1 Chapter 1

1.1 Preliminaries

This course assumes familiarity with basic arithmetic and algebraic operations.

There are several number systems that will be of interest throughout the course.

Definition 1.1.1 (Number systems)

- Natural numbers: $\mathbb{N} = \{1, 2, 3, 4, ...\}$
- "Whole" numbers: $W = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, 4, ...\}$
- Integers: $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
- Rational numbers: $\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\right\}$
- Real numbers: informally, any number that can be expressed as a decimal. Denoted $\mathbb{R}$
- Irrational numbers: any real number which is not also a rational number

The book's terminology of "whole numbers" is not standard, and conventions vary as to whether the natural numbers should start at 0 or at 1. Therefore, I will refer to the positive integers $\{1, 2, 3, ...\}$ and the nonnegative integers $\{0, 1, 2, 3, ...\}$ instead.

1.2 Visualizing and graphing data

1.2.1 One-variable data

One variable data comes in the form of a list of numbers.

Given a set of one-variable data, we may represent it visually on a number line, or find various properties of the data.

Definition 1.2.1 (Size-related properties of data)

- Minimum: smallest value in the data set
- Maximum: largest value in the data set
- Range: difference between the minimum and maximum

Definition 1.2.2 (Mode)

The value that appears most frequently in the data set. (This value is not strictly unique, since we could have multiple values appear at the same frequency.)

Definition 1.2.3 (Mean)

Also known as average. To take the average of n numbers, add them all and then divide this total by n.

Definition 1.2.4 (Median)

Sort the list of data from smallest to largest. If there is an odd number of data points, then the mode is the middle one. Otherwise, take the average of the middle two.

Example 1.2.5 (Textbook 1.2 ex 1)

The following table lists the low temperature, T, in degrees Fahrenheit that occured in Minneapolis, Minnesota for 6 consecutive nights:

T	-12	-4	-8	21	18	9

(a) Plot these temperatures on a number line.

(b) Find the maximum and minimum of these temperatures.

(c) Determine the mean of these six temperatures.

(d) Find the median and interpret the result.

Solution

(a)

(b) Maximum: 21
Minimum: -12

(c) Mean:

$$\frac{(-12)+(-4)+(-8)+21+18+9}{6} = \frac{24}{6} = 4.$$

(d) Median: First we put the data in increasing order:

$$-12, -8, -4, 9, 18, 21.$$

Since there is an even number of data points, we take the average of the middle two:

$$\frac{-4+9}{2} = \frac{5}{2}.$$

This means that half of the nights were warmer than $2.5°$F, and half were colder.

Ask students if they have any questions before moving on.

1.2.2 Two-variable data

Sometimes, there is a relationship between two pieces of data. We call this a relation.

Definition 1.2.6 (Ordered pair)

Suppose we have two related lists of data. An ordered pair, written (x, y), is a pair of numbers where the first number x comes from the first list and the second number y comes from the second.

Definition 1.2.7 (Relation)

A relation is a set of ordered pairs. We can also think of it as two lists of data which are related to each other.

Definition 1.2.8 (Domain and range)

- The domain of a relation is the set of x values from the relation (first components of the ordered pairs)
- The range of a relation is the set of y values from the relation (second components of the ordered pairs)

Example 1.2.9 Textbook 1.2 ex 2

A physics class measured the time y that it takes for an object to fall x feet, as shown in the following table. The object was dropped twice from each height.

x (feet)	20	20	40	40
y (seconds)	1.2	1.1	1.5	1.6

(a) Express the data as a relation S.

(b) Find the domain and range of S.

Solution

(a) A relation is a list of ordered pairs, so we need to list all of the ordered pairs from the table. Each column gives us an ordered pair. Thus,

$$S = \{(20, 1.2), (20, 1.1), (40, 1.5), (40, 1.6)\}.$$

(b) The domain is the set of possible x values:

$$D(S) = \{20, 40\}.$$

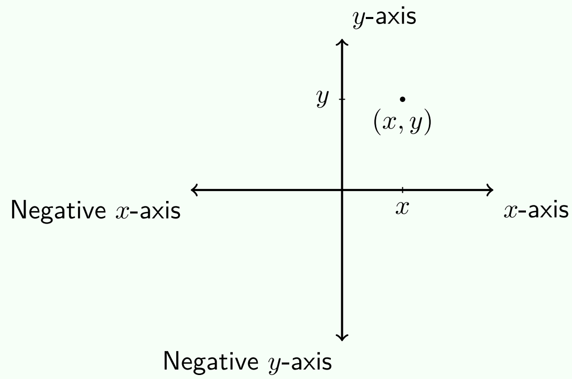
The range is the set of possible y values:

$$R(S) = \{1.2, 1.1, 1.5, 1.6\}.$$

When we have two-variable data, it can be useful to represent this visually.

