Existence of Solutions to Non-Local Uncertain Differential Equations Under The ψ-Caputo Fractional Derivative

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2 juin 2024

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Introduction

Considering the uncertain non-local fuzzy fractional differential equation :

$$\begin{cases} {}^{C}D_{0^+}^{\alpha,\psi}u = f(t,u) + g(t,u) \quad t \in I = [0,1], \\ u(0) = \tilde{0} \in \mathbb{E} \end{cases} \tag{1}$$

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where ${}^{C}D_{0^+}^{\alpha,\psi}u$ is the ψ-Caputo derivative of order $0 < \alpha < 1$ of u, the fuzzy function $f : [0,1] \times \mathbb{E} \to \mathbb{E}$ is continuous and compact, and $g : [0,1] \times \mathbb{E}^c \to \mathbb{E}$ fulfill

$$D(g(t,u), g(t,v)) \leq LD(u,v), \quad u, v \in \mathbb{E}, \text{ for } L \geq 0. \tag{2}$$

Basic Notation

Defintion 1 (Fuzzy Number)

A fuzzy number is a fuzzy set $x : \mathbb{R} \to [0,1]$ that satisfies the following conditions :

1. x is normal, i.e. there is a $t_0 \in \mathbb{R}$ such that $x(t_0) = 1$;
2. x is a fuzzy convex set ;
3. x is upper semi-continuous ;
4. x closure of $\{t \in \mathbb{R}, \quad x(t) > 0\}$ is compact.

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We denote by $\mathbb{E}$ the space of all fuzzy numbers on $\mathbb{R}$.

$$\mathbb{E} = \{x : \mathbb{R} \rightarrow [0, 1], \quad x \quad \text{satisfies } (1-4) \text{ below } \}.$$

- For all $\alpha \in (0,1]$ the α-cut of an element of $\mathbb{E}$ is defined by

$$x^{\alpha} = \{t \in \mathbb{R}, x(t) \geq \alpha\}$$

By the former parcels we can write

$$x^{\alpha} = [\underline{x}(\alpha), \overline{x}(\alpha)].$$

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- The distance between two element of $\mathbb{E}$ is given by

$$d(x,y) = \sup_{\alpha \in (0,1]} \max\{|\bar{x}(\alpha) - \bar{y}(\alpha)|, |\underline{x}(\alpha) - \underline{y}(\alpha)|\}.$$

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- The operations of addition and scalar multiplication of fuzzy numbers on $\mathbb{R}_{\mathcal{F}}$ have the form

$$[x \oplus y]^{\alpha} = [x]^{\alpha} + [y]^{\alpha} \text{ and } [\lambda \odot x]^{\alpha} = \lambda [x]^{\alpha}, \lambda \in \mathbb{R}$$

For $x, y \in \mathbb{E}$, the gH-difference of x and y, denoted by $x \ominus_{gH} y$, is defined as the element $z \in \mathbb{E}$ such that

$$x \ominus_{gH} y = z \Longleftrightarrow \begin{cases} \text{(i)} x = y \oplus z \text{ or} \\ \text{(ii) } y = x \oplus (-1)z \end{cases} .$$

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In terms of α-levels we have

$$(x \ominus_{gH} y)^{\alpha} = [\underline{z(\alpha)}, \overline{z(\alpha)}]$$

with

$$\underline{z(\alpha)} = \min\{\underline{x}(\alpha) - \underline{y}(\alpha), \overline{x}(\alpha) - \overline{y}(\alpha)\}$$
$$\overline{z(\alpha)} = \max\{\underline{x}(\alpha) - \underline{y}(\alpha), \overline{x}(\alpha) - \overline{y}(\alpha)\}$$

Allow $a > 0$. The space of all continuous fuzzy functions defined on $(0, a]$ is denoted as $C((0, a], \mathbb{E})$. Suppose $r \geq 0$ now. We specify

$$C_r([0, a], \mathbb{E}) = \{\Phi \in C((0, a], \mathbb{E}) ; \Phi_r \in C([0, a], \mathbb{E})\}$$

where $\Phi_r(t) = \psi^r(t)\Phi(t)$, for all $\psi \in C([0, a], \mathbb{R})$ and $t \in (0, a]$.

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where $\Phi_r(t) = \psi^r(t)\Phi(t)$, for all $\psi \in C([0, a], \mathbb{R})$ and $t \in (0, a]$. Clearly, $C_r([0, a], \mathbb{E})$ is a metric set that is complete referring to the distance

$$d_r(\Phi, \varphi) = \max_{t \in [0,a]} \psi^r(t) D(\Phi(t), \varphi(t)), \quad \Phi, \varphi \in C_r([0, a], \mathbb{E}).$$

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We indicate $\mathbb{E}^c$ as the sets of fuzzy elements $\Phi \in \mathbb{E}$ with the fact that the mapping $\alpha \mapsto [\Phi]^\alpha$ is continuous referring to the Hausdorff distance on $[0, 1]$. It is commonly known that the metric space $(\mathbb{E}^c, D)$ is complete.

