Dynamical Analysis of Modern Wage Dynamics

Supply and Demand Equations

For a single household, sugar supply and demand equations are

$$S_D = \max\left(S_N, \frac{1}{p}(\omega H_M + m)\right) \tag{1}$$

$$S_S = A H_M^{\gamma} + I. \tag{2}$$

Hours supply and demand equations are

$$H_S = \beta H_{max} + \frac{\alpha}{\omega}\left(p S_N - m\right) \tag{3}$$

$$H_D = \left(\frac{S_D^{t-1}}{A}\right)^{\frac{1}{\gamma}}. \tag{4}$$

Note that the sugar equations depend on one of the two hours formulations, with $H_M = min(H_S, H_D)$, and H_D depends on the previous value of S_D, thus recursively on previous values of H_M. Also note that I will only be positive, while m can be both positive and negative. A negative m will increase hours supplied and decrease sugar demanded by the households, and positive values of m will behave conversely.

Model Solutions

The model adjusts p and ω to reach the desired solution of $S_S - S_D = 0$. The hour equations are more analytically tractable than the sugar equations as both are independent of S_M, and H_D is independent of both p and ω.

The stopping condition for the model, $S_S = S_D$, occurs in four ways given that S_D has a minimum value of S_N and $H_M \in (H_S, H_D)$. For minimum S_D and $H_M = H_S$, $S_S = S_D$ yields

$$A\left(\beta H_{max} + \frac{\alpha}{\omega}\left(p S_N - m\right)\right)^{\gamma} + I = S_N. \tag{5}$$

For maximal S_D and $H_M = H_S$, $S_S = S_D$ is

$$A\left(\beta H_{max} + \frac{\alpha}{\omega}\left(p S_N - m\right)\right)^{\gamma} + I = \frac{1}{p}\left(\omega\left(\beta H_{max} + \frac{\alpha}{\omega}\left(p S_N - m\right)\right) + m\right), \tag{6}$$

For minimum S_D and $H_M = H_D$, $S_S = S_D$ is

$$S_D^{t-1} + I = S_N, \tag{7}$$

and for maximum S_D and $H_M = H_D$,

$$S_D^{t-1} + I = \frac{1}{p}\left(\omega\left(\frac{S_D^{t-1}}{A}\right)^{\frac{1}{\gamma}} + m\right). \tag{8}$$

The $H_S = H_D$ Condition

Setting $H_S = H_D$,

$$\beta H_{max} + \frac{\alpha}{\omega}(pS_N - m) = H_D \tag{9}$$

which can be solved for both p and ω.

$$p = \frac{1}{S_N}\left(\frac{\omega}{\alpha}(H_D - \beta H_{max}) + m\right) \tag{10}$$

and

$$\omega = \frac{\alpha(pS_N - m)}{H_D - \beta H_{max}} \tag{11}$$

for $m \geq 0$. If $H_S > H_D$, then adjusting toward equality would lower p or raise ω. In the case of $H_S < H_D$, adjusting to equality would raise p or lower ω.

	$S_S > S_D$	$S_S < S_D$
$H_S > H_D$	$p = \frac{1}{S_N}\left(\frac{\omega}{\alpha}(H_D - \beta H_{max}) + m\right)$ (decrease p)	$\omega = \frac{\alpha(pS_N - m)}{H_D - \beta H_{max}}$ (increase ω)
$H_S \leq H_D$	$\omega = \frac{\alpha(pS_N - m)}{H_D - \beta H_{max}}$ (decrease ω)	$p = \frac{1}{S_N}\left(\frac{\omega}{\alpha}(H_D - \beta H_{max}) + m\right)$ (increase p)

Table 1: Explicit and qualitative changes to p and ω given relative sugar and hours conditions.

