14/03/22

MODULE : 4

COMPLEX INTEGRATION

I First Evaluation Method

(Indefinite Integration of Analytic function)

Let $f(z)$ be analytic in a simply connected domain D then, there exists an indefinite integral of $f(z)$ in the domain D, i.e, an analytic function $F(z)$ such that $F(z) = f'(z)$.

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$

(i) Integrate $\int_0^{1+i} z^2\,dz$ analytic $\qquad i^3 = -i$

Ans: $\int_0^{1+i} z^2\,dz = \left(\frac{z^3}{3}\right)_0^{1+i} = \frac{(1+i)^3}{3} = \frac{1+i^3+3i-3}{3} = \frac{1}{3}(-2+2i)$

(ii) $\int_{-\pi i}^{\pi i} \cos z\,dz$ analytic $= (\sin z)_{-\pi i}^{\pi i} = \sin(\pi i) - \sin(-\pi i)$ $\qquad -\sin x = \sin x$

$= \sin \pi i + \sin \pi i$

$= 2\sin \pi i = 2i\sinh \pi$

II. Second Evaluation Method.

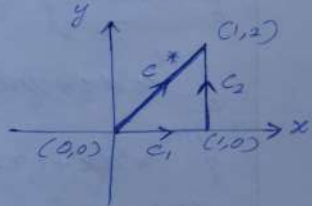
Let c be the piece wise small path, $f(z)$ is continuous then, $\int_c f(z)\,dz = \int_c (u+iv)(dx+i\,dy)$

$= \int_c (u\,dx - v\,dy) + i(u\,dy + v\,dx)$

$= \int_c u\,dx - v\,dy + i\int_c u\,dy + v\,dx$

Qn: Integrate $f(z) = \operatorname{Re} z$ from 0 to $1+2i$ along

(i) c^* (ii) consisting of c_1, c_2



Ans: $f(z) = x = \operatorname{Re} z$ $\qquad \frac{y-0}{2-0} = \frac{x-0}{1-0}$

$\int_c x\,dz$: (1) $c^* \to y = 2x$

$\therefore \int_{c^*} x(dx + i\,dy) \Rightarrow \int_{c^*} x\,dx + i\int x\,dy$

$= \int_0^1 x\,dx + i\int_0^1 x \cdot 2dx$

$y = 2x$
$dy = 2dx$

$\Rightarrow \left.\frac{x^2}{2}\right]_0^1 + i2\left[\frac{x^2}{2}\right]_0^1$

$\Rightarrow \frac{1}{2} + i$

(2) consisting of C_1 & C_2

$\int_C f(z)dz = \int_C x(dx+idy) = \int_{C_1} x(dx+idy) + \int_{C_2} x(dx+idy)$

Along C_1 : $y = 0 \Rightarrow dy = 0$
Along C_2 : $x = 1 \Rightarrow dx = 0$

$\frac{y-0}{2} = \frac{x-1}{1-1}$

$= \int_{C_1} x\,dx + \int_{C_2} x\,i\,dy$

$= \int_0^1 x\,dx + i\int_0^2 1\,dy$

$= \left.\frac{x^2}{2}\right]_0^1 + i\left.y\right]_0^2$

$= \frac{1}{2} + 2i$

Qn: Evaluate $\int_C \mathrm{Re}\, z\,dz$. C is the parabola $y = 1 + \frac{1}{2}(x-1)^2$ from $1+i$ to $3+3i$

Ans: $\int_C x(dx+idy)$

$y = 1+\frac{1}{2}(x-1)^2$
$dy = \frac{1}{2}\times 2(x-1)\,dx$
$dy = (x-1)\,dx$

$\int_C x\,dx + i\int_C x(x-1)\,dx$

$\int_1^3 x\,dx + i\int_1^3 x(x-1)\,dx$

$\Rightarrow \left.\frac{x^2}{2}\right]_1^3 + i\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^3 \Rightarrow \left(\frac{9}{2} - \frac{1}{2}\right) + i\left(3 - \frac{9}{2} - \frac{1}{3} + \frac{1}{2}\right)$

$\Rightarrow 4 + i\left(-\frac{4}{3}\right) = 4 - \frac{4}{3}i$

Qn: Evaluate $\int_0^{1+i} (x^2+iy)\,dz$ along the line $y = x$

Ans: $\int_C f(z)dz = \int_0^{1+i} (x^2+iy)(dx+idy)$

$y = x$
$dy = dx$

$\int_0^{1+i} x^2dx + iy\,dx + ix^2dy + i^2y\,dy$

$\int_0^{1+i} x^2dx + ix\,dx + ix^2dx + i^2x\,dx$

$\int_0^{1+i} (x^2-x)\,dx + i\int_0^{1+i} (x^2+x)\,dx \Rightarrow \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^{1+i} + i\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^{1+i}$

$$\Rightarrow \left[\frac{(1+i)^3}{3} - \frac{(1+i)^2}{2}\right] + i\left[\frac{(1+i)^3}{3} + \frac{(1+i)^2}{2}\right]$$

$$\Rightarrow \left[\frac{1+i+3i-3}{3} - \frac{(1+2i-1)}{2}\right] + i\left[\frac{1+i+3i-3}{3} + \frac{(1+2i-1)}{2}\right]$$

$$\Rightarrow \left[\frac{4i-2}{3} - \frac{2i}{2}\right] + i\left[\frac{4i-2}{3} + \frac{2i}{2}\right]$$

$$\Rightarrow \frac{8i-4-6i}{6} + i\left[\frac{8i-4+6i}{6}\right]$$

$$\Rightarrow \frac{2i}{6} - \frac{4}{6} - \frac{8}{6} - \frac{4i}{6} - 1 \rightarrow \frac{-18}{6} - \frac{2i}{6} \Rightarrow -(3+i/3) = -3 - \frac{1}{3}i$$

Ans: $\int_c f(z)\,dz = \int_0^{1+i} (x^2+iy)(dx+idy)$ ~~along the real axis to x.~~

$y=x$, $z=x+iy$

$dy=dx$, $dz=dx+idy$

$= \int_0^{1+i} (x^2+ix)(dx+idx)$

$= \int_0^{1+i} (x^2+ix)(i+1)\,dx = \left[\frac{x^3}{3} + \frac{ix^2}{2}\right]_0^{1+i} (1+i)$

$= (1+i)\left[\frac{(1+i)^3}{3} + \frac{i(1+i)^2}{2}\right] = (1+i)(-1+1$

$\frac{-1+5i}{6}$

Qn: Evaluate $\int_c z^2\,dz$ where c is a segment from $(1,1)$ to $(2,4)$ and eqn of c t is given by $x=t$, $y=t^2$

