Partimos de que:

$$Vol(V_2) = \int \prod_{i,j=1}^{2} do_{ij}\, \delta\left(\sqrt{o_{11}^2 + o_{21}^2} - 1\right) \delta\left(\sqrt{o_{12}^2 + o_{22}^2} - 1\right) \delta\left(o_{11} o_{12} + o_{21} o_{22}\right)$$

y consideremos el cambio de coordenadas

$o_{11} = r\cos(\theta)$ $\quad o_{12} = R\cos(\varphi)$ $\quad r, R \in [0, \infty)$

$o_{21} = r\sin(\theta)$ $\quad o_{22} = R\sin(\varphi)$ $\quad \theta, \varphi \in [0, 2\pi)$

Cuyo jacobiano es $|-Rr|$ (anexo la comprobación)

Sustituyendo el cambio de variables y multiplicando por el Jacobiano tenemos que:

$$Vol(V_2) = \int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} d\varphi\, d\theta\, dr\, dR\, (|-Rr|)\, \delta(r-1)\, \delta(R-1)\, \delta(rR(\cos\theta\cos\varphi + \operatorname{sen}\theta \operatorname{sen}\varphi))$$

Recordando que $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \operatorname{sen}(\alpha)\operatorname{sen}(\beta)$

$$Vol(V_2) = \int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} d\varphi\, d\theta\, dr\, dR\, (Rr)\, \delta(r-1)\, \delta(R-1)\, \delta(Rr\cos(\theta-\varphi))$$

Aplicando la propiedad $\int F(x)\delta(x-a) = F(a)$ dos veces con las funciones $f_1(r) = r$ y $f_2(R) = R$ tenemos que:

$$Vol(V_2) = \int_0^{2\pi} \int_0^{2\pi} d\varphi\, d\theta\, \delta(\cos(\theta - \varphi))$$

Y utilizando la propiedad

$$\int_0^M \int_0^M f(x-y)\, dy\, dx = \int_{-M}^{M} f(z)(M - |z|)\, dz$$

con $M = 2\pi$ y $f(x-y) = \delta(\cos(\theta - \varphi))$

Llegamos a que:

$$\mathrm{Vol}(V_2) = \int_0^{2\pi}\int_0^{2\pi} d\varphi\, d\theta\, \delta(\cos(\theta-\varphi)) = \int_{-2\pi}^{2\pi} \delta(\cos(\varphi))(2\pi - |\varphi|)\,d\varphi$$

Ahora notemos que $\delta(\cos(\varphi)) = \delta(\cos(-\varphi))$ al igual que

es decir $\delta(\cos(\varphi))$ es par así que:

$(2\pi - |\varphi|)$ entonces su producto es par

$$\mathrm{Vol}(V_2) = \int_{-2\pi}^{2\pi} \delta(\cos(\varphi))(2\pi - |\varphi|)\,d\varphi$$

$$= 2\int_0^{2\pi} \delta(\cos(\varphi))(2\pi - |\varphi|)\,d\varphi$$

Luego notamos que $\cos(\varphi)$ es continuamente diferenciable en $[0, 2\pi]$ y además $\cos(\varphi) = 0$ tiene raices reales en $\pi/2$ y $3\pi/2$ (ambas dentro de $(0, 2\pi)$) en particular además $(\cos(\varphi))' = -\mathrm{sen}(\varphi)$

$|-\mathrm{sen}(\pi/2)| = |-\mathrm{sen}(3\pi/2)| = 1 \neq 0$

Podemos usar el resultado de que

$$\int_\Omega \delta(f(x))\, g(x)\, dx = \sum_k \frac{g(\alpha_k)}{|f'(\alpha_k)|}$$

con $\alpha_1 = \pi/2$ y $\alpha_2 = 3\pi/2$

Con $f(x) = \cos(x)$ y $g(x) = (2\pi - |x|)$

Con lo que obtenemos

$$Vol(V_2) = 2\int_0^{2\pi} f(\cos(\varphi))(2\pi - |\varphi|)\, d\varphi$$

$$= 2\left(\left(\frac{2\pi - \pi/2}{1}\right) + \left(\frac{2\pi - 3\pi/2}{1}\right)\right)$$

$$= 2\left(\frac{3}{2}\pi + \frac{\pi}{2}\right) = 2\left(\frac{4}{2}\pi\right) = 4\pi$$

