MATEMÀTIQUES 2
Exercises

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Prologue

The course MATEMÀTIQUES 2 is divided into two parts. The first part (until the partial exam) studies real functions of one real variable, and the second part is an introduction to real functions of several variables.

This collection of exercises is organized in lessons, each of them corresponding approximately to a topic. The exercises for each lesson have been grouped into three parts: section Problems, section Workshop, and the section You should also do.

The teachers of the course will present classical concepts and solve most of the exercises of the problems section, and together with students in class will find solutions to the exercises workshop section. The exercises in the You should also do section must be solved by the students without a teacher's help, as part of the development of their independent learning in the university education.

Mónica Sánchez Soler
(teacher coordinator)

Contents

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[11] Link to on-line courses from the Massachusetts Institute of Technology (MIT)
http://ocw.mit.edu/OcwWeb/Mathematics/index.htm

[12] Link to the *Calculus with Applications* course from the MIT. It includes interactive lessons with java
http://ocw.mit.edu/ans7870/18/18.013a/textbook/MathML/index.xhtml

[13] Digital book: "Introduction to Real Analysis" by William F. Trench
http://ramanujan.math.trinity.edu/wtrench/misc/index.shtml

[14] Website of the professor Willian Chen with different mathematical courses.
http://www.maths.mq.edu.au/~wchen/ln.html

Part I

Real valued functions of one real variable

Chapter 1

$\mathbb{R}$, the set of real numbers.

1.1 Problems

1 Find all the real numbers x satisfying the following inequalities

$$\text{a) } x^2 > 3x + 4; \qquad \text{b) } 1 < x^2 < 4; \qquad \text{c) } \frac{1}{x} < x.$$

For each section, represent on the real number line the set of solutions and say if that set is bounded from above (bounded from below). If so, find the supremum (infimum).

2 Find all the real numbers x such that:

$$\text{a) } x^3 - 1 \geq 0; \qquad \text{b) } (x-1)\left|x^2 - 2\right| > 0; \qquad \text{c) } |4x - 5| \leq 13;$$

For each section, represent on the real number line the set of solutions and say if that set is bounded from above (bounded from below). If so, find the supremum (infimum).

3 Find all the real numbers x such that:

$$\text{a) } |x-3| = 2; \quad \text{b) } |x+1| < 4; \quad \text{c) } |x-1| + |x+3| = 4; \quad \text{d) } |x+1| + |x+2| < 2.$$

For each section, represent on the real number line the solution set and say if that set has a maximum or minimum.

4 Show that if $|x| \leq 1$, then we have $\left|x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 + \frac{1}{8}x + \frac{1}{16}\right| < 2$.

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1.2 Workshop

5 Solve the following inequalities:

$$\text{a) } \frac{x-1}{x+1} < 0; \qquad \text{b) } \frac{1}{x+3} > \frac{1}{4}; \qquad \text{c) } \frac{x-1}{x+1} \le \frac{x+1}{x-1}; \qquad \text{d) } x^2 + x \le 0.$$

For each section, represent on the real number line the set of solutions and say if that set is bounded from above (bounded from below). If so, find the supremum (infimum).

6 Find all the real numbers x satisfying the following inequalities:

$$\text{a) } |2x+7| \ge 3; \qquad \text{b) } |x^2-1| \le 3; \qquad \text{c) } |x-1| > |x+1|; \qquad \text{d) } |x| + |x+1| < 2.$$

For each section, represent on the real number line the set of solutions and say if that set is bounded from above (bounded from below). If so, find the supremum (infimum).

7 For each of the following sets:

$$\text{a) } \{x \in \mathbb{R} \mid x^3 - 4x < 0\}; \quad \text{b) } \{x \in \mathbb{R} \mid \exists n \in \mathbb{N},\ x = 2^{-n}\}; \quad \text{c) } \{y \in \mathbb{R} \mid \exists x \in \mathbb{R},\ y = 1 + x^2\},$$

determine if the set is bounded from above, from below, if it's bounded, or if it's not. Find the supremum and the infimum, if applicable.

8 Absolute error. If we know $\bar{x}$, the approximate value of a real number instead of the real number x, we say we are making an error. The first measurement of this error is called absolute error, defined by $e_a(x) = \bar{x} - x$. In general, since we don't know the exact values of the errors, we work with error bounds. We say that ϵ_a is an absolute error bound if $|e_a(x)| \le \epsilon_a$.

a) Give absolute error bounds for $x = \pi$ and $\bar{x} = 22/7$.

b) Show that $\bar{x} - \epsilon_a \le x \le \bar{x} + \epsilon_a$ (usually written as $x = \bar{x} \pm \epsilon_a$).

c) If an error bound in $\bar{x} = 3789.405$ is three thousandths, find in which interval is the exact value x.

d) If $x = 3789.405 \pm 0{,}003$ and $y = 2523.106 \pm 0{,}002$, find an interval for $x + y$ and an interval for $x - y$.

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1.3 You should also do

9 Let a and b be real numbers with $a < b$. Show that $a < \dfrac{(a+b)}{2} < b$.

10 Let $a \geq 0$ and $b \geq 0$ be real numbers.

i) Show that $\sqrt{ab} \leq \dfrac{a+b}{2}$.

ii) Show that the inequality is an equality if and only if, $a = b$.

11 Write the following expressions without the absolute values:

a) $|x-1| - |x|$; b) $\big||x| - 1\big|$; c) $|x| - |x^2|$; d) $x - \big|x + |x|\big|$.

12 Show that for every $x \in \mathbb{R}$ the following inequality is true: $|x-1| + |x-2| \geq 1$. In which case this inequality is an equality?

13 Find the real numbers x such that:

a) $|x-1||x+2| = 3$ b) $\dfrac{1}{4} \leq \left|x^2 - 5x + 6\right| \leq 3$.

For each section, represent on the real number line the solution set and say if that set has a maximum or minimum.

14 Solve the following inequalities:

a) $\left|\dfrac{2x-2}{x+4}\right| < 1$; b) $\left|\dfrac{x}{x-2}\right| > 10$; c) $|3x-5| - |2x+3| > 0$;
d) $|2 - x^2| < 1$; e) $x^2 < |2x+8|$; f) $\left|x^2 - 5x\right| > \left|x^2\right| - |5x|$.

For each section, represent on the real number line the solution set and say if that set, in each case, has a maximum or minimum.

15 Let

$$A = (-3, 9], \qquad B = \mathbb{N}, \qquad C = (4, +\infty), \qquad D = \{x \,|\, x \in \mathbb{Q} \wedge 0 \leq x \leq \sqrt{2}\}.$$

Find, in case they exist, the supremum and the infimum for the sets A, B, C, D, $A \cap B$, $A \cap C$, $B \cap (C \cup A)$. Say if these are the maximum and minimum.

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1.4 Solutions

1 a) $A = (-\infty, -1) \cup (4, +\infty)$; unbounded.
b) $B = (-2, -1) \cup (1, 2)$; bounded from above, $\sup(B) = 2$ and bounded from below, $\inf(B) = -2$.
c) $C = (-1, 0) \cup (1, +\infty)$; bounded from below with $\inf(C) = -1$.

2 a) $A = [1, +\infty)$; bounded from below, $\inf(A) = 1$.
b) $B = (1, \sqrt{2}) \cup (\sqrt{2}, +\infty)$; bounded from below, $\inf(B) = 1$.
c) $C = [-2, \frac{9}{2}]$; bounded, $\inf(A) = -2$ and $\sup(A) = \frac{9}{2}$.

3 a) $A = \{1,\ 5\}$; it's a bounded set, $min(A) = 1$ and $max(A) = 5$.
b) $B = (-5, 3)$; it's a bounded set, $\sup(C) = -5$ and $\inf(C) = 3$ and there is not a maximum or minimum.
c) $C = [-3, 1]$; it's a bounded set, $min(D) = -3$ and $max(D) = 1$.
d) $D = (-5/2, -1/2)$; it's a bounded set, $\sup(E) = -1/2$ and $\inf(E) = -5/2$ and there is not a maximum or minimum.

5 a) $A = (-1, 1)$; it's a bounded set, $\inf(A) = -1$ and $\sup(A) = 1$.
b) $B = (-3, 1)$; it's a bounded set, $\sup(B) = 1$ and $\inf(B) = -3$.
c) $C = (-1, 0] \cup (1, \infty)$; it's a set bounded from below with $\inf(C) = -1$.
d) $D = [-1, 0]$; it's a bounded set, $\sup(D) = 0$ and $\inf(D) = -1$.

6 a) $A = (-\infty, -5] \cup [-2, +\infty)$; it's not a bounded set and there is not a supremum or infimum.
b) $B = [-2, 2]$; bounded $\sup(B) = 2$ and $\inf(B) = -2$.
c) $C = (-\infty, 0)$; bounded from above with $\sup(C) = 0$.
d) $D = (-\frac{3}{2}, \frac{1}{2})$; bounded $\sup(D) = 1/2$ and $\inf(D) = -3/2$.

7 a) $A = (-\infty, -2) \cup (0, 2)$; bounded from above, $\sup(A) = 2$; not bounded from below.
b) B it's bounded, $\sup(B) = 1/2$ and $\inf(B) = 0$.
c) $C = [1, +\infty,)$ bounded from below with $\inf(C) = 1$, not bounded from above.

8 a) If we take $\frac{22}{7} \approx 3.1$ an absolute error bound is 0.05; if we take $\frac{22}{7} \approx 3.14$ an absolute error bound is 0.002.
c) $3789.402 \leq \bar{x} \leq 3789.408$.
d) $x + y = 6312.511 \pm 0.005$, $x - y = 1266.299 \pm 0.005$.

11 a) $|x-1| - |x| = \begin{cases} 1 & \text{for } x < 0, \\ -2x+1 & \text{for } 0 \leq x < 1, \\ -1 & \text{for } x \geq 1; \end{cases}$ c) $|x| - |x^2| = \begin{cases} -x - x^2 & \text{for } x < 0, \\ x - x^2 & \text{for } x \geq 0; \end{cases}$

b) $\big||x|-1\big| = \begin{cases} -1-x & \text{for } x<-1, \\ 1+x & \text{for } -1\le x<0, \\ 1-x & \text{for } 0\le x<1, \\ -1+x & \text{for } x\ge 1; \end{cases}$

d) $x-\big|x+|x|\big| = \begin{cases} x & \text{for } x<0, \\ -x & \text{for } x\ge 0. \end{cases}$

12 We get the equality for $1 \le x \le 2$.

13 a) $A = \left\{\frac{-1-\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}\right\}$; it's a bounded set, $\max(A) = \frac{-1+\sqrt{21}}{2}$ and $\min(A) = \frac{-1-\sqrt{21}}{2}$.
b) $B = \left[\frac{5-\sqrt{13}}{2}, \frac{5-\sqrt{2}}{2}\right] \cup \left[\frac{5+\sqrt{2}}{2}, \frac{5+\sqrt{13}}{2}\right] \cup \frac{5}{2}$; ; it's a bounded set, $\max(B) = \frac{5+\sqrt{13}}{2}$, $\min(B) = \frac{5-\sqrt{13}}{2}$.

14 a) $A = (-2/3, 6)$; it's a bounded set, $\inf(A) = -2/3$ and $\sup(A) = 6$ and there is not a maximum or a minimum.
b) $B = (20/11, 2) \cup (2, 20/9)$; it's a bounded set $\inf(B) = 20/11$ and $\sup(B) = 20/9$ and there is not a maximum or a minimum.
c) $C = (-\infty, 2/5) \cup (8, +\infty)$; it's not a bounded set and there is not a maximum or a minimum.
d) $D = (-\sqrt{3}, -1) \cup (1, \sqrt{3})$; it's a bounded set, $\inf(D) = -\sqrt{3}$ and $\sup(D) = \sqrt{3}$ and there is not a maximum or a minimum.
e) $E = (-2, 4)$; it's a bounded set, $\inf(E) = -2$, $\sup(E) = 4$ and there is not a maximum or a minimum.
f) $F = (-\infty, 0) \cup (0, 5)$; it's a bounded set from above, $\sup(F) = 5$ and there is not a maximum or a minimum.

