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Asymptotic and bootstrap prediction regions for vector autoregression

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Abstract

Small sample properties of asymptotic and bootstrap prediction regions for VAR models are evaluated and compared. Monte Carlo simulations reveal that the bootstrap prediction region based on the percentile-*t* method outperforms its asymptotic and other bootstrap alternatives in small samples. It provides the most accurate assessment of future uncertainty under both normal and non-normal innovations. The use of an asymptotic prediction region may result in a serious under-estimation of future uncertainty when the sample size is small. When the model is near non-stationary, the use of the bootstrap region based on the percentile-*t* method is recommended, although extreme care should be taken when it is used for medium to long-term forecasting.

Keywords: VAR model; Prediction regions; Bootstrap; Backward VAR model

1. Introduction

The vector autoregressive (VAR) model (Sims, 1980) has been a popular tool for forecasting a set of economic time series. Many past studies favored VAR models over large scale econometric or other time series models as a forecasting tool (see, among others, Fackler & Krieger, 1986; Trevor & Thorp, 1988; Liu, Gerlow & Irwin, 1994). However, past studies focused on evaluation of point forecasts, while interval forecasting in the VAR context received little attention. For example, Simkins (1995), Bessler and Babula (1987), Litterman (1986) and McNees (1986) used only point forecasts to evaluate forecasting ability of VAR models. This practice is seriously flawed because the use of point forecasts is meaningless if the extent of the associated uncertainty is unknown. As Chatfield (1993) stressed, interval forecasts should be used (together with or instead of point forecasts) as they provide more informative assessment of future uncertainty by giving upper and lower limits of forecasts with a prescribed probability.

An interval forecast for VAR takes the form of ellipsoid (hereafter called prediction ellipsoid and denoted PE). A cubical approximation based on Bonferroni's method (Lütkepohl, 1991) can be made to PE to yield what may be called the prediction cube (PC), as illustrated in Fig. 1 for the bivariate case. The individual intervals (or component intervals) that form the PC are called the prediction intervals (PIs) in this paper. These PIs can be calculated by using the asymptotic formula given in

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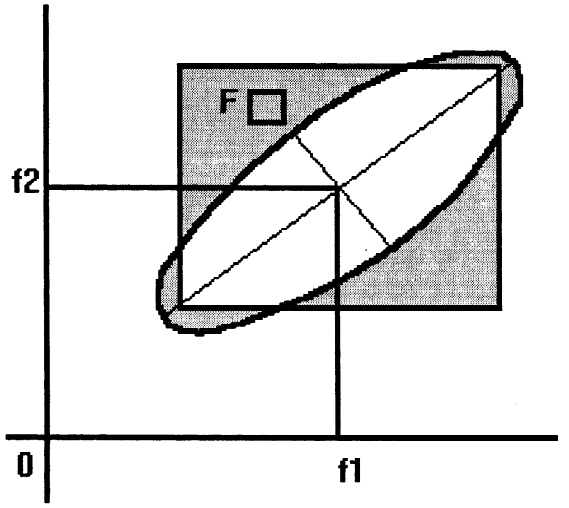

Fig. 1. Prediction ellipsoid and prediction cube.

Lütkepohl (1991).[1] Alternatively, the bootstrap method (Efron, 1979) can be used, which has been applied widely to econometric and time series modelling (see Jeong & Maddala, 1993; Li & Maddala, 1996). In the univariate AR case, Masarotto (1990) and Thombs and Schucany (1990) examined small sample properties of bootstrap PIs to find that they are useful alternatives to the asymptotic one in small samples. More recently, Fachin and Bravetti (1996) and Rilstone and Veall (1996) found that the bootstrap provides a superior small sample alternative to the asymptotic method, in the context of statistical inferences in structural VAR and seemingly unrelated regression models.

It seems that Lütkepohl's (1991) asymptotic formula is not well-known to practitioners or it could be somewhat difficult to implement. From the theoretical point of view, small sample properties of this asymptotic formula are unknown and the bootstrap may provide a superior alternative in small samples. In this paper, Monte Carlo simulations are conducted to examine small sample properties of asymptotic and bootstrap prediction regions for VAR models. Bootstrap prediction regions are constructed based on the percentile and percentile-*t* methods (see Hall, 1988, p. 937; Davison & Hinkley, 1997, p. 194). It should be noted that we use what Hall (1988) referred to as the correct percentile methods.[2] Bootstrap regions based on the incorrect percentile methods (Hall, 1988) and the BC method (Efron, 1981) are evaluated in simulation, but their details are not reported as they are found to be inferior to those based on the correct percentile methods.

The major finding of this paper is that the bootstrap prediction region based on the percentile-*t* method performs better than other alternatives in small samples, providing the most accurate assessment of future uncertainty under both normal and non-normal innovations. In the next section, asymptotic and bootstrap prediction regions for VAR forecasts are introduced. Sections 3 and 4 present experimental design and simulation results. An empirical example is given in Section 5, and the conclusions are drawn in Section 6.

