28/12/21

MODULE : 2

APPLICATION OF PDE

Wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$ ⇒ Assume the solution to be:

$$y(x,t) = X(x) \cdot T(t)$$

$$XT'' = c^2 X'' T$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k$$

$$\frac{X''}{X} = k \qquad \frac{1}{c^2}\frac{T''}{T} = k$$

$$X'' - kX = 0 \qquad T'' - kc^2 T = 0$$

$$\lambda^2 - k = 0 \qquad \lambda^2 - kc^2 = 0$$

Case 1: k is +ve $k = p^2$

$$\lambda^2 - p^2 = 0 \qquad \lambda^2 - p^2c^2 = 0$$

$$\lambda = \pm p \qquad \lambda = \pm pc$$

$$X = c_1 e^{px} + c_2 e^{-px}, \quad T = c_3 e^{pct} + c_4 e^{-pct}$$

Case 2: k is -ve $k = -p^2$

$$\lambda^2 + p^2 = 0 \qquad \lambda^2 + p^2c^2 = 0$$

$$\lambda = \pm ip \qquad \lambda = \pm pci$$

$$X = (c_1 \cos px + c_2 \sin px), \quad T = (c_3 \cos pct + c_4 \sin pct)$$

Case 3: $k = 0$

$$\lambda^2 = 0 \qquad \lambda^2 = 0$$

$$\lambda = 0,0 \qquad \lambda = 0,0$$

$$X = c_1 + c_2 x, \quad T = c_3 + c_4 t$$

① $y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pct} + c_4 e^{-pct})$

② $y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct)$

③ $y(x,t) = (c_1 + c_2 x)(c_3 + c_4 t)$

Since wave motion is a periodic function of x & t, the solution must involve trignometric terms. Therefore, the solution of wave motion is: $y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct)$

Fourier Solution

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Therefore: solution $y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$

The boundary conditions are: $y(0,t) = y(l,t) = 0$

$y(0,t) = c_1(c_3 \cos pct + c_4 \sin pct) = 0$

$\therefore c_1 = 0$

$y(x,t) \Rightarrow c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt) = 0$

$\therefore c_2 \sin pl = 0$

i.e, $\sin pl = 0$

$pl = n\pi$

$p = \frac{n\pi}{l}$

$y(x,t) = \left(c_2 c_3 \cos \frac{n\pi ct}{l} + c_2 c_4 \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$

$c_2 c_3 = a_n \quad , \quad c_2 c_4 = b_n$

$$\therefore y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

Initial conditions : $y(x,0) = f(x)$

$\left.\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$

$a_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$

$b_n = \frac{2}{cn\pi}\int_0^l g(x) \cos \frac{n\pi x}{l}\, dx$

1. Qn: A tightly stretched string with fixed end pts: with $x=0$ & $x=l$ is in a position given by $y = a \times \sin \frac{\pi x}{l}$. If it is released from rest find the displacement $y(x,t)$

Ans: Wave Equation : $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Solution : $y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin cpt)$

Applying the boundary conditions : $y(0,t) = y(l,t) = 0$

$c_1 = 0 \;,\; p = \frac{n\pi}{l}$

$$y(x,t) = \left(c_2c_3 \cos\frac{n\pi ct}{l} + c_2c_4 \sin\frac{n\pi ct}{l}\right) \sin\frac{\pi n x}{l}$$

$$c_2c_3 = a_n \ ; \ c_2c_4 = b_n$$

$$y(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

since; initial conditions $\begin{cases} y(x,0) = a\sin\frac{\pi x}{l} = f(x) \\ \left.\frac{\partial y}{\partial t}\right|_{t=0} = 0 \quad b_n = 0 \end{cases}$

