2. Sequences of real numbers

Contents

2.1 Introduction

A *sequence* (of real numbers) is a mapping $a\colon D \to \mathbb{R}$, whose domain D is an infinite subset of $\mathbb{N}$. The image of $n \in D$ by a is denoted by a_n (instead of $a(n)$, the usual notation for mappings), and is called *nth term* of the sequence. The sequence a is also denoted by (a_n).

The usual way for defining a sequence (a_n) consists in giving explicitly the image of each $n \in D$ (for example, $a_n = n^2 - 3$). Nevertheless, in some contexts is more natural a definition by *recurrence*, which consists in giving the first terms $a_0, \dots, a_{k-1}$ and a relation that, for $n \geq k$, indicates how to calculate a_n from k former terms $a_{n-1}, a_{n-2}, \dots, a_{n-k}$. For instance, an *arithmetic progression* is a sequence such that each term is obtained by adding to the former one a fixed real number d called *difference*. In this case, we have a sequence defined by the first term a_1 and the recurrence $a_n = a_{n-1} + d$ for $n \geq 2$.

2.2 Bounds

Let (a_n) be a sequence. If there exists $k \in \mathbb{R}$ such that $a_n \leq k$ for all n, we say that k is an *upper bound* of (a_n) and that (a_n) is *bounded from above*; in this case the smallest upper bound is called *supremum* of (a_n). If there exists $h \in \mathbb{R}$ such that $h \leq a_n$ for all n, we say that h is a *lower bound* of (a_n) and that (a_n) is *bounded from below*; in this case the greatest lower bound is called *infimum* of (a_n). If (a_n) is bounded from above and from below, we will simply say that (a_n) is *bounded*.

2.3 Limits

The *limit* of a sequence (a_n) is

- the real number ℓ if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - \ell| < \epsilon$ for all $n \geq N$.

- $+\infty$ if for all $M > 0$ there exists $N \in \mathbb{N}$ such that $a_n > M$ for all $n \geq N$.

- $-\infty$ if for all $M < 0$ there exists $N \in \mathbb{N}$ such that $a_n < M$ for all $n \geq N$.

These three cases are denoted, respectively, by

$$\lim a_n = \ell, \qquad \lim a_n = +\infty, \qquad \lim a_n = -\infty.$$

If the limit of (a_n) is a real number ℓ, we say that the sequence (a_n) is *convergent* and it *converges* to ℓ; if the limit is $\pm\infty$, we say that the sequence is *divergent*. Any sequence not convergent or divergent is called *oscillating*. Determining the *behavior* of a sequence consists in finding out whether it is convergent, divergent or oscillating.

A first property of the convergent sequences is that they are bounded sequences. The converse is not true, for instance the sequence $a_n = (-1)^n$ is bounded but not convergent.

The three definitions of limit can be unified. Let $\Box \in \{\ell, +\infty, -\infty\}$ and (a_n) a sequence with domain D. The limit of (a_n) is $\Box$ if for each neighborhood U of $\Box$, there exists a neighborhood $(N, +\infty)$ of $+\infty$ such that, if $n \in (N, +\infty) \cap D$, then $a_n \in U$.

Some properties involve operations with two limits. If both limits are real numbers, the meaning of the operation is clear, but if one or both are $+\infty$ or $-\infty$, then the following rules apply (where the properties commutative and associative are assumed to hold)

- $(+\infty) + \ell = +\infty; \quad (-\infty) + \ell = -\infty.$
  $(+\infty) + (+\infty) = +\infty; \quad (-\infty) + (-\infty) = -\infty.$
- if $\ell > 0, \quad (+\infty) \cdot \ell = +\infty$ and $(-\infty) \cdot \ell = -\infty;$
  if $\ell < 0, \quad (+\infty) \cdot \ell = -\infty$ and $(-\infty) \cdot \ell = +\infty;$
  $(+\infty)(+\infty) = +\infty; \quad (+\infty)(-\infty) = -\infty; \quad (-\infty)(-\infty) = +\infty;$
- if $\ell > 0, \quad (+\infty)^{\ell} = +\infty;$
  $(+\infty)^{+\infty} = +\infty; \quad$ if $1 < \ell, \ \ell^{+\infty} = +\infty; \quad$ if $0 < \ell < 1, \ \ell^{+\infty} = 0.$

Limits of sequences have the following properties.

- If a sequence has limit, this limit is unique.
- If there exist $\lim a_n$ and $\lim b_n$, then $\lim(a_n + b_n) = \lim a_n + \lim b_n$, except for the case $+\infty + (-\infty)$.
- If there exist $\lim a_n$ and $\lim b_n$, then $\lim(a_n \cdot b_n) = \lim a_n \cdot \lim b_n$, except for the cases $0 \cdot (\pm\infty)$.
- If there exist $\lim a_n$ and $\lim b_n = \ell \neq 0$, then $\lim(a_n/b_n) = \dfrac{1}{\ell}\,(\lim a_n)$.
- $\lim |a_n| = +\infty \ \Leftrightarrow \ \lim(1/a_n) = 0.$
- If $\lim a_n = \Box$ and $\lim b_n = \Diamond$, and the sequence $c_n = a_n^{b_n}$ makes sense, then $\lim c_n = \Box^{\Diamond}$, except for the cases $1^{\pm\infty}$, 0^0 and $(+\infty)^0$.

The cases in which the limits of (a_n) and (b_n) are known but this does not allow to calculate directly the limits of $(a_n + b_n)$, $(a_n b_n)$, (a_n/b_n) or $(a_n^{b_n})$ are called *indeterminate forms*, which are usually represented by $\infty - \infty$, $\infty \cdot 0$, ∞/∞, $0/0$, 1^∞, 0^0 and ∞^0.

