blow-up Removes Intersection.* ([Har77, II, Exercise 7.12]) Suppose that X_i, $i = 1, 2$, are two closed subschemes of Y, corresponding to finite type quasicoherent sheaves of ideals $\mathcal{I}_i$, such that neither one is set-theoretically contained in the other. Set $\mathcal{I} := \mathcal{I}_1 + \mathcal{I}_2$, which is the ideal sheaf of the scheme-theoretic intersection $X := X_1 \cap X_2$. Then the proper transforms $\widetilde{X}_i$ of X_i in the blow-up $\widetilde{Y} := \mathrm{Bl}_X Y$ do not intersect.

2.20. **Correspondence.** In this paragraph, X and C are assumed to be k-schemes. The main reference of this paragraph is [Var14, § 1].

2.20.1. A *correspondence* is just a morphism $c = (c_1, c_2) : C \to X \times X$. In case C and X are both k-schemes of finite type , § 2.5 then yields that c is a finite type morphism, and we are always in this situation in the proof of the main theorem.

2.20.2. *Remark.* This is not a standard notion, as usually we require c to be a closed embedding, and we also allow the target to be a product of two different schemes.

2.20.3. A *morphism* from a correspondence $\widetilde{c} : \widetilde{C} \to \widetilde{X} \times \widetilde{X}$ to $c : C \to X \times X$ is a pair of morphisms $(f, f_C) := [f]$, making the diagram

$$\begin{array}{ccccc} \widetilde{X} & \xleftarrow{\widetilde{c}_1} & \widetilde{C} & \xrightarrow{\widetilde{c}_2} & \widetilde{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_C} & & \downarrow{\scriptstyle f} \\ X & \xleftarrow{c_1} & C & \xrightarrow{c_2} & X \end{array}$$

commute. Given two correspondences $c, \widetilde{c}$ and a morphism $f : \widetilde{X} \to X$, we say $\widetilde{c}$ *lifts* c along f, if there exists a morphism $f_C : \widetilde{C} \to C$ such that (f, f_C) is a morphism from $\widetilde{c}$ to c.

2.20.4. *Restrictions* are special kinds of liftings. Let $c : C \to X \times X$ be a correspondence. Let $U \subseteq X$ and $W \subseteq C$ be two open subschemes. Then we have two natural restrictions, denoted by $c|_W : W \to X \times X$ and $c|_U : c_1^{-1}(U) \cap c_2^{-1}(U) \to U \times U$ of c. In diagrams, we have

$$\begin{tikzcd} U \arrow[d, "i"'] & c_1^{-1}(U) \cap c_2^{-1}(U) \arrow[l, "(c|_U)_1"'] \arrow[r, "(c|_U)_2"] \arrow[d, "j"] & U \arrow[d, "i"] \\ X & C \arrow[l, "c_1"'] \arrow[r, "c_2"] & X, \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[d, "\mathrm{id}"'] & W \arrow[l, "(c|_W)_1"'] \arrow[r, "(c|_W)_2"] \arrow[d, "j"] & X \arrow[d, "\mathrm{id}"] \\ X & C \arrow[l, "c_1"'] \arrow[r, "c_2"] & X, \end{tikzcd}$$

where i and j are the inclusions (open embeddings). Note that $c_1^{-1}(U) \cap c_2^{-1}(U) = c^{-1}(U \times U)$.

2.20.5. Now suppose X and C are finite type $\mathbb{F}$-schemes and X is defined over $\mathbb{F}_q$. For any natural number $n \in \mathbb{N}$, let ϕ_{q^n} be the n^{th} power of the Frobenius as in § 2.18.3. If $c : C \to X \times X$ is a correspondence, then the *Frobenius twist* of c is the morphism $\big((\phi_{q^n}) \circ c_1, c_2\big)$, which we denote by $c^{(n)}$. In diagrams, we have

$$\begin{tikzcd} X \arrow[d, "\phi_{q^n}"'] & C \arrow[l, "c_1"'] \arrow[r, "c_2"] \arrow[d, equal] & X \arrow[d, "\mathrm{id}"] \\ X & C \arrow[l, "c_1^{(n)}=(\phi_{q^n})\circ c_1"'] \arrow[r, "c_2^{(n)}=c_2"] & X. \end{tikzcd}$$

2.20.6. *Remark.* We may indeed generalize the definition of a morphism between correspondences a little bit by allowing two different morphisms $\widetilde{X} \to X$. In fact, it is defined as this in [Var07, § 1.1]. By doing so, the Forbenius twist of a correspondence becomes a lifting. However, we will not use this definition in this note.

2.21. **de Jong's Alteration Theorem.** Roughly speaking, de Jong's theorem states that for any variety X over k, there is an alteration $X' \to X$, such that X' is a non-singular variety.

2.21.1. *Definition.* ([Stacks, Tag 0AB0]) Let X be an integral scheme. An *alteration* of X is a proper dominant morphism $f : Y \to X$ such that Y integral and f is *generically finite*, i.e., $f^{-1}(U) \to U$ is finite for some nonempty open $U \subseteq X$.

2.21.2. *Theorem.* ([Jon96, Theorem 4.1 and Remark 4.2]) Let X be a k-variety (§ 2.8) and $Z \subsetneq X$ be a closed subset. Then there is an alteration

$$\phi : X' \to X,$$

and an open embedding $j : X' \to \overline{X'}$, such that (a) $\overline{X'}$ is a projective variety and is regular, and (b) the closed subset $j(\phi^{-1}(Z)) \cup \overline{X'} \setminus j(X')$ is a strict normal crossing divisor in $\overline{X'}$. Moreover, the structure map $\overline{X'} \to \operatorname{Spec} k$ factors through $\operatorname{Spec} k'$ with k'/k a finite extension such that $\overline{X'}$ is geometrically irreducible and smooth over k'. If k is perfect, ϕ may be chosen to be generically étale and $\overline{X'}$ is smooth over k.

2.22. **Deligne's Theorem on Weil Conjectures.** This is the key theorem in [Del74] to prove the analogue of the Riemann Hypothesis for varieties over finite fields.

2.22.1. *Theorem.* Let X be a smooth, proper, separated, finite type $\mathbb{F}$-scheme, that is defined over $\mathbb{F}_q$. Let ϕ_q be the Frobenius endomorphism as in § 2.18.3 and ℓ be a prime number different from p. Then for every i and every embedding $\iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$, all eigenvalues λ of $\phi_q^* : \mathrm{H}^i(X, \mathbb{Q}_\ell) \to \mathrm{H}^i(X, \mathbb{Q}_\ell)$ have absolute value $|\iota(\lambda)| = q^{i/2}$ ([Del74, Théorème 1.6]).

3. Locally Invariant Subsets and Locally Contracting Correspondences

In this section, we define two concepts: locally invariant closed subsets and locally contracting correspondences, and we will explore some properties of these two concepts. Most proofs are set-theoretic. The main result that will be used in the proof of our main theorem is Lemma 3.12.

3.1. In this section, all schemes are assumed to be of *finite type* over a field k and all implicit fiber products are taken over $\operatorname{Spec} k$. Hence all schemes in this section are Noetherian.

3.2. **Definition.** Let $c : C \to X \times X$ be a correspondence (§ 2.20.1) and $Z \subseteq X$ a closed subset (§ 2.12).

(a) The closed subset Z is said to be *c-invariant* if *set-theoretically* $c_1(c_2^{-1}(Z)) \subseteq Z$, equivalently, if $c_2^{-1}(Z) \subseteq c_1^{-1}(Z)$.
(b) The closed subset Z is said to be *locally c-invariant* if for every $x \in Z$, there exists an open neighborhood U of x such that $Z \cap U \subseteq U$ is $(c|_U)$-invariant.

3.3. **Lemma.** Suppose $[f] = (f, f_C)$ is a morphism (§ 2.20.3) from $\widetilde{c} : \widetilde{C} \to \widetilde{X} \times \widetilde{X}$ to $c : C \to X \times X$ and $Z \subseteq X$ is a c-invariant closed subset. Then $f^{-1}(Z)$, as a closed subset of $\widetilde{X}$, is $\widetilde{c}$-invariant. It follows that $f^{-1}(Z)$ is locally $\widetilde{c}$-invariant in case that Z is locally c-invariant.

Proof. Suppose Z is c-invariant. Then

$$\widetilde{c}_2^{-1}(f^{-1}(Z)) = f_C^{-1}(c_2^{-1}(Z)) \subseteq f_C^{-1}(c_1^{-1}(Z)) = \widetilde{c}_1^{-1}(f^{-1}(Z)).$$

So $f^{-1}(Z)$ is $\widetilde{c}$-invariant.

The second assertion is local, so by replacing X by an open subset we reduce the problem to the first one. □

3.4. **Notation.** Suppose $c : C \to X \times X$ is a correspondence and $Z \subseteq X$ is a closed subset. We now introduce two subsets that measure how far the closed subset Z is from being c-invariant or locally c-invariant. Set

$$F(c, Z) := c_2^{-1}(Z) \cap c_1^{-1}(X \setminus Z) \subseteq C,$$

and

$$G(c, Z) := \bigcup_{S \in \operatorname{Irr}(F(c,Z))} [\overline{c_1(S)} \cap \overline{c_2(S)}] \subseteq X,$$

where $\operatorname{Irr}(F(c, Z))$ is the *irreducible components* of $F(c, Z)$ (there are only finitely many since C is Noetherian) and $\overline{c_i(S)}$ denotes the *set-theoretic* closure of $c_i(S)$ in X.

3.5. **Remarks.**

(a) If $S \in \operatorname{Irr}(F(c,z))$, then $S \subseteq c_2^{-1}(Z)$ hence $c_2(S) \subseteq c_2(c_2^{-1}(Z)) \subseteq Z$. So $\overline{c_2(S)} \subseteq \overline{Z} = Z$. Therefore, we can conclude that $G(c,Z) \subseteq Z$.

(b) If Z_1 and Z_2 are both locally c-invariant closed subsets, then the union $Z_1 \cup Z_2$ is also locally c-invariant.

(c) Note that if $(U \cap Z)$ is $(c|_U)$-invariant, then for every $V \subseteq U$, $V \cap Z$ is $c|_V$-invariant. It follows that if Z is locally c-invariant, then for any open subset $U \subseteq X$, $Z \cap U \subseteq U$ is locally $c|_U$-invariant.

3.6. **Lemma.** Let $c : C \to X \times X$ be a correspondence and $Z \subseteq X$ be a closed subset.

(a) The closed subset Z is c invariant if and only if $F(c,Z) = \emptyset$; hence for any open subset $U \subseteq X$, $Z \cap U \subseteq U$ is $c|_U$-invariant if and only if $F(c,Z) \cap c_1^{-1}(U) \cap c_2^{-1}(U) = \emptyset$.

(b) The subset $X \setminus G(c,Z)$ is the largest open subset $U \subseteq X$ such that $Z \cap U$ is locally $c|_U$-invariant. Therefore, Z is locally c-invariant if and only if $G(c,Z) = \emptyset$.

Proof. (a) The first statement follows directly from the definition. The second one follows from the fact that $F(c|_U, Z \cap U) = F(c,Z) \cap c_1^{-1}(U) \cap c_2^{-1}(U)$.

(b) Let $V \subseteq X$ be any open subset of X. We have the following observation:

$$\begin{aligned}
& Z \cap V \text{ is } c|_V\text{-invariant} \\
\iff & F(c,Z) \cap c_1^{-1}(V) \cap c_2^{-1}(V) = \emptyset && \text{(item (a))} \\
\iff & S \cap c_1^{-1}(V) \cap c_2^{-1}(V) = \emptyset, \quad \forall S \in \operatorname{Irr}(F(c,Z)) \\
\iff & S \cap c_1^{-1}(V) = \emptyset \text{ or } S \cap c_2^{-1}(V) = \emptyset, \quad \forall S && (S \text{ is irreducible}) \\
\iff & V \subseteq X \setminus \overline{c_1(S)} \text{ or } V \subseteq X \setminus \overline{c_2(S)}, \quad \forall S.
\end{aligned}$$

Therefore, a point $x \in X$ has an open neighborhood $V \subseteq X$ such that $Z \cap V$ is $c|_V$-invariant if and only if for each $S \in F(c,Z)$, $x \in X \setminus \overline{c_1(S)}$ or $x \in X \setminus \overline{c_2(S)}$. By definition, the largest open subset U of X such that $Z \cap U$ is locally $c|_U$-invariant, consists of all such points. As a result, the largest such open subset is exactly $\cup_{S \in F(c,Z)} X \setminus [\overline{c_1(S)} \cap \overline{c_2(S)}]$. $\square$

3.7. **Corollary.** Let $c : C \to X \times X$ be a correspondence.

(a) For any closed subsets $Z_1, Z_2 \subseteq X$, we have $G(c, Z_1 \cup Z_2) \subseteq G(c,Z_1) \cup G(c,Z_2)$.

(b) For any closed subset $Z \subseteq X$, if c_2 is quasi-finite and $Z \neq \emptyset$, one has $\dim G(c,Z) < \dim Z$, where we set $\dim \emptyset = -1$.

Proof. (a) Set $U_i := X \setminus G(c,Z_i)$. It suffices to show $X \setminus G(c, Z_1 \cup Z_2) \supseteq \big(X \setminus G(c,Z_1)\big) \cap \big(X \setminus G(c,Z_2)\big) = U_1 \cap U_2$. By § 3.6, it suffices to show that $(Z_1 \cup Z_2) \cap (U_1 \cap U_2)$ is locally $c|_{U_1 \cap U_2}$-invariant.

Using § 3.6 again, we know $U_i \cap Z_i$ is locally $c|_{U_i}$-invariant for $i = 1,2$. So Remarks 3.5 (c) implies that $U_1 \cap U_2 \cap Z_i$ is locally $c|_{U_1 \cap U_2}$-invariant for $i = 1,2$. Then Remarks 3.5 (b) implies that $(U_1 \cap U_2) \cap (Z_1 \cup Z_2)$ is locally $c|_{U_1 \cap U_2}$-invariant. So we are done.

(b) Following from item (a), we may without loss of generality assume that Z is irreducible. For any $S \in \operatorname{Irr}(F(c,Z))$, we have by definition $S \subseteq c_2^{-1}(Z)$. Noting that c_2 is quasi-finite, we have $\dim S \leq \dim Z$. We also know that $\overline{c_2(S)} \subseteq Z$ and $c_1(S) \subseteq X \setminus Z$. So $c_1(S) \cap \overline{c_2(S)} = \emptyset$ and $\overline{c_1(S)} \cap \overline{c_2(S)} \subseteq \overline{c_1(S)} \setminus c_1(S)$. Thus

$\dim \overline{c_1(S)} \cap \overline{c_2(S)} < \dim c_1(S) \leq \dim S \leq \dim Z$. Since S is arbitrary, the desired result follows. $\square$

3.8. **Definition.** Let $c : C \to X \times X$ be a correspondence, and let $Z \subseteq X$ be a closed subscheme with ideal sheaf $\mathcal{I}_Z = \mathcal{I}_{Z/X} \subseteq \mathcal{O}_X$.