2. Asymptotic and bootstrap prediction regions

Consider the K-dimensional VAR(p) model given by

$$Y_t = \nu + A_1 Y_{t-1} + \cdots + A_p Y_{t-p} + u_t, \quad t = 0, 1, 2, \ldots \tag{1}$$

where Y_t is a $K\times 1$ random vector, A_is are $K\times K$ matrices of coefficients, ν is a $K\times 1$ vector of intercept terms. The $K\times 1$ vector of i.i.d. innovations u_t is such that $E(u_t)=0$ and $E(u_t u_t')=\Sigma_u$, where Σ_u is a $K\times K$ symmetric positive definite matrix with finite elements. The model is stationary so that all roots of the characteristic equation $\det(I_K - A_1 z - \cdots - A_p z^p)=0$ lie outside the unit circle, where I_K is a $K\times K$ identity matrix and $\det(\cdot)$ denotes the determinant of a matrix.

Given n realizations $(Y_1, \ldots, Y_n)$ of (1), the least squares (LS) method is used to estimate unknown parameters. Let $(\hat{\nu}, \hat{A}_1, \ldots, \hat{A}_p)$ denote the LS estimators for $(\nu, A_1, \ldots, A_p)$ and $\hat{u}_t$s the LS residuals (see, for details of LS estimation,

[1] To the best of my knowledge, Lütkepohl's (1991) method is the only conventional (non-bootstrap) method for VAR interval forecasting proposed in the literature.

[2] The distinction between the correct and incorrect percentile methods was made clear by an anonymous referee.

Lütkepohl, 1991, p. 63). Forecasts for the period $n+h$ made at n are generated as

$$\hat{Y}_n(h) = \hat{\nu} + \hat{A}_1\hat{Y}_{n-1}(h) + \cdots + \hat{A}_p\hat{Y}_{n-p}(h),$$

where $\hat{Y}_n(j) = Y_{n+j}$ for $j \leq 0$. The asymptotic mean squared error (MSE) matrix of $\hat{Y}_n(h)$ is denoted as $\Sigma_Y(h)$ and its details are given in Lütkepohl (1991, p. 86–87). It can be shown that

$$[Y_{n+h} - \hat{Y}_n(h)]'\hat{\Sigma}_Y(h)^{-1}[Y_{n+h} - \hat{Y}_n(h)] \xrightarrow{a} \chi^2(K), \quad (2)$$

where '$\xrightarrow{a}$' indicates 'asymptotically distributed' and $\hat{\Sigma}_Y(h)$ is obtained by replacing unknowns in $\Sigma_Y(h)$ with their LS estimators. Based on (2), what is called the asymptotic PE (APE) with probability content (or nominal coverage) $100(1-\alpha)\%$ can be defined as

$$\text{APE}_{1-\alpha} \equiv \{[Y_{n+h}: [Y_{n+h} - \hat{Y}_n(h)]'\hat{\Sigma}_Y(h)^{-1} \times [Y_{n+h} - \hat{Y}_n(h)] \leq \chi^2_{1-\alpha}(K)\}, \quad (3)$$

where $\chi^2_{1-\alpha}(K)$ indicates the $100(1-\alpha)$th percentile of $\chi^2(K)$ and $0 < \alpha < 1$. The expression (3) defines the interior of an ellipsoid on the K-dimensional plane of forecasts. The asymptotic prediction cube (APC) can be obtained by giving a cubical approximation to (3) based on Bonferroni's method (Lütkepohl, 1991, pp. 34–35). The idea is that if we choose intervals with probability content $100(1-\alpha/K)\%$ for each K component of (3), the resulting region will have probability at least $100(1-\alpha)\%$. It can be defined as

$$\text{APC}_{1-\alpha} \equiv \left\{ Y_{n+h}: Y_{n+h} \in \bigcup_{k=1}^{K} [\hat{Y}_{k,n}(h) \pm z_\tau \hat{\sigma}_k(h)] \right\},$$

where $\hat{Y}_{k,n}(h)$ is the kth component of $\hat{Y}_n(h)$, z_τ is the upper 100τth percentile of the standard normal distribution with $\tau = 0.5(\alpha/K)$ and $\hat{\sigma}_k(h)$ is the square root of the kth diagonal element of $\hat{\Sigma}_Y(h)$. Note that the symbol $\cup_{k=1}^{K}$ indicates 'the union of' or 'the collection' of K components. That is, APC consists of K component intervals called the asymptotic PIs (APIs).

In bootstrapping PIs for univariate AR models, Masarotto (1990) and Thombs and Schucany (1990) used the non-parametric bootstrap based on residual resampling adopted analogously in this paper. It involves generation of a large number of pseudo-datasets by resampling residuals, obtained from the fitted model, with replacement. The bootstrap PIs of Masarotto (1990) are constructed from bootstrap replicates of unconditional forecasts based on the percentile-t method, while those of Thombs and Schucany (1990) are obtained from bootstrap replicates of conditional forecasts based on the percentile method. In the latter case, the conditionality of AR forecasts on past observations is explicitly taken into account by generating pseudo-datasets based on the backward AR representation. Since this conditionality is an important factor which influences small sample properties of AR forecasts (see, for the univariate AR(1) case, Maekawa, 1987), this paper follows Thombs and Schucany (1990) in generating bootstrap replicates.

