Lecture 8. DP and Mechanism Design.

(Reverse game theory)

- Revisit Digital Goods Example
- DP as a tool for truthfulness.

- (If have time): Continual Release

Logistics: HW Posted on Canvas

Due Sunday.

Example 2: Pricing a digital good.

- Selling an app; what price?
- n buyers report their valuations: $\underbrace{\text{"How much are they willing to pay?"}}_{x_i}$
  bidders

Revenue:

$$g(p; x) = p \cdot \#\{i : x_i \geq p\}$$

Toy Example: \$1, \$1, <u>\$2.01</u>, \$4.

↳ Rev = 4.02

Utility Theoretic View.

Each buyer may have a utility function $f: Y \to \mathbb{R}_{\geq 0}$

Notation: (from game theory)

$$(x_{-i}, x_i') = (x_1, x_2, \dots, x_{i-1}, x_i', x_{i+1}, \dots x_n)$$

Unilateral deviation.

$\approx$ "Neighboring datasets"

Proposition (Utility Theoretic View of DP)

An algorithm $A: X^n \to Y$ is ε-DP $\boxed{\text{if and only if}}$ for every utility function $f: Y \to \mathbb{R}_+$ and every $x \in X^n$, $x_i' \in X$:

$$\underbrace{\mathbb{E}_{Y \sim A(x)}[\underset{\substack{\uparrow \\ \text{utility for} \\ \text{bidder } i}}{f}(Y)]}_{\substack{\text{Expected utility.} \\ \text{for truthtelling}}} \geq \exp(-\varepsilon) \underbrace{\mathbb{E}_{A \sim A(x_{-i}, x_i')}[f(Y)]}_{\substack{\text{Expected utility} \\ \text{for mis-reporting}}}$$

Proposition (Utility Theoretic View of DP)

An algorithm $A:\mathcal{X}^n \longrightarrow \mathcal{Y}$ is ε-DP. $\boxed{\text{if and only if}}$ for every utility function $f:\mathcal{Y}\to\mathbb{R}_+$ and every $x\in\mathcal{X}^n$, $x_i'\in\mathcal{X}$:

$$\mathop{\mathbb{E}}_{Y\sim A(x)}\left[f(Y)\right] \geq \exp(-\varepsilon)\mathop{\mathbb{E}}_{A\sim A(x_{-i},x_i')}\left[f(Y)\right] \qquad (*)$$

Proof. DP $\Rightarrow$ $(*)$

$$\mathop{\mathbb{E}}_{Y\sim A(x)}\left[f(Y)\right] = \int_y f(y)\cdot \underline{\mathbb{P}\left[A(x)=y\right]}\, dy$$

$$\geq \int_y f(y) \exp(-\varepsilon)\cdot\mathbb{P}\left[A(x')=y\right] dy$$

$$= e^{-\varepsilon}\int_y f(y)\, \mathbb{P}\left[A(x')=y\right] dy$$

$$= \mathop{\mathbb{E}}_{Y\sim A(x_{-i},x_i')}\left[f(Y)\right]\cdot e^{-\varepsilon}.$$

$(*)$ $\Rightarrow$ DP.

Fix $E\subseteq\mathcal{Y}$, $\quad f(y) = \mathbb{1}\left[y\in E\right]$

$$\mathbb{P}\left[A(x)\in E\right] \geq \exp(-\varepsilon)\,\mathbb{P}\left[A(x')\in E\right].$$

$\square$

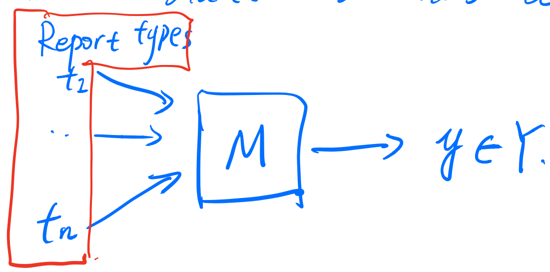
Mechanism Design Basics.

n bidders indexed by i
(agents in general)

Each bidder i has a $\boxed{\text{private type}}$ $t_i \in T_i$

e.g. private valuation preference.

Mechanism



Utility function for agent i: $u_i(y) = u(t_i, y)$

$u: T \times Y \to [0,1]$

Definition (Dominant Strategy Truthfulness)

A mechanism $M: T^n \to Y$ is ε-approximately $\boxed{\text{dominant Strategy}}$ truthful if for all $t \in T^n$ and for all i, $t_i' \in T$

$$u_i(M(t)) \geq u_i(M(t_{-i}, t_i')) - \varepsilon$$

depends on t_i

ε-DP $\Longrightarrow$ ε-approx Dominant Strategy Truthful.

Claim. If M is ε-DP, then M is also ε-approx dominant Strategy truthful.

