3D EM Presentation
Part I
The quasi-static EM:
Forward Operators and Tests

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UBC EOAS 555

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The Forward Problem

Maxwell's equations

Quasi-static in frequency domain

$$\nabla \times \mathbf{E} - i\omega\mu \boldsymbol{H} = 0$$
$$\nabla \times \mathbf{H} - \sigma \boldsymbol{E} = 0$$

Substituting **H** for **E…**

$$\nabla \times \nabla \times \mathbf{E} - i\omega\mu\, \mathbf{E} = 0$$

Using Haber (2000) potential decomposition

$$\mathbf{E} = \mathbf{A} + \nabla\varphi$$

$$\nabla \times \nabla \times \mathbf{A} - i\omega\mu\,(\mathbf{A} + \nabla\varphi) = 0$$

With $\begin{cases} \nabla \cdot \mathbf{A} = 0 \\ \nabla \cdot \nabla\varphi = 0 \end{cases}$

…but since $\nabla \cdot \mathbf{A} = 0$, and $\Delta = (\nabla \times \nabla \times + \nabla\,(\nabla \cdot \mathbf{A})$

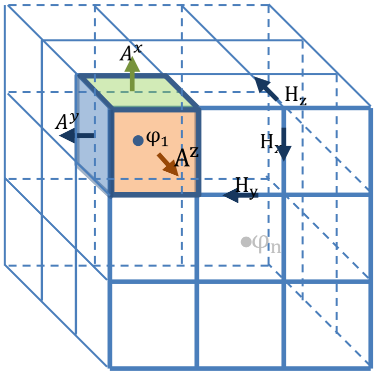
We can substitute for the Laplacian. Turns out to be much simpler to apply Boundary Conditions…

Forward system of equations

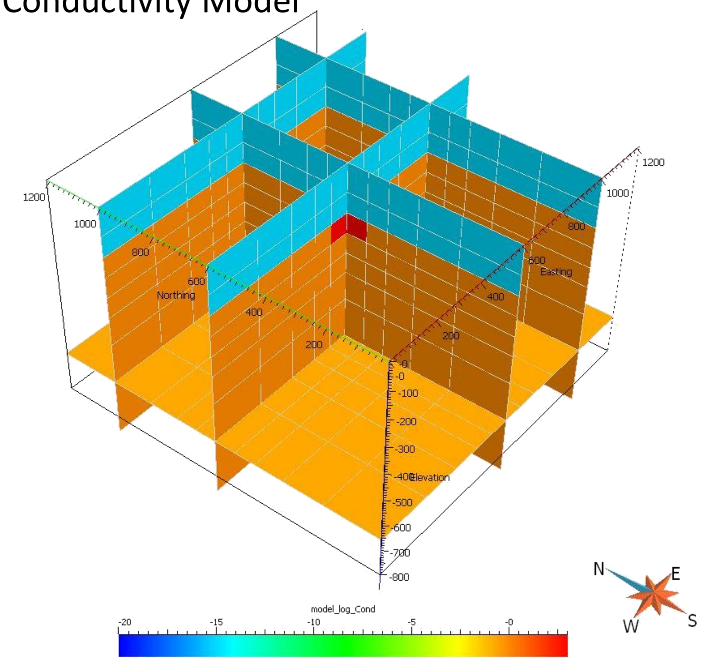
$$\begin{bmatrix} \mathrm{L} - i\omega\mu \mathrm{S} & -i\omega\mu \mathrm{S}\,\nabla \\ \nabla \cdot\ \sigma & \nabla \cdot \sigma\nabla \end{bmatrix} \begin{vmatrix} A \\ \varphi \end{vmatrix} = 0$$

Model Parameters

Cell-center discretization



3D Conductivity Model



Field Decomposition

Setting up the problem…

We first decompose the field in a primary and secondary component

Farquharson and Oldenburg (2002):

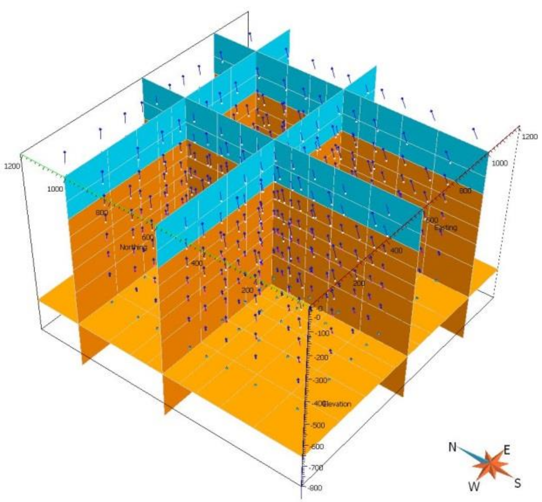
$$(E_{primary}) \begin{bmatrix} \mathrm{L} - i\omega\mu\mathrm{S} & -i\omega\mu\mathrm{S}\,\nabla \\ \nabla\cdot\,\sigma & \nabla\cdot\sigma\nabla \end{bmatrix} \begin{bmatrix} A \\ \varphi \end{bmatrix} = 0$$

