MODULE : 5

Laurent's series

Let $f(z)$ be analytic in a domain containing two concentric circles C_1 & C_2 with centre z_0 and the annulus b/w them, then $f(z)$ can be represented by the Laurent's series as,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

The terms consisting the -ve part of Laurent's series is called the principal part of Laurent's series.

1. Expand $1/1-z$ in non-negative powers & negative powers of z.

Ans: (i) $\frac{1}{1-z}$: positive powers ; $|z| < 1$

$$\Rightarrow (1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots$$

(ii) $\frac{1}{1-z} = \frac{1}{-z(1-1/z)} = \frac{-1}{z}\left(1-\frac{1}{z}\right)^{-1}$ $\quad \frac{1}{|z|} < 1$, $|z| > 1$

$$= \frac{-1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$$

$$= -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots$$

2. Find the Laurent's series expand of $z^{-5}\sin z$ at $z_0 = 0$

Ans: $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$

$$\Rightarrow z^{-5}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

$$= \underbrace{\frac{1}{z^4} - \frac{1}{z^2 3!}}_{\text{principal part}} + \frac{1}{5!} - \cdots$$

3. $z^2 e^{1/z}$; $z_0 = 0$

Ans: $z^2\left(1 + \frac{1/z}{1!} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!}\right)$

$$z^2 + \frac{z}{1!} + \frac{1}{2!} + \frac{1}{z\,3!} \cdots$$

4. $\frac{1}{z^3-z^4}$; $z_0=0$

Ans: $\frac{1}{z^3(1-z)} = \frac{1}{z^3}(1-z)^{-1}$

$\Rightarrow \frac{1}{z^3}(1+z+z^2+z^3+z^4 \ldots)$

$\Rightarrow \frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+z+z^2+\ldots$

5. $\frac{\cos z}{z^4}$; $z_0=0$

Ans: $\frac{1}{z^4}(\cos z) \Rightarrow \frac{1}{z^4}\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\ldots\right)$

$\Rightarrow \frac{1}{z^4}-\frac{1}{z^2 2!}+\frac{1}{4!}-\ldots$

6. $z^3\cosh\frac{1}{z}$, $z_0=0$

Ans: $\Rightarrow z^3\left(1+\frac{(1/z)^2}{2!}+\frac{(1/z)^4}{4!}+\ldots\right)$

$\Rightarrow z^3+\frac{z}{2!}+\frac{1}{z\cdot 4!}+\frac{1}{z^3 6!}+\ldots$

7. $\frac{e^z}{(z-1)^2}$; $z_0=1$

Ans: $\frac{e^z}{(z-1)^2} = \frac{e^{(z-1)+1}}{(z-1)^2} = e\frac{e^{z-1}}{(z-1)^2}$

$\frac{e}{(z-1)^2}\left(1+\frac{(z-1)}{1!}+\frac{(z-1)^2}{2!}+\ldots\right)$

$e\left(\frac{1}{(z-1)^2}+\frac{1}{(z-1)1!}+\frac{1}{2!}+\frac{(z-1)}{3!}+\ldots\right)$

8. $\frac{\sin z}{(z-\pi/4)^3}$; $z_0=\pi/4$

Ans: $\frac{\sin(z-\pi/4+\pi/4)}{(z-\pi/4)^3} = \frac{\sin(z-\pi/4)\cos\pi/4+\cos(z-\pi/4)\sin\pi/4}{(z-\pi/4)^3}$

$\Rightarrow \frac{1}{\sqrt{2}}\left[\frac{\sin(z-\pi/4)}{(z-\pi/4)^3}+\frac{\cos(z-\pi/4)}{(z-\pi/4)^3}\right]$

$\Rightarrow \frac{1}{\sqrt{2}}\left[\frac{1}{(z-\pi/4)^3}\left[(z-\pi/4)-\frac{(z-\pi/4)^3}{3!}-\frac{(z-\pi/4)^5}{5!}\right]+\left[\frac{1}{(z-\pi/4)^3}\left(1-\frac{(z-\pi/4)^2}{2!}+\right.\right.\right.$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{(z-\pi/4)^2} - \frac{1}{8!} - \frac{(z-\pi/4)^2}{5!}\right] + \left[\frac{1}{(z-\pi/4)^3} - \frac{1}{(z-\pi/4)2!} + \frac{(z-\pi/4)^2}{4!}\right]$$

$$= \frac{1}{\sqrt{2}}$$

9. $f(z) = \frac{(z^2+1)}{(z-2)(z-3)}$, (i) when $|z|<3$ & $|z|<2$, (ii) $2<|z|<3$

Here partial fraction is not possible because num deg ≥ din deg

so

$$z^2+5z+6 \overline{) z^2+1 } \; 1$$
$$\ominus\; z^2+5z+6$$
$$-5z-7$$

Ans. $\Rightarrow 1 + \frac{(-5z+7)}{(z-2)(z-3)}$

$-5z+7 = A(z-3) + B(z-2)$

when $z=3$; $B=-8$

When $z=2$; $A=3$

$$= 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

(i) $|z|<3$; $\frac{|z|}{3}<1$ $\qquad |z|<2$; $\frac{|z|}{2}<1$

$$\therefore f(z) = 1 + \frac{3}{2(1+z/2)} - \frac{8}{3(1+z/3)}$$

$$f(z) = 1 + \frac{3}{2}\left(1+\left(\frac{z}{2}\right)\right)^{-1} - \frac{8}{3}\left(1-\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left(1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \cdots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \cdots\right)$$

(ii) $2<|z|<3$; $2<|z|$, $2/|z|<1$ $\quad$ OR $\quad \frac{2}{|z|}<1$

$|z|<3$, $3/|z|>1$ $\qquad \frac{|z|}{3}>1$

$$\therefore f(z) = 1 + \frac{3}{z(1+2/z)} - \frac{8}{3(1+z/3)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \cdots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \cdots\right)$$

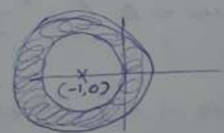
10. $f(z) = \frac{7z-2}{z(z+1)(z-2)}$ when $1<|z+1|<3$