$i \times i = i^2 = -1$

Ans: $\int_c (x+iy)^2(dx+idy)$

$x=t$ $\quad y=t^2$

$dx=dt$ $\quad dy=2t\,dt$

$\int_c (x^2-y^2+i2xy)(dx+idy)$

$\int((x^2-y^2)dx - 2xy\,dy) + i((x^2-y^2)dy + 2xy\,dx$

$t \rightarrow 1$ to 2

$y \rightarrow 1$ to 4

$\int_1^2 (t^2-t^4)\,dt - 2t\,t^2\,2t\,dt) + i\int_1^2((t^2-t^4)\,dt + 2t\,t^2\,dt)$

$$\Rightarrow \left[\frac{t^3}{3} - \frac{t^5}{5}\right]_1^2 - \left[\frac{4t^5}{5}\right]_1^2 + i\left[\frac{t^3}{3} - \frac{t^5}{5}\right]_1^2 + \left[\frac{2t^4}{4}\right]_1^2$$

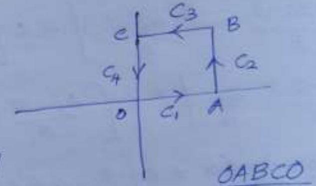
$$\Rightarrow \left[\frac{(2)^3}{3} - \frac{(2)^5}{5} - \frac{4(2)^5}{5} - \frac{1}{3} + \frac{1}{5} + \frac{4}{5}\right] + i\left[\frac{(2)^3}{3} - \frac{(2)^5}{5} + \frac{(2)^4}{2} - \frac{1}{3} + \frac{1}{5} - \frac{1}{2}\right]$$

$$\Rightarrow \frac{8}{3} - \frac{32}{5} - \frac{128}{5} - \frac{1}{3} + 1 + i\left[\frac{8}{3} - \frac{32}{5} + \frac{16}{2} - \frac{1}{3} - \frac{1}{5} - \frac{1}{2}\right]$$

$$\Rightarrow \left(\frac{7}{3} - \frac{160}{5} + 1\right) + i\left[\frac{7}{3} - \frac{33}{5} + \frac{15}{2}\right] \Rightarrow \frac{-86}{3} - 6i$$

Qn: Evaluate integral over c: $\oint_c |z|^2 dz$ where c is a square having vertices (0,0) (1,0) (1,1) & (0,1)

Ans:



$|z| = \sqrt{x^2+y^2}$

$$\int_c (x^2+y^2)(dx+idy)$$

$$\Rightarrow \int_{c_1} + \int_{c_2} + \int_{c_3} + \int_{c_4} \left[x^2+y^2\right]\left[dx+idy\right]$$

Along: $c_1 \rightarrow y=0$; $dy=0$

Along: $c_2 \Rightarrow x=1$; $dx=0$

Along: $c_3 \Rightarrow y=1$; $dy=0$

Along: $c_4 \Rightarrow x=0$; $dx=0$

$\int |z|^2 dz = \int_{c_1} |z|^2 dz + \int_{c_2} + \int_{c_3} + \int_{c_4}$

$$c_1 \Rightarrow \int_0^1 x^2 dx \Rightarrow \left.\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$c_2 \Rightarrow \int_0^1 (1+y^2)\, i\,dy \Rightarrow i\left[y+\frac{y^3}{3}\right]_0^1 = i\left[1+\frac{1}{3}\right] = \frac{4}{3}i$$

$$c_3 \Rightarrow \int_1^0 (x^2+1)dx = \left[\frac{x^3}{3}+x\right]_1^0 \Rightarrow -\frac{1}{3}-1 \Rightarrow \frac{-4}{3}$$

$$c_4 \Rightarrow \int_1^0 (y^2)i\,dy = i\left[\frac{y^3}{3}\right]_1^0 \Rightarrow i\left[-\frac{1}{3}\right] \Rightarrow \frac{-1}{3}i$$

$$\therefore \int_c (x^2+y^2)(dx+idy) \Rightarrow \frac{1}{3}+\frac{4}{3}i-\frac{4}{3}-\frac{1}{3}i \Rightarrow \underline{\underline{-1+i}}$$

$z = re^{i\theta}$

$z = e^{i\theta}$

Qn: Evaluate $\int_c \frac{1}{z} dz$ where c is the unit circle

Ans:

$$\int \frac{1}{z} dz \Rightarrow \int_0^{2\pi} \frac{dz}{z}$$

$|z| = 1$

$$\Rightarrow \int_0^{2\pi} \frac{ie^{i\theta} d\theta}{e^{i\theta}}$$

$z = re^{i\theta}$

$$\Rightarrow i\int_0^{2\pi} d\theta \Rightarrow i(\theta)_0^{2\pi} = 2\pi i$$

$z = e^{i\theta}$

$dz = ie^{i\theta}d\theta$

$\int_c |z| dz$ where c is the left half of the unit circle $|z|=1$ from $z=-i$ to i

Qn: $\int_c |z| dz$ where c is the left half of the unit circle $|z|=1$ from $-i$ to i

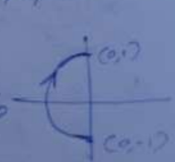
Ans:

$$\int |z| dz \Rightarrow \int_{3\pi/2}^{\pi/2} 1 \cdot i e^{i\theta} d\theta$$

$|z| = 1$

$z = e^{i\theta}$

$dz = ie^{i\theta}d\theta$



$$\Rightarrow i\int_{3\pi/2}^{\pi/2} e^{i\theta} d\theta \Rightarrow i\left.\frac{e^{i\theta}}{i}\right]_{3\pi/2}^{\pi/2} = e^{i\pi/2} - e^{i3\pi/2}$$

$\Rightarrow \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} - \left(\cos\frac{3\pi}{2} + i\sin 3\pi/2\right)$

$\Rightarrow i - (-i) = 2i$

Cauchy's Integral Theorem (C.I.T).

If $f(z)$ is analytic in a simply connected domain D, then for every simply closed path c in D, $\oint_c f(z)dz = 0$

① Evaluate $\oint_c e^z dz$

Ans: By C.I.T ; $f(z) = e^z$ $\qquad$ e^z, analytic

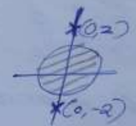
$\oint_c e^z dz = 0$

② $\oint_c \cos z \, dz$

Ans: By C.I.T, $f(z) = \cos z$ $\qquad$ $\cos z$ = analytic

$\oint_c \cos z \, dz = 0$

③ $\oint_{c:|z|=1} \frac{dz}{z^2+4}$



$|z| = 1$

To check whether analytic : $f(z) = \frac{1}{z^2+4}$ $\qquad$ inside/on : analytic

$z^2+4 = 0$

$\therefore z = \pm 2i$

$\therefore f(z)$ is analytic if and on $|z| = 1$

By CIT $\therefore \oint \frac{dz}{z^2+4} = 0$

Cauchy's Integral Formula (C.I.F)