and

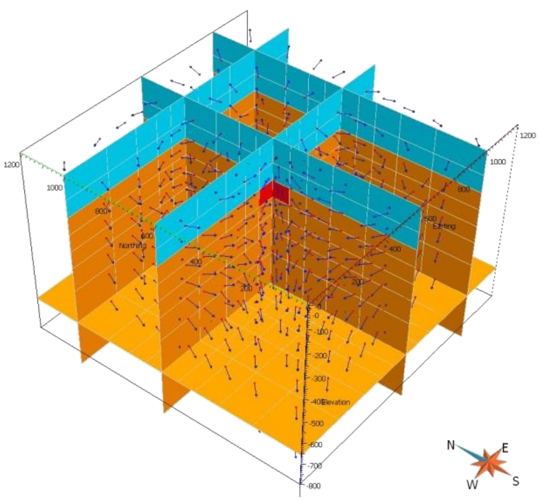
$$(E_{secondary}) \begin{bmatrix} \mathrm{L} - i\omega\mu\mathrm{S} & -i\omega\mu\mathrm{S}\,\nabla \\ \nabla\cdot\,\mathrm{S} & \nabla\cdot\mathrm{S}\nabla \end{bmatrix} \begin{bmatrix} A \\ \varphi \end{bmatrix} = \begin{bmatrix} -i\omega\mu\,\Delta\mathrm{S}\,\mathrm{E_p} \\ -\nabla\cdot\;\Delta\mathrm{S}\;\mathrm{E_p} \end{bmatrix}$$

with : $\Delta\boldsymbol{S} = \mathbf{S} - \mathbf{S_p}$

Step 1: Solve for background model



Step 2: Solve for anomalous field



… and the data

For the MT problem, the inducing field is unknown. We solve for ratios of E and H, or impedances.

$$\begin{bmatrix} E_x{}^x & E_x{}^y \\ E_y{}^x & E_y{}^y \end{bmatrix} = \begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix} \begin{bmatrix} H_x{}^x & H_x{}^y \\ H_y{}^x & H_y{}^y \end{bmatrix}$$

$$Z_{xy} = \frac{E_x{}^x H_y{}^y - E_x{}^y H_y{}^x}{H_y{}^y H_y{}^x - H_y{}^x H_x{}^y} \qquad Z_{xy} = \frac{E_x{}^y H_x{}^x - E_x{}^x H_x{}^y}{H_y{}^y H_y{}^x - H_y{}^x H_x{}^y}$$

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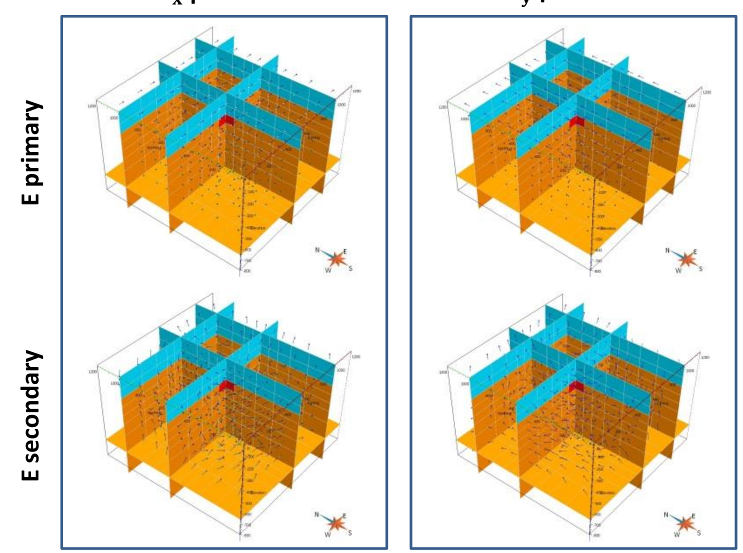
Solving for four unknown requires 4 independent variables. Hence we need to solve for $\left(\boldsymbol{E_p}, \boldsymbol{E_s}\right)$ for two orthogonal polarizations.