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$y(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} = a\sin\frac{\pi x}{l} \qquad a_1 = a$$

$$= a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{2\pi x}{l} + a_3 \sin\frac{3\pi x}{l}$$

$$y(x,t) = a_1 \cos\frac{\pi ct}{l}\sin\frac{\pi x}{l}$$

2 Qn: A tightly stretched string with fixed ends: $x=0$ & $x=l$ is given an initial velocity $V_0\sin^3\frac{\pi x}{l}$ find the displacement $y(x,t)$

Ans: Wave equation : $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$

Solution: $y(x,t) = (c_1\cos px + c_2\sin px)(c_3\cos cpt + c_4\sin cpt)$

Applying boundary conditions : $y(0,t) = y(l,t) = 0$

$$c_1 = 0, \ p = \frac{n\pi}{l}$$

$$y(x,t) = \left(c_2c_3 \cos\frac{n\pi ct}{l} + c_2c_4 \sin\frac{n\pi ct}{l}\right)\sin\frac{\pi n x}{l}$$

$$c_2c_3 = a_n \ , \ c_2c_4 = b_n$$

$$y(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi ct}{l} + b_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

Since; Initial condition $\begin{cases} y(x,0) \Rightarrow \therefore a_n = 0 \\ \left.\frac{\partial y}{\partial t}\right|_{t=0} = V_0\sin^3\frac{\pi x}{l} \end{cases}$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \times \frac{l}{n\pi c}$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} \Rightarrow \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\,\frac{l}{n\pi c} = V_0\sin^3\frac{\pi x}{l}$$

$$\Rightarrow b_1 \frac{l}{\pi c}\sin\frac{\pi x}{l} + b_2\frac{l}{2\pi c}\sin\frac{2\pi x}{l} + b_3\frac{l}{3\pi c}\sin\frac{3\pi x}{l}\cdots = V_0\sin^3\left(\frac{\pi x}{l}\right)$$

$$b_1\frac{l}{\pi c}\sin\frac{\pi x}{l} + b_2\frac{l}{2\pi c}\sin\frac{2\pi x}{l} + b_3\frac{l}{3\pi c}\sin\frac{3\pi x}{l}\cdots = \frac{V_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$

$$\frac{b_1 l}{\pi c} = \frac{3V_0}{4} \quad ; \quad \frac{b_3 l}{3\pi c} = \frac{-V_0}{4}$$

$$b_1 = \frac{3V_0\pi c}{4l} \qquad b_3 = \frac{-3V_0\pi c}{4l}$$

$$\therefore y(x,t) = \frac{3V_0\pi l}{4l}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}\cdot\frac{l}{\pi c} - \frac{3V_0\pi c}{4l}\cdot\frac{l}{3\pi c}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}$$

$$\therefore y(x,t) = \frac{3V_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{V_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}$$

3. A tightly stretched string with fixed ends : $x=0$, $x=l$ is initially kept in a position where; $y = y_0\sin^3\frac{\pi x}{l}$. If it is released from rest find displacement $y(x,t)$.

Ans: Wave equation: $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$

Solution: $y(x,t) = (c_1\cos px + c_2\sin px)(C_3\cos pct + C_4\sin pct)$

Applying boundary conditions: $x=0, x=l$; $y(0,t) = y(l,t) = 0$

$c_1 = 0 \quad p = \frac{n\pi}{l}$

$$y(x,t) = \left(c_2c_3\cos\frac{n\pi ct}{l} + c_2c_4\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

$$y(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi ct}{l} + b_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

since; Initial conditions $\left\{\begin{array}{l} y(x,0) = y_0\sin^3\frac{\pi x}{l} = f(x) \\ \left.\frac{\partial y}{\partial t}\right|_{t=0} = 0,\ b_n = 0\end{array}\right.$

$$y(x,t) = \sum_{n=1}^{\infty} a_n\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$y(x,0) = \sum_{n=1}^{\infty} a_n\sin\frac{n\pi x}{l} = y_0\sin^3\frac{\pi x}{l}$$

$$= \sum_{n=1}^{\infty} a_n\sin\frac{n\pi x}{l} = y_0\left(\frac{3\sin A - \sin 3A}{4}\right)$$

$$a_1\sin\frac{\pi x}{l} + a_2\sin\frac{2\pi x}{l} + a_3\sin\frac{3\pi x}{l} = y_0\left[\frac{3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}}{4}\right]$$