(a) The correspondence c is said to be *contracting near* Z, if $\mathcal{I}_{c_1^{-1}(Z)} \subseteq \mathcal{I}_{c_2^{-1}(Z)}$, and $(\mathcal{I}_{c_1^{-1}(Z)})^n \subseteq (\mathcal{I}_{c_2^{-1}(Z)})^{n+1}$ for some $n \in \mathbb{N}$. The first condition is to say that $c_2^{-1}(Z)$ is scheme-theoretically contained in $c_1^{-1}(Z)$.
(b) We say c is *locally contracting near* Z, if for every $x \in X$, there exists an open neighborhood $U \subseteq X$ of x such that $c|_U$ is contracting near $Z \cap U$.

3.9. **Remark.** Note that if c is (locally) contracting near Z, then Z is (locally) c-invariant by definition.

3.10. In case $c : C \to X \times X$ is a correspondence over $\mathbb{F}$ such that X is defined over $\mathbb{F}_q$, we say a closed subset $Z \subseteq X$ is *locally c-invariant over* $\mathbb{F}_q$ or c is *locally contracting near Z over* $\mathbb{F}_q$, if in the respective definition, the open subsets could be chosen to be defined over $\mathbb{F}_q$.

3.11. **Definition.** Let $f : X \to Y$ be a morphism of Noetherian schemes and Z a closed subset of Y. Then the *ramification degree* of f at Z is

$$\operatorname{ram}(f, Z) := \min \left\{ m \in \mathbb{N} : \left(\sqrt{\mathcal{I}_{f^{-1}(Z)}}\right)^m \subseteq \mathcal{I}_{f^{-1}(Z)} \right\}.$$

The existence of $\operatorname{ram}(f, Z)$ follows from the fact that X is Noetherian.

3.12. **Lemma.** Let $c : C \to X \times X$ be a correspondence over $\mathbb{F}$ such that X is defined over $\mathbb{F}_q$. Assume $n \in \mathbb{N}$ is a natural number such that $q^n > \operatorname{ram}(c_2, Z)$ and Z is locally $c^{(n)}$-invariant over $\mathbb{F}_q$ (§ 3.10), where $c^{(n)}$ is the Frobenius twist (§ 2.20.5) of c. Then the correspondence $c^{(n)}$ is locally contracting near Z over $\mathbb{F}_q$.

Proof. Set $m = \operatorname{ram}(c_2, Z)$. For every open subset $U \subseteq X$ that is defined over $\mathbb{F}_q$, it is easy to see that

$$\operatorname{ram}\left((c|_U)_2, Z \cap U\right) \leq \operatorname{ram}(c_2, Z) = m. \tag{3.1}$$

Hence it suffices to show that $c^{(n)}$ is contracting near Z, under the condition that Z is $c^{(n)}$-invariant.

Let $\varphi_n := \operatorname{id} \times \sigma_{q^n}^{-1}$ as in § 2.18.3 and let Fr_{q^n} be the absolute q^n-Frobenius as in § 2.18. Then we know that $\phi_{q^n} = \operatorname{Fr}_{q^n} \circ \varphi_{q^n}$ (§ 2.18.4). Then for any section s of $\mathcal{I}_Z$ (following convention in § 2.11), it holds that (see Example 2.18.5)

$$(\phi_{q^n})^{-1}(s) = (\operatorname{Fr}_{q^n} \circ \varphi_{q^n})^{-1}(s) = (\varphi_{q^n})^{-1}(\operatorname{Fr}_{q^n})^{-1}(s) = (\varphi_{q^n})^{-1}(s^{q^n}).$$

Therefore, $(\phi_{q^n})^{-1}\mathcal{I}_Z = (\varphi_{q^n})^{-1}(\mathcal{I}_Z)^{q^n}$. So

$$(c_1^{(n)})^{-1}\mathcal{I}_Z = (\phi_{q^n} \circ c_1)^{-1}\mathcal{I}_Z = (c_1)^{-1}(\phi_{q^n})^{-1}\mathcal{I}_Z = (c_1)^{-1}(\varphi_{q^n})^{-1}(\mathcal{I}_Z)^{q^n}.$$

Since Z is $c^{(n)}$-invariant, we have $c_2^{-1}(Z) \subseteq (c_1^{(n)})^{-1}(Z)$. Therefore

$$\mathcal{I}_{(c_1^{(n)})^{-1}(Z)} \subseteq \sqrt{\mathcal{I}_{(c_1^{(n)})^{-1}(Z)}} \subseteq \sqrt{\mathcal{I}_{c_2^{-1}(Z)}}.$$

It follows that

$$\mathcal{I}_{(c_1^{(n)})^{-1}(Z)} = (c_1^{(n)})^{-1}\mathcal{I}_Z \cdot \mathcal{O}_C = (c_1)^{-1}(\varphi_{q^n})^{-1}(\mathcal{I}_Z)^{q^n} \cdot \mathcal{O}_C \subseteq \sqrt{\mathcal{I}_{c_2^{-1}(Z)}}.$$

By taking radicals, we obtain that $(c_1)^{-1}(\varphi_{q^n})^{-1}\mathcal{I}_Z \cdot \mathcal{O}_C \subseteq \sqrt{\mathcal{I}_{c_2^{-1}(Z)}}$. As we have $q^n > m = \mathrm{ram}(c_2, Z)$, we can conclude that

$$\mathcal{I}_{(c_1^{(n)})^{-1}(Z)} = (c_1)^{-1}(\varphi_{q^n})^{-1}(\mathcal{I}_Z)^{q^n} \cdot \mathcal{O}_C \subseteq \sqrt{\mathcal{I}_{c_2^{-1}(Z)}}^{q^n} \subseteq \sqrt{\mathcal{I}_{c_2^{-1}(Z)}}^{m} \subseteq \mathcal{I}_{c_2^{-1}(Z)},$$

and that,

$$(\mathcal{I}_{(c_1^{(n)})^{-1}(Z)})^m = (c_1)^{-1}(\varphi_{q^n})^{-1}(\mathcal{I}_Z)^{mq^n} \cdot \mathcal{O}_C \subseteq \sqrt{\mathcal{I}_{c_2^{-1}(Z)}}^{m(m+1)} \subseteq (\mathcal{I}_{c_2^{-1}(Z)})^{m+1}$$

Hence $c^{(n)}$ is contracting near Z. $\square$

4. Main Technical Result

The main result that is for later use is Corollary 4.4.

4.1. **Setup.** Let k be field and assume that all implicit fiber products are taken over k. Suppose that C and X are two *irreducible* finite type k-schemes of the *same dimension* and $X^0 \subseteq X$ is an nonempty open subset of X. Let $c := (c_1, c_2) : C \to X \times X$ be a correspondence with dominant c_2.

4.2. **Lemma.** There exist non-empty open subsets $V \subseteq U \subseteq X^0$ such that

(a) $c_1^{-1}(V) \subseteq c_2^{-1}(U)$;
(b) the closed subset $U \setminus V$ of U is locally $(c|_U)$-invariant.

Proof. Since $\dim C = \dim X$ and c_2 is dominant, there exists some non-empty open subset $U_0 \subseteq X^0$ such that $c_2|_{c_2^{-1}(U_0)}$ is quasi-finite (§ 2.9). Inductively, define for all $j \in \mathbb{N}$ that

$$\begin{aligned} V_j &:= U_j \setminus \overline{c_1(c_2^{-1}(X \setminus U_j))} \subseteq U_j \\ G_j &:= G(c|_{U_j}, U_j \setminus V_j) \subseteq U_j \setminus V_j \\ U_{j+1} &:= U_j \setminus G_j \subseteq U_j \end{aligned}$$

We will show this proposition according to the following steps:

Step 1: $U_j \neq \emptyset$ and $V_j \neq \emptyset$, for any $j \in \mathbb{N}$;
Step 2: $c_1^{-1}(V_j) \subseteq c_2^{-1}(U_j)$, for any $j \in \mathbb{N}$;
Step 3: $G(c|_{U_{j+1}}, U_{j+1} \setminus V_j) = \emptyset$, i.e., $U_{j+1} \setminus V_j = U_{j+1} \cap (U_j \setminus V_j)$ is locally $(c|_{U_{j+1}})$-invariant (Corollary 3.7), for any $j \in \mathbb{N}$;
Step 4: there exist some j such that $G_j = \emptyset$.

Once we showed the above statements, by setting $U = U_j = U_{j+1}$ and $V = V_j$, we would prove the proposition.

The proof depends heavily on the construction of the subsets U_j and V_j. To get an intuitive idea how the construction works, the Venn diagrams in Figure 1 may be helpful.

4.2.1. *Step 1.* We show the statement by induction on j. Note first $U_0 \neq \emptyset$. Suppose that $U_j \neq \emptyset$, then $c_2^{-1}(X \setminus U_j) \neq C$ since c_2 is dominant. It then follows that $\dim c_2^{-1}(X \setminus U_j) < \dim C$ since C is irreducible. Therefore

$$\begin{aligned} \dim \overline{c_1\big(c_2^{-1}(X \setminus U_j)\big)} &= \dim c_1\big(c_2^{-1}(X \setminus U_j)\big) \leq \dim c_2^{-1}(X \setminus U_j) \\ &< \dim C = \dim X = \dim U_j. \end{aligned}$$

So we obtain

$$V_j = U_j \setminus \overline{c_1(c_2^{-1}(X \setminus U_j))} \neq \emptyset,$$

and

$$U_{j+1} = U_j \setminus G_j = U_j \setminus G(c|_{U_j}, U_j \setminus V_j) \supseteq U_j \setminus (U_j \setminus V_j) = V_j \neq \emptyset.$$

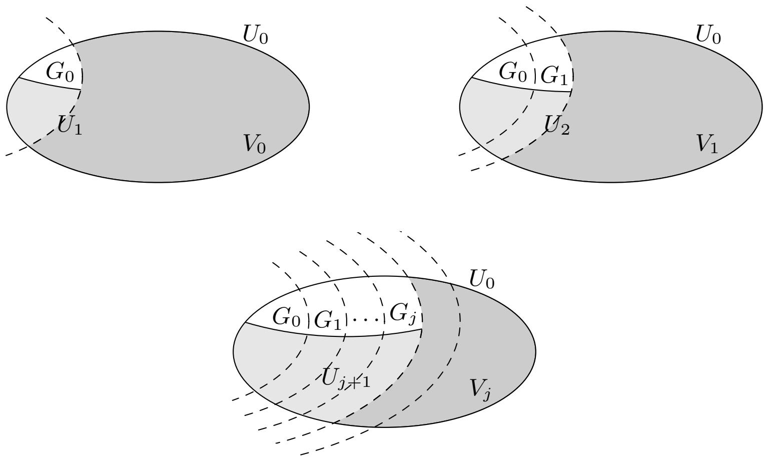

FIGURE 1. Inductive construction of U_j and V_j

So we showed the first claim by induction.
Step 2. By construction,

$$\begin{aligned} V_j \cap \overline{c_1(c_2^{-1}(X \setminus U_j))} = \emptyset &\implies V_j \cap c_1(c_2^{-1}(X \setminus U_j)) = \emptyset \\ &\implies c_1^{-1}(V_j) \cap c_2^{-1}(X \setminus U_j) = \emptyset \\ &\implies c_1^{-1}(V_j) \subseteq c_2^{-1}(U_j). \end{aligned}$$

4.2.2. *Step 3.* This follows directly from Lemma 3.6. Indeed, we konw U_{j+1} is the largest open subset U inside U_j such that $U \cap (U_j \setminus V_j)$ is locally $(c|_U)$-invariant, hence of course, $U_{j+1} \cap (U_j \setminus V_j) = U_{j+1} \setminus V_j$ is locally $(c|_{U_{j+1}})$-invariant.

4.2.3. *Step 4.* To show that there exists some j such that $G_j = \emptyset$, it suffices to show that the dimension of G_j strictly decreases. Note that

$$U_{j+1} \setminus V_{j+1} = (U_{j+1} \setminus V_j) \cup (V_j \setminus V_{j+1}) = (U_{j+1} \setminus V_j) \cup (\overline{V_j \setminus V_{j+1}}).$$

It follows from § 4.2.2 and Corollary 3.7 (a) that

$$\begin{aligned} G_{j+1} = G\big(c|_{U_{j+1}}, U_{j+1} \setminus V_{j+1}\big) &\subseteq \emptyset \cup G\big(c|_{U_{j+1}}, \overline{V_j \setminus V_{j+1}}\big) \\ &= G\big(c|_{U_{j+1}}, \overline{V_j \setminus V_{j+1}}\big) \subseteq \overline{V_j \setminus V_{j+1}}. \end{aligned}$$

If $V_j \setminus V_{j+1} = \emptyset$, we are done. If $V_j \setminus V_{j+1} \neq \emptyset$, since $c_2|_{U_{j+1}}$ is quasi-finite, it follows from Corollary 3.7 (b) that

$$\dim G_{j+1} \leq \dim G\big(c|_{U_{j+1}}, \overline{V_j \setminus V_{j+1}}\big) < \dim \big(\overline{V_j \setminus V_{j+1}}\big) = \dim \big(V_j \setminus V_{j+1}\big).$$

Moreover, we have $X \setminus U_{j+1} = (X \setminus U_j) \cup G_j$ and $G_j \subseteq U_j \setminus V_j$. Hence,

$$\overline{c_1\big(c_2^{-1}(X \setminus U_{j+1})\big)} = \overline{c_1\big(c_2^{-1}((X \setminus U_j) \cup G_j)\big)} = \overline{c_1\big(c_2^{-1}(X \setminus U_j)\big)} \cup \overline{c_1(c_2^{-1}(G_j))},$$

and then

$$\begin{aligned} V_{j+1} &= U_{j+1} \setminus \overline{c_1\big(c_2^{-1}(X \setminus U_{j+1})\big)} \\ &= (U_j \setminus G_j) \setminus [\overline{c_1\big(c_2^{-1}(X \setminus U_j)\big)} \cup \overline{c_1(c_2^{-1}(G_j))}] \\ &= U_j \setminus \big[\overline{c_1\big(c_2^{-1}(X \setminus U_j)\big)} \cup G_j \cup \overline{c_1(c_2^{-1}(G_j))}\big] \\ &= V_j \setminus \big[G_j \cup \overline{c_1(c_2^{-1}(G_j))}\big] \\ &= V_j \setminus \overline{c_1(c_2^{-1}(G_j))}. \end{aligned}$$

Hence $V_j \setminus V_{j+1} \subseteq \overline{c_1(c_2^{-1}(G_j)})$. We know that $G_j \subseteq U_j \subseteq U_0$, so $c_2|_{c_2^{-1}(G_j)}$ is quasi-finite. Therefore $\dim \overline{c_1(c_2^{-1}(G_j))} = \dim c_1(c_2^{-1}(G_j)) \leq \dim c_2^{-1}(G_j) \leq \dim G_j$. All together we have

$$\dim G_{j+1} < \dim(V_j \setminus V_{j+1}) \leq \dim \left(\overline{c_1(c_2^{-1}(G_j))}\right) \leq \dim G_j. \qquad \square$$

4.3. **Proposition.** There exists a non-empty open subset $V \subseteq X^0$ and a blow-up $\pi : \widetilde{X} \to X$, which is an isomorphism over V, such that for every correspondence $\widetilde{c} : \widetilde{C} \to \widetilde{X} \times \widetilde{X}$ lifting c, the closed subset $\widetilde{X} \setminus \pi^{-1}(V) \subseteq \widetilde{X}$ is locally $\widetilde{c}$-invariant.