Calculating limits of sequences consists essentially in studying methods which allow to decide, for these indeterminate forms, whether the limit exists and how to calculate it.

Other limit properties are the following.

- If the limit of (a_n) is not zero, then there exists $m \in \mathbb{N}$ such that a_n has the same sign as the limit, for any $n \geq m$.
- If for some $N \in \mathbb{N}$ one has $a_n \leq b_n \leq c_n$ for all $n \geq N$, and $\lim a_n = \ell$, $\lim b_n = r$, $\lim c_n = s$, then $\ell \leq r \leq s$.
- (Squeeze theorem, also called *sandwich* rule) If for some $N \in \mathbb{N}$ one has $b_n \leq a_n \leq c_n$ for all $n \geq N$, and $\lim b_n = \ell = \lim c_n$, then $\lim a_n = \ell$.
- $\lim a_n = \ell \;\Rightarrow\; \lim |a_n| = |\ell|$; $\quad \lim |a_n| = 0 \;\Leftrightarrow\; \lim a_n = 0$.
- If $\lim a_n = 0$ and (b_n) is a bounded sequence, then $\lim a_n b_n = 0$.
- If $\lim a_n = +\infty$ and (b_n) is bounded from below, then $\lim(a_n + b_n) = +\infty$. Similarly, if $\lim a_n = -\infty$ and (b_n) is bounded from above, then $\lim(a_n + b_n) = -\infty$.
- If $\lim a_n = \pm\infty$ and (b_n) has a positive lower bound, then $\lim a_n b_n = \pm\infty$.

2.4 Monotone sequences

A sequence (a_n) is *increasing* if $a_{n+1} \geq a_n$ for all n, and it is *strictly increasing* if $a_{n+1} > a_n$ for all n. Similarly, (a_n) is *decreasing* if $a_{n+1} \leq a_n$ for all n, and *strictly increasing* if $a_{n+1} < a_n$ for all n. A *monotone* sequence is an increasing or decreasing sequence, and a *strictly monotone* sequence is a strictly increasing or strictly decreasing sequence.

The following theorem holds:

Monotone convergence theorem. Any monotone and bounded sequence is convergent.

In fact, for an increasing and bounded from above sequence, the limit is the supremum, and for a decreasing and bounded from below sequence, the limit is the infimum.

An important example of bounded and monotone sequence is

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

This sequence is strictly increasing, and bounded between 2 and 3. The proof can be consulted in many books. Its limit is an irrational number which is called Euler's number, denoted by e, and its approximate value is $2.71828183\ldots$

Related with this limit, the following properties hold.

- If (a_n) is a sequence and $\lim |a_n| = +\infty$, then

$$\lim \left(1 + \frac{1}{a_n}\right)^{a_n} = e.$$

- If (a_n) and (b_n) are sequences such that

$$\lim a_n = 1, \quad \lim |b_n| = +\infty, \quad \lim b_n(a_n - 1) = L,$$

then

$$\lim (a_n)^{b_n} = \begin{cases} 0 & \text{if } L = -\infty, \\ e^L & \text{if } L \in \mathbb{R}, \\ +\infty & \text{if } L = +\infty. \end{cases}$$

2.5 Criteria for evaluation of limits

Stolz criterium. If (a_n) and (b_n) are sequences and (b_n) is strictly increasing and

$$\lim b_n = +\infty \quad \text{and} \quad \lim \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = L \in \mathbb{R} \cup \{\pm\infty\}, \quad \text{then} \quad \lim \frac{a_n}{b_n} = L.$$

Root criterium. If (a_n) is a sequence such that $\lim \sqrt[n]{|a_n|} = L \in \mathbb{R} \cup \{+\infty\}$, one has

(i) if $L < 1$, then $\lim a_n = 0$;

(ii) if $L > 1$, then $\lim |a_n| = +\infty$.

Quotient criterium. Let (a_n) be a sequence such that for some $N \in \mathbb{N}$ we have $a_n \neq 0$ for all $n > N$. Suppose

$$\lim \frac{|a_n|}{|a_{n-1}|} = L \in \mathbb{R} \cup \{+\infty\}.$$

(i) If $L < 1$, then $\lim a_n = 0$;

(ii) if $L > 1$, then $\lim |a_n| = +\infty$.

The similarity between these two former properties suggests that there must be a relation between $\lim |a_n|/|a_{n-1}|$ and $\lim \sqrt[n]{|a_n|}$. In fact, we have

- Let (a_n) be a sequence and suppose that there exists $N \in \mathbb{N}$ such that $a_n \neq 0$ for all $n > N$. If

$$\lim \frac{|a_n|}{|a_{n-1}|} = L \in \mathbb{R} \cup \{+\infty\}, \quad \text{then} \quad \lim \sqrt[n]{|a_n|} = L.$$

Nevertheless, for some sequences (a_n) it can happen that $(\sqrt[n]{|a_n|})$ has a limit but $(|a_n|/|a_{n-1}|)$ has no limit.

2.6 Subsequences

A subsequence of a sequence (a_n) is a sequence obtained by taking an infinite number of terms of (a_n) keeping their relative position in (a_n).

For instance, if in the sequence $a_n = n^2 - 15$ we take only the terms with even index we obtain the subsequence $a_{2k} = (2k)^2 - 15 = 4k^2 - 15$.
Usually, a subsequence of (a_n) is denoted by (a_{n_k}). In the former example, $n_k = 2k$.

- A sequence is convergent and has limit ℓ if and only if all its subsequences are also convergent and have limit ℓ.

This result is used sometimes for proving that a sequence is not convergent, by finding two subsequences with distinct limits.
For instance, the sequence $a_n = (-1)^n \dfrac{n}{n+1}$ is not convergent, because the subsequences a_{2k} and a_{2k-1} have limits 1 and -1, respectively.