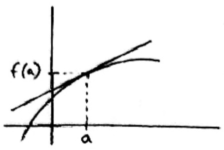
8: PARTIAL AND DIRECTIONAL DERIVATES

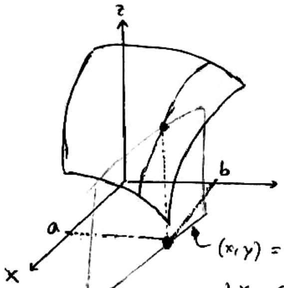
Remember that for 1 variable:



The derivative at a is the slope "at the point $(a, f(a))$"

That is, the slope of the tangent line at this point

For two variables: Different slopes for different directions



$(x,y) = (a,b) + \lambda (v_1, v_2)$

$\begin{cases} x = a + \lambda v_1 \\ y = a + \lambda v_2 \end{cases}$

$z = f(x,y)$

$\vec{v} = (v_1, v_2)$ unitary $\sqrt{{v_1}^2 + {v_2}^2} = 1 \Rightarrow {v_1}^2 + {v_2}^2 = 1$

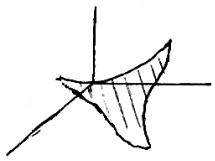
We define the derivative of f at (a,b) along the vector $\vec{v}$ as

$$\lim_{\lambda \to 0} \frac{f((a,b) + \lambda(v_1, v_2)) - f(a,b)}{\lambda} \underset{\text{notation}}{=} \frac{\partial f}{\partial \vec{v}}(a,b) \quad \left(\text{or also } f_{\vec{v}}(a,b)\right)$$

We say that it is a "directional derivative"

Example: $f(x,y) = 2x^2 - y^2 \wedge (a,b) = (1,2) \wedge \vec{v} = \left(\frac{3}{5}, \frac{4}{5}\right)$

$\|\vec{v}\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$. YES, then it's unitary

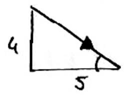


$$\frac{\partial f}{\partial \vec{v}}(1,2) = \lim_{\lambda \to 0} \frac{f((1,2) + \lambda(\frac{3}{5}, \frac{4}{5})) - f(1,2)}{\lambda}$$

$$= \lim_{\lambda \to 0} \frac{f(1 + \frac{3\lambda}{5}, 2 + \frac{4\lambda}{5}) - f(1,2)}{\lambda}$$

$$= \lim_{\lambda \to 0} \frac{2(1 + \frac{3\lambda}{5})^2 - (2 + \frac{4\lambda}{5})^2 - (2-4)}{\lambda}$$

$$= \lim_{\lambda \to 0} \frac{-\frac{4\lambda}{5} + \frac{2\lambda^2}{25}}{\lambda} = \lim_{\lambda \to 0} \left(-\frac{4}{5} + \frac{2\lambda}{25}\right) = -\frac{4}{5}$$



For $n > 2$: $f: D \subset \mathbb{R}^n \to \mathbb{R}$

$\bar{a} \in D \subset \mathbb{R}^n$

$\vec{v} \in \mathbb{R}^n$ unitary $(\|\vec{v}\| = 1)$

Derivative of f at $\bar{a}$ along $\vec{v}$: $\frac{\partial f}{\partial \vec{v}}(\bar{a}) = \lim \frac{f(\bar{a} + \lambda \bar{v}) - f(\bar{a})}{\lambda}$

- Suppose $(n=2) \wedge \vec{v} = (1,0) \Rightarrow \frac{\partial f}{\partial v}(a,b) = \lim_{\lambda \to 0} \frac{f((a,b) + \lambda(1,0)) - f(a,b)}{\lambda} = \lim_{\lambda \to 0} \frac{f(a+\lambda, b) - f(a,b)}{\lambda}$

Is the limit of a function of $\underline{1}$ variable ("y" is constant) $\Rightarrow$ It can be evaluated by using the value and formulae for derivatives of functions of 1 variable

- For $\vec{v} = (0,1) \Rightarrow \frac{\partial f}{\partial y}(a,b) = \frac{\partial f}{\partial v}(a,b) = \lim_{\lambda \to 0} \frac{f((a,b) + \lambda(0,1)) - f(a,b)}{\lambda} = \lim_{\lambda \to 0} \frac{f(a, b+h) - f(a,b)}{\lambda}$ (the same as before)

- $\frac{\partial f}{\partial x}(a,b)$ is the partial derivative of f with respect to x at (a,b)
- $\frac{\partial f}{\partial y}(a,b)$ is the partial derivative of f with respect to y at (a,b)