$a_1 = y_0^3$, $a_1 = -3y_0$, $a_2 = 0$, $a_3 = -y_0$

$$y(x,t) = 3y_0 \cos\frac{\pi ct}{l} \sin\frac{\pi x}{l} - y_0 \cos\frac{3\pi ct}{l} \sin\frac{3\pi ct}{l}$$

4. A tightly stretched string has its end points fixed at $x=0$, & $x=l$ at time 't', the spring is given a shake defined by $F(x) = \mu x(l-x)$ where, μ is a constant and then released. Find the displacement of the string x at any point.

Ans: Wave equation: $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

solution: $y(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos pct + C_4 \sin pct)$

Applying 1st boundary conditions: $C_1 = 0$

Applying 2nd boundary conditions: $p = \frac{n\pi}{l}$

$$y(x,t) = \left(C_2C_3 \cos\frac{n\pi ct}{l} + C_2C_4 \sin\frac{n\pi ct}{l}\right) \sin\frac{n\pi x}{l}$$

$$y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right) \sin\frac{n\pi x}{l}$$

$y(x,0) = F(x) = \mu x(l-x)$

$\therefore$ Initial velocity: $\left.\frac{\partial y}{\partial t}\right|_{t=0}$; $b_n = 0$

$$\therefore y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}$$

$$a_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx \; : \; \frac{2}{l}\int_0^l \mu x(l-x) \sin\frac{n\pi x}{l}\, dx$$

$$a_n = \frac{2}{l}\mu \int_0^l (lx - x^2)\left(\sin\frac{n\pi x}{l}\right) dx$$

$$a_n = \frac{2}{l}\mu \left(lx - x^2\right)\left(-\cos\frac{n\pi x}{l} \times \frac{l}{n\pi}\right) - (l-2x)\left(-\sin\frac{n\pi x}{l} \times \frac{l^2}{(n\pi)^2}\right) - (-2)\left(+\cos\frac{n\pi x}{l} \times \frac{l^3}{(n\pi)^3}\right)$$

$$a_n = \frac{2}{l}\mu (lx - x^2)(-\cos x) + 2\left(\cos x \frac{l}{(n\pi)^2}\right)$$

$$a_n = \frac{2\mu}{l}\left[(l^2 - l^3)\left(-\cos l\, \frac{n\pi \times l}{l\; n\pi}\right) + 2\cos l \frac{n\pi \times l^3}{l\,(n\pi)^3}\right]$$

$$\Rightarrow \frac{2\mu}{l}\left(\frac{-2l^3 \cos n\pi}{n^3\pi^3} + \frac{2l^3}{n^3\pi^3}\right) = \frac{-4\mu l^2}{n^3\pi^3}\left[1 - \cos n\pi\right]$$

5. A tighly stretched string with fixed end points $x=0$ & $x=l$ is initially at rest in its equilibrium position. If it is set vibrating by given two each of its points a velocity $\lambda x(l-x)$ find the displacement of the string at any distance x from one end at any time t.

