Combinatorial Probability

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Why counting is important in probability?

For a symmetric probability space (S, P): $\quad P(A) = \dfrac{N(A)}{N(S)}$.

1. Toss a fair coin <u>twice</u>.
   - $S = \{TT, TH, HT, HH\}$.
   - $P(\text{exactly one head}) = \frac{2}{4} = \frac{1}{2}$.

2. Toss a fair coin <u>20 times</u>.
   - $S =??$
   - $P(\text{exactly one head}) = \frac{\#\text{outcomes with exactly one head}}{N(S)} =?$

Outline

Addition Rule

When $A_1, A_2, \ldots, A_k$ are mutually exclusive events, then:

$$N\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} N(A_i).$$

outcomes of the experiment are possible.

Outline

Example

- At a restaurant, you can choose one of 2 starters:

  soup, salad.

- and you can choose one from 3 main dishes, namely:

  fish, meat, vegetarian.

- How many different two-course meals are possible?

Product Rule

- Suppose an experiment consists of performing k partial experiments.
- The i^{th} partial experiment has n_i possible outcomes $(i = 1, \ldots, k)$,
- no matter what the results of other partial experiments are.
- Then there are total

$$n_1 \times n_2 \times \cdots \times n_k$$

outcomes of the experiment are possible.

How many different ordered arrangements of the letters a, b, and c are possible?

How many different ordered arrangements of n distinct objects are possible?

Factorial

For any positive integer n we define n *factorial* as

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1.$$

Convention: $0! = 1$.

For instance:

- $1! = 1$,
- $2! = 2 \times 1 = 2$,
- $3! = 3 \times 2 \times 1 = 6$,
- $4! = 4 \times 3 \times 2 \times 1 = 24$, etc.

Note, $n! = n \times (n-1)!$.

Suppose there are 3 balls of identical shape - two of them are red (r) and one is blue (b). How many different ordered arrangements of these balls are possible?

Permutation of indistinguishable objects

- Suppose, of n objects, there are k distinct types.
- n_1 of them are of type 1.

  n_2 of them are of type 2.

  $\vdots \qquad\qquad\qquad \vdots$

  n_k of them are of type k.

  $n = n_1 + n_2 + \cdots + n_k$.
- Then there are total

$$\frac{n!}{n_1! n_2! \cdots n_k!} \quad \text{arrangements possible.}$$

Outline

Example: How many different samples?

- Suppose we select 2 objects from the collection $\{a, b, c, d\}$.
- How many different samples are possible?

It depends on:

- whether sampling is done
  - with replacement (WR), or
  - without replacement (WOR),
- whether it is an
  - ordered sampling, or
  - unordered sampling.

How many different samples?

	Without Replacement	With Replacement
Ordered	$\dfrac{n!}{(n-k)!}$	n^k
Unordered	$\dbinom{n}{k}$	$\dbinom{n+k-1}{k}$

Remark: In the "*with replacement unordered sampling scheme*", the resulting sample space is not symmetric.

WR and Ordered sampling

- Suppose we select 2 objects from the collection $\{a, b, c, d\}$ with replacement (WR).
- How many different samples are possible if order matters?

		2^{nd} draw			
		a	b	c	d
1^{st} draw	a	(a, a)	(a, b)	(a, c)	(a, d)
	b	(b, a)	(b, b)	(b, c)	(b, d)
	c	(c, a)	(c, b)	(c, c)	(c, d)
	d	(d, a)	(d, b)	(d, c)	(d, d)

WR Ordered Sampling

- Suppose there are n number of distinct elements.
- A sample of k elements is to be selected with replacement.
- The number of different *ordered* samples possible is

$$n^k.$$

EXAMPLE: If we select 2 elements out of 4 with replacement, then there are

$$4^2 = 16 \text{ different ordered samples possible.}$$

WOR and Ordered sampling

- Suppose we select 2 objects from the collection $\{a, b, c, d\}$ with replacement (WR).
- How many different samples are possible if order matters?

