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MODULE: 3

COMPLEX VARIABLE

A function f defined on a set of complex numbers is a rule that assigns to every z a complex number w, called the value of f at z. ie, $w = f(z)$. Also $f(z)$ can be written as $u+iv$ where u, v are functions of x & y.

$z = x+iy$ → $f(z)$ → $w = u+iv$

$$f(z) = u(x,y) + iv(x,y)$$

1) Let $w = f(z) = z^2 + 3z$. Find u & v. Calculate the value of f when $z = 1+3i$.

$z = x+iy$ (general)

Ans:- $f(z) = (x+iy)^2 + 3(x+iy)$

$f(z) = x^2 + 2xiy - y^2 + 3x + 3iy$

$f(z) = x^2 + 3x - y^2 + i(2xy + 3y)$

$f(z) = u(x,y) + v(x,y)$

$\therefore u = x^2 + 3x - y^2, \quad v = 2xy + 3y$

Given: $z = 1+3i$; $z = x+iy$

$\therefore x = 1, \; y = 3$

$\therefore f(z) = (1)^2 + (3\times1) - (3)^2 + i(2\times1\times3 + 3\times3)$

$f(z) = 4 - 9 + i(6+9)$

$f(z) = \underline{\underline{-5 + 15i}}$

2) Let $w = f(z) = 2iz + 6\bar{z}$, find u & v and value of f when $z = \frac{1}{2} + 4i$

$z = x+iy$

$\bar{z} = x - iy$

Ans: $f(z) = 2i(x+iy) + 6(x-iy)$

$f(z) = 2ix + (-2y) + 6x - 6iy$

$f(z) = 6x - 2y + i(2x - 6y)$

$f(z) = u(x,y) + v(x,y)$

$\therefore u = 6x - 2y, \quad v = 2x - 6y$

$\therefore$ Given: $z = \frac{1}{2} + 4i$; $z = x+iy$

$\therefore x = 1/2, \; y = 4$

$\therefore f(z) = 6(1/2) - 2(4) + i(2\times 1/2 - 6\times4)$

$= 3 - 8 + i(1 - 24)$

Real Imaginery

3' Find R_{ef}, I_{mf} and their values at the point z.

(i) $f(z) = 5z^2 - 12z + 3 + 2i$; $z = 4 - 3i$

(ii) $f(z) = 1/_{1+z}$; $z = (1-i)$; $\frac{1}{(1+x)+iy} \times \frac{1+x-iy}{1+x-iy}$

(iii) $f(z) = \frac{z-1}{z+1}$; $z = 2i$

when z is in deno multiply denominator with conjugate

Analytic Function

A function $f(z)$ is said to be analytic in a domain D, if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be analytic at a point $z = z_0$ in D ; if $f(z)$ is analytic in a neighbourhood of z_0.

Cauchy-Riemann Equations (C.R)

H.W ans:-

(i) $f(z) = 5(x+iy)^2 - 12(x+iy) + 3 + 2i$

$f(z) = 5x^2 + 10xiy - 5y^2 - 12x - 12iy + 3 + 2i$

$f(z) = 5x^2 - 5y^2 - 12x + 3 + i(10xy - 12y + 2)$

$f(z) = u(x,y) + v(x,y)$

$\therefore u = 5x^2 - 5y^2 - 12x + 3$, $v = 10xy - 12y + 2$

Given: $z = 4 - 3i$ $\because z = x + yi$ $\therefore x = 4$, $y = -3$

$\therefore f(z) = 5(4)^2 - 5(3)^2 - 12(4) + 3 + i(10 \times 4 \times -3 - 12 \times -3 + 2)$

$f(z) = 80 - 45 - 48 + 3 + i(-120 + 36 + 2)$

$\therefore f(z) = -10 - 82i$

$R_{ef} = -10$

$I_{mf} = -82$

(ii) $f(z) \Rightarrow \frac{1}{1+x+iy} \times \frac{1+x-iy}{1+x-iy}$

$f(z) = \frac{(1+x)-iy}{[(1+x)+iy][(1+x)-iy]}$

$f(z) = \frac{(1+x)-iy}{(1+x)^2+y^2}$ $\leftarrow (iy)^2 = -y^2$

$f(z) = \frac{1+x}{(1+x)^2+y^2} - i\left(\frac{y}{(1+x)^2+y^2}\right)$

$f(z) = u(x,y) + v(x,y)$

$\therefore$ Given $x = 1$, $y = -1$

$\therefore f(z) = \frac{1+1}{(1+1)^2+(-1)^2} - i\frac{(-1)}{(1+1)^2+(-1)^2}$

$f(z) = \frac{2}{2^2+1} + i\frac{1}{2^2+1}$

$f(z) = \frac{2}{5} + i\frac{1}{5}$

$R_{ef} = \frac{2}{5}$, $I_{mf} = \frac{1}{5}$

(iii) $f(z) = \frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1}$

$f(z) = \frac{(x-1)+iy}{(x+1)+iy}$

$f(z) = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$

$f(z) = \frac{(x+1)(x-1)+iy(x+1)-iy(x-1)+y^2}{(x+1)^2+y^2}$

$f(z) = \frac{x^2-1+\cancel{xy}i+iy-\cancel{iyx}+iy+y^2}{(x+1)^2+y^2}$

$f(z) = \frac{x^2-1+y^2+2iy}{(x+1)^2+y^2}$

$\therefore f(z) = \frac{x^2+y^2-1}{(x+1)^2+y^2} + i\frac{2y}{(x+1)^2+y^2}$

Given: $z = 2i$; $x=0, y=2$

$\therefore f(z) = \frac{(2)^2-1}{1^2+(2)^2} + i\frac{2(2)}{1^2+(2)^2}$

$\therefore f(z) = \frac{3}{5} + i\frac{4}{5}$

$\therefore Re\,f = \frac{3}{5}$

$Im\,f = \frac{4}{5}$

A function $f(z) = u(x,y)+iv(x,y)$ is analytic in a domain v if and only if the 1st order PD's of u & v satisfy the two CR eqns:

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ OR $u_x = v_y$ & $v_y = -v_x$

1. Show that $f(z) = z^2$ is analytic.

Ans: $f(z) = (x+iy)^2 = x^2+2xiy-y^2$

$f(z) = x^2-y^2+i(2xy)$

$u = x^2-y^2$, $v = 2xy$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial v}{\partial y} = 2x$

$\frac{\partial u}{\partial y} = -2y$, $\frac{\partial v}{\partial x} = 2y$

$\therefore u_x = v_y$ & $-u_y = v_x$

$\therefore$ CR eqns are satisfied. Hence $f(z)$ is analytic

2. $f(z) = \bar{z}$

Ans: $u = x$, $v = -y$

$\frac{\partial u}{\partial x} = 1$, $\frac{\partial v}{\partial y} = -1$: $u_x \neq v_y$ $\therefore$ CR eqns are not satisfied

Hence $f(z)$ is non analytic

3. $f(z) = e^x(\cos y + i\sin y)$

Ans: $f(z) = e^x\cos y + i e^x\sin y$

$\therefore u = e^x\cos y \;;\; v = e^x\sin y$

$\frac{\partial u}{\partial x} = e^x\cos y \quad \frac{\partial v}{\partial y} = e^x\cos y$

$\frac{\partial u}{\partial y} = -e^x\sin y \quad \frac{\partial v}{\partial x} = e^x\sin y$

$\therefore U_x = V_y \;\&\; -U_y = V_x$

$\therefore$ CR eqns are satisfied. Hence $f(z)$ is analytic

4. $f(z) = iz\bar{z}$

Ans: $f(z) = i(x+iy)(x-iy)$

$f(z) = (ix \cancel{} - y)(x-iy)$

$f(z) = ix^2 - yx + iy^2 + yx$

$\therefore u = 0 \quad , v = x^2+y^2$

$U_x = 0, \; V_y = 2y$

$\therefore$ CR eqns are not satisfied. Hence $f(z)$ is non analytic.

THEOREM

Let $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some neighbourhood of a point $z = x+iy$ and differentiable at z. Then at that point the first order PD of u & v exist and satisfy the CR eqns.

Proof:

Given; the derivative of $f(z)$ i.e, $f'(z)$ exists and is given by

$$f'(z) = \underset{\Delta z \to 0}{Lt} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \underset{\Delta x + i\Delta y \to 0}{Lt} \frac{f(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - (u(x,y) + iv(x,y))}{\Delta x + i\Delta y}$$

$$= \underset{\Delta x + i\Delta y \to 0}{Lt} \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} - i\,\frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y}$$

Case:(1) : $\Delta z \to 0$ Along a line $\parallel^{al}$ to real axis : $\Delta y = 0$

$$f'(z) = \underset{\Delta x \to 0}{Lt} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \underset{\Delta x \to 0}{Lt} \frac{iv(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} \quad \text{———} (1)$$

Case: (2): $\Delta z \to 0$; Along a line parallel to imaginary axis.
i.e, $\Delta x = 0$

$$f'(z) = \lim_{i\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i\frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

$$\therefore f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{--- (2)}$$

From (1) & (2): Since $f'(z)$ exists (1) & (2) are equal

$\therefore$ Equating the real and imaginary parts

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Qn: Show that $w = \sin z$ is analytic everywhere also find its derivative

Ans:-

$w = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$w = \sin x \cosh y + i \cos x \sinh y$

$u = \sin x \cosh y$, $v = \cos x \sinh y$

$\therefore \frac{\partial u}{\partial x} = \cos x \cosh y$ $\quad \frac{\partial v}{\partial y} = \cosh y \cos x$

$\frac{\partial u}{\partial y} = +\sinh y \sin x$ $\quad \frac{\partial v}{\partial x} = -\sin x \sinh y$

$\therefore u_x = v_y$; $u_y = -v_x$

CR eqns are satisfied. Hence w is analytic

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} \quad \text{--- (1)}$$

$= \cos x \cosh y + i \sin x \sinh y$

$= \cos x \cos iy - \sin x \sin iy$ $\quad \because \cos(x+iy)$

$= \cos(x+iy) = \cos z$

Qn: Show that $w = \cos z$ is analytic everywhere also find its derivative

Ans: $w = \cos z = \cos(x+iy)$

$= \cos x \cdot \cos iy - \sin x \sin iy$

$= \cos x \cdot \cosh y - i \sin x \cdot \sinh y$

$u = \cos x \cdot \cosh y$ $\quad v = -\sin x \sinh y$

$u_x = -\sin x \cosh y$ $\quad v_x = -\cos x \cdot \sinh y$

$u_y = \cos x \cdot \sinh y$ $\quad v_y = -\sin x \cdot \cosh y$

$u_x = v_y$, $-v_x = u_y$

$$f(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$= -\sin x \cosh y + i(\cos x \sinh y)$

$= -(\sin x \cosh y) + i(\cos x \sinh y)$

$= -\sin(x+iy) = -\sin z$

8. Prove that the function $w = e^z$ is analytic. Also find derivative.

Ans: $w = e^{(x+iy)} = e^x \cdot e^{iy}$

$w = e^x(\cos y + i\sin y)$

$u = e^x \cos y$, $v = e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cos y \quad \frac{\partial v}{\partial y} = e^x(+\cos y)$

$\frac{\partial u}{\partial y} = -e^x \sin y$, $\frac{\partial v}{\partial x} = e^x \sin y$

$U_x = V_y$; $U_y = -V_x$

$\therefore$ CR eqns are satisfied. Hence w is analytic

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}$

$= e^x \cos y + i \cdot e^x \sin y$

$= e^x(\cos y + i\sin y)$

$\Rightarrow e^x \cdot e^{iy} = e^z$

$\sinh z = \frac{1}{i}\sin iz$

$= -i\sin iz$

$i\sinh x = \sin ix$

$\cosh z = \cos iz$

11. Prove that $w = \sinh z$:

Ans: $w = \sinh z = -i\sin iz$

$\Rightarrow -i\sin i(x+iy)$

$\Rightarrow -i\sin(ix-y)$

$\Rightarrow -i[\sin ix \cos y - \cos ix \sin y]$

$\Rightarrow -i[i\sinh x \cos y - \cosh x \sin y]$

$\Rightarrow \sinh x \cos y + i\cosh x \sin y$

$\cos iz \Rightarrow \cosh z$

$u = \sinh x \cos y$, $v = \cosh x \sin y$

$\frac{\partial u}{\partial x} = \cosh x \cos y \quad \frac{\partial v}{\partial y} = \cosh x \cos y$

$\frac{\partial u}{\partial y} = -\sinh x \sin y \quad \frac{\partial v}{\partial x} = \sinh x \sin y$

$U_x = V_y$; $U_y = -V_x$ $\therefore$ CR eqns satisfied. Hence w is analytic

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}$

$\Rightarrow \cosh x \cos y + i\sinh x \sin y$

$\Rightarrow \cos ix \cos y + \sin ix \sin y$

$= \cos(ix-y) = \cos(ix+i^2y)$

$= \cos i(x+iy) = \cos iz = \cosh z$

Qn: If $f(z)$ is an analytic function with constant modulus, show that $f(z)$ is a **constant**.

Ans:- $f(z) = u+iv$

$u_x = v_y$

$u_y = -v_x$

$|f(z)| = \sqrt{u^2+v^2} = k$

$|f(z)|^2 = u^2+v^2 = k^2$

$u^2+v^2 = k^2$ —— ①

Differentiating w.r.t x

$2u \cdot u_x + 2v \cdot v_x = 0$

$2(u \cdot u_x + v v_x) = 0$

$u u_x + v v_x = 0$ —— ②

Differentiating w.r.t y

$2u \cdot u_y + 2v \cdot v_y = 0$

$2(u \cdot u_y + v v_y) = 0$

$u \cdot u_y + v v_y = 0$ —— ③

$-u v_x + v u_x = 0$ —— ④

Applying : $②^2 + ④^2$

$(u u_x + v v_x)^2 + (-u v_x + v u_x)^2 = 0$

$u^2 u_x^2 + v^2 v_x^2 + \cancel{2uv u_x v_x} + (-u^2 v_x^2 + v^2 u_x^2 - \cancel{2uv u_x v_x}) = 0$

$u^2(u_x^2+v_x^2) + v^2(v_x^2+u_x^2) = 0$

$(u^2+v^2)(u_x^2+v_x^2) = 0$

$u_x^2 + v_x^2 = 0$

$|f'(z)|^2 = 0$

$f'(z) = 0$

i.e; $f(z) = $ constant

Qn: If $w = \log z$ find $\frac{dw}{dz}$, its derivative. And determine where w is non analytic

$x = r\sin\theta$
$y = r\cos\theta$
$x+iy = r(\cos\theta + i\sin\theta)$
$= re^{i\theta}$

Ans:- $w = \log(x+iy) = \log r \cdot e^{i\theta}$

$= \log r + \log e^{i\theta}$

$= \log r + i\theta$

$= \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$

$\Rightarrow \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$

$u = \frac{1}{2}\log(x^2+y^2)$; $v = \tan^{-1}\frac{y}{x}$

$U_x = \frac{x}{x^2+y^2}$; $V_x = \frac{-y}{x^2+y^2}$

$U_y = \frac{y}{x^2+y^2}$; $V_y = \frac{x}{x^2+y^2}$

$x^2+y^2=r^2$

$r=\sqrt{x^2+y^2}$

$\frac{y}{x}=\tan\theta$

$\theta=\tan^{-1}(y/x)$

The CR eqns are not satisfied at $z=0$, i.e, when both x, y are zero.

$f'(z) = U_x + iV_x$

$f'(z) = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$

$f'(z) = \frac{x-iy}{x^2+y^2} = \frac{1}{x^2+y^2}\bar{z} = \frac{\bar{z}}{x^2+y^2}$

LAPLACE EQUATIONS HARMONIC FUNCTION

If $f(z) = u(x,y) + iv(x,y)$ is an analytic function then it satisfies the CR eqns.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{— (1)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{— (2)}$$

Differentiate (1) w.r.t x

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y\partial x} \quad \text{— (3)}$$

Differentiate (2) w.r.t y

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x\partial y} \quad \text{— (4)}$$

(3) + (4) : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Now. differentiate (1) w.r.t y

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 v}{\partial y^2} \quad \text{— (5)}$$

Differentiate (2) w.r.t x

$$\frac{\partial^2 u}{\partial y\partial x} = -\frac{\partial^2 v}{\partial x^2} \quad \text{— (6)}$$

(5) − (6) : $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Therefore; both the functions u & v satisfy the Laplace equation in two variable.

$\therefore$ u & v are also called harmonic functions.

Hence the real & imaginary parts of an analytic function are harmonic functions.

If two harmonic functions u & v satisfy the CR eqns in a domain D, they are the real and imaginary parts of analytic function, f in D; then v is said to be a harmonic conjugate function of u in D.

Qn: Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plan and find harmonic conjugate function $v(u)$

$x^3 - 3xy^2 - 5y$

Ans: $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = -2y - 1$, $\frac{\partial^2 u}{\partial x^2} = 2$, $\frac{\partial^2 u}{\partial y^2} = -2$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \Rightarrow 2 + (-2) = 0$; $\therefore$ u is a harmonic function

$u_x = 2x \Rightarrow V_y = U_x$

$U_x = V_y$ to find harmonic conjugate

$\therefore V_y = 2x$

Integrate w.r.t y keeping x const.

$v = 2xy + f(x)$

$V_x = 2y + f'(x) = -U_y$

$2y + f'(x) = -(-2y - 1) = 2y + 1$

$\therefore f'(x) = 1$

By integration, $f(x) = x + c$

$\therefore v = 2xy + x + c$

<u>OR</u>

$U_y = -2y - 1 = -V_x$

$V_x = 2y + 1$

$V = 2yx + x + f(y)$

$V_y = 2x + f'(y) = U_x$

$2x + f'(y) = 2x$

$f'(y) = 0$

$\therefore f(y) = 0$

$v = 2xy + x + c$

$u = x^2 - y^2 - y$

$\frac{\partial u}{\partial x} = 2x$; $\frac{\partial u}{\partial y} = -2y - 1$

$\frac{\partial^2 u}{\partial x^2} = 2$; $\frac{\partial^2 u}{\partial y^2} = -2$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore$ Hence harmonic

$U_x = V_y = 2x$

Integrate w.r.t y keeping

$V = 2xy + f(x)$

Differentiate w.r.t x

$V_x = 2y + f'(x) = -U_y$

$V_x = -U_y$

$2y + f'(x) = 2y + 1$

$\therefore f'(x) = 1$

$\therefore V = 2xy + 1$

$f(x) = x + c$

$V = 2xy + x + c$

Q: Find an analytic function whose real part is $e^x \cos y$.

Ans: $u = e^x \cos y$

$\frac{\partial u}{\partial x} = e^x \cos y$; $\frac{\partial u}{\partial y} = -e^x \sin y$, $\frac{\partial^2 u}{\partial y^2} = -e^x \cos y$, $\frac{\partial^2 u}{\partial x^2} = e^x \cos y$

Integrate w.r.t y keeping x const. $\qquad U_x = e^x \cos y = V_y$

$$\frac{\partial^2 u}{\partial x \partial y} = e^x \sin y + f(x) = v$$

$$V_x = e^x \sin y + f'(x) = -U_y$$

$$V_x = e^x \sin y + f'(x) = -(-e^x \sin y)$$

$$\therefore f'(x) = 0, \quad f(x) = 0$$

$$V = e^x \sin y + c$$

$$f(z) = u + iv$$
$$= e^x \cos y + i(e^x \sin y + c)$$
$$= e^x (\cos y + i \sin y) + ic$$
$$= e^x \cdot e^{iy} + ic \Rightarrow e^z + ic$$

$e^x \cos y$

$V_x = e^x \cos y$

$V_y = e^x \sin y$

$\therefore U_x = V + C$

$V = e^x \sin y + C$

$f(z) = u + iv$

$\Rightarrow e^x \cos y + ie^x \sin y + c$

$\Rightarrow e^x(\cos y + i \sin y) + c$

$\Rightarrow e^x e^{iy} + c$

$\Rightarrow e^{x+iy} + c = e^z + c$

Milne Thomson Method

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$
$$= \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}$$
$$= e^x \cos y + i e^x \sin y$$

$$f'(z) = e^x(\cos y + i \sin y) \quad \therefore [x = z, y = 0]$$
$$f'(z) = e^z$$
$$f(z) = e^z + c$$

Q: Show that the function $u = 4xy - 3x + 2$ ~~and~~ is analytic and find its harmonic conjugate

Ans: $\frac{\partial u}{\partial x} = 4y - 3$, $\frac{\partial u}{\partial y} = 4x$, $\frac{\partial^2 u}{\partial x^2} = 0$, $\frac{\partial^2 u}{\partial y^2} = 0$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \therefore U_x = 4y - 3 = V_y$

Integrate w.r.t y keeping x const.

$$\frac{\partial u}{\partial y} = 2y^2 + f(x) - 3y$$

$V_x = f'(x) = -U_y = -4x$

$\therefore f'(x) = -4x$

$\therefore f(x) = -\frac{4x^2}{2} = -2x^2 + C$

$\therefore V = 2y^2 - 3y - 2x^2 + C$

Qn: Find an analytic function whose imaginary part $V = (2x+1)y$

Ans: $V = (2x+1)y = 2xy + y$

$\frac{\partial v}{\partial x} = 2y$; $\frac{\partial v}{\partial y} = 2x+1$; $\frac{\partial^2 v}{\partial x^2} = 2$, $\frac{\partial^2 v}{\partial y^2} = 2$

$\cancel{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}} =$ $V_x = 2y$, $V_y = 2x+1$

Integrate with r.t x keep y constant

$u = x^2y + xy + f(y)$

Taking derivative with r.t y

$U_y = x^2 + x + f'(y) = -V_x$

$x^2 + x + f'(y) = -2y$

$f'(y) = -2y - x^2 - x$

$f(y) = -y^2 - \frac{x^3}{3} - \frac{x^2}{3} + C$

$u = x^2y + xy + x^2$

$U_x = V_y = 2x+1$

Integrate V_y with x

$U = x^2 + x + f(y)$

differentiate with y

$U_y = f'(y) = 2x+1 \quad 2x+y =$

$u = xy^2 + \frac{y^2}{2} + f(x)$

$U_x = y^2 + f'(x) = V_y$

$y^2 + f'(x) = 2x+1$

$f'(x) = 2x - y^2 + 1$

$f(x) = x^2 - y^2x + x + C$

$u = xy^2 + \frac{y^2}{2} + x^2 - y^2x + x + C$

$u = \frac{y^2}{2} + x^2 + x + C$

$V = (2x+1)y = 2xy + y$

$V_x = 2y$, $V_y = 2x+1$

$V_{x^2} = 2$ $V_{y^2} = 2$ $\rightarrow$ $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

Qn: Determine a so that the given function is harmonic & find a harmonic conjugate. Given $u = e^{-\pi x}\cos ay$.

Ans: $u = e^{-\pi x}\cos ay$

$\frac{\partial u}{\partial x} = -\pi e^{-\pi x}\cos ay$, $\frac{\partial^2 u}{\partial x^2} \Rightarrow \pi^2 e^{-\pi x}\cos ay$; $\frac{\partial u}{\partial y} = +ae^{-\pi x}\sin ay$

$\frac{\partial^2 u}{\partial y^2} = -a^2 e^{-\pi x}\cos ay$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\pi^2 \cdot e^{-\pi x}\cos ay + (-a^2 \cdot e^{-\pi x}\cos ay) = 0$

$e^{-\pi x}\cos ay(\pi^2 - a^2) = 0$

$\pi^2 - a^2 = 0$

$\pi = a$

$\therefore u = e^{-\pi x}\cos \pi y$

Integrate w.r.t y: $\frac{\partial u}{\partial x \partial y} = -\frac{\pi}{\pi} e^{-\pi x}\sin \pi y + f(x)$

$V_x = \pi e^{-\pi x}\sin\pi y + f'(x) = -u_y$

$$\pi e^{-\pi x}\sin\pi y + f'(x) = +\pi e^{-\pi x}\sin\pi y$$

$$f'(x) = 0,\ f(x) = 0$$

$$\therefore V = e^{-\pi y} + \sin\pi y + C$$

Qn: Show that $u = x^3 - 3xy^2$ is harmonic, hence find its harmonic conjugate.

Ans: $\frac{\partial u}{\partial x} = 3x^2 - 3y^2$, $\frac{\partial u}{\partial y} = -3x \times 2y$

$\frac{\partial^2 u}{\partial x^2} = 6x$, $\frac{\partial^2 u}{\partial y^2} = -6x$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

$u = x^3 - 3xy^2$

Integrate: u_x w.r.t y : $3x^2y - 3\frac{y^3}{3} + f(x) = \frac{\partial^2 u}{\partial x \partial y}$

$$V_x = 6xy + f'(x) = -u_y$$

$$6xy + f'(x) = +6xy$$

$$\therefore f'(x) = 0,\ f(x) = 0$$

$$V = 6xy + C$$

$u = x^3 - 3xy^2$

Integrate w.r.t y keeping x const.

$\frac{\partial^2 u}{\partial x \partial y} = x^3y - xy^3 + f(x)$

$V_x = 3x^2y - y^3 + f'(x) = -u_y$

$-3x^2y + y^3 = f'(x) - 6xy$

$f'(x) = y^3 - 3x^2y + 6xy$

Integrate V_y

$V = 3x^2y - 3\frac{y^3}{3} + f(x)$

$V = 3x^2y - y^3 + f(x)$

$V_x = 6xy + f'(x)$

$f'(x) = 0$

$\therefore V = 3x^2y - y^3$

Qn: If $v = e^x(x\sin y + y\cos y)$, find an analytic function $f(z) = u + iv$.

Ans: Use Milne Thomson Method

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}$$

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$\frac{\partial v}{\partial y} = -xe^x\cos y + y(\sin y) + \cos y$, $\frac{\partial v}{\partial x} = e^x\sin y + x\sin y\, e^x + e^x y\cos y$

$$\therefore f'(z) = e^x(x\cos y - y\sin y + \cos y) + i\,e^x(\sin y + x\sin y + y\cos y)$$

$$\therefore x = z,\ y = 0$$

$$f'(z) = e^z(z - 0 + 1) + i(e^z 0 + z \times 0 +)$$

$$f'(z) = e^z(z+1) + i(0)$$

$$f'(z) = e^z(z+1)$$

$$f(z) = \int e^z(z+1) \Rightarrow e^z\cancel{\left(\frac{z^2}{2}+z\right)} = \left[(z+1)e^z - (1)e^z\right]$$

$$f(z) \Rightarrow e^z(z+1) - e^z \Rightarrow e^z((z+1)-1) \Rightarrow e^z(z)$$

$$f(z) \Rightarrow z \cdot e^z + c$$

Qn: Show that $v = 3x^2y - y^3$ is harmonic, hence find harmonic co, analytic function $f(z)$.

Ans: By Milne Thomson method

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}$$

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial x} = 6xy \;;\; \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \;,\; \frac{\partial^2 v}{\partial x^2} = 6y,\; \frac{\partial^2 v}{\partial y^2} = -6y$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$6y - 6y = 0$$

$\therefore$ Harmonic

$$f'(z) = \cancel{6xy + i(3x^2 - 3y^2)}\; 3x^2 - 3y^2 + i(6xy)$$

$$x = z,\; y = 0$$

$$f'(z) = 3z^2 + i(6 \times z \times 0)$$

$$f'(z) = 3z^2$$

$$f(z) = \int 3z^2 = 3 \times \frac{z^3}{3} \Rightarrow z^3$$

$$\therefore f(z) = z^3 + c$$

Qn: If $f(z) = u + iv$. Prove that $u =$ constant & $v =$ constant are families of curves cutting orthogonally.

curves are orthogonal $\Rightarrow$ slopes also cut orthogonally i.e $\perp$

Ans: $u(x,y) = c_1$ —— ① $\quad v(x,y) = c_2$ —— ②

Differentiate partially w.r.t x

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0 \;\; ① \qquad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{dy}{dx} = 0 \;\; ②$$

$$m_1 = \frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y} \;—\; ① \qquad m_2 = \frac{dy}{dx} = \frac{-\partial v/\partial x}{\partial v/\partial y} \;—\; ②$$

$$m_1 m_2 = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \times \frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \times \frac{-\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} = -1$$

29/01/22

Continuity

A function $f(z)$ is said to be continuous at $z=z_0$ if $f(z_0)$ is defined at: $\lim_{z\to z_0} f(z) = f(z_0)$