15 $\inf(A) = -3,\ \sup(A) = \max(A) = 9.$
$\inf(B) = \min(B) = 1.$
$\inf(C) = 4.$
$\inf(D) = \min(D) = 0,\ \sup(D) = \sqrt{2},$
$\inf(A\cap B) = \min(A\cap B) = 1,\ \sup(A\cap B) = \max(A\cap B) = 9.$
$\inf(A\cap C) = 4,\ \sup(A\cap C) = \max(A\cap C) = 9.$
$\inf(B\cap(C\cup A)) = \min(B\cap(C\cup A)) = 1.$

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Chapter 2

Sequences of real numbers.

2.1 Problems

1 Find the limit of the following sequences:

$$\text{a) } \alpha^n, \quad \forall \alpha \in \mathbb{R}; \qquad \text{b) } n^\alpha, \quad \forall \alpha \in \mathbb{R}; \qquad \text{c) } \sqrt[n]{n}.$$

2 Find the limit of the following sequences:

$$\text{a) } \frac{6n^3+4n+1}{2n}; \qquad \text{b) } \frac{n^2-6n-2}{3n^2-9n}; \qquad \text{c) } \left(\sqrt{\frac{n+1}{2n+1}}\right)^{\frac{2n-1}{3n-1}}.$$

3 Use the sandwich theorem for finding, if possible, $\lim_{n\to\infty} b_n$ where the general term of $\{b_n\}_{n\in\mathbb{N}}$ is

$$b_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}.$$

4 Find the limit of the following sequences:

$$\text{a) } \lim_{n\to+\infty} \frac{a^n}{n!}, \quad |a|>1; \qquad \text{b) } \lim_{n\to+\infty} \frac{n^\alpha}{a^n}, \quad |a|>1, \quad \alpha \in \mathbb{R}^+.$$

5 Find the limit of the following sequences:

$$\text{a) } \frac{\cos n}{n^2}; \qquad \text{b) } \frac{2^n+3^n}{2^n-3^n}; \qquad \text{c) } \frac{2^n}{n!}; \quad \text{d) } \frac{n^7}{7^n}; \quad \text{e) } \left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{\tfrac{n+1}{2}}.$$

6 Let $\{a_n\}$ be the sequence defined by $a_1 = -2/3$ and $3\,a_{n+1} = 2 + a_n^3$ for $n \geq 1$.

a) Show that $-2 \leq a_n \leq 1$, for all $n \geq 1$.

b) Show that $\{a_n\}$ is an increasing sequence.

c) Show that $\{a_n\}$ is convergent and find its limit.

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2.2 Workshop

7 Find the limit of the following sequences:

a) $\dfrac{n}{n^2+1}+\dfrac{n}{n^2+2}+\cdots+\dfrac{n}{n^2+n}$; b) $\dfrac{1}{n^2}+\dfrac{2}{n^2}+\cdots+\dfrac{n}{n^2}$;

c) $\dfrac{n!}{n^n}$; d) $\left(\dfrac{n^2+n+1}{n^2-n+1}\right)^{\dfrac{n^2+2}{n+1}}$.

8 Find the following limits:

a) $\displaystyle\lim_{n\to+\infty}\frac{2^n\cdot 3^n+5^{n+1}}{(2^n+1)(3^{n-1}-1)}$; b) $\displaystyle\lim_{n\to+\infty}\frac{5(n+1)^{n+1}}{(3n^2+1)n^{n-1}}$;

c) $\displaystyle\lim_{n\to+\infty}\left(\frac{n+2}{2n}\right)^{\sin(1/n)}$; d) $\displaystyle\lim_{n\to+\infty}\left(\sqrt[5]{\frac{2n+3}{3n+4}}\right)^{\left(\frac{n^3+1}{n^3+n}\right)^{n^2+1}}$;

e) $\displaystyle\lim_{n\to+\infty}\frac{\sqrt[n]{(n+1)(n+2)\dots(2n)}}{n}$; f) $\displaystyle\lim_{n\to+\infty}\frac{\sqrt{3n^3+2n+2}-\sqrt{3n^3-2n-1}}{\sqrt{n^3+n^2+3n}-\sqrt{n^3+n^2-3n}}$.

9 Let $\{a_n\}$ be the sequence defined by $a_1=1$ and $a_n=\sqrt{1+a_{n-1}}$ for $n>1$.

a) Show that $0<a_n<2,\quad \forall n\geq 1$.

b) Show that $\{a_n\}$ is increasing.

c) Show that $\{a_n\}$ is convergent and find its limit.

10 Let $\{a_n\}$ be the sequence defined by $a_n=\dfrac{1\cdot 3\cdot 5\dots(2n-1)}{2\cdot 4\cdot 6\dots(2n)}$ for $n\geq 1$.

a) Find a_1, a_2 i a_3. Get a recurrent relation, kind $a_{n+1}=f(a_n)$.

b) Prove that $\{a_n\}$ is decreasing.

c) State the monotone convergence theorem.

d) Prove that $\{a_n\}$ is convergent.

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2.3 You should also do

11 Find a and b in each of the following inequalities:

a) $\lim_{n\to+\infty}\left(\dfrac{1-an^2}{3n^2-2}\right)^{1-bn^2}=\sqrt{e}\,;$

b) $\lim_{n\to+\infty}\left(\dfrac{n+a}{n+2}\right)^{an+b}=\lim_{n\to+\infty}\left(\dfrac{n+b}{n+2}\right)^{2n+a}.$

12 Find $\lim_{n\to+\infty}\dfrac{\sqrt{n+a}-\sqrt{n+b}}{\sqrt{n+c}-\sqrt{n+d}}$, where a, b, c and d are real numbers, with $c \neq d$.

13 Show that the sequence with general term

$$a_n=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}, \quad \text{for all } n\geq 1,$$

is convergent, and give an interval with length less or equal than $1/2$ containing the value of the limit.

14 Let $\{a_n\}$ be the sequence defined by

$$\left\{\sqrt{3}, \sqrt{3+\sqrt{3}}, \sqrt{3+\sqrt{3+\sqrt{3}}}, \ldots\right\}$$

a) Find a recurrence relation satisfied by this sequence.

b) Show that $\{a_n\}$ is bounded.

c) Show that $\{a_n\}$ is increasing.

d) Show that $\{a_n\}$ is convergent and find its limit.

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2.4 Solutions

1 a) $+\infty$ for $\alpha > 1$; 1 for $\alpha = 1$; 0 for $-1 < \alpha < 1$; $\nexists$ for $a = -1$; ∞ for $\alpha < -1$(this means $\lim|\alpha^n| = +\infty$).
b)$+\infty$ for $\alpha > 0$; 1 for $\alpha = 0$; 0 for $\alpha < 0$. c) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

2 a) $+\infty$, b) $1/3$, c) $\sqrt[3]{1/2}$.

3 $\dfrac{n}{\sqrt{n^2+n}} < b_n < \dfrac{n}{\sqrt{n^2+1}}$; $\lim b_n = 1$.

4 a) 0; b) 0.

5 a) 0, b) -1, c) 0, d) 0, e) $\dfrac{\sqrt{2}}{4}$.

6 c) $l = 1$.

7 a) 1, b) $\dfrac{1}{2}$, c) 0, d) e^2.

8 a) 3, b) $\dfrac{5}{3}e$ c) 1 d) $\sqrt[5e]{\dfrac{2}{3}}$, e) $\dfrac{4}{e}$ f) $\dfrac{2}{9}\sqrt{3}$.

9 $\dfrac{1+\sqrt{5}}{2}$.

10 a) $a_{n+1} = \dfrac{2n+1}{2n+2}a_n$.

11 a) $a = -3$ and $b = -1/2$; or $b = 0$ and $a = -3\sqrt{e}$;
b) They must fullfill the condition $2b - 4 = a(a-2)$.

12 $\dfrac{a-b}{c-d}$.

13 $1/2 \le l \le 1$.

14 a) $a_1 = \sqrt{3}$ and $a_n = \sqrt{3 + a_{n-1}}$; d) $\dfrac{1+\sqrt{13}}{2}$

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Chapter 3

Theorems on continuous functions.

3.1 Problems

1 Show that the equation $x^3 - 3x^2 + 1 = 0$ has only one solution in the interval $[0, 2]$.

2 Let $a, b \in \mathbb{R}$, with $a < b$, and let f and g be two continuous functions in $[a, b]$ with $f(a) < g(a)$ and $f(b) > g(b)$. Show that there exists $c \in (a, b)$ verifying $f(c) = g(c)$.

3 Show that if $f : [0, 1] \longrightarrow [0, 1]$ is a continuous function, then there exists at least one $\xi \in [0, 1]$ such that $f(\xi) = \xi$.

4 Can we assure that the function $f(x) = \dfrac{x^3}{4} - \sin \pi x + 3$ takes the value 2.5 in the closed interval $[-2, 2]$?

5 Explain why the equation $e^{-x^2} = 2x$ has a solution in the interval $[0, 1]$ and solve it approximately with a precision of 0.1.

6 a) Separate the two real solutions of the equation $x - 3 \ln x = 0$.

b) Apply the bisection method for evaluating both solutions with precision $0.5 \cdot 10^{-3}$.

c) Apply the secant method for evaluating both solutions with a precision of three correct decimals.

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3.2 Workshop

This workshop session will be held in classrooms with PCs on LCFIB, and together with the session of next chapter

3.3 You should also do

7 Show that for some real value of x the following equality holds: $x^{163} + \dfrac{134}{1 + x^4 + \cos^2 x} = 113$.

8 Is it possible to apply the Bolzano's theorem to the function $f(x) = \dfrac{1}{1 - e^{\frac{1}{x}}}$ in the interval $[-1, 1]$? Show that the equation $f(x) + \dfrac{1}{2} = 0$ has a solution, and find an interval with length less or equal than $1/3$ containing it.

9 Find all the solutions of:

a) The equation $10x^3 - 10x + 1 = 0$ with a precision of 0.000005.

b) The equation $\sin 2x = \cos 3x$ in the interval $[0, 2\pi]$. Which precision do you take?

10 Show that:

a) the equation $x^3 - 2x^2 + 3 = 2$ has a solution in the interval $[-1, 2]$.

b) the equation $\sin x = x - 1$ has a solution in the interval $[1, 2]$.

c) the equation $x^2 = x \cdot \sin x + \cos x$ has a positive solution and a negative solution

d) the equation $2x^4 - 14x^2 + 14x - 1 = 0$ has four solutions.

e) the equation $\ln x = x^2 - 4x$ has a solution in the interval $[1, +\infty)$

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3.4 Solutions

5 0.4.

6 a) $[1, 2]$ and $[4, 5]$.
b) 1.8569 and 4.5366
c) 1.8572 and 4.5364

8 The interval can be, for instance, $[3/4, 1]$.

9 a)-1.046680, 0.101032 and 0.945652. b) There are six solutions. With a precision of 0.00005 They are 0.31416, 1.57080, 2.82743, 4.08407, 4.71239 and 5.34070.

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Chapter 4

Theorems on derivable functions.

4.1 Problems

1 Find the point of the parabola $y = x^2$ at which the tangent line is parallel to the segment AB containing the points $A(1,1)$ and $B(3,9)$.

