Probability: Introduction

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Introduction

1. Toss a coin and observe the resulting face at the top.
2. Roll a die and observe the face coming on top.
3. Invest money in a stock and observe the price of its share a year later.
4. Forecast the weather of tomorrow morning.

etc...

Probability is a measure of uncertainty intrinsic in many real life situations.

Probability is also the foundation of *statistical inference* (or *inferential statistics*).

Outline

Experiment

A **random experiment** (or just **experiment**) is an action that leads to one of several possible outcomes that cannot be predicted.

Sample space

Sample space (S) is the collection of all possible outcomes of an experiment.

Sample points

Outcomes are also known as **sample points**.

Example 1:

- Experiment: Toss a coin and observe the resulting face at the top.
- Sample points: Head (H) and Tail (T).
- Sample space: $S = \{H, T\}$.

Example 2:

- Experiment: Toss a coin twice and observe the outcomes.
- Sample space: $S = \{HH, HT, TH, TT\}$.

Example 3:

- Experiment: Throw a die and observe the face on top.
- Sample space: $S = \{1, 2, 3, 4, 5, 6\}$.

Example 4:

- Experiment: Toss a coin till you get a head.
- Sample space: $S = \{H, TH, TTH, TTTH, \ldots\}$.

Example 5:

- Experiment: Lifetime of a light bulb.
- Sample space: $S = [0, \infty)$.

- Roll a die and observe the face on top.
- How likely it is to get a six?
- What is the chance of getting an even number?

Events

An **event** is a subset of the sample space with which we can attach a probability.

An event with one single outcome is called an *elementary event*.

We say *"the event A occurred"* if an outcome from the event A resulted in the experiment.

EXAMPLE: Rolling a die. $S = \{1, 2, 3, 4, 5, 6\}$.

Events: any subset of S, e.g. $\{1\}$, or $\{2, 4, 6\}$, $\{3, 4, 5, 6\}$ etc.

If the resulting outcome is 2 then

- the event $\{2, 4, 6\}$ occurs,
- but the event $\{1, 5\}$ does not occur.

- Sample space S is the *certain event*.
- Empty set $\emptyset$ is an *impossible event*.
- We typically denote events by capital letters A, B, C etc.

Outline

Symmetric Sample Space

- Suppose the sample space S has finite number of outcomes; say, it contains $N(S)$ number of outcomes.

- Every sample point is considered *equally likely*, if for any $s \in S$,

$$P(\{s\}) = \frac{1}{N(S)}.$$

  Such an S is called a *symmetric sample space*.

- Then, if A is an event with $N(A)$ number of outcomes,

$$P(A) = \frac{N(A)}{N(S)}.$$

- Here (S, P) is called a *symmetric probability space*.

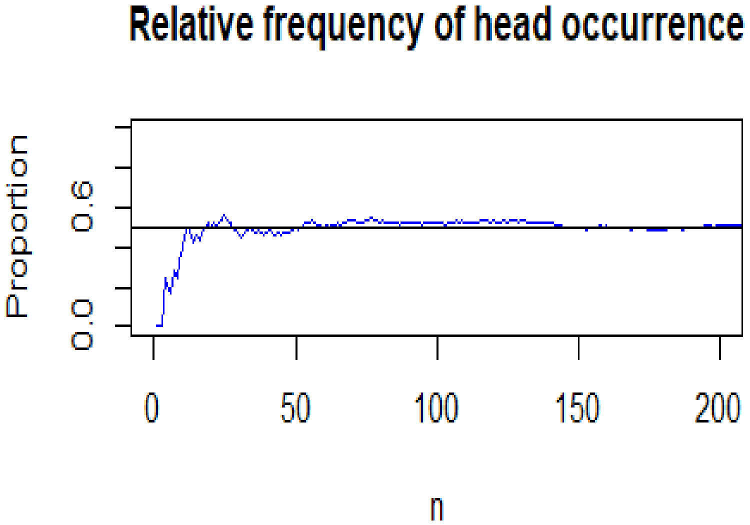
EXAMPLE: Toss a fair coin twice. What is the probability that at least one head occurred?

Symmetric Sample Space: Limitations

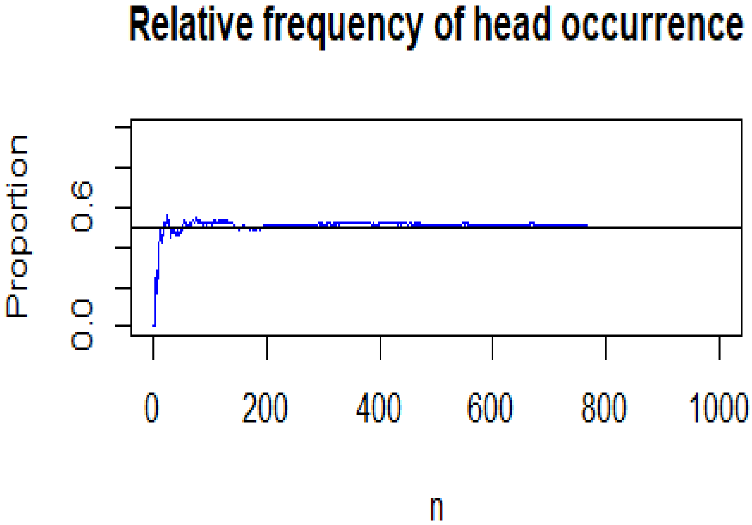
- Cannot be used if S is infinite.
- Considering each sample point being equally likely is not always realistic either.