Let $f(z)$ be analytic in a simply connected domain D, then for any point z_0 in D and any simple closed path c in D that encloses z_0,

$$f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)dz}{z - z_0}$$

or

$$\oint_c \frac{f(z)dz}{z - z_0} = 2\pi i f(z_0)$$

1. $\oint_c \frac{e^z}{z-2}\,dz$ a) where c : any contour where $z_0 = 2$ lies inside
b) where c : any contour where $z_0 = 2$ lies outside

Ans: a) $f(z) = e^z$:→ analytic

not analytic : deno = 0

$z_0 = 2$ lies inside c

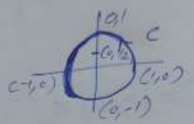
∴ By C.I.F

$$\oint_c \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

$$\oint_c \frac{e^z}{z-2}\,dz = 2\pi i f(2) = 2\pi i e^2$$

b) $f(z)$: $\oint_c \frac{e^z}{z-2}\,dz = 0$: By C.I.T

2. $\oint_c \frac{z^3-6}{2z-i}\,dz$; $c \equiv |z| = 1$



Ans: $2z - i = 0$

$z_0 = i/2$ $(0, 1/2)$

$$\oint_c \frac{z^3-6}{2(z-i/2)}\,dz = \frac{1}{2}\oint_c \frac{z^3-6}{z-i/2}\,dz$$

$f(z) = z^3 - 6$ ⇒ analytic

$f(z) = z^3 - 6$
$f(z) = (\frac{i}{2})^3 - 6$
$f(z) = \frac{i}{8} - 6$

∴ By C.I.F,

$$\oint_c \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

$$\oint_c \frac{z^3-6}{z-i/2}\,dz = 2\pi i f(i/2) = 2\pi i\left[\frac{(i/2)^3-6}{2(i/2)-i}\right]$$

$\Rightarrow 2\pi i\left[i/8 - 6\right] \Rightarrow \frac{1}{2}\left(\frac{-2\pi}{8} - 12\pi i\right) \Rightarrow \frac{\pi}{8} - 12\pi i$

$\oint \frac{\cos 2z}{4z}\,dz$ $c = |z| = 1$

3. $\oint_c \frac{\cos 2z}{4z}\,dz$; $c = |z| = 1$

$f(z) = \cos 2z$
$f(0) = \cos 0 = 1$

Ans: $4z = 0$
$z_0 = (0,0)$

By C.I.F; $\oint_c \frac{\cos 2z}{4z} = \frac{1}{4}\oint_c \frac{f(z)}{z-z_0}$

$\Rightarrow 2\pi i f(z_0) \times \frac{1}{4} \Rightarrow \frac{\pi}{2} i \times f_{z_0}$

$\Rightarrow \frac{\pi}{2} i \times 1 \Rightarrow \frac{\pi}{2} i$

$|z-a| = r$ general

4. Evaluate $\int_c \frac{z^2+1}{z^2-1}\,dz$; $c = |z-1| = 1$

$z=1\ (1,0)$

$z=-1\ (-1,0)$

Ans: $\oint \frac{z^2+1}{(z-1)(z+1)}\,dz = \int \frac{\frac{z^2+1}{z+1}}{z-1}\,dz$

$f(z) = \frac{z^2+1}{z+1}$ $\quad z=1$

By C.I.F

$\oint_c \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$

$\Rightarrow 2\pi i$

$z_0 = 1$

$f(1) = 1$

5. $\int_c \frac{z^2+1}{z^2-1}\,dz$; $c = |z+1| = 1$

Ans: $\oint \frac{z^2+1}{(z-1)(z+1)}\,dz = \oint \frac{\frac{z^2+1}{z-1}}{z+1}\,dz$

$f(z) = \frac{z^2+1}{z-1}$ $\quad z=-1$

$z_0 = -1$

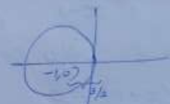
$f(-1) = -1$

By C.I.F

$\oint_c \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$

$= 2\pi i(-1) = -2\pi i$

6. $\int_c \frac{z^2}{z^2-1}\,dz$; $c = |z+1| = 3/2$



$r = 3/2$

Ans: $f(z)$ is analytic; $\oint \frac{\frac{z^2}{z-1}}{z+1}\,dz$

$z = \pm 1$ (non analytic)

By C.I.F

$\int_c \frac{f(z)}{z-z_0}\,dz = 2\pi i(f(z_0)) = 2\pi i \times 1 = 2\pi i$

$f(z) = z^2/z-1$

$z_0 = 1$

$f(1) = 1$

$|1+1| = |2| > 3/2$ (inside)

$|-1+1| = 0 < 3/2$

(lies outside)

7. $\int_c \frac{z^2}{z^2+1}\,dz$; $c: |z-1-i| = \pi/2$

$z = -1$

$|-1-1-i| = |-2-i|$

Ans: $f(z)$ is analytic; By C.I.T

$\int_c \frac{z^2}{z+1}\,dz = 0$

$\Rightarrow \sqrt{4+1} = \sqrt{5}$

$\Rightarrow 2.23 > 1.57$

(lies outside)

8. $\int_c \frac{z^2}{4z-i}\,dz$; $c: |z| = 1$

Ans: $\therefore$ By C.I.F,

$\int \frac{z^2}{4(z-i/4)}\,dz = \frac{1}{4}\int_c \frac{z^2}{z-i/4}\,dz$

(non analytic)

$z = i/4$

$|i/4|$

$\Rightarrow \sqrt{(1/4)^2} = \frac{1}{4} < 1$

(lies inside)

$f(z) = z^2$

$z_0 = i/4$

$f(i/4) \Rightarrow -1/16$

By C.I.F : $\oint_c \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$

$= 2\pi i(-1/16)$

$= \left(\frac{-\pi i}{8}\right) \times \frac{1}{4} = \frac{-\pi i}{32}$

9. $\int_c \frac{z+2}{z-2}\, dz \quad ; \; c : |z-1| = 2$

$r = 2$ $\quad z = 2$

$f(z) = z+2$

$f(2) = 4$

$|2-1| = |1| = \sqrt{1} < 2$ (lies inside)

Ans: By C.I.F

$$\oint_c \frac{f(z)}{z-z_0}\, dz = 2\pi i f(z_0)$$
$$= 2\pi i(4) = 8\pi i$$

10. $\int_c \frac{\ln(z+1)}{z^2+1}\, dz \quad ; \; c : |z-i| = 1.4$

$\cancel{z = -1}$

$\cancel{f(z) = \ln(z+1)}$

$\cancel{f(-1) \Rightarrow}$

$\cancel{|-1-i| = \sqrt{1+1} = \sqrt{2}}$

$\cancel{1.4 = 1.4}$

~~(lies on~~

Ans: $z^2+1 = 0 \; ; \; z^2 = i \quad \therefore z = \pm i$

When $z = i$; $|i-i| = 0 < 1.4$ (lies inside)