Ans: Wave eqn: $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Solution: $(c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct) = y(x,t)$

Applying boundary conditions: $c_1 = 0$, $p = \frac{n\pi}{l}$

$$y(x,t) = \left(c_2 c_3 \cos \frac{n\pi ct}{l} + c_2 c_4 \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

$$y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

$y(x,0) \Rightarrow a_n = 0$

Initial condition: $\left.\frac{\partial y}{\partial t}\right|_{t=0} = \lambda x(l-x)$

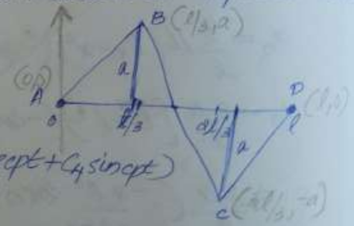
$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \times \frac{l}{n\pi c}$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} \Rightarrow \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \times \frac{l}{n\pi c} = \lambda x(l-x)$$

$$b_1 \frac{l}{\pi c} \sin \frac{\pi x}{l} + b_2 \frac{l}{2\pi c} \sin \frac{2\pi x}{l} + b_3 \frac{l}{3\pi c} \sin \frac{3\pi c}{l} \cdots = \lambda x(l-x)$$

6. The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Find the displacement $y(x,t)$:



Ans: Wave eqn: $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Solution: $y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct)$

Applying boundary conditions: $y(0,t) = y(l,t) = 0$; $c_1 = 0$, $p = \frac{n\pi}{l}$

$$\therefore y(x,t) = \left(c_2 c_3 \cos \frac{n\pi ct}{l} + c_2 c_4 \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

$$y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

$\frac{\partial y}{\partial t} = 0$; $b_n \Rightarrow 0$: given displacement

$y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi c t}{l} \sin\frac{n\pi x}{l}$

$y(x,0) = f(x)$

Eqn to AB: $\frac{x-0}{l/3} = \frac{y-0}{a}$ $\quad\therefore y = \frac{3ax}{l}$

Eqn to BC: $\frac{x-l/3}{2l/3-l/3} = \frac{y-a}{-a-a}$ $\quad\therefore -2ax - \frac{lx}{3} = \frac{yl}{3} - \frac{al}{3}$ check

$-2ax = \frac{l}{3}[x+y-a]$

$-2ax \times \frac{3}{l} = x+y-a$

$\therefore y = a - \frac{6ax}{l} + 2a$

$y = 3a - \frac{6a}{l}x$

$y = \frac{3a}{l}(l-2x)$

Eqn to CD: $\frac{x-2l/3}{l-2l/3} = \frac{y+a}{-a}$

$\therefore -ax + \frac{2l}{3}x = \frac{yl}{3} + \frac{al}{3}$

$9ax + 6lx = l(y+a)3$

$9ax + 6lx = 3ly + 3al$

$\therefore y = \frac{9ax+6lx-3al}{3l}$

$y = \frac{3a}{l}(x-l)$

$y(x,0) = f(x) = \begin{cases} \frac{3ax}{l} ; & 0<x<l/3 \\ \frac{3a}{l}(l-2x) ; & l/3<x<2l/3 \\ \frac{3a}{l}(x-l) ; & 2l/3<x<l \end{cases}$

$\therefore a_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$

$\therefore \frac{6a}{l^2}\left[\int_0^{l/3} x\sin\frac{n\pi x}{l}\,dx + \int_{l/3}^{2l/3}(l-2x)\sin\frac{n\pi x}{l}\,dx + \int_{2l/3}^{l}(x-l)\sin\frac{n\pi x}{l}\,dx\right]$

$\therefore \frac{6a}{l^2}\left[x\left(-\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right) - (1)\left(-\sin\frac{n\pi x}{l}\cdot\frac{l^2}{(n\pi)^2}\right)\right]_0^{l/3} + (l-2x)\left(\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right) - (-2)\left(\sin\frac{n\pi x}{l}\frac{l^2}{(n\pi)^2}\right)$

$\therefore a_n = \frac{18a}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right)$