Ans. $\Rightarrow 7z-2 = A(z+1)(z-2) + B(z)(z-2) + C(z)(z+1)$

When $z=2$; $C=2$

When $z=-1$; $B=-3$

When $z=0$; $A=1$



$$f(z) = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2}$$

$1<|z+1|$: $\frac{1}{|z+1|}<1$; $|z+1|<3$: $\frac{|z+1|}{3}<1$

$$f(z) = \frac{1}{z+1-1} - \frac{3}{z+1} + \frac{2}{z+1-3}$$

$$= \frac{1}{(z+1)(1-1/z+1)} - \frac{3}{(z+1)} + \frac{2}{-3(1-\frac{z+1}{3})}$$

$$= \frac{1}{z+1}\left(1-\frac{1}{z+1}\right)^{-1} - \frac{3}{z+1} - \frac{2}{3}\left(1-\frac{z+1}{3}\right)^{-1}$$

$$= \frac{1}{z+1}\left(1+\frac{1}{z+1}+\left(\frac{1}{z+1}\right)^2+\cdots\right) - \frac{3}{z+1} - \frac{2}{3}\left(1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\cdots\right)$$

11. $f(z) = \frac{1}{z(1-z)}$ $\quad |z+1| > 2$

Ans: $1 = A(1-z) + B(z)$ $\qquad |z+1| > 2$

When $z=0$; $A = 1$ $\qquad \frac{2}{|z+1|} < 1$

When $z=1$; $B = 1$

$$f(z) = \frac{1}{z} + \frac{1}{1-z}$$

$$= \frac{1}{z+1-1} + \frac{1}{1-(z+1-1)}$$

$$= \frac{1}{(z+1)(1-\frac{1}{z+1})} + \frac{1}{2-(z+1)}$$

$$= \frac{1}{(z+1)(1-1/z+1)} + \frac{1}{-(z+1)(1-2/z+1)}$$

$$= \frac{1}{z+1}\left(1-\frac{1}{z+1}\right)^{-1} - \frac{1}{z+1}\left(1-\frac{2}{z+1}\right)^{-1}$$

$$= \frac{1}{z+1}\left(1+\frac{1}{z+1}+\left(\frac{1}{z+1}\right)^2+\cdots\right) - \frac{1}{z+1}\left(1+\frac{2}{z+1}+\left(\frac{2}{z+1}\right)^2+\cdots\right)$$

$$= \left(\frac{1}{z+1}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\cdots\right)\left(-\frac{1}{z+1}-\frac{2}{(z+1)^2}-\frac{4}{(z+1)^3}-\cdots\right)$$

12. $f(z) = \frac{z+4}{(z+3)(z-1)^2}$ in (i) $0<|z-1|<4$ (ii) $|z-1|>4$

Ans: $\frac{z+4}{(z+3)(z-1)^2} = \frac{A}{z+3} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$ $\quad$ OR $\quad \frac{A}{z+3} + \frac{Bz+C}{(z-1)^2}$

$z+4 = A(z-1)^2 + B(z+3)(z-1) + C(z+3)$ $\qquad z+4 \Rightarrow A(z-1)^2 + (Bz+C)(z+3)$

when; $z=1$;

when; $z=1$; $C = 5/4$

when ; $z=-3$; $A = 1/16$

when; $z=0$; $B = -1/16$

a) $0<|z-1|<4 \quad ; \frac{|z-1|}{4}<1$

Ans $\Rightarrow \frac{1/16}{4\left(1+\frac{z-1}{4}\right)} - \frac{1/16}{z-1} + \frac{5/4}{(z-1)^2}$

$= \frac{1}{64}\left(1+\left(\frac{z-1}{4}\right)\right)^{-1} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$

$= \frac{1}{64}\left(1 - \frac{z-1}{4} + \left(\frac{z-1}{4}\right)^2 + \dots - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}\right.$

b) $|z-1|>4 \quad ; \quad \frac{|z-1|}{4}>1 \text{ or } \frac{4}{|z-1|}<1$

Ans $\Rightarrow \frac{1/16}{z-1+4} - \frac{1/16}{(z-1)} + \frac{5/4}{(z-1)^2}$

$\Rightarrow \frac{1/16}{z-1\left(1+\frac{4}{z-1}\right)} - \frac{1/16}{(z-1)} + \frac{5/4}{(z-1)^2}$

$= \frac{1}{16(z-1)}\left(1+\frac{4}{z-1}\right)^{-1} - \frac{1}{16}(z-1)^{-1} + \frac{5}{4}(z-1)^{-2}$

$= \frac{1}{16(z-1)}\left(1-\frac{4}{z-1}+\left(\frac{4}{z-1}\right)^2 \dots\right) - \frac{(z-1)^{-1}}{16} + \frac{5}{4}(z-1)^{-2}$

13. $\frac{1}{z^2+3z+2} \quad ; \quad 1<|z|<2$

Ans $\frac{1}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$

$1 = A(z+2) + B(z+1)$

$1<|z| \qquad |z|<2$

$\frac{1}{|z|}<1 \qquad \frac{|z|}{2}<1$

When; $z=2$; $B = 1/3$

When; $z=1$; $A = 1/3$

$f(z) = \frac{1/3}{z+1} + \frac{1/3}{z+2} \Rightarrow \frac{1/3}{z(1+1/z)} + \frac{1/3}{2\left(\frac{z}{2}+1\right)}$

$f(z) = \frac{1}{3z}(1+1/z)^{-1} + \frac{1}{6}(1+z/2)^{-1}$

$\Rightarrow \frac{1}{3z}\left(1-\frac{1}{z}+\frac{1}{z^2}\dots\right) + \frac{1}{6}\left(1-\frac{z}{2}+\left(\frac{z}{2}\right)^2+\dots\right)$

14. $\frac{z^2-6z-1}{(z-1)(z-3)(z+2)} \quad ; \quad 3<|z+2|<5$

Ans. $z^2-6z-1 = A(z-3)(z+2) + B(z-1)(z+2) + C(z-1)(z-3)$

When; $z=3$; $B = -1$

$z=1$; $A = 1$

$z=-2$; $C = 9$

Singular Point

A function $f(z)$ is singular at a point $z=z_0$ if $f(z)$ is not analytic at $z=z_0$ but at every neighbourhood of $z=z_0$ $f(z)$ is analytic.

Zero of the function

Zero of an analytic function $f(z)$ is a point of z for which $f(z)=0$.

Isolated singularity

$z=z_0$ is an isolated singularity of $f(z)$ if $z=z_0$ has a neighbourhood without further singularities of $f(z)$.

Isolated singularity of $f(z)$ at $z=z_0$ can be classified by Laurent's series

$z=z_0$ is isolated singularity of $f(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

$z=z_0$ is a pole when $f(z)$ is a -ve power

Power series containing -ve power of $z-z_0$ - principal part.

If it has only finite terms, i.e, $\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots \frac{b_m}{(z-z_0)}$ at $(b_m \neq 0)$

then the singularity of $f(z)$ at $z=z_0$ is a pole of order m.