$\mathrm{E_x}$ polarized | **$\mathrm{E_y}$ polarized**



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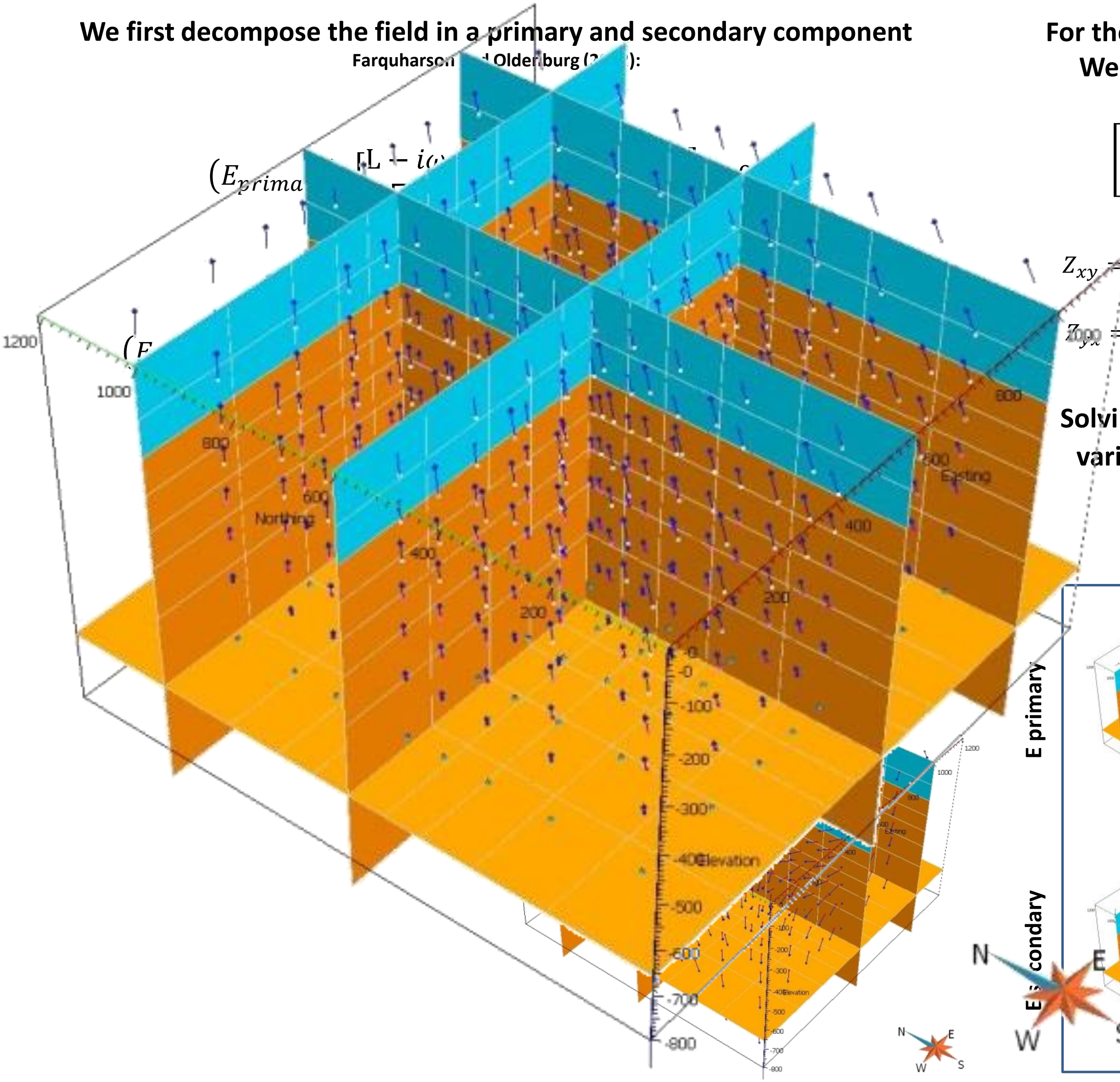
Farquharson [partially hidden] Oldenburg (2[partially hidden]):



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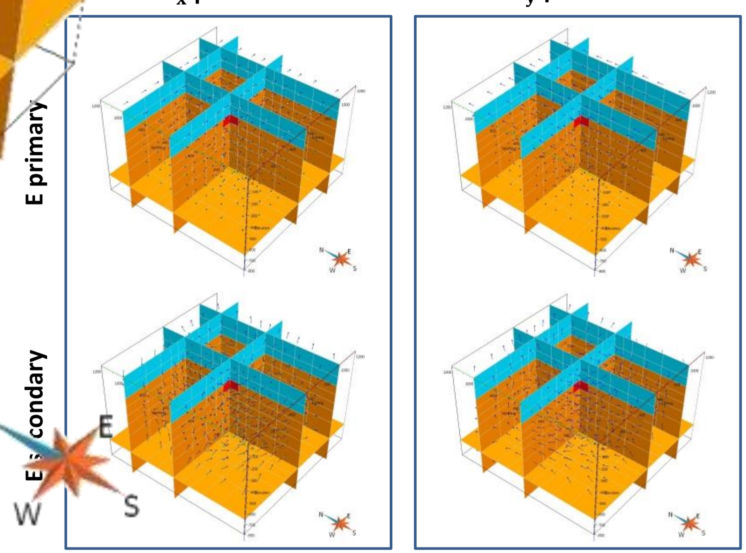
E_x polarized | **E_y polarized**