When $z = -i$; $|-i-i| = |-2i| = \sqrt{4} = 2 > 1.4$ (lies outside)

~~10~~ $\cancel{\oint_c \frac{z \sin z}{2z-1}\, dz}$ When $z = i$ (By C.I.F)

$$\int_c \frac{\frac{\ln(z+1)}{z^2+1}}{z-1} = 2\pi i f(i)$$
$$= 2\pi i \frac{[\ln(i+1)]}{2i}$$

$f(z) = \ln(z+1)/(z+1)$

$f(i) = \frac{\ln(i+1)}{2i}$

$\cancel{z = 1/2}$

$\cancel{f(z) = z \sin z}$

$\cancel{f(1/2) =}$

When $z = -i$; By C.I.T

$$\int_c \frac{\ln(z+1)}{z^2+1} = 0$$

11. $\int_c \frac{z \sin z}{2z-1} \quad ; \; c : |z| = 1$

$\frac{\pi}{2} i \sin \frac{1}{2}$

$z = 1/2$

$f(z) = z \sin z$

$f(1/2) = \frac{1}{2} \sin \frac{1}{2}$

Ans: $\frac{1}{2} \int_c \frac{z \sin z}{z-1} \Rightarrow \int_c \frac{f(z)}{z-z_0}\, dz$ By C.I.F

$$\Rightarrow \frac{1}{2} \int_c \frac{f(z)}{z-z_0} = 2\pi i f(z_0) = 2\pi i \left(\frac{1}{2} \sin 1/2\right)$$
$$\Rightarrow \frac{1}{2}\left(2\pi i \left(\frac{1}{2} \sin 1/2\right)\right) \Rightarrow \frac{\pi}{2} i \sin 1/2$$

12. Evaluate $\oint_c \frac{dz}{z^2+4} \quad ; \; c : 4x^2 + (y-2)^2 = 4$

coordinates $\Rightarrow (0,2) \, \& \, (0,-2)$

Ans: $z^2+4 = 0$, $z^2 = -4$, $z = \pm 2i$

When $z = +2i$ [lies inside]

$$\int \frac{dz}{(z+2i)(z-2i)} \quad : \text{By using C.I.F}$$

when $(0,2)$

$: 4x^2+(y-2)^2 = 4$

$0 < 4$

(lies inside)

$$\int \frac{\frac{dz}{z+2i}}{z-2i} = \oint \frac{f(z)}{z-z_0}\, dz = 2\pi i f(z_0)$$
$$= 2\pi i \times \frac{1}{4i}$$
$$= \frac{\pi}{2}$$

when $(0,-2)$

$: 4x^2+(y-2)^2 = 4$

$16 > 4$

(lies outside)

$f(z) = \frac{1}{z+2i}$

$z_0 = 2i$

$f(z_0) = 1/4i$

13. Evaluate $\int_c \frac{z}{z^2+4z+3}\,dz$; c: a circle with centre (-1) & r = 2

Ans. $c: |z+1| = 2$

$z^2+4z+3 = 0$; $z = -1, -3$

Where $z: -1$, $|-1+1| = 0 < 2$ (lies inside)

When $z: -3$, $|-3+1| = |-2| = 2$ (lies on)

$\int \frac{z}{(z+1)(z+3)}\,dz \Rightarrow \int \frac{f(z)\,dz}{z-z_0}$

$\frac{z}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$

$f(z) = 1$; $z_0 = -1$

$f(z_0) \Rightarrow 1$

$z = A(z+3) + B(z+1)$

When $z = -3$; $B = 3/2$

When $z = -1$; $A = -1/2$

$f(z) = 1$, $z_0 = -3$

$f(z_0) = 1$

$\Rightarrow \int_c \frac{-1/2}{z+1}\,dz + \int_c \frac{3/2}{z+3}\,dz$

$\Rightarrow \frac{-1}{2}\int_c \frac{1}{z-(-1)}\,dz + \frac{3}{2}\int_c \frac{1}{z-(-3)}\,dz$

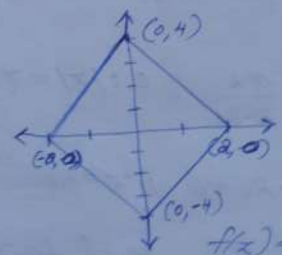
$\Rightarrow -\frac{1}{2}(2\pi i\, f(z_0)) + \frac{3}{2}(2\pi i\, f(z_0))$

$\Rightarrow -\frac{1}{2} \times 2\pi i + \frac{3}{2} \times 2\pi i \Rightarrow -\pi i + 3\pi i = 2\pi i$

14. $\int_c \frac{\cosh(z^2-\pi i)}{z-\pi i}\,dz$; c: boundary of the square with vertices $\pm 2, \pm 4i$

Ans:- $z - \pi i = 0$, $z = \pi i$

πi coordinates $\Rightarrow (0, \pi)$



$\therefore$ By C.I.F

$\int_c \frac{f(z)\,dz}{z-z_0} = 2\pi i(f(z_0))$

$= 2\pi i \cosh(-\pi^2 - \pi i)$

(Only π lies inside)

$f(z) = \frac{\cosh(z^2-\pi i)}{z-\pi i}$

$z_0 = \pi i$

$f(z_0) = \frac{\cosh(\pi i^2 - \pi i)}{\pi i - \pi i}$

Multiply Connected domains

If $f(z)$ is analytic on c_1 & c_2 and in the ring shaped domain bounded by c_1 & c_2 and z_0 is any point in that domain, then

$$\int_{c_1} \frac{f(z)}{z-z_0}\,dz + \int_{c_2} \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

Qn: Evaluate $\int_c \frac{\sin z}{4z^2(z-2i)}\,dz$; where c consist of the boundaries of the square with vertices $\pm 3, \pm 3i$, ± 2; ± 1

Ans: $4z^2(z-2i) = 0$;

$4z^2 = 0$, $z - 2i = 0$ $\therefore$ $z = 0, 0$, $z = 2i$

$f(z) = \frac{\sin z}{z^2}$

$f(2i) = \frac{\sin 2i}{(2i)^2} = \frac{\sin}{-4}$

$$\frac{1}{4}\left\{\int_{c_1} \frac{\frac{\sin z}{z^2}}{z-2i}\,dz + \int_{c_2} \frac{\frac{\sin z}{z^2}}{z-2i}\,dz = 2\pi i f(z_0)\right.$$

$$= \pi i \frac{\sin 2i}{-2}$$

Qn: $\int_c \ln z\,dz$, $|z| = 1$,

$e^{i\theta} = z$

$dz = i e^{i\theta} d\theta$

$$\int_0^{2\pi} \ln e^{i\theta} \times e^{i\theta} d\theta$$

$$\int_0^{2\pi} i\theta \cdot i e^{i\theta} d\theta = -\int_0^{2\pi} \theta e^{i\theta} d\theta$$

Qn: $\int_c \frac{dz}{z-2i}$; $c: |z| = \pi$

Ans: $z - 2i = 0$; $z = 2i$ $|2i| = \sqrt{2^2} = 2 < 3.14$ (lies inside)

$$\int_c \frac{dz}{z-2i} = \int_c \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

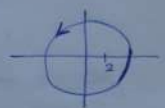
$f(z) = 1/z-2i$

$z = 2i$

$f(2i) = 1$

$\therefore$ By C.I.F : $\oint_c \frac{1}{z-2i}\,dz = 2\pi i(1) = 2\pi i$

Qn: $\int_c \frac{2z-1}{z^2-z}\,dz$, where c:



Ans: $z = 0$, $z = 1$

$$\int_c \frac{2z-1}{z^2-z}\,dz = \int \frac{1}{z}\,dz + \int \frac{dz}{z-1} = 2\pi i + 2\pi i = 4\pi i$$

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Derivatives of analytic function

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are also analytic functions in D. The values of these derivatives at a point z_0 in D are given by the formulas

$$f'(z_0) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^2}\, dz$$

$$f''(z_0) = \frac{2!}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^3}\, dz$$

and in general;

$$f^n(z_0) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

where c is any simple closed path in D that encloses z_0.

Qn: Evaluate: $\int_c \frac{\cos z}{(z-\pi i)^2}\, dz$ c is the contour enclosing the points πi

Ans: $(z-\pi i)^2 = 0$; $z = \pi i, \pi i$

$$f'(z_0) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^2}\, dz \quad [\text{By C.I.F}]$$

$f(z) = \cos z$ $\quad$ $f'(z_0) = -\sin \pi i$

$z_0 = \pi i$ $\quad$ $= -i\sinh \pi$

$$\int_c \frac{f(z)}{(z-z_0)^2}\, dz = 2\pi i\, f'(z_0)$$

$$\int_c \frac{\cos z}{(z-\pi i)^2}\, dz = 2\pi i \times -i\sinh \pi = \underline{\underline{2\pi \sinh \pi}}$$

Qn: Evaluate: $\int_c \frac{z^4 - 3z^2 + 6}{(z+i)^3}\, dz$; c is any contour enclosing $-i$

Ans: $(z+i)^3 = 0$; $z+i = 0$; $z = -i, -i, -i$

$$f''(z_0) = \frac{2!}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^3}\, dz \quad [\text{By C.I.F}]$$

$$\int_c \frac{f(z)}{(z-z_0)^3}\, dz = \frac{2\pi i\, f''(z_0)}{2!}$$

$$\Rightarrow \frac{2\pi i\,(-18)}{2!} \Rightarrow \underline{\underline{-18\pi i}}$$

$f(z) = z^4 - 3z^2 + 6$

$f'(z) = 4z^3 - 6z$

$z_0 = -i$

$f''(z) = 12z^2 - 6 = -12 - 6$

$= \underline{\underline{-18}}$

~~$f'''(z_0) = 24z$~~

~~$f'''(z_0) = 24(-i) = -24i$~~

Qn: Evaluate $\int_C \frac{e^z}{(z-1)^2(z^2+4)}\,dz$ for any contour for which 1 lies inside and $\pm 2i$ lies outside.

Ans: $(z-1)^2(z^2+4)=0$; $z = 1, \pm 2i$

$\int_C \frac{\left(\frac{e^z}{z^2+4}\right)}{(z-1)^2}\,dz$ ∴ By C.I.F

$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^2}$

$\int_C \frac{f(z)\,dz}{(z-z_0)^2} = 2\pi i\, f'(z_0)$

$= \frac{6\pi i\, e}{25}$

$f(z) = (e^z/z^2+4)$

$z_0 = 1$

$f'(z) = \frac{(z^2+4)(ze^z) - e^z(2z)}{(z^2+4)^2}$

$\Rightarrow \frac{z^3e^z+4ze^z-2ze^z}{(z^2+4)^2}$

$\Rightarrow \frac{z^3e^z+2ze^z}{(z^2+4)^2}$

$f'(1) = \frac{e^1(1+2)}{(1+4)^2}$

$f'(1) = 3e/25$

Qn: $\int_C \frac{\sin 2z}{z^4}\,dz$ where c is a unit circle in the counter clockwise direction.

Ans: $z^4=0$, $z=0$

$\int_C \frac{\sin 2z}{(z-0)^4}\,dz$

$f'''(z_0) = \frac{3!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^4}$

$\int_C \frac{f(z)}{(z-z_0)^4}\,dz = \frac{2\pi i}{3!} f'''(z_0) = \frac{2\pi i}{3!}\times(-8)$

$\Rightarrow -\frac{8}{3}\pi i$

$f(z) = \sin 2z$

$z_0 = 0$

$f'(z) = 2\cos 2z$

$f''(z) = -4\sin 2z$

$f'''(z) = -8\cos 2z$

$f'''(z_0) = -8\cos 0$

$f'''(z_0) = -8$

Qn: $\int_C \frac{\sinh 2z}{(z-1/2)^4}\,dz$ where c is a unit circle in the counter clockwise direction.

Ans: $(z-1/2)^4=0$; $z = 1/2$

$\int_C \frac{f(z)}{(z-z_0)^4}\,dz = \frac{2\pi i}{3!} f'''(z_0) = \frac{2\pi i}{3!}(8\cosh 1)$

$= \frac{8}{3}\pi i \cosh 1$

$f(z) = \sinh 2z$

$z_0 = 1/2$

$f'(z) = 2\cosh 2z$

$f''(z) = 4\sinh 2z$

$f'''(z) = 8\cosh 2z$

$f'''(z_0) = 8\cosh 2\times 1/2$

$= 8\cosh 1$

Qn: Evaluate $\int_c \frac{\tan \pi z}{z^2} dz$; $c: 16x^2+y^2=1$ (elipse)

Ans: $z^2=0$, $z=0$

$0<1$ (lies inside)

$f(z) = \tan \pi z$

$z_0 = 0$

$f'(z) = \pi \sec^2 \pi z$

$f'(0) = \pi$

$\int_c \frac{\tan \pi z}{(z-0)^2} dz$

$f^{\prime}(z_0) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^2} dz$

$\int_c \frac{f(z)}{(z-z_0)^2} dz = 2\pi i f'(z_0)$

$= 2\pi i(\pi) = 2\pi^2 i$

Qn: $\int_c \frac{(1+z)\cos z}{(2z-1)^2} dz$; $c: |z-i| = 2$

Ans: $(2z-1)^2=0$; $z = 1/2$

$|1/2 - i| = \sqrt{(1/2)^2+(1)^2}$

$= \sqrt{5/4}$

$= 1.11 < 2$

(lies inside)

$\int_c \cancel{\frac{1+z\cos z}{2z-1}} dz$ $\therefore$ By C.I.F $\int_c \frac{(1+z)\cos z\, dz}{4(z-1/2)^2}$