Proof. Let $V \subseteq U \subseteq X^0$ be the open subsets constructed as in Lemma 4.2. Set $Z := X \setminus V$, $F := F(c, Z) = c_2(Z) \cap c_1^{-1}(V)$ and $G := G(c, Z)$.

For any irreducible component $S \in \mathrm{Irr}(F)$ of F and $i = 1, 2$, let $\mathcal{I}_{\overline{c_i(S)}/X}$ be the ideal sheaf corresponding to the closed subset $\overline{c_i(S)} \subseteq X$ (§ 2.12) and define

$$\mathcal{K}_S := \mathcal{I}_{\overline{c_1(S)}/X} + \mathcal{I}_{\overline{c_2(S)}/X} \subseteq \mathcal{O}_X, \quad \forall S \in \mathrm{Irr}(F).$$

Then set

$$\mathcal{K} := \prod_{S \in \mathrm{Irr}(F)} \mathcal{K}_S \subseteq \mathcal{O}_X.$$

Let $\pi : \widetilde{X} := \mathrm{Bl}_{\mathcal{K}} X \to X$ be the blow up of X with respect to the ideal sheaf $\mathcal{K}$. We claim that the open subset V and the blow-up π satisfy the properties that we want. We will proceed in 3 steps:

Step 1: π is an isomorphism over V.
Step 2: $\overline{\pi^{-1}(c_1(S))} \cap \overline{\pi^{-1}(c_2(S))} = \emptyset$.
Step 3: $G\big(\widetilde{c}, \pi^{-1}(Z)\big) = \emptyset$, hence by Corollary 3.7 (b), $\widetilde{X} \setminus \pi^{-1}(V) = \pi^{-1}(Z) \subseteq \widetilde{X}$ is locally $\widetilde{c}$-invariant for any $\widetilde{c}$ lifting c.

4.3.1. *Step 1.* By definition of $\mathcal{K}$ and Remarks 3.5 (a) we have

$$\mathrm{supp}(\mathcal{O}_X/\mathcal{K}) = \bigcup_{S \in \mathrm{Irr}(F)} \left(\overline{c_1(S)} \cap \overline{c_2(S)}\right) = G(c, Z) = G \subseteq Z = X \setminus V.$$

Therefore π is isomorphic over $V \subseteq X \setminus G$ by properties of blow-ups (§ 2.19.1).

4.3.2. *Step 2.* Since For each $S \in \mathrm{Irr}(F)$, we know $S \subseteq F \subseteq c_1^{-1}(V) \subseteq c_2^{-1}(U)$. Hence $c_1(S) \subseteq V \subseteq U$ and $c_2(S) \subseteq U$. Moreover, by construction, $U \setminus V = U \cap Z$ is locally $c|_U$-invariant, hence $U \subseteq X \setminus G$ by Lemma 3.6. So we have $c_i(S) \subseteq X \setminus G$, hence $c_i(S) \subseteq \overline{c_i(S)} \setminus G$ for all $i = 1, 2$. Then it follows from § 2.19.7 and § 2.19.4 that

$$\overline{\pi^{-1}(c_1(S))} \cap \overline{\pi^{-1}(c_2(S))} \subseteq \overline{\pi^{-1}\big(\overline{c_1(S)} \setminus G\big)} \cap \overline{\pi^{-1}\big(\overline{c_1(S)} \setminus G\big)} = \emptyset.$$

4.3.3. *Step 3.* Let $[\pi] = (\pi, \pi_C)$ be any lift of $c : C \to X \times X$ to $\widetilde{c} : \widetilde{C} \to \widetilde{X} \times \widetilde{X}$, that is, we have a commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\pi"'] & \widetilde{C} \arrow[l, "\widetilde{c}_1"'] \arrow[r, "\widetilde{c}_2"] \arrow[d, "\pi_C"] & \widetilde{X} \arrow[d, "\pi"] \\ X & C \arrow[l, "c_1"'] \arrow[r, "c_2"] & X. \end{tikzcd}$$

To show that $\widetilde{Z} := \widetilde{X} \setminus \pi^{-1}(V) = \pi^{-1}(Z) \subseteq \widetilde{X}$ is locally $\widetilde{c}$-invariant, it suffices to show that $\widetilde{G} := G(\widetilde{c}, \widetilde{Z}) = \emptyset$ by Lemma 3.6. By definition,

$$\widetilde{G} = \bigcup_{\widetilde{S} \in \mathrm{Irr}(\widetilde{F})} \overline{\widetilde{c}_1(\widetilde{S})} \cap \overline{\widetilde{c}_2(\widetilde{S})},$$

where $\widetilde{F} := F(\widetilde{c}, \widetilde{Z})$. So it is enough to show that for any $\widetilde{S} \in \mathrm{Irr}(\widetilde{F})$, $\overline{\widetilde{c}_1(\widetilde{S})} \cap \overline{\widetilde{c}_2(\widetilde{S})} = \emptyset$. In fact,

$$\begin{aligned} \widetilde{F} = F(\widetilde{c}, \widetilde{Z}) &= \widetilde{c}_2^{-1}(\widetilde{Z}) \cap c_1^{-1}(\widetilde{X} \setminus \widetilde{Z}) \\ &= \widetilde{c}_2^{-1}(\pi^{-1}(Z)) \cap \widetilde{c}_1^{-1}(\pi^{-1}(V)) \\ &= \pi_C^{-1}\big(c_2^{-1}(Z) \cap c_1^{-1}(V)\big) \\ &= \pi_C^{-1}(F(c, Z)) = \pi_C^{-1}(F). \end{aligned}$$

Hence $\pi_C(\widetilde{F}) \subseteq F$. So for any $\widetilde{S} \in \mathrm{Irr}(\widetilde{F})$, we have $\pi_C(\widetilde{S}) \subseteq S$ for some $S \in \mathrm{Irr}(F)$. It then follows that for any $i = 1, 2$, $\widetilde{c}_i(\widetilde{S}) \subseteq \pi^{-1}\big(\pi(\widetilde{c}_i(\widetilde{S}))\big) = \pi^{-1}\big(c_i(\pi_C(\widetilde{S}))\big) \subseteq \pi^{-1}(c_i(S))$, then $\overline{\widetilde{c}_i(\widetilde{S})} \subseteq \overline{\pi^{-1}(c_i(S))}$. So

$$\overline{\widetilde{c}_1(\widetilde{S})} \cap \overline{\widetilde{c}_2(\widetilde{S})} \subseteq \overline{\pi^{-1}(c_1(S))} \cap \overline{\pi^{-1}(c_2(S))} = \emptyset.$$

Then we conclude that $\widetilde{Z}$ is locally $\widetilde{c}$-invariant. $\square$

4.4. **Corollary.** Assume that everything is as in § 4.1, with $k = \mathbb{F}$. Moreover, assume that X and X^0 are defined over $\mathbb{F}_q$. Then there exists an open subset $V \subseteq X^0$ and a blow-up $\pi : \widetilde{X} \to X$, which is an isomorphism over V, such that V and π are defined over $\mathbb{F}_q$. And for every correspondence $\widetilde{c} : \widetilde{C} \to \widetilde{X} \times \widetilde{X}$ lifting c, the closed subset $\widetilde{X} \setminus \pi^{-1}(V) \subseteq \widetilde{X}$ is locally $\widetilde{c}$-invariant over $\mathbb{F}_q$.

Proof. As X^0 and X are defined over $\mathbb{F}_q$, there exists $\underline{X}$ with an open subset $\underline{X}^0$ such that $X = \underline{X} \times_{\mathbb{F}_q} \mathbb{F}$ and $X^0 = \underline{X}^0 \times_{\mathbb{F}_q} \mathbb{F}$. Denote by ω the canonical projection from $X = \underline{X} \times_{\mathbb{F}_q} \mathbb{F}$ to $\underline{X}$. As C, X and c are of finite type over $\mathbb{F}$, we know both C and c are defined over $\mathbb{F}_{q^r}$ for some r (§ 2.5), say c is the pullback of $\underline{C} \to \underline{X}_{\mathbb{F}_{q^r}} \times_{\mathbb{F}_{q^r}} \underline{X}_{\mathbb{F}_{q^r}}$, where $\underline{C}$ is of finite type over $\mathbb{F}_{q^r}$. Clearly $\underline{C}$ is of finite type over $\mathbb{F}_q$ as $\mathbb{F}_{q^r}/\mathbb{F}_q$ is a finite field extension. Then $(\omega \times \omega) \circ c : C \to \underline{X} \times_{\mathbb{F}_q} \underline{X}$ factors through $\underline{c} : \underline{C} \to \underline{X} \times_{\mathbb{F}_q} \underline{X}$. Then $\underline{X}$, $\underline{X}^0$, $\underline{C}$ and $\underline{c}$ satisfy the assumptions in § 4.1 with $k = \mathbb{F}_q$.

So by Proposition 4.3, there exists an open subset $\underline{V} \subseteq \underline{X}^0$ and a blow-up $\underline{\pi} : \underline{\widetilde{X}} \to \underline{X}$, such that all the properties in Proposition 4.3 are satisfied. Let V and $\pi : \widetilde{X} \to X$ be the corresponding pullback to $\mathbb{F}$ and denote by $\widetilde{\omega} : \widetilde{X} \to \underline{\widetilde{X}}$ the canonical projection. We now have the following diagram:

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] & X^0 \arrow[r, hook] \arrow[d] & X \arrow[d, "\omega"] & \widetilde{X} \arrow[l, "\pi"'] \arrow[d, "\widetilde{\omega"] & \text{over } \mathbb{F} \\ \underline{V} \arrow[r, hook] & \underline{X}^0 \arrow[r, hook] & \underline{X} & \underline{\widetilde{X}} \arrow[l, "\underline{\pi}"'] & \text{over } \mathbb{F}_q. \end{tikzcd}$$

We claim that V and π satisfy all the desired properties.

In fact, it is clear that π is an isomorphism over V and that V and π are defined over $\mathbb{F}_q$. Now for any $\widetilde{c} : \widetilde{C} \to \widetilde{X} \times \widetilde{X}$ that lifts $c : C \to X \times X$, consider the following diagram

$$
\begin{tikzcd}
& \widetilde{C} \arrow[rr, "\widetilde{c}"] \arrow[dl] \arrow[dd, equal] & & \widetilde{X} \times \widetilde{X} \arrow[dl, "\pi\times\pi"'] \arrow[dd, "\widetilde{\omega}\times\widetilde{\omega}"] \\
C \arrow[rr, "c"] \arrow[dd] & & X \times X \arrow[dd, "\omega\times\omega"' near start] & \\
& \widetilde{C} \arrow[rr, "\widetilde{d}" near end] \arrow[dl] & & \underline{\widetilde{X}} \times \underline{\widetilde{X}} \arrow[dl, "\underline{\pi}\times\underline{\pi}"] \\
\underline{C} \arrow[rr, "\underline{c}"] & & \underline{X} \times \underline{X} &
\end{tikzcd}
$$

where $\widetilde{d} = (\widetilde{\omega} \times \widetilde{\omega}) \circ \widetilde{c} : \widetilde{C} \to \underline{\widetilde{X}} \times_{\mathbb{F}_q} \underline{\widetilde{X}}$. Clearly $\widetilde{d}$ lifts $\underline{c}$. So by construction and Lemma 3.3, $\underline{X} \setminus \underline{V}$ is locally $\underline{c}$-invariant, so $\underline{\pi}^{-1}(\underline{X} \setminus \underline{V}) = \underline{\widetilde{X}} \setminus \underline{\pi}^{-1}(\underline{V}) \subseteq \underline{\widetilde{X}}$ is locally $\widetilde{d}$-invariant. Note also that $\widetilde{c}$ is a lift of $\widetilde{d}$, so

$$\widetilde{\omega}^{-1}(\underline{\widetilde{X}} \setminus \underline{\pi}^{-1}(\underline{V})) = \widetilde{X} \setminus (\underline{\pi} \circ \widetilde{\omega})^{-1}(\underline{V}) = \widetilde{X} \setminus (\omega \circ \pi)^{-1}(\underline{V}) = \widetilde{X} \setminus \pi^{-1}(V)$$

is locally $\widetilde{c}$-invariant. □

5. Geometric Construction by Pink

In this section, we reproduce a geometric construction given in [Pin92]. The proofs are basically based on computations using local coordinates (§ 2.14.3). All results for later use are summarized in Lemma 5.3.4.

5.1. **Setup.** Let X be a finite type k-scheme that is smooth of relative dimension d over k. Let ∂X be a *strict normal crossing divisor* (§ 2.17) with irreducible components X_i, where $i \in I$. As remarked in § 2.17.1, each X_i is also smooth if k is perfect.

5.1.1. For any subset $J \subseteq I$, we introduce the following subschemes:

$$X_J := \bigcap_{j \in J} X_j, \qquad Y_J := X_J \times X_J = \bigcap_{j \in J} X_j \times X_j,$$

$$\partial X_J := \bigcup_{i \in I \setminus J} X_{J \cup \{i\}} = X_J \cap \bigcup_{i \in I \setminus J} X_i, \qquad \partial Y_J := \bigcup_{i \in I \setminus J} Y_{J \cup \{i\}},$$

$$X_J^0 := X_J \setminus \partial X_J, \qquad Y_J^0 := Y_J \setminus \partial Y_J,$$

where we set $X_\emptyset := X$. Moreover, for each $i \in I$, write Y_i for $Y_{\{i\}}$ and set $Y := Y_\emptyset := X \times X$, $X^0 := X_\emptyset^0 = X \setminus \partial X$ and $Y^0 := Y_\emptyset^0 = Y \setminus \cup_{i \in I} Y_i$. Note that $X_J^0 \times X_J^0 \subsetneq Y_J^0$ unless $|J| = 1$.

