Exact Constants in Banach Space Geometry: User's Guide for the Allometry Software

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We describe in this note the MATLAB code that was used to perform the computations of [1]. We work with a finite-dimensional normed space $\mathcal{V}$, which is of the following types

- type 0: the n-dimensional Euclidean space $\mathcal{E}_n$,
- type 1: the n-dimensional sequence space ℓ_p^n,
- type 2: the $(n+1)$-dimensional space $\mathcal{P}_n$ of algebraic polynomials of degree $\leq n$,
- type 3: the $(2n+1)$-dimensional space $\mathcal{T}_n$ of trigonometric polynomials of degree $\leq n$,
- type 4: the 2-dimensional space determined by the vertices of its unit ball,
- type 5: the 2-dimensional space determined by the vertices of its dual unit ball.

The functions available so far are — in lexicographic order — `AbsCdN`, `Allo`, `Allo`, `CanCdN`, `CanDual`, `CdN`, `DualNormAlg`, `DualNormPolygon`, `DualNormPolygon2`, `DualNormTrig`, `Mx2Vc`, `NormAlg`, `NormPolygon`, `NormPolygon2`, `NormTrig`, `Nzed`, `Reg2nGone`, `Signs`, `Signs1`, `SymAllo`, `Vc2Mx`. Note that one can access at any time a description of a function by typing `help` followed by the name of the function at the MATLAB prompt. For instance, typing

```
>> help CanCdN
```

returns

```
>> CANCDN Calculate the canonical condition number
   CANCDN(V) returns the condition number of the frame formed by the columns of
V relatively to its canonical dual frame
```

1 The three main functions

The functions `CdN`, `AbsCdN`, and `Allo` are respectively designed to compute

- the condition number of a system $\underline{v} = (v_1, \ldots, v_N)$ spanning the space $\mathcal{V}$, relatively to a dual system $\underline{\lambda} = (\lambda_1, \ldots, \lambda_N)$ of linear functional on $\mathcal{V}$, i.e. the quantity

$$\kappa_\infty(\underline{v}|\underline{\lambda}) := \sup_{a \in \ell_\infty^N} \frac{\left\| \sum_{j=1}^N a_j v_j \right\|}{\|a\|_\infty} \cdot \sup_{v \in \mathcal{V}} \frac{\left\| (\lambda(v_j)) \right\|_\infty}{\|v\|},$$

- the absolute condition number of a system $\underline{v} = (v_1, \ldots, v_N)$ spanning the space $\mathcal{V}$, i.e. the quantity

$$\kappa_\infty(\underline{v}) := \min \left\{ \kappa_\infty(\underline{v}|\underline{\lambda}), \quad \underline{\lambda} \text{ dual system to } \underline{v} \right\},$$

- the N-th allometry constant of the space $\mathcal{V}$, i.e. the quantity

$$\kappa_\infty^N(\mathcal{V}) := \min\{\kappa_\infty(\underline{v}), \quad \underline{v} = (v_1, \ldots, v_N) \text{ spans } \mathcal{V}\}.$$

2 Input arguments

If a system $\underline{v} = (v_1, \ldots, v_N)$ spanning the n-dimensional space $\mathcal{V}$ needs to be entered, e.g. in `AbsCdN`, it is done so via its matrix in a specific basis $\underline{w} = (w_1, \ldots, w_n)$ for $\mathcal{V}$, i.e. via the $n \times N$ matrix

$$V = \begin{bmatrix} \operatorname{coef}_{w_1}(v_1) & \ldots & \operatorname{coef}_{w_1}(v_N) \\ \vdots & \ldots & \vdots \\ \operatorname{coef}_{w_n}(v_1) & \ldots & \operatorname{coef}_{w_n}(v_N) \end{bmatrix}.$$

If in addition a dual system $\underline{\lambda} = (\lambda_1, \ldots, \lambda_N)$ needs to be entered, it is done so via the $n \times N$ matrix

$$U = \begin{bmatrix} \lambda_1(w_1) & \cdots & \lambda_N(w_1) \\ \vdots & \ldots & \vdots \\ \lambda_1(w_n) & \cdots & \lambda_N(w_n) \end{bmatrix}.$$

Note that the number of elements in the system $\underline{v}$ and the dimension of the space $\mathcal{V}$ are implicitly entered. If we want to calculate the N-th allometry constant of a n -dimensional space $\mathcal{V}$, however, we need to input the integers N and n. For example, we would type

```
>> Allo(3,4,0)
>> Allo(3,4,1,1)
```

to compute the 4-th allometry constant of the 3-dimensional space $\mathcal{E}_3$ — space of type 0 — and then the 4-th allometry constant of the 3-dimensional space ℓ_1^3 — space of type 1, specifying $p = 1$ as the fourth argument. When working with polygonal spaces — spaces of type 4 and type 5 — we provide the vertices of the (dual) unit ball as fourth argument. For instance,

```
>>Hex=[1 1/2 -1/2 -1 -1/2 1/2; 0 sqrt(3)/2 sqrt(3)/2 0 -sqrt(3)/2 -sqrt(3)/2];
>>[Allo(2,2,4,Hex),Allo(2,3,4,Hex)]
```

returns `1.5000` and `1.3333`, which are the 2-cd and 3-rd allometry constants of the 'hexagonal' space and believed to be maximal among all 2-dimensional spaces.

3 Output arguments

If only one output argument is requested, then the result is the numerical value of the quantity invoked by the function. If more than one output argument is requested, then optimal systems are also returned. For example, typing

```
>> [k,OS,OD]=Allo(2,3,0)
```

returns the value of the 3-rd allometry constant of the Euclidean plane $\mathcal{E}_2$, together with a best conditioned 3-frame `OS` and an optimal dual frame `OD`.

References

[1] Foucart, S., Allometry constants of finite-dimensional spaces: theory and computations, Preprint.