Definition 2

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Definition 3

Allow $M \subseteq E^c$. If, for each $\epsilon > 0$, there's $\eta > 0$ such as

$$\mid \alpha - \beta \mid < \eta \text{ implies } D\left([x]^\alpha, [x]^\beta\right) < \epsilon, \forall x \in M,$$

Consequently, at $\beta \in [0, 1]$, we state that M is level-equicontinuous.

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Consequently, at $\beta \in [0, 1]$, we state that M is level-equicontinuous.

We say M is level-equicontinuous on $[0, 1]$ if the condition described in Definition 3 is true for each $\alpha \in [0, 1]$.

Theorem

Assume that M is a compact supported subset of E^c. The set M is relatively compact in (E^c, D) iff it's level equicontinuous on $[0,1]$.

Furthermore, we remember the semi-linear version of the classic Schaüder fixed point theorem.

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Furthermore, we remember the semi-linear version of the classic Schaüder fixed point theorem.

Theorem (Schaüder fixed point theorem)

Assume U is a semi-linear Banach space with cancellation. Assume S is a nonempty, bounded, closed, convex subset of U. Consider $P : S \to S$ to be a compact map. Therefore, in S, P has (at least) one fixed point.

Theorem (Krasnosel'skii fixed point theorem)

Allow $\mathcal{M}$ to be a non empty, closed and convex subset of $C\left(I, \mathbb{E}^c\right)$ and assume that $\mathcal{Q}$ and $\mathcal{H}$ map $\mathcal{M}$ into S and

i) $\mathcal{Q}$ is continuous and compact,

ii) $\mathcal{Q}t + \mathcal{H}s \in \mathcal{M}$, for every $t, s \in \mathcal{M}$,

iii) $\mathcal{H}$ is a contraction mapping.

Then, there exists a fixed point for $\mathcal{Q} + \mathcal{H}$ in $\mathcal{M}$, that is, there is $s \in \mathcal{M}$ for which $\mathcal{Q}s + \mathcal{H}s = s$.

Definition 4

Assume $\alpha > 0, g \in C^{n-1}(I, \mathbb{E})$ and $\psi \in C^{n}(I, \mathbb{R})$ such that $\psi'(t) > 0 \ \forall t \in I$. The ψ-Riemann Liouville fractional integral of level α of the function g is written asy

$$I_{0^+}^{\alpha,\psi} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} g(s) ds. \tag{3}$$

And, The ψ-Caputo fractional derivative of order α of the function g is expressed as

$${}^{C}D_{0^+}^{\alpha,\psi} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1} g_{\psi}^{[n]}(s) ds \tag{4}$$

Where

$$g_{\psi}^{[n]}(s) = \left(\frac{1}{\psi'(s)} \frac{d}{ds} \right)^{n} g(s) \text{ and } n = [\alpha] + 1.$$

And $[\alpha]$ denotes the integer part of the real number α.

Remark

more specifically, if $\alpha \in]0,1[$, so there is

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\begin{aligned}
{}^{C}D_{0^+}^{\alpha,\psi} g(s) &= \frac{1}{\Gamma(\alpha)} \int_0^s (\psi(s) - \psi(t))^{\alpha-1} g'(t) dt \\
&= I_{0^+}^{1-\alpha,\psi} \left(\frac{g'(s)}{\psi'(s)} \right).
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Proposition 1

Allow $\alpha > 0$, if $g \in C^{n-1}(I, \mathbb{E})$, so we've

1. ${}^{C}D_{0^+}^{\alpha,\psi} I_{0^+}^{\alpha,\psi} g(s) = g(s)$.
2. $I_{0^+}^{\alpha,\psi C} D_{0^+}^{\alpha,\psi} g(s) = g(s) \ominus_{gH} \sum_{k=0}^{n-1} \frac{g_{\psi}^{[k]}(0)}{k!} (\psi(s) - \psi(0))^k$.

Uncertain Non-Local Fuzzy Fractional Differential Equation

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where ${}^{C}D_{0^+}^{\alpha,\psi}u$ is the ψ-Caputo derivative of order $0 < \alpha < 1$ of u, the fuzzy function $f : [0,1] \times \mathbb{E} \to \mathbb{E}$ is continuous and compact, and $g : [0,1] \times \mathbb{E} \to \mathbb{E}$ fulfill

$$D(g(t,u), g(t,v)) \leq LD(u,v), \quad u, v \in \mathbb{E}, \text{ for } L \geq 0.$$

Integral Equation

A fuzzy function $u \in C\left((0,1], \mathbb{E}\right) \cap L\left((0,1], \mathbb{E}\right)$ is a solution to problem (5) if and only if u satisfies the integral equation

$$u(t) = \frac{1}{\Gamma(\alpha)} \odot \int_0^t \psi'(s)(\psi(t) \ominus \psi(s))^{\alpha-1}[f(s,u(s)) \oplus g(s,u(s))]ds \tag{6}$$

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Proof

Just apply operator $I_{0^+}^{\alpha,\psi}$ to both sides of equation (5), then use Proposition 1.