Outline

What do we mean by $P(H) = \frac{1}{2}$?



WHAT DO WE MEAN BY $P(H) = \frac{1}{2}$?



Relative Frequency & Empirical Law

- Suppose a random experiment can be repeated infinite number of times, and each time it has the same sample space S.
- Suppose the event A occurred $n(A)$ number of times after n number of experiments. Hence its relative frequency is

$$f_n(A) = \frac{n(A)}{n}.$$

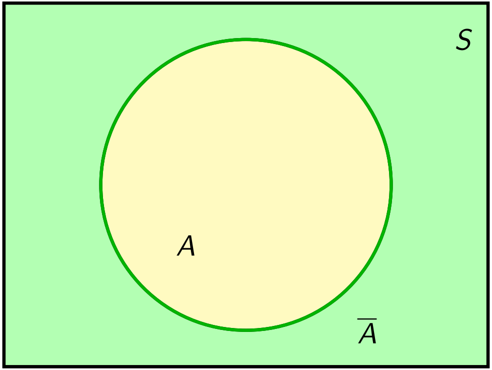
- Then according to empirical law, $P(A) = \lim_{n\to\infty} f_n(A).$

Relative frequency approach: Limitations

- Performing an experiment infinite number of times is not possible.
- Even doing it a very large number of times can be expensive and time consuming.

Outline

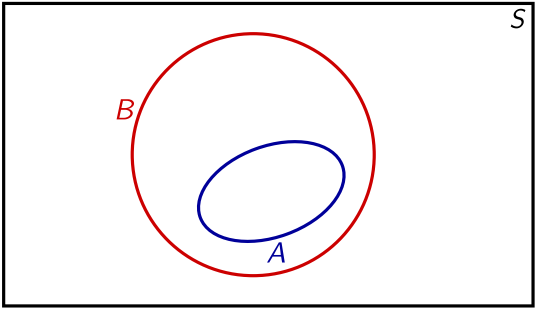
Event and its complement



$\overline{A}$ consists of outcomes which are not in A.

$\overline{A} =$ not A.

SUBSET



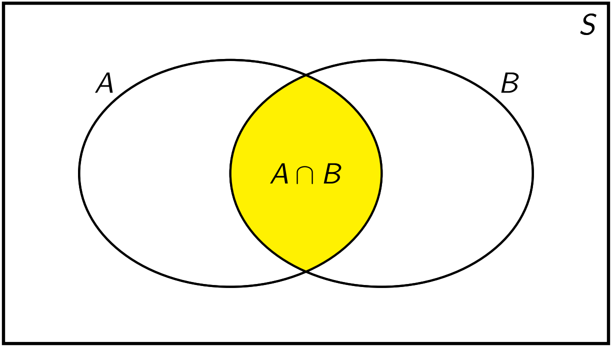
Here A is a subset of B.

NOTATION: $A \subseteq B$ (or $A \subset B$).

If $A \subset B$, but $A \neq B$, we shall write $A \subsetneq B$.

INTERSECTION: $A \cap B$

$A \cap B$ is the event which consists of all outcomes that are both in A and in B.

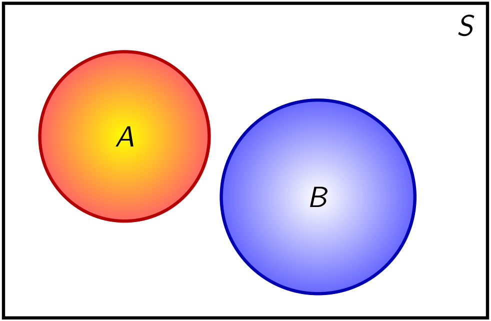


$A \cap B =$ "A and B" $=$ "both A and B".

Mutually Exclusive (Disjoint) Events

A and B are mutually exclusive or disjoint if $A \cap B = \emptyset$,

i.e. A and B have no common outcomes.

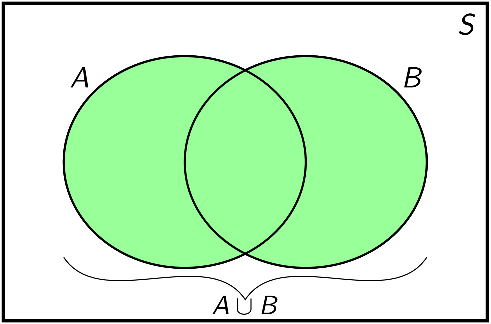


- The events $A_1, A_2, \ldots, A_n$ (or $A_1, A_2, \ldots$) are called *mutually exclusive* (or *pairwise disjoint*) if $A_i \cap A_j = \emptyset$, whenever $i \neq j$.