The backward VAR model associated with the forward one (1) can be written as

$$Y_t = \mu + H_1 Y_{t+1} + \cdots + H_p Y_{t+p} + \nu_t, \quad t = 0, 1, 2, \ldots, \quad (4)$$

where H_is are $K \times K$ matrices of coefficients, μ is a $K \times 1$ vector of intercept terms. The $K \times 1$ random vector v_t is such that $E(v_t) = 0$ and $E(\nu_t \ \nu_t') = \Sigma_\nu$, where Σ_v is a $K \times K$ symmetric positive definite matrix with finite elements. The forward and backward VAR models (1) and (4) are closely related; for example, $H_1' = \Gamma^{-1}A_1\Gamma$ where $\Gamma = E(Y_tY_t')$ for the VAR(1) case. Further details on the relationship between the two VAR representations can be found in Kim (1997, 1998).

Let $(\hat{\mu}, \hat{H}_1, \ldots, \hat{H}_p)$ denote the LS estimators for $(\mu, H_1, \ldots, H_p)$ and $\hat{\nu}_t$s the LS residuals. A pseudo-dataset can be generated recursively based on (4) as

$$Y_t^* = \hat{\mu} + \hat{H}_1 Y_{t+1}^* + \cdots + \hat{H}_p Y_{t+p}^* + \nu_t^*, \quad (5)$$

where the p starting values are set equal to the last p values of the original series and ν_t^* is a random draw from $\{\hat{\nu}_t\}_{t=1}^n$ with replacement. This yields pseudo-data sets whose final p observations, on which VAR forecasts are conditional, are identical to the last p values of the original series. The bootstrap forecast, for the period $n+h$ made at time n, can be generated based on (1) as

$$Y_n^*(h) = \nu^* + A_1^* Y_n^*(h-1) + \cdots + A_p^* Y_n^*(h-p) + u_{n+h}^*,$$

where $Y_n^*(j) = Y_{n+j}$ for $j \leq 0$, u_{n+h}^* is a random draw from $\{\hat{u}_t\}_{t=1}^n$ with replacement and $(\nu^*, A_1^*, \ldots, A_p^*)$ are the LS estimators for $(\nu, A_1, \ldots, A_p)$ obtained from the pseudo-dataset $\{Y_t^*\}_{t=1}^n$. Repeated generation of pseudo-datasets, say B times, will yield the bootstrap forecast distribution $\{Y_n^*(h;i)\}_{i=1}^B$ which is conditional on the last p observations of the original series. The bootstrap counterparts of APE and APC are now introduced. Consider the quadratic form

$$[Y_{n+h} - \hat{Y}_n^*(h)]' \hat{\Sigma}_Y^*(h)^{-1} [Y_{n+h} - \hat{Y}_n^*(h)],$$

where $\hat{\Sigma}_Y^*(h)$ is the LS estimator of $\Sigma_Y(h)$ obtained from $\{Y_n^*(h;i)\}_{i=1}^B$ and $\hat{Y}_n^*(h)$ is the sample mean of $\{Y_n^*(h;\ i)\}_{i=1}^B$. The bootstrap PE (BPE), with probability content $100(1-\alpha)\%$, is defined as

$$\begin{aligned}\mathrm{BPE}_{1-\alpha} \equiv \{Y_{n+h}: [Y_{n+h} - \hat{Y}_n^*(h)]'\ \hat{\Sigma}_Y^*(h)^{-1} \\ \times [Y_{n+h} - \hat{Y}_n^*(h)] \leq Q_{1-\alpha}^*\},\end{aligned} \tag{6}$$

where $Q_{1-\alpha}^*$ is the $100(1-\alpha)$th percentile of the distribution

$$\{[Y_n^*(h;\ i) - \hat{Y}_n^*(h)]' \hat{\Sigma}_Y^*(h)^{-1} [Y_n^*(h;\ i) - \hat{Y}_n^*(h)]\}_{i=1}^B.$$

The bootstrap PC (BPC) can be obtained by giving a cubical approximation to (6) based on Bonferroni's method. The BPC based on the percentile method with probability content at least $100(1-\alpha)\%$ can be defined as

$$\begin{aligned}\mathrm{BPC}_{p,1-\alpha} \equiv \Big\{ Y_{n+h}: Y_{n+h} \\ \in \bigcup_{k=1}^K [2\hat{Y}_{k,n}(h) - Y_{k,n}^*(h, 1-\tau), 2\hat{Y}_{k,n}(h) \\ - Y_{k,n}^*(h, \tau)] \Big\}\end{aligned}$$

where $Y_{k,n}^*(h,\ \tau)$ is the 100τth percentile of $\{Y_{k,n}^*(h;\ i)\}_{i=1}^B$, the kth component of the bootstrap distribution $\{Y_n^*(h;\ i)\}_{i=1}^B$. The BPC based on the percentile-t method, with probability content at least $100(1-\alpha)\%$, is constructed from the studentized bootstrap distribution of the VAR forecasts and defined as