Heat Equation

$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

$u(x,t) = X(x)\,T(t)$

$XT' = c^2 X'' T$

$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = k$

$X'' - Xk = 0 \qquad T' - c^2kT = 0$

$\lambda^2 - k = 0 \qquad \lambda - c^2k = 0$

Case 1: k is +ve, $k = p^2$

$\lambda^2 - p^2 = 0 \qquad \lambda - c^2p^2 = 0$

$\lambda = \pm p \qquad \lambda = c^2p^2$

$X = c_1 e^{px} + c_2 e^{-px} \qquad T = c_3 e^{c^2p^2t}$

Case 2: k is -ve, $k=-p^2$

$\lambda^2-(-p^2)=0 \qquad \lambda+c^2p^2=0$

$\lambda=\pm ip \qquad \lambda=-c^2p^2$

$X=C_1\cos px+C_2\sin px\ ;\ T=C_3e^{-c^2p^2t}$

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Case 3: k is zero, $k=0$

$\lambda^2=0, \qquad \lambda^2=0$

$\lambda=0,0 \qquad \lambda=0,0$

$X=C_1+C_2x \qquad T=C_3$

$\therefore$ ① : $u(x,t)=(C_1e^{px}+C_2e^{-px})C_3e^{c^2p^2t}$

② : $u(x,t)=(C_1\cos px+C_2\sin px)C_3e^{-c^2p^2t}$

③ : $u(x,t)=(C_1+C_2x)C_3$

$\therefore$ solution of heat equation : $u(x,t)=(C_1\cos px+C_2\sin px)C_3e^{-c^2p^2t}$

1) Solve the eqn $\frac{\partial u}{\partial t}=\frac{\partial^2u}{\partial x^2}$ with the conditions $u(x,0)=3\sin n\pi x$, $u(0,t)=0$, $u(1,t)=0$, $0<x<l$, $t>0$.

$\boxed{c^2=1}$

Ans: Solution of heat eqn :> $u(x,t)=(C_1\cos px+C_2\sin px)C_3e^{-p^2t}$

Applying boundary conditions: $u(0,t)\Rightarrow C_1C_3e^{-p^2t}=C_1\times0=0$

$u(x,t)\Rightarrow C_2\sin px\, C_3e^{-p^2t}$

$u(l,t)\Rightarrow C_2\sin p(1)C_3e^{-p^2t}=0$

$\sin p=0,\ p=n\pi$

$u(x,t)=C_2C_3\sin n\pi x\, e^{-n^2\pi^2t}$

$C_2C_3=a_n$

$u(x,t)=\sum_{n=1}^{\infty}a_n\sin n\pi x\, e^{-n^2\pi^2t}$

$u(x,0)=\sum_{n=1}^{\infty}a_n\sin n\pi x=3\sin n\pi x$

$\therefore a_n=3$

$\therefore u(x,t)=\sum_{n=1}^{\infty}3\sin n\pi x\cdot e^{-n^2\pi^2t}$

2) Find the temperature distribution in a rod of length 2m whose end points are maintained at $x=0$ and initial temp $f(x)=100(2x-x^2)$

Ans: solution :> $u(x,t)=(C_1\cos px+C_2\sin px)(C_3e^{-p^2t})$

Applying boundary conditions: } $c_1=0$, $p=\frac{n\pi}{2}$

$$u(x,t) = c_2c_3 \sin\frac{n\pi x}{2}\cdot e^{-\frac{c^2n^2\pi^2}{4}t}$$

$$c_2c_3 = a_n$$

$$u(x,t)=\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{2}\cdot e^{-\frac{c^2n^2\pi^2}{4}t}$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$$

$$\cancel{u(x,0)=\sum_{n=1}^{\infty} a_n \sin n\pi x}$$

$$a_n = \int_0^2 100(2x-x^2)\sin\frac{n\pi x}{2}\,dx$$

$$a_n = 100\Big[(2x-x^2)\Big(-\cos\frac{n\pi x}{2}\cdot\frac{2}{n\pi}\Big) - (2-2x)\Big(-\sin\frac{n\pi x}{2}\cdot\frac{4}{n^2\pi^2}\Big) + (-2)\Big(\cos\frac{n\pi x}{2}\cdot\frac{8}{n^3\pi^3}\Big)\Big]_0^2$$

$$a_n = 100\Big[0-(-2)\Big(-\sin n\pi\cdot\frac{4}{n^2\pi^2}\Big) + (-2)\Big(\cos n\pi\cdot\frac{8}{n^3\pi^3}\Big)\Big] - \Big[0-2\Big(\frac{8}{n^3\pi^3}\Big)\Big]$$

$$a_n = 100\Big[\Big(-2\underbrace{\sin n\pi}_{0}\frac{4}{n^2\pi^2}\Big) - 2\cos n\pi\frac{8}{n^3\pi^3} + \frac{16}{n^3\pi^3}\Big]$$

$$a_n = 100\Big[-\frac{16\cos n\pi}{n^3\pi^3} + \frac{16}{n^3\pi^3}\Big]$$

$$a_n = \frac{1600}{n^3\pi^3}\big[-\cos n\pi + 1\big]$$

$$a_n = \frac{1600}{n^3\pi^3}\big[1-\cos n\pi\big]$$

Q: An insulated rod of length 'l' has its lengths A and B, 0°C and 100°C respectively until steady state (independent of time) conditions prevail. If B is suddenly reduced to 0°C and is maintained, find the temp. at a distance x from A at time t. Solve the above problem if the change consists of raising the temp of A to 20°C & reducing the temp of B to 80°C.

Ans:- $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$

$\frac{\partial u}{\partial t}=0$, $\frac{\partial^2 u}{\partial x^2}=0$: $\lambda^2=0$, $\lambda=0,0$

∴ $u(x) = c_1x+c_2$ [solution]

At $x=0$; $u(0) = c_1 \times 0 + c_2 = 0$

$\therefore c_2 = 0$

$u(x) = c_1 x$

At $x=l$; $u(l) = c_1 \times l = 100$

$\therefore c_1 = \frac{100}{l}$

$u(x) \Rightarrow \frac{100}{l} x$

$u(0,t) = 0°c$, $u(l,t) = 0°c$

$u(x,0) \Rightarrow \frac{100x}{l}$

$u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2 p^2 t}$

$c_1 = 0$, $p = \frac{n\pi}{l}$

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 c^2 t}{l^2}}$$

$$a_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$a_n = \frac{2}{l}\int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l} dx$$

$$a_n = \frac{200}{l^2}\int_0^l x \sin \frac{n\pi x}{l} dx$$

$$a_n \Rightarrow \frac{200}{l^2}\left[x\left(-\cos \frac{n\pi x}{l} \cdot \frac{l}{n\pi}\right) - 1\left(-\sin \frac{n\pi x}{l} \cdot \frac{l^2}{n^2\pi^2}\right)\right]_0^l$$

$$a_n = \frac{200}{l^2}\left[-l \cos n\pi \cdot \frac{l}{n\pi} + \sin n\pi \cdot \frac{l^2}{n^2\pi^2}\right]_0^l$$

$$a_n = \frac{200}{l^2} \cdot \left[\frac{l^2}{n\pi}\right]\left(-\cos n\pi + \cancel{\sin n\pi \cdot \frac{1}{n\pi}}\right)_0$$

$$a_n = \frac{200}{n\pi}(-\cos n\pi) = \frac{200}{n\pi}(-1)^{n+1}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200}{n\pi}(-1)^{n+1} \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2\pi^2 c^2 t}{l^2}}$$

NOTE: A long insulated rod with ends at non zero temp, if the ends points of the rod, are maintained at any temp then the temp distribution on the rod is given by $u(x,t) = u_s(x) + u_t(x,t)$ where $u_s(x)$ is a solution which is independent of time and $u_t(x,t)$ is the transient part of the temp which decreases with the increase of time.

Note 1: If $u_s(x)$ is the non-zero temp at the end points then the variable $u_t(x,t) = u(x,t) - u_s(x)$, will be such that

$u_t(x,t) = 0$ at the end points.

Note 2: The boundary conditions with non-zero temperatures is known as non-homogeneous boundary conditions.

b) $u(x,t) = u_t(x,t) + u_s(x)$

under steady state

$\frac{\partial x}{\partial t} = 0 \quad \frac{\partial^2 u}{\partial x^2} = 0$

20°C 80°C

$u_s(x) = c_1 x + c_2$

At $x = 0$,

$u_s(0) = c_1 \times 0 + c_2 = 20$

$c_2 = 20$

At $x = l$

$u_s(l) = c_1 \times l + 20 = 80$

$c_1 l = 60$

$c_1 = \frac{60}{l}$

$u_s(x) \Rightarrow \frac{60}{l} x + 20$

$u_t(x,t) = u(x,t) - u_s(x)$

$u_t(0,t) = u(0,t) - u_s(0)$

$= 20 - 20 = 0$

$u_t(l,t) = u(l,t) - u_s(l)$

$= 80 - 80 = 0$

$u_t(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-\frac{n^2\pi^2 \alpha}{l^2} t}$

Boundary conditions: $c_1 = 0$; $p = \frac{n\pi}{l}$

$u_t(x,0) = u(x,0) - u_s(x) = \frac{100x}{l} - \left(\frac{60}{l}x + 20\right) = \frac{40x}{l} - 20 = f(x)$

$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 c^2 t}{l^2}}$

$a_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$

$a_n = \frac{2}{l}\int_0^l \left(\frac{40x}{l} - 20\right) \sin \frac{n\pi x}{l}\, dx$

$a_n = \frac{40}{l}\int_0^l \left(\frac{20x}{l} - 1\right) \sin \frac{n\pi x}{l}\, dx$

$a_n = \frac{40}{l}\left[\left(\frac{20x}{l} - 1\right)\left(-\cos \frac{n\pi x}{l} \cdot \frac{l}{n\pi}\right) - \frac{20}{l}\left(-\sin \frac{n\pi x}{l} \cdot \frac{l^2}{n^2\pi^2}\right)\right]_0^l$

$a_n = \frac{40}{l}\left[-19 \cos n\pi \frac{l}{n\pi} + \frac{20}{l} \sin n\pi \cdot \frac{l^2}{n^2\pi^2} - \frac{l}{n\pi}\right]$

$a_n = \frac{40}{l}\left[\frac{l}{n\pi}(-19\cos n\pi - 1)\right] = \frac{40}{n\pi}\left(19(-1)^{n+1} - 1\right)$

$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}\left(19(-1)^{n+1} - 1\right) \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 c^2 t}{l^2}}$

10/12/21

D' Alembert's solution of Wave eqn:

wave eqn: $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Let the independent variables be $u(x,t)$ & $v(x,t)$ so that $y(x,t)$ becomes functions of u & v.

$\therefore$ D'Alembert's solution of the wave eqn is;

$$y(x,t) = \frac{1}{2}\{f(x+ct)+f(x-ct)\}$$

① Find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial ~~deflection~~ displacement $f(x) = k(\sin x - \sin 2x)$.

$\sin(A+B)+\sin(A-B)$
$\Rightarrow 2\left(\sin\frac{(A+B)}{2}\cdot\cos\frac{(A-B)}{2}\right)$

Ans: Wave eqn: $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$:

D'Alembert's solution: $y(x,t) = \frac{1}{2}\{f(x+ct)+f(x-ct)\}$

$$y(x,t) = \frac{k}{2}\{(\underbrace{\sin(x+ct)}_{A}-\sin 2(x+ct)) + (\underbrace{\sin(x-ct)}_{B}-\sin 2(x-ct)\}$$

$$y(x,t) = \frac{k}{2}\{(\sin(x+ct)+\sin(x-ct)) - (\sin(2x+2ct)+\sin(2x-2ct)\}$$

$$y(x,t) = \frac{k}{2}\{2\sin x\cos ct - \sin 2x\cdot\cos 2ct\}$$

$$\frac{\partial y}{\partial t} = k\{\sin x\cdot c(-\sin ct) - \sin 2x \times 2c(-\sin ct)\}$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$$

(3). Find the solution of the wave eqn: $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ corresponding to triangular initial deflection; $f(x) \begin{cases} = \frac{2k}{l}x, & 0<x<l/2. \\ = \frac{2k}{l}(l-x) & l/2<x<l \end{cases}$
Initial velocities are zero.

