VAR PLS

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Outline

- Motivation
- PLSAR and PLS
- Quick overview of VAR Models
- Definition for VAR-PLS
- Bootstrap for VAR-PLS
- Conclusions

Motivation

- PLS is a technique that has had a big impact on many applications such as quality control for batch processes in Chemical industries, medical image analysis, microarrays, path modeling, classification, discrimination, spatio-temporal PLS models, just to mention some, with authors such as McGregor, Nomikos, MacIntosh, V. Esposito Vinz, P. Garthwaite, among others.
- It can be used in univariate and multivariate data as well.
- *It has been shown that gives better predictions even when the standard assumptions are met.*
- Phillip Hans Franses (2006) proposes a methodology to construct the forecast h steps ahead in an optimal way, through an autoregressive order p model: An *Autoregressive Partial Least Squares Model denoted as PLSAR*(h, p).

Case Study

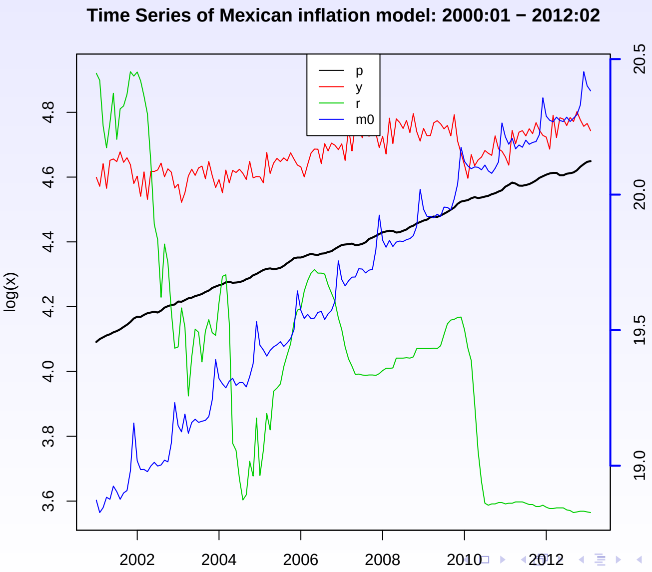
- To develop a model to predict the Mexican inflation, as precise as possible.
- The model has to consider, as the principal source of the Mexican inflation, the growth and variation of the monetary condition of the country.
- Irrespective of all possible discussions, there seems to be a common understanding that the inflationary process, in the lung run, is a purely monetary phenomena.
- Here, we will not discuss whether or not such relationship exists but we will show its empirical properties with a model that is tested out of sample via its error prediction measure.

Case Study

We work with 4 indexes (we built them) from January 2000 to Feb 2012

- p: Consumer price index.
- $m0$: Monetary base.
- r: Equilibrium interest rate (28 days).
- y: Industrial production index.

Case Study


Time Series of Mexican inflation model: 2000:01 − 2012:02
log(x)
p
y
r
m0
3.6
3.8
4.0
4.2
4.4
4.6
4.8
2002
2004
2006
2008
2010
2012
19.0
19.5
20.0
20.5

Case Study

- We generalize the work proposed by Franses in the following way:

  1. Give a multivariate representation based on the flexibility of VAR models, model that we will call $VAR - PLS(h, p)$.
  2. Extend the model to consider deterministic variables (dummies, trend, etc.) as well as exogenous variables.
  3. Bootstrap prediction intervals.
  4. Compare the forecast capabilities between $VAR - PLS(h, p)$ models and $VAR(p)$ models explicitly built for prediction (integral predictor method).

$$PLSAR(h, p)$$

$PLSAR(h, p)$

Franses considers three different ways to construct a forecast for an $AR(p)$:

1- A single model for all horizons

$$AR(p) : y_{T+h} = \mu + \rho_1 y_{T+h-1} + \rho_2 y_{T+h-2} + \cdots + \rho_p y_{T+h-p} + \epsilon_T$$

For the $AR(p)$ this is the classical iterative procedure to get h steps ahead forecast, plus the fact that we estimate the parameters via OLS.

