15/11/21
Monday

MODULE : 1

PARTIAL DIFFERENTIAL EQUATIONS

Let $z = f(x, y)$, where x and y are independent and z is dependent. The notatitions used are :-

$$\frac{\partial z}{\partial x} = p \quad , \quad \frac{\partial z}{\partial y} = q \quad , \quad \frac{\partial^2 z}{\partial x^2} = r \quad , \quad \frac{\partial^2 z}{\partial x \partial y} = s \quad , \quad \frac{\partial^2 z}{\partial y^2} = t$$

Formation of PDE

PDE can be formed either by the elimination of arbitary constants or by the elimination of arbitary functions from a relation involving two or more functions.

1. Derive a PDE from the relation : $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Ans: $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ —— ① arbitary constants : a, b

Differentiating ① w.r.t x

$$2 \cdot \frac{\partial z}{\partial x} = \frac{2x}{a^2}$$

$$p = \frac{\partial z}{\partial x} = \frac{x}{a^2} \quad ; \quad \therefore \frac{p}{x} = \frac{1}{a^2} \quad —— ②$$

Differentiating ① w.r.t y

$$2 \cdot \frac{\partial z}{\partial y} = \frac{2y}{b^2}$$

$$q = \frac{\partial z}{\partial y} = \frac{y}{b^2} \quad ; \quad \therefore \frac{q}{y} = \frac{1}{b^2} \quad —— ③$$

Substituting ② & ③ in eqn ①

$$2z = x^2 \cdot \frac{p}{x} + y^2 \cdot \frac{q}{y}$$

$$2z = px + qy$$

OR

$$2z = \frac{\partial z}{\partial x} \cdot x + \frac{\partial z}{\partial y} \cdot y$$

2. Find PDE, $z = f(x^2 - y^2)$

independendent variable : x, y

dependent variable : z

Ans: $z = f(x^2-y^2)$ —— ①

Differentiating ① w.r.t x

$$\frac{\partial z}{\partial x} = p = f'(x^2-y^2) \times 2x \quad \text{—— ②}$$

Differentiating ① w.r.t y

$$\frac{\partial z}{\partial y} = q = f'(x^2-y^2) \times (-2y) \quad \text{—— ③}$$

Dividing eqn ② by eqn ③

$$\frac{p}{q} = \frac{2x}{(-2y)}$$

$$-py = qx$$

$$qx + py = 0$$

$$\frac{\partial z}{\partial y} \cdot x + \frac{\partial z}{\partial x} \cdot y = 0$$

3. Form the PDE from the relation $z = f\left(\frac{xy}{z}\right)$

Ans: $z = f\left(\frac{xy}{z}\right)$ —— ①

Differentiating ① w.r.t x

$$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right)\left(\frac{zy - yxp}{z^2}\right) = p \quad \text{—— ②}$$

Differentiating ① w.r.t y

$$\frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right)\left(\frac{zx - xyq}{z^2}\right) = q \quad \text{—— ③}$$

Dividing eqn ② by eqn ③

$$\frac{p}{q} = \frac{zy - xyp}{z^2} \times \frac{z^2}{zx - xyq}$$

$$\frac{p}{q} = \frac{zy - xyp}{zx - xyq}$$

$$p(zx - xyq) = q(zy - xyp)$$

$$pzx - \cancel{pxyq} = qzy - \cancel{qxyp}$$

$$pzx - qzy = 0$$

$$px - qy = 0$$

Formation of PDE

(i) If the no. of arbitary constants to be eliminated is equal to the no. of independent variable, the process of elimination results in a PDE of the first order.

(ii) If the no. of arbitary constants to be eliminated is more than the no. of independent variables, the process of elimination will lead to a PDE of second or higher order.

(iii) If the PDE is formed by elimination arbitary functions the order of the equation will be equal to the no. of arbitary functions eliminated.

4. Derive the PDE from the relation $z = f(x+at) + g(x-at)$

Ans: $z = f(x+at) + g(x-at)$ —— ①

independent variable : x, t

a is constant

Differentiate ① w.r.t x

$$\frac{\partial z}{\partial x} = p = f'(x+at)\cdot 1 + g'(x-at)\cdot 1 \quad \text{—— ②}$$

Differentiate ① w.r.t t

$$\frac{\partial z}{\partial t} = f'(x+at)\cdot a + g'(x-at)\cdot(-a) \quad \text{—— ③}$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at) \quad \text{—— ②*}$$

$$\frac{\partial^2 z}{\partial t^2} = f''(x+at)a^2 + g''(x-at)a^2 \quad \text{—— ③*}$$

$$\frac{\partial^2 z}{\partial t^2} = a^2\left(f''(x+at) + g''(x-at)\right)$$

$$\frac{\partial^2 z}{\partial t^2} = a^2 \cdot \frac{\partial^2 z}{\partial x^2}$$

5. $z = f(x) + e^y \cdot g(x)$ —— ①

Ans: Differentiate ① w.r.t x

$$p = f'(x) + e^y \cdot g'(x) = \frac{\partial z}{\partial x} \quad \text{—— ②}$$

Differentiate ① w.r.t y

$$\frac{\partial z}{\partial y} = 0 + e^y g(x) \quad \text{—— ③}$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x) + e^y \cdot g''(x) \quad \text{—— ②*}$$

$$\frac{\partial^2 z}{\partial y^2} = g(x)\cdot e^y \quad \text{——} \quad (3^*)$$

$$\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2} \quad ; \quad q = t$$

6. $z = f_1(x)\cdot f_2(y)$

Ans: Differentiate w.r.t x

$$\frac{\partial z}{\partial x} = p = \cancel{f_1(x)\cdot f_2'(y)} * f_2(y)\cdot f_1'(x)$$

Differentiate w.r.t y

$$\frac{\partial z}{\partial y} = q = f_1(x)\cdot f_2'(y)$$

$$\frac{\partial^2 z}{\partial x^2} = f_1''(x)\cdot f_2(y)$$

$$\frac{\partial^2 z}{\partial y^2} = f_1(x)\cdot f_2''(y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = f_1'(x) \cdot f_2'(y)$$

$$\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} = f_1'(x) \cdot f_2'(y) \cdot f_1(x) \cdot f_2(y)$$

$$\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial x \partial y} \cdot z$$

$$\therefore \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial x \partial y} \times z$$

