Best Covering, Sensor networks and Maximum Polarization

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I. Introduction

Wireless sensor networks have various applications from cellphone coverage, to intruder detection, to preventative measures and monitoring of wild fires. The mathematical model leads to the so-called best covering problem, namely how do you cover a region in the plane with a fixed number of identical circles, of least radius. It is a limiting case of the recently introduced maximum polarization problem, see for example [1, Chapter 14].

Polarization is an optimization problem. Say we are given a region Ω in the plane and want to find n points, $x_1,\ x_2, ...,\ x_n$ in Ω that maximize the minimum quantity, namely determine

$$\max_{x_i \in \Omega} \min_{x \in \Omega} \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^s}. \tag{1}$$

That is, we want to find points which have the largest possible minimum in (1). This "energy function" depends on the parameter s. And as we increase s towards infinity, this energy problem becomes equivalent to covering the region with n discs of minimal radius. As s increases, the points closer to the x_i grow in importance, the value of the point furthest away from all the x_i will plummet.

Since we care first about the minimum, this is what we want to prevent. We want to move the x_i closer to the furthest away point to increase its value. This has the consequence that other points will move further away, so we must only move the x_i enough that the points on either side are near equal distance away. "Near equal" comes from the fact that some points will be surrounded by points, and will have more contribution than points near the edges. As s increases, this distinction falls away, and this "near equal" becomes exactly equal. So we want the furthest points away to all be the same smallest distance, which is the same as covering with circles.

II. Best Covering of 4 Points

Proposition 1 *The best covering of the square with four points is when the centers of the circles are the midpoints between the center of the square and the four vertices.*

Proof: Suppose we have a unit square on the plane with vertices

$$(0,0),\ (0,1),\ (1,0),\ (1,1).$$

The unit square can be covered with four circles of radius, $r = \sqrt{2}/4$, with centers

$$(0.25, 0.25),\ (0.75, 0.25),\ (0.25, 0.75),\ (0.75, 0.75).$$

Suppose the unit square can be covered with four circles of radius $x < \sqrt{2}/4$. Let A, B, C, D, E be disks of radius x centered at $(0,0), (0,1), (1,0), (1,1)$ and $(0.5, 0.5)$ respectively. Hence, for the point $(0,0)$ to be covered, one of the centers of the four circles must lie on disk A. Likewise, each of the disks $B, C, D,$ and E must contain one of the centers of the four circles. Since, $x < \sqrt{2}/4$, the union of any two disks is empty. Hence, one of the five A, B, C, D, E disks does not contain any of the centers of the four circles.