$$\begin{aligned}\mathrm{BPC}_{pt,1-\alpha} \equiv \Big\{ Y_{n+h}: Y_{n+h} \\ \in \bigcup_{k=1}^K [\hat{Y}_{k,n}(h) - z_{k,n}^*(h, 1-\tau)\hat{\sigma}_k(h), \hat{Y}_{k,n}(h) \\ - z_{k,n}^*(h, \tau)\hat{\sigma}_k(h)] \Big\},\end{aligned}$$

where $z_{k,n}^*(h,\ \tau)$ is the 100τth percentile of

$$\{z_{k,n}^*(h;\ i)\}_{i=1}^B,\ z_{k,n}^*(h;\ i) = \frac{Y_{k,n}^*(h;\ i) - \hat{Y}_{k,n}(h)}{\hat{\sigma}_k^*(h)}$$

and $\hat{\sigma}_k^*(h)$ is the bootstrap counterpart of $\hat{\sigma}_k(h)$. Note that bootstrap PIs (BPIs) based on the percentile and percentile-t methods are obtained as K component intervals of the respective BPCs defined above.

It can easily be shown that the bootstrap forecasts converge to the true future values in distribution, by extending Theorem 3.1 of Thombs and Schucany (1990) to the VAR case. This property justifies the use of the bootstrap as a small sample alternative to the asymptotic method for VAR interval forecasting. In what follows, Monte Carlo simulations are conducted to examine which method is superior when VAR models with sample sizes frequently employed in time series econometrics are considered.

3. Experimental design

Small sample properties of the APC and BPCs are evaluated by simulating a number of bivariate VAR models of orders 1 and 2. The models are chosen so that important parts of the parameter space — such as stationarity and near non-stationarity, and the cases of real and complex characteristic roots — are systematically dealt with. The VAR(1) models considered are $\mathrm{vec}(A_1) = (0.5,\ -0.6,\ 0.3,\ \beta)'$ and $\beta \in \{-0.4,\ 0.5,\ 1.3\}$, where vec denotes the column stacking operator. As β varies from -0.4 to 1.3, the moduli of the roots change from (5, 10), (1.53, 1.53) to (1.10, 1.10). Note that the roots are complex when $\beta = 0.5$. The VAR(2) models considered are labelled M1 to M4 and presented in Table 1 with their characteristic roots. Models M1 to M3 have their roots, some of which are complex, not so close to the non-stationarity region; and model M4 is near non-stationary with two of its roots fairly close to the unit circle. All VAR(2) models given in Table 1 have an off-diagonal element of $(I_K - A_1 z - \cdots - A_p z^p)$ equal to zero. This is only for simplicity; additional simulations for VAR(2) models with all elements of $(I_K - A_1 z - \cdots - A_p z^p)$ non-zero provide similar results.

All simulations are conducted with $n = 25$, 50 and

Table 1
Design of VAR(2) models

	α_1	α_2	β_1	β_2	Characteristic roots
M1[a]	0.9	−0.2	−0.7	−0.1	2.5, 2, −2, −5
M2[a]	0.9	−0.2	−0.5	−0.125	2.5, 2, $-2\pm2i$
M3[a]	0.2	−0.5	−0.5	−0.125	$0.2\pm1.4i$, $-2\pm2i$
M4[a]	0.4	0.45	1.4	−0.45	2, −2, 1.11, 1.11

[a] Note. Models M1 to M4 are based on the following form:

$$I_2 - A_1B - A_2B^2 = \begin{bmatrix} 1-\alpha_1B-\alpha_2B^2 & 0 \\ 0.5B-0.8B^2 & 1-\beta_1B-\beta_2B^2 \end{bmatrix}$$

where B is the backward shift operator.

100; and $h=1$, 4 and 8. The vector of intercepts ν is set to $(0, 0)'$, but is assumed to be unknown and estimated. The structure of the innovation covariance matrix considered is restricted to $\text{vech}(\Sigma_u)=(1, 0.5, 1)'$, where vech is the column stacking operator that stacks elements on and below the diagonal only. Preliminary simulations revealed that the results do not change qualitatively if more general structures of Σ_u are employed. The values of nominal coverage $100(1-\alpha)\%$ chosen for PCs are 90% and 95%, so that the corresponding PIs have nominal coverages 95% and 97.5%, respectively. The number of Monte Carlo iterations is set to 500, and the number of bootstrap iterations B to 999 (this choice was suggested by a referee and is to avoid the discreteness problem; see Booth and Hall, 1994). In generating VAR processes, the initial values are set to the vector of zeros; and 100 pre-sample values are generated to reduce the impact of the initial values. Computations are conducted on an IBM Pentium PC using GAUSS version 3.2.[3] A simulation run for a VAR model with the sample size 50 takes about 1.5 h.