Definition 1.2.10 (Cartesian coordinates)

To graph, we use the xy-plane, also called Cartesian coordinates. The horizontal axis is called the x-axis, and the vertical axis is the y-axis.



Definition 1.2.11 (Graphing two-variable data)

- Graphing each point in a relation gives a scatterplot
- If a relation only has one point per x value, we can make a line graph by connecting consecutive points in the scatterplot with line segments

Example 1.2.12 (Textbook 1.2 ex 3)

Complete the following for the relation

$$S = \{(5, 10), (5, -5), (-10, 10), (0, 15), (-15, -10)\}.$$

(a) Find the domain and range of the relation.

(b) Determine the maximum and minimum of the x-vales and then of the y-values.

(c) Label appropriate scalse on the $x-$ and y-axes.

(d) Plot the relation as a scatterplot.

Solution

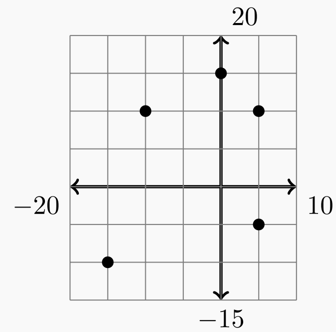
(a) The domain is the set of x values: $D = \{5, -10, 0, -15\}$.
The range is the set of y values: $R = \{10, -5, 15, -10\}$.

(b)

	Maximum	Minimum
x-values	5	-15
y-values	15	-10

(c) The axes need to cover the full x and y range. It's also customary to include slightly more on each end. Since the data is all multiples of 5, we might have each tick represent 5. We can have the x axis range from -20 to 10, and the y axis range from -15 to 20.

(d)



Note that we can't turn this into a line graph because there are multiple points with the same x coordinate, so it isn't clear which points we would consider consecutive.

1.2.3 Midpoint between points

If we have two data points and want to find the point halfway between them, we can use the midpoint formula.

Formula 1.2.13 (Midpoint formula)

The midpoint of the line segment between points(x_1, y_1) and (x_2, y_2) is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

In other words, its coordinates are the average of the coordinates of the two endpoints.

Example 1.2.14 (Textbook 1.2 ex 7)

Find the midpoint of the line segment connecting the points $(6, -7)$ and $(-4, 6)$.

Solution

We have $(x_1, y_1) = (6, -7)$ and $(x_2, y_2) = (-4, 6)$. Plugging these into the midpoint formula, we get

$$\begin{aligned}\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) &= \left(\frac{6 + (-4)}{2}, \frac{-7 + 6}{2}\right) \\ &= \left(\frac{2}{2}, \frac{-1}{2}\right) \\ &= \left(1, -\frac{1}{2}\right).\end{aligned}$$

This can be useful for making estimates in real-world scenarios.

Example 1.2.15

Suppose the weather forecast calls for 2 inches of snow to fall between midnight and 6 am. Use the midpoint formula to estimate how much snow will have fallen by 3 am.

Solution

First, we need to find the endpoints. At midnight, it has been 0 hours since snow started falling, and 0 inches of snow have fallen. We can interpret this as the point $(0, 0)$. At 6 am, it has been 6 hours since snow started falling, and 2 inches have fallen; this gives us the point $(6, 2)$.
Now we plug the points into the formula:

$$\begin{aligned}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) &= \left(\frac{0+6}{2}, \frac{0+2}{2}\right) \\ &= \left(\frac{6}{2}, \frac{2}{2}\right) \\ &= (3, 1).\end{aligned}$$

Back in the original context, this means that 3 hours after the snowfall starts, we expect there to be 1 inch of snow on the ground.

1.2.4 Distance between points

Sometimes, we want to be able to find the distance between two points on the plane.

