Probability Models, Chapter 2:

Solutions to Various Exercises on Random Variables

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$\boxed{\textbf{P.}}$ The Gamma Function is defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx.$$

Show that, if $n \in \mathbb{N}$, then $\Gamma(n) = (n-1)!$.

Proof: First note that

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty e^{-x} x^{1-1} dx \\
&= \int_0^\infty e^{-x} dx \\
&= 1 \\
&= 0!.
\end{aligned}$$

Now suppose that $\Gamma(k) = (k-1)!$, for some integer $k \geq 1$. Then

$$\begin{aligned}
\Gamma(k+1) &= \int_0^\infty e^{-x} x^k dx \\
&= -\left[e^{-x} x^k\right]_0^\infty - \int_0^\infty -e^{-x} k x^{k-1} dx \\
&= 0 + k \int_0^\infty e^{-x} x^{k-1} dx \\
&= k\Gamma(k) \\
&= k(k-1)! \\
&= k!.
\end{aligned}$$

Therefore, by the Principle of Mathematical Induction, it can be concluded that $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$.

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$\boxed{\textbf{16.}}$ An airline knows that 5 percent of the people making reservations on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. What is the probability that there will be a seat available for every passenger who shows up?

Solution: Let X be a random variable which represents the number of ticket-buyers who don't show up. Then X is a binomial random variable with $p = 0.05$ and $n = 52$. There will be a seat available for every passenger that shows up, as long as $X \geq 2$. The probability of this occurring is

$$\begin{aligned}
P(X \geq 2) &= 1 - P(X < 2) \\
&= 1 - P(X = 0) - P(X = 1) \\
&= 1 - \binom{52}{0} 0.95^{52} - \binom{52}{1} (0.05)(0.95)^{51} \\
&\approx 0.7405.
\end{aligned}$$

Thus, there is only a 74.05% chance that everyone who shows up gets a seat. *Prima facie*, it looks like the airline is taking a great risk in implementing this policy, though there may be justification for it when all relevant details are considered (e.g., the typical response of a customer who is told that there isn't enough room for him on the flight he was expecting to take).

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23. Suppose a trial results in success with probability p and failure with probability $1 - p$. Let X be a random variable that represents the number of independent trials needed to obtain the rth success. Find the probability mass function for the distribution of X. This distribution is called the Negative Binomial Distribution.

Solution: Suppose n trials are needed to obtain the rth success, with $n \geq r$. Clearly, the probability of the occurrence of any particular sequence of n trial outcomes is

$$p^r (1 - p)^{n-r},$$

since each trial is independent of the others. There are, however, multiple sequences that yield this probability, and they must be counted in order to find $P(X = n)$. Note that trial n must result in a success. Therefore, among the previous $n - 1$ trials, there must be distributed $r - 1$ other successes. The number of ways of distributing $r - 1$ things over $n - 1$ spots is obviously $\binom{n-1}{r-1}$. Therefore, the probability that the rth success occurs on the nth trial is

$$P(X = n) = \binom{n-1}{r-1} p^r (1 - p)^{n-r}.$$

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30. Let X be a Poisson random variable with parameter λ. Show that $P(X = i)$ increases monotonically and then decreases monotonically as i increases, reaching its maximum when i is the largest integer not exceeding λ.

Proof: To show that $P(X = i)$ increases monotonically for $i \leq \lambda$, it must be shown that $P(X = i) \geq P(X = i - 1)$, i.e., that $\frac{P(X=i)}{P(X=i-1)} \geq 1$. But

$$\begin{aligned}
\frac{P(X = i)}{P(X = i - 1)} &= \left(\frac{e^{-\lambda} \lambda^i}{i!} \right) \left(\frac{(i - 1)!}{e^{-\lambda} \lambda^{i-1}} \right) \\
&= \frac{\lambda}{i}.
\end{aligned}$$

Since $i \le \lambda$, it must be the case that $\frac{\lambda}{i} \ge 1$, i.e., which means $P(X = i)$ increases monotonically on $i \le \lambda$. If, on the other hand, $i > \lambda$, then $\frac{\lambda}{i} < 1$, which indicates that on $i > \lambda$, $P(X = i)$ decreases monotonically. Certainly, the combination of these two conclusions proves that $i = \lfloor \lambda \rfloor$ is where the maximum of $P(X = i)$ occurs.

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34. Let the probability density of X be given by

$$f(x) = \begin{cases} c(4x - 2x^2), & \text{if } 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the value of c and $P\left(\frac{1}{2} < X < \frac{3}{2}\right)$.