The empirical coverage and volume are used as measures of comparing small sample performances of alternative PCs. The empirical coverage of a PC is defined as

$$\text{EC(PC)} = \#\{Y_{n+h}(i)\colon Y_{n+h}(i) \in \text{PC}\}/F, \quad i=1,\ldots,F,$$

where $\#$ indicates 'the number of' and $\{Y_{n+h}(i)\}_{i=1}^{F}$ are the true future values generated based on (1) using the last p values of the original series as starting values. The value of F is set equal to 100. The volume of a PC can be calculated by multiplying the lengths of the corresponding PIs; and the volume of an $\text{APE}_{1-\alpha}$ with the known MSE matrix, denoted V(PE), by using the formula

$$\text{V(PE)} = [\pi^{0.5K}/\Gamma(0.5K+1)][\chi^2_{1-\alpha}(K)]^{0.5K}\{\det[\Sigma_Y(h)^{-1}]\}^{-0.5},$$

where $\Gamma(\cdot)$ is the gamma function.

Since the assumption of normality may not be justifiable in small samples, it is of interest to examine how prediction regions perform under non-normal innovations. This paper also examines small sample properties of VAR prediction regions under non-normal innovations. The bivariate Student-t distribution with five degrees of freedom is used as a representative of distributions with a fat-tail behavior, and the (centered) bivariate chi-squared distribution with four degrees of freedom as a representative of asymmetric distributions. These non-normal distributions are adjusted so that the innovation variance-covariance structure satisfies $\text{vech}(\Sigma_u)=(1, 0.5, 1)'$.

When innovations are non-normal, the backward innovations ν_ts are uncorrelated but dependent. As a result, resampling backward residuals $\{\hat{\nu}_t\}_{t=1}^{n}$ as in (5) is mis-specified. One way of circumventing this problem is to resample forward residuals $\{\hat{u}_t\}_{t=1}^{n}$, whose underlying innovations are independent, and obtain backward residuals by using the relationship between the forward and backward VAR representations. McCullough (1994) adopted a similar idea in his empirical study for the bootstrap PI's in the univariate AR case. For example, in the VAR(1) case, forward and backward innovations are related as

$$\begin{aligned}\nu_t &= -H_1(I_K - H_1^{-1}B)(I_K - A_1B)^{-1}u_{t+1} \\ &= -H_1(I_K + C_1B + C_2B^2 + \ldots)u_{t+1},\end{aligned}$$

where B is the backshift operator, $C_1 = A_1 - H_1^{-1}$ and $C_k = A_1C_{k-1}$ for $k \geq 2$. The infinite polynomial in B above can be truncated at some point j where the contribution of H_1C_j becomes negligible. How-

[3] The Gauss codes for computation of asymptotic and bootstrap PIs are available from the author on request.

ever, the use of the above relation can involve a large number of estimated parameters, which can add substantial extra sampling variability to the bootstrap procedure. In this paper, direct resampling of backward residuals is adopted.

4. Simulation results

Table 2 presents the mean coverages and volumes of PCs under normal innovations when the nominal coverage is 90% (or the nominal coverage for individual PIs is at least 95%). It can be seen that APC under-estimates the nominal coverage, especially when the sample size is small or the model is near non-stationary. BPC_p tends to under-estimate the nominal coverage, and its coverage properties are in general inferior to those of BPC_{pt}. It is evident that BPC_{pt} outperforms the other alternatives with its mean coverages closest to the nominal coverage. It under-estimates the nominal coverage to a degree when $n = 25$ or the model is near non-stationary, but the extent of under-estimation is much smaller than those associated with APC and BPC_p. When the sample size is 100, all PCs exhibit desirable coverage properties even when the model is near non-stationary. Comparing the volume properties, it can be seen that APC is smaller than BPCs; and, in general, BPC_{pt} is larger than BPC_p. The differences can be substantial when the sample size is small and the model is near non-stationary. The differences in volumes become larger as the model tends to non-stationarity and smaller as sample size increases. These volume properties, combined with the coverage properties observed, suggest that BPC_{pt} provides the most accurate assessment of future uncertainty in small samples. APC, on the other hand, is often too small and under-estimates the future uncertainty.

The use of PC as an approximation to PE can result in a serious over-estimation of future uncertainty, especially when forecasts are more than moderately correlated. Fig. 1 illustrates the case where forecasts are moderately correlated in the bivariate case. It can be seen that the volume (or area) of the PC is much larger than that of the PE. Suppose we want to evaluate whether the region F is likely to be realized in the future. The use of PC is apparently misleading in this case as the region F is located outside the PE but inside the PC. The extent of over-estimation increases as the correlation between components of VAR forecasts increases. Note that forecasts from near non-stationary models can be highly correlated contemporaneously. The last column of Table 2 reports the values of V(PE), which can be compared with the mean volumes of PCs in order to measure the extent of over-estimation. The extent of over-estimation is small or moderate when β is -0.4 and 0.5, but it is serious when model is near non-stationary with $\beta = 1.3$ even when the sample size is 100. It can also be observed that the extent of over-estimation increases as h increases. It is advisable in practice to check if a future scenario contained in a PC is also contained in the corresponding PE. If the future values of interest contained in PC satisfy the inequalities in (3) or (6), it is more assuring that these future values are also contained in the PE. This check is important especially when the model is near non-stationary and the forecast lead is long.

