Probability Models, Chapter 4:

Solutions to Various Exercises on Markov Chains

Nolan R. H. Gagnon

July 26, 2017

$\boxed{\textbf{1.}}$ Three white and three black balls are distributed in two urns in such a way that each contains three balls. We say that the system is in state i, $i = 0, 1, 2, 3$, if the first urn contains i white balls. At each step, we draw one ball from each urn and place the ball drawn from the first urn into the second, and conversely with the ball from the second urn. Let X_n denote the state of the system after the nth step. Explain why $\{X_n, n = 0, 1, 2, \ldots\}$ is a Markov chain and calculate its transition probability matrix.

Solution: Clearly $\{X_n, n = 0, 1, 2, \ldots\}$ takes on a finite number of values, i.e., $0, 1, 2$ or 3. Furthermore, the probability that the next state is j depends only on the present state. For example, suppose that the process is currently in state 2, i.e., the first urn contains two white balls. This information is sufficient for determining the probability that the process will be in state j after the next step. To determine $P_{2,1}$, for instance, it is sufficient to know that, presently, the first urn contains 2 white balls and 1 blue ball, while the second urn contains 1 white ball and two blue balls. Thus, the probability of transitioning from state 2 to state 1 is equal to the probability of drawing a white ball from urn one and a blue ball from urn two, i.e., $\frac{2}{3}\frac{2}{3} = \frac{4}{9}$. No past information was needed for this calculation.

As for the transition probability matrix,

$$\mathbf{P} = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ 0 & 0 & 1 & 0 \end{Vmatrix}$$

As an example, row three of $\mathbf{P}$ will be calculated. For row three, the beginning state is 2. This means there are 2 white balls and 1 blue ball in the first urn, and 1 white ball and 2 blue balls in the second urn. It is impossible to transition from state 2 to state 0 because only one ball is drawn from each earn at each step. Therefore, $P_{2,0} = 0$. In order to transition from state 2 to state 1, a white ball must be drawn from the first urn and a blue ball must be drawn from the second urn. Therefore, $P_{2,1} = \frac{2}{3}\frac{2}{3} = \frac{4}{9}$. There are two ways to remain in state 2. Either draw a white ball from urn one and a white ball from urn 2, or draw a blue ball from urn one and a blue ball from urn two. Therefore, $P_{2,2} = \frac{2}{3}\frac{1}{3} + \frac{1}{3}\frac{2}{3} = \frac{4}{9}$. Finally, in order to transition to state 3, a blue ball must be drawn from urn one and a white ball must be drawn from urn two. Thus, $P_{2,3} = \frac{1}{3}\frac{1}{3} = \frac{1}{9}$.

---

2. Suppose that whether or not it rains today depends on previous weather conditions through the last three days. Show how this system may be analyzed by using a Markov chain. How many states are needed?

Solution: Let a string of length 3 consisting of the symbols R and $\overline{R}$ represent a particular state. For example, the string RRR means that it has rained today and during the past two days, whereas $\overline{R}RR$ means that it didn't rain today, but it rained during the previous two days. The number of such strings is

$$\binom{3}{3} + \binom{3}{2} + \binom{3}{1} + \binom{3}{0} = 1 + 3 + 3 + 1 = 8.$$

Thus, if 8 states are created properly, this weather system can be analyzed with a Markov Chain. Let the states be

state 0: RRR

state 1: $RR\overline{R}$

state 2: $R\overline{R}R$

state 3: $\overline{R}RR$

state 4: $R\overline{R}\overline{R}$

state 5: $\overline{R}R\overline{R}$

state 6: $\overline{R}\overline{R}R$

state 7: $\overline{R}\overline{R}\overline{R}$

If $X_n, n \geq 0$ takes on values in the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$, then it is a Markov chain.

---

3. In Exercise 2, suppose that if it has rained for the past three days, then it will rain today with probability 0.8; if it did not rain for any of the past three days, then it will rain today with probability 0.2; and in any other case the weather today will, with probability 0.6, be the same as the weather yesterday. Determine $\mathbf{P}$ for this Markov chain.

Solution: