MATHEMATICS 1

Theoretical questions
QP18

In the Graph Theory mid-term exam and in the optional part F1 of the final exam, one of the questions will be chosen among the following ones:

1. - Give the definition of the incidence matrix of a graph.
   - State and prove the Handshaking Lemma.
2. Let u and v be two different vertices of a graph G. Prove that if G has a u-v walk of length k, then it also has a u-v path of length at most k.
3. Prove that if a connected graph has order n and size m, then $m \geq n - 1$.
4. - Give the definition of a cut vertex.
   - State and prove the characterization of cut vertices.
5. State Ore's and Dirac's theorems on Hamiltonian graphs. For each of them, give an example showing that the condition in the theorem is not necessary, and an example showing that the inequality that appears in the condition cannot be improved.
6. - Give the definitions of trail and of Eulerian trail.
   - Prove that a connected graph has an Eulerian trail if and only if it has exactly two vertices of odd degree.
7. - Give the definition of a tree.
   - Prove that a graph T is a tree if and only if for each pair of vertices u, v there is a unique u-v path in T.
8. - State the characterization theorem for trees.
   - Prove that a tree of order $n \geq 2$ has at least two leaves.

In part F2 of the final exam, corresponding to the second part of the course (Linear Algebra), one of the questions will be chosen among the following ones:

1. - Define vector subspace.
   - Let S and S' be vector subspaces of a vector space E. Prove that $S \cap S'$ is a vector subspace of E and give an example showing that $S \cup S'$ need not be a vector subspace.

2. - Define linear independence and linear combination.
   - Let $u_1, \ldots, u_k$ be vectors in a vector space E and let us assume that u_1 is a linear combination of $u_2, \ldots, u_k$. Prove that $\langle u_1, u_2, \ldots, u_k \rangle = \langle u_2, \ldots, u_k \rangle$.

3. - Define linear independence and linear combination.
   - Prove that a set of vectors is linearly dependent if and only if one of them can be written as a linear combination of the others.

4. - Define basis of a vector space.
   - Let $B = \{b_1, \ldots, b_n\}$ be a basis of a vector space E. Prove that any vector in E can be written in a unique way as a linear combination of the elements of B.

5. Let E and F be vector spaces and let $f\colon E \to F$ be a map.
   - Explain what conditions should f satisfy in order to be a linear map.
   - Prove that if U is a vector subspace of E and f is a linear map, then $f(U)$ is a vector subspace of F.

6. Let E and F be vector spaces and let $f\colon E \to F$ be a map.
   - Explain what conditions should f satisfy in order to be a linear map.
   - Prove that f is uniquely determined by the image of any basis $\{b_1, \ldots, b_n\}$ of E.

7. Let $f : E \to F$ be a linear map between vector spaces.
   - Define the matrix of f in bases $B = \{b_1, \ldots, b_n\}$ and $W = \{w_1, \ldots, w_m\}$ of E and F, respectively.
   - Write out and explain the formula relating the matrices of f in different bases of E, F.

8. Let $f : E \to E$ be an endomorphism of a vector space E.
   - Define eigenvalue of f and eigenvector of f of eigenvalue λ.
   - Prove that $E_\lambda = \{u \in E : f(u) = \lambda u\}$ is a vector subspace of E.