Probability Models, Chapter 1:

Solutions to Various Exercises

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10. Prove that Boole's Inequality holds, i.e., that

$$P\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} P(E_i).$$

Proof: This proof is broken into two parts.

Part 1: It can be shown that $\bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} F_i$, where $F_1 = E_1$ and $F_i = E_i \bigcap_{j=1}^{i-1} \overline{E_j}$ for $i \geq 2$. To this end, let $x \in \bigcup_{i=1}^{n} E_i$. Then x is an element of at least one of the E_i. For the sake of reaching a contradiction, suppose that $x \notin \bigcup_{i=1}^{n} F_i$. The implication of this supposition is that $x \in \bigcap_{i=1}^{n} \left(\overline{E_i} \bigcup_{j=1}^{i-1} E_j\right)$. Thus, $x \in \overline{E_1}$, and $x \in \overline{E_2} \bigcup E_1$, and $x \in \overline{E_3} \bigcup E_2 \bigcup E_1$, and so on. But considering each of these unions in order leads to the conclusion that $x \notin E_1$ and $x \notin E_2$ and $x \notin E_3$, and so on. Therefore, x cannot be an element of the set $\bigcup_{i=1}^{n} E_i$. This is a contradiction, so $x \in \bigcup_{i=1}^{n} F_i$.

The reverse set inclusion is trivially proven. Thus, $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i$.

Part 2: Using the result of part one of this proof, $P\left(\bigcup_{i=1}^{n} E_i\right)$ can be written as

$$P\left(\bigcup_{i=1}^{n} E_i\right) = P\left(\bigcup_{i=1}^{n} F_i\right) = P\left(\bigcup_{i=1}^{n}\left[E_i \bigcap_{j=1}^{i-1} \overline{E_j}\right]\right).$$

Clearly, the F_i are mutually exclusive. Therefore, the σ-additivity axiom of probability distributions applies, i.e.,

$$P\left(\bigcup_{i=1}^{n}\left[E_i \bigcap_{j=1}^{i-1} \overline{E_j}\right]\right) = \sum_{i=1}^{n} P\left(E_i \bigcap_{j=1}^{i-1} \overline{E_j}\right) \tag{1}$$

$$= P(E_1) + P\left(E_2\overline{E_1}\right) + \ldots + P\left(E_n\overline{E_{n-1}}\cdots\overline{E_1}\right) \tag{2}$$

$$= \sum_{i=1}^{n} P(E_i) + \sum_{j=1}^{n-1}\left[P\left\{\overline{\bigcup_{k=1}^{j} E_k}\right\} - P\left(E_{j+1} \bigcup \left\{\overline{\bigcup_{k=1}^{j} E_k}\right\}\right)\right]. \tag{3}$$

Now, clearly the set $E_{j+1} \bigcup \left\{\overline{\bigcup_{k=1}^{j} E_k}\right\}$ is larger than the set $\overline{\bigcup_{k=1}^{j} E_k}$. Hence, the second summation in equation (3) is negative. It can, therefore, be concluded that $P\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} P(E_i)$. This completes the proof.

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