$PLSAR(h, p)$

2- One model for each horizon, the variance will vary within each horizon and *there are different models for each step*

$$AR_h(p) : y_{t+h} = \mu + \rho_{1,h} y_t + \rho_{2,h} y_{t-1} + \cdots + \rho_p y_{t-p} + \epsilon_{t,h}$$

The $AR_h(p)$ is an alternative to the $AR(p)$ because

- OLS minimizes the sum of squares of ϵ_t's but there is no way to assure it will remain minimum for all the h steps in the future.
- For stationary time series recall the forecast of an $AR(p)$ model quickly converges to the inconditional mean (and variance, for the interval prediction error) clearly depending on $h \geq p$.
- For more details on this type of models see : Pesaran & Pick (2010), Marcellino, Stock & Watson (2004), Carreiro, Kapetorios & Marcellino (2010), Tiao & Xu (1993) among others.

$PLSAR(h, p)$

3- Something in between: *PLSAR*, this model behaves like *in the middle* between an $AR(p)$ and a $AR_h(p)$

$$PLSAR(h, p) : \hat{Y} = XB_{PLS}$$

- It is clear that there exists adjacent correlation between the time series, and neither one of the above models takes it into account. In other words, we know that (y_t, y_{t-j}) are correlated and so are (y_{T+h}, y_{T+h-j}). Therefore we would like to jointly predict $(y_{t+h}, y_{t+h-1}, y_{t+h-2}, \ldots, y_{t+1})$ through $(y_t, y_{t-1}, y_{t-2}, \ldots, y_{t-p})$. PLS is a technique very appealing to do so.

$PLSAR(h, p)$

3- Franses proposed to arrange the information as

$$(y_t, y_{t-1}, y_{t-2}, \ldots, y_{t-p}) \text{ as the predictor matrix } X$$

$$(y_{t+h}, y_{t+h-1}, y_{t+h-2}, \ldots, y_{t+1}) \text{ as the predicted matrix } Y$$

- Apply the *PLS* algorithm to get the latent variables with the relevant information given in X and Y.
- His simulations show that the $PLSAR(h, p)$ is quite competitive with respect to the classical models in the literature.

PLS

PLS

- PLS can be seen from different viewpoints. For us, the relationship between its linear expression will be the best one, in order to relate it with a Vector Autoregressive model

$$Y = XB + U,$$

where Y is a $N \times k$ matrix, X is $N \times p$, B is a $p \times k$ matrix, and U is $N \times k$.

PLS

- The basic procedure maximizes a covariance

$$\max cov(X\alpha, Y\beta)^2$$

under certain restrictions,

$$\alpha'(S_{xx}^* + \lambda_x)\alpha = 1 \quad \text{and} \quad \beta'(S_{yy}^* + \lambda_y)\beta = 1$$

where $S_{xx}^* = (1 - \lambda_x)S_{xx}$ and $S_{yy}^* = (1 - \lambda_y)S_{yy}$.

  - $(X\alpha, Y\beta)$ are the linear combinations that maximize the covariance (actually, the squared covariance) (the sign is not important, just the direction).
  - S_{xx} and S_{yy} the covariance matrices, $\beta'\beta = 1$ and $a'S_{xx}a = 1$.

PLS

- We maximize the objective function:

$$\mathcal{L} = (\alpha' S_{xy} \beta)^2 - \gamma(\alpha'(S_{xx} + \lambda_x)\alpha - 1) - \mu(\beta'(S_{yy} + \lambda_y)\beta - 1).$$

- After some algebra we get the scores for X and Y, $t = Xw = Ew$ and $u = Yq = Fq$.
- Normalizing the scores $t = t/\sqrt{t't}$, and after simplifications and more algebra we get the loadings for X and Y: $p = E't$ and $q = F't$.