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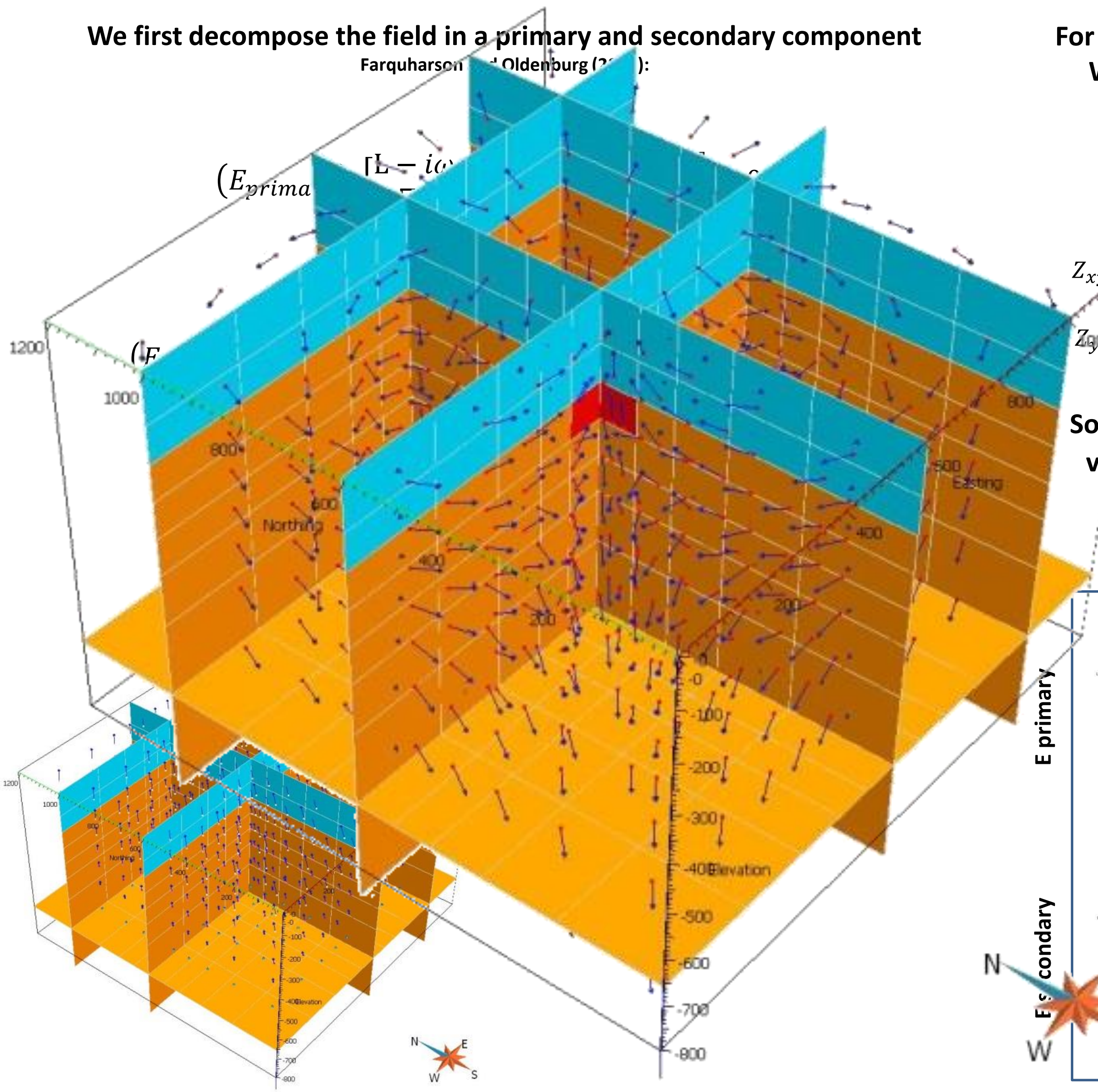
Farquharson and Oldenburg (2…):



