ASYMMETRIC DOLPH-CHEBYSHEV, SARAMÄKI, AND TRANSITIONAL WINDOWS FOR FRACTIONAL DELAY FIR FILTER DESIGN

Timo I. Laakso[1], *Tapio Saramäki*[2] *and Gerald D. Cain*[1]

[1]*Centre for Microelectronic Systems Applications*
University of Westminster
115 New Cavendish Street, W1M 8JS London, UK
e-mails: timol@westminster.ac.uk,
cain@cmsa.westminster.ac.uk

[2]*Signal Processing Laboratory,*
Tampere University of Technology, PO Box 553
FIN-33101 Tampere, Finland
e-mail: ts@cs.tut.fi

ABSTRACT

An approach for the design of fractionally shifted (asymmetric) windows is presented. This approach is based on bandlimited sinc interpolation combined with oversampling techniques. The design methods are illustrated with practical examples, demonstrating that shifting of symmetric windows can be achieved while essentially maintaining the magnitude response of the window spectrum.

I. INTRODUCTION

Windowing an ideal infinite-length discrete-time impulse response is a well-known technique to design linear-phase FIR filters. Usually *symmetric* windows are used, with one sample in the middle if the filter length is odd [1]. However, in some applications a fractionally shifted window is needed, e.g. when it is desired to design a fractional-delay FIR filter approximating a delay that is a noninteger multiple of the sample interval [2], [3]. Even if the desired fractional delay filter is linear phase, and thus its impulse response is in principle symmetric, the centre of symmetry is shifted somewhere between the samples so that a corresponding shift is required for the window function as well.

Recently work on fractional-delay filters, which are useful for adjusting rendezvous delays in demodulation, control, and synchronization applications, has shown that asymmetric windowing is a powerful and simple way to improve the quality of such delay elements [4], [5].

Some windows like rectangular, Bartlett (triangular), Hanning, Hamming, Blackman (all of the raised-cosine type), and Kaiser (a parametric window) are defined as a function of a continuous-time variable which, although usually taking integer values n, can also be chosen to take fractionally shifted values $n-d$ where d is the fractional shift, $0 < d < 1$. However, the the spectrum of the discrete-time window is always somewhat different from that of the original continuous-time window due to aliasing effects. The aliasing is usually a problem for small window lengths, as demonstrated in [6].

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However, the best adjustable windows, like Dolph-Chebyshev (DC) [1], [7], [8], Saramäki (S) [8], [9], and Transitional (T) [7], [8] are defined in the frequency domain. The time-domain window coefficients can be obtained either via inverse Discrete Fourier Transform of the sample spectrum or via simple recursive formulas presented in [7], [8] (which, however, are applicable only for symmetric windows).

In this paper we propose an efficient technique for obtaining fractionally shifted DC, S, and T windows (or any other windows defined in frequency domain). We employ sinc interpolation, enhanced by an optional oversampling in the window design phase. The generation of shifted windows as well as the properties of the resulting filters are illustrated with examples.

II. PROBLEM FORMULATION

The most straightforward technique for designing FIR filters is to use a window function to truncate and smooth the impulse response of an ideal zero-phase infinite-impulse response filter (see e.g. [1], [7], [8]). For example, the impulse response of the ideal lowpass filter with a cutoff frequency ω_c is

$$h_{id}(n) = \sin(\omega_c n)/(\pi n) , \tag{1}$$

The windowed zero-phase filter coefficients are obtained via

$$h(n) = w(n)h_{id}(n) , \tag{2}$$

where $w(n)$ is an $N = (2M+1)$-length window function, being nonzero only for $-M \le n \le M$. The impulse response of the corresponding causal filter is $h(n-M)$. For simplicity of notation, we will keep the filters and windows in the zero-phase format of (2) for the rest of the paper.

In the frequency domain, the response of $h(n)$ is related to those of $w(n)$ and $h_{id}(n)$ via

$$H(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{id}(\omega)W(\omega-\theta)d\theta. \quad (3)$$

In order to maintain the spectrum of the ideal filter, the window spectrum $W(\omega)$ should be as close to an impulse function as possible. There are several alternatives for the formulation of the optimal window (see e.g. [1], [10], [11]). Window designers typically try to minimize the width of the main lobe and the peaks or the energy of the sidelobes of $W(\omega)$, resulting in a reasonable compromise between these two. Some windows are adjustable, like Kaiser, DC, S, and T windows [7], so that one can control the ripples and the transition bandwidth of the final filter with one parameter in an implicit manner.