$\Rightarrow \frac{1}{4}\int_c \frac{(1+z)\cos z}{(z-1/2)^2} dz$

$f(z) = (1+z)\cos z$, $z_0 = 1/2$

$\int_c \frac{f(z)}{(z-z_0)^2} dz = \frac{1}{4} \times 2\pi i f'(z_0)$

$\Rightarrow \frac{\pi}{2} i f'(z_0)$

$\Rightarrow \frac{\pi}{2} i \cos\frac{1}{2} - \frac{3}{2}\sin\frac{1}{2}$

$f'(z) = \cos z - (1+z)\sin z$

$f'(z_0) = \cos\frac{1}{2} - \frac{3}{2}\sin\frac{1}{2}$

Qn: $\int_c \frac{\ln z}{(z-4)^2} dz$; $c: |z-3| = 2$

$|4-3| = |1| = 1 < 2$

(lies inside)

Ans: $(z-4)^2=0$; $z=4$

$\int_c \frac{f(z)}{(z-z_0)^2} dz = 2\pi i f'(z_0) = 2\pi i\left(\frac{1}{4}\right) = \frac{\pi i}{2}$

$f(z) = \ln z$

$z_0 = 4$

$f'(z) = 1/z$

$f'(z_0) = 1/4$

Qn: $\int_c \frac{z}{(z-1)^3} dz$; $c: |z| = 2$

$|1| = 1 < 2$

(lies inside)

Ans: $(z-1)^3=0$; $z=1$

$\int_c \frac{f(z)}{(z-z_0)^3} dz = \frac{2\pi i}{2!} f''(z_0) = \pi i f''(z_0) = 0$

$f(z) = z$

$z_0 = 1$

$f'(z) = 1$

$f''(z) = 0$

Qn: $\int_C \frac{dz}{(z-3)^2}$; $C: |z| = 1$

Ans: $(z-3)^2 = 0$; $z = 3$ $|3| = 3 > 1$ (lies outside)

∴ By C.I.T ;

$$\int_C \frac{dz}{(z-3)^2} = 0$$

Qn: $\int_C \frac{z}{(z-1)(z-2)^2}\,dz$; $C: |z-2| = 1/2$

Ans: $z = 1, 2$ $|1-2| = |-1| = 1 > 1/2$ (lies outside)
$|2-2| = 0 < 1/2$ (lies inside)

$\int_C \frac{\frac{z}{(z-1)}}{(z-2)^2}\,dz$ ∴ By C.I.F

$f(z) = z/z-1$
$f'(z) = \frac{(z-1)-z}{(z-1)^2}$
$z_0 = 2$
$f'(z_0) = 1-1 = 0$

$$\int_C \frac{f(z)\,dz}{(z-z_0)^2} = 2\pi i f'(z_0) = 0$$

Qn: $\int_C \frac{e^{2z}}{(z+1)^4}\,dz$; $C: |z| = 2$

Ans: $(z+1)^4 = 0$; $z = -1$ $|-1| = 1 < 2$ (lies inside)

$$\int_C \frac{f(z)}{(z-z_0)^4}\,dz = \frac{2\pi i}{3!} f'''(z_0) = \frac{\pi i}{3} \times 8e^{-2}$$

$$\Rightarrow \frac{8}{3}\pi i e^{-2}$$

$f(z) = e^{2z}$
$f(z_0) = -1$
$f'(z_0) = 2e^{2z}$
$f''(z) = 4e^{2z}$
$f'''(z) = 8e^{2z}$
$f'''(z_0) = 8e^{-2}$

Qn: $\int_C \frac{5z+7}{z^2+2z-3}\,dz$; $C: |z-2| = 2$

UP

Ans: $z = 1, -3$; when $z = 1$; $|1-2| = |-1| = 1 < 2$ (lies inside)
when $z = -3$; $|-3-2| = |-5| = 5 > 2$ (lies outside)

$$\int_C \frac{5z+7}{(z-1)(z+3)}\,dz \Rightarrow \int_C \frac{\frac{5z+7}{(z+3)}}{z-1}\,dz$$

$f(z) = \frac{5z+7}{z+3}$
$z_0 = 1$
$f(z_0) = \frac{5+7}{1+3} = \frac{12}{4} = 3$

By C.I.F : $\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0) = 2\pi i \times 3$
$\Rightarrow 6\pi i$

Qn: $\int_c \frac{\sin^2 z}{(z-\pi/6)^3}\, dz$; $c: |z| = 1$

Ans: $(z-\pi/6)^3 = 0$, $z = \pi/6$ when: $|\pi/6| = 0.5 < 1$ (lies inside)

$\int_c \frac{f(z)}{(z-z_0)^3}\, dz = \frac{2\pi i}{2!} f''(z_0) \Rightarrow \pi i(1)$

$\Rightarrow \underline{\underline{\pi i}}$

$f(z) = \sin^2 z$
$z_0 = \pi/6$
$\sin 2z = f'(z) = 2\sin z \cos z$
$f''(z) = \cos 2z \times 2$
$f''(z_0) = 2 \times \frac{1}{2} = 1$

Qn: $\int_c \tan z\, dz$; $c: |z| = 1$

$\int_c \frac{\sin z}{\cos z}\, dz$ $\cos z = 0$, $\therefore z = \frac{\pi}{2}$ $\left|\frac{\pi}{2}\right| = 1.5 > 1$ (lies outside)

$\therefore$ By C.I.T; $\int_c \tan z\, dz = \underline{\underline{0}}$ $\tan z$ is analytic within & on c

Qn: $\int_c \frac{z^2}{z^3 - z^2 - z + 1}\, dz$ (i) $c: |z+1| = 3/2$
(ii) $c: |z-1-i| = \pi/2$

Ans: $z = -1, 1, 1$ (i) when $z = -1$; $|-1+1| = 0 < 3/2$ (lies inside)
when $z = 1$; $|1+1| = |2| = 2 > 3/2$ (lies outside)

$\int_c \frac{z^2}{(z+1)(z-1)^2}\, dz \rightarrow \int_c \frac{\frac{z^2}{(z-1)^2}}{z+1}\, dz$

$f(z) = \frac{z^2}{(z-1)^2}$
$z_0 = -1$
$f(z_0) = \frac{(-1)^2}{(-1-1)^2} = \frac{1}{4}$

By C.I.F,
$\int_c \frac{f(z)}{z-z_0}\, dz = 2\pi i f(z_0) = 2\pi i \times \frac{1}{4}$
$\Rightarrow \underline{\underline{\frac{\pi i}{2}}}$

(ii) When $z = -1$; $|-1-1-i| = |-2-i| = \sqrt{2^2+1^2} = \sqrt{5} = 2.2 > 1.5$ (lies outside)
When $z = 1$; $|1-1-i| = |-i| = \sqrt{1^2} = 1 < 1.5$ (lies inside)