5.1.2. Let $\mathcal{K}_i := \mathcal{I}_{Y_i/Y}$ be the ideal sheaf corresponding to the closed subscheme $Y_j \subseteq Y$. For each $J \subseteq I$, set $\mathcal{K}_J := \prod_{j \in J} \mathcal{K}_j$ and $\mathcal{K} := \mathcal{K}_I$. Denote by

$$\pi : \widetilde{Y} := \mathrm{Bl}_{\mathcal{K}} Y \to Y$$

the blow-up of Y with respect to the ideal sheaf $\mathcal{K}$ and let $\widetilde{Y}_J := \mathrm{Bl}_{\mathcal{K}_J} Y$ be the blow-up of Y with respect to $\mathcal{K}_J$ for each $J \subseteq I$. Recall § 2.19.4 and § 2.19.1 that there is a canonical projection $\widetilde{Y} \to \widetilde{Y}_J$, which is an isomorphism over the preimage of $\big(Y \setminus (\cup_{i \in I \setminus J} Y_i)\big) \subseteq Y$ in $\widetilde{Y}_J$.

5.1.3. For each $J \subseteq I$, let E_J be the total transform of Y_J, i.e., $E_J := \pi^{-1}(Y_J) \subseteq \widetilde{Y}$. So we have the following Cartesian diagram

$$\begin{tikzcd} E_J \arrow[r, hook, "i_J"] \arrow[d, "\pi_J"'] & \widetilde{Y} \arrow[d, "\pi"] \\ Y_J \arrow[r, hook] & Y \end{tikzcd} \tag{5.1}$$

where $i_J : E_J \hookrightarrow \widetilde{Y}$ is the inclusion and $\pi_J := \pi|_{E_J} : E_J \to Y_j$ is the restriction of π to E_J.

5.1.4. If $c : C \to Y = X \times X$ is a correspondence, denote by $\widetilde{c} : \widetilde{C} \to \widetilde{Y}$ the proper transform of c.

5.1.5. If we assume that $k = \mathbb{F}$, and X and X_i's are all separated and defined over $\mathbb{F}_q$, then we can consider the Frobenius endomorphism ϕ_q on X. As everything is defined over $\mathbb{F}_q$, the Frobenius ϕ_q then restricts to X_J and X_J^0 for all $J \subseteq I$, which we still denote by ϕ_q. We can introduce the following *closed* subschemes:

$\Gamma \subseteq Y = X \times X$	graph of ϕ_q
$\widetilde{\Gamma} \subseteq \widetilde{Y}$	proper transform of Γ
$\Gamma_J \subseteq X_J \times X_J = Y_J$	graph of ϕ_q
$\Gamma_J^0 \subseteq X_J^0 \times X_J^0 \subsetneq Y_J^0$	graph of ϕ_q

5.2. **Basic (Local) Case.** Let $I = \{1, 2, \dots, m\}$ where $m = |I|$. Consider

$$X := \mathbb{A}^I := \operatorname{Spec} k[x_1, x_2, \dots, x_m].$$

Set $X_i := V(x_i) = \operatorname{Spec} k[x_1, \dots, x_m]/(x_i)$ to be the coordinate hyperplane defined by x_i. Then $\partial X := \cup X_i = \operatorname{Spec} k[x_1, \dots, x_m]/(x_1 x_2 \dots x_m)$ is the union of all coordinate hyperplanes and is a *strict normal crossing divisor* (§ 2.17). For any $J \subseteq I$, we have

$$X_J = \cap_{j \in J} X_j = \cap_{j \in J} \operatorname{Spec} \frac{k[x_1, \dots, x_m]}{(x_j)} = \operatorname{Spec} \frac{k[x_1, \dots, x_m]}{(x_j)_{j \in J}} \cong \mathbb{A}^{I \setminus J},$$

$$Y_J = X_J \times X_J = \operatorname{Spec} \frac{k[x_1, y_1, \dots, x_m, y_m]}{(x_j, y_j)_{j \in J}} \cong (\mathbb{A}^2)^{I \setminus J}.$$

Then it is easy to see that the resulting blow-up is

$$\pi : \widetilde{Y} = \mathrm{Bl}_{\mathcal{K}} Y \cong (\widetilde{\mathbb{A}^2})^I \longrightarrow (\mathbb{A}^2)^I,$$

where $\widetilde{\mathbb{A}^2}$ is the blow-up of the affine plane with respect to the origin, or more precisely, with respect to the ideal $(x, y) \subseteq k[x, y]$. Explicitly we have

$$\widetilde{Y} = \left(\operatorname{Proj}_{k[x_i, y_i]} \frac{k[x_i, y_i, a_i, b_i]}{(a_i y_i - b_i x_i)} \right)^I \subseteq (\mathbb{A}^2 \times \mathbb{P}^1)^I. \tag{5.2}$$

We also know that

$$\Gamma = \left(\operatorname{Spec} \frac{k[x_i, y_i]}{(y_i - x_i^q)} \right)^I \subseteq (\mathbb{A}^2)^I = Y.$$

Moreover, we have for each $J \subseteq I$,

$$E_J := \pi^{-1}(Y_J) \cong (\widetilde{\mathbb{A}^2})^{I \setminus J} \times (\mathbb{P}^1)^J, \tag{5.3}$$

and $\pi_J : E_J \to Y_J$ is given by the composition

$$(\widetilde{\mathbb{A}^2})^{I \setminus J} \times (\mathbb{P}^1)^J \to (\widetilde{\mathbb{A}^2})^{I \setminus J} \to (\mathbb{A}^2)^{I \setminus J}. \tag{5.4}$$

5.2.1. *Lemma.* $E_J \subseteq \widetilde{Y}$ is smooth of dimension $2|I| - |J| = 2\dim X - |J|$.

Proof. This follows immediately from eq. (5.3). $\square$

5.2.2. *Lemma.* Suppose $k = \mathbb{F}$. Assume that $c : C \to X \times X = \mathbb{A}^I \times \mathbb{A}^I$ is contracting near $\partial X = \partial \mathbb{A}^I$ over $\mathbb{F}_q$ (§ 3.10). With the notations in § 5.1.5, we have

$$\widetilde{c}(\widetilde{C}) \cap \widetilde{\Gamma} \subseteq \pi^{-1}(X^0 \times X^0).$$

Moreover, the projection $\pi|_{\widetilde{\Gamma}} : \widetilde{\Gamma} \to \Gamma$ is an isomorphism.

Proof. Get down to equations. Let $\widetilde{Y}'$ be the open subscheme of $\widetilde{Y} = (\widetilde{\mathbb{A}}^2)^I$ where all a_i in the explicit form (5.2) do not vanish, that is

$$\widetilde{Y}' := (D_+(a_i))^I \cong \left(\operatorname{Spec} \frac{k[x_i, y_i, b'_i]}{(y_i - b'_i x_i)}\right) \subseteq (\mathbb{A}^3)^I. \tag{5.5}$$

Then consider the closed subscheme

$$\widetilde{\Gamma}' := \left(\operatorname{Spec} \frac{k[x_i, y_i, b'_i]}{(y_i - x_i^q, b'_i - x_i^{q-1})}\right)^I \subseteq \widetilde{Y}'.$$

Clearly, $\pi : \widetilde{Y} \to Y$ induces an isomorphism $\widetilde{\Gamma}' \to \Gamma$, in particular, it is proper. Note that π is proper and $\Gamma \subseteq Y$ is closed (as X is separated), so $\widetilde{\Gamma}'$ is closed in $\widetilde{Y}$, hence $\widetilde{\Gamma}'$ has to coincide with $\widetilde{\Gamma}$. Therefore $\widetilde{\Gamma} \subseteq \widetilde{Y}'$.

Consequently, it suffices to show that $\widetilde{c}(\widetilde{C}) \cap \widetilde{Y}' \subseteq \pi^{-1}(X^0 \times X^0)$, which in turn will follow from

$$\widetilde{C}' := \widetilde{c}^{-1}(\widetilde{Y}') \subseteq \widetilde{c}^{-1}\big(\pi^{-1}(X^0 \times X^0)\big). \tag{5.6}$$

Denote by τ the induced map from $\widetilde{C} \to C$. We then have a commutative diagram

$$\begin{tikzcd}
\widetilde{C}' \arrow[r] \arrow[d, hook] & \widetilde{Y}' \arrow[d, hook] & \widetilde{Y}' \cap \pi^{-1}(Y^0) \arrow[l, hook'] \arrow[d, hook] \\
\widetilde{C} \arrow[r, "\widetilde{c}"] \arrow[d, "\tau"'] & \widetilde{Y} \arrow[d, "\pi"] & \pi^{-1}(Y^0) \arrow[l, hook'] \arrow[d] \\
C \arrow[r, "c"] & Y & Y^0 \arrow[l, hook']
\end{tikzcd} \tag{5.7}$$

with all hooked arrows being open embeddings. Recall the coordinate rings of Y and $\widetilde{Y}'$ (eq. (5.2) and eq. (5.5)). Set $x := \prod_{i \in I} x_i$ and $y := \prod_{i \in I} y_i$, which are regular functions on Y, $\widetilde{Y}$ and $\widetilde{Y}'$ (here and later we slightly abuse notations by using the same x, but this will not make too much confusion), and set $b' := \prod_{i \in I} b'_i$, which is a regular function on $\widetilde{Y}'$. Then to prove (5.6), it suffices to show the functions x and y on Y pull back to invertible functions on $\widetilde{C}'$, in other words, to show $\widetilde{c}^{-1}(x)$ and $\widetilde{c}^{-1}(y)$ are invertible over $\widetilde{C}'$.

Recall that c is assumed to be contracting near ∂X, which means there exists a natural number $n > 0$, and a regular function f on C, such that $c^{-1}(x)^n = c^{-1}(y)^{n+1} \cdot f$ over C. Therefore, by pulling back to $\widetilde{C}'$ along τ, and using that $c \circ \tau = \pi \circ \widetilde{c}$, we get

$$\widetilde{c}^{-1}(x)^n = \widetilde{c}^{-1}(y)^{n+1} \cdot \tau^{-1}(f). \tag{5.8}$$

So it is enough to show $\widetilde{c}^{-1}(x)$ is invertible on $\widetilde{C}'$.

Recall that by definition, the proper transform $\widetilde{C}$ of C, is the closure in $C \times_Y \widetilde{Y}$ of

$$(C \times_Y \widetilde{Y}) \setminus \big(C \times_Y \partial Y \times_Y \widetilde{Y}\big) = C \times_Y Y^0 \times_Y \widetilde{Y}.$$

It follows that $\widetilde{C}' = \widetilde{C} \times_{\widetilde{Y}} \widetilde{Y}'$ is the closure of $C \times_Y (Y^0 \times_Y \widetilde{Y}')$ in $C \times_Y \widetilde{Y}$. Therefore it is enough to show $\widetilde{c}^{-1}(x)$ is invertible on $C \times_Y (Y^0 \times_Y \widetilde{Y}') = C \times_Y \big(\pi^{-1}(Y^0) \cap \widetilde{Y}'\big) \subseteq C \times_Y \widetilde{Y}'$, which will in turn follow from the fact that x is invertible on $\pi^{-1}(Y^0) \cap \widetilde{Y}' \subseteq \widetilde{Y}'$. Observing that for any closed point P in $\pi^{-1}(Y^0) \cap \widetilde{Y}'$, we know $\pi(P) \in Y^0$ hence for all $i \in I$, $x_i(\pi(P)) \neq 0$ or $y_i(\pi(P)) \neq 0$. Noting also that on $\widetilde{Y}'$, $y_i(P) = b_i'(P)x_i(P)$, we can then conclude that $x_i(P) \neq 0$ for all i. Hence $x = \prod_{i \in I} x_i$ is invertible on $\pi^{-1}(Y^0) \cap \widetilde{Y}'$. So we are done. □

5.2.3. *Remark.* There is an "intuitive" way to see $\widetilde{\Gamma} \subseteq \widetilde{\Gamma}'$ according to an elementary description of the blow-up of $\mathbb{A}^2$ with center $(0,0)$, see Figure 2 for example. Besides, that $\widetilde{\Gamma} \cong \Gamma$ could also be seen from the blow-up closure lemma (§ 2.19.2) and the fact blow-up with a center that is an effective Cartier divisor is an isomorphism (§ 2.19.1).

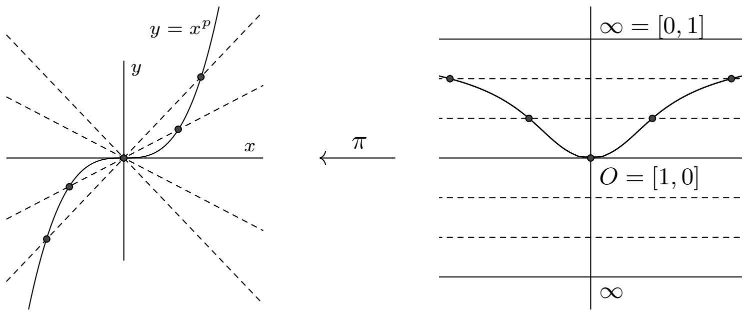

FIGURE 2. $\widetilde{\Gamma} \subseteq \widetilde{\Gamma}'$

5.2.4. *Lemma.* Suppose $k = \mathbb{F}$. For each $J \subseteq I$, $\overline{\pi_J^{-1}(\Gamma_J^0)} \subseteq E_J$ is smooth of dimension $|I|$ over $\mathbb{F}$. And the scheme-theoretic preimage $\pi_J^{-1}(\Gamma_J) \subseteq E_J$ is a scheme-theoretic union of $\overline{\pi_{J'}^{-1}(\Gamma_{J'}^0)}$ with J' running over all subsets of I that contain J.

Proof. Recall eq. (5.4) that π_J is given by $(\widetilde{\mathbb{A}^2})^{I\setminus J} \times (\mathbb{P}^1)^J \to (\widetilde{\mathbb{A}^2})^{I\setminus J} \to (\mathbb{A}^2)^{I\setminus J}$ where the first map is smooth of relative dimension $|J|$ and the second map is denoted by π'_J. Then $\pi_J^{-1}(\Gamma_J^0) = \big((\pi'_J)^{-1}(\Gamma_J^0)\big) \times (\mathbb{P}^1)^J$ and similar for $\pi_J^{-1}(\Gamma_J)$. So the original statement for $X = \mathbb{A}^I$ with blow-up π follows from that for $X = \mathbb{A}^{I\setminus J}$ with blow-up π'_J. So we reduce the problem to the case $J = \emptyset$.

In this case, $\Gamma^0 = \Gamma^0_\emptyset$ and $\overline{\pi^{-1}(\Gamma^0)}$ is the proper transform of Γ, hence isomorphic to Γ as proved in § 5.2.2 and is smooth of dimension $|I|$.

The second assertion reduces to the case $|I| = 1$ immediately. Note that $\Gamma = \operatorname{Spec} k[x,y]/(y - x^q)$. So

$$\begin{aligned}\pi^{-1}(\Gamma) &= \operatorname{Proj}_{k[x,y]} \frac{k[x,y,a,b]}{(y - x^q, ax^q - bx)} \\ &= \operatorname{Proj}_{k[x,y]} \frac{k[x,y,a,b]}{(y - x^q, ax^{q-1} - b)} \cup \operatorname{Proj}_k k[a,b] \\ &\cong \widetilde{\Gamma} \cup \mathbb{P}^1\end{aligned}$$

is the scheme-theoretic union of $\overline{\pi^{-1}(\Gamma^0)} = \widetilde{\Gamma}$ and $\overline{\pi_I^{-1}(\Gamma_I^0)} = \mathbb{P}^1$. □

5.3. **General Case.** For the general scheme X, we may use the following two observations to reduce the problem to the basic case.

5.3.1. *Observation 1.* As ∂X is a strict normal crossing divisor (§ 2.16), and X is smooth over k, as the construction in § 2.14.3, we know that for any point $P \in X_I = \cap_{i\in I} X_i \subseteq X$, there are an open neighborhood $U_P \subseteq X$ of P, and regular functions $\psi_i \in \mathcal{O}_X(U)$, such that $\psi := (\psi_i) : U \to \mathbb{A}^I$ is smooth of relative dimension $d - |I|$ and $X_i \cap U = \psi^{-1}(V(x_i))$, where $V(x_i)$ denotes the vanishing locus of x_i in $\mathbb{A}^I$.

5.3.2. *Observation 2.* For any two points $P, Q \in X_I$, let ψ_P, ψ_Q, U_P, U_Q are the morphisms and neighborhoods of P and Q respectively as in § 5.3.1. Then $\psi := (\psi_P, \psi_Q) : U := U_p \times U_Q \to (\mathbb{A}^2)^I$ is smooth of relative dimension $2d - 2|I|$, so in particular flat. Then from the fact that blow-ups commute with flat base change (§ 2.19.5) and § 2.19.1, it follows that ψ induces an isomorphism $\pi^{-1}(U) = \widetilde{Y} \times_Y U \to (\widetilde{\mathbb{A}^2})^I \times_{\mathbb{A}^2} U$.

5.3.3. Using these two observations, and noting that the statements in § 5.2 are all local, we could reduce the proof for general X to that for $\mathbb{A}^n$. We will make this precise as follows.

(a) § 5.2.1 in the general case: The same statement holds.

Proof. It suffices to show that for each closed point $P \in Y_J$, there is an open neighborhood $U \subset Y$ such that $\pi^{-1}(U) \cap E_J$ is smooth of dimension $2d - |J|$. We do the following tricks:

i) If $P \in \partial Y_J$, then $P \in \partial Y_{J\cup\{i\}}$ for some $i \in I \setminus J$. So either $P \in \partial Y_{J\cup\{i\}}$ or $P \in Y^0_{J\cup\{i\}}$. If the former holds, we continue this argument until reducing to the case $P \in Y_I$; or we will arrive at some $J \subseteq \widetilde{J} \subseteq I$, such that $P \in Y^0_{\widetilde{J}}$. Then we move on to the next step.
ii) If $P \in Y^0_J$, then $P \in Y \setminus (\cup_{i\in I\setminus J} Y_i)$. Recall § 5.1.2, we can make a successive blowup $\widetilde{Y} \to \widetilde{Y}_J \to Y$. It is enough to show that there is a neighborhood $U \subseteq P$ of P, such that the preimage of U in $\widetilde{Y}_J$ is smooth of dimension $2d - |J|$. So by replacing I by J, we can also assume that we are in the case $P \in Y_I$.

Now applying § 5.3.2, we reduce the problem to the basic case. Notice that $U \to (\mathbb{A}^2)^I$ is of relative dimension $2d - 2|I|$, we finally obtain that $\dim E_J = (2d - 2|I|) + (2|I| - |J|) = 2d - |J|$. $\square$

(b) § 5.2.2 in the general case: Assume further that X and X_i's are defined $\mathbb{F}_q$, and relax the condition that c is contacting to that c is *locally* contracting near ∂X over $\mathbb{F}_q$. Then the same conclusion holds.

Proof. Let $\widetilde{Q} \in \widetilde{c}(\widetilde{C}) \cap \widetilde{\Gamma} \subseteq \widetilde{Y}$ be a closed point. Write $Q = (P_1, P_2) := \pi(\widetilde{Q}) \in \Gamma \subseteq Y$, i.e., $P_i := \mathrm{pr}_i(\pi(\widetilde{Q})) \in X$, where $\mathrm{pr}_i : X \times X = Y \supseteq \Gamma \to X$ are the two projections. It follows that $P_2 = \phi_q(P_1)$. So to show $\pi(Q) \in X^0 \times X^0$, it suffices to prove that $P_1 \in X^0$. Since c is locally contracting near ∂X over $\mathbb{F}_q$, there is an open neighborhood $U \subseteq X$ of P_1 defined over $\mathbb{F}_q$ such that $c|_U$ is contracting near $U \cap \partial X$. So by replacing c by $c|_U$ and X by U, we can assume c is contracting near ∂X over $\mathbb{F}_q$. Now suppose $P_1 \notin X^0$, then using the trick as item (a), we may assume that $P_1 \in X_I$, so $P_2 = \phi_q(P_1) \in X_I$:

i) As $P_1 \in \partial X$, so $P_1 \in X_i$ for some $i \in I$. Then there are two cases, $P_1 \in \partial X_i$ or $P_1 \in X^0_i$. If the former holds, continue using this argument until reducing to $P \in X_I$; else we will stop at some $J \subseteq I$ such that $P_1 \in X^0_J$, then move on to the next step.

ii) Now suppose $P_1 \in X_J^0$. Then $P_2 = \phi_q(P_1) \in X_J^0$, hence $Q = (P_1, P_2) \in X_J^0 \times X_J^0 \subseteq Y_J^0$. In this case, consider the successive blowup $\widetilde{Y} \to \widetilde{Y}_J \to Y$ as in item (a), we can reduce to the case $Q \in Y_I$ so $P_1 \in X_I$.

Using § 5.3.1, we know there is an open neighborhood U of P_1, and a smooth morphism $\psi : U \to \mathbb{A}^I$ such that $X_i = \psi^{-1}\big(V(x_i)\big)$ for all $i \in I$. As X is defined over $\mathbb{F}_q$, we may assume ψ and U are both defined over $\mathbb{F}_q$. So $P_2 = \phi_q(P_1) \in U$. By replacing c by $c|_U$, we may assume that $U = X$. Then consider the composition $c' := (\psi, \psi) \circ c : C \to \mathbb{A}^I \times \mathbb{A}^I$. Clearly that c' is contracting near $\partial\mathbb{A}^I := \cup_{i\in I} V(x_i)$ over $\mathbb{F}_q$ if c is contracting near ∂X over $\mathbb{F}_q$. In this we reduce the proof to the basic case, i.e., the case where $X = \mathbb{A}^I$. □

(c) § 5.2.4 in the general case: Assume further X and X_i's are defined over $\mathbb{F}_q$. Then the same conclusion holds.

Proof. Assume $Q = (P_1, P_2) \in \Gamma_J^0 \subseteq X_J^0 \times X_J^0 \subsetneq Y_J^0$ (resp. $Q \in \Gamma_J \subseteq X_J \times X_J = Y_J$). The same argument in item (b) leads us to an open subset $U \subseteq X$ and a smooth morphism $\psi : U \to \mathbb{A}^I$ of relative dimension $d - |I|$, with $X_i = \psi^{-1}(V(x_i))$ for all i and ψ, U both defined over $\mathbb{F}_q$, so $P_2 = \phi_q(P_1) \in U$. Then we may replace X be U. Then as in § 5.3.2, ψ induces an isomorphism $\widetilde{Y} \cong Y \times_{(\mathbb{A}^2)^I} (\widetilde{\mathbb{A}^2})^I$. So everything follows from the basic case. □

5.3.4. *Lemma.* Assume everything in § 5.1. We conclude this section with the following facts:

(a) E_J is smooth of dimension $2d - |J|$;
(b) Assume X and X_i's are defined over $\mathbb{F}_q$ and $c : C \to X \times X$ is *locally contracting* near ∂X over $\mathbb{F}_q$. Then $\widetilde{c}(\widetilde{C}) \cap \widetilde{\Gamma} \subseteq \pi^{-1}(X^0 \times X^0)$;
(c) Assume X and X_i's are defined over $\mathbb{F}_q$. The scheme-theoretic preimage $\pi_J^{-1}(\Gamma_J) \subseteq Y_J$ is a scheme-theoretic union of $\overline{\pi_{J'}^{-1}(\Gamma_{J'}^0)}$ with J' running over all subsets of I that containing J. And each $\overline{\pi_J^{-1}(\Gamma_J^0)} \subseteq E_J$ is smooth of dimension d.

6. An Analogue of the Lefschetz Trace Formula

6.1. **Setup.** Recall the notations in § 5.1 with $k = \mathbb{F}$. We assume that X is a smooth and proper $\mathbb{F}$-scheme that is defined over $\mathbb{F}_q$. Let $\partial X := \cup_{i\in I} X_i$ be a strict normal crossing divisor with irreducible components X_i, and assume that ∂X is also defined over $\mathbb{F}_q$. Let $C \hookrightarrow Y = X \times X$ be an integral closed subscheme of dimension d with $C \cap Y^0 \neq \emptyset$ and $\widetilde{C}$ be its proper transform under the blow-up constructed in § 5.1.2.

6.1.1. *Cycles.* Recall Appendix A.1. Let $[C] \in A_d(Y)$ and $[\widetilde{C}] \in A_d(\widetilde{Y})$ be cycles associated with C and $\widetilde{C}$. For each subset $J \subseteq I$, we know Y_J and E_J are smooth and proper over k (§ 2.19.6 and § 5.3.4). Recall diagram (5.1), then define

$$[\widetilde{C}]_J := (\pi_J)_* i_J^* [\widetilde{C}] \in A_{d-|J|}(Y_J).$$

Let $[\Gamma] \in A_d(Y)$, $[\widetilde{\Gamma}] \in A_d(\widetilde{Y})$ and $[\Gamma_J] \in A_{d-|J|}(Y_J)$ be the cycles associated with Γ, $\widetilde{\Gamma}$ and Γ_J, where the dimension of each cycle is either obvious or computed in § 5.3.4.

6.2. **Lemma.** As in situation of § 6.1, we have

$$[\widetilde{C}] \cdot [\widetilde{\Gamma}] = [C] \cdot [\Gamma] + \sum_{J \neq \emptyset} (-1)^{|J|} [\widetilde{C}]_J \cdot [\Gamma_J] \tag{6.1}$$

Proof. Noting that $[C] = \pi_*[\widetilde{C}]$ (Appendix A.2), and $[\widetilde{C}]_J := (\pi_J)_* i_J^*[\widetilde{C}]$, we can conclude from the projection formula (Appendix A.7) that

$$[C]\cdot[\Gamma] = [\widetilde{C}]\cdot\pi^*[\Gamma], \quad \text{and} \quad [\widetilde{C}]_J\cdot[\Gamma_J] = [\widetilde{C}]\cdot(i_J)_*\pi_J^*[\Gamma_J].$$

Therefore to prove eq. (6.1), it is enough to prove

$$[\widetilde{\Gamma}] = \pi^*[\Gamma] + \sum_{J\neq\emptyset}(-1)^{|J|}(i_J)_*\pi_J^*[\Gamma_J]. \tag{6.2}$$

Observe that E_J, Y_J and Γ_J are all smooth. So § 2.15.1 and Appendix A.6.1 together imply that

$$\pi_J^*[\Gamma_J] = [\pi_J^{-1}(\Gamma_J)].$$

Moreover, by Lemma 5.3.4, we know that $[\pi_J^{-1}(\Gamma_J)] = \sum_{J'\supseteq J}[\overline{\pi_{J'}^{-1}(\Gamma_{J'}^0)}]$, so

$$(i_J)_*\pi_J^*[\Gamma_J] = \sum_{J'\supseteq J}[\overline{\pi_{J'}^{-1}(\Gamma_{J'}^0)}], \tag{6.3}$$

where we also use Appendix A.2 and the fact that i_J is a closed embedding. In particular if $J=\emptyset$, we have

$$\pi^*[\Gamma] = [\widetilde{\Gamma}] + \sum_{J'\neq\emptyset}[\overline{\pi_{J'}^{-1}(\Gamma_{J'}^0)}]. \tag{6.4}$$

Combining eq. (6.3), eq. (6.4) and the fact that

$$\sum_{J'\supseteq J}(-1)^{|J|} = 0, \quad \forall J'\neq\emptyset,$$

we can conclude eq. (6.2) hence eq. (6.1). □

6.3. **Corollary.** As in situation of § 6.1, assume each $c_i : C \hookrightarrow Y \twoheadrightarrow X$ is dominant. Then for sufficiently large n, we have

$$[\widetilde{C}]\cdot[\widetilde{\Gamma}_{q^n}] \neq 0,$$

hence $\widetilde{C}\cap\widetilde{\Gamma}_{q^n}\neq\emptyset$.

Proof. Recall eq. (B.6), for all J and n we have,

$$[C]\cdot[\Gamma_{q^n}] = \mathrm{Tr}\left((\phi_q^*)^n\circ \mathrm{H}^*([C])\,\middle|\,\mathrm{H}^*(X,\mathbb{Q}_\ell)\right),$$
$$[\widetilde{C}]_J\cdot[\Gamma_{J,q^n}] = \mathrm{Tr}\left((\phi_q^*)^n\circ \mathrm{H}^*([\widetilde{C}]_J)\,\middle|\,\mathrm{H}^*(X_J,\mathbb{Q}_\ell\right),$$

where Tr stands for the alternating trace as in eq. (B.3). Now we have the following observations according to Deligne's theorem § 2.22.

(a) Recall eq. (B.5) that $\mathrm{H}^{2d}([C]) = \deg(c_1)\,\mathrm{id}$. Note also that ϕ_q^* acts as multiplication by q^d on $\mathrm{H}^{2d}(X,\mathbb{Q}_\ell)$ ([Mil80, VI, Theorem 12.6]). Therefore,

$$[C]\cdot[\Gamma_{q^n}] = \deg(c_1)(q^d)^n + (\text{lower order (in } n\text{) terms}) \sim \deg(c_1)q^{dn}.$$

(b) Let A be a $d\times d$ matrix with entries in $\mathbb{C}$. Suppose A has Jordan form and all eigenvalues of A have modulus less than a with $a>1$. Then we know that all entries of A^n is of magnitude $O(n^{d-1}a^n)$. In fact, each entry of A^n is either 0 or $C_n^k\lambda^{n-k}$ for some $0\le k<d$ and some eigenvalue λ of A. As a consequence, for all $J\neq\emptyset$,

$$[\widetilde{C}]_J\cdot[\Gamma_J] = O\big(n^{d-|J|-1}(q^{2(d-|J|)/2})^n\big) = O\big(n^{d-|J|-1}q^{(d-|J|)n}\big).$$

Then Lemma 6.2 implies $[\widetilde{C}]\cdot[\widetilde{\Gamma}]\sim\deg(c_1)q^{dn}$, hence for sufficiently large n, $[\widetilde{C}]\cdot[\widetilde{\Gamma}_{q^n}]\neq 0$. □

7. Proof of the Theorem

In this section, we give the full proof of our main theorem 1.2. We start with the assumptions as stated in the main theorem.