While hours updating yields a single update equation each for p and ω, unacceptable negative values can be obtained for p when $H_D < \beta H_{max}$ or m is negative and $|m| > \frac{\omega}{\alpha}(H_D - \beta H_{max})$, and for ω when $H_D < \beta H_{max}$ or $pS_N < m$ (unless both are true). These cases can be handled computationally, setting a minimum value for H_D as

$$H_{D,\min} = \left(\frac{nS_N}{A}\right)^{\frac{1}{\gamma}}. \tag{12}$$

Algorithmic Results

We explore three price and wage setting algorithms,

1. Explicit setting of p an ω using the equations in Table 2 (explicit),
2. Making a fractional adjustment to the explicit values for p and ω based on the equations in Table 2 (fractional),
3. Adjusting p and ω by a fixed percentage based on the qualitative changes in Table 2 (fixed), and
4. Using simulated annealing by adjusting p and ω by a decreasing percentage determined by $\delta_0 e^{-\theta x}$ and the qualitative changes in Table 2 (annealing). Decay rates as a function of θ are shown in Figure 1.

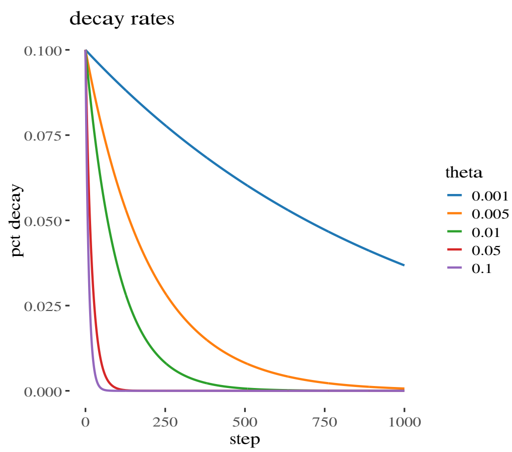

Figure 1: Decay rates over time by θ values with $\delta_0 = .1$.

We see in Figures 2 and 3 that explicit-based algorithms land on either $S_M = 0$ or $S_M = S_N$. For $S_M = 0$, β values are low, less than .2, $S_S \neq S_D$, $m > 0$ and $H_S < H_D$. For $S_M = S_N$, results include both high and low β values, $S_S = S_D$, $m < 0$ and $H_S > H_D$.

For qualitative algorithms, Figures 4 and 5 demonstrate S_M solutions greater than S_N. The fixed algorithm shows a resonance between higher and lower values of S_M, while the annealing algorithm results suggest four solution regimes: 0, S_N, a high value around 4000, and various intermediate values between S_N and the high value. Note that for annealing, the highest value isn't obtained with $\theta = .1$. Highest values are obtained for $H_S = H_D = H_{max}$, and intermediate values for $H_S > H_D$, but in each case $\beta > .5$, $m = 0$ and $S_S = S_D$. For highest values of S_M, $H_S = H_{max}$ is nearly equal to H_D, and β values are high. Like the explicit-based algorithms, $S_M = 0$ when $S_S < S_D$ and $H_S < H_D$ and $m > 0$, and $S_M = S_N$ when $S_S = S_D$, $H_S > H_D$ and $m < 0$.

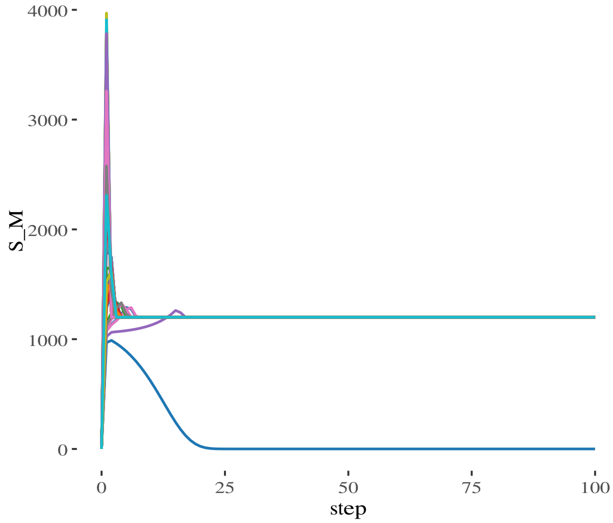

Figure 2: Results for S_M over time for the explicit algorithm for 100 simulation runs.