Table 3 reports mean coverages and volumes of PCs for selected cases of VAR(2) models under different types of innovations when the nominal coverage is 95%. It can be seen that BPCs are superior to APC in coverage properties; and, in particular, BPC_{pt} outperforms the others in most cases. As before, APC tends to under-estimate the nominal coverage more than its bootstrap alternatives. These features are evident under both normal and non-normal innovations. In volume properties, APC is always smaller than its bootstrap alternatives. It is interesting to observe that, for both APC and BPCs, the mean volumes under non-normal innovations are larger than those under normal. This means that higher uncertainty is associated with VAR forecasts when innovations are generated from non-normal distributions. Moreover, when PC is used as an approximation to PE, the extent of over-estimation would be larger under non-normal innovations than under normal.

The case of near non-stationary models is worth mentioning. Although it can be observed from Tables 2 and 3 that BPC_{pt} performs most desirably in this case, the mean coverages of all PCs, including BPC_{pt}, deteriorate substantially as h increases. BPC_{pt} exhibits reasonably good performance only when $h = 1$. This suggests that, although BPC_{pt} should be

Table 2
Coverages and volumes of prediction regions for VAR(1) models (nominal coverage 90%, normal innovations)

h	Mean coverage			Mean volume			V(PE)
	APC	BPC_p	BPC_{pt}	APC	BPC_p	BPC_{pt}	
$n=25,\ \beta=-0.4$							
1	87.7	87.2	89.5	16.4	17.4	19.4	13.5
4	89.1	90.2	90.2	28.0	30.3	30.9	24.5
8	89.5	90.8	90.5	28.0	30.8	31.1	24.5
$n=25,\ \beta=0.5$							
1	87.7	87.4	89.6	16.4	17.6	19.7	13.5
4	88.3	88.9	89.2	30.6	31.9	33.7	27.3
8	89.0	89.9	89.6	31.3	33.5	34.0	27.3
$n=25,\ \beta=1.3$							
1	85.1	87.4	89.4	16.1	19.2	21.2	13.5
4	80.6	85.9	87.1	71.4	103.7	108.9	60.2
8	73.1	78.7	79.2	187.7	346.1	348.4	120.7
$n=50,\ \beta=-0.4$							
1	89.4	88.8	89.9	16.0	16.6	17.4	13.0
4	89.7	90.3	90.3	26.5	27.8	27.8	24.0
8	89.4	90.1	90.1	26.5	27.9	27.9	24.0
$n=50,\ \beta=0.5$							
1	89.1	88.8	89.9	15.9	16.7	17.5	13.0
4	89.3	89.7	89.7	28.5	29.4	29.8	26.0
8	89.3	90.0	89.7	29.3	30.8	30.5	26.2
$n=50,\ \beta=1.3$							
1	88.8	88.8	89.8	15.9	17.0	17.8	13.0
4	87.9	89.0	89.8	66.4	73.5	76.2	52.7
8	85.2	85.8	87.0	177.8	196.0	217.4	102.9
$n=100,\ \beta=-0.4$							
1	90.0	89.6	90.1	15.6	15.9	16.3	12.77
4	89.9	90.3	90.2	25.6	26.2	26.3	23.74
8	89.9	90.2	90.1	25.6	26.2	26.3	23.74
$n=100,\ \beta=0.5$							
1	90.0	89.7	90.2	15.6	15.9	16.3	12.8
4	90.0	90.0	90.1	27.5	27.9	28.1	25.3
8	89.8	90.1	90.0	28.4	29.0	29.0	25.8
$n=100,\ \beta=1.3$							
1	89.8	89.8	90.3	15.5	16.0	16.4	12.8
4	90.3	90.5	90.8	62.6	64.3	65.6	48.9
8	89.8	89.3	90.3	169.5	170.8	181.9	93.8

V(PE) indicates the volume of APE with the known MSE matrix.

APC, asymptotic prediction cube; BPC_p, bootstrap prediction cube based on the percentile method; BPC_{pt}, bootstrap prediction cube based on the percentile-t method; h, forecasting lead.

Table 3
Coverage and volumes of PCs for VAR(2) models (nominal coverage: 95%)

