Assignment 2 (ML for TS) - MVA

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1 Introduction

Objective. The goal is to better understand the properties of AR and MA processes and do signal denoising with sparse coding.

Warning and advice.

- Use code from the tutorials as well as from other sources. Do not code yourself well-known procedures (e.g., cross-validation or k-means); use an existing implementation.
- The associated notebook contains some hints and several helper functions.
- Be concise. Answers are not expected to be longer than a few sentences (omitting calculations).

Instructions.

- Fill in your names and emails at the top of the document.
- Hand in your report (one per pair of students) by Sunday 7$^{\text{th}}$ December 11:59 PM.
- Rename your report and notebook as follows:
  `FirstnameLastname1_FirstnameLastname1.pdf` and
  `FirstnameLastname2_FirstnameLastname2.ipynb`.
  For instance, `LaurentOudre_ValerioGuerrini.pdf`.
- Upload your report (PDF file) and notebook (IPYNB file) using this link: https://forms.gle/J1pdeHspSs9zNfWAA.

2 General questions

A time series $\{y_t\}_t$ is a single realisation of a random process $\{Y_t\}_t$ defined on the probability space $(\Omega, \mathcal{F}, P)$, i.e. $y_t = Y_t(w)$ for a given $w \in \Omega$. In classical statistics, several independent realizations are often needed to obtain a "good" estimate (meaning consistent) of the parameters of the process. However, thanks to a stationarity hypothesis and a "short-memory" hypothesis, it is still possible to make "good" estimates. The following question illustrates this fact.

Question 1

An estimator $\hat{\theta}_n$ is consistent if it converges in probability when the number n of samples grows to ∞ to the true value $\theta \in \mathbb{R}$ of a parameter, i.e. $\hat{\theta}_n \xrightarrow{\mathcal{D}} \theta$.

- Recall the rate of convergence of the sample mean for i.i.d. random variables with finite variance.
- Let $\{Y_t\}_{t\geq 1}$ a wide-sense stationary process such that $\sum_k |\gamma(k)| < +\infty$. Show that the sample mean $\bar{Y}_n = (Y_1 + \cdots + Y_n)/n$ is consistent and enjoys the same rate of convergence as the i.i.d. case. (Hint: bound $\mathbb{E}[(\bar{Y}_n - \mu)^2]$ with the $\gamma(k)$ and recall that convergence in L_2 implies convergence in probability.)

Answer 1

- Let $(X_i)_{i\geq 1}$ be i.i.d. with mean μ and variance $\sigma^2 < \infty$, and let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Then

  $$\mathbb{E}[\bar{X}_n] = \mu, \qquad \mathrm{Var}(\bar{X}_n) = \frac{\sigma^2}{n}.$$

  Hence $\bar{X}_n \to \mu$ in probability and the typical size of the error is $\sqrt{\mathrm{Var}(\bar{X}_n)} = \sigma/\sqrt{n}$. Moreover, by the Central Limit Theorem,

  $$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

  so the convergence rate of $\bar{X}_n$ toward μ is $n^{-1/2}$.

- By stationarity, $\mathbb{E}[\bar{Y}_n] = \mu$. Moreover,

  $$\mathrm{Var}(\bar{Y}_n) = \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \gamma(j-i) = \frac{1}{n}\Big(\gamma(0) + 2\sum_{k=1}^{n-1}\Big(1 - \frac{k}{n}\Big)\gamma(k)\Big).$$

  Let $A_n := \gamma(0) + 2\sum_{k=1}^{n-1}(1-\frac{k}{n})\gamma(k)$; we show $A_n \to A < \infty$.