We can compute the vertical and horizontal distances between the points as the difference between the y coordinates and x coordinates, respectively. Then the Pythagorean theorem allows us to find the distance between the two points:

Formula 1.2.16 (Pythagorean Theorem)

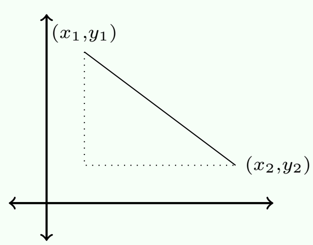
Let $\triangle ABC$ be a right triangle with sides a, b, c. If c is the hypotenuse, then we have

$$a^2 + b^2 = c^2.$$

Taking the square root of both sides, we get the relationship

$$c = \sqrt{a^2 + b^2}.$$

Definition 1.2.17 (Distance between points)



The distance between (x_1, y_1) and (x_2, y_2) is the length of the hypotenuse of a triangle whose sides are the change in x, $x_2 - x_1$, and the change in y, $y_2 - y_1$. Thus it is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1.2.18

Find the distance between the points $(1,4)$ and $(5,1)$.

Solution

We have

$$(x_1, y_1) = (1,4)$$
$$(x_2, y_2) = (5,1).$$

Then we plug these into the formula and simplify to find the distance:

$$\begin{aligned}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} &= \sqrt{(5-1)^2+(1-4)^2}\\ &= \sqrt{(4)^2+(-3)^2}\\ &= \sqrt{16+9}\\ &= \sqrt{25}\\ &= 5.\end{aligned}$$

Therefore, the two points are 5 units apart.

Example 1.2.19 (Textbook 1.2 ex 6)

Suppose that at noon car A is traveling south at 20 miles per hour and is located 80 miles north of car B. Car B is traveling east at 40 miles per hour.

(a) Let $(0,0)$ be the initial coordinates of car B in the xy-plane, where units are in miles. Plot the location of each car at noon and at 1:30 pm.

(b) Approximate the distance between the cars at 1:30 pm.

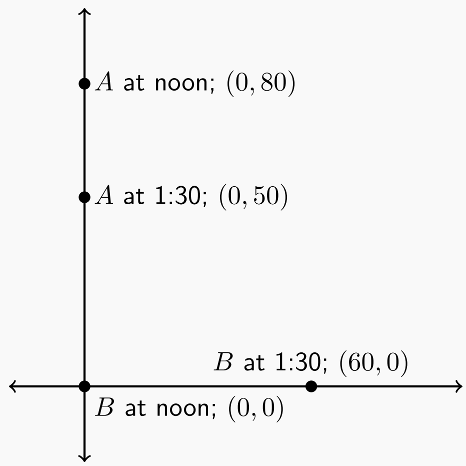
Solution

(a) First we need to figure out where the points are.

We know that car B is at $(0,0)$ at noon, and A is 80 miles north of B. Therefore, car A is at $(0,80)$ at noon.

Between noon and 1:30 pm is 1.5 hours. Car A is moving 20 mph south, so travels $20\frac{\text{miles}}{\text{hour}} \cdot 1.5$ hours $= 30$ miles south. Therefore, car A is at $(0,50)$ at 1:30 pm.

Car B is moving 40 mph east, so travels $40\frac{\text{miles}}{\text{hour}} \cdot 1.5$ hours $= 60$ miles east. Therefore, car B is at $(60,0)$ at 1:30 pm.



(b) We need to find the distance between the points $(x_1, y_1) = (0, 50)$ and $(x_2, y_2) = (60, 0)$. Plugging these into the formula, we get

$$\begin{aligned}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} &= \sqrt{(60-0)^2 + (0-50)^2} \\ &= \sqrt{(60)^2 + (-50)^2} \\ &= \sqrt{3600 + 2500} \\ &= \sqrt{6100} \\ &= 10\sqrt{61}.\end{aligned}$$

1.2.5 Circles

Ask students how to characterize a circle.

Definition 1.2.20 (Circle)

A circle is the set of all points at a set distance from a fixed center. We call the distance from each point to the center the radius of the circle.
Denote the center by the point (h, k) and the radius by r.

The distance equation tells us that every point (x, y) on the circle with radius r and center (h, k) satisfies

$$\sqrt{(x-h)^2 + (y-k)^2} = r.$$

Squaring both sides gives us the standard equation of a circle:

Formula 1.2.21 (Standard equation for a circle)

The equation for the circle with radius r and center (h, k) is

$$(x-h)^2 + (y-k)^2 = r^2.$$

Example 1.2.22

Write the equation for a circle with radius 5 and center $(1, 0)$.