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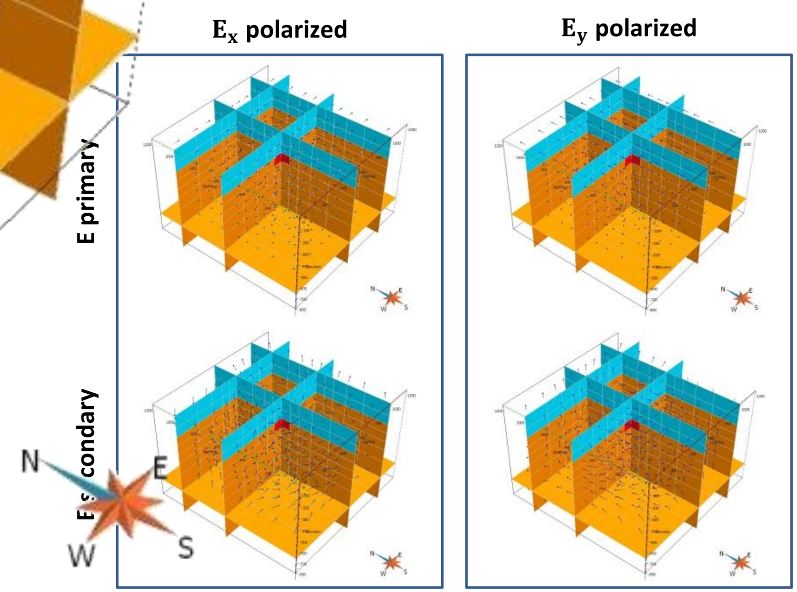


Operators : Divergence

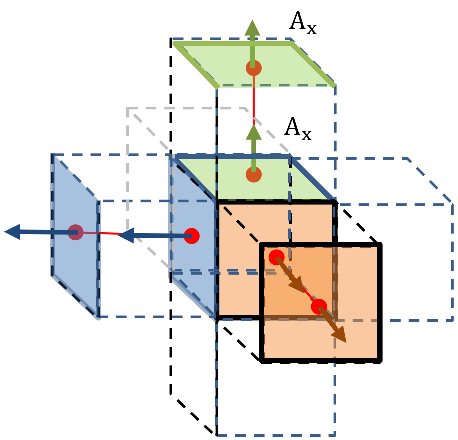
In theory…

$$\nabla \cdot \mathbf{A} = \frac{\partial \mathrm{A}^{x}}{\partial x} + \frac{\partial \mathrm{A}^{y}}{\partial y} + \frac{\partial \mathrm{A}^{z}}{\partial z}$$

$$\frac{\partial \mathrm{A}^{x}}{\partial x} = \frac{({A^{x}}_{n+1} - A^{x})}{h_n}$$

$$\frac{\partial \mathrm{A}^{y}}{\partial y} = \frac{({A^{y}}_{n+1} - A^{y})}{h_n}$$

$$\frac{\partial \mathrm{A}^{z}}{\partial z} = \cdots$$



In Matlab form…

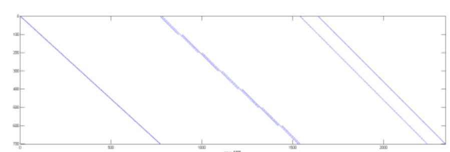
$$\frac{\partial}{\partial x} = \begin{bmatrix} 1/\sqrt{h_1} & & \cdots & 0 \\ -1/\sqrt{h_2} & 1/\sqrt{h_2} & \ddots & \vdots \\ 0 & & \cdots & -1/\sqrt{h_{end}} \end{bmatrix}$$

$$\frac{\partial \mathrm{A}^{x}}{\partial x} = \mathrm{kron}\ (\ \mathrm{kron}\ (\mathrm{speye(nz)}, \mathrm{speye(ny)}), \frac{\partial}{\partial x})$$

Same idea for the 2 other partial derivatives yielding…

Divergence Operator

$$\mathbf{DIV} = [\mathbf{D_x} \quad \mathbf{D_y} \quad \mathbf{D_z}]$$



Boundary Conditions
(Primary Field)
x-y polarized

$$\frac{\partial \mathrm{A}^{x}}{\partial x}\ and\ \frac{\partial \mathrm{A}^{y}}{\partial x}\bigg|_{\partial\Omega} = 0$$

$$\frac{\partial \mathrm{A}^{x}}{\partial y}\ and\ \frac{\partial \mathrm{A}^{y}}{\partial y}\bigg|_{\partial\Omega} = 0 \qquad \Delta A^{z}\big|_{\partial\Omega} = 0$$

$$\frac{\partial \mathrm{A}^{x}}{\partial z}\ and\ \frac{\partial \mathrm{A}^{y}}{\partial z}\bigg|_{\partial\Omega} = g(x, y, z)$$