7. Find the PDE of all spheres of fixed radius having their centres in the xy plane.

Ans: $(x-a)^2 + (y-b)^2 + z^2 = r^2$ —— ①

Differentiating ① w.r.t x

$$2(x-a)\cdot 1 + 2z\cdot\frac{\partial z}{\partial x} = 0$$

$$2(x-a) + 2zp = 0 \quad \text{——} \quad ②$$

Differentiating ① w.r.t y

$$2(y-b)\cdot 1 + 2z\cdot\frac{\partial z}{\partial y} = 0$$

$$2(y-b) + 2zq = 0 \quad \text{——} \quad ③$$

Eqn ②: $-\cancel{z}(x-a) = \cancel{z}pz$

Eqn ③: $-\cancel{z}(y-b) = \cancel{z}qz$ spheres : imp

$$(-zp)^2+(-zq)^2+z^2=r^2$$

$$z^2p^2+z^2q^2+z^2=r^2$$

8. Find PDE of the sphere of fixed radius having their centres in the z plane.

Ans:- $x^2+y^2+(z-c)^2=r^2$ —— ① $(0,0,c)$

Differentiating ① partially w.r.t x

$$2x+2(z-c)p=0$$

$$2x=-2(z-c)p$$

$$x=-(z-c)p \text{ —— ②}$$

Differentiating ① partially w.r.t y

$$2y+2(z-c)q=0$$

$$2y=-2(z-c)q$$

$$y=-(z-c)q \text{ —— ③}$$

Dividing ② by ③

$$\frac{x}{y}=\frac{p}{q}$$

$$qx-py=0$$

9. Form PDE for the relation, $z=y^2+2f\left(\frac{1}{x}+\ln y\right)$

Ans:- $z=y^2+2f\left(\frac{1}{x}+\ln y\right)$ —— ①

Differentiating ① partially w.r.t x

$$p=2f'\left(\frac{1}{x}+\ln y\right)\cdot\left(\frac{-1}{x^2}\right) \text{ —— ②}$$

Differentiating ① partially w.r.t y

$$q=2y+2f'\left(\frac{1}{x}+\ln y\right)\cdot\left(\frac{1}{y}\right) \text{ —— ③}$$

Rearranging eqn ② ; Rearranging eqn ③

$$-px^2=2f'\left(\frac{1}{x}+\ln y\right) \qquad qy=2y^2+2f'\left(\frac{1}{x}+\ln y\right)$$

Substituting ② in ③

$$qy=2y^2-px^2$$

$$px^2+qy-2y^2=0$$

10. Find the PDE of all planes which are at a constant distance 'a' from the origin.

$$\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

$(x_1,y_1,z_1)\to(0,0,0)$

Ans: Let $z = lx+my+n$ be the equation to the plane.

$$a = \frac{|n|}{\sqrt{l^2+m^2+1}} = \frac{\pm n}{\sqrt{l^2+m^2+1}}$$

$$a\sqrt{l^2+m^2+1} = \pm n \quad ; \; n = \pm a\sqrt{l^2+m^2+1}$$

$$z = lx+my \pm a\sqrt{l^2+m^2+1}$$

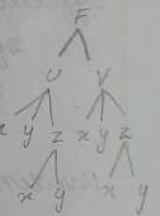
$$p = l \quad , q = m$$

$$z = px+qy \pm a\sqrt{l^2+m^2+1}$$

11. Form the PDE of the function; $F(x+y+z, \; x^2+y^2+z^2)=0$

Ans: $F(u,v) = 0$ —— ①

$u = x+y+z, \quad v = x^2+y^2+z^2$



Differentiating w.r.t x

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0$$

$$\frac{\partial F}{\partial u}(1+1\cdot p) + \frac{\partial F}{\partial v}(2x+2z\cdot p) = 0 \quad \text{—— ②}$$

Differentiating w.r.t y

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0$$

$$\frac{\partial F}{\partial u}(1+1\cdot q) + \frac{\partial F}{\partial v}(2y+2z\cdot q) = 0 \quad \text{—— ③}$$

By the method of determinants

$$\begin{vmatrix} 1+p & 2x+2zp \\ 1+q & 2y+2zq \end{vmatrix} = 0$$

Expanding: $(1+p)(2y+2zq) - (1+q)(2x+2zp) = 0$

$$2y+2zq+2yp+\cancel{2zqp} - 2x - 2zp - 2qx - \cancel{2zqp} = 0$$

$$2y-2x+2zq-2zp+2yp-2qx = 0$$

$$2y+2zq+2yp = 2x+2zp+2qx$$

$$y+zq+yp = x+zp+qx$$

$$y+py-pz = qx-qz+x$$

$$p(y-z) - q(x-z) + y - x = 0$$

12. $z=(x+y)\phi(x^2-y^2)$ —— ①

H.W

1) $z=f(x^2+y^2)+x+y$

2) $xyz=\phi(x+y+z)$

3) $F(xy+z^2, x+y+z)=0$

Ans:- Differentiate partially w.r.t x

$$p=(x+y)\cdot\phi_1'(x^2-y^2)\cdot 2x+\phi(x^2-y^2)$$

Differentiate partially w.r.t y

$$q=(x+y)\cdot\phi_1'(x^2-y^2)\cdot(-2y)+\phi(x^2-y^2)$$

$\phi(x^2-y^2)=\dfrac{z}{(x+y)}$: substituting

$$p=(x+y)\cdot 2x\cdot\phi_1'(x^2-y^2)+\frac{z}{x+y}$$

$$p-\frac{z}{x+y}=2x(x+y)\cdot\phi_1'(x^2-y^2) \text{ —— ②}$$

$$q=(x+y)\cdot -2y\cdot\phi_1'(x^2-y^2)+\frac{z}{x+y}$$

$$q-\frac{z}{x+y}=-2y(x+y)\cdot\phi_1'(x^2-y^2) \text{ —— ③}$$

② ÷ ③

$$\frac{p-\frac{z}{x+y}}{q-\frac{z}{x+y}}=\frac{2x}{-2y} \quad ; \quad p-\frac{z}{x+y}(-y)=q-\frac{z}{x+y}(x)$$

$$\frac{zy}{x+y}-py=xq-\frac{zx}{x+y}$$

$$zy-pyx-py^2=x^2q+qxy-zx$$

$$py(x+y)+qx(x+y)-z(x+y)=0$$

$$\therefore py(x+y)+qx(y+x)-z(x+y)=0$$

$$\therefore py+qx=z$$

18. $(x-a)^2+(y-b)^2=z^2\cot^2\alpha$ —— ①

Ans: Differentiating w.r.t x

$$2(x-a)=2z\frac{\partial z}{\partial x}\cot^2\alpha \text{ —— ②}$$

Differentiating w.r.t y

$$2(y-b)=2z\frac{\partial z}{\partial y}\cot^2\alpha \text{ —— ③}$$

substituting ② & ③ in ①

$$z^2p^2\cot^4\alpha+z^2q^2\cot^4\alpha=z^2\cot^2\alpha$$

$$z^2\cot^2\alpha(p^2\cot^2\alpha+q^2\cot^2\alpha)=z^2\cot^2\alpha$$

$$p^2+q^2(\cot^2\alpha)=1$$

$$p^2+q^2=\frac{1}{\cot^2\alpha}$$

$$p^2+q^2=\tan^2\alpha$$

14. $z = f(x^2+y^2)+x+y$

Ans: Differentiating w.r.t x

$$\frac{\partial z}{\partial x} = f'(x^2+y^2)\cdot 2x + 1 \quad \text{—— ①}$$

Differentiating w.r.t y

$$\frac{\partial z}{\partial y} = f'(x^2+y^2)\cdot 2y + 1 \quad \text{—— ②}$$

From ①; $p-1 = f'(x^2+y^2)\cdot 2x$ —— ③

From ②; $q-1 = f'(x^2+y^2)\cdot 2y$ —— ④

Dividing ③ & ④

$$\frac{p-1}{q-1} = \frac{x}{y}$$

$$(p-1)y = (q-1)x$$

$$py - y = qx - x \qquad \text{i.e; } py - qx = y - x$$

$$\therefore (q-1)x - (p-1)y = 0$$

15. $xyz = \phi(x+y+z)$

Ans: Differentiating w.r.t x

$$y\left(x\cdot\frac{\partial z}{\partial x} + z\cdot 1\right) = \phi'(x+y+z)\cdot(1+p)$$