		2^{nd} draw			
		a	b	c	d
1^{st} draw	a		(a, b)	(a, c)	(a, d)
	b	(b, a)		(b, c)	(b, d)
	c	(c, a)	(c, b)		(c, d)
	d	(d, a)	(d, b)	(d, c)	

WOR Ordered Sampling

- Suppose there are n number of distinct elements.
- A sample of k elements is to be selected without replacement ($k \leq n$).
- The number of different *ordered* samples possible is

$$\frac{n!}{(n-k)!}.$$

EXAMPLE: If we select 2 elements out of 4 without replacement, then there are

$$\frac{4!}{(4-2)!} = 12 \text{ different ordered samples possible.}$$

WOR and Unordered sampling

- Suppose we select 2 objects from the collection $\{a, b, c, d\}$ without replacement (WOR).
- How many different samples are possible if order does not matter?

$$
\begin{array}{cc|cccc|}
 & & \multicolumn{4}{c}{\textbf{2}^{nd}\textbf{ draw}} \\
 & & a & b & c & d \\
\hline
\multirow{4}{*}{\textbf{1}^{st}\textbf{ draw}} & a & & (a,b) & (a,c) & (a,d) \\
 & b & (b,a) & & (b,c) & (b,d) \\
 & c & (c,a) & (c,b) & & (c,d) \\
 & d & (d,a) & (d,b) & (d,c) & \\
\hline
\end{array}
$$

WOR and Unordered sampling

- Suppose we select 2 objects from the collection $\{a, b, c, d\}$ without replacement (WOR).
- How many different samples are possible if order does not matter?

		2^{nd} draw			
		a	b	c	d
1^{st} draw	a		(a, b)	(a, c)	(a, d)
	b	(b, a)		(b, c)	(b, d)
	c	(c, a)	(c, b)		(c, d)
	d	(d, a)	(d, b)	(d, c)	

Unordered sample

$\{a, b\}$	$\{a, c\}$	$\{a, d\}$
	$\{b, c\}$	$\{b, d\}$
		$\{c, d\}$

WOR Unordered Sampling

- Suppose there are n number of distinct elements.
- A sample of k elements is to be selected without replacement $(k \leq n)$.
- The number of different *unordered* samples possible is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The symbol $\binom{n}{k}$ is often called n *choose* k.

EXAMPLE: If we select 2 elements out of 4 without replacement, then there are

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = 6 \text{ different unordered samples possible.}$$

WR and Unordered sampling

- Suppose we select 2 objects from the collection $\{a, b, c, d\}$ with replacement (WR).
- How many different samples are possible if order does not matter?

		2^{nd} **draw**			
		a	b	c	d
1^{st} **draw**	a	(a, a)	(a, b)	(a, c)	(a, d)
	b	(b, a)	(b, b)	(b, c)	(b, d)
	c	(c, a)	(c, b)	(c, c)	(c, d)
	d	(d, a)	(d, b)	(d, c)	(d, d)

WR and Unordered sampling

- Suppose we select 2 objects from the collection $\{a, b, c, d\}$ with replacement (WR).
- How many different samples are possible if order does not matter?

		2^{nd} draw				**Unordered sample**			
		a	b	c	d				
1^{st} draw	a	(a, a)	(a, b)	(a, c)	(a, d)	$\{a, a\}$	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$
	b	(b, a)	(b, b)	(b, c)	(b, d)		$\{b, b\}$	$\{b, c\}$	$\{b, d\}$
	c	(c, a)	(c, b)	(c, c)	(c, d)			$\{c, c\}$	$\{c, d\}$
	d	(d, a)	(d, b)	(d, c)	(d, d)				$\{d, d\}$

N.B.: Sample space is NOT symmetric.

i.e., outcomes are not equally likely.

WR Unordered Sampling

- Suppose there are n number of distinct elements.
- A sample of k elements is to be selected with replacement.
- The number of different *unordered* samples possible is

$$\binom{n+k-1}{k}.$$

EXAMPLE: If we select 2 elements out of 4 with replacement, then there are

$$\binom{4+2-1}{2} = \binom{5}{2} = 10$$

different unordered samples possible.

Outline

Example

If we toss a fair coin 20 times, what is the probability of getting exactly one head?

Example

If we toss a fair coin 20 times, what is the probability of getting exactly k heads?

Example

If we randomly pick two cards from a pack of cards without replacement, what is the chance that both selected cards are diamonds?

Example

If we randomly pick two cards from a pack of cards without replacement, what is the chance that one diamond and one spade are selected?

Example

If we randomly pick 2 cards from a pack of cards without replacement, what is the chance that exactly one diamond is selected?