If it has only one term, i.e, $\frac{b_1}{z-z_0}$ then it is called pole of order one or simple pole

If the principal part has infinitly many terms, then the isolated singularity $z=z_0$ is called essential singularity

Removable singularity

A function $f(z)$ has a removable singularity at $z=z_0$ if $f(z)$ is not analytic at $z=z_0$ but can be made analytic by assigning a suitable value of $f(z_0)$

Identify the singularities and specify its nature

1. $f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$

Ans $z=0$, $\rightarrow$ pole of order 5
$z=2$ $\rightarrow$ pole of order 1

3. $f(z) = e^{1/z}$

Ans: $z=0 \longrightarrow$ essential singularity $= 1 + \frac{1}{z\,1!} + \frac{1}{z^2\,2!} + \frac{1}{z^3\,3!}$

3. $f(z) = \sin 1/z$

Ans: $z=0 \longrightarrow$ essential singularity $= \frac{1}{z} - \frac{(1/z)^3}{3!} + \frac{(1/z)^5}{5!}$

4. $f(z) = z^{-5}\sin z$

Ans: $z=0 \longrightarrow$ pole of order 4

$\frac{\sin z}{z^5} = \frac{1}{z^5}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$

$\Rightarrow \frac{1}{z^4} - \frac{1}{z^2\,3!} + \frac{1}{5!} - \cdots$

5. $f(z) = \sin z/z$

Ans: $\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = \frac{1}{1} - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$ no z term in deno

$z=0 \longrightarrow$ removable singularity

6. $f(z) = \cot\frac{\pi}{z}$

Ans: $\cot \pi/z = \frac{\cos \pi/z}{\sin \pi/z}$

$\sin\frac{\pi}{z} = 0 = n\pi$; $n = \pm 1, \pm 2 \cdots$

$\frac{\pi}{z} = n\pi \quad \therefore z = \frac{1}{n} \quad = -1, -1, \frac{1}{2}, \frac{-1}{2} \cdots$

Isolated singularity

7. $f(z) = \frac{1-\cos z}{z}$

Ans: $z=0 \longrightarrow$ removable singularity

$1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)/z$ no z term in deno

$\Rightarrow \left(\frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)/z$

$\Rightarrow \left(\frac{z}{2!} + \frac{z^3}{4!} - \cdots\right)$

8. $f(z) = \frac{1}{\sin z - \cos z}$

Ans: $\sin z = \cos z \quad \therefore z = \frac{\pi}{4}$

Pole of order 1 or simple pole

9. $f(z) = \frac{\cot \pi/z}{(z-a)^2}$

Ans: $z = \frac{\cos \pi z}{\sin \pi z} \times \frac{1}{(z-a)^2}$; $\sin \pi z (z-a)^2 = 0$

$\therefore z = a$, $z = n\pi = 0, \pm 1, \pm 2 \cdots$

Isolated singularity

10. $f(z) = \dfrac{z - \sin z}{z^3}$

Ans: $\dfrac{z-\sin z}{z^3} = \dfrac{1}{z^3}\left[z - \left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots\right)\right]$

$\Rightarrow \dfrac{1}{z^3}\left(\dfrac{z^3}{3!} - \dfrac{z^5}{5!} + \dfrac{z^7}{7!} - \cdots\right)$

$\Rightarrow \dfrac{1}{3!} - \dfrac{z^2}{5!} + \dfrac{z^4}{7!} - \cdots$

$z=0$: $\longrightarrow$ removable singularity

11. $f(z) = (1+z^2)$ find the zeroes

Ans: $1+z^2=0$, $z = \pm i$

Residue Integration Method

If $f(z)$ has a singularity at a point $z=z_0$ inside c then $f(z)$ can be expanded as Laurent's series.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

The coefficient of 1st -ve powers of $z-z_0$ in the Laurent's series is b_1, then $b_1 = \dfrac{1}{2\pi i}\int_c f(z)\,dz$.

$\therefore \int_c f(z)\,dz = 2\pi i \times b_1$

The coefficient b_1 is called the residue of $f(z)$ at $z=z_0$ and we denote it by $b_1 = \operatorname{Res}_{z=z_0} f(z)$

1. Integrate $\dfrac{\sin z}{z^4}\,dz$ over c, where c is a unit circle taken counter-clockwise

Ans: $z^4=0$, $z=0$

0 lies inside the circle

$\dfrac{\sin z}{z^4} = \dfrac{z - z^3/3! + z^5/5! - \cdots}{z^4}$

$f(z) = \dfrac{1}{z^3} - \dfrac{1}{z\,3!} + \dfrac{z}{5!} - \dfrac{z^3}{7!} + \cdots$

$\int_c f(z)\,dz = 2\pi i \times b_1 = 2\pi i \times \left(-\dfrac{1}{3!}\right) = \dfrac{-\pi i}{3}$

2. Integrate $f(z) = \dfrac{1}{z^3 - z^4}$, clockwise around the circle; $c: |z| = \frac{1}{2}$

Ans: $z^3(1-z) = 0$; $\therefore z = 0, 1$

$\int_c f(z)dz = 2\pi i \operatorname{Res}_{z=0} f(z)$

$f(z) = \frac{1}{z^3(1-z)} = \frac{1}{z^3}(1-z)^{-1} = \frac{1}{z^3}(1+z+z^2+z^3+\cdots)$

$f(z) = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + \cdots\cdots$

$\int_c \frac{1}{z^3 - z^4} dz = 2\pi i \times b_1 \overset{\text{clockwise}}{=} -2\pi i \times 1 = \underline{\underline{-2\pi i}}$

Formulas for calculating Residues

simple pole at $z=z_0$

If $f(z)$ is a simple pole at $z=z_0$,

1. $\operatorname{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z-z_0) f(z)$

If $f(z) = \frac{p(z)}{q(z)}$, then

$$\operatorname{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)} \quad \text{where } q'(z_0) \neq 0,\ p(z_0) \neq 0$$

Pole of order m at $z=z_0$

$$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$$

Residue Theorem

Let $f(z)$ be analytic inside a simple closed path c. If $z_1, z_2, \cdots z_k$ are singular points inside c then,

Counterclockwise Direction.

$$\int_c f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}_{z=z_j} f(z_j)$$

Q: Find all the singularities in the finite plane and the corresponding residues.

(i) $f(z) = \frac{9z+i}{z^3+z}$

Ans: $z(z^2+1) = z$; $z^2 = -1$, $z = 0, \pm i$

All are simple poles.