2 Show that the equation $3^{-x} = x$ has only one solution. Which is the integer part of this solution?

3 Show that the polynomial function $f_m(x) = x^3 - 3x + m$ doesn't have two roots in $[0,1]$, $\forall m \in \mathbb{R}$.

4 Consider the equation

$$e^{-x} = \ln x. \tag{4.1}$$

a) Show that the equation (4.1) has a solution in the set $[1, +\infty)$.

b) Give an interval of length 0.1 containing this solution.

c) Reason why the given equation can't have two solutions in $[1, +\infty)$.

d) Apply Newton-Raphson method with the initial value $x_0 = 1$ to determine the positive root. Stop the calculation when the difference between two consecutive iterations is less than 10^{-4}. How many iterations do we need in this case?

5 Evaluate the following limits: a) $\displaystyle\lim_{x\to+\infty} \frac{\ln x}{\sqrt{x}}$; b) $\displaystyle\lim_{x\to 0^+} x \ln x$

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4.2 Workshop (I)

This workshop session will be held in classrooms with PCs on LCFIB.

6 Consider the polynomial functions

a) $f(x) = x^3 - x + 5$; b) $g(x) = x^5 + 4x^3 - 2x + 2$; c) $h(x) = 4x^4 - 5x - 1$.

i) For each one of them find the number k such that the interval $[k, k+1]$ contains some solution of the equation $f(x) = 0$.

ii) By applying the bisection method, find for each case a solution of $f(x) = 0$ with one decimal precision ($\eta = 0.05$).

iii) By applying the secant method, find for each case a solution of $f(x) = 0$ with a precision of 0.0005.

iv) By applying the tangent method (Newton–Raphson's method), find the smallest positive solution for each case with a precision of $\eta = 0.0005$.

4.3 Workshop (II)

7 Let $f : [0,1] \to [0,1]$ be a continuous and derivable function such that $f'(x) \neq 1$ for all $x \in [0,1]$. Show that there exists an unique $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

8 Consider the equation

$$e^x = \frac{1}{2}x + 2. \tag{4.2}$$

a) Show that the equation (4.2) has a positive and a negative solution in the interval $[-5, 2]$.

b) Show that the equation (4.2) has only two real solutions.

c) Determine, without doing **any** iteration, the number of iterations needed in the the bisection method for evaluating the positive solution of the equation (4.2) with an absolute error less than 10^{-8}.

9 Find the following limits by using l'Hôpital's rule:

a) $\lim\limits_{x\to+\infty} \dfrac{e^x}{x^5}$; b) $\lim\limits_{x\to+\infty} x^{\frac{1}{x}}$; c) $\lim\limits_{x\to0^+} x^{\sin x}$;

d) $\lim\limits_{x\to0} \left(\dfrac{a^x + b^x}{2}\right)^{\frac{1}{x}}$; e) $\lim\limits_{x\to0} \dfrac{\ln(1+x)}{x}$; f) $\lim\limits_{x\to0} \dfrac{1}{x} \ln\sqrt{\dfrac{1+x}{1-x}}$;

g) $\lim\limits_{x\to0} (\cos x)^{1/x}$; h) $\lim\limits_{x\to+\infty} x^{\tan\frac{1}{x}}$; i) $\lim\limits_{x\to+\infty} \dfrac{\ln(1+x^\alpha)}{\ln(1+x^\beta)}$ si $(\alpha > \beta > 0)$.

10 Find the following limits without using l'Hôpital's rule:

$$\text{a) } \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\sin x}; \qquad\qquad \text{b) } \lim_{x\to\infty} \frac{x+\sin x}{x-\sin x}.$$

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4.4 You should also do

11 Use the mean value theorem for showing that the following statements are true

$$\text{a) } \arctan x > \frac{x}{1+x^2} \quad \text{for} \quad x>0\,; \qquad \text{b) } \ln(1+x) \leq x \quad \text{for} \quad x \geq 0\,;$$

$$\text{c) } \arccos x > \frac{\pi}{2} - \frac{x}{\sqrt{1-x^2}} \quad \text{for} \quad 0<x<1\,.$$

12 Study the number of real roots of the polynomial $f(x) = 2x^3 - 3bx^2 + 8$ according to the values of the parameter b. For $b = 3$, find the approximate value of the smallest root with two correct decimal numbers.

13 Let a, b and c be real numbers and suppose $a^2 < 3b$. Show that the graph of the function $f(x) = x^3 + ax^2 + bx + c$ intersects exactly once the abscissa axis.

14 Apply the iterative Newton–Raphson's method for evaluating the two solutions of the equation $x - 3\ln x = 0$, with six decimals precision.

15 By using the Newton–Raphson's method, solve the following equations getting the results with six decimals precision:

a) $x - \cos x = 0$;	b) $10^x = 6x + 30$;	c) $x^2 - 1 = \sin x$;
d) $x^x = 10$;	e) $5\sin x = x + \dfrac{1}{x}$;	f) $xe^x = 1$;
g) $e^x = 5x + 10$,	h) $\ln x = 1 + \dfrac{1}{x}$;	i) $2x^2 - 10x + 10 + \cos\dfrac{x}{9} = 0$.

Problems from pre-universitary courses

16 Determine the point of the graph of the function $f(x) = (x-2)^2$ where the tangent is perpendicular to the line with equation $2x - y + 2 = 0$.

17 Find a, b and c such that the graphs of the functions $f(x) = x^2 + ax + b$ and $g(x) = x^3 - c$ intersect at the point $(1, 2)$ and have the same tangent line at that point.

18 Evaluate and simplify the derivatives of the following functions:

a) $y = \ln \dfrac{x^2-1}{x^2+1}$; b) $y = \arctan \sqrt{\dfrac{1-x}{1+x}}$; c) $y = \dfrac{1}{\sqrt{3}} \arctan \dfrac{2x+1}{\sqrt{3}}$; d) $y = \arcsin \dfrac{1}{\sqrt{x}}$;

e) $y = \ln \sqrt{x^2+x+1}$; f) $y = e^{\sin((2x)^{\frac{1}{3}})}$; g) $y = \sin(\ln x)$; h) $y = x^{x^{\ln x}}$.

19 Determine and classify the critical points of the function $f(x) = e^{\,8x-a(x^2+16)}$ according to the values of the parameter a ($a \in \mathbb{R}$). Does f have asymptotes? Find them according to the values of a.

20 Decompose the number 100 into two addends so that the sum of their cubes is minimal.

21 Find the absolute extrema of the following functions at the indicated intervals:

a) $f(x) = x^5 + 4x^4 + x^3 - 10x^2 - 4x + 8$ in $[-1, 2]$;

b) $f(x) = \dfrac{x}{4} + \sqrt[3]{x^2}$ in $[-20, 1]$; c) $f(x) = \dfrac{\ln x}{x}$ in $[1, e^2]$.

22 Determine the intervals of increasing and decreasing, and the local extrema, if they exist, for each one of the following functions in their domains of definition:

a) $f(x) = \ln(x^2 - 9)$, $|x| > 3$; b) $f(x) = x^{2/3}(x-1)^4$, $0 \le x \le 1$.

23 Find the intervals of increasing, concavity and convexity of the function $f(x) = \dfrac{x^3}{12 + x^2}$.

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4.5 Solutions

1 $(2, 4)$

2 0

4 *Mid-semester exam, 9/11/2011.*

5 a) 0, b) 0.

6 i) a) $[-2, -1]$, b) $[-1, 0]$, c)$[-1, 0]$ and $[1, 2]$.
ii) a) -1.91, b) -0.91, c)-0.22 and 1.16.
iii) a) -1.9042 b) -0.9256 c)-0.1988 and 1.1370.
iv) a) -1.9042 b) -0.9256 c)-0.1988 and 1.370.

8 *Mid-semester exam, 13/04/2011.*

9 a) $+\infty$, b) 1, c) 1, d) $\sqrt{ab}$, e) 1, f) 1 g) 1, h) 1, i) α/β.

10 a) 0, b) 1.

12 For $b < 2$, there's one solution; for $b = 2$ there are two solutions and for $b > 2$ there are three
If $b = 3$, the approximate value of the smallest root with precision of two decimals is -0.864.

14 1.8571839 and 4.5364037.

15 a) 0.7390851, b) 1.5975328, and 4.9999983, c) 1.4096240 and -0.6367327,
d) 2.5061841, e) ± 0.5141686 and ± 2.5189457, f) 0.5671433,
g) 3.2718121 and -1.9721691, h) 3.5911215, i) 1.6292701 and 3.3859854.

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Chapter 5

Taylor's theorem

5.1 Problems

1 Use a 2–degree Taylor's polynomial of the function $f(x) = \sqrt[3]{1728 + x}$ for evaluating $\sqrt[3]{1731}$. Bound the error made.

2 Consider the function $f(x) = \ln(1 - x)$.

a) Find the five first non-null terms of the Taylor polynomial centered at the origin and the corresponding Lagrange remainder.

b) Determine the degree of the Taylor polynomial of $f(x)$ for obtaining the value of $\ln 0.75$ with an error less than 10^{-3}.

3 Give an upper bound for the error in the formula $e \approx 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$ by using the Taylor formula for e^x.

4 By using the error propagation formula for functions of one variable:

a) Find the approximate value of the area of a rectangle of height 0.5cm, when the length of the basis is $r = 2.5\,cm \pm 0.05\,cm$. Give an upper bound of the error of this approximation.

b) Find the approximate value of the area of a rectangle of height 1cm, when the length of the basis is $r = 2.5\,cm \pm 0.05\,cm$. Give an upper bound of the error of this approximation.

c) Find the approximate value of the area of a rectangle of height 2cm, when the length of the basis is $r = 2.5\,cm \pm 0.05\,cm$. Give an upper bound of the error of this approximation.

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5.2 Workshop

5 Find the five first non-null terms of the Taylor polynomial centered at the origin and the Lagrange remainder of the following functions:

$$\begin{array}{lll} \text{a) } f(x) = e^x; & \text{b) } f(x) = \sin(x); & \text{c) } f(x) = \cos(x); \\ \text{d) } f(x) = \ln(1+x); & \text{e) } f(x) = \sqrt{1+x}; & \text{f) } f(x) = \sinh x; \\ \text{g) } f(x) = \dfrac{1}{1-x}; & \text{h) } f(x) = \dfrac{1}{1+x}; & \text{i) } f(x) = \dfrac{1}{(1-x)^2}; \end{array}$$

Represent graphically, using some mathematical software, the function and the Taylor polynomials obtained.

6 Evaluate with three correct decimals ($error \leq \frac{1}{2}10^{-3}$) the following amounts:

a) $e^{0.25}$; b) $\sin(-0.2)$; c) $\cos(0.9)$; d) $\ln(1.1)$; e) $\ln(0.9)$; f) $\sqrt{1.05}$; g) $\sqrt{0.97}$; h) $1/\sqrt{e}$.

7 Which precision should be used to measure the radius of a circle $R \approx 30.5\,cm$ so that the circle's surface, $S = \pi R^2$, has an error less than 0.1% ?
Note: π is represented exactly and the percentage refers to the relative error.

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5.3 You should also do

8 Let $f(x) = \sqrt{1+x}$.

a) Find the 2–degree Taylor polynomial for the function $f(x)$ at $x = 0$.

b) By using the polynomial of section **a)** calculate an approximate value of $\sqrt{1.02}$.

c) Give an upper bound on the error made in the calculation of section **b)**.

