Log-concavity of Discrete Distributions:

Various Basic Proofs

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$\boxed{\textbf{1. Theorem}}$ If $p(t)$ is the probability mass function (PMF) of a discrete random variable on $\mathbb{Z}^+$, then $p(t)$ is log-concave if $\frac{p(t+1)}{p(t)}$ is a decreasing sequence.

Proof: Let $i \in \mathbb{Z}^+$. If $\frac{p(t+1)}{p(t)}$ is a decreasing sequence, then it must be the case that

$$\frac{p[(i+1)+1]}{p(i+1)} \leq \frac{p(i+1)}{p(i)},$$

since $i+1 > i$. Now, as p is a PMF, it must satisfy $p(t) > 0$ for all $t \in \mathbb{Z}^+$. Thus, multiplying by $p(i)p(i+1)$ on both sides of the above inequality cannot change the direction of the inequality sign. Performing this multiplication yields

$$p(i)p(i+2) \leq p^2(i+1).$$

This inequality can be rearranged to

$$p^2(i+1) - p(i)p(i+2) \geq 0.$$

Therefore, p is log-concave.

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$\boxed{\textbf{2. Note}}$ A determinant for log-concavity of a PMF $p(t)$ can be defined by

$$\Delta\eta(t) := \frac{p(t+1)}{p(t)} - \frac{p(t+2)}{p(t+1)}.$$

When $\Delta\eta(t) > 0$, the PMF is log-concave. When $\Delta\eta(t) < 0$, the PMF is log-convex.

$\boxed{\textbf{3. Definition}}$ The Extended Katz family of distributions are distributions whose PMFs satisfy

$$\frac{p(t+1)}{p(t)} = \frac{\alpha + \beta t}{\gamma + t},$$

where $\alpha > 0, \beta < 1, \gamma > 0$.

$\boxed{\textbf{4. Theorem}}$ Let $p(t)$ be the PMF of a distribution within the Extended Katz family. Let the parameters of the distribution be $\alpha > 0, \beta < 1, \gamma > 0$. If $\alpha - \beta\gamma > 0$, then $p(t)$ is log-concave.

Proof: Since $p(t)$ is the PMF of a distribution within the Extended Katz family, it must satisfy the equation

$$\frac{p(t+1)}{p(t)} = \frac{\alpha + \beta t}{\gamma + t}.$$

In this case,

$$\begin{aligned}
\Delta\eta(t) &= \frac{\alpha + \beta t}{\gamma + t} - \frac{\alpha + \beta(t+1)}{\gamma + t + 1} \\
&= \frac{(\alpha + \beta t)(\gamma + t + 1) - (\alpha + \beta t + \beta)(\gamma + t)}{(\gamma + t)(\gamma + t + 1)} \\
&= \frac{(\alpha + \beta t)(\gamma + t) + (\alpha + \beta t) - (\alpha + \beta t)(\gamma + t) - \beta(\gamma + t)}{(\gamma + t)(\gamma + t + 1)} \\
&= \frac{\alpha + \beta t - \beta\gamma - \beta t}{(\gamma + t)(\gamma + t + 1)} \\
&= \frac{\alpha - \beta\gamma}{(\gamma + t)(\gamma + t + 1)}.
\end{aligned}$$

Notice that the denominator is always positive, since $\gamma > 0$ and $t > 0$ (since the support is restricted to the non-negative integers). Therefore, the determinant $\Delta\eta(t)$ is strictly positive only if $\alpha - \beta\gamma > 0$. Hence, $p(t)$ is log-concave only if $\alpha - \beta\gamma > 0$.

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$\boxed{\textbf{5. Lemma}}$ Let $n, m, k \in \mathbb{Z}^+$, with $n > m + k$. Then

$$\frac{\binom{n}{m+k}}{\binom{n}{m+k-1}} = \frac{n - m - k + 1}{m + k}.$$

Proof: Clearly,

$$\begin{aligned}
\frac{\binom{n}{m+k}}{\binom{n}{m+k-1}} &= \frac{\left(\frac{n!}{(n-m-k)!(m+k)!}\right)}{\left(\frac{n!}{(n-m-k+1)!(m+k-1)!}\right)} \\
&= \frac{n!(n-m-k+1)!(m+k-1)!}{n!(n-m-k)!(m+k)!} \\
&= \frac{n - m - k + 1}{m + k}.
\end{aligned}$$

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$\boxed{\textbf{6. Theorem}}$ The Binomial Distribution is a member of the Extended Katz family.