Solution

We have $r = 5$ and $(h, k) = (1, 0)$. Plugging this into the standard form, we get

$$(x - 1)^2 + (y - 0)^2 = 5^2,$$

or equivalently

$$(x - 1)^2 + y^2 = 25.$$

Example 1.2.23 (Textbook 1.2 ex 9)

Find the center and radius of the circle with the given equation. Graph each circle.

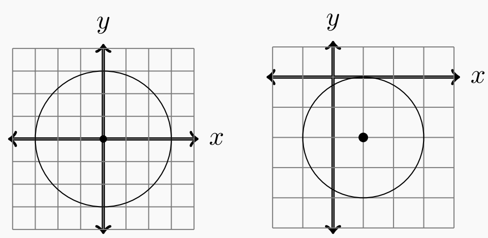
(a) $x^2 + y^2 = 9$

(b) $(x - 1)^2 + (y + 2)^2 = 4$

Solution

(a) Since $x = x - 0$ and $y = y - 0$, we can equivalently write this as $(x - 0)^2 + (y - 0)^2 = 4$. Then the center is $(-0, -0) = (0, 0)$ and the radius is $\sqrt{9} = 3$.

(b) The center is $(-(-1), -2) = (1, -2)$ and the radius is $\sqrt{4} = 2$.



Sometimes, we might know the center and one point on the circle. Then we can find the radius by using the distance formula to determine the distance between the two points.

We might also be given two endpoints of a diameter of a circle. Then the center is the midpoint, which we can find with the midpoint formula, and the radius can be found using the distance formula with one of the endpoints and the center we found.

1.2.6 Completing the square

Whenever we have a term like $(x - h)^2 = (x - h)(x - h)$, we can use FOIL to multiply it out.

Ask students if they remember what FOIL is. Remind students that FOIL stands for First, Outer, Inner, Last and demonstrate multiplying out $(x - h)^2 = x^2 - 2xh + h^2$.

In particular, we can manipulate the equation of a circle by multiplying out the squared terms to get something in the form of a general equation of a circle.

Formula 1.2.24 (General equation of a circle)

The general equation of a circle has the form

$$x^2 + ax + y^2 + by = c$$

for some constants a, b, c.

When we want to determine the center and/or radius of a circle given its equation in general form, we need to use a process called completing the square.

Completing the square

We start with an expression of the form $x^2 + ax$ and want to get an expression resembling $(x + b)^2$. Observe that $(x + \frac{a}{2})^2 = x^2 + ax + \left(\frac{a}{2}\right)^2$. Therefore, $x^2 + ax = (x + \frac{a}{2})^2 - \left(\frac{a}{2}\right)^2$.
We call this completing the square because the extra term $\left(\frac{a}{2}\right)^2$ allows us to make a perfect square: \`
TODO: add pic of completing the square

Example 1.2.25

Complete the square with the following equation: $y^2 - 6y = 18$.

Solution

In this equation, we use the variable y instead of x; however, the process still works in exactly the same way. We have $a = -6$, so get that $y^2 - 6y = (y + \frac{-6}{2})^2 - (-6/2)^2$. Substituting this into the original equation, we have $(y - 3)^2 + 9 = 18$, which turns into $(y - 3)^3 = 9$ when we move all constant terms to the right.

Ask students if this makes sense. If not, do another example.

Example 1.2.26 (Textbook 1.2 ex 12)

Find the center and radius of the circle given by $x^2 + 4x + y^2 - 6y = 5$.

Solution

First, we need to transform the equation into standard form. We will do this by completing the square with both the x terms and the y terms.
The x terms are $x^2 + 4x$. Completing the square, we get $x^2 + 4x = (x + \frac{4}{2})^2 - \left(\frac{4}{2}\right)^2$.
The y terms are $y^2 - 6y$. Completing the square, we get $y^2 - 6y = (y + \frac{-6}{2})^2 - \left(\frac{-6}{2}\right)^2$.
Substituting both of these into the original equation and simplifying, we get

$$\begin{aligned}(x^2 + 4x) + (y^2 - 6y) &= 5\\ ((x+2)^2 - 4) + ((y-3)^2 - 9) &= 5\\ (x+2)^2 + (y-3)^2 - 4 - 9 &= 5\\ (x+2)^2 + (y-3)^2 &= 5 + 4 + 9 = 18.\end{aligned}$$

Now we can find the center and radius. The center is $(-2, 3)$ and the radius is $\sqrt{18}$.