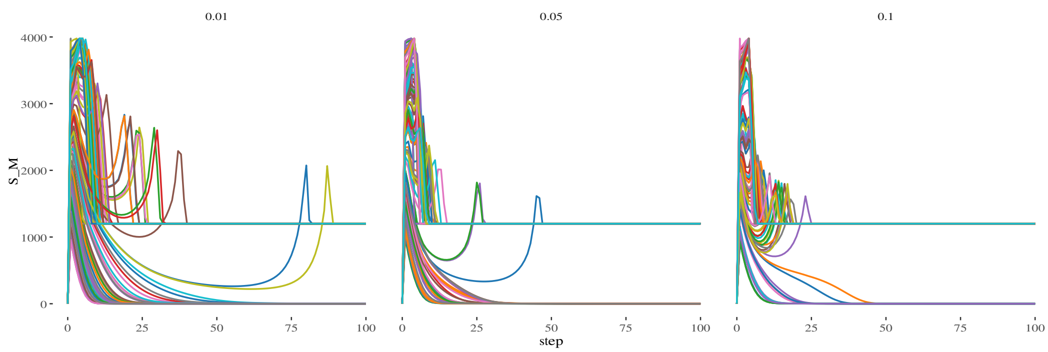

Figure 3: Results for S_M over time for the fractional algorithm for 100 simulation runs.

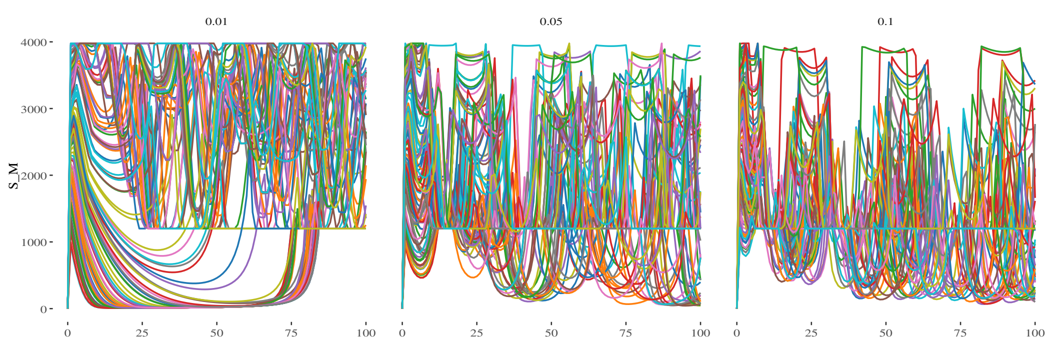

Figure 4: Results for S_M over time for the fixed algorithm for 100 simulation runs.

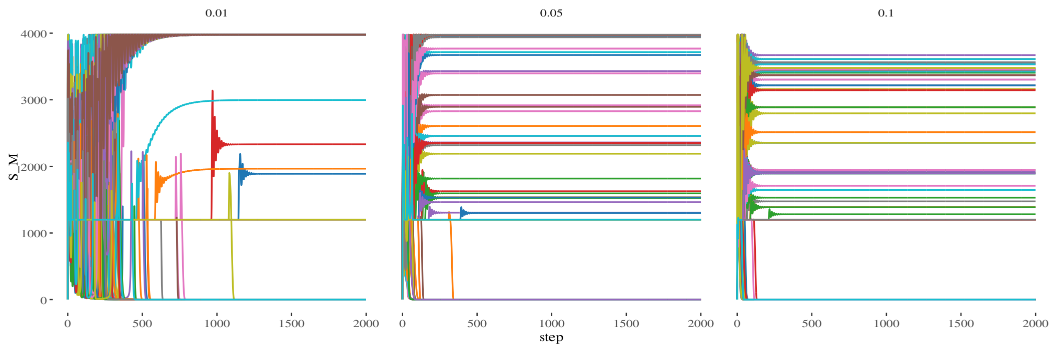

Figure 5: Results for S_M over time for the annealing algorithm for 100 simulation runs.