h	Mean coverage			Mean volume		
	APC	BPC_p	BPC_{pt}	APC	BPC_p	BPC_{pt}
Model M3, normal innovations, n=25						
1	91.9	90.7	93.3	23.3	23.9	29.0
4	94.1	95.0	94.4	47.2	52.3	53.0
8	95.1	96.6	95.3	51.2	61.0	57.1
Model M3, Student-t innovations, n=25						
1	90.2	90.4	92.6	23.0	27.0	33.0
4	92.2	93.8	93.1	47.3	58.8	60.4
8	93.4	95.5	93.9	51.8	72.8	64.6
Model M3, Chi-squared innovations, n=25						
1	91.6	89.1	91.5	29.1	31.0	38.0
4	93.0	93.8	93.1	59.2	70.1	71.3
8	93.6	95.0	93.6	64.5	82.7	76.4
Model M4, Normal innovations, n=25						
1	88.6	91.0	93.1	21.2	26.2	30.5
4	79.1	86.4	86.8	71.2	122.9	133.2
8	70.3	81.4	79.5	137.6	470.3	319.3
Model M4, Student-t innovations, n=25						
1	86.5	90.2	92.0	21.1	28.6	33.6
4	74.6	84.4	84.2	69.3	127.4	138.0
8	65.7	78.6	75.8	135.9	473.7	351.7
Model M3, Chi-squared innovations, n=50						
1	87.8	89.7	91.8	26.4	34.0	40.2
4	77.1	85.2	85.7	86.1	148.3	162.3
8	69.2	79.3	77.2	163.7	453.1	374.6
Model M1, Normal innovations, n=25						
1	94.3	93.6	94.6	21.7	22.5	24.1
4	93.3	93.6	93.8	58.2	61.8	63.1
8	93.4	94.3	93.8	59.7	64.9	64.0
Model M1, Chi-squared innovations, n=50						
1	93.2	89.0	90.1	26.8	29.1	31.1
4	92.9	92.5	92.5	72.2	79.6	81.5
8	93.1	93.1	92.6	74.1	84.3	83.1
Model M2, Normal innovations, n=50						
1	94.3	93.5	94.5	21.7	22.5	24.1
4	93.6	94.0	94.0	50.0	53.2	54.0
8	93.7	94.4	94.1	50.8	55.1	54.5
Model M3, Student-t innovations, n=50						
1	92.4	93.4	94.1	21.8	28.8	30.4
4	92.5	93.6	93.6	51.5	60.9	63.5
8	93.0	94.4	94.2	52.3	63.0	64.5

used for near non-stationary models, extreme care should be taken when it is used for medium to long term forecasting. Other general findings are listed below.

1. As n increases, mean coverage converges to the nominal value and mean volume decreases for all PCs.
2. As h increases, mean coverage of a PC tends to

the nominal value in general, except for near non-stationary models where mean coverage deteriorates. It is plausible to observe that mean volume of a PC increases as h increases.

3. The volumes of PCs are far larger when the model is near non-stationary than otherwise. The volume of a PC increases as h increases at a much faster rate when the model is near non-stationary than otherwise. This indicates that substantially higher uncertainty is associated when forecasting is conducted with near non-stationary models.
4. When the sample size is 100, APC performs reasonably well for all models considered under both normal and non-normal innovations, exhibiting similar coverage and volume properties to those of BPC_{pt}. Hence, when the sample size is 100 or more, APC may be preferred to BPCs as it is computationally more efficient.
5. Simulation results associated with the nominal coverage 90% are found to be similar to those associated with 95%.
6. Thombs and Schucany (1990) found, in the univariate AR case, that asymptotic and bootstrap PIs (based on the percentile method) exhibit little difference when the model is near non-stationary. In the VAR case, however, it is found that BPC_{pt} performs much better than its asymptotic counterpart when a near non-stationary model is considered.

5. Empirical example

As an application, the VAR model considered in Bessler and Babula (1987) is used. It is a four-dimensional model based on the US data on wheat prices, wheat export sales, wheat export shipments and exchange rates. Their major concern is to determine if wheat sales and shipments are dynamically influenced by exchange rates. They compared out-of-sample point forecasts from univariate AR and VAR models to find that univariate AR forecasts are more accurate than VAR forecasts using the root mean squared error (RMSE) criterion. They regard this as evidence that exchange rates have little influence on wheat sales and shipments, because univariate AR models for wheat shipments and sales do not include past values of exchange rates. However, comparison of point forecasts may be inadequate for this purpose. According to Chatfield (1993), interval forecasts should be used to compare forecasts from different methods more thoroughly. In this section, the findings of Bessler and Babula (1987) are re-evaluated using interval forecasts.

The data, given in Bessler and Babula (1987), are monthly from January 1974 to March 1985; and transformed to natural logarithms and deseasonalized using seasonal dummy variables. Based on the first 107 data points, Bessler and Babula (1987) found the VAR order to be 3 using an order selection criterion; and, in a similar way, the univariate AR orders for sales and shipments are found to be 1 and 2, respectively. Based on these models, 8 out-of-sample point and prediction intervals for shipments and sales are generated using the asymptotic and bootstrap methods. Bessler and Babula (1987) used updated forecasts based on the Kalman-filter routine with a wider range of forecasting leads. Here we use usual VAR forecasts with forecasting lead 8 for simplicity and to be consistent with the simulation design adopted in this paper.

For wheat export shipments, the RMSE values for 8 point forecasts are 0.40 and 0.37, respectively, for VAR and AR; and, for wheat export sales, they are 0.55 and 0.59. That is, for wheat export shipments, the univariate AR model provides more accurate point forecasts than the VAR. On this basis, one may conclude that exchange rates exert little influence on wheat export shipments, as in Bessler and Babula (1987). The lengths of API and BPIs with the nominal coverage 95% for wheat shipments and sales are reported in Table 4. The BPIs are constructed based on 4999 bootstrap iterations. It can be seen that, for all cases, the VAR provides shorter PIs than the univariate AR for both wheat shipments and sales. This indicates that higher uncertainty is associated with univariate AR forecasts. This is in conflict with the RMSE results for wheat shipments. This example illustrates that evaluation of point forecasts only can be misleading.