Qn: Show that $f(z)$ is continuous at $z=0$, if $f(0)=0$, $z\neq 0$; The func f is equal to $\frac{Re\, z^2}{|z|}$

Ans: $\lim_{z\to 0} f(z) = f(0) = 0$

$z = x+iy$

$Re\, z^2 = Re\,(x+iy)^2$

$= Re\,((x^2-y^2)+i2xy)$

$\Rightarrow x^2-y^2$

$|z| = \sqrt{x^2+y^2}$

$y = mx$

$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{Re\, z^2}{|z|}$

$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{x^2-y^2}{\sqrt{x^2+y^2}}$

$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{x^2-(mx)^2}{\sqrt{x^2+(mx)^2}}$

$\Rightarrow \lim_{x\to 0} \frac{x^{2}(1-m^2)}{x\sqrt{1+m^2}} \neq 0 = f(0)$ ∴ Not continuous;

x tends to zero

Qn: Test the continuity at $z=0$ if $f(z) = \frac{Im\, z}{|z|}$ when $z\neq 0$; if $f(z)=0$, $z=0$

Ans: $\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{Im\, z}{|z|}$

$Im: z = x+iy$

$y = mx$

$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{y}{\sqrt{x^2+y^2}}$

$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{mx}{\sqrt{x^2+(mx)^2}}$

$\Rightarrow \lim_{x\to 0} \frac{mx}{x\sqrt{1+m^2}} = \lim_{x\to 0} \frac{m}{\sqrt{1+m^2}} \neq 0$

∴ The function is not continuous at $z=0$

Qn: Show that $f(z) = Re\,\frac{z}{z}$, $z\neq 0$, $f(z)=0$, $z=0$, is not continuous at $z=0$

$Re\ z = x+iy$

Ans: $\lim_{z\to 0} f(z) = \lim_{x\to 0} Re\,\frac{z}{z}$

$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{x}{}$

03/01/22

CONFORMAL MAPPING

A mapping $w=f(z)$ is called conformal if it preserves angles b/w oriented curves in magnitude as well as in direction. The mapping $w=f(z)$ by an analytic function f is conformal at critical points. (the points at which the derivative $f'=0$.)

$w=z^2 \Rightarrow u=x^2-y^2$

$w=\sin z$

$w=1/z$

$w=z^2$

$f(z)=\cos z$

$f'(z)=\sin z=0$

$z=n\pi$

e^z

$z^2 \Rightarrow x^2-y^2$

① $w=f(z)=z^2$

$f(z)=u+iv=z^2$

→ Using cartesian coordinates

$f(z)=(x+iy)^2=x^2-y^2+i2xy$

$\boxed{u=x^2-y^2 \ , \ v=2xy}$

Under the transformation/mapping

→ Using Polar coordinates

$z=re^{i\theta} \ , \ w=Re^{i\phi}$

$r=\sqrt{x^2+y^2}$

$\theta=\tan^{-1} y/x$

∴ The mapping $w=z^2=(re^{i\theta})^2=r^2e^{i2\theta}=Re^{i\phi}$

$\boxed{\therefore R=r^2; \ \phi=2\theta}$

05/01/22

Qn: Find the image under the mapping $w=z^2$

Ans: (i) the real axis → the eqn of real axis (x axis) → $y=0$

$\therefore u=x^2-y^2=x^2$

$v=2xy=0$

∴ The image of real axis in u-v plane is mapped into the real axis of u-v plane.

Under the mapping $w=z^2$, the points on the real axis of the z plane are mapped into the +ve real axis of w plane.

(ii) the imaginary axis → the eqn of imaginary axis (y axis) → $x=0$

$\therefore u=x^2-y^2=-y^2$

$v=2xy=0$

under the mapping $w=z^2$ the points on the imaginary axis of the z plane are mapped into the -ve real axis of w plane.

(iii) $0 < \theta < \pi/2$

$\Rightarrow 0 < 2\theta < \pi$

$\Rightarrow 0 < \phi < \pi$

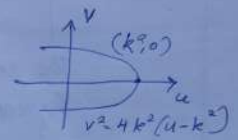
The first +ve quadrant of z plane is mapped to first and second (or upper part of) quadrant of uv plane.

(iv) lines parallel to the imaginary axis

$\Rightarrow$ eqn of line parallel to imaginary axis $\Rightarrow x = k$

$\therefore u = x^2 - y^2 = k^2 - y^2$

$v = 2xy = 2ky \qquad \therefore y = \frac{v}{2k}$



$$u = k^2 - \left(\frac{v}{2k}\right)^2 = k^2 - \frac{v^2}{4k^2} = \frac{4k^4 - v^2}{4k^2}$$

$$v^2 = 4k^4 - 4k^2u = 4k^2(k^2 - u) = -4k^2(u-k^2)$$

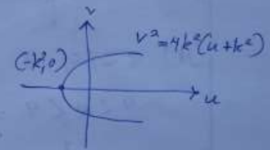
$\therefore$ The lines parallel to the imaginary axis of z plane are mapped into parabolas in w plane.

(v) lines parallel to the real axis

$\rightarrow$ eqn : $y = k$

$\therefore u = x^2 - y^2 = x^2 - k^2$

$v = 2xy = 2xk \qquad \therefore x = \frac{v}{2k}$



$\therefore u = \left(\frac{v}{2k}\right)^2 - k^2 = \frac{v^2 - 4k^4}{4k^2}$

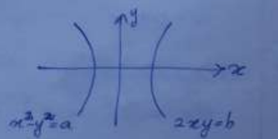
$\therefore v^2 = 4k^2(u+k^2)$

$\therefore$ The lines parallel to the real axis are mapped to parabolas in w plane.

(vi) Rectangular hyperbola

$\Rightarrow x^2 - y^2 = a$

$\therefore u = a$



$2xy = b$

$\therefore v = b$

$\therefore$ Rectangular hyperbolas are mapped to lines parallel to real and imaginary axis of w plane.

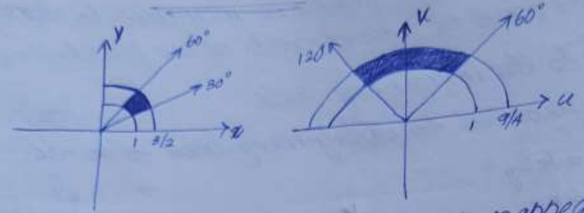
(vii) $1 \le |z| \le 3/2$, $\pi/6 < \theta < \pi/3$

$\Rightarrow |z| = \sqrt{x^2+y^2} = r$ $\quad \frac{\pi}{6} < \theta < \pi/3$

$1 \le r \le 3/2$ $\quad \pi/3 < 2\theta < 2\pi/3$

$1 \le r^2 \le 9/4$ $\quad \pi/3 < \phi < 2\pi/3$

$1 \le R \le 9/4$, $\pi/3 < \phi < 2\pi/3$



∴ The small region in z plane is mapped into a larger region of w plane.

(viii) $|z| \le 1/2$, $-\pi/8 < \text{Arg } z < \pi/2$

$\Rightarrow r \le 1/2$ $\quad -\pi/8 < \theta < \pi/2$

$R \le 1/4$ $\quad -\pi/4 < \phi < \pi$

(ix) $2 < |z| < 3$, $-\pi/4 < \theta < \pi/4$

$4 < R < 9$, $-\pi/2 < \phi < \pi/2$

* $w = f(z) = e^z$

$w = u + iv$

→ Using cartesian coordinates

$f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy}$

$\Rightarrow e^x(\cos y + i\sin y) = e^x\cos y + ie^x\sin y$

$\boxed{u = e^x\cos y \; ; \; v = e^x\sin y}$

→ Using Polar coordinates

$w = Re^{i\phi} = u + iv$

$R = \sqrt{u^2+v^2}$, $\phi = \tan^{-1}\frac{v}{u}$

$$R = \sqrt{(e^x\cos y)^2 + (e^x \sin y)^2} = \sqrt{e^{2x}(\cos^2 y + \sin^2 y)}$$

$$R = \sqrt{e^{2x}} = e^{2x \times 1/2} = e^x$$

$$\phi = \tan^{-1}\frac{e^x \sin y}{e^x \cos y} = \tan^{-1}\tan y = y$$

$$\boxed{\therefore\ R = e^x\ ;\ \phi = y}$$

Qn: Find the image of the following under the mapping $w = e^z$

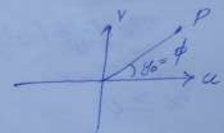
(i) vertical lines : $x = x_0$ [lines // to y axis]

$\therefore R = e^x = e^{x_0}$

$\therefore$ The vertical lines in the z plane are mapped to circles of radius e^{x_0}.

(ii) horizontal lines: $y = y_0$ [lines // to x axis]

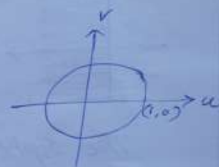
$\therefore \phi = y = y_0$



$\therefore$ The horizontal lines in the z plane are mapped to radial lines with $\phi = y_0$

(iii) imaginary axis : $x = 0$

$\therefore R = e^x = e^0 = 1$



$\therefore$ The imaginary axis in the z plane is mapped to unit circle in uv plane

(iv) real axis : $y = 0$

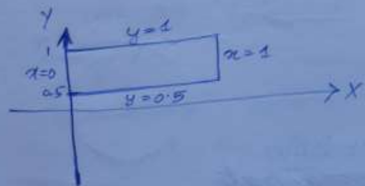
$\therefore \phi = y = 0$

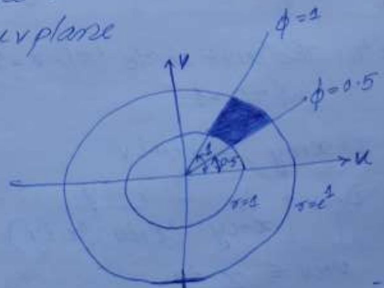
$\therefore u = e^x \cos y = e^x$ (always +ve)

$v = e^x \sin y = 0$

$\therefore$ The real axis in the z plane is mapped to +ve real axis of uv plane

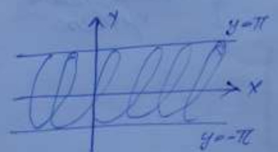
(v)

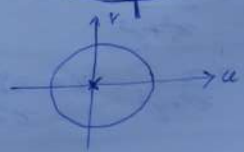




$-\pi < \phi \le -\pi$

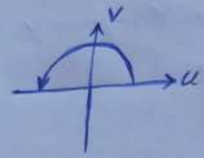
(vi) $-\pi < y \le \pi$



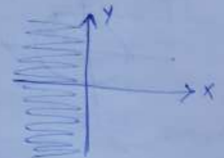


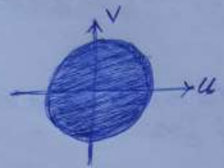
It is mapped to the entire ω plane without the origin.

(vii) $0<y<\pi$ $\qquad\qquad$ $0<\phi<\pi$



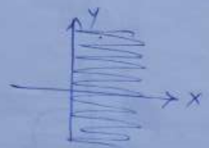
(vii) $x \le 0$ $\qquad$; $R=e^{x} \le e^{0}=1$ $\quad$; $R \le 1$

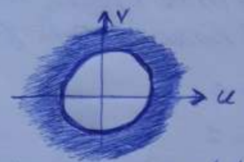




The left half plane is mapped to inside of unit disc

$x \ge 0$ $\qquad$; $R=e^{x} \ge e^{0}=1$; $R \ge 1$





The right half plane is mapped to outside of unit disc

③ ★ $\omega = \frac{1}{z} = f(z)$

$z = re^{i\theta}$

$\omega = Re^{i\phi}$

$\Rightarrow$ $Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$

$\boxed{R = \frac{1}{r} \quad \phi = -\theta}$ $\quad$ } Polar coordinate system

NOTE: The unit circle $|z|=r=1$ is mapped onto the unit circle $|\omega|=R=1$.

$\Rightarrow$

$z = x+iy$, $\omega = u+iv$

$u+iv = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$

$u+iv = \frac{x-iy}{x^2+y^2}$ $\quad$ } Cartesian coordinate system

$\boxed{u = \frac{x}{x^2+y^2} \quad v = \frac{-y}{x^2+y^2}}$

$w = 1/z \quad , \quad z = 1/w$

$$x+iy = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$\boxed{x = \frac{u}{u^2+v^2} \; ; \; y = -\frac{v}{u^2+v^2}}$$

Qn: $w = 1/z$ map every straight line/circle onto circle/straight line.

Ans: $\therefore A(x^2+y^2) + B(x) + Cy + D = 0$ $\quad A \neq 0 \rightarrow$ circle; $A = 0 \rightarrow$ straight line

$$A(z\bar{z}) + B\left(\frac{z+\bar{z}}{2}\right) + C\left(\frac{z-\bar{z}}{2i}\right) + D = 0$$

$$A\left(\frac{1}{w}\frac{1}{\bar{w}}\right) + \frac{B}{2}\left(\frac{1}{w}+\frac{1}{\bar{w}}\right) + \frac{C}{2i}\left(\frac{1}{w}-\frac{1}{\bar{w}}\right) + D = 0$$

$$A + \frac{B}{2}(\bar{w}+w) + \frac{C}{2i}(\bar{w}-w) + Dw\bar{w} = 0$$

$w = u+iv$, $\bar{w} = u-iv$; $w\bar{w} = u^2+v^2$

$w+\bar{w} = 2u$; $\bar{w}-w = -2iv$

$$A + \frac{B}{2}(2u) + \frac{C}{2i}(-2iv) + D(u^2+v^2) = 0$$

When $D \neq 0$; circle

When $D = 0$; straight line

Eqn of a circle or straight line of z plane can be written as;

$A(x^2-y^2)+Bx+Cy+D=0$

$z = x+iy$

$\bar{z} = x-iy$

$z-\bar{z} = 2iy$

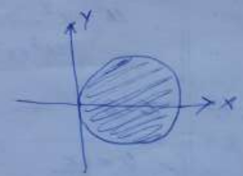
Qn: Find the image under the transformation $w = 1/z$