  Since $\sum_{k=-\infty}^{\infty}|\gamma(k)| < \infty$, in particular $\sum_{k=1}^{\infty}|\gamma(k)| < \infty$. Hence for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

  $$\sum_{k=K+1}^{\infty}|\gamma(k)| < \varepsilon.$$

  Then, for all n,

  $$\sum_{k=1}^{n-1}\Big(1-\frac{k}{n}\Big)\gamma(k) = \sum_{k=1}^{K}\Big(1-\frac{k}{n}\Big)\gamma(k) + \sum_{k=K+1}^{n-1}\Big(1-\frac{k}{n}\Big)\gamma(k).$$

  The first finite sum converges to $\sum_{k=1}^{K}\gamma(k)$ by dominated convergence (since $|1-k/n| \leq 1$). For the tail,

  $$\Big|\sum_{k=K+1}^{n-1}(1-\tfrac{k}{n})\gamma(k)\Big| \leq \sum_{k=K+1}^{\infty}|\gamma(k)| < \varepsilon.$$

  Thus the whole sum converges to $\sum_{k=1}^{\infty}\gamma(k)$, and

  $$A_n \xrightarrow[n\to\infty]{} A := \gamma(0) + 2\sum_{k=1}^{\infty}\gamma(k) < \infty.$$

  Therefore $\mathrm{Var}(\bar{Y}_n) \sim \frac{A}{n}$, which implies $\mathrm{Var}(\bar{Y}_n) \to 0$ and hence $\bar{Y}_n \xrightarrow{\mathbb{P}} \mu$, with rate $n^{-1/2}$ as in the i.i.d. case.

3 AR and MA processes

Question 2 *Infinite order moving average MA(∞)*

Let $\{Y_t\}_{t\geq 0}$ be a random process defined by

$$Y_t = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \cdots = \sum_{k=0}^{\infty} \psi_k \varepsilon_{t-k} \tag{1}$$

where $(\psi_k)_{k\geq 0} \subset \mathbb{R}$ ($\psi = 1$) are square summable, i.e. $\sum_k \psi_k^2 < \infty$ and $\{\varepsilon_t\}_t$ is a zero mean white noise of variance σ_ε^2. (Here, the infinite sum of random variables is the limit in L_2 of the partial sums.)

- Derive $\mathbb{E}(Y_t)$ and $\mathbb{E}(Y_t Y_{t-k})$. Is this process weakly stationary?
- Show that the power spectrum of $\{Y_t\}_t$ is $S(f) = \sigma_\varepsilon^2 |\phi(e^{-2\pi i f})|^2$ where $\phi(z) = \sum_j \psi_j z^j$. (Assume a sampling frequency of 1 Hz.)

The process $\{Y_t\}_t$ is a moving average of infinite order. Wold's theorem states that any weakly stationary process can be written as the sum of the deterministic process and a stochastic process which has the form (1).

Answer 2

- Since the series converges in L^2 and $\mathbb{E}[\varepsilon_t] = 0$,

  $$\mathbb{E}[Y_t] = \mathbb{E}\Big[\sum_{k=0}^{\infty} \psi_k \varepsilon_{t-k}\Big] = \sum_{k=0}^{\infty} \psi_k \mathbb{E}[\varepsilon_{t-k}] = 0.$$

  For $h \in \mathbb{Z}$,

  $$\mathbb{E}[Y_t Y_{t-h}] = \mathbb{E}\Big[\sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \sum_{m=0}^{\infty} \psi_m \varepsilon_{t-h-m}\Big] = \sum_{j,m\geq 0} \psi_j \psi_m \mathbb{E}[\varepsilon_{t-j}\varepsilon_{t-h-m}].$$

  By whiteness, $\mathbb{E}[\varepsilon_{t-j}\varepsilon_{t-h-m}] = \sigma_\varepsilon^2 \mathbf{1}_{\{t-j=t-h-m\}}$, i.e. nonzero only if $m = j - h$. Using the convention $\psi_k = 0$ for $k < 0$, we obtain

  $$\gamma_Y(h) := \mathbb{E}[Y_t Y_{t-h}] = \sigma_\varepsilon^2 \sum_{j=0}^{\infty} \psi_j \psi_{j-h},$$

  which depends only on the lag h. Moreover, by Cauchy–Schwarz,

  $$\sum_{j=0}^{\infty} |\psi_j \psi_{j-h}| \leq \Big(\sum_{j=0}^{\infty} \psi_j^2\Big)^{1/2} \Big(\sum_{j=0}^{\infty} \psi_{j-h}^2\Big)^{1/2} = \sum_{j=0}^{\infty} \psi_j^2 < \infty,$$

  so $\gamma_Y(h)$ is finite for all h. Hence $(Y_t)_t$ is weakly stationary with mean 0 and autocovariance $\gamma_Y(h)$.