Example 1: Partial derivatives of $f(x,y) = x^2 - 2y^2$ at $(1,2)$

$$\frac{\partial f}{\partial x} = 2x - 0 = 2x \quad \wedge \quad \frac{\partial f}{\partial y} = 0 - 4y = -4y \quad \Big\} \text{ At } (1,2): \frac{\partial f}{\partial x}(1,2) = 2 \quad \wedge \quad \frac{\partial f}{\partial y}(1,2) = -8$$

Example 2: $f(x,y) = \arctan\left(\frac{x}{y}\right)$ at any (x,y)

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{1}{y + \frac{x^2}{y^2} \cdot y} = \frac{1}{y + \frac{x^2}{y}} = \frac{y}{y^2+x^2} = \frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{-x}{y^2} = \frac{-x}{y^2 + \frac{x^2}{y^2} y^2} = \frac{-x}{x^2+y^2} \quad \wedge \quad \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{-x}{y^2}$$

For more variables:

$\bar{a} = (a_1, a_2, \ldots, a_n)$

$\vec{v} = (0, \ldots, 0, 1, 0, \ldots, 0) = \vec{e}_i$ (1 at position i)

$$\frac{\partial f}{\partial x_i}(\bar{a}) = \frac{\partial f}{\partial e_i}(\bar{a}) = \lim \frac{f(a_1, a_2, \ldots, a_i + \lambda, a_{i+1}, \ldots, a_n) - f(a_1, \ldots, a_n)}{\lambda}$$

(Again this is a derivative of 1 variable)

Example 3: Evaluate the partial derivatives of $f(x,y,z,t) = 3x^2yz^4t - 2xyz$

$f_x = \frac{\partial f}{\partial x} = 6xyz^4t - 2yz$ $\qquad f_z = 12x^2yz^3t - 2xy$

$f_y = \frac{\partial f}{\partial y} = 3x^2z^4t - 2xz$ $\qquad f_t = 3x^2yz^4$

More notation:

Sometimes we will put:

f_x instead of $\frac{\partial f}{\partial x}$

f_y instead of $\frac{\partial f}{\partial y}$

f_{x_i} instead of $\frac{\partial f}{\partial x_i}$

For $f: D \subset \mathbb{R}^n \to \mathbb{R}$ we define the 'gradient vector of f' at $\bar{a} \in D$ as $\vec{\nabla} f(\bar{a}) = \left(\frac{\partial f}{\partial x_1}(\bar{a}), \frac{\partial f}{\partial x_2}(\bar{a}), \ldots, \frac{\partial f}{\partial x_n}(\bar{a})\right)$ (also $\vec{\text{grad}}\, f(\bar{a})$)

∇ "nabla"

For $f: D \subset \mathbb{R}^n \to \mathbb{R}^m$

Jacobian matrix of f at $\bar{a}$: $\quad Jf(\bar{a}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\bar{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\bar{a}) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(\bar{a}) & \cdots & \frac{\partial f_n}{\partial x_n}(\bar{a}) \end{pmatrix}$ Note that the row j corresponds to ∇f_j

Example: Jacobian matrix for $f(x,y) = (x^2y, xy^3, \ln(x,y))$ if $\begin{pmatrix} 2xy & x^2 \\ y^3 & 3xy^2 \\ 1/x & 1/y \end{pmatrix}$ $\left\{ \begin{array}{l} f_1(x,y) = x^2y \\ f_2(x,y) = xy^3 \\ f_3(x,y) = \ln(xy) \end{array} \right.$ $\quad \frac{\partial f}{\partial x} = \frac{1}{xy} \cdot y$, $\quad \frac{\partial f}{\partial y} = \frac{1}{xy} \cdot x$

Straight lines in $\mathbb{R}^3$

Given $P=(a,b,c) \wedge \vec{v}=(v_1,v_2,v_3) \Rightarrow$ straight line r

$(x,y,z)-(a,b,c)=\lambda(v_1,v_2,v_3)$

$$\left.\begin{array}{l} x=a \\ y=b \\ z=c \end{array}\right\} \text{Parametric equations for } r$$

$$\boxed{\frac{x-a}{v_1}=\frac{y-b}{v_2}=\frac{z-c}{v_3}}$$

Example: Tangent plane to $z=2x^2-y^2$ at $(1,2,-2)$

$(1,2,-2)$ belongs to this surface, because $\underset{x}{2\cdot 1^2}-\underset{y}{2^2}=\underset{z}{-2} \leftarrow c$ $\Rightarrow f_x=4x \rightarrow f_x(1,2)=4=p$
$f_y=-2y \rightarrow f_y(1,2)=-4=q$