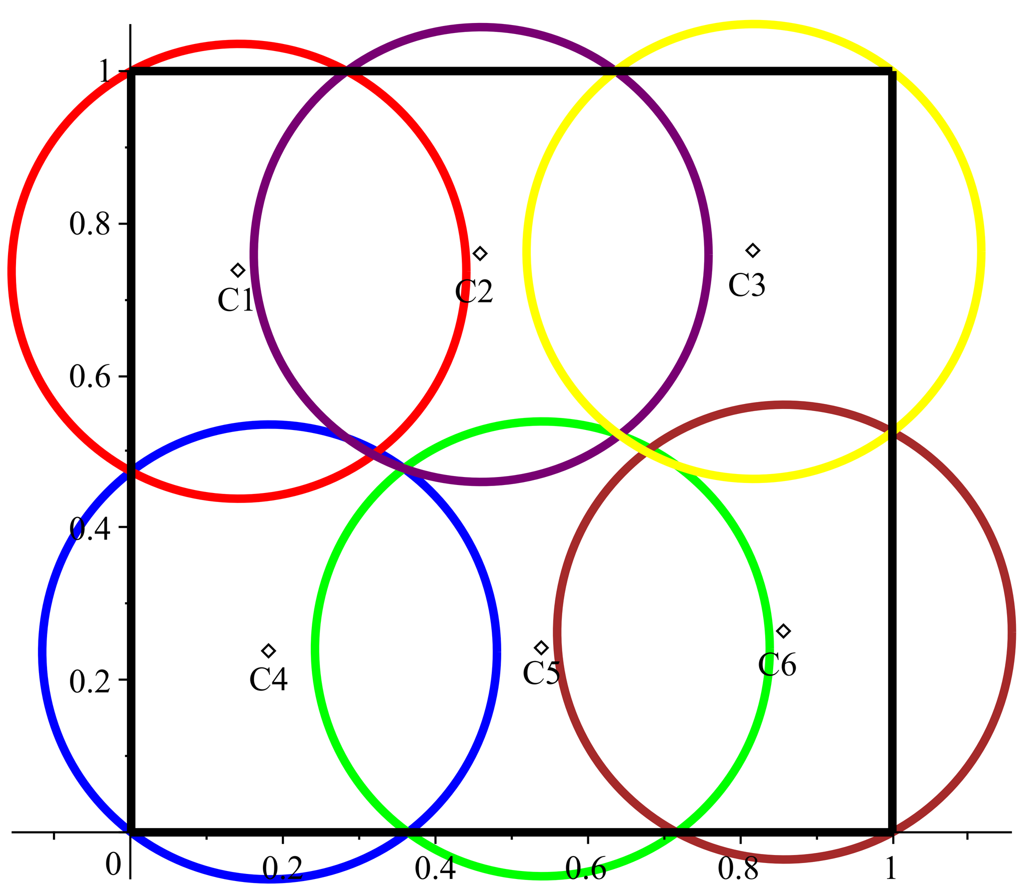
Therefore, by the Pigeon Hole Principle one of the points $(0,0), (0,1), (1,0), (1,1)$ and $(0.5, 0.5)$ is not covered by the four circles. This is a contradiction. Thus, $r = \sqrt{2}/4$, is the minimum radius which can cover a unit square with four circles of equal radius.

Furthermore, there is unique collection of four points that can serve as centers of the best covering solution. □

It is known that when $s = 0$ the four points are all located at the center of the square. Numerical computations we performed confirm the following conjecture:

Conjecture: The solution of the Maximal Polarization problem on the unit square with four points for $0 < s < \infty$ is given by four points that are equidistant to the square's center with distance increasing from 0 to $\sqrt{2}/4$ as s changes from 0 to ∞.

III. On N points and 6 Points



Consider a covering of the unit square with N equal circles of minimum radius.

Lemma 1. Any optimal covering can be rearranged such that every corner is incident with a circle.

Proof. Suppose we have an optimal covering, with a corner contained in the interior of circle C. Let A and B be the points where C intersects the boundary of the square. Let L be the line containing the center of C perpendicular to the line segment AB. Translate the center of C until the corner is incident with C. All points in the interior of C before the translation will remain covered. Hence, the new covering remains optimal. This process can be repeated to each corner. □

Lemma 2. Any point which lies on the boundary of the square and is not contained in the interior of a circle, must be incident with at least two circles.

Proof. Let A be a point which lies on the boundary of the square and is not contained in the interior of any circle. Thus, A is incident with some circle C and any neighborhood of A must contain another point on the boundary of the square which is not contained in C. Therefore, A must be incident with a second circle. □

III. cont.

Lemma 3. Any point contained in the interior of the square which is not contained in the interior of a circle, must be incident with at least three circles.

Proof. Let A be a point contained in the interior of the square which is not contained in the interior of a circle. Thus, A is incident with some circle C. Let L be the line tangent to C containing A. Hence, any neighborhood of A must contain a point on L which is in the interior of the square which also is not contained in C. Thus, A must be incident with another circle D. Hence, D is either tangent to C or intersects C at two points.

Case 1. Suppose circle D is tangent to L at A. Thus, any neighborhood of A must contain a point on L which is in the interior of the square and is neither contained in C nor contained in D. Thus, A must be incident with a third circle.