1.3 Functions and Their Representations

1.3.1 What is a function?

Definition 1.3.1 (Function)

A function is a process that receives inputs and produces outputs. For each input, the function must give exactly one output.
We write "$f(x) = y$" to mean "on input x, the function f returns output y".
Alternatively, a function is a relation where every element of the domain corresponds to exactly one element of the range. All functions are relations, but not all relations are functions (see **??**).

Let's consider an example of a function. There is a relationship between distance from a lightning strike and seconds between seeing the lightning and hearing the thunder. We could have a function that takes the number of seconds between the lightning and thunder, and returns the distance in miles from the lightning strike.

Write on board:
Example function: takes seconds between lightning and thunder, and returns distance in miles to the lightning strike.

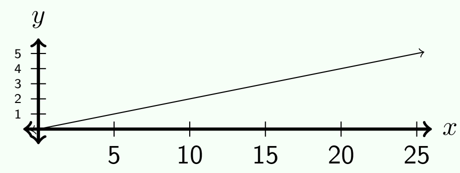
Definition 1.3.2 (Representations of functions)

There are many ways to represent the data of a function.

- Verbal representation: verbally describe the computation of a function.
  Ex: "divide x by 5".
- Numerical representation: table of values. Since these generally contain only a portion of the possible domain of the function, we call it a partial numerical representation.
  Ex:

x	0	5	10	15	20
y	0	1	2	3	4

- Graphical representation: a plot of ordered pairs belonging to the relation.
  Ex:



- Symbolic representation: a formula telling us what to evaluate for each input.
  Ex: $f(x) = \frac{x}{5}$

Example 1.3.3 (Textbook 1.3 ex 7)

When the relative humidity is less than 100%, air cools at $3.6°F$ for every 1000-foot increase in altitude. Let f be a function that computes this change in temperature for an increase in altitude of x thousand feet, with domain $0 \leq x \leq 6$. Represent f

(a) verbally

(b) symbolically

(c) graphically

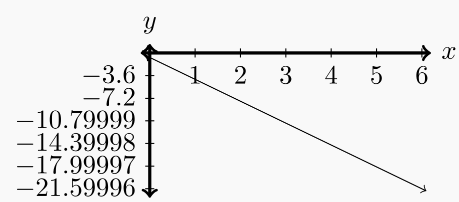
(d) numerically

Solution

(a) Multiply the input, x, by -3.6 to get the change in temperature, y.

(b) $f(x) = -3.6x$

(c)



(d)

Increase in altitude (ft)	0	1000	2000	3000
Change in temperature ($°F$)	0	-3.6	-7.2	-10.8

Example 1.3.4 (Textbook 1.3 ex 3)

The function f computes the revenue in dollars per unique user for different technology companies. This function is defined by $f(A) = 189, f(G) = 24, f(Y) = 8, f(F) = 4$, where A is Amazon, G is Google, Y is Yahoo, and F is Facebook.

(a) Write f as a set of ordered pairs.

(b) Give the domain and range of f.

(c) Interpret $f(A) = 189$.

Solution

(a) From $f(x) = y$, we get the ordered pair (x, y). Therefore, we have

$$f = \{(A, 189), (G, 24), (Y, 8), (F, 4)\}.$$

(b) Now that we have f written as a list of ordered pairs, we can find the domain and range just like we did when working with relations.

$$D_f = \{A, G, Y, F\}$$
$$R_f = \{189, 24, 8, 4\}$$

(c) f takes a company as input and returns the revenue in dollars per unique user. Therefore, $f(A) = 189$ means Amazon receives $189 in revenue per unique visitor.

1.3.2 Notating sets of real numbers

For functions with finite domains, we can write the entire domain as a set. However, many functions have infinite domains, so we need more sophisticated ways to describe their domains.

Previously, we have notated everything as a set.

Definition 1.3.5 (Sets)

A set is a collection of elements. We notate them surrounded by curly braces and separated by commas: $\{a, b, c\}$. The order of items within the braces does not matter; an item can only be in a set once (so repetitions do not matter).