9 Show, by using the Taylor formula at 0, the equivalences between the pairs of the following equivalent infinitesimals:

$$\begin{array}{lll} \text{a) } \sin x \cong x \quad (x \to 0)\,; & \text{b) } 1-\cos x \cong \dfrac{x^2}{2} \quad (x \to 0)\,; & \text{c) } \tan x \cong x \quad (x \to 0)\,; \\ \text{e) } e^x - 1 \cong x \quad (x \to 0)\,; & \text{f) } a^x - 1 \cong x \ln a \quad (x \to 0)\,; & \text{g) } \ln(1+x) \cong x \quad (x \to 0)\,. \end{array}$$

$\Big($ The expression $f(x) \cong g(x)$ $(x \to a)$ means that $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = 1$ and $\displaystyle\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0\Big)$.

10 Find the following limits by using Taylor's theorem and/or infinitesimal equivalences:

a) $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x}$; b) $\displaystyle\lim_{x\to a} \frac{\sin x - \sin a}{x^2 - 4ax + 3a^2}$; c) $\displaystyle\lim_{x\to 1/2} (2x^2 - 3x + 1)\tan(\pi x)$.

d) $\displaystyle\lim_{x\to 0^+} \frac{\sin 3x}{2 - 2\cos x}$; e) $\displaystyle\lim_{x\to 0} \frac{(1-\cos x)\arctan x}{x\sin^2 x}$; f) $\displaystyle\lim_{x\to 0} \frac{(\cosh x - 1)\ln(1+x) - \frac{x^3}{2}}{x(\sin x - \arcsin x)}$.

11 Find the Taylor polynomials

a) of degree 3 at the origin for the function $f(x) = e^x \tan x$.

b) of degree 6 at the origin for the function $f(x) = \sqrt{1-x^2}$.

c) of degree 3 at the origin for the function $f(x) = e^{\cos x}$.

d) of degree 2 at $x = \dfrac{\pi}{2}$ for the function $f(x) = \ln(\sin x)$.

12 A string fixed by its extremes describes a curve named *catenary*, with the equation $f(x) = a\cosh(x/a)$. Show that, for small values of x, the catenary can be approximated by the parabola $y = a + x^2/(2a)$.

13 Consider the function $f(x) = \dfrac{1}{\sqrt{1-x}}$.

a) Find the 1–degree Taylor polynomial for the function $f(x)$ at the point $x_0 = 0$. Justify your answer.

b) Write the remainder term for the error made when considering the 1–degree Taylor polynomial obtained instead of the irrational function $f(x)$.

c) Give an upper bound on the error for $|x| < \dfrac{1}{16}$ in the approximation $\dfrac{1}{\sqrt{1-x}} \approx 1 + \dfrac{x}{2}$.

d) In the special theory of relativity the mass m of a particle that moves with speed v is defined as $m = \dfrac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$, where m_0 is the particle's mass in rest and c is the speed of light. Using the sections **(a)** and **(c)** justify that for small speeds compared to the speed of light $m \approx m_0\left(1 + \dfrac{v^2}{2c^2}\right)$.

14 Find the approximate value of a sphere's volume,

$$V = \frac{4}{3}\pi r^3,$$

if the radius measures $r = 2.5\,cm \pm 0.05\,cm$ and π is represented exactly. Give an upper bound for the error made in this approximation. (General error propagation formula).

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5.4 Solutions

1 $P_2(x) = 12 + \dfrac{x}{432} - \dfrac{x^2}{2239488}, \quad \epsilon \leq 0.4 \cdot 10^{-8}.$

2 a) $P_5(x) = -x - x^2/2 - x^3/3 - x^4/4 - x^5/5$, b) $n \geq 4$.

3 $1/80$.

4 a) area $\approx 1.25 \pm 0.025$, .b) area $\approx 2.5 \pm 0.05$, c) area $\approx 5 \pm 0.1$,

5 a) $P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4, \quad R_4(x) = \frac{x^5}{120}e^c;$

b) $P_9(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9, \quad R_{10}(x) = -\frac{x^{11}}{39916800}\cos(c);$

c) $P_8(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8, \quad R_9(x) = -\frac{x^{10}}{3628800}\cos(c);$

d) $P_5(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^5 + \frac{1}{5}x^5, \quad R_5(x) = -\frac{x^6}{6(c+1)^6};$

e) $P_4(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4, \quad R_4(x) = \frac{7\,x^5}{256(c+1)^{9/2}};$

f) $P_9(x) = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \frac{1}{362880}x^9, \quad R_{10}(x) = -\frac{x^{11}}{39916800}\cosh(c);$

g) $P_4(x) = 1 + x + x^2 + x^3 + x^4, \quad R_4(x) = \frac{x^5}{(1-c)^6};$

h) $P_4(x) = 1 - x + x^2 - x^3 + x^4, \quad R_4(x) = -\frac{x^5}{(1+c)^6};$

i) $P_4(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4, \quad R_4(x) = \frac{6x^5}{(1-c)^7}.$

6 a) 1.2840, b) -0.1987, c) 0.6216, d) 0.0950, e) -0.1050, f) 1.0247, g) 0.9849, h) 0.6065.

7 $1.5 \cdot 10^{-2}$.

8 *Mid-semester exam, 16/11/2010, afternoon group*

10 a) 1, b) $\dfrac{-\cos a}{2a}$, c) $\dfrac{1}{\pi}$, d) $+\infty$, e) $1/2$, f) $+\infty$.

11 a) $x + x^2 + \frac{5}{6}x^3$, b) $1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6$, c) $e - \frac{e}{2}x^2$, d) $-\frac{1}{2}\left(x - \frac{\pi}{2}\right)^2$.

13 *Final exam, 17/01/2012.*

14 $V \approx 20.83\pi \pm 1.25\pi$.

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Chapter 6

Integration.

6.1 Problems (I)

1 Show that the function $y = E[x]$ (integer part of x) is integrable on $[0,5]$ and evaluate $\displaystyle\int_0^5 E[x]\,dx$.

2 Find the derivative of the following functions:

a) $f(x) = \displaystyle\int_3^x \sin\ln t\,dt,\ x > 3;$ b) $g(x) = \displaystyle\int_x^{10} \sin\ln t\,dt,\ x > 0;$

c) $h(x) = \displaystyle\int_0^{\ln(x)} \sin t^3\,dt,\ x > 0;$ d) $s(x) = \displaystyle\int_{x^2+3x}^{x^4+2x+1} e^{\sin t}\,dt.$

3 Find the following limits:

a) $\displaystyle\lim_{x\to 0} \frac{\displaystyle\int_0^{x^2} \sin\sqrt{t}\,dt}{x^3};$ b) $\displaystyle\lim_{x\to 0} \frac{x\displaystyle\int_0^x e^{t^2}\,dt}{\displaystyle\int_0^x e^{t^2}\sin t\,dt}.$

4 Let $f : (0,+\infty)\setminus\{1\} \to \mathbb{R}$ defined by $f(x) = \displaystyle\int_x^{x^2} \frac{dt}{\ln t}$. Show that f is strictly increasing on $(0,1)$ and also on $(1,+\infty)$.

5 Let $F(x) = \displaystyle\int_{-x}^x t^2 e^{t^2}\,dt$. Show that F' is an even function and study the concavity of F.

6 Consider the parabolas $y = x^2$ and $y = x^2 - 4x + 6$.

a) Show that the line $y = x - 1/4$ is tangent to both of them.

b) Find the area between the two parabolas and the line $y = x - 1/4$.

7 Evaluate the following integral $I = \displaystyle\int_0^4 (1 - e^{\,x/4})\; dx\,.$

(a) By using the Barrow's rule (fundamental theorem of calculus).

(b) By using the trapezoidal method with 4 sub-intervals.

(c) By using the Simpson's method with 4 sub-intervals.

(d) Evaluate the absolute errors produced in sections **(b)** and **(c)** to approximate the value of the integral I. Discuss the results obtained.

(e) Find upper bounds on the errors made in the calculations of sections **(b)** and **(c)** by using the methods' error formulae. Discuss the obtained results.

8 Consider the integral $I = \displaystyle\int_{0.6}^{1.0} \big(\sin(x)\cos(x)\big)^{4/3}\; dx\,.$

(a) Knowing that $0 < f^{(4)}(x) < 20$, $\forall x \in [0.6, 1.0]$, find the number of sub-intervals needed to get the value of the integral using the Simpson's rule with a precision of at least 4 decimal digits $(0.5 \cdot 10^{-4})$.

(b) Give an approximate value of the integral I with the precision required in section **a)**.

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6.2 Workshop (I)

9 Find the following immediate integrals:

a) $\displaystyle\int \left(\frac{1-x}{x}\right)^2 dx;$ b) $\displaystyle\int \frac{x^3}{x^4+1}\, dx;$ c) $\displaystyle\int \sqrt{\frac{\arcsin x}{1-x^2}}\, dx;$ d) $\displaystyle\int x\sqrt{x}\, dx;$

e) $\displaystyle\int \frac{1}{x\,\ln x}\, dx;$ f) $\displaystyle\int x5^{2x^2}\, dx;$ g) $\displaystyle\int \frac{1}{1+16x^2}\, dx$ h) $\displaystyle\int \tan^2 x\, dx.$

10 Find the area of the region between the parabola $y = x^2 + 7$ and the line $y = 10$.

11 Find the area of the enclosed region between the abscissae axis, the curves $y = e^x$ and $y = e^{-x}$ and the lines $x = 2$ and $x = -2$.

12 Calculate by parts:

a) $\int e^{2x}\sin x\,dx;$ b) $\int \frac{\ln x}{\sqrt{x}}\,dx;$ c) $\int \arcsin x\,dx;$ d) $\int x\sin 2x\,dx.$

13 Find the area in the fourth quadrant region limited by the curve $y=(x^2-x)e^{-x}$ and the axis of abscissas.

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6.3 Workshop (II)

This workshop session will be held in classrooms with PCs on LCFIB.

14 Use the trapezoidal method and the Simpson's method with 4 sub-intervals for evaluating the integrals

$$a)\ \int_1^{\pi/2}\sqrt{\sin x}\,dx\,; \qquad b)\ \int_0^{0.5}\cos x^2\,dx.$$

15 Use the trapezoidal method and the Simpson's method with 4 sub-intervals for evaluating the following integrals with an error less than 10^{-3}. Give also later a better upper bound for the error on each calculation.

$$\text{a)}\ \int_0^2 (1+x^4)^{\frac{1}{2}}\,dx\,; \qquad \text{b)}\ \int_0^1 \cos(x^2)\,dx\,; \qquad \text{c)}\ \int_0^1 \frac{1}{1+x^3}\,dx\,.$$

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6.4 You should also do

16 Compute the following integrals by substitution (change of variable):

$$\text{a)}\ \int \frac{\sqrt{x}}{\sqrt{x}-x^{1/3}}\,dx; \qquad \text{b)}\ \int \frac{e^x}{\sqrt{4-e^{2x}}}\,dx; \qquad \text{c)}\ \int \frac{\cos x}{1+\sin^2 x}\,dx.$$

17 Evaluate the following integrals:

$$\text{a)}\ \int_0^1 \left(x^2+\frac{1}{1+x^2}\right)dx\,; \qquad \text{b)}\ \int_2^4 x\ln x\,dx\,; \qquad \text{c)}\ \int_0^{\pi} \cos x\cosh x\,dx\,.$$

$$\text{d)}\ \int_0^1 \sinh x\cosh x\,dx\,; \qquad \text{e)}\ \int_1^3 x\,\mathrm{e}^x\,dx\,.$$

18 Find the area between the curves $y = x^3/2$, $y = x^2 - 2x + 4$ and the ordinate axis.

19 Find the area enclosed by the curve $y = |x^2 - 4x + 3|$ and the lines $x = 0,\ x = 4$ and $y = 0$.

20 Find the area of the region limited by the graphs of:

a) $y = 1/x^2,\ y = 0,\ 1 \le x \le 4;$

b) $y = e^{-x},\ y = 0,\ 0 \le x \le 1;$

c) $y = (x-1)(x-2)(x-3),\ y = 0,\ x = 1$ i $x = 3.$

21 Find the area between the curves $y = 2/(1+x^2)$ and $y = x^2$.

22 Find the area of one of the regions between the curves $y = \sin x$ and $y = \cos x$.

23 Compute, with an error less than 10^{-3}, the following integrals

a) $\displaystyle\int_0^1 \sin^3 x\,dx$; b) $\displaystyle\int_{-1}^1 \ln(1+x^4)\,dx$; c) $\displaystyle\int_2^4 \frac{x}{2+x^6}\,dx$;

d) $\displaystyle\int_2^3 x\tanh x\,dx$; e) $\displaystyle\int_3^7 \ln x \sin x\,dx$; f) $\displaystyle\int_0^{\frac{\pi}{2}} \ln(1+\sin x)\,dx$.