Existence and Uniqueness Results via Schaüder fixed point theorem

Theorem (Schaüder fixed point theorem)

Assume U is a semi-linear Banach space with cancellation. Assume S is a nonempty, bounded, closed, convex subset of U. Consider $P : S \to S$ to be a compact map. Therefore, in S, P has (at least) one fixed point.

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proof

If u is continuous, the function $\underline{u}^{\alpha}$ and $\overline{u}^{\alpha}$ are continuous and bounded on $[0, 1]$. Then $D(u, \tilde{0}) \leq \max \{|\underline{u}^{\alpha}|, |\bar{u}^{\alpha}|\}$ is bounded.

Below we define the operator $\mathcal{M} : C\left([0,1], \mathbb{E}^c\right) \to C\left([0,1], \mathbb{E}^c\right)$ by

$$\begin{aligned} [\mathcal{M}u](t) &= \frac{1}{\Gamma(\alpha)} \odot \int_0^t \psi'(s)(\psi(t) \ominus \psi(s))^{\alpha-1} u_{-r}(s)ds, \\ &= \frac{1}{\Gamma(\alpha)} \odot \int_0^t \psi'(s)(\psi(t) \ominus \psi(s))^{\alpha-1} \psi^{-r}(s)u(s)ds, \end{aligned} \tag{7}$$

for $u \in C\left([0,1], \mathbb{E}^c\right)$ and $0 \leq r < \alpha < 1$.
And the operator $\mathcal{N} : \Omega \to C\left([0,1], \mathbb{E}^c\right)$ by

$$\left[\mathcal{N}u\right](t) = \psi^r(t)\left(f(t, u(t)) + g(t, u(t))\right), \quad t \in [0,1], \tag{8}$$

where

$$\Omega = \left\{u \in C\left([0,1], \mathbb{E}^c\right) \mid d_0(u, \tilde{0}) \leq R_0\right\}.$$

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Indeed, Firstly, we demonstrate that $\mathcal{M}$ is well defined. That is, for a fixed $u \in C\left([0,1], \mathbb{E}^{c}\right)$, ensure that $\mathcal{M}u \in C\left([0,1], \mathbb{E}^{c}\right)$. In reality, we demonstrate $\mathcal{M}u$ is uniformly continuous on $[0,1]$. Allow $t_1, t_2 \in [0,1], t_1 < t_2$, and let R such as

$$D(u(s), \tilde{0}) \leq R, \quad \forall s \in [0,1].$$

Then, by computing $D\left(\mathcal{M}u\left(t_1\right), \mathcal{M}u\left(t_2\right)\right)$, we find

$$D\left(\mathcal{M}u\left(t_1\right),\mathcal{M}u\left(t_2\right)\right) \leq R\left[\frac{\Gamma(-r+1)}{\Gamma(-r+q+1)}\left(\psi^{q-r}(t_1)-\psi^{q-r}(t_2)\right)\right.$$
$$\left.+\frac{2}{\Gamma(q)}\int_{t_1}^{t_2}\psi'(s)\left(\psi(t_1)\ominus\psi(s)\right)^{\alpha-1}\psi^{-r}(s)ds\right]$$

Therefore, through the continuity of ψ, we have $D\left(\mathcal{M}u\left(t_1\right),\mathcal{M}u\left(t_2\right)\right) \to 0$ when $|t_1 - t_2| \to 0$.

$$D\left(\mathcal{M}u\left(t_1\right),\mathcal{M}u\left(t_2\right)\right)\leq R\left[\frac{\Gamma(-r+1)}{\Gamma(-r+q+1)}\left(\psi^{q-r}(t_1)-\psi^{q-r}(t_2)\right)\right.$$
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Following that, we demonstrate the continuity of $\mathcal{M}$. Allow $u_n\to u$ as $n\to\infty$ in $C\left([0,1],\mathbb{E}^c\right)$, That is, $D\left(u_n,u\right)\to 0$ as $n\to\infty$. Then, by computing $D\left(\mathcal{M}u_n,\mathcal{M}u\right)$, we have

$$D\left(\mathcal{M}u_n,\mathcal{M}u\right)\leq\frac{\Gamma(-r+1)}{\Gamma(-r+q+1)}D\left(u_n,u\right)\sup_{t\in[0,1]}\psi^{q-r}(t)$$

Therefore $\mathcal{M}u_n\to\mathcal{M}u$ as $n\to\infty$ in $C\left([0,1],\mathbb{E}^c\right)$.

Remark

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Indeed, Obviously, for $v \in G$ we obtain

$$D(\mathcal{M}v(t), \tilde{0}) \leq \sup_{t\in[0,1]} D(v(t), \tilde{0}) \frac{\Gamma(1-r)}{\Gamma(1-r+q)} \psi^{q-r}(t)$$

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Lemma 3

If $G \subseteq C\left([0,1], \mathbb{E}^c\right)$ is bounded, so $\mathcal{M}(G)$ is equicontinuous in $C\left([0,1], \mathbb{E}^c\right)$.