Union: $A \cup B$

$A \cup B$ contains outcomes which are either in A, or in B, or in both.

On Venn diagram, it is the region that covers A and B.

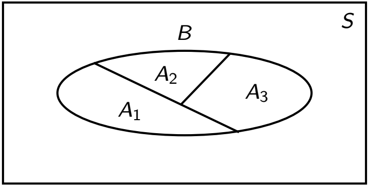


$A \cup B =$ "A or B" $=$ "either A or B or both" $=$ "at least one of A and B".

Partition

The sequence of events $\{A_i\}$ is a **partition** of the event B if

- A_i's are mutually exclusive, and
- $\bigcup_i A_i = B$.



Notice:

1. A and $\overline{A} \cap B$ form a partition of $A \cup B$.
2. $A \cap B$ and $\overline{A} \cap B$ form a partition of B.

Consider the event “exactly one of A and B occurs”.

- How do you show that on Venn diagram?
- What is its relation to “only A” and “only B”?
- How is it related to $A \cup B$ and $A \cap B$?

Some important properties of set algebra

1. $A \cup B = B \cup A$,

   $A \cap B = B \cap A$.

2. $(A \cup B) \cup C = A \cup (B \cup C)$,

   $(A \cap B) \cap C = A \cap (B \cap C)$.

3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
5. **De Morgan's law:**

$$\overline{A \cup B} = \overline{A} \cap \overline{B}, \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}.$$

Outline

Kolmogorov's Axioms

Consider an experiment with sample space S.

A function P which assigns a real number $P(A)$ to every event $A \subseteq S$, is called a **probability** (or **probability measure**) on S if:

1. $P(A) \geq 0, \quad$ for every event A.
2. $P(S) = 1$.
3. for every countable sequence of mutually exclusive events $A_1, A_2, \ldots, A_n \quad (\text{or } A_1, A_2, \ldots), \quad$ then
$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i).$$

The pair (S, P) is called a **probability space**.

Probability properties

For any events A, B and C the following hold:

1. $P(\emptyset) = 0$.

2. $P(\overline{A}) = 1 - P(A)$.

3. If $A \subseteq B$, then $\quad P(A) \leq P(B)$.

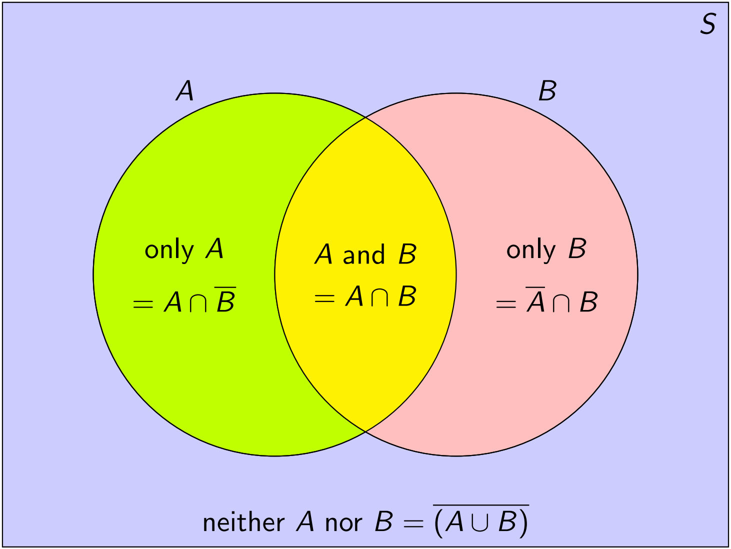
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

5. $$\begin{aligned} P(A \cup B \cup C) = & \ P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C). \end{aligned}$$

EXAMPLE: A survey found out that 56% US university students live in a campus dorm, 62% participate in a campus meal program and 42% do both.

1. What is the probability that a randomly selected university student either lives on campus dorm or has a campus meal plan?

Identifying the regions



EXAMPLE: A survey found out that 56% US university students live in a campus dorm, 62% participate in a campus meal program and 42% do both. What is the probability that a randomly selected university student

2. neither lives on campus nor have a campus meal plan?

3. lives on campus but does not have a campus meal plan?

4. does not live on campus but has a campus meal plan?

5. either lives on campus or has a meal plan, but not both?

Warning: common mistakes

- Although $P(S) = 1$, but $P(A) = 1$ **does not mean** $A = S$.

  That means there may exist set A $(\neq S)$, such that $P(A) = 1$.

- Similarly, $P(A) = 0$ **does not mean** $A = \emptyset$, although $P(\emptyset) = 0$.

- A and B are mutually exclusive means $A \cap B = \emptyset$, and hence $P(A \cap B) = 0$.

  However, $P(A \cap B) = 0$ **does not imply** A and B are mutually exclusive.