Case 2. Suppose circle D intersects C at two points. Let M be the line containing these two points. Hence, any neighborhood of A must contain a point on M which is in the interior of the square and is neither contained in C nor contained in D. Thus, A must be incident with a third circle. □

IV. Computational Results

In order to show that we are at a local minimum, we could perturb the intersection of C_1, C_4 and the left edge, and we perturb the intersection of C_3, C_6 and the right edge. Due to the symmetry there are only two cases we care about. The perturbation is in the same direction, or in opposite directions. We obtain equations for each intersection. Algebraically, this is difficult to work with, but it is completely feasible for a computer to handle.

The points were assumed to have $180 \deg$ rotational symmetry around the center of the square. The computer simulation had been performed by a binary search up to the precision determined by the coordinates of C_4.

$$r = .29872706223691915877$$

	x	y
C_1	.14164494343118383162	.7369894152071500613
C_2	.45978280218118383162	.75898517918281784339
C_3	.8181377589025	.7630105846758236451
C_4	.1818622410975	.2369894153241763549
C_5	.54021729744901824136	.24101482086463304984
C_6	.85835505635151824	.2630105847928499387

V. Other (Failed) Approaches

Like we had with the optimal covering for 4 circles, we could try to find special points for a covering with 6 circles, so that the same sort of restriction gives us an optimal covering.

With 4 circles, we said that a covering of the whole square must cover 5 distinguished points, the corners and the center, and then we only focused on covering these, ignoring if it didn't cover some other points. The Pigeonhole principle was able to give us a lower bound on the optimal radius, $\sqrt{2}/4$, since we cannot cover these 5 points with 4 circles if their radius is $< \sqrt{2}/4$. Further, this same covering told us exactly where at least one center would need to be which achieve this radius. We need one of the centers to be within 2 circles, at radius $\sqrt{2}/4$ this is one of the midpoints of the center and a corner. Using all those points, we achieve the lower bound, and so have found the minimal configuration for 4 points.

Expanding on this idea, 2 opposing corners gives us the optimal radius (and the optimal covering) for 1 circle. Taking the corners, the midpoints and the center gives us the optimal radius for 2 circles. Unfortunately, it is not easy to find circles which work for any particular N. We have only found them for $N = 1, 2, 4$. For $N = 3$, taking the corners gives a lower bound for the radius of $.5$, but the optimal radius for 3 is $\sqrt{65}/16 \approx .50389$. Likewise, for other N this can be used to imperfectly bound the radius, but getting the most out of it is difficult.

So instead of finding special points we could try to take points without too much structure and look for a more naive bound. This can be made equivalent to an unknown graph theory problem. Given a weighted graph G, can we find N full sub-graphs that cover every vertex with smallest maximum weight.

Instead of taking N circles and their overlaps, which are somewhat unwieldy, we could instead take N convex polygons that cover the square and consider their vertices, and the diameter of each polygon. The Voronoi Diagram of the centers. This is an imperfect correspondence, but is robust when furthest vertices correspond to diametrically opposite points. However, there are many different configurations of N polygons that can cover the square. Further, each vertex corresponds to one of the 4 corners, which has no variables associated to it; to an point on the edge which has 1 variable; or to a point in the interior of the square, which has 2 variables. After summing all of these, optimizing any individual configuration is already a problem in a fairly high dimension.

But the hope is that these configurations partition the local minimums into more easily handled families.

Conclusions

The optimal covering of the square is a difficult problem without much infrastructure built around it. There are not many problems like it that can be used in a solution, though there are some, they differ in important ways. Lloyd's algorithm for example, gives a very similar approach to the Voronoi configuration minimization, but the specifics about what Lloyd's algorithm actually minimizes are very different. There are opportunities to try and connect these approaches to the covering problem. For future research we could investigate finding distinguished points which give better lower bounds for arbitrary N. Or we could try to deal with the perturbation equations more systematically. We could take a closer look at the possible Voronoi configurations and try to solidify the correspondence.

Acknowledgment

We thank our faculty advisor Prof. Peter Dragnev for his guidance during the semester.

References

[1] Edward B. Saff Sergiy V. Borodachov Douglas P. Hardin. *Discrete Energy on Rectifiable Sets*. Springer Monographs in Mathematics. Springer, 2019. ISBN: 9780387848082.