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Indeed, Allow $d_0(u, \tilde{0}) \leq M$, for all $u \in G$ and $t_1, t_2 \in [0,1], t_1 < t_2$. According to the previous calculations of the lemma 2, we get, $\forall u \in G$,

$$D\left(\mathcal{M}u\left(t_1\right), \mathcal{M}u\left(t_2\right)\right) \leq M\left[\frac{\Gamma(-r+1)}{\Gamma(-r+q+1)}\left(\psi^{q-r}(t_1)-\psi^{q-r}(t_2)\right)\right.$$
$$\left.+\frac{2}{\Gamma(q)}\int_{t_1}^{t_2}\psi'(s)\left(\psi(t_2)-\psi(s)\right)^{q-1}\psi^{-r}(s)ds\right],$$

which tends to 0 as $|t_1 - t_2| \to 0$. As a result, $\mathcal{M}(G)$ is equicontinuous in $C\left([0,1], \mathbb{E}^c\right)$.

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Proposition 2

If $G\subseteq C\left([0,1],\mathbb{E}^c\right)$ in such a way that $\{v(s)\mid v\in G, s\in[0,1]\}$ is compact supported in $\mathbb{E}^c$, therefore G is bounded.

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Indeed, Obviously, there's a compact set K in $\mathbb{R}$ such as $\left\{[u(s)]^0 \mid u \in G, s \in [0,1]\right\} \subseteq K$. in contrast, for $u \in G$, we have $D(u, \tilde{0}) < +\infty$ Well, $G \subseteq C\left([0,1], \mathbb{E}^c\right)$ is bounded.

Lemma 4

If $G \subseteq C\left([0,1], \mathbb{E}^c\right)$ is like $\{u(s) \mid u \in G, s \in [0,1]\}$, is compactly supported and level equicontinuous, so $\mathcal{M}(G)$ is relatively compact in $C([0,1],\ \mathbb{E}^c)$.

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If $G \subseteq C\left([0,1], \mathbb{E}^c\right)$ is like $\{u(s) \mid u \in G, s \in [0,1]\}$, is compactly supported and level equicontinuous, so $\mathcal{M}(G)$ is relatively compact in $C([0,1],\ \mathbb{E}^c)$.

Indeed, so $\{u(s) \mid u \in G, s \in [0,1]\}$ is compactly supported so according to proposition 2, G is bounded. Moreover by Lemma 3 and because G is bounded we get that $\mathcal{M}(G)$ is equicontinuous.So, we still need to demonstrate that $\mathcal{M}(G)(t)$ is is relatively compact in $\mathbb{E}^c$ for all $t \in [0,1]$. Viewed at Theorem 1, that is comparable to demonstrating $\mathcal{M}(G)(t)$ is a compactly supported subset of $\mathbb{E}^c$ and level equicontinuous on $[0,1]$ for all $t \in [0,1]$.

As, $\{u(s) \mid u \in G, s \in [0,1]\}$ is compactly supported, there's a compact space $K \subseteq \mathbb{R}$ such as $[u(s)]^0 \subseteq K$ for each $s \in [0,1]$ and $u \in G$. Therefore, $\forall u \in G$ and $t \in [0,1]$,

$$[\mathcal{M}(u)(t)]^0 \subseteq \frac{\Gamma(-r+1)\sup_{t\in[0,1]}\psi(t)}{\Gamma(-r+q+1)}K$$

Hence, $\mathcal{M}(G)(t)$ is compactly supported for any $t \in [0,1]$.
Furthermore, to demonstrate level equicontinuity, we fix $t \in [0,1]$ and $\varepsilon > 0$. If $w \in \mathcal{M}(G)$, we get for $u \in G$ such as $w = \mathcal{M}(u)(t)$,

$$[w]^\alpha = \frac{1}{\Gamma(q)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{q-1}\psi^{-r}(s)[u(s)]^\alpha ds.$$

Then

$$d_H\left([w]^\alpha,[w]^\beta\right) \leq \frac{1}{\Gamma(q)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{q-1}\psi^{-r}(s)d_H\left([u(s)]^\alpha,[u(s)]^\beta\right)ds.$$

Since $\{u(s) \mid u \in G, s \in [0,1]\}$ is level equicontinuous, as well as for provided $\varepsilon \dfrac{\Gamma(-r+q+1)}{2\lambda\Gamma(-r+1)} > 0$ with $\lambda = \sup\limits_{t\in[0,1]} \psi^{q-r}(t)$, there's $\delta > 0$ satisfies $|\alpha - \beta| < \delta$, well

$$d_H\left([u(s)]^\alpha, [u(s)]^\beta\right) < \frac{\varepsilon\Gamma(-r+q+1)}{2\lambda\Gamma(-r+1)} \quad u \in G, \quad s \in [0,1].$$

So

$$d_H\left([w]^\alpha, [w]^\beta\right) \leq \frac{\varepsilon\Gamma(-r+q+1)}{2\lambda\Gamma(-r+1)} \frac{\Gamma(-r+1)}{\Gamma(-r+q+1)} \psi^{q-r}(t) \leq \frac{\varepsilon}{2} < \varepsilon$$

Thus, $\mathcal{M}(G)(t)$ is level equicontinuous in $\mathbb{E}^c$ on $[0,1]$, $\forall t \in [0,1]$.