The first method we have is a refinement of the sets we were already using, and is called set builder notation.

Definition 1.3.6 (Set-builder notation)

The pattern for set builder notation is $\{x \in \text{[big set]} : \text{[conditions on } x\text{]}\}$. Sometimes, a pipe is used as a separator instead of a colon. If we don't specify the big set, then it typically means x is a real number.

Example 1.3.7

- We can exclude specific values
  Ex: $\{x : x \neq 1, x \neq 2, x \neq 3\}$
- We can set a minimum or maximum value
  Ex: $\{x : x < -2\}$
  $\{x : x \geq 7\}$
- We can specify a range
  Ex: $\{x : \frac{1}{2} \leq x < 7\}$
- We can combine multiple different types of condition
  Ex: $\{x : x > 0, x \neq 1\}$

The second tool we have is called interval notation.

Definition 1.3.8 (Interval notation)

Interval notation consists of the two endpoints of the interval written inside parentheses or square brackets, depending on whether each endpoint is included or not. A square bracket means that endpoint is included, and a parenthesis mean it is not. There are 4 possible cases:

- $a < x < b$ becomes (a, b)
- $a < x \leq b$ becomes $(a, b]$
- $a \leq x < b$ becomes $[a, b)$
- $a \leq x \leq b$ becomes $[a, b]$

If there is no lower endpoint, we write $-\infty$ (cases where $x < b$ or $x \leq b$). If there is no upper endpoint, we write ∞ (whenever $a < x$ or $a \leq x$). We always use parentheses with infinity.

Sometimes, we may have sets that consist of two intervals.

Definition 1.3.9 (Union)

The union of two sets is the set which consists of all elements from either set. It is denoted $\cup$.
We can take the union of intervals, such as $(-\infty, -1) \cup [3, 5] \cup (9, \infty)$.

1.3.3 Domain, range, and evaluation

Definition 1.3.10 (Domain)

We think of the domain as the set of all valid inputs which make sense when plugged in. More formally, the domain of a function f is the set of all real numbers for which its formula is defined. Sometimes we refer to this as the implied domain.

Example 1.3.11

- $f(x) = x$: it makes sense to plug in any real number, so the domain is all real numbers.
  $\{x \in \mathbb{R}\}$ or $(-\infty, \infty)$
- $g(x) = \frac{1}{x}$: we can plug in any number except for 0, so the domain is everything but 0.
  $\{x : x \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$
- $h(x) = \sqrt{3x}$: we can plug in any nonnegative number, so the domain is $x \geq 0$.
  $\{x : x \geq 0\}$ or $[0, \infty)$

Example 1.3.12 (Textbook 1.3 ex 4)

Let $f(x) = \frac{x}{x-1}$.

(a) If possible, evaluate

(i) $f(2)$

(ii) $f(1)$

(iii) $f(a + 1)$

(b) Find the domain of f. Use set-builder notation.

Solution

(a) (i) We substitute in $x = 2$ in the definition of $f(x)$, then simplify:

$$f(2) = \frac{2}{2-1} = 2.$$

(ii) Substitute:

$$f(1) = \frac{1}{1-1} = \frac{1}{0} = textundefined.$$

This means we cannot evaluate $f(1)$, or that 1 is outside the domain of f.

(iii) Substitute:

$$f(a+1) = \frac{a+1}{(a+1)-1} = \frac{a+1}{a}.$$

This is defined for any $a \neq 0$.

(b) $f(x)$ is defined as long as the denominator is nonzero. $x - 1 = 0$ when $x = 1$, so the domain of f is everything except 1. In set-builder notation, this is $\{x : x \neq 1\}$. In interval notation, this is $(-\infty, 1) \cup (1, \infty)$.

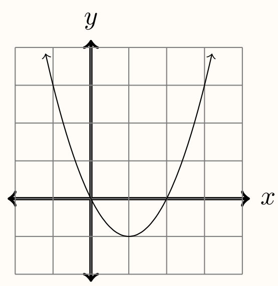
Example 1.3.13 (Textbook 1.3 ex 5)

Let $g(x) = x^2 - 2x$. The graph of g is shown below.

(a) Find the domain and range of g. Use interval notation.

(b) Evaluate $g(-1)$ using the definition.