	$\theta = .01$	$\theta = .05$	$\theta = .01$
max value	9	1	0
intermediate	19	34	50
S_N	33	34	18
0	39	31	32

Table 2: Of 100 annealing simulations, the number of runs that resolve to each solution regime by θ.

When is $S_D > S_N$?

Ideally, the model would like to explore the solution region with $S_M > S_N$. Under what conditions does this occur for $H_M = H_S$?

$$\frac{1}{p}\left(\omega\left(\beta H_{max} + \frac{\alpha}{\omega} p S_N - \frac{\alpha}{\omega} m\right) + m\right) > S_N \tag{13}$$

$$\frac{1}{p}\left(\omega\beta H_{max} + \alpha p S_N - \alpha m + m\right) > S_N \tag{14}$$

$$\frac{\omega\beta}{p} H_{max} + \alpha S_N + \frac{\beta m}{p} > S_N \tag{15}$$

$$\frac{\omega\beta}{p} H_{max} + \frac{\beta m}{p} > S_N - \alpha S_N \tag{16}$$

$$\frac{\omega\beta}{p} H_{max} + \frac{\beta m}{p} > \beta S_N \tag{17}$$

$$\frac{\omega}{p} H_{max} + \frac{m}{p} > S_N \tag{18}$$

and we see, perhaps oddly, that the condition does not depend on β, the preference for income. We do see this lack of β dependence in annealing results, demonstrated in Figure 6. For $m = 0$, we find the condition to be true for

$$\frac{\omega}{p} > \frac{S_N}{H_{max}}. \tag{19}$$

For $m \neq 0$,

$$\frac{\omega}{p} > \frac{S_N}{H_{max}} - \frac{m}{pH_{max}}, \tag{20}$$

which implies that for negative m, the wage-price ration increases, and for positive m, the ratio decrease, which is the opposite of expectation, since with more m, households offer fewer hours, and will need higher wages to be enticed to work, and conversely for indebted households, will offer hours even at lower wages, but which could explain the S_N result.

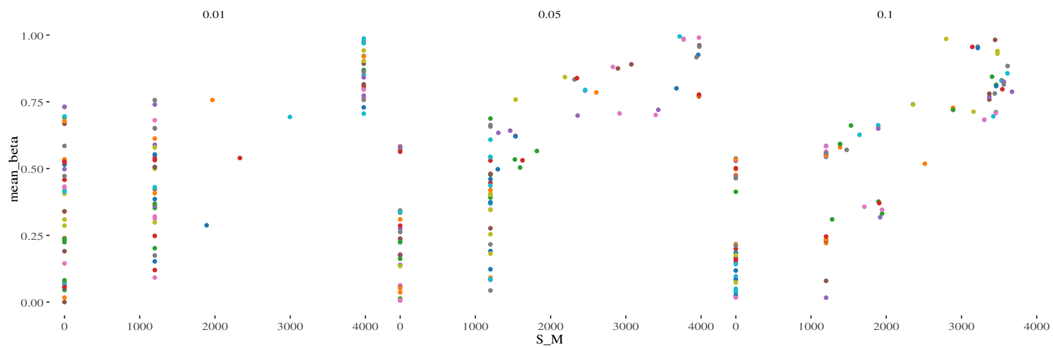

Figure 6: Beta values by S_M for 100 simulations using the annealing algorithm with various θ values.