(i) $|z - 1/2| \leq 1/2$ $\qquad |x+iy| = \sqrt{x^2+y^2}$

$|x+iy-1/2| \leq 1/2$

$|(x-1/2)+iy)| \leq 1/2$

$\sqrt{(x-1/2)^2+y^2} \leq 1/2$



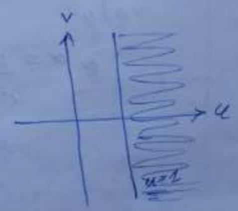
$(x-1/2)^2+y^2 \leq 1/4$

$x^2+\frac{1}{4}-x+y^2 \leq \frac{1}{4}$

$x^2+y^2-x \leq 0$

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - \frac{u}{u^2+v^2} \leq 0$$

Taking LCM



$u^2+v^2-u(u^2+v^2) \leq 0$

$u^2+v^2(1-u) \leq 0$

$1-u \leq 0$

$u \geq 1$

(ii) $|z-2i|=2$

$|x+iy-2i|=2$

$\sqrt{x^2+(y-2)^2}=2$

$x^2+(y-2)^2=4$

$x^2+y^2+4-4y=4$

$x^2+y^2-4y=0$

$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2+\frac{4v}{u^2+v^2}=0$

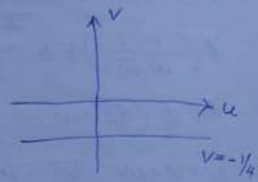
$u^2+v^2+4v(u^2+v^2)=0$

$u^2+v^2(1+4v)=0$

$1+4v=0$

$4v=-1$

$v=-1/4$



(iii) $|z-1|=1$

$|x+iy-1|=1$

$\sqrt{(x-1)^2+y^2}=1$

$(x-1)^2+y^2)=1$

$x^2-2x+1+y^2=1$

$x^2+y^2-2x=0$

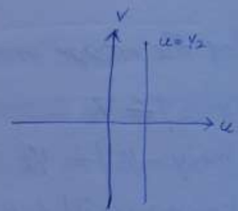
$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2-\frac{2u}{u^2+v^2}=0$

$u^2+v^2-2u(u^2+v^2)=0$

$u^2+v^2(1-2u)=0$

$1-2u=0$

$u=1/2$



(iv) $x=2$

$u=\frac{x}{x^2+y^2}$, $v=\frac{-y}{x^2+y^2}$

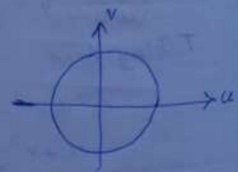
$u=\frac{2}{4+y^2}$; $v=\frac{-y}{4+y^2}$

$x=\frac{u}{u^2+v^2}=2$

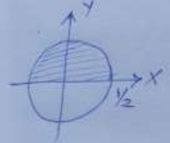
$2(u^2+v^2)=u$

$2(u^2+v^2)-u=0$

$u=0$



(v) $|z| < \frac{1}{2}$, $\text{Im} z \geq 0$



Ans: $|x+iy| \leq 1/2$

$\sqrt{x^2+y^2} < 1/2$

$x^2+y^2 < 1/4$

$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 < 1/4$

$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} < 1/4$

$\cancel{u^2(u^2+v^2)} \quad \frac{u^2+v^2}{(u^2+v^2)^2} < \frac{1}{4}$

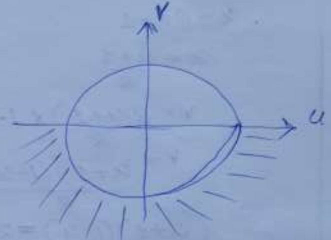
$4(u^2+v^2) < (u^2+v^2)^2$

$4(u^2+v^2) - (u^2+v^2)^2 < 0$

$[4-(u^2+v^2)](u^2+v^2) < 0$

$4-(u^2+v^2) < 0$

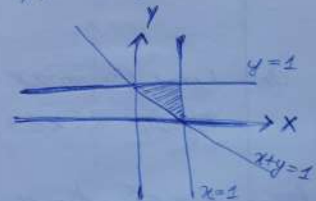
$u^2+v^2 > 4$



$\text{Im} z \geq 0 \Rightarrow y \geq 0$

$\frac{-v}{x^2+y^2} \geq 0$, $-v > 0$; $v < 0$

Qn: Determine the region of the w-plane to which the region bounded by $x=1$, $y=1$, $x+y=1$. is mapped by the transformation $w = z^2$.
UQ



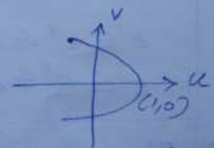
Ans: $w = z^2$

$u = x^2 - y^2$; $v = 2xy$

When $x = 1$,

$u = 1 - y^2$; $v = 2y$

$y = v/2$



$u = 1 - (v/2)^2$; $v^2 = 4 - 4u$

$4u = 4 - v^2$; $v^2 = 4(1-u)$; ~~u = 1/2~~

$v = \cancel{\sqrt{4(1-u)}}$

$\therefore$ $x=1$ st line in z plane is transformed into parabola with vertex (1,0) in w plane

Vertical lines are mapped to parabolas

$w = z^2$

when $y = 1$; $u = x^2 - 1$, $v = 2x$

$x = v/2$

$u = (v/2)^2 - 1$

$4u = v^2 - 4$

$v^2 = 4u + 4$

$v^2 = 4(u+1)$

$u + 1 = 0$

$u = -1$

Parabola with vertex $(-1, 0)$

$y^2 = 4ax$

$(y-x)^2 = 4a(x-p)$

$x + y = 1$

$y = 1 - x$

$u = x^2 - (1-x)^2$ $\qquad$ $v = 2x(1-x)$

$u = x^2 - 1 \cancel{+} x^2 + 2x$ $\qquad$ $v = 2x - 2x^2$

$u = \cancel{2x^2} - 1 + 2x$ $\qquad$ $v = 2x(1-x)$

$u = 2x - 1$ $\qquad$ $1 - x = 0$

$x = \frac{u+1}{2}$ $\qquad$ $x = 1$

$v = 2\frac{(u+1)}{2} \times 1 - \frac{(u+1)}{2}$

$v = \frac{(u+1)(1-u)}{2}$

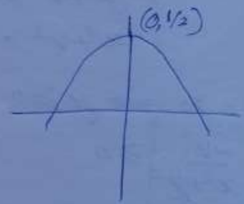
$(u+1)(1-u) = 2v$

$1 - u^2 = 2v$

$-u^2 = 2v - 1$

$u^2 = 1 - 2v$

$v^2 = -2(v - 1/2)$

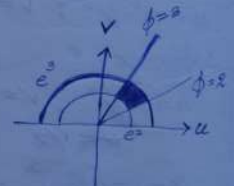


Qn: Find the image and draw a rough sketch of a mapping of the region $2 \le x \le 3$, $2 \le y \le 3$. under the mapping $w = e^z$

Ans: $R = e^x$, $\phi = y$

$e^2 \le e^x \le e^3$ $\quad$; $2 \le \phi \le 3$

$e^2 \le R \le e^3$



Qn: Find the image of $x \ge 1$ for the mapping $w = 1/z$

Ans: $u = \frac{x}{x^2+y^2}$, $v = \frac{-y}{x^2+y^2}$

$x = \frac{u}{u^2+v^2} \ge 1 \Rightarrow u \ge u^2 + v^2$

$u^2 + v^2 - u \le 0$

u^2+v^2 = circle

u^2 or v^2 = parabola

Qn: Find the image ~~of~~ of the region $|z - 1/3| \le 1/3$ under the transformation $w = 1/z$