$$y(xp+z) = \phi'(x+y+z)\cdot(1+p) \quad \text{—— ①}$$

Differentiating w.r.t y

$$x\left(y\cdot\frac{\partial z}{\partial y} + z\cdot 1\right) = \phi'(x+y+z)\cdot(1+q)$$

$$x(yq+z) = \phi'(x+y+z)\cdot(1+q) \quad \text{—— ②}$$

Dividing ① by ②

$$\frac{y(xp+z)}{x(yq+z)} = \frac{1+p}{1+q}$$

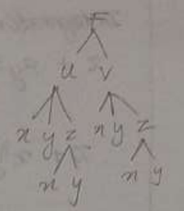
Rearranging; $y(xp+z)(1+q) = x(yq+z)(1+p)$

$$yxp + yz + \cancel{yxpq} + yzq = xyq + xz + p\cancel{xyq} + pxz$$

$$p(xy - xz) + yz = q(xy - yz) + xz$$

$$px(y-z) - qy(x-z) + z(y-x) = 0$$

16. $F(xy+z^2,\ x+y+z)=0$



Ans: $F(u,v)=0 \quad \text{——} ①$

$u=xy+z^2 \qquad v=x+y+z$

Differentiating w.r.t x

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right]+\frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right]=0$$

$$\frac{\partial F}{\partial u}\left[y+2zp\right]+\frac{\partial F}{\partial v}\left[1+p\right]=0$$

Differentiating w.r.t y

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right]+\frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right]=0$$

$$\frac{\partial F}{\partial u}\left[x+2zq\right]+\frac{\partial F}{\partial v}\left[1+q\right]=0$$

By the method of determinants

$$\begin{vmatrix} y+2zp & 1+p \\ x+2zq & 1+q \end{vmatrix}=0$$

Expanding: $(1+q)(y+2zp)-(1+p)(x+2zq)=0$

$$y+2zp+qy+2zpq-x-2zq-px-2zpq=0$$

$$y+2zp+qy=x+2zq+px$$

$$y+2zp-px=x+2zq-qy$$

$$y+p(2z-x)=x+q(2z-y)$$

$$p(2z-x)-q(2z-y)+y-x=0$$

29/11/21 **Solution of PDE**

* Equations solvable by direct integration

These equations which contain only one PD can be solved by direct integration. In place of the constants of integration we use arbitary functions of the variables kept as constant.

variable function is arbitary const

1. $\dfrac{\partial^3 z}{\partial x^2 \partial y}+18xy^2+\sin(2x-y)=0$

Ans: Integrating w.r.t to y keeping x const.

$$\frac{\partial^2 z}{\partial x^2}+18x\frac{y^3}{3}-\frac{\cos(2x-y)}{-1}=f(x)$$

Integrating w.r.t to x keeping y const.

$$\frac{\partial z}{\partial x}+6y^3\frac{x^2}{2}+\frac{\sin(2x-y)}{2}=\int f(x)+g(y)$$

Integrating w.r.t x keeping y const

$$z + 3y^3 \cdot \frac{x^3}{3} - \frac{\cos(2x-y)}{4} = \iint f(x) + \int g(y) + h(y)$$

$$z + x^3y^3 - \frac{\cos(2x-y)}{4} = \overset{2^{nd}}{f_1(x)} + x\overset{1^{st}}{(g(y))} + h(y)$$

2. Solve $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$

Ans: Integrating w.r.t y keeping x const

$$\frac{\partial z}{\partial x} = x \ln y + ay + f(x)$$

Integrating w.r.t x keeping y const

$$z = \frac{x^2}{2} \ln y + axy + \int f(x) + g(y)$$

3. Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$. Given $\frac{\partial z}{\partial y} = -2 \sin y$: when $x=0$, & $z=0$: when y is an odd multiple of $\pi/2$.

Ans: Integrating w.r.t x keeping y const

$$\frac{\partial z}{\partial y} = -\sin y \cos x + f(y)$$

$$-2\sin y = -\sin y + f(y)$$

$$\therefore f(y) = -2\sin y + \sin y = -\sin y$$

i.e, $\frac{\partial z}{\partial y} = -\sin y \cos x - \sin y$

Integrating w.r.t y keeping x const

$$z = \cos x \cos y + \cos y + g(x)$$

y is odd : so : $y = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$

$\cos \frac{\pi}{2} = 0$

$$0 = \cos x \cos 0 + \cos 0 + g(x)$$

$$\therefore g(x) = 0$$

i.e, $z = \cos y(\cos x + 1)$

4. Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$, for $x=0$: $z = e^y$ & $\frac{\partial z}{\partial x} = 1$

Ans: By using characteristics eqn : $\lambda^2 + 1 = 0$: $\lambda = \pm i$

$\therefore$ Solution $= f(x)\cos x + g(y) \sin x$

$$z = f(x) \cos x + g(y) \sin x$$

$$e^y = f(x) \qquad \text{when } x=0$$

$$\frac{\partial z}{\partial x} = -f(x)\sin x + g(y) \cos x$$

$$1 = g(y) \qquad \text{(condition)}$$

$\therefore z = e^y \cos x + \sin x$

5. $\dfrac{\partial^2 z}{\partial y^2} = z$, $z = e^x$, $\dfrac{\partial z}{\partial y} = e^{-x}$, $y=0$

Ans: $\dfrac{\partial^2 z}{\partial y^2} - z = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$$z = f(x)e^y + g(x)e^{-y}$$

$$e^x = f(x)e^0 + g(x)e^{-0}$$

$$e^x = f(x) + g(x) \quad \text{——} \ ①$$

$$\frac{\partial z}{\partial y} = yf(x)e^y - yg(x)e^{-y}$$

$$e^{-x} = f(x) - g(x) \quad \text{——} \ ②$$

①+② :

$$e^x + e^{-x} = 2f(x)$$

$$f(x) = \frac{e^x + e^{-x}}{2} = \cosh x$$

①-② :

$$e^x - e^{-x} = 2g(x)$$

$$g(x) = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\therefore z = \cosh x \cdot e^y + \sinh x \cdot e^{-y}$$

Lagrange's Linear PDE

A first order linear PDE of the form $Pp + Qq = R$ where P, Q, R are function of x, y, z.