- Writing in matrix form w, t, p and q we get $R = W(P'W)^{-1}$ and finally
$$Y = XB + U, \text{ then } \hat{Y} = XB_{PLS},$$
where $B_{PLS} = R(T'T)^{-1}T'Y = RQ'$.

- For a nice introduction see P.H. Garthwaite (1994). An Interpretation of Partial Least Squares. JASA Vol 89, No 425, pp 122-127 and A. Hoskuldsson (1988). PLS Regression Methods. Journal of Chemometrics, Vol 2, pp 221-228.

  Note: Franses shows that if the B_{PLS} matrix has full rank it implies a different model for each of the columns of Y, and hence a model like $AR_{h,p}$. In the exceptional case that B_{PLS} has rank 1, then the $AR(p)$ appears.

Vector Autoregressive models and PLS

$VAR - PLS$

A $VAR(p)$ processes is defined as

$$y_t = A_1 y_{t-1} + A_2 y_{t-2} + \cdots + A_p y_{t-p} + CD_t + u_t,$$

where

- A_i, $i = 1, 2, \ldots, p$, the coefficient matrix.
- u_t a white noise process with variance covariance given by $\Sigma_u = E(u_t, u_t')$.
- C, matrix of regressor coefficients for deterministic factors
- D_t, vector of deterministic factors.

We also know that a $Var(p)$ can be written as a $Var(1)$ as follows

$$Y_t = AY_{t-1} + V_t,$$

with

$$Y_t = \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix}, \quad V_t = \begin{pmatrix} u_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

If the eigenvalues of A are less than one, then the $VAR(p)$ is stable.

- We use the VAR representation to determine the order of the model.
- The general procedure is quite standard. Order $p = 0, \ldots, p_{\max}$ and choose the value of p that minimize some criteria. The criteria are usually written as:

$$IC(p) = \log |\Sigma(\hat{p})| + C_T \varphi(K, p),$$

where

  - $\Sigma(\hat{p}) = T^{-1} \sum_{i=1}^{T} \hat{u}_t' \hat{u}_t$.
  - C_T an indexed sequence of the size T.
  - $\varphi(K, p)$ is a penalty function that involves the order of the $VAR(p)$.

- The most common information criteria are: Akaike (AIC), Schwarz-Bayesian (BIC), Hannan-Quinn (HQ) and Final Prediction Error (FPE):
  - Akaike: $AIC(p) = |\Sigma(\hat{p})| + \frac{2}{t} pK^2$
  - Schwartz-Bayesian: $BIC(p) = |\Sigma(\hat{p})| + \frac{\log T}{t} pK^2$
  - Hannan-Quinn: $HQ(p) = |\Sigma(\hat{p})| + \frac{2 \log T}{t} pK^2$
  - Final Prediction Error: $FPE(p) = \left(\frac{T+p^*}{T-p^*}\right)^K \det(\Sigma(\hat{p}))$
- The AIC asymptotically overestimates the order of the model with a positive probability whereas BIC and HQ are consistent estimators of the order if the true value is less than or equal than p_{max}.

As for the univariate case, we can build the forecast with a recursive method:

$$y_{T+h|T} = A_1 y_{T+h-1|T} + \cdots + A_p y_{T+h-p|T} + CD_{T+h|T}$$

- We estimate A_i through OLS

$$vec(\hat{A}) = \begin{pmatrix} \hat{A}_1 \\ \vdots \\ \hat{A}_p \end{pmatrix}.$$

- Under stationarity and ergodicity conditions for the VAR model (see Hamilton (1994), Lütkepohl (1991) among others), $vec(\hat{A})$ is consistent and asymptotically distributed with covariance matrix given by

$$\widehat{var}\left(vec(\hat{A})\right) = \hat{\Sigma} \otimes (Z'Z)^{-1},$$

where

$$\hat{\Sigma} = \frac{\sum_{t=1}^{T} \hat{\epsilon}_t' \hat{\epsilon}_t}{T - K}$$

and

$$\hat{\epsilon}_t = Y_t - \hat{A}' Z_t = Y_t - \hat{A}' Y_t$$

the OLS residual at time t.