$\operatorname{Res}(f(0)) : \lim_{z\to 0} (z-0) \frac{9z+i}{z(z+i)(z-i)} = \frac{i}{i(-i)} = i$

$\operatorname{Res} f(i) : \lim_{z\to i} (z-i) \frac{9z+i}{z(z+i)(z-i)} = \frac{10i}{i(2i)} = \frac{10}{-2} i = -5i$

$Res\, f(-i) = \lim_{z \to -i} (z+i) \dfrac{9z+i}{z(z+i)(z-i)} = \dfrac{9(-i)+i}{i(i-i)} = \dfrac{-8i}{-2} = \underline{\underline{4i}}$

Qn $f(z) = \dfrac{\sin 2z}{z^6}$

Ans $z^6 \Rightarrow z = 0$; pole of order 6

$Res\, f(z) = \dfrac{1}{(m-1)!} \lim_{z \to z_0} \dfrac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$ $\qquad m = 6$

$Res\, f(0) = \dfrac{1}{5!} \lim_{z \to 0} \dfrac{d^5}{dz^5} z^6 \dfrac{\sin 2z}{z^6}$

$Res\, f(0) = \dfrac{1}{5!} \lim_{z \to 0} \dfrac{d^5}{dz^5} \sin 2z$

$Res\, f(0) = \dfrac{1}{5!} \lim_{z \to 0} 32 \cos 2z$

$Res\, f(0) = \dfrac{1}{5!} 32 = \dfrac{32}{120} = \underline{\underline{\dfrac{4}{15}}}$

$f(x) = \sin 2x$
$f'(x) = 4\cos 2x$
$f''(x) = -4\sin 2x$
$f'''(x) = -8\cos 2x$
$f^{iv}(x) = 16 \sin 2x$
$f^{v}(x) = 32 \cos 2x$

Qn Evaluate the following integral counterclockwise along any simple closed path : c ; $\oint_z \dfrac{4-3z}{z^2-z} dz$

(i) 0 & 1 are inside c

(ii) 0 is inside, 1 is outside

(iii) 1 is inside, 0 is outside

(iv) 0 & 1 are outside c.

Ans: $z(z-1)$; $z = 0, +1$ [denominators equating to zero)

$z=0$; simple pole

$z=1$; pole of order 1

$\oint_c f(z)\,dz = 2\pi i \sum Res\, f(z)$

$z_0 = 0 \rightarrow Res\, f(z) = \lim_{z \to 0} (z-0) \dfrac{4-3z}{z(z-1)}$

$\Rightarrow \dfrac{4-0}{0-1} = \dfrac{4}{-1} = \underline{\underline{-4}}$

$z_0 = 1 \rightarrow Res\, f(1) = \lim_{z \to 1} (z-1) \dfrac{4-3z}{z(z-1)}$

$= \dfrac{4-3}{1} = \underline{\underline{1}}$

(ii) 0 & 1 are inside : $\oint \frac{4-3z}{z^2-z}\,dz = 2\pi i\,(\text{Res}\,f(0) + \text{Res}\,f(1))$
$= 2\pi i\,(-4+1) = -6\pi i$

(iii) 0 inside, 1 outside : $\oint \frac{4-3z}{z^2-z}\,dz = 2\pi i\,(\text{Res}\,f(0)) = 2\pi i\,(-4)$
$= -8\pi i$

(iv) 1 inside, 0 outside : $\oint \frac{4-3z}{z^2-z}\,dz = 2\pi i\,(\text{Res}\,f(1)) = 2\pi i\,(1)$
$= 2\pi i$

(v) 0 & 1 are outside : $\oint \frac{4-3z}{z^2-z}\,dz = 2\pi i\,(0)$ ∵ By C.I.T

Q: Evaluate $\int_c \frac{\tan z}{z^2-1}\,dz$ counter clockwise around $c : |z| = \frac{3}{2}$

Ans: $z^2 = 1$, $z = 1, -1$: poles of 1

$z = 1, -1 \Rightarrow 1, -1 < 3/2$ lies inside c.

$\text{Res}\,f(1) = \lim_{z\to 1} (z-1)\,\frac{\tan z}{(z+1)(z-1)} = \lim_{z\to 1} \frac{\tan 1}{2}$

$\text{Res}\,f(-1) = \lim_{z\to -1} (z+1)\,\frac{\tan z}{(z+1)(z-1)} = \lim_{z\to -1} \frac{\tan(-1)}{-2} = \frac{\tan 1}{2}$

$\therefore \oint_c \frac{\tan z}{z^2-1}\,dz = 2\pi i\,(\text{Res}\,f(1) + \text{Res}\,f(-1))$
$= 2\pi i\left(\frac{\tan 1}{2} + \frac{\tan 1}{2}\right)$
$= 2\pi i \tan 1$

Q: Evaluate $\int_c \frac{z-23}{z^2-4z-5}\,dz$; $|z-2-i| = 3.2$

Ans: $z^2-4z-5 = 0$; $z = -1, 5$: poles of order 1

$z=5$; $|z-2-i| = |5-2-i| = |3-i| = \sqrt{9+1} = \sqrt{10}$: inside the region

$z=-1$; $|z-2-i| = |-1-2-i| = \sqrt{3^2+1} = \sqrt{10} = 3.1 < 3.2$: inside the region.

Pole of order 1

$\text{Res}\,f(5) = \lim_{z\to 5} (z-5)\,\frac{z-23}{(z-5)(z+1)} = -3$

$\text{Res}\,f(-1) = \lim_{z\to -1} (z+1)\,\frac{z-23}{(z-5)(z+1)} = 4$

$\therefore \int_c \frac{z-23}{z^2-4z-5}\,dz = 2\pi i\,[\text{Res}\,f(5) + \text{Res}\,f(-1)]$
$= 2\pi i\,(-3+4) = 2\pi i$

Qn: Evaluate $\int_C \left(\frac{z(e^{\pi z})}{z^4-16} + z(e^{\pi/z}) \right) dz$ where $C: 9x^2+y^2=9$

Ans:

$z^4=16$, $(z^2)^2=4^2$; $((z^2)^2-4^2)=0$; $(z^2+4)(z^2-4)=0$

$\therefore$ Roots are $\Rightarrow \pm 2, \pm 2i$

simple poles

When, $z=2 \rightarrow x+iy$: $x=2, y=0$; $(2,0)$

$9(2)^2+0^2=9$

$36>9$ (outside)

When, $z=-2 \rightarrow x+iy$: $(-2,0)$

$9(-2)^2+0^2=9$

$36>9$ (outside)

When, $z=+2i$; $(0,2)$

$9(0)+2^2=9$

$4<9$ (inside)

~~$z(e^{\pi/z}) \Rightarrow 0$ (∴ By CIT)~~

When, $z=-2i$; $(0,-2)$

$9(0)+(-2)^2=9$

$4<9$ (inside)

$\int_C \frac{z(e^{\pi z})}{z^4-16} + \cancel{z(e^{\pi/z})}$:) $\text{Res } f(2i) = \lim_{z\to 2i} (z-2i) f(z)$