Working Method

(1) Form the auxillary eqns: $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

Solve the auxillary eqns by the method of grouping, multiplier or both to get two independent solutions $u=a$, $v=b$ then $\phi(u,v)=0$, or $u=f(v)$ is the general solutions of $Pp+Qq=R$

H.W

1. $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$

Ans: Integrating w.r.t t keeping x const.

$$\frac{\partial u}{\partial x} = -t \cdot e^{-t}\cos x + f(x)$$

Integrating w.r.t x keeping t const.

$$u = -t \cdot e^{-t} \sin x + \int f(x) + g(t)$$

$\frac{\partial^2 z}{\partial x^2} = a^2 z$

2. $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$

Ans: Integrating with r.t y keeping x const.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\sin(2x+3y)}{3} + f(x)$$

Integrating with r.t x keeping y const.

$$\frac{\partial z}{\partial x} = -\frac{\cos(2x+3y)}{6} + \int f(x) + g(y)$$

Integrating w.r.t x keeping y const.

$$z = -\frac{\sin(2x+3y)}{12} + \iint f(x) + \int g(y) + h(y)$$

$$z + \frac{\sin(2x+3y)}{12} = f_1(x) + x(g(y)) + h(y)$$

3. $\frac{\partial^2 z}{\partial x^2} = a^2 z$

Ans: Integrating w.r.t x keep $\frac{\partial^2 z}{\partial x^2} - a^2 z = 0$

By using characteristic eqn: $\lambda^2 - a^2 = 0$; $\lambda^2 = a^2$, $\therefore \lambda = \pm a$

$\therefore$ Solution $\Rightarrow$ $z = f(y)e^{ax} + g(y)e^{-ax}$

questions (LPDE)

1. Solve $xp + yq = 3z$

Ans: $Pp + Qq = R$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\underbrace{\frac{dx}{x} = \frac{dy}{y}}_{1} = \underbrace{\frac{dz}{3z}}_{2} \quad \text{grouping}$$

1: $\frac{dx}{x} = \frac{dy}{y}$

$\ln x = \ln y + \ln a$

$\ln x - \ln y = \ln a$

$\ln \frac{x}{y} = \ln a$

$\therefore \frac{x}{y} = a$

$\Rightarrow [u = a]$

2: $\frac{dy}{y} = \frac{dz}{3z}$

$\ln y = \frac{1}{3} \ln z + \ln b$

$\ln y = \ln z^{1/3} \cdot b$

$y = z^{1/3} \cdot b$

$\therefore \frac{y}{z^{1/3}} = b$

$\rightarrow [v = b]$

$$\therefore \phi(x/y,\ y/z^{1/3}) = 0$$

2. $(z-y)p + (x-z)q = y-x$

Ans: $$\frac{dx}{(z-y)} = \frac{dy}{(x-z)} = \frac{dz}{(y-x)} = \frac{dx+dy+dz}{z-y+x-z+y-x} = \frac{dx+dy+dz}{0}$$

Multipliers

$\therefore \; dx + dy + dz = 0$

$x + y + z = a$

$\Rightarrow [u = a]$

Using multipliers x, y, z

$$\frac{x\,dx}{zx - yx} = \frac{y\,dy}{xy - zy} = \frac{z\,dz}{zy - xz} = \frac{x\,dx + y\,dy + z\,dz}{zx - yx + xy - zy + zy - xz}$$

$x\,dx + y\,dy + z\,dz = 0$

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = b$

$\Rightarrow [v = b]$

$\therefore \phi\left(x + y + z, \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}\right) = 0$

3. $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{dz}{dy} = ly - mx$

Ans:- $$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Using multipliers x, y, z

$$\frac{x\,dx}{xmz - nyx} = \frac{y\,dy}{nxy - lzy} = \frac{z\,dz}{lyz - mxz} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$x\,dx + y\,dy + z\,dz = 0 \quad \Rightarrow [u = a]$

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = a$

Using multipliers l, m, n

$$\frac{l\,dx}{lmz - lny} = \frac{m\,dy}{mnx - lmz} = \frac{n\,dz}{nly - mnx} = \frac{l\,dx + m\,dy + n\,dz}{0}$$

$l\,dx + m\,dy + n\,dz = 0$

$lx + my + nz = b \Rightarrow [v = b]$

$\therefore \phi\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}, lx + my + nz\right) = 0$

4. $(x^2 - y^2 - z^2)p + q\,2xy = 2xz$

Ans:- $$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$ grouping

$\frac{dy}{2xy} = \frac{dz}{2xz} \Rightarrow \frac{dy}{y} = \frac{dz}{z}$; $\ln y = \ln z + \ln a$

$\ln y - \ln z = \ln a$

$\frac{y}{z} = a \Rightarrow [u = a]$

using multipliers x, y, z

$$\frac{x\,dx}{x^3-xy^2-xz^2} = \frac{y\,dy}{2xy^2} = \frac{z\,dz}{2xz^2} = \frac{x\,dx+y\,dy+z\,dz}{x^3-xy^2-xz^2+2xy^2+2xz^2}$$

compare

$$= \frac{x\,dx+y\,dy+z\,dz}{x(x^2+y^2+z^2)} = \frac{dz}{2xz} = \frac{2x\,dx+2y\,dy+2z\,dz}{x^2+y^2+z^2} = \frac{dz}{z}$$

$$\Rightarrow \ln(x^2+y^2+z^2) = \ln z$$

$$x^2+y^2+z^2 = z$$

$$\therefore \phi\left(y/z,\ x^2+y^2+z^2\right) = 0$$

H.W

1) $p\tan x + q\tan y = \tan z$

Ans: $\underbrace{\frac{dx}{\tan x} = \frac{dy}{\tan y}}_{①} \underbrace{= \frac{dz}{\tan z}}_{②}$

① $\frac{dx}{\tan x} = \frac{dy}{\tan y}$

$\ln(\sin x) = \ln(\sin y) + \ln a$

$\ln(\sin x) - \ln(\sin y) = \ln a$

$\ln \frac{\sin x}{\sin y} = \ln a$

$\frac{\sin x}{\sin y} = a$: $[v=a]$

② $\frac{dy}{\tan y} = \frac{dz}{\tan z}$

$\ln(\sin y) = \ln(\sin z) + \ln b$

$\ln(\sin y) - \ln(\sin z) = \ln b$

$\ln \frac{\sin y}{\sin z} = \ln b$

$\frac{\sin y}{\sin z} = b$: $[v=b]$

$$\therefore \phi\left(\frac{\sin x}{\sin y};\ \frac{\sin y}{\sin z}\right) = 0$$

2. $y^2 z p + x^2 z q = y^2 x$

Ans: $\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{y^2 x}$

using multipliers x, y, z.