Solution: If f is a probability density function, it must be the case that

$$\int_0^2 c(4x - 2x^2)dx = 1.$$

Therefore,

$$\begin{aligned} \left[4c\frac{x^2}{2} - 2c\frac{x^3}{3}\right]_0^2 &= 8c - \frac{16c}{3} \\ &= c\frac{8}{3} \\ &= 1. \end{aligned}$$

Thus, $c = \frac{3}{8}$. Now $P\left(\frac{1}{2} < X < \frac{3}{2}\right)$ can be calculated as

$$\begin{aligned} P\left(\frac{1}{2} < X < \frac{3}{2}\right) &= \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{3}{8}(4x - 2x^2)dx \\ &= \frac{6}{8}\frac{9}{4} - \frac{1}{4}\frac{27}{8} - \frac{6}{8}\frac{1}{4} + \frac{1}{4}\frac{1}{8} \\ &= \frac{22}{32} \\ &\approx 0.6875 \end{aligned}$$

P. Derive the moment generating function (MGF) for the Bernoulli Distribution with parameter p.

Solution: Let X have a Bernoulli Distribution with parameter p. Then the MGF of X is

$$\begin{aligned} E\left[e^{tX}\right] &= \sum_{x=0}^{1} e^{tx} P(X = x) \\ &= (1 - p) + pe^t \\ &= q + pe^t. \end{aligned}$$

P. Derive the MGF for the Binomial Distribution.

Solution: Let X be a binomial random variable with parameters p and n. The MGF of X is then

$$\begin{aligned} E\left[e^{tX}\right] &= \sum_{x=0}^{n} e^{tx} P(X=x) \\ &= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} \left(pe^t\right)^x q^{n-x} \\ &= \left(pe^t + q\right)^n . \end{aligned}$$

P. Derive the MGF of the Poisson Distribution.

Solution: Let X be a Poisson random variable with parameter $\lambda > 0$. The MGF of X is

$$\begin{aligned} E\left[e^{tX}\right] &= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\left(e^t\lambda\right)^x}{x!} \\ &= e^{-\lambda} e^{e^t\lambda} \\ &= e^{\lambda(e^t-1)}. \end{aligned}$$

P. Derive the MGF of the Negative Binomial Distribution.

Solution: Let X be a negative binomial random variable with parameters p and r. The MGF of X is

$$\begin{aligned} E\left[e^{tX}\right] &= \sum_{x=r}^{\infty} e^{tx} \binom{x-1}{r-1} p^r (1-p)^{x-r} && (1) \\ &= \sum_{x=r}^{\infty} \binom{x-1}{r-1} \left(\frac{p}{1-p}\right)^r \left[e^t(1-p)\right]^x && (2) \\ &= \left(\frac{p}{1-p}\right)^r \sum_{x=r}^{\infty} \binom{x-1}{r-1} \left[e^t(1-p)\right]^x . && (3) \end{aligned}$$

This can be vastly simplified. Let $k = x - r$. Then

$$\begin{aligned} \left(\frac{p}{1-p}\right)^r \sum_{x=r}^{\infty} \binom{x-1}{r-1} \left[e^t(1-p)\right]^x &= \left(\frac{p}{1-p}\right)^r \sum_{k=0}^{\infty} \binom{k+r-1}{r-1} [e^t(1-p)]^{k+r} && (4) \\ &= \left(pe^t\right)^r \sum_{k=0}^{\infty} \binom{k+r-1}{r-1} [e^t(1-p)]^k. && (5) \end{aligned}$$

For simplicity's sake, let $y = e^t(1-p)$, and let $a = r - 1$. The summation in (4) can now be written as

$$\sum_{k=0}^{\infty} \binom{k+a}{a} y^k.$$

This summation is computed as follows: First multiply and divide by $a!$. This yields

$$\begin{aligned}
\sum_{k=0}^{\infty}\binom{k+a}{a}y^k &= \frac{1}{a!}\sum_{k=0}^{\infty}a!\binom{k+a}{a}y^k && (6)\\
&= \frac{1}{a!}\sum_{k=0}^{\infty}\left[\frac{(k+a)!}{k!}\right]y^k && (7)\\
&= \frac{1}{a!}\sum_{k=0}^{\infty}\left(\prod_{j=0}^{a-1}[(a-j)+k]\right)y^k && (8)\\
&= \frac{1}{a!}\sum_{k=0}^{\infty}\frac{d^a}{dy^a}\left[y^{a+k}\right] && (9)\\
&= \frac{1}{a!}\frac{d^a}{dy^a}\left[\sum_{k=0}^{\infty}y^{a+k}\right] && (10)\\
&= \frac{1}{a!}\frac{d^a}{dy^a}\left[\frac{1}{1-y}-\sum_{j=0}^{a-1}y^j\right] && (11)\\
&= \frac{1}{a!}\left[\frac{d^a}{dy^a}\left(\frac{1}{1-y}\right)-\frac{d^a}{dy^a}\sum_{j=0}^{a-1}y^j\right] && (12)\\
&= \frac{1}{a!}\left(\frac{a!}{(1-y)^{a+1}}\right) && (13)\\
&= \frac{1}{(1-y)^{a+1}} && (14)
\end{aligned}$$

Note that it must be assumed that $|y| < 1$ for the above equations to be valid. Substituting (13) into (4) and putting everything in terms of r, p, and t yields

$$E[e^{tX}] = \frac{(pe^t)^r}{[1-(1-p)e^t]^r},$$

with the requirement that $(1-p)e^t < 1$, i.e., $t < \ln\left(\frac{1}{1-p}\right)$.