No boundary condition required since operates from face to center…

Operators : Gradient

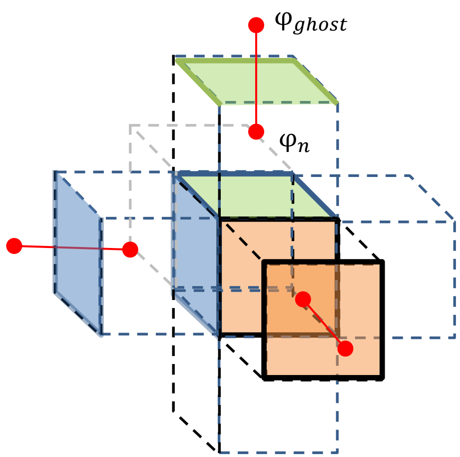
In theory…

$$\nabla\varphi = \left\langle \frac{\partial\varphi}{\partial x}; \frac{\partial\varphi}{\partial y}; \frac{\partial\varphi}{\partial z} \right\rangle$$

$$\frac{\partial\varphi}{\partial x} = \frac{(\varphi_{yzx-1} - \varphi_{zyx})}{h_{xmid}}$$

$$\frac{\partial\varphi}{\partial y} = \frac{(\varphi_{xzy-1} - \varphi_{xzy})}{h_{ymid}}$$

$$\frac{\partial\varphi}{\partial z} = \cdots$$



In Matlab form…

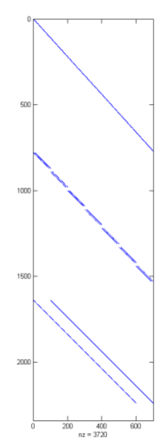
$$\frac{\partial}{\partial x} = \begin{bmatrix} 1/\sqrt{h_{mid\,x1}} & & \cdots & 0 \\ -1/\sqrt{h_{mid\,x2}} & 1/\sqrt{h_{mid\,x2}} & \ddots & \vdots \\ 0 & & \cdots & -1/\sqrt{h_{end}} \end{bmatrix}$$

$$\frac{\partial\varphi}{\partial x} = \mathrm{kron}\ (\ \mathrm{kron}\ (\mathrm{speye(nz)}, \mathrm{speye(ny)}), \frac{\partial}{\partial x})$$

Same idea for the 2 other partial derivatives yielding…

Gradient Operator

$$\mathbf{DIV} = [\mathbf{D_x};\quad \mathbf{D_y};\quad \mathbf{D_z}]$$

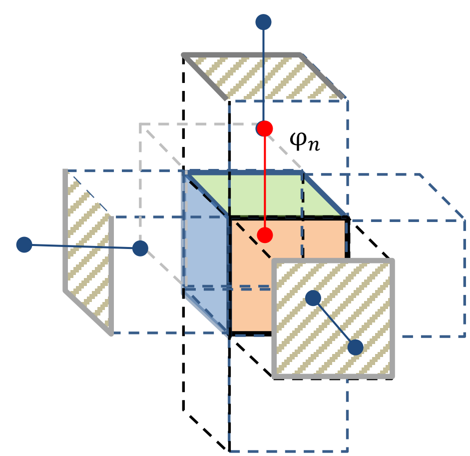


Boundary Conditions
(Primary Field)
x-y polarized

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = 0$$

Boundary conditions requires that derivative operator is 0 on the edges…

$$\frac{\partial\varphi}{\partial n} = \mathrm{kron}\ (\ \mathrm{kron}\ ((\mathrm{nz}), (\mathrm{ny})), dn([1{:}\,end] = 0)$$

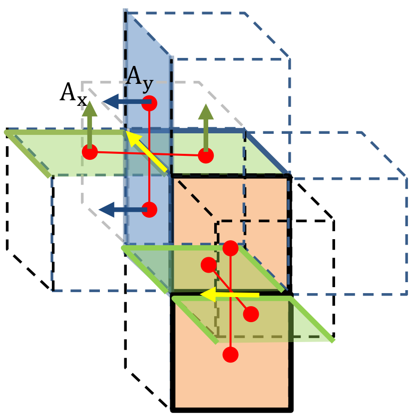


Operators : Curl

In theory…

$$\nabla \times \mathrm{A} = \begin{bmatrix} 0 & \partial_z{}^y & -\partial_x{}^z \\ -\partial_z{}^x & 0 & \partial_x{}^z \\ \partial_y{}^x & -\partial_x{}^y & 0 \end{bmatrix}$$