$$\frac{x\,dx}{xy^2 z} = \frac{y\,dy}{x^2 y z} = \frac{z\,dz}{xy^2 z} = \frac{x\,dx+y\,dy+z\,dz}{2xy^2z + x^2yz}$$

$$\frac{x\,dx}{xy^2 z} = \frac{z\,dz}{xy^2 z} \Rightarrow x\,dx = z\,dz$$

Integrate; $\frac{x^2}{2} - \frac{z^2}{2} = a$; $[a=u]$

use multipliers x^2, y^2, z^2

$$\frac{x^2dx}{x^2y^2z} = \frac{y^2dy}{x^2y^2z} = \frac{z^2dz}{y^2xz^2}$$

$$\frac{x^2dx}{x^2y^2x} = \frac{y^2dx}{x^2y^2z} \longrightarrow x^2dx = y^2dy$$

Integrate ;

$$\frac{x^3}{3} - \frac{y^3}{3} = b \quad [b=v]$$

$$\phi\left(x^2/_2 - z^2/_2 ,\ x^3/_3 - y^3/_3\right) = 0$$

5. solve $(x^2-yz)p + (y^2-zx)q = z^2 xy$

Ans: $\frac{dx}{(x^2-yz)} = \frac{dy}{(y^2-zx)} = \frac{dz}{(z^2-xy)}$

using multipliers x, y, z

$$\frac{xdx}{x^3-xyz} = \frac{ydy}{y^3-xyz} = \frac{zdz}{z^3-xyz} = \frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}$$

using multipliers 1, 1, 1

$$\frac{dx+dy+dz}{x^2+y^2+z^2-yz-xy-zx} = \frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}$$

$$\frac{dx+dy+dz}{x^2+y^2+z^2-yz-xy-zx} = \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$dx+dy+dz\,(x+y+z) = xdx+ydy+zdz$$

$$(x+y+z)\,d(x+y+z) = xdx+ydy+zdz$$

$$\frac{(x+y+z)^2}{2} + a = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}$$

$$\frac{x^2+y^2+z^2}{2} - \frac{x+y+z}{2} = a$$

Using multiplier 1, -1

$$\frac{dx-dy}{x^2-y^2+zx-yz} = \frac{dy-dz}{(y^2-z^2)+xy-zx}$$

$$\frac{dx-dy}{(x^2-y^2)+z(x-y)} = \frac{dy-dz}{(y^2-z^2)+x(y-z)}$$

$$\frac{dx-dy}{(x-y)[(x+y+z)]} = \frac{dy-dz}{(y-z)[(x+y+z)]}$$

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

$$\ln(x-y) = \ln(y-z) + \ln k$$

$$\ln\left(\frac{x-y}{y-z}\right) = \ln k \quad [v=b] \qquad \frac{x-y}{y-z} = b$$

$\therefore \phi($

7. Solve $xp + yq = y^2 - x^2$

Ans: $\frac{dx}{x} = \frac{dy}{+y} = \frac{dz}{y^2-x^2}$

Grouping : $\frac{dx}{x} = \frac{dy}{+y}$

$$\ln(x) = +\ln(y) + \ln(a)$$

$$\ln(x) - \ln(y) = \ln(a)$$

$$\ln(x/y) = \ln(a)$$

$$\therefore x/y = a \rightarrow [u=a]$$

Adding & comparing

$$\frac{dx+dy}{x-y} = \frac{dz}{y^2-x^2}$$

$$\frac{dx+dy}{(x-y)} = \frac{dz}{-(x^2-y^2)}$$

$$\frac{dx+dy}{x-y} = \frac{-dz}{(x-y)(x+y)}$$

$$(\overset{u}{x+y})(\overset{du}{dx+dy}) = -dz$$

$$\frac{(x+y)^2}{2} + z = b$$

$$\therefore \phi\left(x/y, \frac{(x+y)^2}{2} + z\right) = 0$$

8. Solve $x(y^2-z^2)p + y(z^2-x^2)q - z(x^2-y^2) = 0$

Ans: $Pp + Qq = r$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$$

Using multipliers $1/x, 1/y, 1/z$

$$\frac{1/x\,dx}{y^2-z^2} = \frac{1/y\,dy}{z^2-x^2} = \frac{1/z\,dz}{x^2-y^2} = \frac{1/x\,dx + 1/y\,dy + 1/z\,dz}{0}$$

$$1/x\,dx + 1/y\,dy + 1/z\,dz = 0$$

$$\ln(x) + \ln(y) + \ln(z) = \ln(a)$$

$$\ln(xyz) = \ln(a)$$

$$xyz = a \quad = [u=a]$$

using multiplier x, y, z

$$\frac{x\,dx}{z^2y^2 - x^2z^2} = \frac{y\,dx}{z^2y^2 - x^2y^2} = \frac{z\,dz}{z^2x^2 - z^2y^2} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$x\,dx + y\,dy + z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = b \quad ; [v=b]$$

$$\therefore \phi\left(xyz,\ \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}\right) = 0$$

9. Solve $y^2p - xyq = x(z - 2y)$

Ans: $$\underbrace{\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}}$$

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$-x\,dx = y\,dy$$

$$x\,dx + y\,dy = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} = a \Rightarrow [u=a]$$

$$\frac{dx + dz}{y^2 - xy} = \frac{dz}{zx - 2y}$$

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)} =$$

subtract:

$$\frac{dy - dz}{-y - z + 2y} = \frac{dy - dz}{y - z} = \frac{dy}{-y}$$

$$\ln(y-z) = -\ln(y) + \ln(b)$$

$$\ln(y-z) + \ln(y) = \ln(b)$$

$$\ln(y-z)y = \ln b$$

$$y^2 - zy = b \Rightarrow [v=b]$$

$$\therefore \phi\left(\frac{x^2}{2} + \frac{y^2}{2},\ y^2 - zy\right) = 0$$

1) $p\cos(x+y) + q\sin(x+y) = z$

2) $p\sqrt{x} + q\sqrt{x} = \sqrt{z}$

3) $pyz + qzx = xy$

4) $p - q = \ln(x+y)$

5) $(y+z)p - (z+x)q = x - y$

10. Solve $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$

Ans: $$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + zx} = \frac{dz}{xy - zx}$$

using multipliers x, y, z

$$\frac{x\,dx}{xz^2 - 2xy - y^2x} = \frac{y\,dy}{xy^2 + zxy} = \frac{z\,dz}{zxy - z^2x} = \frac{x\,dx + y\,dy + z\,dz}{-2xy + 2xyz}$$

$$x\,dx + y\,dy + z\,dz = 0 \ ; \ \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = a \ ; [u=a]$$

Grouping;
$$\frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$$
$$\frac{dy}{y+z} = \frac{dz}{y-z}$$
$$(y-z)\,dy = (y+z)\,dz$$
$$y\,dy - z\,dz = 2yz$$
$$\frac{y^2}{2} - \frac{z^2}{2} - \cancel{2yz} = b$$
$$\therefore \phi\left(x^2+y^2+z^2, \frac{y^2}{2} - \frac{z^2}{2} - 2yz\right) = 0$$

H.W ①
$$p\cos(x+y) + q\sin(x+y) = z$$

Ans: $P_p + Q_q = R$
$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}$$
$$\sin(x+y)\,dx = \cos(x+y)\,dy$$

Comparing, $\sin(x+y)\,dx = \frac{1}{z}\,dz$
$$-\cos(x+y) - \log z = a$$
$$[u=a]$$