6. Concluding remarks

The use of a bootstrap prediction region based on the percentile-t method is strongly recommended in

Table 4
Length of prediction intervals from univariate AR and VAR models

h	API		BPI_p		BPI_{pt}	
	VAR	AR	VAR	AR	VAR	AR
Wheat export shipments						
1	0.61	0.97	0.63	0.85	0.64	1.36
4	0.83	1.05	0.91	1.30	0.90	1.61
8	0.93	1.06	1.04	1.44	1.06	1.57
Wheat export sales						
1	1.75	2.50	1.90	2.14	1.77	2.89
4	1.91	2.60	2.16	3.62	1.92	3.72
8	1.94	2.61	2.39	3.58	2.02	3.69

API, asymptotic prediction interval; BPI_p, bootstrap prediction interval based on the percentile method; BPI_{pt}, bootstrap prediction interval based on the percentile-*t* method.

practice, as it provides the most accurate assessment of future uncertainty as compared with the other alternatives considered in this paper. It is found to be particularly attractive when the sample size is small or the model is near non-stationary. For near non-stationary VAR models, however, extreme care should be taken when a bootstrap region is used for medium to long-term forecasting. The use of an asymptotic prediction region may result in a serious under-estimation of future uncertainty when the sample size is small or the model is near non-stationary. In constructing a prediction region for a VAR, a cubical approximation is made to an ellipsoidal figure, and this can result in a serious over-estimation of future uncertainty especially when the model is near non-stationary. A simple check is proposed in this paper which can easily be adopted in practice.

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References

Bessler, D. A., & Babula, R. A. (1987). Forecasting wheat exports: do exchange rates matter? *Journal of Business and Economic Statistics 5*, 397–406.

Booth, J. G., & Hall, P. (1994). Monte Carlo approximation and the iterated bootstrap. *Biometrika 81*, 331–340.

Chatfield, C. (1993). Calculating interval forecasts. *Journal of Business and Economic Statistics 11*, 121–135.

Davison, A. C., & Hinkley, D. V. (1997). Bootstrap Methods and Their Application, Cambridge University Press, Cambridge.

Efron, B. (1979). Bootstrap method: another look at the jackknife. *The Annals of Statistics 7*, 1–26.

Efron, B. (1981). Nonparametric standard errors and confidence intervals. *The Canadian Journal of Statistics 9*, 139–172.

Fachin, S., & Bravetti, L. (1996). Asymptotic and bootstrap inference in structural VAR analysis. *Journal of Forecasting 15*, 329–341.

Fackler, J. S., & Krieger, S. C. (1986). An application of vector time series techniques to macroeconomic forecasting. *Journal of Business and Economic Statistics 4*, 71–80.

Hall, P. (1988). Theoretical comparison of bootstrap confidence intervals. *The Annals of Statistics 16*, 927–953.

Jeong, J., & Maddala, G. S. (1993). A perspective on application of bootstrap methods in econometrics. In: Maddala, G. S. et al. (Ed.), Handbook of Statistics, Vol. 11, North-Holland, Amsterdam.

Kim, J. H. (1997). Relationship between the forward and backward representations of the stationary VAR model, Problem 97.5.2. *Econometric Theory 13*, 899–990.

Kim, J. H. (1998). Relationship between the forward and backward representations of the stationary VAR model, Solution 97.5.2. *Econometric Theory 14*, 691–693.

Li, H., & Maddala, G. S. (1996). Bootstrapping time series models. *Econometric Reviews 15*, 115–158.

Litterman, R. B. (1986). Forecasting with Bayesian vector autoregressions — Five years of experience. *Journal of Business and Economic Statistics 4*, 25–38.

Liu, T. -R., Gerlow, M. E., & Irwin, S. W. (1994). The performance of alternative VAR models in forecasting exchange rates. *International Journal of Forecasting 10*, 419–433.

Lütkepohl, H. (1991). Introduction to Multiple Time Series Analysis, Springer-Verlag, Berlin.

Maekawa, K. (1987). Finite sample properties of several predictors from an autoregressive model. *Econometric Theory 3*, 359–370.

Masarotto, G. (1990). Bootstrap prediction intervals for autoregressions. *International Journal of Forecasting 6*, 229–239.

McCullough, B. D. (1994). Bootstrapping forecast intervals: an application to AR(p) models. *Journal of Forecasting 13*, 51–66.

McNees, S. K. (1986). Forecasting accuracy of alternative techniques: a comparison of U.S. macroeconomic forecasts. *Journal of Business and Economic Statistics 4*, 5–15.

Rilstone, P., & Veall, M. (1996). Using bootstrapped confidence intervals for improved inferences in seeming unrelated regression equations. *Econometric Theory 11*, 569–580.

Simkins, S. (1995). Forecasting with vector autoregressive (VAR) models subject to business cycle restrictions. *International Journal of Forecasting 11*, 569–583.

Sims, C. (1980). Macroeconomics and reality. *Econometrica 48*, 1–48.

Thombs, L. A., & Schucany, W. R. (1990). Bootstrap prediction intervals for autoregression. *Journal of the American Statistical Association 85*, 486–492.

Trevor, R. G., & Thorp, S. J. (1988). VAR forecasting models of the Australian economy. *Australian Economic Papers 27*, 108–125.

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