VAR – *PLS*

- The i-th element of $vec(\hat{A})$ is asymptotically normal (for a stable VAR) and the standard errors are the square roots of the diagonal elements of $\hat{\Sigma} \otimes (Z'Z)^{-1}$.
- The t-tests for the estimated coefficients are asymptotically correct.
- Other important situation for the VAR models is the presence of one or more unit roots for the y_j's. From the point of view of the Economic theory it means the study of a long run behavior plus the temporal dynamic of the series.

- **Cointegration:** The components of a k-dimensional vector y_t are cointegrated of order (d, b), denoted by $y_t \sim CI(d, b)$, if

  1. all components of y_t are $I(d)$.
  2. there exist a vector $\beta \neq 0$ such that $z_t = \beta' y_t \sim I(d - b)$, $b > 0$. The vector β is called the cointegration vector (Lütkepohl, 1991).

The $VAR(p)$ model can be written as a Transitory - Vector Error Correction Model ($VECM$)

$$\Delta y_t = \Pi y_{t-1} + \Gamma_1 \Delta y_{t-1} + \cdots + \Gamma_{p-1} \Delta y_{t-p+1} + CD_t + u_t,$$

where $\Gamma_i = -(A_{i+1} + \cdots + A_p)$, for $i = 1, \ldots, p-1$ and $\Pi = -(I - A_1 - A_2 - \cdots - A_p)$.

Or as a Long run - Vector Error Correction Model ($VECM$)

$$\Delta y_t = \Pi y_{t-p} + \Gamma_1 \Delta y_{t-1} + \cdots + \Gamma_{p-1} \Delta y_{t-p+1} + CD_t + u_t,$$

where $\Gamma_i = -(I - A_1 - A_2 - \cdots - A_i)$, for $i = 1, \ldots, p-1$ and $\Pi = -(I - A_1 - A_2 - \cdots - A_p)$.

The matrix Π has the following characteristics:

1. If $rk(\Pi) = n$, the n linear combinations are stationary; in other words the *VECM* is no more than a *VAR* model (in levels).
2. If $rk(\Pi) = 0$, there is no linear combination that makes $\Pi y_{(t-1)}$ stationary, except for the trivial solution *i.e.*, it becomes a $VAR(p-1)$ in first differences.
3. If $0 < rk(\Pi) < n$, then $\Pi = \alpha\beta'$ (α and β with dimensions $n \times r$) and $\beta' y_{t-1}$ is stationary. Each column of β represent a long run relationship.

If the objective is to forecast series that are integrated or cointegrated, working with a VAR representation is quite appropriated (see Lütkepohl 2006)

VAR Example

- For the Mexican inflation example we specify the order of the model through the final error prediction criteria, it was $p = 2$.
- We also performed Johansen test to determine the presence of a long run relationship. Finding the following relationship (significant at a 1% level)

$$price + 9.68 - 0.43m - 0.89y + 0.1r = 0$$

- This relationship is congruent with the Economic theory behind it. The inflationary movement increases with monetary growth, the exceed on demand and with the reduction of the cost of money.

$$VAR - PLS(h, p)$$

$VAR - PLS(h, p)$

- The VAR model will give us the DGP
- The VAR model will provide the autoregressive process that we will use to build the PLS regression
- We then form the matrices in a natural way as:

$VAR - PLS(h, p)$

- For the X matrix we include the lag vector

$$X = Y_{t-1} = \begin{pmatrix} y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p} \end{pmatrix},$$

- For Y, the observation until time t :

$$Y = Y_t = \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{pmatrix},$$

using the X matrix with all the lags considered for the DGP.