The filter *h(n)* of (2) is also exactly zero-phase so that the implemented filter *h(n–M)* is linear phase with constant group delay of *M* at all frequencies. However, in various applications in synchronization for communications, speech coding, modeling of musical instruments etc. (for a tutorial, see [2]), the delay of the actual filter cannot be assumed to be a multiple of the sampling interval, but any real-valued delay *D* with a fractional part *d* should be attainable. This can be reduced to a problem of *fractional delay filter* design which, using the windowing approach, can be expressed as designing

$$h_d(n) = h(n-d) = w(n-d)h_{id}(n-d). \quad (4)$$

Note that this filter no longer has symmetric impulse response, as the centre of symmetry generally lies somewhere between two samples.

There is no problem in fractional shifting of the ideal response (1) as the interpolating sinc function can be considered as a continuous one, but the windows are not always straightforward to shift. The DC, S, and T windows [7], [8], [9] are defined in the frequency domain, and although recursive formulas are known for the time-domain coefficients in closed form, those apply for symmetric impulse responses only and cannot be used for offset windows of the type (4).

Fractional shifting of (essentially) continuous-time Hamming and von Hann windows was considered in [5], demonstrating that, particularly for short window lengths, it is essential to use shifted windows.

In this paper we address the problem of shifting windows defined in the frequency domain. The basic design objective is to avoid disturbance of the magnitude spectrum of the shifted window so as to meet the criteria that the original symmetric window was designed to meet.

III. SHIFTING FREQUENCY DOMAIN WINDOWS

The spectrum of an odd-length symmetric window like DC, S, or T window can be expressed as

$$W(\omega) = \sum_{k=-M}^{M} w(k)e^{-jk\omega} = w(0) + \sum_{k=1}^{M} w(k)\cos(k\omega), \quad (5)$$

where *w(n)* are the time-domain window coefficients. The time-domain window coefficients are formally obtained from the spectrum (2) via inverse Fourier transform as

$$w(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(\omega)e^{jn\omega}. \quad (6)$$

A. Window Shifting via Sinc Interpolation

Let us assume that we know the coefficients of a symmetric DC, S, or T window *w(n)*. The coefficients of the shifted window $w_d(n)$ are then obtained via (6) as

$$w_d(n) = w(n-d) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[W(\omega)e^{-jd\omega}\right]e^{jn\omega}, \quad (7)$$

which is seen to be equivalent to taking the IFT of the original window spectrum multiplied by a linear phase factor $e^{-jd\omega}$. This transform thus completely retains the original magnitude spectrum of the window! Substituting (5) into (7) yields the coefficients of the shifted window as

$$w_d(n) = w_c(n-d), \quad (8)$$

where

$$w_c(t) = \sum_{k=-M}^{M} w(k)\operatorname{sinc}(t-k), \quad (9)$$

which is seen to be an expansion of the familiar sinc functions, defined as

$$\operatorname{sinc}(x) = \sin(\pi x)/(\pi x). \quad (10)$$

Hence, Eq. (8) gives a discrete-time window with exactly the same bandlimited spectrum as the original window, except for the phase factor due to the shift.

However, due to infinite-length sinc functions, the window (8) is of infinite length and needs some windowing itself. The most straighforward way is to simply truncate, i.e. to use the rectangular window by picking up samples $w_d(n)$ for $n=-M, -M+1,\ldots,M$.

The arrival at an expansion of sinc functions may first appear to be going in circles. However, the 'windowing of the window' is essentially a much more friendly problem than windowing the original lowpass filter response, as a typical window spectrum is smooth and its Fourier series truncation will thus not lead into as bad Gibbs phenomenon effects as those from the truncation of the Fourier series of an ideal lowpass filter. This will be apparent from the examples in Section IV.

Another error source for the direct truncation comes from the fact that $w_c(t)$ is practically nonzero for $-(M+1) < t < (M+1)$ and zero otherwise. Therefore, if *d* is positive, we disregard an essentially nonzero sample $w_d(M+1) = w_c(M+1-d)$. Similarly, for a negative *d*, a nonzero sample $w_d(-M-1) = w_c(-M-1-d)$ is disregarded. This error source is also considered in connection with the examples of Section IV.