Define $f_r(t,u) = \psi^r(t)f(t,u), t \in [0,a]$. Let we now consider

$$S = \left\{x \in \mathbb{E}^c \mid D(x,\tilde{0}) \leq R\right\}.$$

As a consequence, we get the next result :

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Lemma 5

Allow $f_r + g_r : [0,1] \times \mathbb{E}^c \to \mathbb{E}^c$ to be uniformly continuous and bounded in $[0,1] \times S$. Hence, the operator $\mathcal{N}$ is continuous and bounded in $C\left([0,a], \mathbb{E}^c\right)$.

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Proof

Indeed, For continuity, it suffices to take a sequence $u_n \in \Omega$ converging to u on Ω and show that $\mathcal{N}u_n \to \mathcal{N}u$ in $C\left([0,a], \mathbb{E}^c\right)$.

In contrast, let set B is bounded in Ω, so $\forall u \in B$ we have

$$d_0(u, \tilde{0}) \leq M, \quad i.e., \quad \sup_{t\in[0,1]} D(u(t), \tilde{0}) \leq M.$$

Then,

$$\begin{aligned} d_0(\mathcal{N}u, \tilde{0}) &= \sup_{t\in[0,1]} D(\mathcal{N}u(t), \tilde{0}) \\ &= \sup_{t\in[0,1]} D\left(f_r(t, u(t)) + g_r\left(t, u(t)\right), \tilde{0}\right), \forall u \in B. \end{aligned}$$

Since f_r is bounded in $[0, a] \times S$ and g_r is continuous, then there's a $K > 0$ such that $d_0(\mathcal{N}u, \tilde{0}) \leq K, \forall u \in B$. Hence, $\mathcal{N}(B)$ is bounded.

Lemma 6

If $\{(f_r + g_r)(s, x) \mid (s, x) \in [0, a] \times S\}$ is compacly supported and level equicontinuous, so

$$\{u(s) \mid u \in N(\Omega), s \in [0, a]\}$$

is relatively compact.

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is relatively compact.

Indeed, Because

$$\begin{aligned}\{(f_r + g_r)(s, u(s)) \mid u \in \Omega, s \in [0, a]\} &= \{(\mathcal{N}u)(s) \mid u \in \Omega, s \in [0, a]\} \\ &= \{u(s) \mid u \in \mathcal{N}(\Omega), s \in [0, a]\}\end{aligned}$$

is compact supported and level equicontinuous, it is relatively compact.

Lemma 7

Allow $f_r(t,u)$ and $g_r(t,u)$ to be a continuous functions on $[0,1] \times S$. Thus, they're compact on $[0,1] \times S$ iff the set $\{\psi^r(t)\,(f(t,u)+g(t,u)) \mid t \in [0,1], u \in S\}$ is compact supported and level equicontinuous.

Lemma 7

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Indeed Firstly, consider $f_r(t,x)$ and $g_r(t,u)$ as two functions continuous and compact on $[0,1] \times S$. Then by Theorem 1, we have

$$\{f_r(t,u) + g_r(t,u) \mid t \in [0,1], u \in S\}$$

is compact supported and level equicontinuous.
Now let $\{f_r(t,u) + g_r(t,u) \mid t \in [0,1], u \in S\}$ is compact supported and level equicontinuous. By Theorem 1 once again, it is relatively compact. Hence f_r an g_r are compact on $[0,1] \times S$. because for any bounded set $B \subseteq [0,1] \times S$, the set $\{(f_r + g_r)(t,u) \mid (t,u) \in B\}$ is relatively compact.

Main Theorem

Let $0 \leq r < \alpha < 1$ and let f and g be two continuous applications from $(0, 1] \times \mathbb{E}^c$ to $\mathbb{E}^c$. If $f_r + g_r$ is compact and uniformly continuous on $[0, 1] \times \mathbb{E}^c$, As a result, for a sufficient $0 < \delta \leq 1$, the fuzzy integral equation has at least one continuous solution specified on $[0, \delta]$.

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Proof. We have

$$u(t) = \frac{1}{\Gamma(\alpha)} \odot \int_0^t \psi'(s)(\psi(t) \ominus \psi(s))^{\alpha-1}[f(s,u(s)) \oplus g(s,u(s))]ds$$

let's take the set

$$\Omega = \left\{u \in C\left([0,1], \mathbb{E}^c\right) \mid d_0(u, \tilde{0}) \leq R_0\right\}$$

It is straightforward to demonstrate that Ω is a closed, bounded, convex subset of the semilinear Banach space $C\left([0,1], \mathbb{E}^c\right)$.

we define the operator $\mathcal{A} : \Omega \to C\left([0,1], \mathbb{E}^c\right)$ as

$$\begin{aligned}(\mathcal{A}u)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(f(s,u(s)) + g(t,u(s)))ds \\ &= \left[\mathcal{M} \circ \mathcal{N}\left(u\right)\right](t).\end{aligned}$$

As, $f_r + g_r$ is continuou and compact, $\{(f_r + g_r)(t,u) \mid t \in [0,1], u \in S\}$ is compact-supported and level-equicontinuous according to Lemma 7. Thus, according to Lemma 6 , $\{u(s) \mid u \in N(\Omega), s \in [0,1]\}$ is compact supported and level equicontinuous. As a result, according to Lemma 4, $\mathcal{M}(\mathcal{N}(\Omega))$ is relatively compact in $C\left([0,1], \mathbb{E}^c\right)$. The operator $\mathcal{A}$ is then compacted on Ω.