(c) Use the graph of g to evaluate $g(-1)$.



Solution

(a) The formula for $g(x)$ is defined for all real numbers, so the domain is $(-\infty, \infty)$. If we look at the graph, the minimum value g takes is -1. g continues to increase as x goes to infinity, so the range is everything greater than or equal to -1; in interval notation this is $[-1, \infty)$

(b) We substitute in $x = -1$ and simplify:

$$g(-1) = (-1)^2 - 2(-1) = 1 - (-2) = 3.$$

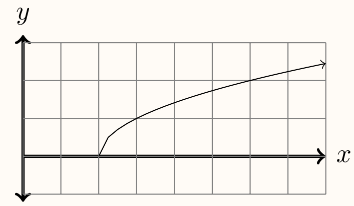
(c) On the graph, we want to find the y coordinate for the point on the graph of g where $x = -1$. This point is $(-1, 3)$, so we get $g(-1) = 3$ from the graph.

Example 1.3.14 (Textbook 1.3 ex 6)

A graph of $f(x) = \sqrt{x-2}$ is shown below.

(a) Evaluate $f(1)$.

(b) Find the domain and range of f. Use both set-builder and interval notation.



Solution

(a) First, we evaluate $f(1)$ graphically. Find 1 on the x axis, and look up/down to see where we intersect the graph. Since we don't ever intersect the graph, $f(1)$ is undefined. To evaluate $f(1)$, we substitute $x = 1$ into the (symbolic) definition of f:

$$f(1) = \sqrt{1-2} = \sqrt{-1}.$$

Since $-1 < 0$, we cannot take its square root, so we again find that $f(1)$ is undefined.

(b) The domain is every real number which it makes sense to plug into f. Looking at the graph, this appears to be everything greater than or equal to 2. We can evaluate f as long as $x - 2 \geq 0$, or $x \geq 2$. In set-builder notation, this is $\{x : x \geq 2\}$. In interval notation, this is $[2, \infty)$. The range is every possible output. Looking at the graph, this is all nonnegative numbers: $\{x : x \geq 0\}$ or $[0, \infty)$.

1.3.4 Identifying functions

As we noticed in the definition of a function, all functions are relations but not all relations are functions. We can use this to identify which relations are functions, and which are not.

Example 1.3.15 (Textbook 1.3 ex 8)

Determine if each set of ordered pairs represents a function.

(a) $A = \{(-2, 3), (-1, 2), (0, -3), (-2, 4)\}$

(b) $B = \{(1, 4), (2, 5), (-3, -4), (-1, 7), (0, 4)\}$

Solution

(a) A is not a function because there is an element of the domain which corresponds to two elements of the range: $(-2, 3)$ and $(-2, 4)$ tell us that -2 in the domain corresponds to both 3 and 4 in the range.

(b) B is a function because each element of the domain corresponds to exactly one element of the range. There is an element of the range which corresponds to two elements of the domain, but this is allowed.

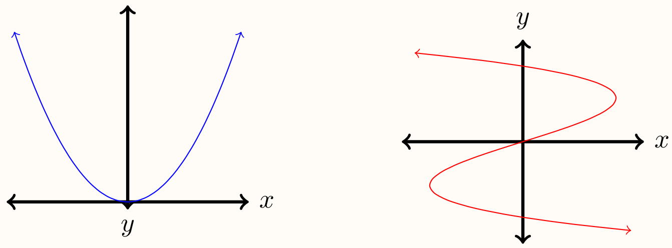
We can also determine whether a graph is a representation of a function.

Definition 1.3.16 (Vertical line test)

If every vertical line intersects a graph at no more than one point, then the graph represents a function.

Example 1.3.17 (Textbook 1.3 ex 9)

Use the vertical line test to determine if the graphs below represent functions.



Solution

(a) Every vertical line we can draw on the first graph intersects at only 1 point, so this passes the vertical line test and is a function.

(b) The line $x = 0$ intersects the graph at 3 points, so this fails the vertical line test and is not a function.

Sometimes, equations define functions.

Example 1.3.18

Suppose we have the equation $x + y = 1$. By subtracting x from both sides, we get $y = 1 - x$, which defines an equation $f(x) = 1 - x$.

However, the equation must pass the vertical line test. This means that some equations, like the equation for a circle we saw last section, do *not* define functions.