$\text{Res } f(2i) = \lim_{z\to 2i} (z-2i) \frac{(ze^{\pi z})}{(z^2-4)(z+2i)(z-2i)}$

$\text{Res } f(2i) = \lim_{z\to 2i} \frac{2i\, e^{2\pi i}}{((2i)^2-4)(2i+2i)}$ $\quad e^{i\theta} = \cos\theta + i\sin\theta$

$\text{Res } f(2i) = \lim_{z\to 2i} \frac{2i e^{2\pi i}}{(-8)(4i)} = \frac{-e^{2\pi i}}{16} = \frac{-1}{16}$

$\text{Res } f(-2i) = \lim_{z\to(-2i)} (z+2i) f(z)$

$\text{Res } f(-2i) = \lim_{z\to(-2i)} (z+2i) \frac{(ze^{\pi z})}{(z^2-4)(z+2i)(z-2i)}$

$\text{Res } f(-2i) = \lim_{z\to(-2i)} \frac{-2i\, e^{-2\pi i}}{((-2i)^2-4)((-2i)-2i)}$ $\quad e^{-i\theta} = \cos\theta - i\sin\theta$

$\text{Res } f(-2i) \Rightarrow \lim_{z\to(-2i)} \frac{-2i\, e^{-2\pi i}}{(-8)(-4i)} \Rightarrow \frac{e^{-2\pi i}}{16} = \frac{-1}{16}$ $\quad e^{-i\theta} \Rightarrow \frac{1}{16}(0-1)$

$\int_C \frac{z(e^{\pi z})}{z^4-16} dz = 2\pi i \left(\frac{-1}{16} - \frac{1}{16} \right) = \frac{-\pi i}{4}$

$\int_C z e^{\pi/2} dz :\Rightarrow \operatorname{Res}_{z=0} f(z) : \quad z\left(1+\frac{\pi}{2/1!}+\frac{(\frac{\pi}{2})^2}{2!} \cdots\right)$

$\Rightarrow z+\frac{1}{z}$

Qn: $\int_C e^{1/z} dz \; ; \; c: |z| = 1$

Ans: $e^{1/z} = 1+\frac{1}{z}+\frac{1}{z^2/2!}+\cdots$

$\therefore \operatorname{Res}_{z=0} f(z) = b_1 = 1$

$\therefore \int_C e^{1/z} dz = 2\pi i(1) = 2\pi i$

Qn: $\int_C \tan 2\pi z \, dz \; ; \; c: |z-0.2| = 0.2$

Ans: $\int_C \frac{\sin 2\pi z}{\cos 2\pi z} : \quad \cos 2\pi z = 0$

$2\pi z = \pm(2n+1)\pi/2 \qquad n = 0,1,2,3 \cdots$

$z = \pm\frac{(2n+1)}{4} \qquad n=0,1,2 \cdots$

$\therefore z = \pm\frac{1}{4}, \pm\frac{3}{4}, \cdots$

All are simple poles.

when $z = 1/4$; $|(0.25-0.2)| \Rightarrow \sqrt{(0.05)^2} = 0.05 < 0.2$ (lies inside)

$\operatorname{Res}_{z=1/4} f(z) = \frac{\sin 2\pi(1/4)}{-\sin 2\pi \times \frac{1}{4} \times 2\pi} = \frac{1}{-1\times 2\pi} = \frac{-1}{2\pi}$ formula: $\operatorname{Res}_{z=z_0} f(z) \frac{p(z_0)}{q'(z_0)}$

Qn: $\int_C e^{-z^2}/\sin 4z \; ; \; c: |z| = 1.5$

Ans: $\sin 4z = 0 \; ; \; z = \pm\frac{n\pi}{4} \; ; n = 0,1,2,3 \cdots$ $z = 0 < 1.5$ Inside; $z = \pm\pi/4 <$ inside

$4z = n\pi \quad \therefore z = \frac{\pi}{4}, \frac{2\pi}{4}, \cdots$

$\operatorname{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{e^{-z^2}}{\sin 4z} \Rightarrow = \frac{e^{-z^2}}{4\cos 4z} = \frac{e^0}{4} = \frac{1}{4}$ formula: $\operatorname{Res}_{z=z_0} f(z) \frac{p(z_0)}{q'(z_0)}$

$\operatorname{Res}_{z=\pi/4} f(z) = \lim_{z\to\pi/4} \frac{e^{-z^2}}{\sin 4z} = \frac{e^{(\pi/4)^2}}{4\cos 4\times\frac{\pi}{4}} = \frac{-e^{-\pi^2/16}}{4}$

$\operatorname{Res}_{z=-\pi/4} f(z) = \lim_{z\to-\pi/4} \frac{e^{-z^2}}{\sin 4z} = \frac{e^{(\pi/4)^2}}{4\cos(\frac{\pi}{4})\times 4} = \frac{e^{\pi^2/16}}{4}$

$\cosh x = \cos ix$

Q: $\int_c \frac{z\cosh\pi z}{z^4+13z^2+36}\,dz \;;\; c:|z|=\pi$

Ans:- $z^2 = t$

$z^4+13z^2+36 \Rightarrow t^2+13t+36 \;:\; t=(-9,-4)$

$\therefore z = \pm 3i, \pm 2i$

$\therefore$ All are simple poles

when $z=3i \;;\; |3i| = \sqrt{3^2} = 3 < 3.14$ (inside)

When $z=-3i \;;\; |-3i| = \sqrt{(-3)^2} = 3 < 3.14$ (inside)

When $z=2i \;;\; |2i| = \sqrt{(2)^2} = 2 < 3.14$ (inside)

When $z=-2i \;;\; |-2i| = \sqrt{(-2)^2} = 2 < 3.14$ (inside)

$\Rightarrow$ deno $(z^2+9)(z^2+4)$

$\therefore \operatorname{Res} f(3i) = \lim_{z\to 3i} \cancel{(z-3i)} \frac{z\cosh\pi z}{\cancel{(z-3i)}(z+3i)(z^2+4)}$

$\operatorname{Res} f(3i) = \frac{3i\cosh\pi 3i}{(3i+3i)((3i)^2+4)} = \frac{3i\cosh\pi 3i}{6i(-5)} = \frac{\cosh\pi 3i}{-10}$

$\therefore \operatorname{Res} f(-3i) = \lim_{z\to -3i} \cancel{(z+3i)} \frac{z\cosh\pi z}{\cancel{(z+3i)}(z-3i)(z^2+4)}$