Besides that, given $0 \leq t \leq \delta \leq 1$ such as $\psi(t) \leq \delta$, we obtain

$$D\left(\mathcal{M}u(t), \tilde{0}\right) \leq \frac{\Gamma(1-r)}{\Gamma(1-r+\alpha)} \delta^{\alpha-r} \|u\|_0$$

which implies that

$$\begin{aligned} D\left(\mathcal{A}u(t), \tilde{0}\right) &\leq \frac{\Gamma(1-r)}{\Gamma(1-r+\alpha)} \delta^{\alpha-r}(t) \|\mathcal{N}(u)\|_0 \\ &\leq \frac{2\max\{L, \|f\|_0\}\Gamma(1-r)}{\Gamma(1-r+\alpha)} \delta^{\alpha} \|u\|_0 \end{aligned}$$

Hence, we have

$$\|\mathcal{A}u\|_0 \leq \varepsilon \|u\|_0,$$

Where we are able to reduce $\delta > 0$ to make $\varepsilon > 0$ as small as we wish, we get

$$\mathcal{A}(\Omega) \subseteq \Omega.$$

Then, Schaüder fixed point theorem guarantees that the operator $\mathcal{A}$ has at least one fixed point. As a result, Eq. (5) has at least one solution u given on $[0, \delta]$, where $\delta > 0$ and $\delta \leq 1$.

Existence of solution via Krasnosel'skii fixed point theorem

Let $I_\delta := [0, \delta]$, where $\delta \leq 1$, and $r_0 > 0$ such as

$$R := \sup\left\{D(\tilde{0}, f_r(t, x)) \mid t \in I_{\delta_0}, D(x, \tilde{0}) \leq r_0\right\} < +\infty.$$

It is also feasible to pick δ to be minimal enough so that

$$\frac{R\Gamma(-r+1)}{\Gamma(-r+\alpha+1)}\psi^{\alpha-r}(\delta) \leq r_0$$

. Set the set

$$\Phi := \left\{x \in C\left(I_\delta, \mathbb{E}^c\right) \mid D(\tilde{0}, x(t)) \leq r_0, \text{ for all } t \in I_\delta, \text{ and } x(0) = \tilde{0}\right\}.$$

We can readily verify that $\Phi \subseteq C\left(I_\delta, \mathbb{E}^c\right)$ is bounded, closed, and convex. Also, keep in mind that $C\left(I_\delta, \mathbb{E}^c\right)$ is a semi-linear Banach set. To begin, take the mapping $\mathcal{F} : \Phi \to C\left(I_\delta, \mathbb{E}^c\right)$ is given as

$$(\mathcal{F}u)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^{-r}(s)f_r(s, u(s))ds, \text{ for } t \in I_\delta. \quad (9)$$

- (H) The the preceding inequality occurs for any pair of points $(t,u),(t,v) \in [0,a] \times C\left([0,1],\mathbb{E}^c\right)$.

$$D(f(t,u(t)),f(t,v(t))) \leq p(t)w\left(d_0(u,v)\right),$$

where w is a real continuous function on $[0,\infty)$ with $w(0)=0$ and $p:[0,1] \to \mathbb{R}^+$ fulfil $I^\alpha p(t) < N$ for every $t \in [0,1]$.

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Lemma 8

Let f to fulfill (H) on $[0,a] \times \mathbb{E}^c$. The following assumptions are thus verified :

i) $\mathcal{F}$ is well-defined on $C\left(I_\delta,\mathbb{E}^c\right)$.

ii) $\mathcal{F}$ is a continuous mapping on $C\left(I_\delta,\mathbb{E}^c\right)$

Proof. To begin, we demonstrate that the map $\mathcal{F}$ is quite defined. For any $u \in \Phi$, $(\mathcal{F}u)(0) = \tilde{0}$ is evident by construction. With a given $u \in \Phi$, we prove that $\mathcal{F}u \in C\left(I_\delta, \mathbb{E}^c\right)$. Furthermore, we demonstrate that $\mathcal{F}$ is a uniformly continuous function on the interval I_δ. We choose a determined $t, t' \in I_\delta$, with $t < t'$. We get

$$\begin{aligned} & D\left((\mathcal{F}u)(t), (\mathcal{F}u)\left(t'\right)\right) \\ & \leq \frac{R\Gamma(-r+1)}{\Gamma(-r+\alpha+1)}\left(2\left(\psi(t') - \psi(t)\right)^{\alpha-r} + \psi^{\alpha-r}(t) - \psi^{\alpha-r}(t')\right) \end{aligned}$$