Ans: Wave eqn: $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$

D'Alemberts solution: $y(x,t) = \frac{1}{2}\{f(x+ct)+f(x-ct)\}$

$u(x,t) = (C_1\cos px + C_2 \sin px)(C_3\cos cpt + C_4 \sin cpt)$

Applying boundary conditions $\}$ $c_1 = 0$, $p = \frac{n\pi}{l}$

$\therefore u(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$

$u(x,0)\begin{cases} x, & 0<x<50 \\ 100-x, & 50<x<100\end{cases}$

Given: $\left.\frac{\partial y}{\partial t}\right|_{t=0} : 0 \quad \therefore b_n = 0$

$u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$

$a_n = \frac{2}{l}\int_0^{l/2}\frac{2kx}{l}\sin\frac{n\pi x}{l}\,dx + \frac{2}{l}\int_{l/2}^{l}\frac{2k}{l}(l-x)\sin\frac{n\pi x}{l}\,dx$

$a_n = \frac{2}{l}\left[\frac{2k}{l}\left[x\left(-\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right) + (1)\sin\frac{n\pi x}{l}\cdot\frac{l^2}{n^2\pi^2}\right]_0^{l/2} + \left[\frac{2k}{l}(l-x)\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} - (-1)\left(-\sin\frac{n\pi x}{l}\,\frac{l^2}{n^2\pi^2}\right)\right]_{l/2}^{l}\right] \Longrightarrow \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$

$\therefore u(x,t) = \sum_{n=1}^{\infty}\frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$

Qn: Find the temp. $u(x,t)$ in a homogenous bar of heat conducting material of length l, whose ends are kept at temp $0°c$ & initial temp is given by $\frac{ax(l-x)}{l^2}$.

Ans: ~~Wave eqn~~:

$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$

$(C_1\cos px + C_2\sin px)C_3 e^{-c^2p^2t}$

$u(x,t) = (C_1\cos px + C_2 \sin px)C_3 e^{-c^2p^2t}$

$u(x,t) = a_n \sin\frac{n\pi x}{l}$

Applying boundary conditions $\}$ $c_1 = 0$; $p = \frac{n\pi}{l}$

$\therefore u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2c^2}{l^2}t}$

$a_n = \frac{2}{l}\int_0^l \frac{ax(l-x)}{l^2}\sin\frac{n\pi x}{l}\,dx$

$a_n = \frac{2a}{l^3}\left[(xl-x^2)\left(-\cos\frac{n\pi x}{l}\times\frac{l}{n\pi}\right) - (l-2x)\left(-\sin\frac{n\pi x}{l}\times\frac{l^2}{n^2\pi^2}\right) + (-2)\left(\cos\frac{n\pi x}{l}\times\frac{l^3}{n^3\pi^3}\right)\right]_0^l$

$a_n = \frac{2a}{l^3}\left[\left(-2\cos n\pi\cdot\frac{l^3}{n^3\pi^3} - \left(-2\cdot\frac{l^3}{n^3\pi^3}\right)\right)\right] = \frac{4a}{n^3\pi^3}\left(1-(-1)^n\right)$

$\therefore u(x,t) = \sum_{n=1}^{\infty}\frac{4a}{n^3\pi^3}\left(1-(-1)^n\right)e^{-\frac{n^2\pi^2c^2}{l^2}t}$