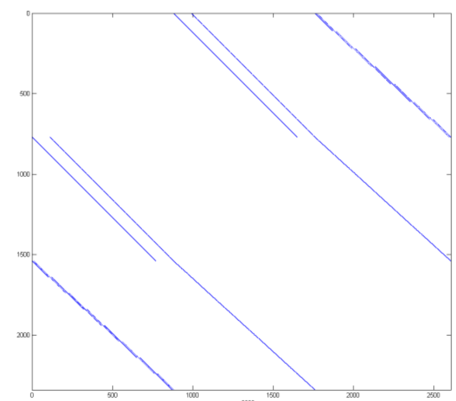
$$\nabla \times \mathrm{A}^x = \frac{(\mathrm{A}_{xyz-1}{}^y - \mathrm{A}_{xyz}{}^y)}{h_{zmid}} - \frac{(\mathrm{A}_{xzy-1}{}^z - \mathrm{A}_{xzy}{}^z)}{h_{ymid}}$$



In Matlab form…

CURL Operator

$$\mathbf{CURL} = \begin{bmatrix} 0 & D_z{}^y & -D_x{}^z \\ -D_z{}^x & 0 & D_x{}^z \\ D_y{}^x & -D_x{}^y & 0 \end{bmatrix}$$



Boundary Conditions
(Primary Field)
x-y polarized

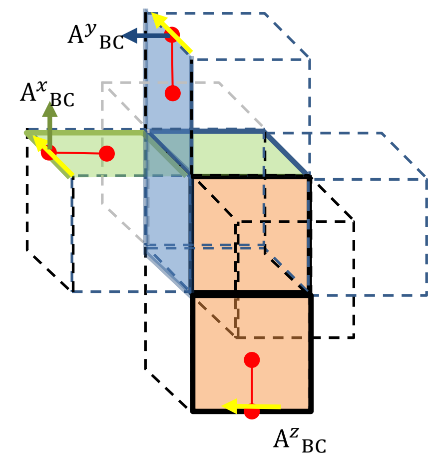
$$\left.\frac{\partial \mathrm{A}^x}{\partial x} \; and \; \frac{\partial \mathrm{A}^y}{\partial x}\right|_{\partial\Omega} = 0$$

$$\left.\frac{\partial \mathrm{A}^x}{\partial y} \; and \; \frac{\partial \mathrm{A}^y}{\partial y}\right|_{\partial\Omega} = 0$$

$$\left.\frac{\partial \mathrm{A}^x}{\partial z} \; and \; \frac{\partial \mathrm{A}^y}{\partial z}\right|_{\partial\Omega} = g(x,y,z)$$

Tricky to apply boundary matrix since we need to put value at the right location…

Much simpler if the field vanishes at the boundary.



Operators : Vector Laplacian

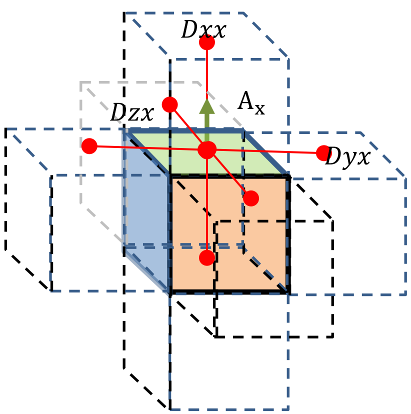
In theory…

$$\Delta\, A_x = \frac{\partial^2 A^x}{\partial x^2} + \frac{\partial^2 A^x}{\partial y^2} + \frac{\partial^2 A^x}{\partial z^2}$$

$$\frac{\partial^2 A_x}{\partial x^2} = \frac{\dfrac{(A^x{}_{n+1} - A^x)}{h_{n-1}} - \dfrac{(A^x - A^x{}_{n-1})}{h_n}}{h_{mid}}$$

$$\frac{\partial^2 A_x}{\partial y^2} = \frac{\dfrac{(A^x{}_{n+1} - A^x)}{h_{mid}} - \dfrac{(A^x - A^x{}_{n-1})}{h_{mid}}}{h}$$

$$\frac{\partial^2 A_x}{\partial z^2} = \ \ldots$$



In Matlab form…

$$\frac{\partial}{\partial x} = \begin{bmatrix} 1/\sqrt{h_1} & \cdots & 0 \\ -1/\sqrt{h_2} \quad 1/\sqrt{h_2} & \ddots & \vdots \\ 0 & \cdots & -1/\sqrt{h_{end}} \end{bmatrix}$$