- The spectral density is the Fourier series of $\gamma_Y(h)$:

  $$S(f) = \sum_{h=-\infty}^{\infty} \gamma_Y(h) e^{-2\pi i f h}, \qquad f \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

Using $\gamma_Y(h) = \sigma_\varepsilon^2 \sum_{j\geq 0} \psi_j \psi_{j-h}$ and absolute convergence (from $\sum_k \psi_k^2 < \infty$), we can interchange the sums:

$$S(f) = \sigma_\varepsilon^2 \sum_{h=-\infty}^{\infty} \sum_{j=0}^{\infty} \psi_j \psi_{j-h} e^{-2\pi i f h} = \sigma_\varepsilon^2 \sum_{j,m\geq 0} \psi_j \psi_m e^{-2\pi i f (j-m)},$$

where we set $m = j - h$. Thus

$$S(f) = \sigma_\varepsilon^2 \Big(\sum_{j\geq 0} \psi_j e^{-2\pi i f j} \Big) \Big(\sum_{m\geq 0} \psi_m e^{2\pi i f m} \Big) = \sigma_\varepsilon^2 \big| \varphi(e^{-2\pi i f}) \big|^2,$$

where $\varphi(z) = \sum_{j=0}^{\infty} \psi_j z^j$.

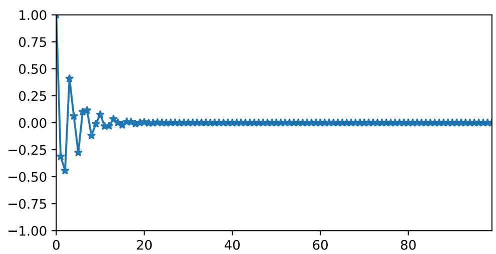
Question 3 *AR(2) process*

Let $\{Y_t\}_{t\geq 1}$ be an AR(2) process, i.e.

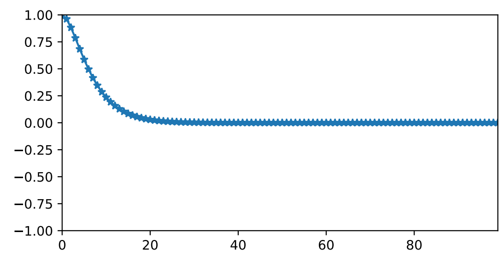
$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t \tag{2}$$

with $\phi_1, \phi_2 \in \mathbb{R}$. The associated characteristic polynomial is $\phi(z) := 1 - \phi_1 z - \phi_2 z^2$. Assume that ϕ has two distinct roots (possibly complex) r_1 and r_2 such that $|r_i| > 1$. Properties on the roots of this polynomial drive the behavior of this process.

- Express the autocovariance coefficients $\gamma(\tau)$ using the roots r_1 and r_2.
- Figure 1 shows the correlograms of two different AR(2) processes. Can you tell which one has complex roots and which one has real roots?
- Express the power spectrum $S(f)$ (assume the sampling frequency is 1 Hz) using $\phi(\cdot)$.
- Choose ϕ_1 and ϕ_2 such that the characteristic polynomial has two complex conjugate roots of norm $r = 1.05$ and phase $\theta = 2\pi/6$. Simulate the process $\{Y_t\}_t$ (with $n = 2000$) and display the signal and the periodogram (use a smooth estimator) on Figure 2. What do you observe?