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6.5 Solutions

1 10.

2 a) $\sin(\ln x)$; b) $-\sin(\ln x)$; c) $\dfrac{1}{x}\sin((\ln x)^3)$;

d) $(4x^3+2)e^{\sin(x^4+2x+1)} - (2x+3)e^{\sin(x^2+3x)}$.

3 a) $\dfrac{2}{3}$, b) 2 .

5 If $x<0$, $F''(x)<0$; if $x>0$, $F''(x)>0$; $x=0$ is an inflection point.

6 b) $\dfrac{2}{3}$.

7 *Final exam, 17/01/2012.*

8 *Mid-semester exam, 16/11/2010 morning group.*

9 a) $x-\dfrac{1}{x}-2\ln x+k$; b) $\dfrac{1}{4}\ln(x^4+1)+k$; c) $\dfrac{2}{3}(\arcsin x)^{\frac{3}{2}}+k$; d) $2\dfrac{\sqrt{x^5}}{5}+k$;

e) $\ln(\ln(x))+k$; f) $\dfrac{5^{2x^2}}{4\ln 5}+k$; g) $\dfrac{1}{4}\arctan(4x)+k$; h) $\tan(x)-x+k$.

10 $4\sqrt{3}$.

11 $2\left(1-\dfrac{1}{e^2}\right)$.

12 a) $\dfrac{1}{5}e^{2x}(2\sin x-\cos x)+k$; b) $2\sqrt{x}\ln(x)-4\sqrt{x}+k$;

c) $\sqrt{1-x^2}+x\arcsin x+k$; d) $\dfrac{1}{4}\sin(2x)-\dfrac{x}{2}\cos(2x)+k$.

13 $\dfrac{3-e}{e}$.

14 a) $T(4)=0.555$, $S(4)=0.555162$.
b) $T(4)=0.497$, $S(4)=0.49688$.

15 a) $T(44) = 3.654$ and $\epsilon_{44} \leq 0.00097$; $S(8) = 3.653$ and $\epsilon_8 \leq 0.00061$;
b) $T(18) = 0.904$ and $\epsilon_{18} \leq 0.00099$; $S(4) = 0.905$ and $\epsilon_4 \leq 0.00092$;
c) $T(13) = 0.835$ and $\epsilon_{13} \leq 0.00086$; $S(4) = 0.836$ and $\epsilon_4 \leq 0.00073$;

16 a) $6\sqrt[6]{x} + 3\sqrt[3]{x} + 2\sqrt{x} + \frac{3}{2}\sqrt[3]{x^2} + \frac{6}{5}\sqrt[6]{x^5} + x + 6\ln(\sqrt[6]{x} - 1) + k$;

b) $\arcsin\left(\frac{e^x}{2}\right) + k$; c) $\arctan(\sin x) + k$.

17 a) $\frac{1}{3} + \arctan 1$; b) $14\ln 2 - 3$; c) $-\frac{1}{2}\sinh\pi$; d) $-\frac{1}{2}\left(1 - \cosh(1)^2\right)$; e) $2e^3$.

18 $\frac{14}{3}$.

20 a) $\frac{3}{4}$, b) $1 - \frac{1}{e}$, c) $\frac{1}{2}$.

23 a) $T(13) = 0.180$, $S(4) = 0.179$; b) $T(18) = 0.323$, $S(6) = 0.322$;
c) $T(12) = 0.015$, $S(6) = 0.014$; d) $T(4) = 2.463$, $S(2) = 2.463$;
e) $T(96) = -2.597$, $S(12) = -2.598$; f) $T(18) = 0.743$, $S(12) = 0.743$.

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Part II

Real valued functions of several variables

Chapter 7

Functions of several variables.

7.1 Problems

1 Consider the sets:

$$A = \left\{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 < 1\right\}.$$

$$B = \left\{(x, y) \in \mathbb{R}^2 : \ |y| \le x^2, \ y \ne 0, \ x \in [-2, 2]\right\}.$$

a) Draw these sets.

b) Find the boundary, the interior and the closure of these sets.

c) Are they open sets? Are they closed sets?

d) Are they compact sets?

2 Find and draw the domain of the following functions:

a) $f(x, y) = \ln(1 + xy)$; b) $g(x, y) = \sqrt{y \sin x}$.

3 For each of the following functions, draw the level curves (also called *contour lines*)

a) $z(x, y) = x^2 - y^2$; b) $z(x, y) = 1 - |x| - |y|$,

corresponding to levels $z = -2, -1, 0, 1, 2$.

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7.2 Workshop

4 Draw the following subsets of $\mathbb{R}^2$:

a) $A = \big\{(x, y) \in \mathbb{R}^2 :\ |x - 3| < 2,\ |1 - y| \leq 5\big\}$;

b) $B = \big\{(x, y) \in \mathbb{R}^2 :\ |x^2 + 4x + 1| = -x^2 - 4x - 1,\ |y - 2| < 10\big\}$;

c) $C = \big\{(x, y) \in \mathbb{R}^2 :\ x^2 + y^2 \leq 1,\ x < y\big\}$.

5 Consider the sets:

$$A = \big\{(x, y) \in \mathbb{R}^2 :\ x^2 - y^2 < 1\big\} ;$$

$$B = \big\{(x, y) \in \mathbb{R}^2 :\ x > 0,\ y > 0,\ xy \leq 1\big\} ;$$

$$C = \big\{(x, y, z) \in \mathbb{R}^3 :\ x + y + z = 1,\ x^2 + y^2 + z^2 \leq 1\big\} .$$

a) Draw these sets.

b) Find the boundary, the interior and the closure of these sets.

c) Which of these sets are open? Which ones are closed? Which ones are compact?

6 Find and draw the domain of the following functions:

a) $f(x, y) = x^2 - y^2$; b) $g(x, y) = \sqrt{1 - x^2 - y^2}$; c) $h(x, y) = \ln(x + y)$.

7 For each one of the following functions, draw their level curves

a) $z(x, y) = x^2 y$; b) $z(x, y) = x^2 + y^2 - 1$; c) $z(x, y) = |x + y| + |x - y|$;

corresponding to levels $z = -2, -1, 0, 1, 2$.

8 Check that the parabola $y = ax^2$ is a level curve for the function

$$f(x, y) = \frac{x^4 y^4}{(x^4 + y^2)^3} .$$

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7.3 Solutions

1 a) In this picture, the points in the set are in black and grey, and in red those points that are not.

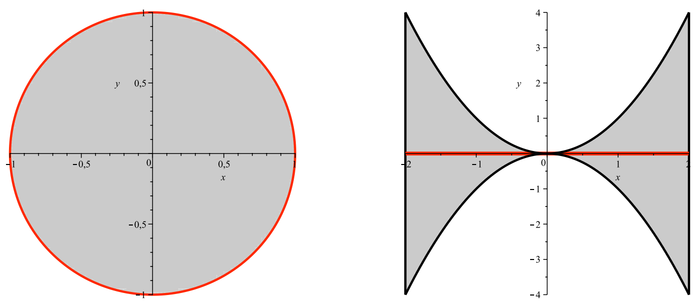


Figure 7.1: Set A (left) and set B (right).

b) $\text{int}(A) = A$, $\text{adh}(A) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, $\text{Fr}(A) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
$\text{int}(B) = \{(x,y) \in \mathbb{R}^2 : |y| < x^2,\, y \neq 0,\, x \in (-2,2)\}$,
$\text{adh}(B) = \{(x,y) \in \mathbb{R}^2 : |y| \leq x^2,\, x \in [-2,2]\}$,
$\text{Fr}(B) = \{(x,y) \in \mathbb{R}^2 : |y| = x^2 \wedge x \in [-2,2]\} \cup \{(x,y) \in \mathbb{R}^2 : x = \{-2,2\} \wedge -4 \leq y \leq 4)\} \cup \{(x,y) \in \mathbb{R}^2 : y = 0 \wedge -2 \leq x \leq 2)\}$.

c) The set A is open, not closed. The set B is neither open or closed.

d) Neither the set A nor the set B are compact.

2 In this picture, the points in the set are in black and grey, and in red those points that are not.

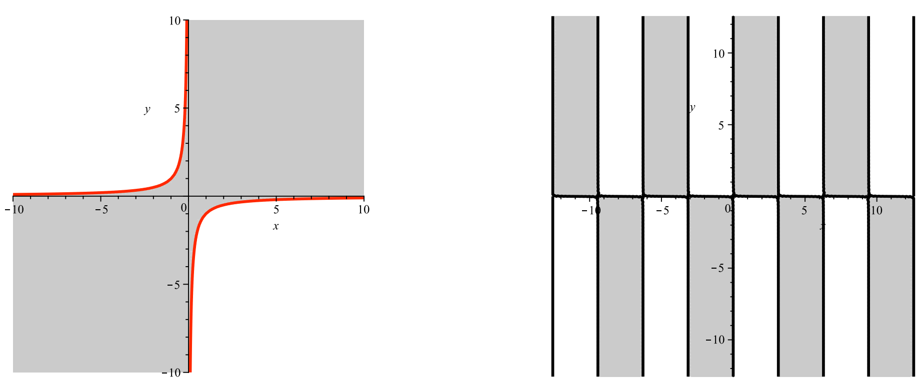


Figure 7.2: $Dom(f)$ (left) and $Dom(g)$ (right).

a) $Dom(f) = \{(x,y) \in \mathbb{R}^2 : x > 0, y > -1/x\} \cup \{(x,y) \in \mathbb{R}^2 : x < 0, y < -1/x\} \cup$
$\cup \{(x,y) \in \mathbb{R}^2 : x = 0\}$.
b) $Dom(g) = \{(x,y) \in \mathbb{R}^2 : 2n\pi \leq x \leq (2n+1)\pi,\ y \geq 0,\ \forall n \in \mathbb{N}\} \cup$
$\cup \{(x,y) \in \mathbb{R}^2 : (2n-1)\pi \leq x \leq (2n)\pi,\ y \leq 0,\ \forall n \in \mathbb{N}\}$.

3 In the graphical representation, each contour line is drawn in a different colour.

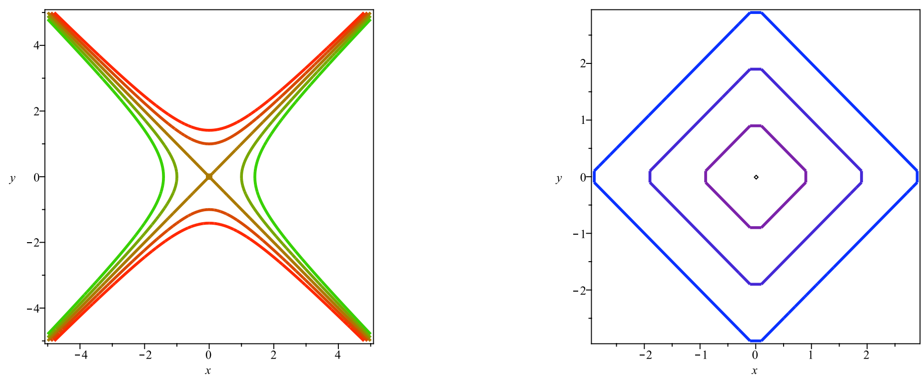


Figure 7.3: Sets $z = x^2 - y^2$ (left) and sets $z = 1 - |x| - |y|$ (right).