B. Discrete Interpolation

The performance of the shifted window as well as that of the resulting filter can be improved by first using a discrete-time interpolation for the window function, that is, designing a longer window in such a way that its subsamples approximate very closely those of the original unshifted window. Then, the above-mentioned sinc interpolation is applied to this longer window and proper samples are selected.

In order to carry out the interpolation in an appropriate manner, we recall that for the S, DC, and T windows considered in [7] and [8] the basic adjustable parameter, denoted by β, is directly the ratio between the sidelobe of the window and that of the rectangular window. If it is desired to padd K-1 zero-valued samples between the existing samples and K_{end} additional samples at both ends, then half the order of the resulting longer window, denoted by $\hat{w}(n)$, is $\hat{M} = MK + K_{end}$ and Eq. (8) is replaced by

$$w_d(n) = w_{ci}(n-d), \qquad (11)$$

where

$$w_{ci}(t) = \sum_{k=-\hat{M}}^{\hat{M}} \hat{w}(k/K)\mathrm{sinc}\big[(t-k)/K\big]. \qquad (12)$$

If it is desired that the transition bandwith of the resulting decimated window $w_d(n)$ is $4\beta\pi/(2M+1)$ (the transition bandwidth of the rectangular window is $\beta\pi/(2M+1)$), the value of β for the interpolated window has to be chosen as $(2\hat{M}+1)\beta/[K(2M+1)]$. With this selection, the mainlobe width of $\hat{w}(n)$ is $\beta\pi/[K(2M+1)]$ which is increased to the desired value $\beta\pi/(2M+1)$ after decimating by K. In the special case d=L/K, $w_d(n) = \hat{w}(nK-L)$ so that no sinc interpolation is needed.

It has been observed experimentally that for long window lengths (more than 40) the above discrete interpolation does not significantly improve the results. For window lengths shorter than 40, using a small value of K (2 or 4) and $K_{end} = K-1$ results in an improvement in the performance of the window.

IV. EXAMPLES

Consider first the use of shifted windows for synthesizing filters of length 33 (M=16) in such a way that the cutoff frequency of the ideal filter is π and the maximum deviation from unity in the filter passband is $\delta = 0.01$.

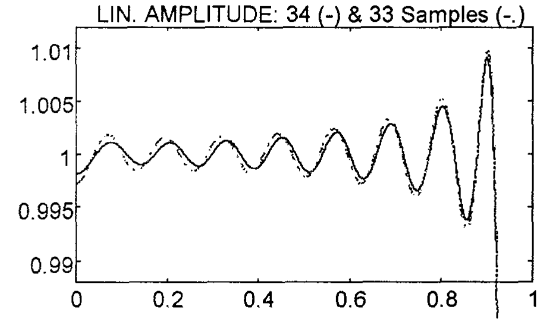

Figure 1. The effect of discarding of the last sample on the filter amplitude response when using a length-33 Saramäki window.

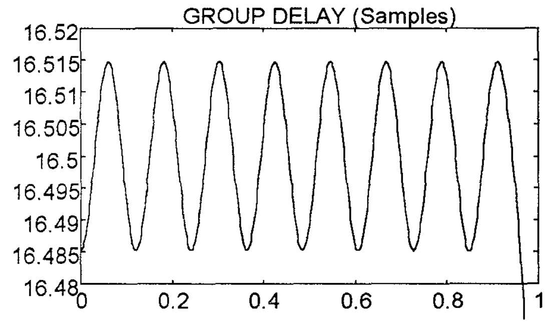

Figure 2. Group delay response for the wideband filter designed using a shifted length-33 Saramäki window.

The fractional delay is chosen as d=0.5 which is the worst case when $-0.5 \le d \le 0.5$.

When the sinc interpolation formula of Eq. (8) is applied, for n=–16,–15,...,16, the minimum value for the adjustable parameter for the Saramäki window is for $A_s = 20\log\hat{\delta} = 46.6$ dB. Therefore, $\hat{\delta} = 0.00649 = \delta/\alpha$ with α=2.13. This means that the resulting ripple is α times that of the filter in conventional FIR filter design problems.

Figure 1 shows the response of the resulting filter as well as that of the filter where the sample at n=17 is included. For the latter filter, the length is 34 and the ripple is approximately $2\hat{\delta}$. This can be explained to be true in all cases based on the theory of half-band FIR filters. The increase of α from 2 to 2.13 for the first filter is a consequence of the fact that the last impulse response value is disregarded. The passband edge angle for the filter of length 33 is at 0.91923π and the group delay oscillates in this region around 16.5 in an equiripple manner with the maximum deviation being 0.0148, as seen in Figure 2. The filter for d=–0.5 is obtained by simply time-reversing the impulse response. (The frequency range in all the figures is from zero to the Nyquist frequency).