Comparing; $\cos(x+y)\,dy = \frac{1}{z}\,dz$
$$\sin(x+y) - \log z = b$$
$$[v=b]$$

$\therefore$ Solution : $\phi(-\cos(x+y) - \log z, \sin(x+y) - \log z) = 0$

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}$$
$$\sin(x+y)\,dx = \cos(x+y)\,dy$$
check

② $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

Ans: $\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$

$x^{1/2}dx = y^{1/2}dy$: $\frac{x^{1/2}}{1/2} = \frac{y^{1/2}}{1/2} + a$: $\sqrt{x} - \sqrt{y} = a$

$y^{1/2}dy = z^{1/2}dz$: $\frac{y^{1/2}}{1/2} = \frac{z^{1/2}}{1/2} + b$: $\sqrt{y} - \sqrt{z} = b$

$\therefore$ Solution : $\phi(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$

③ $pyz + qxz = xy$

Ans: $Pp + Qq = r$

$$\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$$

using multipliers x, y, z

$$\underbrace{\frac{y\,dy}{xyz} - \frac{x\,dx}{xyz}}_{①} \underbrace{= \frac{z\,dz}{xyz}}_{②}$$

Comparing ①:

$$\frac{x\,dx}{xyz} = \frac{y\,dy}{xyz}$$

$x\,dx - y\,dy = 0$

Integrating:

$x^2 - y^2 = 2a$

$[u = a]$

Comparing ②:

$$\frac{y\,dy}{xyz} = \frac{z\,dz}{xyz}$$

$y\,dy - z\,dz = 0$

Integrating:

$y^2 - z^2 = 2b$

$[v = b]$

$\therefore \phi(x^2 - y^2, y^2 - z^2) = 0$

④ $p - q = \ln(x+y)$

Ans: $Pp + Qq = r$

$$\underbrace{\frac{dx}{1} = \frac{dy}{-1}}_{①} \underbrace{= \frac{dz}{\ln(x+y)}}_{②}$$

comparing ①: $dx = -dy$

$dx + dy = 0$

Integrating: $x + y = a$

$[u = a]$

Comparing ②: $+dy(\ln(x+y)) = -dz$

$\ln(x+y)\,dy - dz = 0$

$y\ln(x+y) - z = b$

$[v = b]$

$\therefore$ solution: $\phi(x+y, y\ln(x+y) - z) = 0$

⑤ $(y+z)p - (z+x)q = x - y$

Ans: $Pp + Qq = r$

$$\frac{dx}{y+z} = \frac{dy}{-(z+x)} = \frac{dz}{x-y}$$

using multipliers x, y, z

$$\frac{x\,dx}{xy + xz} = \frac{y\,dy}{-zy - xy} = \frac{z\,dz}{zx - yz} = \frac{x\,dx + y\,dy + z\,dz}{2zx - 2zy}$$

Comparing:

$$\frac{2z\,dz}{zx - yz} = \frac{x\,dx + y\,dy + z\,dz}{zx - zy}$$

$x\,dx + y\,dy - z\,dz = 0$

Integrate

$$\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = a$$

$[u = a]$

$$\frac{dx+dy+dz}{x-y-x-z+y+z} = dx+dy+dz = 0$$

$$\therefore x+y+z = b$$

$$\therefore \text{solution}: \phi(x^2+y^2-z^2, x+y+z) = 0$$

Charpits Method

Consider a non linear first order PDE of the form $f(x,y,z,p,q)=0$. The auxillary eqns. are;

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{-dp}{f_x+pf_z} = \frac{-dq}{f_y+qf_z}$$

From the auxillary eqn; we solve for p & q, then the solution can be obtained from the eqn: $dz = pdx + qdy$

① Solve $p^2y + q^2y = zq$

Ans: $(p^2+q^2)y - qz = 0$

Auxillary eqn; $\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{-dp}{f_x+pf_z} = \frac{-dq}{f_y+qf_z}$

$$\therefore \frac{dx}{2py} = \frac{dy}{2qy-z} = \frac{dz}{p(2py)+q(2qy-z)} = \frac{-dp}{-pq} = \frac{-dq}{(p^2+q^2)+q(-q)}$$

$$\frac{dp}{pq} = \frac{-dq}{p^2}$$

$$\frac{dp}{q} = -\frac{dq}{p}$$

$$pdp + qdq = 0$$

$$\frac{p^2}{2} + \frac{q^2}{2} = a$$

$$p^2+q^2 = c^2$$

substituting in qn;

$$c^2y - qz = 0$$

$$c^2y = qz$$

$$\therefore q = \frac{c^2y}{z}$$

$$p^2 + \frac{c^4y^2}{z^2} = c^2$$

$$p^2 = c^2 - \frac{c^4y^2}{z^2}$$

$$p^2 = c^2\left(1 - \frac{c^2y^2}{z^2}\right)$$

$$p^2 = \frac{c^2}{z^2}(z^2 - c^2y^2)$$

$$\therefore p = \frac{c}{z}(z^2 - c^2y^2)$$

$\therefore$ Solution: $dz = pdx + qdy$

$$dz = \frac{c}{z}(z^2-c^2y^2)dx + \frac{c^2y}{z}dy$$

$$zdz = c(z^2-c^2y^2)dx + c^2y\,dy$$

$$zdz - c^2y\,dy = c(z^2-c^2y^2)dx$$

$$\frac{z\,dx - c^2 y\,dy}{\sqrt{z^2 - c^2y^2}} = c\,dx$$

$$\frac{\frac{du}{2}}{\sqrt{u}} = cx + a$$

$$\frac{1}{2} \times 2\sqrt{u} = cx + a$$

$$\sqrt{x^2 + c^2y^2} = cx + b$$

② $q + xp = p^2$

Ans: $q + xp - p^2 = 0$

Auxillary eqns; $\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{-dp}{f_x + pf_z} = \frac{-dq}{f_y + qf_z}$

$$\therefore \frac{dx}{x-2p} = \frac{dy}{1} = \frac{dz}{px - 2p^2 + q} = \frac{-dp}{p + p(0)} = \frac{-dq}{0}$$

Grouping ①: $\frac{-dp}{p} = \frac{-dq}{0}$; $dq = 0$ ∴ $q = a$

Grouping ②: $\frac{dy}{1} = \frac{-dp}{p}$; $y = -\ln p + \ln b$

$y = \ln \frac{b}{p}$

$e^y = \frac{b}{p}$ ∴ $p = \frac{b}{e^y}$

∴ Solution: $dz = \frac{b}{e^y}\,dx + a\,dy$

Integrating; $z = be^{-y}x + ay + c$

③ Solve $2zx - px^2 - 2qxy + pq = 0$

Ans: Auxillary eqn; $\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{-dx}{f_x + pf_z} = \frac{-dy}{f_y + qf_z}$

$$\therefore \frac{dx}{q - x^2} = \frac{dy}{p - 2xy} = \frac{dz}{pq - x^2p + qp - 2qxy} = \frac{-dx}{2z - 2px - 2qy + p2z} = \frac{-dy}{-2qx + q2z}$$