4 In this picture, the points in the set are in black and grey, and in red those points that are not.

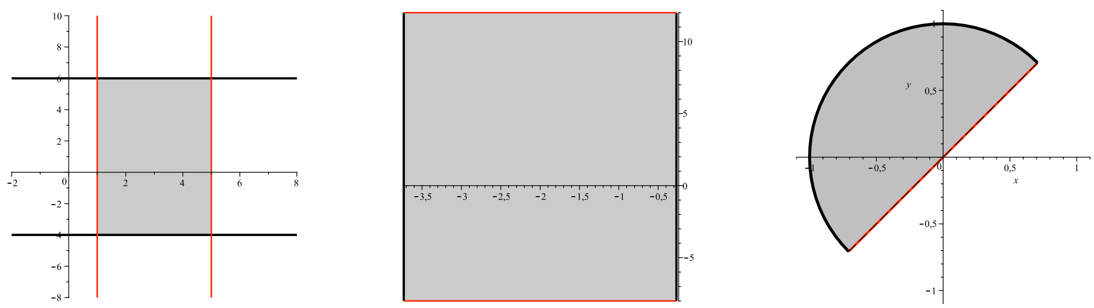
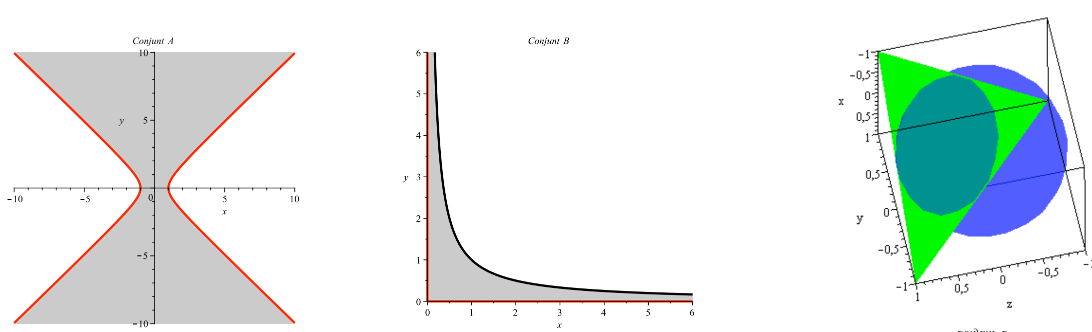


Figure 7.4: Set A (left), set B (middle) and set C (right).

5 a) Set A (left), set B (middle) and set C (right).



b) $\mathrm{Fr}(A) = \left\{(x,y) \in \mathbb{R}^2 :\ x^2 - y^2 = 1\right\}$; $\mathrm{int}(A) = A$, $\mathrm{adh}(A) = \left\{(x,y) \in \mathbb{R}^2 :\ x^2 - y^2 \le 1\right\}$.
$\mathrm{Fr}(B) = \left\{(x,y) \in \mathbb{R}^2 :\ x > 0,\ y > 0,\ xy = 1\ \text{ or }\ x = 0,\ y \ge 0\ \text{ or }\ y = 0,\ x \ge 0\right\}$;

$\operatorname{int}(B) = \left\{(x,y) \in \mathbb{R}^2 : x > 0,\ y > 0,\ xy < 1\right\}$, $\operatorname{adh}(B) = \left\{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ xy \leq 1\right\}$.
$\operatorname{Fr}(C) = \operatorname{adh}(C) = C$ i $\operatorname{int}(C) = \emptyset$.
c) The set A is open, the set B is neither open or closed and the set C is closed and compact.

6 a) All the points of $\mathbb{R}^2$.
b) $\operatorname{Dom}(g) = \left\{(x,y) \in \mathbb{R}^2 :\ x^2 + y^2 \leq 1\right\}$.
c) $\operatorname{Dom}(h) = \left\{(x,y) \in \mathbb{R}^2 :\ x + y > 0\right\}$.

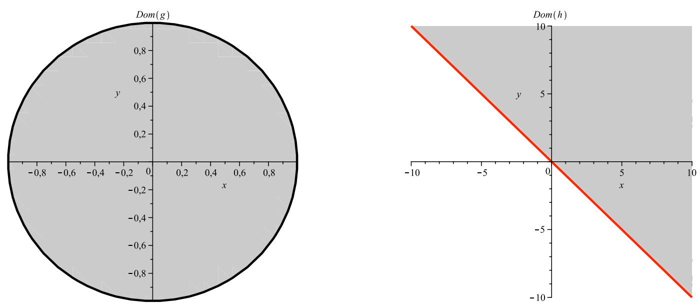


Figure 7.5: Set Dom(g) (left) and set Dom(h) (right).

7 In this picture, each contour line has a different colour.

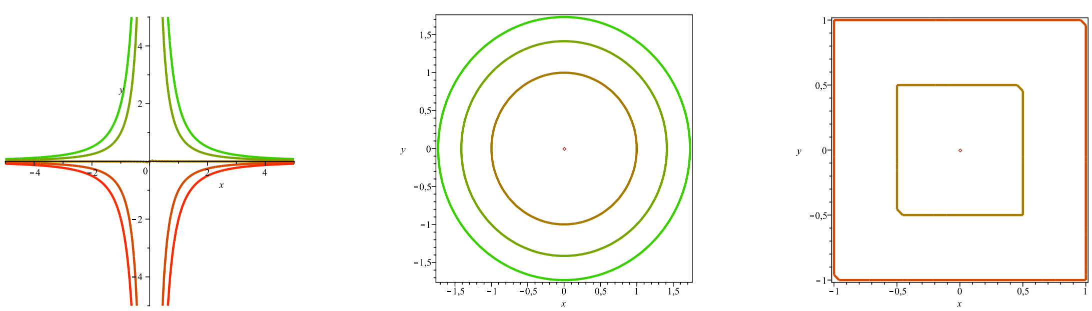


Figure 7.6: Sets $z = x^2 y$ (left), sets $z = x^2 + y^2 - 1$ (middle) and sets $z = |x+y| + |x-y|$ (right).

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Chapter 8

Derivatives. Gradient.

8.1 Problems

1 Find the first order partial derivatives of the function $f(x, y) = (\sin x)^{\sin y}$.

2 Given $f(x, y) = x^2+y^2$, compute the directional derivative of the function f at the point $P = (2, 3)$ along the vector $\overrightarrow{v} = (3/5, 4/5)$.

3 Find the derivative of the function $z = x^2 - y^2$ at the point $M(1, 1)$ in the direction that forms an angle of $\pi/3$ with the positive direction of the OX axis.

4 Compute the values of a, b, c such that the value of the directional derivative of the function

$$f(x, y, z) = axy^2 + byz + cz^2x^3$$

at the point $(1, 2, -1)$ has a maximum value of 64 in a parallel direction to the OZ axis.

5 Write the equations of the tangent plane and normal line to:

a) the surface $z = x^2 + y^2$, at the point $M = (1, 2, 5)$;

b) the surface $z = \arctan \dfrac{y}{x}$, at the point $M = \left(1, 1, \dfrac{\pi}{4}\right)$.

6 Let $f(x, y) = 4x + 2y - x^2 + xy - y^2$. Find the points of the surface $z = f(x, y)$ in which the tangent plane is parallel to the plane XY.

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8.2 Workshop

7 Find the gradient of the following functions:

a) $f(x,y,z) = \ln(z + \sin(y^2 - x))$ at the point $(1,-1,1)$;

b) $f(x,y,z) = e^{3x+y}\sin(5z)$ at the point $(0,0,\pi/6)$;

c) $f(x,y,z) = \displaystyle\int_x^{xy+z^2} \frac{\sin t}{t}\,dt$ at the point $(\pi/2,1,0)$.

8 Consider the function $f(x,y) = x^2 + (y-1)^2 - 1$.

a) Do a sketch of the level curves of $z = f(x,y)$ corresponding to levels $z = -2,-1,0,3$.

b) Which is the direction where $f(x,y)$ increases faster at the point $P = (-1,3)$? Find the directional derivative of $f(x,y)$ in this direction.

c) Which is the direction where $f(x,y)$ decreases faster at the point $P = (-1,3)$? Find the directional derivative of $f(x,y)$ in this direction.

d) Which is the direction where $f(x,y)$ is constant at the point $P = (-1,3)$? Find the directional derivative of $f(x,y)$ in this direction.

9 A temperature distribution on the plane is given by the function

$$f(x,y) = 10 + 6\cos x\cos y + 3\cos 2x + 4\cos 3y\,.$$

Find the direction of the biggest temperature increase and of the biggest decrease at the point $(\pi/3,\pi/3)$.

10 Find the normal line and the tangent plane to

a) the surface $z = \dfrac{2xy}{x^2+y}$ at the point $(2,-2,-4)$.

b) the surface $z = \sin x + 2\cos y$ at the point $(\pi/2,0,3)$.

11 Show that all the tangent planes to the surface $z = x\sin(x/y)$ contain the origin.

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8.3 You should also do

12 Determine the equation of the tangent plane to the surface with equation $z = e^{x\cos y}$ at the point $(-1, \pi/4, e^{-\sqrt{2}/2})$.

13 Find the derivative of the function $z = x^2 - xy + y^2$ at the point $M(1,1)$ in the direction that forms an angle α with the direction positive at the OX axis. In which direction this derivative:

a) attains its maximum value?

b) attains its minimum value?

c) is equal to 0?

14 Which is the difference between the gradient of the function

$$f(x,y,z) = x + y + z$$

at the point $M(1,2,2)$ and the gradient of the function

$$g(x,y,z) = x + y + z + 0.001\sin(10^6\pi\sqrt{x^2+y^2+z^2})$$

at the same point?

15 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function with continuous partial derivatives such that $\dfrac{\partial f}{\partial x}(0,0) = 1$ and $f(x,x) = 3$ for all $x \in \mathbb{R}$.

a) Show that the derivative of f at the point $(0,0)$ in the direction of the first quadrant bisector is zero.

b) By using the result of the previous section, determine in which direction does the directional derivative of f at the origin is maximum and the value of this derivative.

16 Which is the direction where $f(x,y) = x^2 + xy + y^2$ increases faster at the point $(-1,1)$? Find the directional derivative of $f(x,y)$ in this direction.

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8.4 Solutions

1 $f'_x(x,y) = \cos x \, \sin y \, (\sin x)^{\sin y - 1}, \quad f'_y(x,y) = \cos y \, \ln(\sin x) \, (\sin x)^{\sin y}$

2 $\dfrac{36}{5}$.

3 $1 - \sqrt{3}$.

4 $(a,b,c) = (6, 24, -8)$ or $(a,b,c) = (-6, -24, 8)$.

5 a) $2x + 4y - z - 5 = 0, \quad \dfrac{x-1}{2} = \dfrac{y-2}{4} = \dfrac{z-5}{-1}$.
b) $x - y + 2z - \dfrac{\pi}{2} = 0, \quad 2(x-1) = 2(1-y) = z - \dfrac{\pi}{4}$.

6 $\left(\dfrac{10}{3}, \dfrac{8}{3}, \dfrac{28}{3}\right)$.

7 a) $(-1, -2, 1)$, b) $\left(\dfrac{3}{2}, \dfrac{1}{2}, -\dfrac{5\sqrt{3}}{2}\right)$, c) $(0, 1, 0)$.