Proof.

$$\mathbb{E}_{Y\sim M(t)}[u_i(Y)] \geq \exp(-\varepsilon)\cdot \mathbb{E}_{Y\sim M(t_{-i},t_i')}[u_i(Y)] \geq (1-\varepsilon)\cdot \mathbb{E}_{Y\sim M(t_{-i},t_i')}[u_i(Y)]$$

from claim before

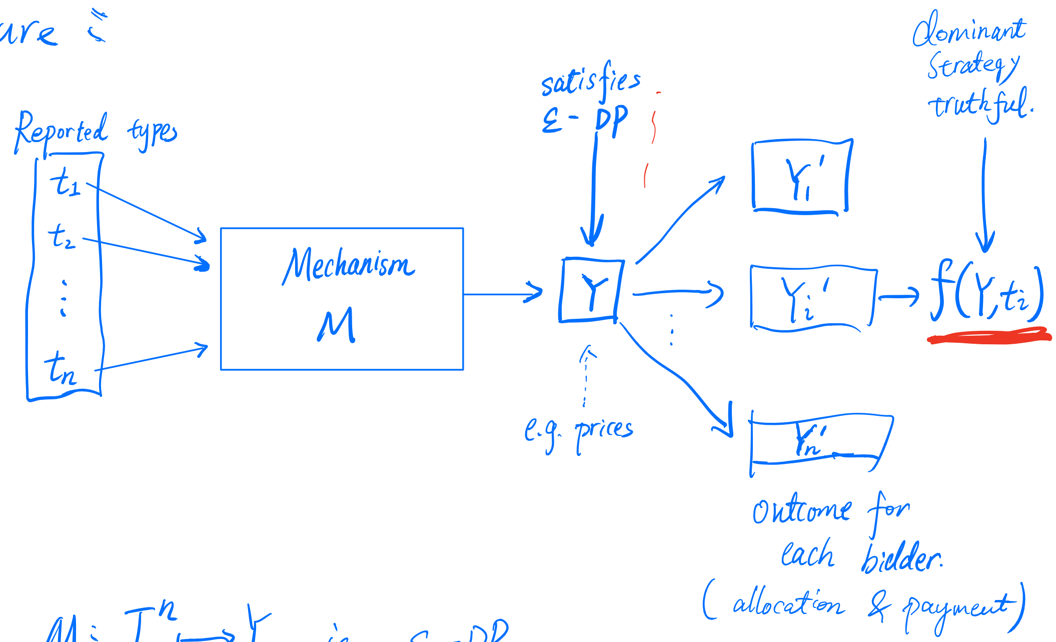
$e^{-\varepsilon} \geq (1-\varepsilon)$

$$\xrightarrow{u_i\in[0,1]} \geq \mathbb{E}_{Y\sim M(t_{-i},t_i')}[u_i(Y)] - \varepsilon \qquad \square$$

Observations:

1. Group privacy $\Rightarrow$ collusion proof.
   (k people collude $\to$ $k\varepsilon$-DST)
2. Every Strategy is approx dominant Strategy.
   (not just being truthful)

Structure :



Claim. If $M: T^n \mapsto Y$ is ε-DP

and $f: Y \times T \to Y_i'$ is dominant strategy truthful. $\forall y \in Y$

Then truthful reporting is ε-approx dominant strategy truthful.

for the mechanism $M': T^n \to Y'$ that returns

$f(M(t), t_i)$ for agent i.

Pricing a digital good.

private type $t_i \in [0,1]$ (re-scaled)

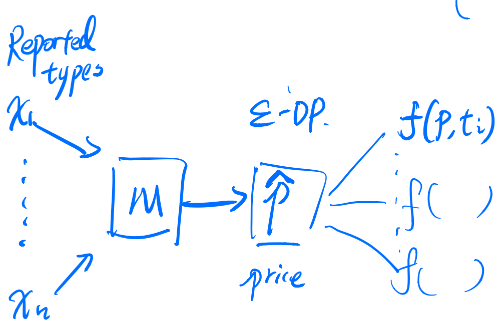
Outcome space $Y = 2^{[n]} \times [0,1]$

"who gets the item" price.

For any (S, p)

$$u_i(S, p) = \mathbb{1}[i \in S] \cdot (t_i - p)$$

$$\equiv \begin{cases} t_i - p & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$$



① M: Run Exp mech. (Report noisy max)

w/ revenue as the quality score.

② $f(\hat{p}, x_i) \equiv \mathbb{1}[x_i \geq p]$

$i \in S$ reported if and only if $x_i \geq p$.

$\Longrightarrow$ ε - approx DS truthful.

$Y_\alpha = \{\alpha, 2\alpha \cdots, 1\}$

$q = \#\{x_i \geq \hat{p}\} \cdot \hat{p}$.