4. $p(q^2+1)+(b-z)q=0$

Ans: Auxillary eqn: $\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{-dp}{f_x+pf_z}=\frac{-dq}{f_y+qf_z}$

$$:\frac{dx}{q^2+1}=\frac{dy}{b-z}=\frac{dz}{p(q^2+1)+q(b-z)}=\frac{-dp}{p(-q)}=\frac{-dq}{q(-q)}$$

$\frac{dp}{pq}=\frac{dq}{q^2}$ (Grouping)

$\frac{dp}{p}=\frac{dq}{q}$

$\ln p=\ln q+\ln a$

$\ln p-\ln q=\ln a$

$\frac{p}{q}=a \qquad p=aq$

Substituting in qn:

$q^3a+qa+qb-zq=0$

$q(q^2a+a+b-z)=0$

$q^2=\frac{z-a-b}{a} = q=\sqrt{\frac{z-a-b}{a}}$

$p=a\sqrt{\frac{z-a-b}{a}}$

$dz=pdx+qdy$

$dz=a\sqrt{\frac{z-a-b}{a}}dx+\sqrt{\frac{z-a-b}{a}}dy$

$\frac{dz}{\sqrt{\frac{z-a-b}{a}}}=adx+dy$

$2\sqrt{\frac{z-a-b}{a}}=ax+y+c$

$\frac{z-a-b}{b}=\frac{a^2x^2+y^2+d}{4}$

$z=b\left[\frac{a^2x^2+y^2+d}{4}+\frac{a+b}{b}\right]$

5. $z^2=pqxy$

Ans: $z^2-pqxy=0$

Auxillary eqn: $\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{-dp}{f_x+pf_z}=\frac{-dq}{f_y+qf_z}$

$$:\frac{dx}{-qxy}=\frac{dy}{-pxy}=\frac{dz}{p(-qxy)+q(-pxy)}=\frac{-dp}{-pqy+p(2z)}=\frac{-dq}{-pqx+q(2z)}$$

$$\frac{dx}{-qxy} = \frac{dy}{-pxy} = \frac{dz}{-pqxy-pqxy} = \frac{dp}{pqy-2pz} = \frac{dq}{pqx-2qz}$$

$$\frac{\frac{dx}{x}}{-qy} = \frac{\frac{dy}{y}}{-px} = \frac{\frac{dp}{p}}{qy-2z} = \frac{\frac{dq}{q}}{px-2z}$$

Substrating ① & ②

$$\frac{\frac{dx}{x}-\frac{dy}{y}}{-qy+px} = \frac{\frac{dp}{p}-\frac{dq}{q}}{qy+px}$$

$$-\ln x+\ln y = \ln p - \ln q + \ln a^2$$

$$-\ln x+\ln q = \ln p - \ln y + \ln a^2$$

$$\ln xq = \ln p\frac{x}{y}a^2$$

$$qy = pxa^2$$

$$z^2 = pqxy = px\,pxa^2 = p^2x^2a^2$$

$$p^2 = \frac{z^2}{x^2a^2}, \quad p = \frac{z}{xa}$$

$$z^2 = \frac{q^2y^2}{a^2} \Rightarrow q^2 = \frac{z^2a^2}{y^2} ; \; q = \frac{za}{y}$$

$$dz \Rightarrow p\,dx + q\,dy$$

$$dz = \frac{z}{xa}dx + \frac{za}{y}dy$$

$$\frac{dz}{z} = \frac{1}{a}\frac{dx}{x} + a\times\frac{dy}{y}$$

$$\ln z = \tfrac{1}{a}\ln x + a\ln y + \ln b$$

$$z = x^{1/a}y^a + b$$

6. Solve $1+p^2 = qz$

Ans: $1+p^2-qz = 0$

Auxillary eqn: $\frac{dx}{2p} = \frac{dy}{-z} = \frac{dz}{2p^2-qz} = \frac{-dp}{-pq} = \frac{-dq}{q(-q)}$

grouping $\frac{dp}{pq} = \frac{dq}{q^2}$

$$\frac{dp}{p} = \frac{dq}{q}$$

$$\ln p = \ln q + \ln a$$

$$\frac{p}{q} = a \quad ; p = aq$$

$ax^2+bx+c=0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Substituting in qn:

$1+a^2q^2 - qz = 0$

$a^2q^2 - zq + 1 = 0$

$\therefore q = \frac{z \pm \sqrt{z^2-4a^2}}{2a^2}$

$1+p^2 - \frac{p}{a}z = 0$

$a + ap^2 - pz = 0$

$ap^2 - pz + a = 0$

$\therefore p = \frac{+z \pm \sqrt{z^2-4a^2}}{2a}$

$\therefore dz = p\,dx + q\,dy$

$dz = \frac{+z \pm \sqrt{z^2-4a^2}}{2a}dx + \frac{z \pm \sqrt{z^2-4a^2}}{2a^2}dy$

$\frac{dz}{z \pm \sqrt{z^2-4a^2}} = \frac{1}{2a}dx + \frac{1}{2a^2}dy$

$\frac{dz(z \pm \sqrt{z^2-4a^2})}{(4a^2)^2} = \frac{1}{2a}dx + \frac{1}{2a^2}dy$

$\int \sqrt{x^2-a^2}\,dx$

$\Rightarrow \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$

OR

$\Rightarrow \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\frac{x}{a}$

$\frac{1}{(4a^2)^2}\left(\frac{z^2}{2} + \frac{z}{2}\sqrt{z^2-4a^2} - \frac{4a^2}{2}\cosh^{-1}\frac{z}{2a}\right) = \frac{1}{2a}x + \frac{1}{2a^2}y$

7. $z = p^2x + q^2y$

Ans: Auxillary eqn: $z - p^2x - q^2y = 0$

$\frac{dx}{-2px} = \frac{dy}{-2qy} = \frac{dz}{-2p^2x - 2q^2y} = \frac{-dp}{-p^2+p(1)} = \frac{-dq}{-q^2+q(1)}$

$\frac{dx}{-2px} = \frac{dy}{-2qy} = \frac{dz}{-(2p^2x+2q^2y)} = \frac{dp}{p^2-p} = \frac{dq}{q^2-q}$

$\frac{dx}{-2px} = \frac{dp}{p(p-1)}$

$\frac{dx}{-2x} = \frac{dp}{p-1}$

$-\frac{1}{2}\ln x = \ln(p-1) + \ln c$

$\ln x^{1/2} = -\ln(1-p)c$

$x^{1/2} = (1-p)c \quad \therefore x^{1/2} = (p-1)c$

02/12/21

METHOD OF SEPARATION OF VARIABLES

1. Solve by the method of separation of variables: $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

Assume soln to be product of 2 variables

Ans: $z = X(x) \cdot Y(y)$

$\frac{\partial z}{\partial x} = X'Y$, $\frac{\partial^2 z}{\partial x^2} = X''Y$, $\frac{\partial z}{\partial y} = Y'X$