Supplemental Utility Derivations

Utility for households is based on leisure time and optional consumption.

$$U(H_O, S_O) = (H_{max} - H_N - H_O)^{\alpha}(S_O)^{\beta}, \tag{21}$$

where

$$H_N = \max\left(0, \frac{pS_N - m}{\omega}\right) \tag{22}$$

because negative tribute hours have no meaning in the model, so for $m > pS_N$, $H_N = 0$. For negative m values tribute hours increase to accommodate debt obligations and $0 < m < pS_N$ will decrease tribute hours. Optional consumption is

$$S_O = \frac{\omega H_O}{p} + \max\left(0, \frac{m}{p} - S_N\right) \tag{23}$$

where $m > pS_N$ increases S_O and all other m values have no effect. Notating the first

$$U(H_O, S_O) = \left(H_{max} - \max\left(0, \frac{pS_N - m}{\omega}\right) - H_O\right)^{\alpha}\left(\frac{\omega H_O}{p} + \max\left(0, \frac{m}{p} - S_N\right)\right)^{\beta} \tag{24}$$

which is of the form $U(H_O, S_O) = a^{\alpha}b^{\beta}$ and

$$\frac{\partial U}{\partial H_O} = \alpha a^{\alpha-1}\frac{\partial a}{\partial H_O}b + \beta b^{\beta-1}\frac{\partial b}{\partial H_O}a. \tag{25}$$

Setting the partial of U wrt H_O equal to zero to maximize U in terms of H_O,

$$\alpha \frac{\partial a}{\partial H_O} b + \beta \frac{\partial b}{\partial H_O} a \tag{26}$$

$$\text{with } \frac{\partial a}{\partial H_O} = -1 \quad \text{and} \quad \frac{\partial b}{\partial H_O} = \frac{\omega}{p} \tag{27}$$

$$\alpha b = \frac{\omega}{p} \beta a \tag{28}$$

$$\alpha \frac{\omega H_O}{p} + \alpha \max\left(0, \frac{M}{p} - S_N\right) = \beta \frac{\omega}{p} H_{max} - \beta \frac{\omega}{p} \max\left(0, \frac{pS_N - m}{\omega}\right) - \beta \frac{\omega}{p} H_O \tag{29}$$

$$\alpha \frac{\omega H_O}{p} + \beta \frac{\omega H_O}{p} = \frac{\beta\omega}{p} H_{max} - \frac{\beta\omega}{\max}\left(0, \frac{pS_N - m}{\omega}\right) - \alpha \max\left(0, \frac{M}{p} - S_N\right). \tag{30}$$

Solving for H_O,

$$H_O = \beta H_{max} - \beta \max\left(0, \frac{pS_N - m}{\omega}\right) - \alpha \frac{p}{\omega} \max\left(0, \frac{M}{p} - S_N\right). \tag{31}$$

To address the maximum functions, we examine two cases, one where $m > pS_N$ and the other where $m \leq pS_N$, and solve for $H_S = H_N + H_O$. In the first case, $m > pS_N$,

$$H_O = \beta H_{max} - \frac{\alpha}{\omega}(m - pS_N) \tag{32}$$

$$H = \beta H_{max} - \frac{\alpha}{\omega}(m - pS_N) + 0 \tag{33}$$

$$H = \beta H_{max} - \frac{\alpha}{\omega}(pS_N - m). \tag{34}$$

In the second case, $m \leq pS_N$,

$$H_O = \beta H_{max} + \beta\left(\frac{m}{\omega} - \frac{pS_N}{\omega}\right) \tag{35}$$

$$H_S = \beta H_{max} + \beta\left(\frac{m}{\omega} - \frac{pS_N}{\omega}\right) + \frac{m}{\omega} - \frac{pS_N}{\omega} \tag{36}$$

$$H_S = \beta H_{max} + \frac{\alpha}{\omega}(pS_N - m) \tag{37}$$

and H_S for each household is the same for both cases.