$\int_c \frac{\frac{z^2}{(z+1)^2}}{(z-1)^2}\, dz \Rightarrow \int_c \frac{f(z)}{(z-z_0)^2}\, dz = 2\pi i f'(z_0)$

$\Rightarrow 2\pi i \times \frac{3}{4} = \underline{\underline{\frac{3}{2}\pi i}}$

$f(z) = \frac{z^2}{(z+1)^2}$
$z_0 = +1$
$f'(z_0) = \frac{(z+1)^2(2z) - z^2(2(z+1))}{(z+1)^4}$
$f'(z_0) \Rightarrow \frac{2(2) - 1}{(2)^2} \Rightarrow \frac{3}{4}$

Qn: $\int_c \frac{4-3z}{z(z-1)}\, dz$; $c: |z| = 3/2$

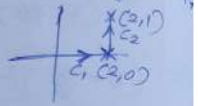
Ans: $z = 0, 1$ when $z = 0$; $0 < 3/2$ (lies inside)
when $z = 1$; $1 < 1.5$ (lies inside)

(i) $\int_c \frac{\frac{4-3z}{z}}{z-1} \Rightarrow 2\pi i \times 1$
$\Rightarrow \underline{\underline{2\pi i}}$

(ii) $\int_c \frac{\frac{4-3z}{z-1}}{z} \Rightarrow 2\pi i \times -4$
$\Rightarrow \underline{\underline{-8\pi i}}$

(i) $f(z) = 4-3z/z$
$z_0 = 1$
$f(z_0) = 1$
(ii) $f(z) = 4-3z/z-1$
$z_0 = 0$
$f(z_0) = \frac{4}{-1} = -4$

Qn: $\int_0^{2+i} (\bar{z})^2\, dz$ along the real axis to 2 and then vertically to 2,



Ans: $\int_C (x-iy)^2 (dx+idy) = \int_C (x^2-y^2-2ixy)(dx+idy)$

Along C_1: $y=0$; $dy=0$
Along C_2: $x=2$; $dx=0$

$\Rightarrow \int_{C_1} (x^2-y^2)dx + 2xy\,dy + i\int_{C_2} (x^2-y^2)dy - 2xy\,dx$

$\Rightarrow \int_0^2 x^2 dx + 0 + i\int_0^1 (4-y^2)dy - 2\times 2y \times 0$

$\Rightarrow \int_0^2 x^2 dx + i\int_0^1 (4-y^2)dy$

$\Rightarrow \frac{x^3}{3}\Big]_0^2 + i\left[\frac{4y^2}{2}-\frac{y^3}{3}\right]_0^1 \Rightarrow \frac{8}{3} - \frac{11}{3}i$

Qn: If $F(a) = \int_C \frac{z^2+7z-1}{z-a}\, dz$; where c is the circle $|z|=2$
(i) Find the values of $F(3)$, (ii) $F(-i)$, (iii) $F'(1-i)$, (iv) $F''(1-i)$

Ans: $F(a) = \int_C \frac{z^2+7z-1}{z-a}\, dz \Rightarrow \int \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$

(i) $F(3)$: $F(a) = 2\pi i f(a)$
$F(3) = 2\pi i f(3)$ — 3 lies outside (analytic)

$\therefore$ By C.I.T: $\int_C \frac{z^2+7z-1}{z-3}\, dz = 0$

$f(z) = z^2+7z-1$
$f(-i) = -1-7i-1$
$f(-i) = -7i-2$

(ii) $F(-i)$: $(0,-1)$; lies inside

$F(-i) = 2\pi i f(-i) = 2\pi i(-2-7i)$
$= 14\pi - 4\pi i$

(iii) $F'(1-i)$: $F'(a) = 2\pi i f'(a)$
$F'(1-i) = 2\pi i f'(1-i)$: $(1,-1)$: $\sqrt{1^2+1^2} = \sqrt{2}$
$1.14 < 2$
(lies inside)
$F'(1-i) = 2\pi i(9-2i)$
$= 4\pi + 18\pi i$

$f'(z) = 2z+7$
$f'(1-i) = 2(1-i)+7$
$= 9-2i$

(iv) $F''(1-i)$: $F''(a) = 2\pi i f''(a)$
$F''(1-i) = 2\pi i f''(1-i)$
$F''(1-i) = 2\pi i \times 2 = 4\pi i$

$f''(1-i) = 2$

Taylor's and Maclaurin Series

The Taylor series of a function $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

where $a_n = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^{n+1}} dz$.

Here z_0 is a point that lies within c, also $f(z)$ is analytic within and on c.

A Maclaurin series is a taylor series with $z_0 = 0$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\frac{1}{1-z} = (1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots$$

$$\frac{1}{(1-z)^2} = (1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

1. Find the Maclaurin series for $f(z) = \frac{1}{1+z^2}$

Ans: $f(z) = \frac{1}{(1+z^2)} = (1+z^2)^{-1} = 1 - z^2 + (z^2)^2 - (z^2)^3$

$= (1-(-z^2))^{-1} = 1 - z^2 + z^4 - z^6 + \cdots \qquad |z| < 1$

2. $f(z) = \sin \frac{z^2}{2}$

Ans: $\sin \frac{z^2}{2} = \frac{z^2}{2} - \left(\frac{z^2}{2}\right)^3 / 3! + \left(\frac{z^2}{2}\right)^5 / 5!$

$2\sin^2\theta = 1 - \cos 2\theta$

3. $2\sin^2 \frac{z}{2} = 1 - \cos z \Rightarrow 1 - 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$

$$\Rightarrow \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots$$

4. $\int_0^z e^{-t^2} dt$

Ans: $\int_0^z \left(1 - \frac{t^2}{1!} + \frac{t^4}{2!} - \cdots\right) dt$

$\Rightarrow \left[t - \frac{t^3}{3} + \frac{t^5}{2!5} - \cdots\right]_0^z$

$\Rightarrow z - \frac{z^3}{3} + \frac{z^5}{5 \cdot 2} - \cdots$

5. $\int_0^z \frac{\sin t}{t} dt$

expand as power series

Ans: $\int_0^z \frac{1}{t}\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right) dt$

$\Rightarrow \int_0^z \left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \cdots\right) dt$

$\Rightarrow \left[t - \frac{t^3}{3 \cdot 3!} + \frac{t^5}{5 \cdot 5!} - \cdots\right]_0^z$

$\Rightarrow z - \frac{z^3}{3 \cdot 3!} + \frac{z^5}{5 \cdot 5!} - \cdots$

6. Find the Taylor series for the function $f(z) = \cos z$ about $z_0 = \pi$

Ans:

