Probability Models, Chapter 1:

Solutions to Various Exercises

Nolan R. H. Gagnon

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10. Prove that Boole's Inequality holds, i.e., that

$$P\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} P(E_i).$$

Proof: This proof is broken into two parts.

Part 1: It can be shown that $\bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} F_i$, where $F_1 = E_1$ and $F_i = E_i \bigcap_{j=1}^{i-1} \overline{E_j}$ for $i \geq 2$. To this end, let $x \in \bigcup_{i=1}^{n} E_i$. Then x is an element of at least one of the E_i. If $x \in E_1$, then it is clearly in $\bigcup_{i=1}^{n} F_i$, since $E_1 = F_1$. But suppose $x \in E_k$, for some $k \geq 2$. Two cases must be considered.

Case 1: If $x \in E_k \bigcap_{j=1}^{k-1} \overline{E_j}$, then x must be in $\bigcup_{i=1}^{n} F_i$, since $E_k \bigcap_{j=1}^{k-1} \overline{E_j}$ is part of this union.

Case 2: If, on the other hand, $x \notin E_k \bigcap_{j=1}^{k-1} \overline{E_j}$, then $x \in \overline{E_k \bigcap_{j=1}^{k-1} \overline{E_j}}$, which, by DeMorgan's Law, implies that $x \in \overline{E_k} \bigcup_{j=1}^{k-1} E_j$. By assumption, $x \in E_k$, so the previous conclusion reduces to $x \in \bigcup_{j=1}^{k-1} E_j$. Now, for the sake of reaching a contradiction, suppose that $x \notin \bigcup_{i=1}^{n} F_i$. The implication of this supposition is that $x \in \bigcap_{i=1}^{n} \left(\overline{E_i} \bigcup_{j=1}^{i-1} E_j\right)$. Thus, $x \in \overline{E_1}$, and $x \in \overline{E_2} \bigcup E_1$, and $x \in \overline{E_3} \bigcup E_2 \bigcup E_1$, and so on. But considering each of these unions in order leads to the conclusion that $x \notin E_1$ and $x \notin E_2$ and $x \notin E_3$, and so on. Therefore, x cannot be an element of the set $\bigcup_{j=1}^{k-1} E_j$. This is a contradiction; hence $x \in \bigcup_{i=1}^{n} F_i$.

In both cases, $x \in \bigcup_{i=1}^{n} F_i$. Therefore, $\bigcup_{i=1}^{n} E_i \subseteq \bigcup_{i=1}^{n} F_i$.

The reverse set inclusion is trivial and omitted from the proof. Thus, $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i$.

Part 2: Using the result of part one of this proof, $P\left(\bigcup_{i=1}^{n} E_i\right)$ can be written as

$$P\left(\bigcup_{i=1}^{n} E_i\right) = P\left(\bigcup_{i=1}^{n} F_i\right) = P\left(\bigcup_{i=1}^{n} \left[E_i \bigcap_{j=1}^{i-1} \overline{E_j}\right]\right).$$

Clearly, the F_i are mutually exclusive. Therefore, the σ-additivity axiom applies, i.e.,

$$P\left(\bigcup_{i=1}^{n}\left[E_i\bigcap_{j=1}^{i-1}\overline{E_j}\right]\right)=\sum_{i=1}^{n}P\left(E_i\bigcap_{j=1}^{i-1}\overline{E_j}\right) \tag{1}$$

$$=P(E_1)+P\left(E_2\overline{E_1}\right)+\ldots+P\left(E_n\overline{E_{n-1}}\cdots\overline{E_1}\right) \tag{2}$$

$$=\sum_{i=1}^{n}P(E_i)+\sum_{j=1}^{n-1}\left[P\left\{\overline{\bigcup_{k=1}^{j}E_k}\right\}-P\left(E_{j+1}\bigcup\left\{\overline{\bigcup_{k=1}^{j}E_k}\right\}\right)\right]. \tag{3}$$

Now, clearly the set $E_{j+1}\bigcup\left\{\overline{\bigcup_{k=1}^{j}E_k}\right\}$ is larger than the set $\overline{\bigcup_{k=1}^{j}E_k}$. Hence, the second summation in equation (3) is negative. It can, therefore, be concluded that $P\left(\bigcup_{i=1}^{n}E_i\right)\leq\sum_{i=1}^{n}P(E_i)$. This completes the proof.

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