7.1. **Some Preparations.**

7.1.1. Suppose (f_{C^0}, f) is a morphism (§ 2.20.3) from $c : C \to X \times X$ to $c^0 : C^0 \to X^0 \times X^0$, where X is a finite type $\mathbb{F}$-scheme defined over $\mathbb{F}_q$ and $f : X \to X^0$ is defined over $\mathbb{F}_q$. In other words, we have a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{c} & X \times X \\ {\scriptstyle f_{C^0}}\downarrow & & \downarrow{\scriptstyle f\times f} \\ C^0 & \xrightarrow{c^0} & X^0 \times X^0. \end{array}$$

Denote by Γ_{q^n} the graph of the Frobenius endomorphism $\phi_{q^n,X}$ on X. Then $\Gamma_{q^n} \subseteq (f \times f)^{-1}(\Gamma^0_{q^n})$ (f is defined over $\mathbb{F}_q$). Consequently,

$$c^{-1}(\Gamma_{q^n}) \subseteq c^{-1}\big((f \times f)^{-1}(\Gamma^0_{q^n})\big) = (f_{C^0})^{-1}\big((c^0)^{-1}(\Gamma^0_{q^n})\big). \tag{7.1}$$

Therefore $c^{-1}(\Gamma_{q^n}) \neq \emptyset$ implies that $(c^0)^{-1}(\Gamma^0_{q^n}) \neq \emptyset$. In other words, Theorem 1.2 for c^0 follows from that for c.

As an immediate consequence, if $U \subseteq X^0$ is any open subscheme of X^0 that is defined over $\mathbb{F}_q$, then we can reduce the proof of Theorem 1.2 for c^0 to that for $c^0|_U$. Similarly, if $W \subseteq C^0$ is open in C^0, then Theorem 1.2 for $c^0|_W$ implies that for c^0. (§ 2.20.4)

7.1.2. Conversely, if f_{C^0} is surjective and $f = \mathrm{id}_{X^0}$,

$$\begin{array}{ccc} C & \xrightarrow{c} & X^0 \times X^0 \\ {\scriptstyle f_{C^0}}\twoheaddownarrow & & \downarrow{\scriptstyle \mathrm{id}_{X^0}\times \mathrm{id}_{X^0}} \\ C^0 & \xrightarrow{c^0} & X^0 \times X^0, \end{array}$$

then eq. (7.1) becomes

$$c^{-1}(\Gamma^0_{q^n}) = (f_{C^0})^{-1}\big((c^0)^{-1}(\Gamma^0_{q^n})\big).$$

Hence if $(c^0)^{-1}(\Gamma^0_{q^n}) \neq \emptyset$, then $c^{-1}(\Gamma_{q^n}) \neq \emptyset$ too; that is, Theorem 1.2 for c^0 also implies that for c in this case.

7.1.3. Moreover, for any $m \in \mathbb{N}$, consider the Frobenius twist $(c^0)^{(m)}$ of c^0:

$$\begin{array}{ccc} C^0 & \xrightarrow{c^0} & X^0 \times X^0 \\ {\scriptstyle \mathrm{id}_{C^0}}\downarrow & & \downarrow{\scriptstyle \phi_{q^m}\times \mathrm{id}_{X^0}} \\ C^0 & \xrightarrow{(c^0)^{(m)}} & X^0 \times X^0. \end{array}$$

Then it holds that $\big(\phi_{q^m} \times \mathrm{id}_{X^0}\big)^{-1}(\Gamma^0_{q^n}) = \Gamma^0_{q^{m+n}}$. Hence

$$\big((c^0)^{(m)}\big)^{-1}(\Gamma^0_{q^n}) = (c^0)^{-1}(\Gamma^0_{q^{m+n}}). \tag{7.2}$$

That is to say, Theorem 1.2 for c^0 follows from that for $(c^0)^{(m)}$.

7.1.4. Let's fix an $r \in \mathbb{N}$. Then any $n \in \mathbb{N}$ can be written as $n = rn' + m$ with $m \in \{0, 1, \dots, r-1\}$. It follows from eq. (7.2) that

$$((c^0)^{(m)})^{-1}(\Gamma_{q^{rn'}}) = (c^0)^{-1}(\Gamma_{q^{rn'+m}}).$$

So Theorem 1.2 holds for c^0 over $\mathbb{F}_q$ if and only if it holds for $(c^0)^{(m)}$ over $\mathbb{F}_{q^r}$ for all $m \in \{0, 1, \dots, r-1\}$.

7.2. **Reduce to the case with "nice" schemes and correspondence.**

7.2.1. *Can assume X^0 and C^0 are reduced.* Consider the natural diagram (§ 2.12 and § 2.6):

$$\begin{tikzcd} C^0_{\mathrm{red}} \arrow[r, "c^0_{\mathrm{red}}"] \arrow[d] & X^0_{\mathrm{red}} \times X^0_{\mathrm{red}} \arrow[d] \\ C^0 \arrow[r, "c^0"] & X^0 \times X^0. \end{tikzcd}$$

So by § 7.1.1, by replacing c^0 by c^0_{red}, we can assume our schemes are reduced, hence also integral by assumption of Theorem 1.2.

7.2.2. *Can assume X^0 is quasi-projective and* $\dim C^0 = \dim X^0$. Note that c^0_i's are both dominant, so we know $r := \dim C^0 - \dim X^0 \geq 0$. Recall § 2.9, there are open affine subsets $U_i \subseteq X^0$, such that $(c^0_i)^{-1}(U_i) \to U_i$ factors as $(c^0_i)^{-1}(U_i) \to \mathbb{A}^r \times U_i \to U_i$ where $(c^0_i)^{-1}(U_i) \to \mathbb{A}^r \times U_i$ is dominant and $\dim(\mathbb{A}^r \times U) = \dim\big((c^0_i)^{-1}(U)\big) = \dim C^0$. Set $U := U_1 \cap U_2$ which is quasi-projective. So according to § 7.1.1, we may replace X^0 by $\mathbb{A}^r \times U$ and C^0 by $c_1^{-1}(U) \cap c_2^{-1}(U)$, so assume X^0 is quasi-projective and $\dim C^0 = \dim X^0$.

7.2.3. *Can assume c^0 is a locally closed embedding.* Assume we are in the situation of § 7.2.2. Let C' be the scheme-theoretic image of c^0, i.e., the scheme-theoretic closure $\overline{c^0(C^0)}$ of $c^0(C^0)$ in $X^0 \times X^0$. As for each i, c^0_i is dominant, recalling again § 2.9, we know that $c^0_i(C^0) \subseteq X^0$ contains an open dense subset U_i of X^0. Hence $c^0(C^0)$ contains an open subset U of C', such that the images of U in X^0 under the canonical projections $\mathrm{pr}_i : X^0 \times X^0 \to X^0$ are dense. So the composition $U \hookrightarrow C' \hookrightarrow X^0 \times X^0$ is a locally closed embedding (§ 2.6). Set $W := (c^0)^{-1}(U)$ and consider the following two lifts:

$$\begin{tikzcd} W \arrow[r, "c^0|_W"] \arrow[d, hook] & X^0 \times X^0 \arrow[d, "\mathrm{id} \times \mathrm{id}"] \\ C^0 \arrow[r, "c^0"] & X^0 \times X^0, \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} W \arrow[r, "c^0|_W"] \arrow[d, two heads] & X^0 \times X^0 \arrow[d, "\mathrm{id} \times \mathrm{id}"] \\ U \arrow[r, hook, "i"] & X^0 \times X^0. \end{tikzcd}$$

Applying § 7.1.1 to the first diagram and § 7.1.2 to the second one, we can replace c^0 by the locally embedding $i : U' \hookrightarrow X^0 \times X^0$.

7.2.4. *Can assume c^0 is a closed embedding.* Assume we are in the situation of § 7.2.3. Note that $\dim(C' \setminus U) < \dim C^0 = \dim X^0$. Then $(X^0)' := X^0 \setminus \big(\overline{\mathrm{pr}_1(C' \setminus U)} \cup \overline{\mathrm{pr}_1(C' \setminus U)}\big)$ is non-empty and open in X^0. Applying § 7.1.1 to the restriction $c^0|_{(X^0)'}$, we may further reduce from a locally closed embedding $c : C^0 \to X^0 \times X^0$ to a closed embedding $c^0|_{(X^0)'}$.

7.2.5. *Can assume c^0 is a lift of a "nice" correspondence.* Assume we are in the situation of § 7.2.4. As X^0 is quasi-projective and integral, by § 2.6, there is an open dense embedding $j : X^0 \to X$ over $\mathbb{F}_q$ with X projective and integral. Let $C := \overline{(j \times j)(c^0(C^0))}$ be the closure of $c^0(C^0)$ in $X \times X$. Then we obtain a Cartesian diagram

$$\begin{tikzcd} C^0 \arrow[r, "c^0"] \arrow[d, "j_C"'] & X^0 \times X^0 \arrow[d, "j\times j"] \\ C \arrow[r, "c"] & X \times X, \end{tikzcd} \tag{7.3}$$

where the inclusion $c : C \hookrightarrow X \times X$ is a closed embedding hence finite. Note also for each i, c_i is dominant since X^0 open dense in X, c_i^0 is dominant and the diagram diagram 7.3 commutes.

7.2.6. To summarize, we are now in a good situation with a Cartesian diagram eq. (7.3), such that besides the assumptions of Theorem 1.2, we also have

(a) Every scheme in diagram 7.3 is integral and has the same dimension.
(b) X^0 is quasi-projective, X is projective and $j : X^0 \to X$ is an open embedding. Moreover X and j are defined over $\mathbb{F}_q$.
(c) $c = (c_1, c_2)$ is finite with c_i dominant.

We will see later that all the above conditions will survive in the following reduction. But remember that, until now, c, as well as c^0, is still a closed embedding.

7.3. **Reduce to a "better" situation where ∂X is "nice".** Now we assume we are in the situation of § 7.2.6.

7.3.1. *Can assume ∂X is locally c-invariant over $\mathbb{F}_q$.* It follows from Corollary 4.4 that there exists an open subset $V \subseteq X^0$ and a blow-up $\pi : \widetilde{X} \to X$, which is an isomorphism over V, such that V and π are defined over $\mathbb{F}_q$ and for every correspondence $\widetilde{c} : \widetilde{C} \to \widetilde{X} \times \widetilde{X}$ lifting c, the closed subset $\widetilde{X} \setminus \pi^{-1}(V) \subseteq \widetilde{X}$ is locally $\widetilde{c}$-invariant. Set $W := (c_1^0)^{-1}(V) \cap (c_2^0)^{-1}(V)$ and consider the following diagram:

$$\begin{tikzcd} & W \arrow[rrr, "c^0|_V"] \arrow[dd] & & & V \times V \arrow[dd] \\ \pi^{-1}(W) \arrow[ur, "\cong"] \arrow[rr, "\widetilde{c}^0"] \arrow[ddd] & & \pi^{-1}(V) \times \pi^{-1}(V) \arrow[urr, "\cong"'] \arrow[ddd] & & \\ & C^0 \arrow[rrr, "c^0"] \arrow[d, "j_C"] & & & X^0 \times X^0 \arrow[d, "j\times j"'] \\ & C \arrow[rrr, "c"] & & & X \times X \\ \widetilde{C} \arrow[ur] \arrow[rr, "\widetilde{c}"] & & \widetilde{X} \times \widetilde{X} \arrow[urr, "\pi\times\pi"'] & & \end{tikzcd}$$

where $\widetilde{C} := \overline{\pi^{-1}(W)} \subseteq \widetilde{X} \times \widetilde{X}$, and $\widetilde{c}$ is the inclusion. By § 7.1.1 we may reduce from the proof for c^0 to that for $c^0|_V$, and further to that for $\widetilde{c}^0 : \pi^{-1}(W) \to \pi^{-1}(V) \times \pi^{-1}(V)$. Replacing c by $\widetilde{c}$ and c^0 by $\widetilde{c}^0$, we then obtain a new Cartesian diagram as diagram 7.3, with $\partial X := X \setminus j(X^0)$ locally c-invariant, such that every property in § 7.2.6 holds. For our convenience, from now on we will not distinguish $j(X^0)$ and X^0, so will just write $\partial X = X \setminus X^0$.

7.3.2. *Can assume that X is smooth and ∂X is a strict normal crossing divisor that is defined over $\mathbb{F}_q$.* Assume now we are in the situation of § 7.3.1. Since X is projective and defined over $\mathbb{F}_q$, we know $X = \underline{X} \times_{\mathbb{F}_q} \mathbb{F}$ for some finite type projective $\underline{X}$ over $\mathbb{F}_q$. Applying de Jong's theorem on alterations (§ 2.21) to $\underline{X}$ then pulling back to $\mathbb{F}$, we know there exists a proper dominant generically finite and generically étale morphism $\psi : \widetilde{X} \to X$, such that (a) $\widetilde{X}$ is integral, projective and smooth over $\mathbb{F}$; (b) $\psi^{-1}(\partial X) \subseteq \widetilde{X}$ is a strict normal crossing divisor with irreducible components $\widetilde{X}_i$; and (c) there is an integer $r \geq 1$, such that $\widetilde{X}$, $\widetilde{X}_i$ and ψ are defined over $\mathbb{F}_{q^r}$ and $\widetilde{X}$, $\widetilde{X}_i$ are smooth over $\mathbb{F}_{q^r}$ and $\widetilde{X}$ is geometrically irreducible over $\mathbb{F}_{q^r}$.

Following § 7.1.4, to prove Theorem 1.2 for c^0, it is enough to prove it for $(c^0)^{(m)}$ over $\mathbb{F}_{q^r}$ for all $m = 0, 1, \ldots, r-1$. Note that by replacing c^0 by $(c^0)^{(m)}$, c by $c^{(m)}$ and $\mathbb{F}$ by $\mathbb{F}_{q^r}$, we are still in the situation § 7.3.1; however, now the new c is only finite but not a closed embedding. So we may assume $\widetilde{X}$, $\widetilde{X}_i$ and π are all defined over $\mathbb{F}_q$.

As for each $i = 1, 2$, c_i is dominant, there exists a unique irreducible component $\widetilde{C}$ of $(\psi \times \psi)^{-1}(C) = C \times_{X \times X} (\widetilde{X} \times \widetilde{X})$, such that each projection $\widetilde{c}_i : \widetilde{C} \to \widetilde{X}$ is dominant. Set $\widetilde{X}^0 := \psi^{-1}(X^0)$, $\widetilde{C}^0 = \psi^{-1}(C^0)$ so we have the following diagram:

$$\begin{tikzcd}
& C^0 \arrow[rr, "c^0"] \arrow[dd] & & X^0 \times X^0 \arrow[dd, "j \times j"] \\
\widetilde{C}^0 \arrow[ur] \arrow[rr, "\widetilde{c}^0"] \arrow[dd] & & \widetilde{X}^0 \times \widetilde{X}^0 \arrow[ur] \arrow[dd] & \\
& C \arrow[rr, "c"] & & X \times X. \\
\widetilde{C} \arrow[ur] \arrow[rr, "\widetilde{c}"] & & \widetilde{X} \times \widetilde{X} \arrow[ur, "\psi \times \psi"'] &
\end{tikzcd}$$

Then we can replace the original diagram of c and c^0 by the diagram of $\widetilde{c}$ and $\widetilde{c}^0$. The boundary ∂X is still locally c-invariant by Lemma 3.3. So we are in a "nicer" case with smooth X and locally c-invariant ∂X, which is also a strict normal crossing divisor with smooth components defined over $\mathbb{F}_q$.

7.3.3. *Can assume that c is locally contracting near ∂X.* Assume we are in the situation § 7.3.2. Choose $m \in \mathbb{N}$, such that $q^m > \mathrm{ram}(c_2, \partial X)$. As ∂X is defined over $\mathbb{F}_q$ and is locally c-invariant over $\mathbb{F}_q$, it is easy to see that it is also locally $c^{(m)}$-invariant. We then know from Lemma 3.12 that $c^{(m)}$ is locally contracting near ∂X over $\mathbb{F}_q$. Applying § 7.1.3, we may replace c^0 by $(c^0)^{(m)}$ and c by $c^{(m)}$.

7.3.4. To summarize, we are now in a good situation where besides the conditions in § 7.2.6, we have

(d) X is smooth and ∂X is a strict normal crossing divisor with smooth irreducible components, that is defined over $\mathbb{F}_q$;
(e) c is locally contracting near ∂X over $\mathbb{F}_q$.

7.4. **Conclude the intersection is non-empty.** Suppose now that we are in the situation of § 7.3.4 with notations as in § 5.1. Since c is locally contracting near ∂X, it follows from Lemma 5.3.4 that $\pi\big(\widetilde{c}(\widetilde{C}) \cap \widetilde{\Gamma}_{q^n}\big) \subseteq X^0 \times X^0$ for all $n \in \mathbb{N}$. As $\pi_C(\widetilde{C}) \subseteq C$ and $\pi(\widetilde{\Gamma}_{q^n}) \subseteq \Gamma_{q^n}$, we have

$$\pi(\widetilde{c}(\widetilde{C}) \cap \widetilde{\Gamma}_{q^n}) \subseteq \big(c(C) \cap \Gamma_{q^n}\big) \cap (X^0 \times X^0) = c^0(C^0) \cap \Gamma^0_{q^n} = c^0\big((c^0)^{-1}(\Gamma_{q^n})\big).$$

Since c is finite, the scheme-theoretic image $c(C) \subseteq X \times X = Y$ is a closed integral subscheme of dimension d, and the proper transform $\widetilde{c(C)} \subseteq \widetilde{Y}$ is exactly $\widetilde{c}(\widetilde{C})$. So by Corollary 6.3, applying to $c(C)$, for sufficiently large n, $\widetilde{c}(\widetilde{C})\cap\widetilde{\Gamma}_{q^n}\neq\emptyset$. Therefore, $(c^0)^{-1}(\Gamma^0_{q^n})\neq\emptyset$.

Appendix A. Intersection Theory

Throughout this part, schemes are assumed to be of finite type over a field k. The main reference are [Ful98] and [Stacks, Tag 0AZ6].

A.1. **Cycles.** An i-cycle on X is a finite formal sum $\sum n_j[Z_j]$ where each Z_j is an integral closed subscheme of X of dimension i and $n_j\in\mathbb{Z}$. The group of i-cycles is denoted by $Z_i(X)$. Set $Z(X):=\oplus Z_i(X)$, it is a graded $\mathbb{Z}$-module ([Ful98, § 1.3]).

For any closed subscheme Z of X of pure dimension i, denote by $[Z]$ the i-cycle associated with Z. Precisely,

$$[Z]:=\sum \ell(\mathcal{O}_{X,Z_i})[Z_i]$$

where Z_i are the irreducible components of Z and $\ell(\mathcal{O}_{X,Z_i})$ is the *length* of $\mathcal{O}_{X,Z_i}$ as an $\mathcal{O}_{X,Z_i}$-module, which is Artinian. In particular, if Z is integral of pure dimension i, then the i-cycle associated with Z is just $[Z]$ ([Ful98, § 1.5]).

A.2. **Proper Pushforward.** Suppose $f:X\to Y$ is a proper morphism between two schemes. Then for any integral closed subscheme Z, we know that $f(Z)$ is an irreducible closed subset of Y. With the reduced structure, $f(Z)$ is then an integral closed subscheme of Y. The pushforward map is defined as

$$\begin{aligned} f_*: A_i(X) &\longrightarrow A_i(Y)\\ [Z] &\longmapsto \begin{cases} 0 & \text{if } \dim f(Z)<\dim Z\\ d\cdot[f(Z)] & \text{if } \dim f(Z)=\dim Z\end{cases}\end{aligned}$$

where Z is an integral closed subscheme and $d=\big[K(Z):K(f(Z))\big]$ is the extension degree of function fields ([Ful98, § 1.4]). The pushforward f_* is functorial: $(g_*\circ f_*)=(g\circ f)_*$ for any two proper morphisms f,g that could be composed.

In particular, if $p:X\to\operatorname{Spec}k$ is proper, then the push forward defines a map $\deg:=p_*:A_0(X)\to A_0(\operatorname{Spec}k)\cong\mathbb{Z}$ ([Ful98, Def. 1.4]).

A.3. **Flat Pullback.** Suppose $f:X\to Y$ is a flat morphism between smooth schemes and is of relative dimension r. Assume Y is of pure dimension n. Then for any integral closed subscheme Z of Y of dimension $n-i$, we know the scheme-theoretic preimage $f^{-1}(Z)$ is of dimension $r+n-i$. The pullback map is defined as

$$\begin{aligned} f^*: A_{n-i}(Y)=A^i(Y) &\longrightarrow A^i(X)=A_{r+n-i}(X)\\ [Z] &\longmapsto [f^{-1}(Z)],\end{aligned}$$

where Z is an integral closed subscheme and $[f^{-1}(Z)]$ is the cycle associated with $f^{-1}(Z)$ ([Ful98, § 1.7]). The pullback f^* is functorial: $f^*\circ g^*=(g\circ f)^*$ for any two flat morphisms f,g whose composition makes sense.

A.4. **Rational Equivalence.** Suppose $\alpha:=\sum n_j[W_j]$ is an $(i+1)$-cycle on $X\times\mathbb{P}^1$ where W_j is an integral closed subscheme for each j. Let a_j,b_j be pairs of distinct closed points of $\mathbb{P}^1$. Assume $X\times a_j, X\times b_j$ and W_j intersect properly ([Stacks, 0AZQ]):

$$\dim(W_j\cap X\times a_j)\le i,\quad \dim(W_j\cap X\times b_j)\le i.$$

We can view the fibers W_{r,a_r}, W_{r,b_r} under the projection $X \times \mathbb{P}^1 \to \mathbb{P}^1$ as closed subschemes of $X \cong X \times a_j \cong X \times b_j$. Then a cycle *rationally equivalent to zero* is any cycle of the form

$$\sum_j n_j\big([W_{j,a_j}] - [W_{j,b_j}]\big).$$

This is an i-cycle. Two i-cycles α and β are *rationally equivalent*, denoted by $\alpha \sim_{\text{rat}} \beta$ if $\alpha - \beta$ is rationally equivalent to zero. Define

$$A_i(X) = Z_i(X)/\sim_{\text{rat}},$$

and the *Chow group* $A(X) = \oplus A_i(X)$ ([Ful98, § 1.6] and [Stacks, Tag 0AZG]).

More importantly, the proper pushforward and the flat pullback induce homomorphisms between Chow groups ([Ful98, Thm. 1.4 and Thm. 1.7]).

A.5. **Intersection Product.** Suppose that X is smooth of pure dimension n. Let $W, V \subseteq X$ be two integral closed subschemes of dimension r and s, respectively. We say that W and V *intersect properly* if $\dim(V \cap W) \leq r + s - \dim(X)$ ([Stacks, 0AZQ]). In this case the sheaves $\mathcal{T}or_j^{\mathcal{O}_X}(\mathcal{O}_W, \mathcal{O}_V)$ are coherent, supported on $V \cap W$, and vanish if $j < 0$ or $j > \dim(X)$. Define

$$W \frown V := \sum_Z \Bigg(\sum_j (-1)^j \ell_{\mathcal{O}_{X,Z}}\big(\operatorname{Tor}_j^{\mathcal{O}_{X,Z}}(\mathcal{O}_{W,Z}, \mathcal{O}_{V,Z})\big)\Bigg)[Z],$$

where Z runs over all irreducible components (with reduced structure) of $W \cap V \subseteq X$ ([Stacks, Tag 0AZR]). Write $A^i := A_{\dim X - i}$. Then we have a well-defined intersection product

$$\begin{aligned} \frown \colon A^i(X) \times A^j(X) &\longrightarrow A^{i+j}(X) \\ (\alpha, \beta) &\longmapsto \alpha \frown \beta. \end{aligned}$$

If particular, if X is proper over $\operatorname{Spec} k$, we then have

$$\begin{aligned} \deg \circ \frown \colon A_i(X) \times A^i(X) &\longrightarrow \mathbb{Z} \\ (\alpha, \beta) &\longmapsto \alpha \cdot \beta := \deg(\alpha \frown \beta). \end{aligned}$$

A.6. **General Pullback.** Suppose that $f : X \to Y$ is a morphism between two smooth schemes. Define

$$\begin{aligned} f^* : A_i(Y) &\longrightarrow A_{i+\dim X - \dim Y}(X) \\ \alpha &\longmapsto f^*(\alpha) := (\mathrm{pr}_X)_*\big(\Gamma_f \frown \mathrm{pr}_Y^*(\alpha)\big), \end{aligned}$$

where $\Gamma_f \subseteq X \times Y$ is the graph of f and $\mathrm{pr}_X, \mathrm{pr}_Y$ are the two projections. This is a well defined ring homomorphism ([Ful98, § 8.1] and [Stacks, 0B0H]).

A.6.1. If $f : X \to Y$ is a morphism between two smooth connected schemes, suppose $Z \hookrightarrow Y$ and $f^{-1}Z = Z \times_Y X \hookrightarrow X$ are both regular embeddings, then $f^*([Z]) = [f^{-1}(Z)]$ ([Ful98, Theorem 6.2(a)]).

A.7. **Projection Formula.** Suppose that $f : X \to Y$ is a proper morphism between smooth schemes of relative dimension r. Suppose $\alpha \in A_i(X)$ and $\beta \in A_j(Y)$ are two cycles. Then ([Stacks, 0B2X])

$$f_*\alpha \frown \beta = f_*(\alpha \frown f^*\beta).$$

In particular, if X and Y are both proper over $\operatorname{Spec} k$, $\alpha \in A_i(X)$ and $\beta \in A^i(Y)$, then

$$f_*\alpha \cdot \beta = \alpha \cdot f^*\beta.$$

Appendix B. Étale Cohomology and ℓ-adic Cohomology

In this section, schemes are all k-scheme for a fixed fixed field k. The main references are [Mil13], [Mil80], [SGA 4, (III)], [SGA 4½, Cycle] and [Poo15, Chapters 6 and 7]. Throughout this section, ℓ is a prime number that is different from the characteristic of k.

B.1. **Étale Morphism.** A morphism $f : X \to Y$ is *étale* if it is smooth of relative dimension 0, equivalently, if it is flat and unramified (so locally of finite presentation).

B.2. **Étale Site.** Let X be a scheme. Let $\mathtt{Et}_X$ be the category of étale morphisms to X. In other words, the objects in $\mathtt{Et}_X$ are schemes U together with an étale morphisms $U \to X$, and arrows in $\mathtt{Et}_X$ are morphisms over X. These arrows will automatically be étale ([Mil80, I, Corollary 3.6]). A family of arrows $(\pi : U_i \to U)$ in $\mathtt{Et}_X$ is an *covering* if set-theoretically $\cup_i \pi_i(U_i) = U$. The category $\mathtt{Et}_X$ together with the coverings is called the *(small) étale site*, denoted by $X_{\text{ét}}$.

B.3. **Sheaves on Étale Site.** A *presheaf* of abelian groups $\mathcal{F}$ on $X_{\text{ét}}$ is just a functor

$$\begin{aligned} \mathcal{F} : \mathtt{Et}_X^{opp} &\longrightarrow \mathtt{AbGrp} \\ U &\longmapsto \mathcal{F}(U), \end{aligned}$$

where $\mathtt{AbGrp}$ is the category of abelian groups. A *sheaf* $\mathcal{F}$ of abelian groups on $X_{\text{ét}}$ is a presheaf such that

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

is exact for all coverings $(U_i \to U)$.

Let $\mathcal{F}$ be a (pre)sheaf on $X_{\text{ét}}$, and $x : \operatorname{Spec} k^{sep} \to X$ a geometric point. Then the *stalk* of $\mathcal{F}$ at x is defined as

$$\mathcal{F}_x := \varinjlim \mathcal{F}(U),$$

where U runs over all commutative diagrams

$$\begin{tikzcd} U \arrow[r, "\text{étale}"] & X \\ \operatorname{Spec} k^{sep}. \arrow[u] \arrow[ur, "x"'] & \end{tikzcd}$$

As in the case for classical topology, there is a canonical way to obtain from a presheaf $\mathcal{F}$ a unique (up to isomorphism) sheaf $\mathcal{F}^\dagger$, called the sheafification of $\mathcal{F}$.

The category $\mathtt{Sh}(X_{\text{ét}})$ of sheaves of abelian groups on $X_{\text{ét}}$ is an abelian category with enough injectives ([Mil80, III, Proposition 1.1]).

A sheaf $\mathcal{F}$ on $X_{\text{ét}}$ is called a torsion sheaf, if all stalks of $\mathcal{F}$ are torsion groups, or equivalently, $\mathcal{F}(U)$ is a torsion group for all étale morphism $U \to X$.

B.3.1. *Example.* Let Λ be an abelian group and set $\mathcal{F}_\Lambda(U) := \Lambda^{\pi_0(U)}$, where $\pi_0(U)$ denotes the set of connected components of U. Then we obtain a sheaf $\mathcal{F}_\Lambda$, called the *constant sheaf* associated with Λ.

B.4. **Operations on Sheaves.** Everything in this subsection is explained in [Mil13, I, § 8]. Suppose $f : X \to Y$ is a morphism between two k-schemes. Let $\mathcal{F} \in \mathtt{Sh}(X_{\text{ét}})$ be a sheaf on $X_{\text{ét}}$ and $\mathcal{G} \in \mathtt{Sh}(Y_{\text{ét}})$ a sheaf on $Y_{\text{ét}}$. The *direct image sheaf* $f_*\mathcal{F}$ is defined via

$$f_*\mathcal{F}(U) := \mathcal{F}(U \times_X Y), \quad \forall\, U \to Y \text{ étale}.$$

Then $f_*\mathcal{F}$ becomes a sheaf.