$f(z) = \cos z \quad f(\pi) = -1$

$f'(z) = -\sin z \quad f'(\pi) = 0$

$f''(z) = -\cos z \quad f''(\pi) = 1$

$f'''(z) = \sin z \quad f'''(\pi) = 0$

$f^{iv}(z) = \cos z \quad f^{iv}(\pi) = -1$

$f(z) = f(z_0) + \frac{z - z_0}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots$

$f(z) = \cos z = f(\pi) + \frac{(z-\pi)}{1!} f'(\pi) + f''(\pi) \frac{(z-\pi)^2}{2!} + \cdots$

$= -1 + \frac{(z-\pi)^2}{2!} \times 1 + \frac{(z-\pi)^4}{4!} \times (-1) \cdots$

$= -1 + \frac{(z-\pi)^2}{2!} - \frac{(z-\pi)^4}{4!} + \cdots$

7. $f(z) = \sin z \; ; \; z_0 = \pi/4$

Ans:

$f'(z) = \cos z \; ; \; f'(\pi/4) = 1/\sqrt{2}$

$f''(z) = -\sin z \; ; \; f''(\pi/4) = -1/\sqrt{2}$

$f'''(z) = -\cos z \; ; \; f'''(\pi/4) = -1/\sqrt{2}$

$f^{iv}(z) = \sin z \; ; \; f^{iv}(\pi/4) = 1/\sqrt{2}$

$f(\pi/4) = 1/\sqrt{2}$

$$f(x) = \sin z = f(\pi/4) + \frac{(z-\pi/4)}{1!} f'(\pi/4) + f''(\pi/4)\frac{(z-\pi/4)^2}{2!} + \cdots$$

$$= \frac{1}{\sqrt{2}} + (z-\pi/4)\frac{1}{\sqrt{2}} + \frac{(z-\pi/4)^2}{2!}\left(\frac{-1}{\sqrt{2}}\right) + \cdots \qquad \frac{1+2}{4}$$

$$= \frac{1}{\sqrt{2}}\left(1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right)$$

$= 20+$

8. $f(z) = \frac{\sin z}{z-\pi}$; $z_0 = \pi$

Ans: $f(z) = \sin z$; $f(\pi) = 0$

$f'(z) = \cos z$; $f'(\pi) = -1$

$f''(z) = -\sin z$; $f''(\pi) = 0$

$f'''(z) = -\cos z$; $f'''(\pi) = 1$

$$f(z) = \frac{\sin z}{z-\pi} = \frac{1}{z-\pi}\left[0 + \frac{(z-\pi)}{1!}\times(-1) + \frac{(z-\pi)^3}{3!} - \cdots\right]$$

$$= -1 + \frac{(z-\pi)^2}{3!} - \frac{(z-\pi)^4}{5!} + \cdots$$

or: $z_0 = i$ model

9. $f(z) = \frac{1}{1+z}$; $z_0 = -i$

$(1+z)^{-1} = 1 - z + z^2 - z^3 \cdots$

$(1-z)^{-1} = 1 + z + z^2 + \cdots$

Ans: $$\frac{1}{1+z} = \frac{1}{1+z+i-i} = \frac{1}{1-i+(z+i)}$$

$$\Rightarrow \frac{1}{(1-i)\left[1+\frac{(z+i)}{(1-i)}\right]} = \frac{1}{(1-i)}\left[1 + \frac{z+i}{1-i}\right]^{-1}$$

$$= \frac{1}{1-i}\left(1 - \frac{(z+i)}{1-i} + \frac{(z+i)^2}{(1-i)^2} - \frac{(z+i)^3}{(1-i)^3} + \cdots\right)$$

$$\boxed{(1+z)^{-m} = 1 - mz + \frac{m(m+1)}{2!}z^2 - \frac{m(m+1)(m+2)}{3!}z^3 + \cdots}$$

10. $f(z) = \frac{1}{(z-i)^2}$; $z_0 = -i$

$(-2i)^2 = 4(-1)$

Ans: $$\frac{1}{(z+i-i-i)^2} = \frac{1}{((z+i)-2i)^2}$$

$$\Rightarrow \frac{1}{-4\left(1-\frac{z+i}{2i}\right)^2} = \frac{-1}{4}\left(1 - \left(\frac{z+i}{2i}\right)\right]^{-2}$$

$$\Rightarrow \frac{-1}{4}\left(1 + 2\left(\frac{z+i}{2i}\right) + 3\left(\frac{z+i}{2i}\right)^2 + \cdots\right)$$

11. $f(z) = \frac{z-1}{z^2}$ about $z_0 = 1$

$f(z) = \frac{z-1}{z+1}$, $z_0 = 1$

Ans: $\frac{z-1}{z^2} = \frac{z-1}{(z-1+1)^2} = \frac{z-1}{((z-1)+1)^2} = (z-1)(1+(z-1))^{-2}$

$\Rightarrow (z-1)(1-2(z-1)+3(z-1)^2+4(z-1)^3 \dots$

$\Rightarrow (z-1)-2(z-1)^2+3(z-1)^3-4(z-1)^4+\dots$

12. $f(z) = \frac{z-1}{z+1}$ about $z_0 = 1$

Ans: $\frac{z-1}{z+1+1-1} \Rightarrow \frac{z-1}{z-1+2} = \frac{z-1}{(z-1)+2} = \frac{z-1}{2(1+\frac{z-1}{2})}$

$= \left(\frac{z-1}{2}\right)\left(1+\left(\frac{z-1}{2}\right)\right)^{-1}$

$\Rightarrow \frac{z-1}{2}\left(1-\left(\frac{z-1}{2}\right)+\frac{(z-1)^2}{4}-\dots\right)$

$\Rightarrow \frac{z-1}{2}-\frac{(z-1)^2}{8}+\frac{(z-1)^3}{18}-\dots$

13. Find the Maclaurin series of $f(z) = \frac{-2z+3}{z^2-3z+2}$

Ans: $\frac{-2z+3}{(z-1)(z-2)} \rightarrow \frac{A}{z-1}+\frac{B}{z-2}$

$-2z+3 = A(z-2)+B(z-1)$

When $z=2$; $B=-1$

When $z=1$; $A=-1$

$\Rightarrow \frac{-1}{z-1}-\frac{1}{z-2} = \frac{1}{1-z}-\frac{1}{-2(1-z/2)}$

$\Rightarrow (1-z)^{-1}+\frac{1}{2}(1-z/2)^{-1}$

$\Rightarrow (1+z+\frac{z^2}{2}+\dots)+\frac{1}{2}(1+\frac{z}{2}+(\frac{z}{2})^2+\dots)$

14. Maclaurin series: $f(x) = \arctan z$

Ans: $\tan^{-1} z = f(z)$

$f'(z) = \frac{1}{1+z^2} = (1+z^2)^{-1} = 1-z^2+z^4-z^6+\dots$

$f(z) =$ Integral $f'(z) \Rightarrow z-\frac{z^3}{3}+\frac{z^5}{5}-\frac{z^7}{7}+\dots$