$\operatorname{Res} f(-3i) = \frac{-3i\cosh\pi(-3i)}{(-3i-3i)((-3i)^2+4)} = \frac{-3i\cosh\pi(-3i)}{(-6i)(-5)} = \frac{-\cosh\pi(-3i)}{+10}$

$\therefore \operatorname{Res} f(2i) = \lim_{z\to 2i} \cancel{(z-2i)} \frac{z\cosh\pi z}{\cancel{(z-2i)}(z+2i)(z^2+9)}$

$\operatorname{Res} f(2i) = \frac{2i\cosh\pi 2i}{(2i+2i)(-4+9)} = \frac{2i\cosh\pi 2i}{4i(+5)} = \frac{\cosh\pi 2i}{+10}$

$\therefore \operatorname{Res} f(-2i) = \lim_{z\to -2i} (z+2i) \frac{z\cosh\pi z}{(z+2i)(z-2i)(z^2+9)}$

$\operatorname{Res} f(-2i) = \frac{2i\cosh\pi(-2i)}{(-2i-2i)((-2i)^2+9)} = \frac{2i\cosh\pi(-2i)}{-4i(-4+9)} = \frac{\cosh\pi(-2i)}{-10}$

$\therefore \operatorname{Res} f\left[\frac{z\cosh\pi z}{z^4+13z^2+36}\right] = 2\pi i\left[\frac{\cosh\pi 3i}{-10} - \frac{\cosh\pi(-3i)}{10} + \frac{\cosh\pi 2i}{10} - \frac{\cosh\pi(-2i)}{10}\right]$

$\Rightarrow 2\pi i$

Qn. Evaluate $\int_c \tan z\, dz$; $c|z| = \pi$

Ans. $\frac{\sin z}{\cos z} dz$; $\cos z = 0$; $z = \cos^{-1} 0 = \pm(2n+1)\frac{\pi}{2}$: $n = 0, 1, 2 \cdots$

$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \pm \ldots$

$\therefore$ All are simple poles

when $z = \pm\pi/2$: $|\pi/2| = \sqrt{(\pi/2)^2} =$ < 3.14 (lies inside)

when $z = \pm 3\pi/2$; $|3\pi/2| = \sqrt{(3\pi/2)^2} =$ $\not< 3.14$ (lies outside)

$\text{Res } f(\pi/2) = \lim_{z\to\pi/2} \frac{\sin \pi/2}{-\sin \pi/2} = -1$ Formula $= \frac{p(z_0)}{q'(z_0)}$

$\text{Res } f(-\pi/2) = \lim_{z\to -\pi/2} \frac{\sin(\pi/2)}{-\sin(-\pi/2)} = -1$

$\therefore \text{Res } f[\tan z\, dz] = 2\pi i(-1-1) = -4\pi i$

Qn. $\int_c \frac{z-1}{(z+1)^2(z-2)} dz$; $c : |z+i| = 2$

Ans: $(z+1)^2 = 0$, $(z-2) = 0$ $z = -1$ (Pole of order 2) $z = 2$ (Simple pole)

$\therefore z = -1, -1, 2$

when; $z = -1$, $|-1+i| = \sqrt{(1)^2+(1)^2} = \sqrt{2} < 2$ (lies inside)

when; $z = 2$, $|2+i| = \sqrt{(2)^2+1^2} = \sqrt{5} > 2$ (lies outside)

$\text{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{(m-1)}} (z-z_0)^m f(z)$ $m = 2$

$\text{Res}_{z=(-1)} f(z) \Rightarrow \frac{1}{1} \lim_{z\to -1} \frac{d^{(2-1)}}{dz^{(2-1)}} (z+1)^2 \cdot \frac{z-1}{(z+1)^2(z-2)}$

$\text{Res}_{z=-1} f(z) \Rightarrow \lim_{z\to(-1)} \frac{d}{dz}\left(\frac{z-1}{z-2}\right) \Rightarrow \lim_{z\to(-1)} \frac{(z-2)(1) - (z-1)(1)}{(z-2)^2}$

$\Rightarrow \frac{(-1-2)-(-1-1)}{(-1-2)^2} \Rightarrow \frac{-3+2}{(-3)^2} = \frac{-1}{9}$

$\therefore \text{Res}\left[\frac{z-1}{(z+1)^2(z-2)}\right] = 2\pi i \times \left(\frac{-1}{9}\right) = \frac{-2\pi i}{9}$

H.W

1. Qn: $\int_c \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)} dz$; $c \cdot |z| = 3$

2. Qn. $\int_c \frac{\sin z}{(z-1)^2(z^2+9)} dz$; $c : |z-3i| = 1$

11/2/22

Integration of Rational functions of $\sin\theta$ & $\cos\theta$

Consider integrals of the type $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$ where $F(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ & $\sin\theta$.

Set $e^{i\theta} = z$, $\therefore ie^{i\theta} d\theta = dz$; $d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$; $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ then the given integral takes the form $\int_c F(z)\, dz$ where $c = |z| = 1$ which can be evaluated by residue theorem

1 Qn: Evaluate $\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta}$

Ans: $\int_c \frac{dz/iz}{\sqrt{2} - \frac{1}{2}(z + 1/z)} = \int_c \frac{dz/iz}{\sqrt{2} - \frac{z^2+1}{2z}} = \int \frac{dz/iz}{2\sqrt{2}z - (z^2+1)} = \frac{-2}{i}\int_c \frac{dz}{2\sqrt{2}z - (z^2+1)}$

2 Qn: Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\, d\theta$; $e^{i\theta} = z$; $d\theta = dz/iz$ $\quad \cos\theta = \frac{1}{2}(z + 1/z)$

$e^{i2\theta} = \cos 2\theta + i\sin 2\theta$

Ans: $\int_0^{2\pi} \frac{1 - \cos 2\theta/2}{5 - 4\cos\theta}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1 - \cos 2\theta}{5 - 4\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{1 - (R.P)e^{i2\theta}}{5 - 4\cos\theta}\, d\theta$

$\Rightarrow \frac{1}{2}\int_0^{2\pi} \frac{1 - z^2}{5 - 4 \times \frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz} = \frac{1}{2i}\int_0^{2\pi} \frac{1 - z^2}{-2z^2 + 5z - 2}\, dz = \frac{-1}{2i}\int_0^{2\pi} \frac{1 - z^2}{2z^2 - 5z - 2}$

$\Rightarrow \frac{-1}{2i}\int_0^{2\pi} \frac{1 - z^2}{2(z-2)(z-1/2)}$: $z = 2, 1/2$ simple poles

When; $z = 1/2$ it lies inside

$\text{Res} f(z) = \lim_{z \to 1/2} \frac{(z - 1/2)(1 - z^2)}{2(z-2)(z-1/2)} = \lim_{z \to 1/2} \frac{(1 - z^2)}{2(z-2)}$