The *inverse image sheaf* $f^{-1}\mathcal{G}$ is defined the sheaf associated with the presheaf given by

$$f^{-1}\mathcal{G}(V) := \varinjlim \mathcal{G}(U), \quad \forall\, V \to X \text{ étale},$$

where the limit is taken over all commutative diagrams

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with $U \to Y$ étale.

Let $j : U \to X$ be an open embedding and $\mathcal{F}$ a sheaf on $U_{\text{ét}}$. The *extension by zero* of $\mathcal{F}$ is the sheaf associated with the presheaf given by

$$(j_!\mathcal{F})(V) := \begin{cases} \mathcal{F}(V) & \text{if } \phi(V) \subseteq U, \\ 0 & \text{otherwise,} \end{cases}$$

where $\phi : V \to X$ is étale.

Indeed, any operation on sheaves in the classical topology generalizes to sheaves on sites, for example the $\mathcal{H}om$ sheaf and the tensor product.

B.5. **Étale Cohomology.** Let X be a scheme. The global section functor

$$\begin{aligned} \Gamma(X, -) : \mathtt{Sh}(X_{\text{ét}}) &\longrightarrow \mathtt{AbGrp} \\ \mathcal{F} &\longmapsto \Gamma(X, \mathcal{F}) := \mathcal{F}(X) \end{aligned}$$

is left exact. Its i^{th} right derived functor is denoted by $\mathrm{H}^i_{\text{ét}}(X, -)$:

$$\begin{aligned} \mathrm{H}^i_{\text{ét}}(X, -) : \mathtt{Sh}(X_{\text{ét}}) &\longrightarrow \mathtt{AbGrp} \\ \mathcal{F} &\longmapsto \mathrm{H}^i_{\text{ét}}(X, \mathcal{F}) \end{aligned}$$

and $\mathrm{H}^i_{\text{ét}}(X, \mathcal{F})$ is called the i^{th} étale cohomology group of $\mathcal{F}$.

For a k-scheme X and a *torsion sheaf* $\mathcal{F}$ on $X_{\text{ét}}$, we can also define the étale cohomology groups with compact support (more logically, proper support) as

$$\mathrm{H}^i_{\text{ét},c}(X, \mathcal{F}) := \mathrm{H}^i_{\text{ét}}(Y, j_!\mathcal{F}),$$

where $j : X \to Y$ is an open embedding such that Y is proper over k (Theorem B.5.1) and $j_!\mathcal{F}$ is extension by zero of the sheaf $\mathcal{F}$ (B.4). This definition is independent of the choice of the open embedding ([Mil80, VI, Proposition 3.1]). In particular, if X itself is proper over k, then the the cohomology with compact support coincide with the usual étale cohomology.

B.5.1. *Theorem.* ([Con07]) Let $f : X \to S$ be a separated morphism of finite type. Suppose further that S is quasi-compact and quasi-separated, e.g., S is Noetherian. Then there exists an open embedding $j : X \to \overline{X}$ over S such that $\overline{X} \to S$ is proper.

B.6. **ℓ-adic Cohomology.** Denote by $\mathbb{Z}/\ell^n\mathbb{Z}$ the constant sheaf (B.3) on $X_{\text{ét}}$ associated with the group $\mathbb{Z}/\ell^n\mathbb{Z}$. Define

$$\mathrm{H}^i(X, \mathbb{Z}_\ell) := \varprojlim \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/\ell^n\mathbb{Z}).$$

When equipped with profinite topology, this is a continuous $\mathbb{Z}_\ell$-module. Define

$$\mathrm{H}^i(X, \mathbb{Q}_\ell) := \mathrm{H}^i(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

We can also defined the ℓ-adic cohomology groups $\mathrm{H}^i_c(X, \mathbb{Q}_\ell)$ with compact support in the same way as in B.5.

If k is separably closed and X is of dimension d, then $\mathrm{H}^i(X, \mathbb{Q}_\ell) = 0$ unless $0 \leq i \leq 2d$ ([SGA 4, (III), X, Corollaire 4.3]). If moreover X is proper, then $\mathrm{H}^i(X, \mathbb{Q}_\ell)$ is a finite dimensional vector space for all $i \in \mathbb{N}$ ([Mil80, V, Lemma 1.11 and VI, Corollary 2.8] and [Mil13, I, Theorems 19.1 and 19.2]).

B.7. **Tate Twist.** For each integer m, define the twisted sheaf $(\mathbb{Z}/n\mathbb{Z})(m)$ as

$$(\mathbb{Z}/n\mathbb{Z})(m) := \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } m = 0, \\ (\mu_n)^{\otimes m} & \text{if } m > 0, \\ \mathcal{H}om((\mu_n)^{\otimes(-m)}, \mathbb{Z}/n\mathbb{Z}) & \text{if } m < 0, \end{cases}$$

where μ_n is the constant sheaf associated with the group of n^{th}-roots of unity of k. As in B.6, we could define

$$\mathrm{H}^i\left(X, \mathbb{Z}_\ell(m)\right) := \varprojlim \mathrm{H}^i_{\text{ét}}\left(X, (\mathbb{Z}/\ell^n\mathbb{Z})(m)\right),$$
$$\mathrm{H}^i\left(X, \mathbb{Q}_\ell(m)\right) := \mathrm{H}^i\left(X, \mathbb{Z}_\ell(m)\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Of course, we can defined the twisted ℓ-adic cohomology $\mathrm{H}^i_c(X, \mathbb{Q}_\ell(m))$ with compact support in the same way. We know $\mathrm{H}^i(X, \mathbb{Q}_\ell(m))$ and $\mathrm{H}^i_c(X, \mathbb{Q}_\ell(m))$ coincide if X itself is proper over k.

B.8. **Poincaré Duality.** ([Mil80, VI, Corollary 11.2]) Suppose that X is a smooth connected scheme of dimension d over a separably closed field k.

(a) There is a natural isomorphism:

$$\mathrm{Tr}_{X/k} : \mathrm{H}^{2d}_c(X, \mathbb{Q}_\ell(d)) \xrightarrow{\cong} \mathbb{Q}_\ell,$$

which is called the trace map;

(b) The cup product ([Mil80, V, Example 1.17] defines a perfect pairing

$$\begin{aligned} \mathrm{H}^r_c(X, \mathbb{Q}_\ell(i)) \times \mathrm{H}^{2d-r}(X, \mathbb{Q}_l(d-i)) &\longrightarrow \mathrm{H}^{2d}_c(X, \mathbb{Q}_\ell(d)) \cong \mathbb{Q}_\ell \\ (\alpha, \beta) &\longmapsto \alpha \smile \beta \mapsto \mathrm{Tr}_{X/k}(\alpha \smile \beta) \end{aligned}$$

for all integers r and i in $\mathbb{Z}$.

B.9. **Cycle Class.** Let X be a d-dimensional smooth irreducible over k. To each integral closed subscheme Z of codimension i, we can associated a class $cl(Z) \in \mathrm{H}^{2i}\left(X, \mathbb{Q}_\ell(i)\right)$. It corresponds to the map

$$\mathrm{H}^{2d-2i}_c\left(X, \mathbb{Q}_\ell(d-i)\right) \xrightarrow{\iota^*} \mathrm{H}^{2d-2i}_c\left(Z, \mathbb{Q}_\ell(d-i)\right) \xrightarrow{\mathrm{Tr}_{Z/k}} \mathbb{Q}_\ell.$$

where $\iota : Z \to X$ is the inclusion. That is to say

$$\mathrm{Tr}_{X/k}(x \smile cl(Z)) := \mathrm{Tr}_{Z/k}(\iota^*(x)), \quad \forall\, x \in \mathrm{H}^{2d-2i}_c\left(X, \mathbb{Q}_\ell(d-i)\right). \tag{B.1}$$

Assume further that X is quasi-projective, then ([Mil80, VI, Corollary 10.7 and Remark 10.8]) this construction gives rise to a homomorphism of graded rings $cl : A^*(X) \to \oplus_i \mathrm{H}^{2i}\left(X, \mathbb{Q}_\ell(i)\right)$, called the *cycle class map*. That is to say,

$$cl(\alpha \frown \beta) = cl(\alpha) \smile cl(\beta), \quad \forall\, \alpha \in A^i(X),\ \beta \in A^j(X). \tag{B.2}$$

B.10. **Grothendieck-Lefschetz Trace Formula.** Set $Y := X_1 \times X_2$, where each X_i is smooth connected proper schemes over k of dimension d_i. Let $\mathrm{pr}_i : Y \to X_i$ be the canonical projection. Suppose that $\alpha \in \mathrm{H}^{2d_1}\big(Y, \mathbb{Q}_\ell(d_1)\big)$ is a cohomology class. Then α induces a morphism

$$\mathrm{H}^i(\alpha) : \mathrm{H}^i(X_1, \mathbb{Q}_\ell) \to \mathrm{H}^i(X_2, \mathbb{Q}_\ell).$$

Indeed, there is a natural map

$$\begin{aligned} \mathrm{H}^i(X_1, \mathbb{Q}_\ell) \times \mathrm{H}^{2d_2-i}(X_2, \mathbb{Q}_\ell(d_2)) &\longrightarrow \mathrm{H}^{2d_2}\big(Y, \mathbb{Q}_\ell(d_2)\big) \\ (x, y) &\longmapsto x \boxtimes y := \mathrm{pr}_1^* x \smile \mathrm{pr}_2^* y. \end{aligned}$$

Then by Poincaré duality $\mathrm{H}^i(\alpha)$ corresponds to the map

$$\begin{aligned} \mathrm{H}^i(X_1, \mathbb{Q}_\ell) \times \mathrm{H}^{2d_2-i}(X_2, \mathbb{Q}_\ell(d_2)) &\longrightarrow \mathbb{Q}_\ell \\ (x, y) &\longmapsto \mathrm{Tr}_{Y/k}(\alpha \smile (x \boxtimes y)). \end{aligned}$$

Similarly, any $\beta \in \mathrm{H}^{2d_2}\big(Y, \mathbb{Q}_\ell(d_2)\big)$ induces a map $\mathrm{H}^i(\beta) : \mathrm{H}^i(X_2, \mathbb{Q}_\ell) \to \mathrm{H}^i(X_1, \mathbb{Q}_\ell)$.

If $f^i : H^i \to H^i$ is an endomorphism on the vector spaces H^i, $i \in \mathbb{N}$, then we define the alternating trace

$$\mathrm{Tr}(f^*|H^*) := \sum_i (-1)^i \,\mathrm{Tr}(f^i|H^i) \tag{B.3}$$

with $\mathrm{Tr}(f^i|H^i)$ being the trace of the linear endomorphism f^i on H^i, and sometimes we omit H^* and just write $\mathrm{Tr}(f^*)$ for short.

Note that $\alpha \smile \beta \in \mathrm{H}^{2(d_1+d_2)}\big(Y, \mathbb{Q}_\ell(d_1 + d_2)\big)$. It was shown in [SGA 4½, Cycle, Proposition 3.3] that

$$\mathrm{Tr}_{Y/k}(\alpha \smile \beta) = \mathrm{Tr}(\mathrm{H}^*(\beta) \circ \mathrm{H}^*(\alpha)). \tag{B.4}$$

Now if $C \subseteq X_1 \times X_2$ is a closed integral subscheme of dimension d_2, equivalently, of codimension d_1, then C gives rise to $cl(C) \in \mathrm{H}^{2d_1}\big(Y, \mathbb{Q}_\ell(d_1)\big)$ and induces a homomorphism $\mathrm{H}^i([C]) := \mathrm{H}^i(cl(C)) : \mathrm{H}^i(X_1, \mathbb{Q}_\ell) \to \mathrm{H}^i(X_2, \mathbb{Q}_\ell)$.

Denote by $c = (c_1, c_2) : C \to X_1 \times X_2$ the inclusion. Using Poincaré duality, define the push-forward $(c_2)_* : \mathrm{H}^i(C, \mathbb{Q}_\ell) \to \mathrm{H}^i(X_2, \mathbb{Q}_\ell)$ to be the one that corresponds to $\mathrm{H}^i(C, \mathbb{Q}_\ell) \times \mathrm{H}^{2d_2-i}\big(X_2, \mathbb{Q}_\ell(d_2)\big) \to \mathbb{Q}_\ell$, $(x, y) \mapsto \mathrm{Tr}_{C/k}(x \smile (c_2)^* y)$. That is to say

$$\mathrm{Tr}_{X_2/k}\big((c_2)_* x \smile y\big) := \mathrm{Tr}_{C/k}\big(x \smile (c_2)^* y\big), \quad \forall\, y \in \mathrm{H}^{2d_2-i}\big(X_2, \mathbb{Q}_\ell(d_2)\big).$$

Then by definition and eq. (B.1), we can see that $\mathrm{H}^i([C]) = (c_2)_* \circ (c_1)^*$. Subsequently, we conclude that if $f : X_2 \to X_1$ is a morphism and X_2 is separated, then $\mathrm{H}^i([\Gamma_f]) = f^* = \mathrm{H}^i(f)$, where Γ_f denotes the graph of f. Besides, in case $X_1 = X_2 = X$ are of dimension d and c_j are dominant for all $j = 1, 2$, we can conclude that

$$\mathrm{H}^{2d}([C]) = \deg(c_1)\,\mathrm{id}, \tag{B.5}$$

as $(c_1)^*$ acts as multiplication by $\deg(c_1)$ and $(c_2)_*$ act as identity ([Mil13, I, Remark 24.2]).

Similarly, if the closed integral subscheme $C \subseteq X_1 \times X_2$ is of codimension d_2, then it gives rise to a cycle class $[C] \in A^{d_2}(Y)$ and a morphism $\mathrm{H}^i([C]) : \mathrm{H}^i(X_2, \mathbb{Q}_\ell) \to \mathrm{H}^i(X_1, \mathbb{Q}_\ell)$. So by eqs. (B.2) and (B.4) we could conclude that

$$[C] \cdot [\Gamma_f] = \mathrm{Tr}\big(f^* \circ \mathrm{H}^*([C])\big). \tag{B.6}$$

A consequence of the above discussion is the following theorem.

B.10.1. *Theorem.* ([SGA 4½, Cycle, Corollaire 3.7]) Suppose that X is a smooth proper scheme over $k = \bar{k}$ and $f : X \to X$ is an endomorphism. Then

$$[\Delta] \cdot [\Gamma_f] = \mathrm{Tr}\left(f^* \middle| \mathrm{H}^*(X, \mathbb{Q}_\ell)\right).$$

If moreover f is nondegenerate (i.e., $1 - df_x$ is invertible on the Zariski tangent space $T_x X$ at x) for all fixed point $x \in X(k)$, then $[\Delta] \cdot [\Gamma_f]$ equals to the number of fixed points of f in $X(k)$.

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Selbstständigkeitserklärung

Name	Hao
Vorname	Yun
geb.am	27.10.1990
Matr.Nr.	4934580

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