Therefore, $D\left((\mathcal{F}u)(t), (\mathcal{F}u)\left(t'\right)\right) \to 0$, when $|t - t'|$ tends to 0, so $\mathcal{F}u$ is a continuous mapping on I_δ, for $u \in \Phi$.In addition, we have

$$D((\mathcal{F}u)(t), \tilde{0}) \leq r_0,$$

$\forall t \in I_\delta$. As a result, $\mathcal{F}$ is a self-mapping $\mathcal{F} : \Phi \to \Phi$.

We next demonstrate that $\mathcal{F}$ is a continuous map. Consider $y_n, y \in \Phi, n = 1, 2, \ldots$ such as $y_n \underset{n\to\infty}{\longrightarrow} y$, with convergence in the set $C([0,1], \mathbb{E}^c)$, i.e, fulfill $d_0(y_n, y) \underset{n\to\infty}{\longrightarrow} 0$. As a result, we may derive that for any $t \in J := [0,1]$,

$$d_0(\mathcal{F}y_n, \mathcal{F}y) \leq \sup_{t\in I_{\delta_0}} I^\alpha p(t) w(\rho_0(y_n, y)).$$

Because, according to the criteria in $(\mathrm{H}), w(0) = 0$ and w is continuous on its domain $[0, \infty)$, it is obvious that $w(r) \underset{r\to 0^+}{\longrightarrow} 0$. Due to $\rho_0(y_n, y) \underset{n\to\infty}{\longrightarrow} 0$ and the estimation $I^\alpha p(t) < N$, we get to the conclusion that $\mathcal{F}y_n \to \mathcal{F}y$ as $n \to \infty$, i.e., $\mathcal{F}$ is continuous.

Main Theorem

Take a given $u \in (0,1)$ and assume that the fuzzy function $f : [0,1] \times \mathbb{E}^c \to \mathbb{E}^c$ is continuous and compact, fulfilling (H), and suppose that $g : [0,1] \times \mathbb{E}^c \to \mathbb{E}^c$ fulfill (2). Given these constraints, the non-local UDE (5) has (at least) one solution specified on $[0, \delta]$. This is a continuous function, with δ being an acceptable positive integer $\delta < 1$.

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Proof. Let's ask

$$\mathcal{F}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \psi^{-r}(s) f_r(s, u(s)) ds$$

and

$$\widetilde{\mathcal{F}}(u(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \psi^{-r}(s) g_r(s, u(s)) ds.$$

So,

$$\mathcal{A}(u(t)) = \mathcal{F}(u(t)) + \widetilde{\mathcal{F}}(u(t))$$

Using Lemma 8 , we have already established that $\mathcal{F}$ is continuous. Now let's demonstrate the compactness of $\mathcal{F}$. This amounts to showing that $\mathcal{F}(\Phi)(t)$ is relatively compact, which means, according to Theorem 1, that $\mathcal{F}(\Phi)(t)$ is equicontinuous and has compact support. If we choose an arbitrary determined $u \in \Phi$ and let $t, t' \in I_\delta$ with $t \leq t'$, so

$$\begin{aligned} & D\left((\mathcal{F}u)(t), (\mathcal{F}u)\left(t'\right)\right) \\ & \leq \frac{R\Gamma(-r+1)}{\Gamma(-r+\alpha+1)}\left(2\left(\psi(t')-\psi(t)\right)^{\alpha-r}+\psi^{\alpha-r}(t)-\psi^{\alpha-r}(t')\right). \end{aligned}$$

This means that $\mathcal{F}(\Phi)$ is equicontinuous in $C\left(I_\delta, \mathbb{E}^c\right)$.

Given that f is a compact mapping, $f(I_\delta \times \Phi)$ is a relatively compact in $\mathbb{E}^c$. According to Theorem 1, $f(I_\delta \times \Phi)$ is a level equicontinuous. As a result, we can assert the existence of $\delta > 0$ for any $\varepsilon > 0$.

$$D\left([f(s, y(s))]^{\alpha_1}, [f(s, y(s))]^{\alpha_2}\right) < \frac{\Gamma(-r+\alpha+1)}{\psi^{\alpha-r}(\delta)\Gamma(-r+1)}\varepsilon, \forall (s, y) \in I_{\delta_0} \times \Phi,$$

provided that $|\alpha_1 - \alpha_2| < \delta$. Therefore, similarly , for $|\alpha_1 - \alpha_2| < \delta$ and $u \in \Phi$, we deduce

$$\begin{aligned} D\left([u]^{\alpha_1}, [u]^{\alpha_2}\right) &= D\left([\mathcal{F}(y)(t)]^{\alpha_1}, [\mathcal{F}(y)(t)]^{\alpha_2}\right) \\ &\leq \frac{\psi^{\alpha-r}(\delta)\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} D\left([f(s, y(s))]^{\alpha_1}, [f(s, y(s))]^{\alpha_2}\right) \\ &\leq \varepsilon. \end{aligned}$$

Thus, $\mathcal{F}(\Phi)(t)$ is level equicontinuous in $\mathbb{E}^c$.