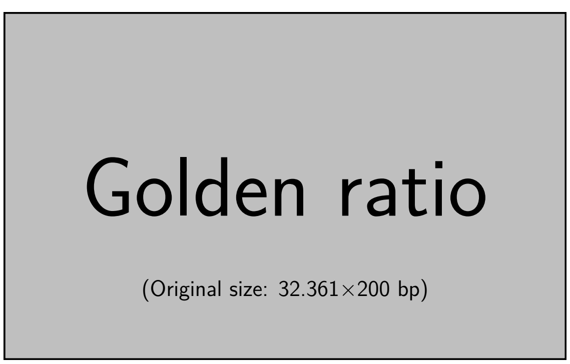
Correlogram of the first AR(2)



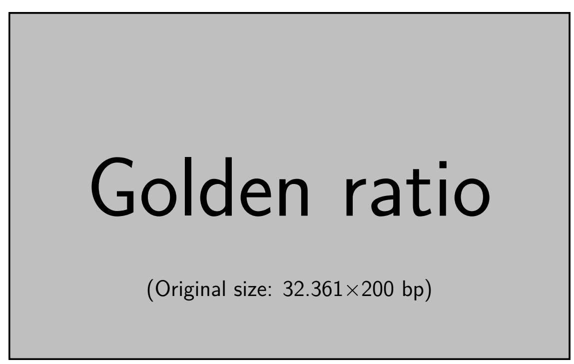
Correlogram of the second AR(2)

Figure 1: Two AR(2) processes

Answer 3



Signal



Periodogram

Figure 2: AR(2) process

4 Sparse coding

The modulated discrete cosine transform (MDCT) is a signal transformation often used in sound processing applications (for instance, to encode an MP3 file). A MDCT atom $\phi_{L,k}$ is defined for a length 2L and a frequency localisation k ($k = 0, \ldots, L-1$) by

$$\forall u = 0, \ldots, 2L-1, \quad \phi_{L,k}[u] = w_L[u]\sqrt{\frac{2}{L}}\cos[\frac{\pi}{L}\left(u + \frac{L+1}{2}\right)(k+\frac{1}{2})] \tag{3}$$

where w_L is a modulating window given by

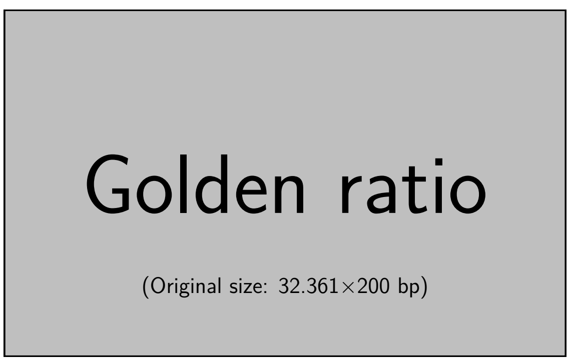
$$w_L[u] = \sin\left[\frac{\pi}{2L}\left(u+\frac{1}{2}\right)\right]. \tag{4}$$

Question 4 *Sparse coding with OMP*

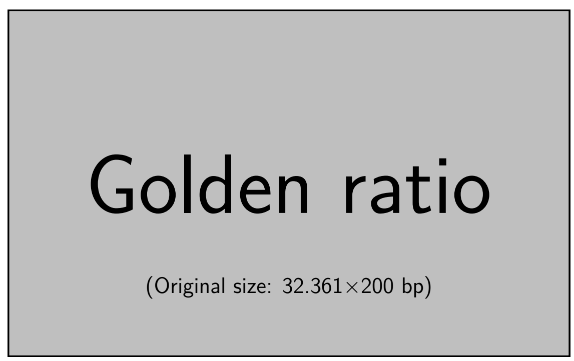
For the signal provided in the notebook, learn a sparse representation with MDCT atoms. The dictionary is defined as the concatenation of all shifted MDCDT atoms for scales L in $[32, 64, 128, 256, 512, 1024]$.

- For the sparse coding, implement the Orthogonal Matching Pursuit (OMP). (Use convolutions to compute the correlation coefficients.)
- Display the norm of the successive residuals and the reconstructed signal with 10 atoms.

Answer 4



Norms of the successive residuals



Reconstruction with 10 atoms

Figure 3: Question 4