Log-concavity of Discrete Distributions:

Various Basic Proofs

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Theorem 1 If $p(t)$ is the probability mass function (PMF) of a discrete random variable on $\mathbb{Z}^+$, then $p(t)$ is log-concave if $\frac{p(t+1)}{p(t)}$ is a decreasing sequence.

Proof: Let $i \in \mathbb{Z}^+$. If $\frac{p(t+1)}{p(t)}$ is a decreasing sequence, then it must be the case that

$$\frac{p[(i+1)+1]}{p(i+1)} \leq \frac{p(i+1)}{p(i)},$$

since $i+1 > i$. Now, as p is a PMF, it must satisfy $p(t) > 0$ for all $t \in \mathbb{Z}^+$. Thus, multiplying by $p(i)p(i+1)$ on both sides of the above inequality cannot change the direction of the inequality sign. Performing this multiplication yields

$$p(i)p(i+2) \leq p^2(i+1).$$

This inequality can be rearranged to

$$p^2(i+1) - p(i)p(i+2) \geq 0.$$

Therefore, p is log-concave.

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Note 2 A determinant for log-concavity of a PMF $p(t)$ can be defined by

$$\Delta\eta(t) := \frac{p(t+1)}{p(t)} - \frac{p(t+2)}{p(t+1)}.$$

When $\Delta\eta(t) > 0$, the PMF is log-concave. When $\Delta\eta(t) < 0$, the PMF is log-convex.

Definition 3 The Extended Katz family of distributions are distributions whose PMFs satisfy

$$\frac{p(t+1)}{p(t)} = \frac{\alpha + \beta t}{\gamma + t},$$

where $\alpha > 0, \beta < 1, \gamma > 0$.

Theorem 4 Let $p(t)$ be the PMF of a distribution within the Extended Katz family. Let the parameters of the distribution be $\alpha > 0, \beta < 1, \gamma > 0$. If $\alpha - \beta\gamma > 0$, then $p(t)$ is log-concave.

Proof: Since $p(t)$ is the PMF of a distribution within the Extended Katz family, it must satisfy the equation

$$\frac{p(t+1)}{p(t)} = \frac{\alpha + \beta t}{\gamma + t}.$$

In this case,

$$\begin{aligned}
\Delta\eta(t) &= \frac{\alpha + \beta t}{\gamma + t} - \frac{\alpha + \beta(t+1)}{\gamma + t + 1} \\
&= \frac{(\alpha + \beta t)(\gamma + t + 1) - (\alpha + \beta t + \beta)(\gamma + t)}{(\gamma + t)(\gamma + t + 1)} \\
&= \frac{(\alpha + \beta t)(\gamma + t) + (\alpha + \beta t) - (\alpha + \beta t)(\gamma + t) - \beta(\gamma + t)}{(\gamma + t)(\gamma + t + 1)} \\
&= \frac{\alpha + \beta t - \beta\gamma - \beta t}{(\gamma + t)(\gamma + t + 1)} \\
&= \frac{\alpha - \beta\gamma}{(\gamma + t)(\gamma + t + 1)}.
\end{aligned}$$

Notice that the denominator is always positive, since $\gamma > 0$ and $t > 0$ (since the support is restricted to the non-negative integers). Therefore, the determinant $\Delta\eta(t)$ is strictly positive only if $\alpha - \beta\gamma > 0$. Hence, $p(t)$ is log-concave only if $\alpha - \beta\gamma > 0$.

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Lemma 5 Let $n, m, k \in \mathbb{Z}^+$, with $n > m + k$. Then

$$\frac{\binom{n}{m+k}}{\binom{n}{m+k-1}} = \frac{n - m - k + 1}{m + k}.$$

Proof: Clearly,

$$\begin{aligned}
\frac{\binom{n}{m+k}}{\binom{n}{m+k-1}} &= \frac{\left(\frac{n!}{(n-m-k)!(m+k)!}\right)}{\left(\frac{n!}{(n-m-k+1)!(m+k-1)!}\right)} \\
&= \frac{n!(n-m-k+1)!(m+k-1)!}{n!(n-m-k)!(m+k)!} \\
&= \frac{n-m-k+1}{m+k}.
\end{aligned}$$

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Theorem 6 The Binomial Distribution is a member of the Extended Katz family.

Proof: Let $\phi(t)$ denote the PMF of a binomial distribution with probability of success p and number of trials n. If it can be shown that there exist $\alpha > 0, \beta < 1, \gamma > 0$ such that

$$\frac{\phi(t+1)}{\phi(t)} = \frac{\alpha + \beta t}{\gamma + t}$$

then the proof will be complete. The ratio on the left-hand side of the equation is

$$\frac{\phi(t+1)}{\phi(t)} = \frac{\binom{n}{t+1}p^{t+1}(1-p)^{n-t-1}}{\binom{n}{t}p^t(1-p)^{n-t}} \tag{1}$$

$$= \left(\frac{p}{1-p}\right)\left(\frac{n-t}{t+1}\right), \tag{2}$$

where Lemma 5 was used in the simplification. Let $\gamma = 1$, $\beta = \frac{-p}{1-p}$, and $\alpha = \frac{np}{1-p}$. Clearly, $\beta < 1$, $\alpha > 0$, and $\gamma > 0$. Using these parameters, the expression in (2) can be rewritten as

$$\left(\frac{p}{1-p}\right)\left(\frac{n-t}{t+1}\right) = \frac{\alpha+\beta t}{\gamma+t}.$$

Therefore, the Binomial Distribution is a member of the Extended Katz family.

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Theorem 7 The PMF of the Binomial Distribution is log-concave.

Proof: Let $\phi(t)$ be the PMF of a binomial distribution with probability of success p and number of trials n. Let $\theta = \frac{p}{1-p}$. Let $\alpha = n\theta$, $\beta = -\theta$, and $\gamma = 1$, as in the previous proof. Since the Binomial Distribution is a member of the Extended Katz family, it must be shown that $\alpha - \beta\gamma > 0$. But

$$\alpha - \beta\gamma = n\theta + \theta$$

$$= \theta(n+1) > 0.$$

Therefore, $\phi(t)$ is log-concave.

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