The original and the shifted windows with 33 samples and the corresponding magnitude spectra are shown in Figure 3. Fig. 3(a) demonstrates the effect of shifting in the time domain, and the magnitude spectra of Fig. 3(b) show that

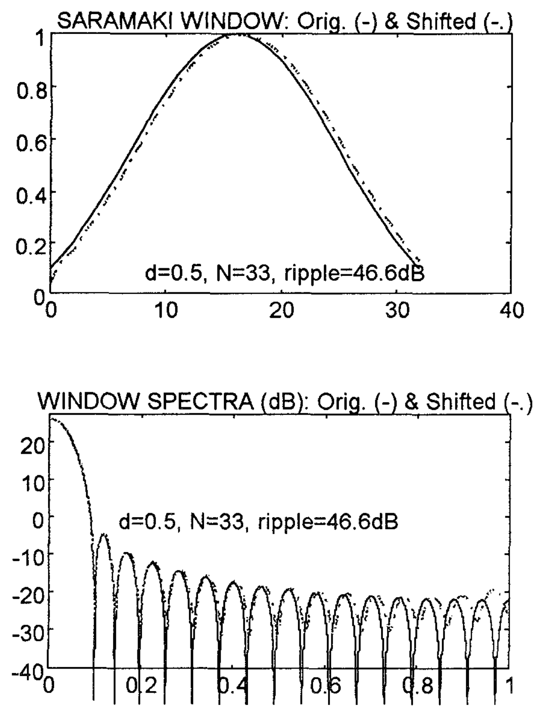

Figure 3. Responses for original and shifted Saramäki windows of length 33.

the sidelobes of the shifted window are only slightly higher than those of the original. Figure 4 compares the magnitude spectrum of the shifted filter of length 34 with the original one of length 33. It is seen that the first sidelobes of these windows are practically the same, whereas the peaks of the last sidelobes are even lower for the shifted window.

When the interpolation formula of Eqs. (11) and (12) is applied with K=2 and $K_{end}=1$, $w_{0.5}(n)=\hat{w}(2n-1)$ and the passband edge becomes slightly larger (0.92001π compared to 0.91923π). In this case, α =2.02.

For the corresponding filter designed by the transitional window (α =2.18) the passband region is wider (0.92516π compared to 0.91923π) at the expense of a higher group delay ripple (0.0426 compared to 0.0148). The minimum length of the filter designed by the Saramäki window to achieve the same passband region is 37, showing that the transitional window gives the best results in the case where the group delay ripple is not the quantity to be optimized. The passband edge for the filter designed using a shifter length-33 Dolph-Chebyshev filter (α =2.14) is only 0.91766π.

When the absolute value of d is decreased, also the value of α reduces. For instance, for the filters designed using the Saramäki window for M=16 and δ=0.01, α is 0.19, 0.69, 1.35, and 1.91 for d=0.1, d=0.2, d=0.3, and d=0.4, respectively. For d=0.05, d=0.15, d=0.25, d=0.35, and

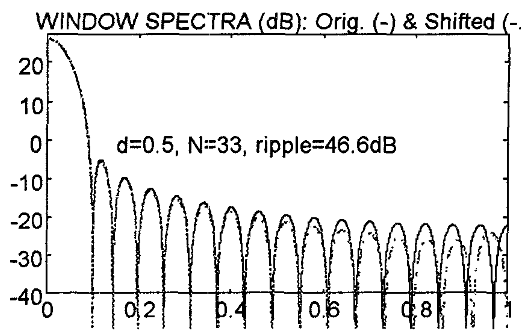

Figure 4. Magnitude spectra for the original Saramäki window of length 33 and the shifted window with 34 samples.

d=0.45, the corresponding values are 0.06, 0.40, 1.01, 1.66, and 2.07. Simultaneously, the passband bandwidth increases. For instance for d=0.25, the edge moves from 0.91923π (d=0.5) to 0.93216π.

V. CONCLUSIONS

A general approach for shifting symmetric windows using sinc interpolation techniques was presented. The results show that shifted windows which maintain the spectral properties of the original ones with high accuracy are obtained using the proposed approach.

Future work is devoted to characterizing the performance of the filters designed by the above shifted windows in more detail and to finding simple design rules.

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