8 (*Final exam 14/06/2011.*)

9 The direction of the biggest temperature increase is $(-3, -1)$ and of the biggest decrease is $(3, 1)$.

10 a) The equation of the normal line is $\dfrac{x-2}{6} = \dfrac{y+2}{4} = \dfrac{z+4}{-1}$ and the equation of the tangent plane is $z = 6x + 4y - 8$.
b) The equation of the normal line is $x = \pi/2$, $y = 0$ and the equation of the tangent plane is $z = 3$.

12 $z = e^{-\sqrt{2}/2} + \dfrac{\sqrt{2}}{2} e^{-\sqrt{2}/2}(x+1) + \dfrac{\sqrt{2}}{2} e^{-\sqrt{2}/2}\left(y - \dfrac{\pi}{4}\right)$.

13 $cos\alpha + sin\alpha$; a) $(1,1)$; b) $(-1,-1)$; c) $(1,-1)$ and $(-1,1)$.

16 The direction is $(-1, 1)$ and the value of the derivative is $\sqrt{2}$.

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Chapter 9

Taylor's theorem. Local extrema.

9.1 Problems

1 Given the function $f(x,y) = \ln(1+2x+3y)$:

a) Write the 2–degree Taylor polynomial for f at the point $(0,0)$.

b) By using the obtained polynomial, compute an approximate value for $f(1/10, 1/10)$ and bound the error.

2 You are asked to,

a) Write the equation of the tangent plane to the surface of $\mathbb{R}^3$ defined by the equation: $z = \sqrt[3]{x\,y}$, at the point $P\,(1,\,1,\,1)$.

b) Evaluate approximately, by using a 1-degree Taylor polynomial, the number $\sqrt[3]{0.99 \cdot 1.01}$. Find the error of this approximation, in other words, give an upper bound of the remainder term of this calculation.

c) Let $a = 1 \pm 0.01$ and $b = 1 \pm 0.01$. Use the error propagation formula for determinig an upper bound of the error made in the approximation $\sqrt[3]{a \cdot b} \approx 1$.

3 Find the local extrema of the following functions. In some of the critical points, the determinant of the hessian matrix is zero, and therefore, it is required to compute the type of the critical point by using directly the definitions of maximum, minimum or saddle point.

a) $f(x,y) = x^3 + y^3 - 9xy + 27\,;$

b) $f(x,y) = (x^2 - 2x + 4y^2 - 8y)^2\,;$

c) $f(x,y) = y^2 - x^3\,;$

d) $f(x,y) = x^2y^2\;(1 - x - y)\,.$

4 Find the values of a and b for which the function $f(x,y) = ax^3 + 3bxy^2 - 15a^2x - 12y + 5$ has a local minimum at $(2,1)$.

9.2 Workshop (I)

5 Find the first and second order partial derivatives of the following functions:

a) $x^4 + y^4 - 4x^2y^2$; b) $\ln(x^2 + y^2)$; c) $x\,y + \dfrac{x}{y}$; d) $\arctan \frac{x}{y}$;

e) $x \sin(x + y)$; f) $(x^2 + y^2)e^{x+y}$; g) $x^{\frac{y}{z}}$; h) $x\,y\,z\,e^{x+y+z}$.

6 Find the 2–degree Taylor polynomial for the function $f(x, y) = xy^2 + \sin xy$ at the point $(1, \pi/2)$.

7 Let $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$ be defined by

$$f(x, y) = 1 + x^3 + y^2 + 2\int_0^{3x} \sqrt{1 + t^2}\, dt + x\int_0^{y^2} e^{t^2/2}\, dt.$$

Find the 2–degree Taylor polynomial for f at $(0, 0)$.

8 By using 2–degree Taylor polynomials, compute approximately:

a) $\sqrt{1.03 + 2.98}$; b) $\sqrt[3]{0.98 \times 1.02}$; c) $0.95^{2.01}$.

9 The measures of the sides of a rectangle are $a = 5\,m \pm 0.1\,m$ i $b = 20\,m \pm 0.5\,m$. Find the approximate area of the rectangle, $S = a\,b$, and an upper bound of the error in the approximation. (General formula of error propagation)

10 The measures of the sides of a rectangle are $a \approx 5\,m$ and $b \approx 20\,m$. What is the limit of the permissible absolute error in these mesures (the same for both) so that the rectangle's area can be determined with an error bound of $1\ m^2$?

⋄⋄⋄⋄⋄

9.3 Workshop (II)

This workshop session MAY be held in classrooms with PCs on LCFIB.

11 Check that $(0, 0)$ is a saddle point for the function $f(x, y) = \left(x^2 + (y - 1)^2 - 1\right)\left(x^2 - 2y\right)$.

12 Find and classify the critical points of the following functions:

a) $f(x, y) = x^2 + y^2 + x + y + xy$;

b) $f(x, y) = \sin x\ \sin y$;

c) $f(x,y) = \dfrac{x+y}{1+x^2+y^2}$;

d) $f(x,y) = (x-1)^4 + (x-y)^4$;

e) $f(x,y) = x^4 + y^4 + 4xy - 2x^2 - 2y^2$;

f) $f(x,y) = x^3 - x^2y + 3y^2$;

g) $f(x,y) = xy^2\,(3-x-y)$.

13 Let $f(x,y) = e^{\lambda x + y^2} + \mu \sin(x^2+y^2)$ with $\lambda, \mu \in \mathbb{R}$. Determine the values of the parameters λ and μ such that f has at $(0,0)$ a local extreme, and that the second degree Taylor polynomial for f at the origin takes the value 6 at the point $(1,2)$. With the results obtained, which kind of extreme is the point $(0,0)$ for f?

14 Given the function $f(x,y) = \sqrt{x^2+y^2} - xy$,

a) Find the local extrema of f on $\mathbb{R}^2 \setminus \{(0,0)\}$.

b) By analyzing the expression of f, find out if f has a local extremum at $(0,0)$.

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9.4 You should also do

15 Find the 2–degree Taylor polynomials at the origin for the following functions:

a) $f(x,y) = e^{x+y}$; b) $f(x,y) = \sin(x)\sin(y)$; c) $f(x,y) = \cos(x)\sin(y)$;

d) $f(x,y) = \sin(x\,y)$; e) $f(x,y) = \ln(1+x+y)$; f) $f(x,y) = \frac{y}{1-x}$.

Represent graphically, by using some mathematical software, the function and the Taylor polynomials obtained.

16 Find the 2–degree Taylor polynomials for the following functions at the indicated point:

a) $f(x,y) = \arctan y/x$ at $(1,1)$; b) $f(x,y) = \cos xy$ at $(0,0)$;

c) $f(x,y) = e^{xy}$ at $(0,0)$; d) $f(x,y) = x\sin y + y \sin x$ at $(0,0)$;

e) $f(x,y) = \ln(1+xy)$ at $(0,0)$; f) $f(x,y) = e^{2x}\cos y$ at $(0,0)$;

17 The falling speed of a parachutist can be calculated by the expression

$$v(t) = \frac{g\,M}{c}\left(1 - e^{-\dfrac{c}{M}t}\right).$$

Use the error propagation formulae for computing v in the following cases:

a) $t = 6\,\mathrm{s}$ if $g = 9.8\,\mathrm{m}\cdot\mathrm{s}^{-2}$, $M = 65\,\mathrm{Kg}$ and $c = 12.5 \pm 2.0\,\mathrm{Kg}\cdot\mathrm{s}^{-1}$.

b) $t = 7\,\mathrm{s}$ if $g = 9.8 \pm 0.1\,\mathrm{m}\cdot\mathrm{s}^{-2}$, $M = 55\,\mathrm{Kg}$ i $c = 12.5 \pm 1.1\,\mathrm{Kg}\cdot\mathrm{s}^{-1}$.

c) $t = 8\,\mathrm{s}$ if $g = 9.8 \pm 0.05\,\mathrm{m}\cdot\mathrm{s}^{-2}$, $M = 60 \pm 1.0\,\mathrm{Kg}$ and $c = 12.5 \pm 1.1\,\mathrm{Kg}\cdot\mathrm{s}^{-1}$.

d) $t = 12 \pm 0.5\,\mathrm{s}$ if $g = 9.8 \pm 0.05\,\mathrm{m}\cdot\mathrm{s}^{-2}$, $M = 60 \pm 1.0\,\mathrm{Kg}$ and $c = 12.5 \pm 1.1\,\mathrm{Kg}\cdot\mathrm{s}^{-1}$.

18 Find the local extrema of the following functions. In some of the critical points, the determinant of the hessian matrix is zero, and therefore, it is required to compute type of the critical point using directly the definitions of maximum, minimum or the saddle point.

a) $f(x,y) = y^2 + x^2y + x^4\,;$

b) $f(x,y) = x^2 - 2xy^2 + y^4 - y^5;$

c) $f(x,y) = (y - x^2)(y - 2x^2)\,;$

d) $f(x,y) = (x^2 + y^2)^2 - 2a^2(x^2 - y^2)\,,$ with $a \neq 0\,;$

e) $f(x,y) = (a\cos x + b\cos y)^2 + (a\sin x + b\sin y)^2\,;$

f) $f(x,y) = \dfrac{xy^2}{x^2 + y^4}\,;$

g) $f(x,y) = x^3 + 3xy^2 - 15x - 12y\,;$

h) $f(x,y) = 9x^2 + 6xy + y^2 + 12x + 4y\,;$

i) $f(x,y) = (x + y - 1)(x^4 + y^4)\,;$

j) $f(x,y) = \sin x + \sin y + \cos(x + y)\,.$

19 Find the following partial derivatives:

a) $\dfrac{\partial^3 u}{\partial x^2 \partial y}$, for $u = x\ln(xy)$;

b) $\dfrac{\partial^6 u}{\partial x^3 \partial y^3}$, for $u = x^3\sin y + y^3 \sin x$;

c) $\dfrac{\partial^3 u}{\partial x \partial y \partial z}$, for $u = e^{xyz}$;

d) $\dfrac{\partial^{p+q} u}{\partial x^p \partial y^q}$, for $u = (x - x_0)^p (y - y_0)^q$.

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9.5 Solutions

1 a) $P_2(x,y) = 2x + 3y - 2x^2 - 6xy - \dfrac{9}{2}y^2$.
b) An upper bound is 0.05 and the approximate value for $f(1/10, 1/10)$ is 0.38.

2 (*Final exam 14/06/2011.*)

3 a) Local minimum at $(3,3)$, and $(0,0)$ is a saddle point.
b) Local maximum at $(1,1)$ and local minima in all the points of the curve $x^2 - 2x + 4y^2 - 8y = 0$.
c) $(0,0)$ is a saddle point
d) Local maxima at $\left(\dfrac{2}{5}, \dfrac{2}{5}\right)$ and at the points of the rays $x = 0,\ y > 1$ and $y = 0,\ x > 1$.
Local minima at the points of the rays $x = 0,\ y < 1$ and $y = 0,\ x < 1$.
The saddle points are $(1,0)$ and $(0,1)$.

4 $a = 1$, $b = 1$.

6 $P_2(x,y) = \left(\dfrac{\pi}{2}\right)^2 + 1 + \left(\dfrac{\pi}{2}\right)^2 (x-1) + \pi\left(y - \dfrac{\pi}{2}\right) - \dfrac{\pi^2}{8}(x-1)^2 + \dfrac{\pi}{2}(x-1)\left(y - \dfrac{\pi}{2}\right) + \dfrac{1}{2}\left(y - \dfrac{\pi}{2}\right)^2$.

7 $P_2(x,y) = 1 + 6x + y^2$.

8 a) 2.00250, b) 0.999867, c) 0.902.