Plane equation: $z=-2+4(x-1)-4(y-2)=4x-4y+2 \Rightarrow 4x-4y-z+2=0 \Rightarrow (4,-4,-1)$ is the "normal vector"

And the "normal line" at this point will be $(1,2,-2)$, vector $(4,-4,-1) \Rightarrow \boxed{\frac{x-1}{4}=\frac{y-2}{-4}=\frac{z+2}{-1}}$

Differentiability:

We will say that f is differentiable at (a,b) if there is a plane containing the point $(a,b,f(a,b))$ such that the surface $z=f(x,y)$ is approximately equal to this plane when (x,y) is close to (a,b). This plane is called the tangent plane, $z=f(x,y)$ at the point $(a,b,f(a,b))$ and it can be shown that the plane is given by the equation $z=c+p(x-a)+q(y-b)$, where $c=f(a,b) \wedge p=\frac{\partial f}{\partial x}(a,b) \wedge q=\frac{\partial f}{\partial y}(a,b)$

Properties of the gradient vector

Suppose f differentiable. It can be shown that

1) For $\vec{v}$ unitary: $\frac{\partial f}{\partial v}(\vec{a})=\vec{\nabla} f(\vec{a})\cdot\vec{v}$

Example: $f(x,y)=2x^2-y^2$, $(a,b)=(1,2) \wedge \vec{v}=\left(\frac{3}{5},\frac{4}{5}\right) \Rightarrow \frac{\partial f}{\partial v}(1,2)=\lim \ldots = -\frac{4}{5}$

Now, by using this property: $\vec{\nabla} f(1,2)=?$

$$\left.\begin{array}{l} f_x=4x \\ f_y=-2y \end{array}\right\} \Rightarrow \left.\begin{array}{l} f_x(1,2)=4 \\ f_y(1,2)=-4 \end{array}\right\} \Rightarrow \vec{\nabla} f(1,2)=(4,-4)$$

$\vec{\nabla} f(1,2)\cdot\left(\frac{3}{5},\frac{4}{5}\right)=$
$=(4,-4)\cdot\left(\frac{3}{5},\frac{4}{5}\right)=$
$=4\cdot\frac{3}{5}-4\cdot\frac{4}{5}=\frac{12}{5}-\frac{16}{5}=-\frac{4}{5}$

Remember that: $\vec{u}\cdot\vec{v}=\sum_{i=1}^{n} u_i\cdot v_i=u_1v_1+u_2v_2+\ldots+u_nv_n \Rightarrow \vec{u}\cdot\vec{v}=\|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos\alpha$

$\|\vec{v}\|=1$, $\frac{\partial f}{\partial v}(\vec{a})=\vec{\nabla} f(\vec{a})\cdot\vec{v}=\|\nabla f(\vec{a})\|\cdot\|\vec{v}\|\cdot\cos\alpha=\|\vec{\nabla} f(\vec{a})\|\cdot\cos\alpha$

For given f and $\vec{a}$, the maximal directional derivative occurs for $\cos\alpha=1$, that is, for $\alpha=0$. This means that $\vec{v}$ must have the direction (and sense) of $\vec{\nabla} f(\vec{a}) \Rightarrow$
$\Rightarrow \vec{\nabla} f(\vec{a})$ indicates the direction (and sense) of maximal directional derivative, and the value of the directional derivative is exactly $\|\vec{\nabla} f(\vec{a})\|$

Example: $f(x,y) = 2x^2 - y^2 \wedge \vec{\nabla} f(1,2) = (4,-4)$: The max. direc. derivative occurs along the vector $(4,-4)$

and its value is $\|(4,-4)\| = \sqrt{4^2+(-4)^2} = \sqrt{2 \cdot 16} = 4\sqrt{2}$

2) It can be shown also that the gradient is perpendicular to the level curves. More precisely, $\vec{\nabla} f(\vec{a})$ is orthogonal to the tangent line to the level curve passing through $\vec{a}$

Example: For $f(x,y) = y - x^2$ at $(1,2)$ check that $\vec{\nabla} f(1,2)$ is orthogonal to the tangent line to the level curve at $(1,2)$

$f(1,2) = 2-1 = 1$, the level curve containing $(1,2)$ is the curve $y - x^2 = 1 \Rightarrow y = 1 + x^2$