Proof: Let $\phi(t)$ denote the PMF of a binomial distribution with probability of success p and number of trials n. If it can be shown that there exist $\alpha > 0, \beta < 1, \gamma > 0$ such that

$$\frac{\phi(t+1)}{\phi(t)} = \frac{\alpha + \beta t}{\gamma + t}$$

then the proof will be complete. The ratio on the left-hand side of the equation is

$$\begin{aligned}\frac{\phi(t+1)}{\phi(t)} &= \frac{\binom{n}{t+1}p^{t+1}(1-p)^{n-t-1}}{\binom{n}{t}p^{t}(1-p)^{n-t}} && (1)\\ &= \left(\frac{p}{1-p}\right)\left(\frac{n-t}{t+1}\right), && (2)\end{aligned}$$

where Lemma 5 was used in the simplification. Let $\gamma = 1$, $\beta = \frac{-p}{1-p}$, and $\alpha = \frac{np}{1-p}$. Clearly, $\beta < 1$, $\alpha > 0$, and $\gamma > 0$. Using these parameters, the expression in (2) can be rewritten as

$$\left(\frac{p}{1-p}\right)\left(\frac{n-t}{t+1}\right) = \frac{\alpha+\beta t}{\gamma+t}.$$

Therefore, the Binomial Distribution is a member of the Extended Katz family.

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$\boxed{\textbf{7. Theorem}}$ The PMF of the Binomial Distribution is log-concave.

Proof: Let $\phi(t)$ be the PMF of a binomial distribution with probability of success p and number of trials n. Let $\theta = \frac{p}{1-p}$. Let $\alpha = n\theta$, $\beta = -\theta$, and $\gamma = 1$, as in the previous proof. Since the Binomial Distribution is a member of the Extended Katz family, it must be shown that $\alpha - \beta\gamma > 0$. But

$$\begin{aligned}\alpha - \beta\gamma &= n\theta + \theta\\ &= \theta(n+1) > 0.\end{aligned}$$

Therefore, $\phi(t)$ is log-concave.

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$\boxed{\textbf{8. Theorem}}$ The PMF of the Negative Binomial Distribution with parameter k and probability of success p is log-concave if $k > 1$, and log-convex if $k < 1$.

Proof: The proof can be simplified by first showing that the Negative Binomial Distribution is a member of the extended Katz family. Let $\phi(t)$ be the PMF of a Negative Binomial Distribution with parameter k and probability of success p. Then

$$\begin{aligned}\frac{\phi(t+1)}{\phi(t)} &= \frac{\binom{k+t}{t+1}p^{k}(1-p)^{t+1}}{\binom{k+t-1}{t}p^{k}(1-p)^{t}}\\ &= \left(\frac{k+t}{t+1}\right)(1-p)\\ &= \frac{\alpha+\beta t}{\gamma+t},\end{aligned}$$

where $\alpha = k(1-p)$, $\beta = 1-p$, and $\gamma = 1$. Since the parameter k must be greater than 0, and $0 \le p \le 1$, it follows that $\alpha > 0$. Furthermore $\beta \le 1$. Therefore, the Negative Binomial Distribution is a member of the extended Katz family.

Now let $k > 1$. Then

$$\alpha - \gamma\beta = k(1-p) - (1-p)$$
$$= (1-p)(k-1) > 0.$$

Thus, $\phi(t)$ is log-concave. Now, clearly, if $k < 1$, then $(1-p)(k-1) < 0$, in which case $\phi(t)$ is log-convex.

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9. Definition The Conway-Maxwell Poisson (CMP) Distribution is defined by the PMF

$$P(X = t) = \frac{\theta^t}{(t!)^\nu} \frac{1}{Z(\theta, \nu)}$$

where $0 \leq t \in \mathbb{Z}$, and $\theta, \nu > 0$. The function $Z(\theta, \nu)$ is defined as

$$Z(\theta, \nu) = \sum_{j=0}^{\infty} \frac{\theta^j}{(j!)^\nu}.$$

The CMP distribution is over-dispersed if $\nu < 1$, and under-dispersed if $\nu > 1$. If $\nu = 1$, the CMP distribution reduces to the standard Poisson Distribution, where the mean and variance are equal (i.e., there is neither over-dispersion nor under-dispersion).

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10. Theorem The CMP distribution approaches a Bernoulli Distribution with probability of success $p = \frac{\theta}{1+\theta}$ as ν tends to infinity.

Proof: Let X be a CMP-distributed random variable and let its parameter $\nu \to \infty$. Then

$$Z(\theta, \nu) = \sum_{j=0}^{\infty} \frac{\theta^j}{(j!)^\nu}$$
$$= 1 + \frac{\theta}{1^\nu} + \frac{\theta^2}{2^\nu} + \ldots$$
$$\to 1 + \theta.$$

Therefore, $P(X = 0) \to \frac{1}{1+\theta}$ and $P(X = 1) \to \frac{\theta}{1+\theta}$. Clearly, $P(X = 0) + P(X = 1) = 1$. So the requirements for a Bernoulli-distributed random variable with probability of success equal to $\frac{\theta}{1+\theta}$ have been satisfied.