9 $S = 100 \pm 4.5$.

10 The upper bound of error is 0.04.

12 a) Local minimum at $\left(-\dfrac{1}{3}, -\dfrac{1}{3}\right)$.
b) Local maxima at $\left(\dfrac{\pi}{2}(2n+1), \dfrac{\pi}{2}(2k+1)\right)$ if $n + k$ is even.
Local minima a $\left(\dfrac{\pi}{2}(2n+1), \dfrac{\pi}{2}(2k+1)\right)$ if $n + k$ is odd.
The saddle points are $(n\pi, k\pi)$.
c) Local maximum at $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$ and Local minimum at $\left(-\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$.
d) Local minimum at $(1,1)$.
e) Local minima $(\pm\sqrt{2}, \mp\sqrt{2})$ and a saddle point at $(0,0)$.
f) The saddle points are $(0,0)$ and $\left(9, \dfrac{27}{2}\right)$.
g) Local maxima at $\left(\dfrac{3}{4}, \dfrac{3}{2}\right)$, and at the rays $y = 0,\ x < 0$ and $y = 0,\ x > 3$.
Local minima at the points of the segment $y = 0,\ 0 < x < 3$.

The saddle points are $(0,0)$, $(0,3)$ and $(3,0)$.

13 $\lambda = 0\,,\ \mu = 1/5\,.$ Relative minimum.

14 a) There isn't any local extrema. The points $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$ are saddle points.

b) $(0,0)$ it's a local minimum.

15 a) $P_2(x,y) = 1 + x + y + \dfrac{x^2}{2} + xy + \dfrac{y^2}{2}\,,$
b) $P_2(x,y) = xy\,,$
c) $P_1(x,y) = y\,,$
d) $P_2(x,y) = xy\,,$
e) $P_2(x,y) = x + y - \dfrac{x^2}{2} - xy - \dfrac{y^2}{2}\,,$
f) $P_2(x,y) = y + xy\,.$

16 a) $P_2(x,y) = \dfrac{\pi}{4} - \dfrac{1}{2}(x-1) + \dfrac{1}{2}(y-1) + \dfrac{1}{4}(x-1)^2 - \dfrac{1}{4}(y-1)^2\,,$
b) $P_1(x,y) = 1\,,$
c) $P_2(x,y) = 1 + xy\,,$
d) $P_2(x,y) = xy\,,$
e) $P_2(x,y) = xy,.$
f) $P_3(x,y) = 1 + 2x + 2x^2 - \dfrac{1}{2}y^2 + \dfrac{4}{3}x^3 - \dfrac{1}{3}xy^2\,.$

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Chapter 10

Optimization of functions of several variables.

10.1 Problems

1 Study the extremes of the function $f(x,y) = x^2 + y^2$ when the variables (x,y) are linked by the condition
$$y + x^2 = 1.$$

2 Determine the conditional extrema of the following functions:

a) $f(x,y) = x + 2y$, under the condition $x^2 + y^2 = 5$;

b) $f(x,y,z) = x^2 + y^2 + z^2$, under the conditions $\left\{ \begin{array}{l} x^2 + y^2 = 1\ , \\ x + y + z = 1\ . \end{array} \right.$

3 Find the absolute extrema values of the function $f(x,y) = x^2 + y^2 - 12x - 8y + 50$ on the domain defined by the inequality $x^2 + y^2 - 4x - 2y \leq 20$.

4 Determine the points where the function $f(x,y) = x^2 + y^2 - xy + x + y$ takes the maximum and minimum absolute values on the compact set $D = \{\,(x,y) \in \mathbb{R}^2 :\ x \leq 0, y \leq 0, x + y \geq -3\}$.

5 The temperature in centigrade degrees of a board at any point (x,y) is given by the function $T(x,y) = 25 + 4x^2 - 4xy + y^2$. A thermic alarm placed over the points of the circle $x^2 + y^2 = 25$, acts if the temperature is over 180 degrees or below 20 degrees. Will this alarm act?

6 Find the minimum distance from the origin to the ellipse defined by
$$\mathcal{E} = \{(x,y) \in \mathbb{R}^2 :\ 5x^2 + 5y^2 - 6xy = 4\}.$$

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10.2 Workshop

7 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = x^2 + y^2$,

a) Find and classify the critical points of f on its domain.

b) Justify the existence of absolute extrema of f on the set

$$\mathcal{K} = \left\{(x, y) \in \mathbb{R}^2 : \, y \leq 1 - x^2 \,, \quad y \geq x - 1\right\}.$$

c) Determine all the candidates in which the absolute maximum and the minimum of f on $\mathcal{K}$ can be attained.

d) Find the points in which the absolute maximum and minimum are attained and say which are the maximum and minimum values of f on $\mathcal{K}$.

8 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = x^4 + y^2$,

a) Find and classify the critical points of f on its domain.

b) Justify the existence of absolute extrema of f on the set

$$\mathcal{K} = \left\{(x, y) \in \mathbb{R}^2 : \, x^2 + y^2 \leq 1 \,, \quad y \geq \frac{1}{2}\right\}.$$

c) Determine the absolute maximum and the absolute minimum of f on the set $\mathcal{K}$.

9 Find the points of the circle $x^2 + y^2 - 2x - 2y = 16$ such that the sum of its coordinates is maximum and minimum, respectively.

10 Find the points of the curve intersection between the surface $x^2 - xy + y^2 - z^2 = 1$ and the surface $x^2 + y^2 = 1$ that are nearer the origin.

11 Three brothers aged 40,45 and 50 respectively have to distribute a 20.000.000 euro inheritance. The successions tax law of their country says that the taxes to pay by each brother are proportional to their age and the square of the received amount. Find the distribution of the inheritance so that the whole amount payed to the tax office by the three brothers is minimal.

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10.3 You should also do

12 Determine the conditional extrema of the following functions:

a) $f(x, y) = xy$, under $x + y = 1$;

b) $f(x, y) = x^2 + y^2$, under $x/2 + y/3 = 1$;

c) $f(x, y, z) = x - 2y + 2z$, under $x^2 + y^2 + z^2 = 9$;

d) $f(x, y, z) = 3x^2 + 3y^2 + z^2$, under $x + y + z = 1$.

13 Let a, b be real parameters and $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = x^2 + y^2 + bxy + az.$$

a) Find the relation between the parameters a and b that is a necessary condition so that the point $(1, 1, 1)$ is a local extreme of f on the sphere $x^2 + y^2 + z^2 = 3$.

b) If we suppose that the previous condition is fulfilled, study for which values of a and b we will have at the point $(1, 1, 1)$ a local maximum or a local minimum.

14 Find the maximum and minimum of $f(x, y) = x^2 + y^2 - 6x - 8y + 25$ on the domain

$$D = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 100\}.$$

15 a) Let a be a real number. Compute and classify the critical points of the function

$$f(x, y) = \frac{1}{x} + axy + \frac{1}{y}.$$

b) Let $a = 2$. Find the absolute extrema of f on the compact subset of $\mathbb{R}^2$ defined by the inequalities: $x \geq 1/2,\ y \geq 1/2,\ x + y \leq 2$.

16 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = 4x + 3y + 1$

a) Let K be any compact subset of $\mathbb{R}^2$. Justify that f admits absolute extrema on K and that they are attained on the boundary of K.

b) Let $K = \left\{(x, y) \in \mathbb{R}^2 : r_1^2 \leq x^2 + y^2 \leq r_2^2\right\}$ with $0 < r_1 < r_2$. Find the absolute extrema of f on this compact K. (If you want, you can use the result of the previous section).

17 With two pieces of a string of length L we want to build a square and a circle. Find the lengths of these pieces so that the total area is maximum, or either minimum.

18 Find the absolute maximum and minimum of $f(x, y) = (x^2 + y^2) \ln(x + y)$ at the region

$$\left\{(x, y) \in \mathbb{R}^2 : \, 2 \leq x^2 + y^2 \leq 4, \, x \geq 0, \, y \geq 0\right\}.$$

19 Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = (ax^2 + by^2)\, e^{-(x^2+y^2)}$ with $a > b > 0$ has absolute extrema. Find them.

20 Find the triangle with given perimeter $2p$ having the maximum area. Remind, the formula relating the area S, the semiperimeter p and the sides of the triangle $a, \ b, \ c$ is $S^2 = p(p-a)(p-b)(p-c)$.

21 From all the rectangular parallelepipeds with total area S, find the one with the maximum volume and, from all with volume V, find the one with minimum area.

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10.4 Solutions

1 Conditional maximum at $(0,1)$. Conditional minimum at $\left(\pm\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$.

2 a) Conditional maximum at $(1,2)$. Conditional minimum at $(-1,-2)$.

b) Conditional maximum at $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1+\sqrt{2}\right)$. Conditional minima at $(1,0,0)$, and at $(0,1,0)$.

3 Absolute maximum at $(-2,-2)$ with value 98. Absolute minimum at $(6,4)$ with value -2.

4 Absolute maximum at $(0,-3)$ and $(-3,0)$. Absolute minimum at $(-1,-1)$.

5 The alarm won't act, because the absolute maximum is 150 and the absolute minimum is 25.

6 (*Final exam 17/01/2013.*)

7 (*Final exam 17/01/2012.*)

8 (*Final exam 11/01/2011.*)

9 (*Final exam 14/06/2011.*)

11 The three parts ordered by age are $\frac{900}{121}$, $\frac{800}{121}$ and $\frac{720}{121}$ million euro respectively.

12 a) Conditional maximum at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

b) Conditional minimum at $\left(\frac{18}{13}, \frac{11}{13}\right)$.

c) Conditional maximum at $(1,-2,2)$. Conditional minimum at $(-1,2,-2)$.

d) Conditional minimum at $\left(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}\right)$.

13 a) $a-b-2=0$.

14 Absolute maximum at $(-6,-8)$ with value 225. Absolute minimum at $(3,4)$ with value 0.

15 a) If $a>0$, local maximum at $\left(\frac{1}{\sqrt[3]{a}}, \frac{1}{\sqrt[3]{a}}\right)$ and it's $3\sqrt[3]{a}$.
If $a<0$, local minimum at $\left(\frac{1}{\sqrt[3]{a}}, \frac{1}{\sqrt[3]{a}}\right)$ and it's $3\sqrt[3]{a}$.
If $a=0$, there isn't any local extremum. b) Absolute minimum at $(1,\frac{1}{2})$, $(\frac{1}{2},1)$, value $\frac{7}{2}$. Absolute maximum at $(\frac{1}{2},\frac{1}{2})$, value $\frac{9}{2}$.

16 b) Absolute maximum at $\left(\frac{4}{5}r_2, \frac{3}{5}r_2\right)$, with value $1+5r_2$.
Absolute minimum at $\left(-\frac{4}{5}r_2, -\frac{3}{5}r_2\right)$, with value $1-5r_2$.

17 To get the minimum total area, the side of the square has to be of length $\frac{L}{\pi+4}$ and the radius of the circle has to be $\frac{L}{2(\pi+4)}$.
To get the maximum total area, the side of the square has to be of length 0 and the radius of the circle has to be $\frac{L}{2\pi}$.

18 Absolute maximum at $(\sqrt{2},\sqrt{2})$ and it's $6\ln 2$. Absolute minima at $(0,\sqrt{2})$, $(\sqrt{2},0)$ and they are $\ln 2$.

19 Absolute maximum at $(\pm 1, 0)$ with value $\frac{a}{e}$.

20 The maximum is for the equilateral triangle with a side length of $\frac{2p}{3}$.

21 The rectangular parallelepiped of total area S and maximum volume is the cube with side $\sqrt{S/6}$.
The rectangular parallelepiped of volume V with minimal area is the cube with side $\sqrt[3]{V}$.

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