Tangent line to $y = 1 + x^2$ at $(1,2)$: $g(x) = 1 + x^2 \Rightarrow g'(x) = 2x \Rightarrow g'(1) = 2$

Tangent line: $y = 2 + 2(x-1) = 2 + 2x - 2 = y = 2x$

Remember: Given $y = mx + n$ $\begin{cases} \text{for } x = 0 \Rightarrow y = n \\ \text{for } x = 1 \Rightarrow y = m+n \end{cases}$ Points $(1, m+n), (0,n) \Rightarrow \vec{v} = (1, m+n) - (0,n) = (1,m)$

In our case $y = 2x \wedge \vec{v} = (1,2)$. On the other hand, $\vec{\nabla} f(1,2) = ? \wedge f(x,y) = y - x^2 \wedge \begin{cases} f_x = -2x \\ f_y = 1 \end{cases} \Rightarrow \vec{\nabla} f(1,2) = (-2,1)$

And we can check that $(1,2) \cdot (-2,1) = 1(-2) + 2 \cdot 1 = \boxed{0}$ ok

Page 39, 6/ $f(x,y) = 4x + 2y - x^2 + xy - y^2$. Find the points at which the tangent plane is paralled to xy

Any horizontal plane: $z = k$ (constant) $\Rightarrow$ Tangent plane at (a,b): $z = f(a,b) + p(x-a) + q(y-b)$ is horizontal if and only if $\begin{cases} p = 0 \\ q = 0 \end{cases}$

We search points (a,b) such that $f_x(a,b) = 0 \wedge f_y(a,b) = 0 \Rightarrow f_x = 4 - 2x + y = 0 \wedge f_y = 2 + x - 2y = 0 \Rightarrow \begin{cases} -2x + y = -4 \\ x - 2y = -2 \end{cases} \Rightarrow \begin{cases} x = 10/3 \\ y = 8/3 \end{cases}$

$z = 4x + 2y - x^2 + xy - y^2 = \frac{40}{3} + \frac{16}{3} - \frac{100}{9} + \frac{80}{9} - \frac{64}{9} = \frac{28}{3} \Rightarrow$ Solution: The point $\left(\frac{10}{3}, \frac{8}{3}, \frac{28}{3}\right)$

$f: D \subset \mathbb{R}^n \to \mathbb{R}$

$f_{x_1}, f_{x_2}, \ldots, f_{x_n}$ partial derivatives (we have n of them). If f_{x_i} admits partial derivatives, the partial derivatives of f_{x_i} are called "second partial derivatives" of f. Notation:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j x_i} \quad \text{or also} \quad f(x_i)_{x_j} = f_{x_i x_j}$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_i^2} \quad \text{or also} \quad f(x_i)_{x_i} = f_{x_i x_i}$$

We have n^2

In an analogous way we define third partial derivatives: $\frac{\partial}{\partial x_k}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = \frac{\partial^3 f}{\partial x_k \partial x_i \partial x_j}$ or $f_{x_i x_j x_k}$; $\frac{\partial}{\partial x_k}\left(\frac{\partial^2 f}{\partial x_k \partial x_j}\right) = \frac{\partial^3 f}{\partial x_k^2 x_j}$ ... (etc.)

we have n^3

Example: $f(x,y) = 3x^7y^4 - 2x^5y^3$

$f_x = 21x^6y^4 - 10x^4y^3 \Rightarrow \begin{cases} f_{xx} = 126x^5y^4 - 40x^3y^3 \\ f_{xy} = 84x^6y^3 - 30x^4y^2 \end{cases}$

$f_y = 12x^7y^3 - 6x^5y^2 \Rightarrow \begin{cases} f_{yx} = 84x^6y^3 - 30x^4y^2 \\ f_{yy} = 36x^7y^2 - 12x^5y \end{cases}$

Schwarz theorem

If f_{xy} and f_{yx} are continuous functions then $f_{xy} = f_{yx}$

Note that if the third partial derivatives of f are continuous, we will have $(n=2)$: $f_{xxy} = (f_x)_{xy} = (f_x)_{yx} = f_{xyx} = (f_{xy})_x = (f_{yx})_x = f_{yxx}$

If f has continuous partial derivatives until order k, we will say that f is of class C^k. $C^k \subset C^{k-1} \subset C^{k-2} \subset C^{k-3} \ldots \subset C^1 \subset C^0$

If f is in the class C^k for any $k \in \mathbb{N}$ we say that f is in the class C^∞