Ans: $|z - 1/3| \le 1/3$

$|x + iy - 1/3| \le 1/3$

$\sqrt{(x - 1/3)^2 + y^2} \le 1/3$

$(x - 1/3)^2 + y^2 \le 1/9$

$x^2 - \frac{2}{3}x + \frac{1}{9} + y^2 \le \frac{1}{9}$

$x^2 - \frac{2}{3}x + y^2 \le 0$

$\left(\frac{u}{u^2+v^2}\right)^2 - \frac{2}{3}\left(\frac{u}{u^2+v^2}\right) + \left(\frac{-v}{u^2+v^2}\right)^2 \le 0$

$u^2 + v^2 - \frac{2}{3}(u^2+v^2) \le 0$

$(u^2+v^2)\left[1 - \frac{2u}{3}\right] \le 0$

$1 - \frac{2}{3}u \le 0$

$\frac{2u}{3} \ge 1$

$u \ge \frac{3}{2}$

$|z - 2i| = 2$

$|x + i(y-2)| = 2$

$\sqrt{x^2 + (y-2)^2} = 2$

$x^2 + (y-2)^2 = 4$

$x^2 + y^2 - 4y + 4 = 4$

$x^2 + y^2 = 4y$

$x = \frac{u}{u^2+v^2}$; $y = \frac{v}{u^2+v^2}$

substitute

Qn: Find the image of lines $x=1$, $y=2$ and $x>0, y<0$ under the mapping $w = z^2$.

Ans:

$x = 1$

$u = x^2 - y^2$, $v = 2xy$

$u = 1 - y^2$; $v = 2y$: $y = v/2$

$u = 1 - (v/2)^2$ $\quad v^2 = 4(1-u)$

$4u = 4 - v^2$

$y = 2$

$u = x^2 - 4$; $v = 4x$ $\quad x = v/4$

$u = \left(\frac{v}{4}\right)^2 - 4$; $4u = v^2 - 16$

$v^2 = 16 + 4u = 4(4+u)$

$x > 0 \Rightarrow u > 0$

$y < 0 \Rightarrow y < 0$

18/01/22

Fixed Points

* $w=f(z)$ are points that are mapped onto themselves or kept fixed under the mapping $w=f(z)$.

ie, $w=f(z)=z$.

Qn: Find the fixed points for the mapping $w=f(z)=\bar{z}$

Ans:

$\bar{z}=z$

$x-iy=x+iy$

$x-iy-x-iy=0$

$-2iy=0 \quad \therefore y=0$

All the points on the real axis are the fixed points

Qn: Find the fixed points for the mapping $w=f(z)=1/z$

$w=(a+ib)z^2$

Ans:

$\frac{1}{z}=z$

$\frac{1}{x+iy}=x+iy$

$1=(x+iy)^2$

$z^2=1$

$z^2-1=0$

$z=\pm 1$

Qn: $w=z-3i$

Ans:

$z-3i=z$

$z-z-3i=0$

$-3i=0$

$\therefore$ No fixed points

Qn: $w=\frac{iz+4}{2z-5i}$

Ans:

$\frac{iz+4}{2z-5i}=z$

$iz+4=2z^2-5iz$

$-2z^2+4+6iz=0$

$-z^2+2+3iz=0$

$\cancel{z=\sqrt{2+3iz}}$

$\Rightarrow 2z^2-6iz-4=0$

$\rightarrow \frac{6i\pm\sqrt{-36+32}}{2\times 2}=\frac{6i\pm 2i}{4}=(2i, i)$

$w = \sin z$

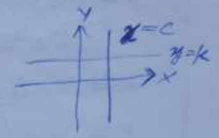
$w = \sin(x+iy)$

$w \Rightarrow \sin x \cos iy + \cos x \sin iy$

$w \Rightarrow \sin x \cosh y + i \cos x \sinh y = u + iv$

$\therefore u = \sin x \cosh y \quad v = \cos x \sinh y$

Qn: Find the image under the mapping $w = \sin z$



(i) $x = c$

$\therefore u = \sin c \cosh y$

$\therefore v = \cos c \sinh y$

$\cosh y = \frac{u}{\sin c}, \quad \sinh y = \frac{v}{\cos c}$

$\cosh^2 y - \sinh^2 y = 1$

$\frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1$

$\frac{x^2}{a^2} - \frac{v^2}{b^2}$

$\therefore$ ~~vert~~ ~~horizontal~~ vertical line is mapped to hyperbola in the w plane.

(ii) $y = k$

$\therefore u = \sin x \cosh k$

$v = \cos x \sinh k$

$\cancel{\sinh k}\ \sin x = \frac{u}{\cosh k}, \quad \cos x = \frac{v}{\sinh k}$

$\sin^2 x + \cos^2 x = 1$

$\frac{x^2}{a}$

$\frac{u^2}{\cosh^2 k} + \frac{v^2}{\sinh^2 k} = 1$

$\therefore$ Horizontal line is mapped into elipse in the w plane.

(iii) $x = 0$ (imaginery axis)

$u = \sin 0 \cosh y = 0$

$v = \cos 0 \sinh y = \sinh y$

(iv) $y = 0$ (real axis)

$u = \sin x \cos 0 = \sin x$

$v = \cos x \sin 0 = 0$

Qn: Find the image of the semicircle $y=\sqrt{4-x^2}$ under the transformation $w=z^2$

Ans:

$u=x^2-y^2$; $v=2xy$

$u=x^2-4+x^2=-4+2x^2 \Rightarrow 2(x^2-2)$

$v=2x\sqrt{4-x^2}$

$x^2=\frac{u}{2}+2$

$y^2=4-x^2$

$x^2+y^2=4$

$v^2=4x^2(4-x^2)$

$v^2=16x^2-(2x^2)^2$

$v^2=16\left(\frac{u+4}{2}\right)-4\left(\frac{u+4}{2}\right)^2$

$v^2=8u+32-u^2-8u-16$

$v^2=16-u^2$

$v=4-u$

$R=r$; $\phi=2\theta$

$\therefore R=2$

$u^2+v^2=16$

$x^2+y^2=r^2$

$\therefore r=2$

Qn: Find the image of $0<x<1$, $1/2<y<1$ under the transformation $w=e^z$

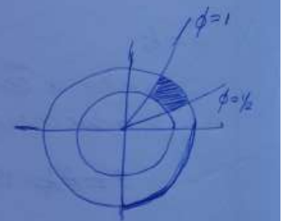
Ans:

$u=e^x\cos y$, $v=e^x\sin y$; $R=e^x$, $\phi=y$

$\therefore 0<x<1$; $1/2<y<1$

$R \Rightarrow e^0<e^x<e^1$ $\phi \Rightarrow 1/2<\phi<1$

$\Longrightarrow 1<R<e$



turn behind

Find the image of lines $x=c$, $y=k$ under transform : $w=\sin z$

Show that $f(z)=|z|^2$ is differential at $z=0$. Hence it is nowhere analytic.

Show that $(u+iv)_n$ is const $u=c$, $\frac{du}{dx}=0=\frac{dv}{dy}$; derivative is zero