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11. Theorem The CMP distribution is log-concave.

Proof: Let $p(t)$ be the PMF of a CMP distribution with parameters θ and ν. The ratio

$$\frac{p(t+1)}{p(t)} = \left(\frac{\theta^{t+1}}{([t+1]!)^\nu} \frac{1}{Z(\theta, \nu)} \right) \left(Z(\theta, \nu) \frac{(t!)^\nu}{\theta^t} \right)$$
$$= \frac{\theta}{(t+1)^\nu}.$$

The ratio

$$\frac{p(t+2)}{p(t+1)} = \left(\frac{\theta^{t+2}}{([t+2]!)^{\nu}}\frac{1}{Z(\theta,\nu)}\right)\left(Z(\theta,\nu)\frac{([t+1]!)^{\nu}}{\theta^{t+1}}\right)$$
$$= \frac{\theta}{(t+2)^{\nu}}.$$

Therefore,

$$\Delta\eta(t) = \frac{\theta}{(t+1)^{\nu}} - \frac{\theta}{(t+2)^{\nu}}$$
$$= \theta\left[\frac{(t+2)^{\nu}-(t+1)^{\nu}}{[(t+1)(t+2)]^{\nu}}\right],$$

which is larger than 0. Thus, the CMP distribution is log-concave.

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12. Definition The Generalized CMP (GCMP) distribution is defined by the PMF

$$P(X=t) = \frac{[\Gamma(\nu+t)]^{r}\,\theta^{t}}{t!C(r,\nu,\theta)}$$

with $0 \leq t \in \mathbb{Z}$ and parameters satisfying either $r<1$ and $\theta,\nu>0$ or $r=1$ and $\nu>0$ and $0<\theta<1$. The normalizing constant C is given by

$$C(r,\nu,\theta) = \sum_{j=0}^{\infty}\frac{[\Gamma(\nu+j)]^{r}\,\theta^{j}}{j!}.$$

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13. Theorem When $r \leq 0$ or $0<r<1$ and $\nu \geq 1$, the GCMP distribution is log-concave.

Proof: Let p be the PMF of a GCMP distribution with parameters r, ν, and θ. First, an expression for $\Delta\eta(t)$ will be found. The ratio

$$\frac{p(t+1)}{p(t)} = \left(\frac{[\Gamma(\nu+t+1)]^{r}\,\theta^{t+1}}{(t+1)!C(r,\nu,\theta)}\right)\left(\frac{[t!C(r,\nu,\theta)]}{[\Gamma(\nu+t)]^{r}\theta^{t}}\right)$$
$$= \frac{[\Gamma(\nu+t)]^{r}(\nu+t)^{r}\theta}{[\Gamma(\nu+t)]^{r}(t+1)}$$
$$= \frac{(\nu+t)^{r}\theta}{t+1}.$$

The ratio

$$\frac{p(t+2)}{p(t+1)} = \frac{(v+t+1)^{r}\theta}{t+2}.$$

Therefore,

$$\Delta\eta(t) = \frac{(\nu+t)^{r}\theta}{t+1} - \frac{(v+t+1)^{r}\theta}{t+2} \tag{3}$$
$$= \frac{\theta(\nu+t)^{r}}{t+1}\left[1-\frac{t+1}{t+2}\left(\frac{\nu+t+1}{\nu+t}\right)^{r}\right]. \tag{4}$$

Clearly (4) will be positive as long as

$$\frac{t+1}{t+2}\left(\frac{\nu+t+1}{\nu+t}\right)^{r} < 1. \tag{5}$$

The expression in parentheses is clearly larger than 1. Therefore, if $r \leq 0$, the entire left-hand side of the inequality will be less than 1. Now, suppose $0 < r < 1$ and $\nu \geq 1$. Notice that, since $\frac{t+1}{t+2} < 1$, it must be the case that

$$\left(\frac{t+1}{t+2}\right)^{r} < \frac{t+1}{t+2}. \tag{6}$$

Furthermore, as $\nu \geq 1$, it follows that

$$\frac{t+1}{t+2} \leq \frac{\nu+t}{\nu+t+1}. \tag{7}$$

Therefore,

$$\begin{aligned}
\frac{t+1}{t+2}\left(\frac{\nu+t+1}{\nu+t}\right)^{r} &\leq \frac{t+1}{t+2}\left(\frac{t+2}{t+1}\right)^{r} \\
&< \frac{t+1}{t+2}\left[\left(\frac{t+2}{t+1}\right)^{\frac{1}{r}}\right]^{r} \\
&= 1.
\end{aligned}$$

Thus the PMF of the GCMP distribution is log-concave when $r \leq 0$ or when $0 < r < 1$ and $\nu \geq 1$.

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