$X''Y - 2X'Y + XY' = 0$

$(X'' - 2X')Y = -XY'$

$\frac{X'' - 2X'}{X} = \frac{-Y'}{Y} = k$

$\frac{X'' - 2X'}{X} = k$ & $\frac{-Y'}{Y} = k$

$X'' - 2X' = kX$ $\quad +Y' + kY = 0$

$X'' - 2X' - kX = 0$ $\quad$ ch. eqn $= \lambda^2 + k = 0$

ch eqn: $\lambda^2 - 2\lambda - k = 0$ $\quad \lambda = \sqrt{-k}$

$\lambda = \frac{2 \pm \sqrt{4+4k}}{2}$ $\quad Y = c_3 e^{\sqrt{-k}y}$

$\lambda = 1 \pm \sqrt{1+k}$

$X = c_1 e^{1+\sqrt{1+k}} + c_2 e^{1-\sqrt{1+k}}$

$\therefore z = (c_1 e^{1+\sqrt{1+k}} + c_2 e^{1+\sqrt{1+k}}) c_3 e^{\sqrt{-k}y}$

2. $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$; $u(x,0) = 6e^{-3x}$

Ans: $u(x,t) = X_{(x)} \cdot T_{(t)}$

$X'T = 2XT' + XT$

$X'T = X(2T' + T)$

$\frac{X'}{X} = \frac{2T' + T}{T} = k$

$\frac{X'}{X} = k$ $\quad \frac{2T'+T}{T} = k$

$X' = kX$ $\quad 2T' + T = kT$

$X' - kX = 0$ $\quad 2T' + T - kT = 0$

$\lambda - k = 0$ $\quad 2\lambda + (1-k) = 0$

$\lambda = k$ $\quad \lambda = \frac{k-1}{2}$

$X = c_1 e^{kx}$ $\quad T = c_2 e^{k-1/2\, t}$

$\therefore u(x,t) = c_1 e^{kx} \cdot c_2 e^{k-1/2\, t}$

Giving values given

$\therefore u(x,0) = c_1 e^{kx} \cdot c_2 e^{0}$

$6e^{-3x} = c_1 c_2 e^{kx}$

$\therefore c_1 c_2 = 6 \quad ; k = -3$

$u(x,t) = 6e^{-3x} \cdot e^{-2t}$

3\. Solve $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$

Ans: $u(x,y) = X(x) \cdot Y(y)$

$4X'Y + Y'X = 3XY$

$4X'Y = 3XY - XY'$

$4X'Y = X(3Y - Y')$

$\frac{4X'}{X} = \frac{3Y - Y'}{Y} = k$

$\frac{4X'}{X} = k \qquad \frac{3Y - Y'}{Y} = k$

$4X' - kX = 0 \qquad 3Y - Y' - kY = 0$

$4\lambda - k = 0 \qquad 3 - \lambda - k = 0$

$4\lambda = k \qquad 3 - k = \lambda$

$\lambda = k/4$

$X = c_1 e^{k/4\, x} \qquad Y = c_2 e^{(3-k)y}$

$\therefore u(x,y) = c_1 e^{k/4\, x} \cdot c_2 e^{(3-k)y} = \underline{\underline{c_1 c_2\, e^{k/4} \cdot e^{(3-k)y}}}$

4\. Solve: $py^3 + qx^2 = 0$

Ans $\frac{\partial z}{\partial x} y^3 + \frac{\partial z}{\partial y} x^2 = 0$

Assume the solution to be: $z(x,y) = X(x) \cdot Y(y)$

$X'Y \cdot y^3 + XY'x^2 = 0$

$X'Yy^3 = -XY'x^2$

$\frac{Yy^3}{Y'} = \frac{-Xx^2}{X'} = k$

$Yy^3 - kY' = 0 \qquad Xx^2 + kX' = 0$

$y^3 = k\lambda \qquad x^2 = -k\lambda$

$\therefore \lambda = y^3/k \qquad \therefore \lambda = -x^2/k$

Sol: $Y = c_1 e^{y^3/k \cdot y} \qquad X = c_2 e^{-x^2/k \cdot x}$

$\therefore z = \underline{\underline{c_1 c_2 e^{y^4/k} \cdot e^{-x^3/k}}}$

OR $-\frac{Y'}{Yy^3} = \frac{X'}{Xx^2} = k$

$-Y' = kYy^3$

$-\frac{dY}{y} = ky^3 dy$

$\ln Y = -k\frac{y^4}{4} + \ln b$

$Y = b \cdot e^{-ky/4}$

$X' = kXx^2$

$\frac{dX'}{x} = kx^2 dx$

$\ln X = k\frac{x^3}{3} + \ln a$

$X = ae^{kx/3}$

$\therefore z = c_1 c_2 \cdot$

5. $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} = 0$ [Assume solution to be $u(x,y) = X(x)\cdot Y(y)$]

Ans: $x^2X'Y + y^2XY' = 0$

$x^2X'Y = -y^2XY'$

$\frac{x^2X'}{X} = \frac{-y^2Y'}{Y} = k$

$\frac{x^2X'}{X} = k$ $\quad$ $\frac{-y^2Y'}{Y} = k$

$x^2X' = kX$ $\quad$ $-y^2Y' = kY$

$x^2X' - kX = 0$ $\quad$ $y^2Y' + kY = 0$

$x^2\lambda - k = 0$ $\quad$ $y^2\lambda + k = 0$

$x^2\lambda = k$ $\quad$ $y^2\lambda = -k$

$\lambda = k/x^2$ $\quad$ $\lambda = -k/y^2$

Sol: $X = c_1e^{k/x^2 \times x}$ $\quad$ $Y = c_2e^{-k/y^2 \times y}$

$\therefore$ $u(x,y) = c_1c_2e^{k/x}\cdot e^{-k/y}$

6. $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$ $\quad$ $u(x,0) = 4e^{-x}$

Ans: Assume solution to be $u(x,y) = X_x \cdot Y_y$

$3X'Y + 2Y'X = 0$

$3X'Y = -2Y'X$

$\frac{3X'}{X} = \frac{-2Y'}{Y} = k$

$\frac{3X'}{X} = k$ $\quad$ $\frac{-2Y'}{Y} = k$

$3X' - kX = 0$ $\quad$ $+2Y' + kY = 0$

$3\lambda - k = 0$ $\quad$ $2\lambda + k = 0$

$3\lambda = k$ $\quad$ $2\lambda = -k$

$\lambda = k/3$ $\quad$ $\lambda = -k/2$

$X = c_1e^{k/3\,x}$ $\quad$ $Y = c_2e^{-k/2\,y}$

$\therefore$ $u(x,y) = c_1c_2e^{k/3\,x}\cdot e^{-k/2\,y}$

Applying given values;

$u(x,0) = c_1c_2e^{k/3\,x}\cdot e^0 = 4e^{-x}$

$u(x,0) = c_1c_2e^{k/3\,x} = 4e^{-x}$

$\therefore$ $c_1c_2 = 4$ & $\frac{kx}{3} = -x$; $k = -3$

$\therefore$ $u(x,y) = 4e^{-x}\cdot e^{3y/2}$