$\text{Res} f(1/2) \Rightarrow \frac{1 - (1/2)^2}{2(1/2 - 2)} \Rightarrow \frac{1 - 1/4}{2(-3/2)} = \frac{3/4}{-6/2} = \frac{1/2}{-2/1} = -1/4$

$\therefore \frac{-1}{2i}\int_0^{2\pi} \text{Res} f(1/2) = \frac{-1}{2i} \times 2\pi i \times \frac{-1}{4} = \frac{\pi}{4}$

$(z+1)(z+2)=0$, $z=-1,-2$ (simple poles)

When $z=-1$, $|(-1)|=\sqrt{(1)^2}=1<3$ (inside)
When $z=-2$, $|(2)|=\sqrt{(2)^2}=2<3$ (inside)

$\text{Res } f(-1)=\lim_{z\to(-1)}(z+1)\times\frac{\cos\pi z^2+\sin\pi z^2}{(z+1)(z+2)}$

$\text{Res } f(-1)=\lim_{z\to-1}\frac{\cos\pi z^2+\sin\pi z^2}{(z+2)}=\frac{\cos\pi(-1)^2+\sin\pi(-1)^2}{(-1+2)}=\frac{-1}{1}=-1$

$\text{Res } f(2)=\lim_{z\to-2}(z+2)\times\frac{\cos\pi z^2+\sin\pi z^2}{(z+2)(z+1)}$

$\text{Res } f(-2)=\lim_{z\to-2}\frac{\cos\pi z^2+\sin\pi z^2}{z+1}=\frac{\cos\pi(4)+\sin\pi(4)}{-2+1}=\frac{1}{-1}=-1$

$\therefore \text{Res } f\left[\frac{\cos\pi z^2+\sin\pi z^2}{(z+2)(z+1)}\right]=2\pi i\,[-1-1]=-4\pi i$

$(z-1)^2(z^2+9)=0$

$z=1,1,-3i+3i$: when $z=1$: Pole of order : 2
when $z=\pm3i$: simple pole

When $z=1$, $|1-3i|=\sqrt{1^2+(-3)^2}=\sqrt{10}>1$; lies outside
When $z=+3i$, $|3i-3i|=0<1$; lies inside
When $z=-3i$, $|-3i-3i|=\sqrt{(-6)^2}=6>1$; lies outside

$\text{Res } f(+3i)=\lim_{z\to3i}(z-3i)\frac{\sin z}{(z-1)^2(z-3i)(z+3i)}=\lim_{z\to3i}\frac{\sin z}{(z-1)^2(z+3i)}$

$\text{Res } f(3i)=\frac{\sin3i}{(3i-1)^2(6i)}=\frac{\sin3i}{(-9+6i+1)6i}=\frac{\sin3i}{-54i+6i-36}=\frac{\sin3i}{-(48i+36)}$

$\int_0^{2\pi}\frac{1+\sin\theta}{3+\cos\theta}\,d\theta$

$e^{i\theta}=z$
$d\theta=dz/iz$
$\sin\theta=\frac{1}{2i}\left(z-\frac{1}{z}\right)$
$\cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)$

$\int_0^{2\pi}\frac{1+\frac{1}{2i}\left(z-\frac{1}{z}\right)}{3+\frac{1}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz}\Rightarrow\int_0^{2\pi}\frac{1+\left(\frac{z^2-1}{2iz}\right)}{3+\left(\frac{z^2+1}{2z}\right)}\frac{dz}{iz}=\int_0^{2\pi}\frac{\frac{2iz+z^2-1}{2iz}}{\frac{6z+z^2+1}{2z}}$

$\frac{1}{i}\int_0^{2\pi}\frac{2iz+z^2-1}{6z+z^2+1}\frac{dz}{iz}\Rightarrow\frac{1}{(i)^2}\int_0^{2\pi}\frac{z^2+2iz-1}{z^3+6z^2+z}\Rightarrow-1\int_0^{2\pi}\frac{z^2+2iz-1}{z(z^2+6z+1)}$

when; $z=0,-3\pm2\sqrt2$; $z=0$,
$z=-3+2\sqrt2$

$\text{Res } f(0)=\lim_{z\to0}(z-0)\frac{z^2+2iz-1}{(z-0)(z^2+6z+1)}=\frac{z^2+2iz-1}{z^2+6z+1}=\frac{-1}{1}=-1$

$\text{Res } f(-3+2\sqrt2)=\lim_{z\to-3+2\sqrt2}(z+(3-2\sqrt2))\frac{z^2+2iz-1}{(z-0)(z-3+2\sqrt2)(z+(3-2\sqrt2))}\Rightarrow\frac{z^2+2iz-1}{(z)(z-3+2\sqrt2)}$

$$\operatorname{Res}(-3+2\sqrt{2}) \Rightarrow \lim_{z\to(-3+2\sqrt{2})} \frac{z^2+2iz-1}{z^2-3z+2\sqrt{2}z} \Rightarrow \frac{(-3+2\sqrt{2})^2+2i(-3+2\sqrt{2})-1}{(-3+2\sqrt{2})(-3+2\sqrt{2}-3+2\sqrt{2})}$$

$$\Rightarrow \frac{(-3+2\sqrt{2})^2-(6i-4\sqrt{2}i)-1}{2(-3+2\sqrt{2})(-3+2\sqrt{2})} = \frac{16-12\sqrt{2}-6i+4\sqrt{2}i}{16-12\sqrt{2}}$$

$$\Rightarrow \frac{(16-12\sqrt{2})-i(6-4\sqrt{2})}{16-12\sqrt{2}} \Rightarrow 1-\frac{i2(3-2\sqrt{2})}{2\cdot 4(4-3\sqrt{2})}$$

$$\Rightarrow 1-\frac{i(3-2\sqrt{2})}{2(4-3\sqrt{2})}$$

4. $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

$e^{i\theta}=z \;;\; d\theta = dz/iz$

$\cos\theta = \frac{1}{2}(z+1/z)$

Ans. $\displaystyle\int_0^{2\pi} \frac{1}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz} = \frac{1}{i}\int_0^{2\pi}\frac{dz}{z\left(2+\frac{z^2+2}{4z}\right)}$

$\displaystyle\frac{1}{i}\int_0^{2\pi}\frac{2dz}{z(4z+z^2+1)/z} \Rightarrow$, $z=-2\pm\sqrt{3}$

$\cancel{\operatorname{Res} f(0) = \lim_{z\to 0}} (z-0)\dfrac{1}{(z-0)(z+(2+\sqrt{3}))(z+(2-\sqrt{3}))} = \dfrac{1}{(2+\sqrt{3})(+2-\sqrt{3})} = \dfrac{1}{1} = 1$