- We can introduce exogenous variables with a C matrix, then

$$X = Y^*_{t-1} = \begin{pmatrix} y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p} \\ D_t \end{pmatrix},$$

and then, the matrix of coefficients

$$A^* = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p & C \\ I & 0 & \cdots & 0 & 0 & 0 \\ 0 & I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \cdots & I & 0 & 0 \end{pmatrix}$$

- Those are the basic ingredients for a $VAR - PLSX$ that will allow us to predict h steps ahead

$VAR - PLSX(h, p)$ Example

For the $VAR(p = 3) - PLSX(h = 24, j)$

- We kept 24 observation to have a long horizon of possible comparisons
- We consider dummies for the monthly effects
- For the optimal p we estimate the $VAR - PLSX$ model. We use R to fit

  $$Y_{t,119\times 4} = X_{t,119\times 23} B_{23\times 4} + U_{t,119\times 4}$$

  and estimate and predict for the $VAR(3) - PLSX(24, j)$ in a recursive way as in the $VAR(p)$. (The last component agrees with the VAR(p) OLS estimate)
- For the $pK + g = 23$ components and the 24 out of sample observations we get the MAPE to make the comparison

$VAR(p)$ forecast

For the model $VAR(p = 2)$

- Combining all the variables estimate our $VAR_j(p)$, $j = 1, 2, \ldots, 3696$
- For the 24 steps out of sample we use 7 different criteria to measure the behavior of the forecast (Hyndman & Koehler 2006):
    - MAPE: Mean Absolute Percentage Error
    - MdAPE: Median Absolute Percentage Error
    - RMSPE: Root Mean Square Percentage Error
    - RMdSPE: Root Median Square Percentage Error
    - MRAE: Mean Relative Absolute Error
    - MdRAE: Median Relative Absolute Error
    - GMRAE: Geometric Mean Absolute Error

$VAR(p)$

- For the last 3 we need to work with a benchmark model (autoregressive order 1) and for $i = 1, \ldots, 24$ ($h = 24$) obtain the statistic:

$$test = \frac{Y_{t-i} - Y^{f}_{t+i,VAR_j(p)}}{Y_{t+i} - Y^{f}_{t+i,AR(1)}}$$

- With the 7 criteria, integrate the forecasts into one by taking their medians for the given horizon, repeat this idea for their upper and lower confidence limits to get the combined prediction interval

Prediction Interval: VAR-PLS

We also have to construct the prediction intervals for the VAR-PLS model.

We use a similar procedure to the one proposed by Pascual, Ruíz and Fresoli (2011). Bootstrap forecast of multivariate VAR models without using the backward representation. Working Paper 11-34, Statistics and Econometrics Series.
They use the seminal ideas of Kim (2001) and some results from a previous work, Pascual, L., J. Romo, and E. Ruiz (2004a). Bootstrap predictive inference for ARIMA processes, Journal of Time Series Analysis, 25, 449-465

$VAR - PLS(h,p)$: Bootstrap

For the VAR model they proposed a method that copes with:

1. The uncertainty given by the estimation of the parameter, building confidence regions using a bootstrap method
   - *This regions are valid under Gaussian assumptions (Lütkepolh et al, 1991), even though do not reflect, for small sample sizes, the asymmetric distribution of the predicted values (under estimated parameters)*
2. The backward representation makes calculations quite complicated, more in the case of the $VAR(p)$ representation and p taking values greater than 5 or 6, which are very common.
   - *"Pascual et al shows that the backward representation can be avoided without loosing the good properties of the bootstrap procedure"*

$VAR - PLS(h,p)$: Bootstrap

Of course we needed to adequate the procedure for the $VAR - PLS$ representation

1. Fit the model to get $Y_t = X_t \hat{B}_{PLS}$.
2. Obtain the standarized residuals $\hat{U}_t^*$ and the empirical distribution of the residuals
3. With the p initial values $Y_0 = \{Y_p, \ldots, Y_1\}$ and the results from Steps 1 and 2 generate Y_t^*, the bootstrap values, through the $\hat{U}_t^*$'s, which are independently drawn from its empirical distribution

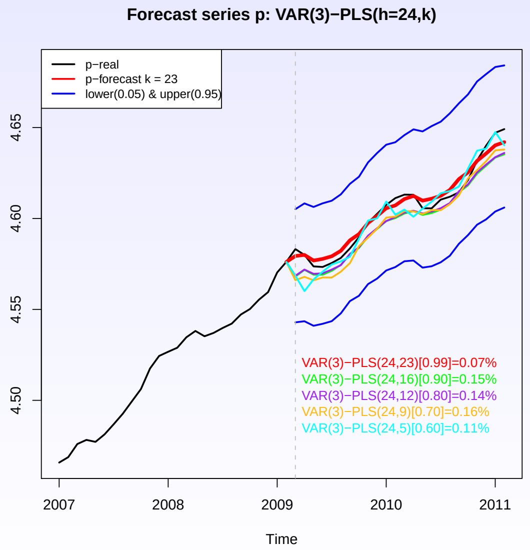
$$Y_t^* = X_t \hat{B}_{PLS} + \hat{U}_t^*, \quad t = 1, \ldots, n-p$$

4. Proceed in this way to get $\hat{Y}_{T+h}^*$, replicating steps 2 to 4 for $n = 1, \ldots, N$.
5. For each one of the n variables and the set of N forecast we get

$$CI_{T+h} = \{y_{n,T+k} | y_{n,T+k} \in [q_B^*(\tau), q_B^*(\tau - 1)]\},$$

where $q_B^*(\tau)$ is the τ-th quantile of $G_{n,B}^*(x) = \#\left(y_{n,T+k}^{*(b)} \leq x\right)/N$

VAR – *PLS* Example



VAR – *PLS* Example

- From the Economic point of view, the approximation is excellent.
- We observe that either using, $k = 23$ (0.99 of the variability) with an error percentage of 0.07% or with 70% explanation with an error percentage of 0.16%, the true value and the predicted one, for practical purposes, are almost identical.
- The bootstrap interval is very well behaved.

Note: The objective is to forecast Price, however since it is a multivariate model we also get forecasts for the other 3 variables with a forecast error (average of MAPE) of: 0.20% for the monetary base, 0.60% for the industry production index and 12.03% for the equilibrium interest rate.

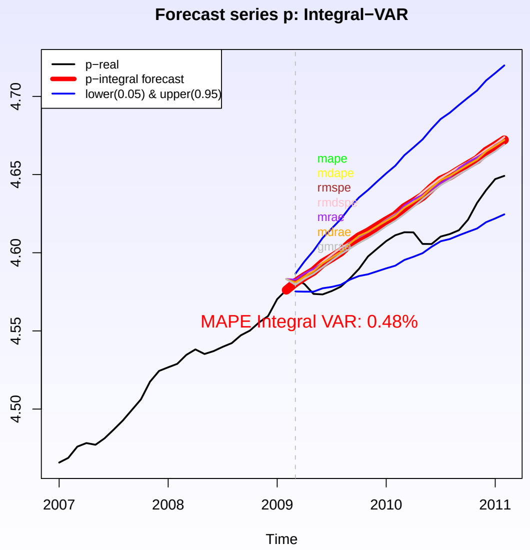
$VAR - PLS$ Example

For the Integral-VAR, the optimum $VAR_J(P)$ are

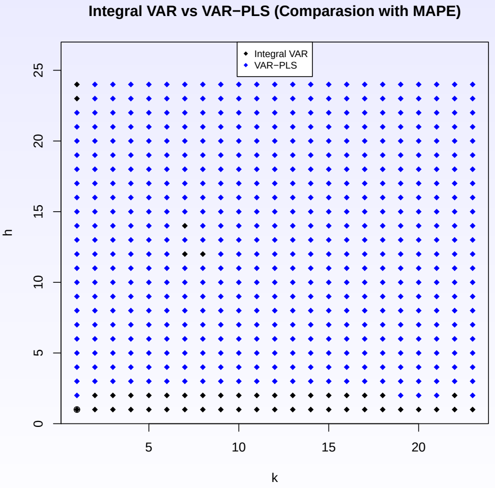
Criterium	MAPE	MdAPE	RMSPE	RMdSPE
Statistic	0.16	0.12	0.19	0.12
Variable	r	r	r	r
Lags	3	2	3	2
Stationality	9	11	9	11
Specification	none	none	none	none

Criterium	MRAE	MdRAE	GMRAE
Statistic	0.16	0.12	0.11
Variable	r	r	r
Lags	2	3	5
Stationality	6	9	11
Specification	none	none	none

VAR – PLS Example



VAR – *PLS* Example



In average, 91.67% of times, the PLS representation performed better than the integral VAR forecasts over all the components. It makes sense that the last components are less effective.

CONCLUSIONS

- The $VAR - PLS$ seems to be an attractive competitor against the integral VAR which is constructed for prediction purposes.
- One advantage is that the bootstrap prediction intervals include the uncertainty due to parameter estimation
- A second advantage is that the forecast reflects the trends and seasonal effects of the original series even for large number of steps ahead

Thanks !

References

- Bjorn-Helge Mevik and Ron Wehrens The pls Package: Principal Component and Partial Least Squares Regression in R, Journal of Statistical Software January 2007, Volume 18, Issue 2.
- Carreiro, Kapetorios and Marcellino (2010), Forecasting Government Bond Yields with large Bayesian VAR's. Working papers No. 662 School of Economics and Finance.
- Esposito Vinzi, V., W.W. Chin, J. Henseler and H. Wang (2007), Handbook of Partial Least Squares, Berlin: Springer.
- Philip Hans Franses (2006) Forecasting 1 to h steps ahead using partial least squares. Econometric Institute, Erasmus University Rotterdam, Econometric Institute Report 2006-47
- P.H. Garthwaite (1994). An Interpretation of Partial Least Squares. JASA Vol 89, No 425, pp122-127
- Hoskuldsson (1988). PLS Regression Methods. Journal of Chemometrics, Vol 2, pp 221-228

References

- Kim, J.H. (2001), Bootstrap after bootstrap prediction intervals for autoregressive models, Journal of Business & Economic Statistics, 19(1), 117-128.
- Lutkepolh, H. (1991), Introduction to Multiple Time Series Analysis, 2nd ed., Springer- Verlag, Berlin.
- Lutkepolh, H. (2006), Forecasting with VARMA models, in Elliot, G., C.W.J. Granger and Timmerman (eds.), Handbook of Economic Forecasting, Vol. 1, 287-325.
- McIntosh A.R., Bookstein F.L., Haxby J.V.and Grady C.L. (1996). Spatial Pattern Analysis of Functional Brain Images Using Partial Least Squares NeuroImage, Volume 3, Number 3, pp. 143-157(15). Academic Press
- Marcellino, Stock and Watson (2004), A Comparison of Direct and Iterated Multistep AR Methods for Forecasting Macroeconomic Time Series.NBER
- Paul Nomikos and John F. MacGregor (1995) Multivariate SPC Charts for Monitoring Batch Processes Technometrics Vol. 37, No. 1, pp. 41-59

References

- Pascual, Ruiz y Fresoli (2011). Bootstrap forecast of multivariate VAR models without using the backward representation. Working Paper 11-34, Statistics and Econometrics Series,
- Pascual, L., J. Romo, and E. Ruiz (2004). Bootstrap predictive inference for ARIMA processes, Journal of Time Series Analysis, 25, 449-465
- Pesaran, H. H., Pick, A., Timmermann, A. (2010). Variable Selection and Inference for Multi-period Forecasting Problems. Journal of Econometrics
- Tiao, G.C., and D. Xu (1993), Robustness of maximum likelihood estimates for multi-step predictors: the exponential case, Biometrika, 80, 623-641

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