Teniendo que

$$Vol(V_2) = 4\pi$$

Considerando el cambio de coordenadas

$$O_{11} = r\cos(\theta) \qquad O_{12} = R\cos(\varphi)$$
$$O_{21} = r\sin(\theta) \qquad O_{22} = R\sin(\varphi)$$

Este tiene Jacobiano igual a $|-Rr|$ por que:

$$\begin{vmatrix} \frac{\partial O_{11}}{\partial r} & \frac{\partial O_{11}}{\partial \theta} & \frac{\partial O_{11}}{\partial R} & \frac{\partial O_{11}}{\partial \varphi} \\ \frac{\partial O_{12}}{\partial r} & \frac{\partial O_{12}}{\partial \theta} & \frac{\partial O_{12}}{\partial R} & \frac{\partial O_{12}}{\partial \varphi} \\ \frac{\partial O_{21}}{\partial r} & \frac{\partial O_{21}}{\partial \theta} & \frac{\partial O_{21}}{\partial R} & \frac{\partial O_{21}}{\partial \varphi} \\ \frac{\partial O_{22}}{\partial r} & \frac{\partial O_{22}}{\partial \theta} & \frac{\partial O_{22}}{\partial R} & \frac{\partial O_{22}}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\,\mathrm{sen}\,\theta & 0 & 0 \\ 0 & 0 & \cos(\varphi) & -R\,\mathrm{sen}(\varphi) \\ \sin(\theta) & r\cos(\theta) & 0 & 0 \\ 0 & 0 & \mathrm{sen}(\varphi) & R\cos(\varphi) \end{vmatrix}$$

$$= \left| \cos(\theta) \begin{vmatrix} 0 & \cos(\varphi) & -R\,\mathrm{sen}(\varphi) \\ r\cos(\theta) & 0 & 0 \\ 0 & \mathrm{sen}(\varphi) & R\cos\varphi \end{vmatrix} + \mathrm{sen}(\theta) \begin{vmatrix} -r\,\mathrm{sen}(\theta) & 0 & 0 \\ 0 & \cos(\varphi) & -R\,\mathrm{sen}(\varphi) \\ 0 & \mathrm{sen}(\varphi) & R\cos(\varphi) \end{vmatrix} \right|$$

$$= \left| \cos(\theta)(-r\cos(\theta)) \begin{vmatrix} \cos(\varphi) & -R\,\mathrm{sen}(\varphi) \\ \mathrm{sen}(\varphi) & R\cos(\varphi) \end{vmatrix} - r\,\mathrm{sen}(\theta)\,\mathrm{sen}(\theta) \begin{vmatrix} \cos(\varphi) & -R\,\mathrm{sen}(\varphi) \\ \sin(\varphi) & +R\cos(\varphi) \end{vmatrix} \right|$$

$$= \left| -r\cos^2(\theta)\left(R\cos^2(\varphi) + R\,\mathrm{sen}^2(\varphi)\right) - r\,\mathrm{sen}^2(\theta)\left(R\cos^2(\varphi) + R\,\mathrm{sen}^2(\varphi)\right) \right|$$

$$= \left| -r\cos^2(\theta)\left(R\left(\cos^2(\varphi) + \mathrm{sen}^2(\varphi)\right)\right) - r\,\mathrm{sen}^2(\theta)\left(R\left(\cos^2(\varphi) + \mathrm{sen}^2(\varphi)\right)\right) \right|$$

$$= \left| -r\cos^2(\theta)R - r\,\mathrm{sen}^2(\theta)R \right| = \left| -rR\left(\cos^2(\theta) + \mathrm{sen}^2(\theta)\right) \right|$$

$$= |-Rr| = Rr \quad \text{pues } 0 \leq R, r \leq \infty$$