$\operatorname{Res} f(-2+\sqrt{3}) = \lim_{z\to -2+\sqrt{3}} z+(2-\sqrt{3})\dfrac{2}{(z+(2+\sqrt{3}))(z+(2-\sqrt{3}))} = \dfrac{2}{(-2+\sqrt{3}+2+\sqrt{3})} = \dfrac{2\sqrt{3}}{2} = 1/\sqrt{3}$

$\operatorname{Res} f(-2-\sqrt{3}) = \lim_{z\to -2-\sqrt{3}} (z+(2+\sqrt{3}))\dfrac{1}{(z+2-\sqrt{3})(z+(2+\sqrt{3}))} = \dfrac{2}{(-2-\sqrt{3}+2-\sqrt{3})} = -1/\sqrt{3}$

$\operatorname{Res} f(z) = 2\pi i \times \dfrac{1}{i}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{2\pi}{\sqrt{3}}$

An integral of the type $\int_{-\infty}^{\infty} f(x)\,dx$ as the interval of integration is not finite, it is called improper integral. To evaluate $\int_{-\infty}^{\infty} f(x)\,dx$ consider $\int_c f(z)\,dz$, where c is the contour consisting of the semicircle together with the diameter that closes it.

Let $f(z)$ has finitely many poles in the upper half plane, then by residue theorem.

$$\int_c f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z)$$

As R tends to ∞, $\int_{C_R} f(z)\,dz \longrightarrow$ zero.

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z)$$

1. Evaluate $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$

Ans $\int_C \frac{z^2}{(z^2+1)(z^2+4)}\,dz$: $z \Rightarrow \pm i, \pm 2i$ (All are simple poles)

$z = +i, +2i$ lies inside:  : +ve y axis.

$\text{Res} f(i) = \lim_{z\to i} (z-i) \frac{z^2}{(z-i)(z+i)(z^2+4)} = \frac{i^2}{(i+i)(i^2+4)} = \frac{-1}{(2i)(+3)} = \frac{-1}{6i}$

$\text{Res} f(2i) = \lim_{z\to 2i} (z-2i) \frac{z^2}{(z-2i)(z+2i)(z^2+1)} = \frac{(2i)^2}{(2i+2i)((2i)^2+1)} = \frac{-4}{4i(-3)} = \frac{1}{3i}$

$\int_C f(z)\,dz = 2\pi i\left(\frac{-1}{6i} + \frac{1}{3i}\right) = 2\pi\left(\frac{1}{3} - \frac{1}{6}\right) = 2\pi \times \frac{3}{18} = \frac{3\pi}{9} = \frac{\pi}{3}$

$\therefore \int_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx$;

as $R \to \infty$, $\int_{C_R} f(z)\,dz \to 0$

$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$: Hence, $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}$

2. Evaluate $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3}\,dx$

Ans $\int_C \frac{1}{(z^2+1)^3}\,dz$: $(z^2+1)^3 = 0$; $z = \pm i, \pm i, \pm i$ [Poles of order 3]

Only $z : +i$ lies inside.

$\text{Res} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$ $m = 3$.

$\text{Res} f(i) = \frac{1}{(2)!} \lim_{z\to i} \frac{d^2}{dz^2} (z-i)^3 \frac{1}{(z^2+1)^3}$

$\text{Res} f(i) = \frac{1}{2} \lim_{z\to i} \frac{d^2}{dz^2} (z-i)^3 \frac{1}{(z-i)^3(z+i)^3}$

$z \to \frac{1}{(z+i)^3}$

$\frac{d}{dz} = -3(z+i)^{-4}$

$\frac{d^2}{dz^2} = 12(z+i)^{-5}$

$\text{Res} f(i) = \frac{1}{2} \lim_{z\to i} \frac{d^2}{dz^2} \frac{1}{(z+i)^3}$

$\text{Res} f(i) = \frac{1}{2} \lim_{z\to i} 12(z+i)^{-5}$

$\text{Res} f(i) = \frac{1}{2} \lim_{z\to i} 12(i+i)^{-5} \Rightarrow \frac{1}{2} \cdot 12(2i)^{-5}$

$= 6(32i)^{-1} \Rightarrow 6 \times \frac{1}{(2i)^5} \Rightarrow \frac{6}{32i} \Rightarrow \frac{3}{16i}$

$\int_C f(z)\,dz = 2\pi i\left(\frac{6}{32i}\right) \Rightarrow \frac{3\pi}{8}$

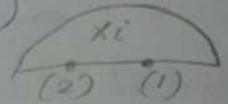
$\therefore \int_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx$

As $R \to \infty$, $\int_{C_R} f(z)\,dz \to 0$

Q3. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)}$

2 is on the circle
1 is on the diameter
z=1

Ans: consider; $C: \displaystyle\int_C \frac{dz}{(z^2-3z+2)(z^2+1)}$: $z \Rightarrow \pm i, +1, +2$ (simple poles)



So only; $z = +i, +1$ lies inside

$\text{Res}\, f(i) = \lim_{z \to i} (z-i) \dfrac{dz}{(z-i)(z+i)(z-1)(z-2)} = \dfrac{1}{(2i)(i-1)(i-2)} = \dfrac{1}{2i+6}$

$\Rightarrow \dfrac{1}{2i+6} \dfrac{(6-2i)}{(6-2i)} \Rightarrow \dfrac{3-i}{20}$

$\text{Res}\, f(1) = \lim_{z \to 1} (z-1) \dfrac{dz}{(z-1)(z^2+1)(z-2)} = \dfrac{1}{(2)(1-2)} \Rightarrow \dfrac{-1}{4}$

$\text{Res}\, f(2) = \lim_{z \to 2} (z-2) \dfrac{dz}{(z-2)(z-1)(z^2+1)} = \dfrac{1}{(2-1)(4+1)} = \dfrac{1}{5}$

$\text{Res}\, f(z) = 2\pi i \times \dfrac{3i}{20} + \pi i \left[\dfrac{-1}{4} + \dfrac{1}{5} \right]$

$= \dfrac{2\pi(i)^2}{20_{10}} + \pi i \left[\dfrac{-1}{20} \right] = \dfrac{-\pi}{10} - \dfrac{\pi i}{20} = \dfrac{-2\pi - \pi i}{20} = \dfrac{-\pi(2+i)}{20}$

If we have poles on the real axis (diameter),

$\displaystyle\int_C f(z)\,dz = 2\pi i \times \sum \text{Res}\, f(z)$ (poles that lie inside semicircle) $+ \pi i \sum \text{Res}\, f(z)$ (poles on the diameter)