Therefore, given the relative compactness of $f\left(I_\delta \times \Phi\right)$, For any $(s, y) \in I_\delta \times \Phi$, we can ensure the existence of a compact set $M \subset \mathbb{R}$ with $[f(s, y(s))]^0 \subseteq M$. As a result, we receive

$$\begin{aligned} & \left[\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \psi^{-r}(s) f_r(s, y(s)) ds\right]^0 \\ & \quad \subseteq \frac{\psi^{\alpha-r}(t) \Gamma(-r+1)}{\Gamma(-r+\alpha+1)} M \end{aligned}$$

As a consequence, as a subset of $\mathbb{E}^c, \mathcal{F}(\Phi)(t)$ is compact supported. Hence, the relative compactness of $\mathcal{F}(\Phi)$ in $C\left([0, \delta], \mathbb{E}^c\right)$ is inferred using the Arzelà-Ascoli principle.

It remains to show that $\widetilde{\mathcal{F}}$ is contractive. So, for all $x, y \in \Phi$, we have Therefore,

$$D((\widetilde{\mathcal{F}}x)(t), (\widetilde{\mathcal{F}}y)(t)) \leq \frac{L\psi^{\alpha-r}(\delta)\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} D(x, y).$$

As a result, if δ is short enough that $L\psi^{\alpha-r}(\delta)\Gamma(-r+1) < \Gamma(-r+\alpha+1)$, therefore the mapping $\widetilde{\mathcal{F}}$ is contraction. As a result, invoking Krasnosel'skii fixed point theorem, the mapping $\mathcal{F} + \widetilde{\mathcal{F}}$ has at least one fixed point in Φ. As a consequence, for a given $0 < \delta \leq a$, there's (at least) one fuzzy solution u for (5) on I_δ.

An illustrative example.

Application

Let's consider the following physical problem

$$\begin{cases} {}^{C}D_{0^+}^{\alpha,\psi}u(t) = \eta v(t) \oplus \rho w(t)(x(t)\sin(\xi) \ominus v(t)\cos(\xi)) \ominus \frac{1}{T_2}u(t) \\ {}^{C}D_{0^+}^{\alpha,\psi}v(t) = (-\eta)u(t) \ominus w(t) \oplus \rho w(t)(v(t)\sin(\xi) \oplus u(t)\cos(\xi)) \ominus \frac{1}{T_2}v(t) \\ {}^{C}D_{0^+}^{\alpha,\psi}w(t) = v(t) \ominus \rho\sin(\xi)\left(u^2(t) \oplus v^2(t)\right) \ominus \frac{1}{T_1}(w(t) - 1) \end{cases} \tag{10}$$

where $\alpha \in (0,1)$, $\eta = -0.4\pi$, $\rho = 30$, $\xi = 0.173$, $T_1 = 5$ and $T_2 = 2.5$.

Ce problème physique est lié à la résonance magnétique nucléaire (RMN) et à la manière dont les impulsions radiofréquence (RF) perturbent le mécanisme de rotation.
Dans la RMN, les atomes sont exposés à des impulsions RF de courte durée qui perturbent leur rotation. Ces impulsions sont beaucoup plus longues que la période de la fréquence de résonance mais beaucoup plus courtes que les temps de relaxation T_1 et T_2. Ainsi, nous devons être capables d'exprimer l'équation de Bloch avec une dérivée que nous pouvons activer et désactiver de manière impulsive ou en échelon. Dans de telles conditions, le choix de la dérivée de Caputo est préférable à celle de la dérivée de Riemann-Liouville, car elle permet de décrire plus efficacement le phénomène étudié, notamment en évitant la nécessité de conditions initiales d'ordre fractionnaire que nous ne pouvons pas fournir.

L'équation initiale de ce système se rapporte au cadre considéré. Alternativement, des résultats comparables peuvent être obtenus pour l'espace $C([0,1], \mathbb{E} \times \mathbb{E} \times \mathbb{E})$ avec des modifications notables ; ainsi, il pourrait être possible de reformuler le problème (10) comme une extension de dimension supérieure du problème (5), en choisissant

$$f(t,u,v,w) = \begin{pmatrix} \eta v \oplus \rho w(x \sin(\xi) \ominus v \cos(\xi)) \\ \\ (-\eta)u \ominus w \oplus \rho w(v \sin(\xi) \oplus u \cos(\xi)) \\ \\ v \ominus \rho \sin(\xi)\left(u^2 \oplus v^2\right) \end{pmatrix}$$

et

$$g(t,u,v,w) = \left(\ominus \frac{1}{T_2} u, \ominus \frac{1}{T_2} v, \ominus \frac{1}{T_1}(w \ominus 1)\right)$$

qui répondent aux exigences pertinentes si g est bien défini.

Thank you for your attention