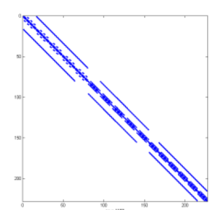
$$\frac{\partial^2}{\partial x^2} = \mathrm{H_{mid}} \left(\frac{\partial}{\partial x}\right)^{\mathrm{T}} \frac{\partial}{\partial x}$$

$$\frac{\partial^2 A^x}{\partial x^2} = \mathrm{kron}\ (\ \mathrm{kron}\ (\mathrm{speye(nz)}, \mathrm{speye(ny)}), \frac{\partial^2}{\partial x^2})$$

Same idea for the 8 other partial derivatives yielding…

Laplacian Operator

$$\mathbf{L} = \begin{matrix} \frac{\partial^2 Ax}{\partial x^2} + \frac{\partial^2 Ax}{\partial y^2} + \frac{\partial^2 Ax}{\partial z^2} & 0 & 0 \\ 0 & \frac{\partial^2 Ay}{\partial x^2} + \frac{\partial^2 Ay}{\partial y^2} + \frac{\partial^2 Ay}{\partial z^2} & 0 \\ 0 & 0 & \frac{\partial^2 Az}{\partial x^2} + \frac{\partial^2 Az}{\partial y^2} + \frac{\partial^2 Az}{\partial z^2} \end{matrix}$$



Boundary Conditions

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$$\frac{\partial A^x}{\partial y}\ and\ \frac{\partial A^y}{\partial y}\Big|_{\partial\Omega} = 0 \qquad \Delta A^z\Big|_{\partial\Omega} = 0$$

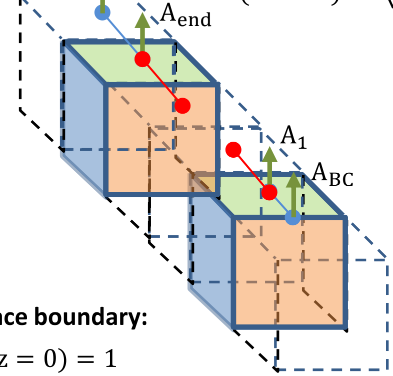
$$\frac{\partial A^x}{\partial z}\ and\ \frac{\partial A^y}{\partial z}\Big|_{\partial\Omega} = g(x,y,z)$$

Defining g(x,y,z) as the 1D problem for the x and y components:

$$A^x(z) = e^{i\sqrt{i\,\omega\mu\sigma}}$$

For bottom boundary:

$$A^x(z = -L) = i\sqrt{i\,\omega\mu\sigma} A^x - A^x$$



For surface boundary:

$$A^x{}_{BC}\ (z = 0) = 1$$

Testing, testing …

…the operators

Use the same function as Haber (2000) for the vector field:

$$A = \left\langle \begin{array}{l} -z \cdot y \cdot e^{-5(x^2+y^2+z^2)}\,\vec{i}; \\ -x \cdot z \cdot e^{-5(x^2+y^2+z^2)}\,\vec{j}; \\ -x \cdot y \cdot e^{-5(x^2+y^2+z^2)}\,\vec{k}; \end{array} \right\rangle$$

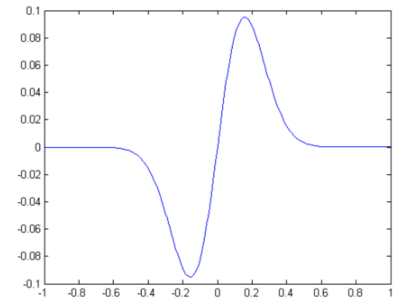
On the interval [-1 , 1]. The function has a simple analytical solution for the Laplacian, curl and divergence.

	Mesh size	Residual
Laplacian test:	2.50e-001	1.29e-001
	1.25e-001	9.24e-002
	6.25e-002	1.10e-002
	3.13e-002	8.05e-004
	1.56e-002	9.99e-004
Curl test:	2.50e-001	1.11e-002
	1.25e-001	4.22e-002
	6.25e-002	1.18e-002
	3.13e-002	3.95e-003
	1.56e-002	9.88e-004
Divergence test:	2.50e-001	2.70e-002
	1.25e-001	4.04e-002
	6.25e-002	7.85e-003
	3.13e-002	2.77e-003
	1.56e-002	7.31e-004

$$f(x,y,z) = \left(tanh\,(x) + tanh\,(y) + tanh\,(z)\right) * \; e^{-10(x^2+y^2+z^2)}$$

Gradient test:

1.25e-001	1.09e-001
6.25e-002	3.66e-002
3.13e-002	1.12e-002
1.56e-002	7.89e-003



…the forward model

We want to compare the numerical solution of the forward operator with some analytical solution. As featured in Haber (1999), we computed:

$$(\,Conductivity\ model\,)\ \ \gamma = tanh\left(a\left(\varepsilon + \frac{1}{4}\right)\right) - tanh\left(a\left(\varepsilon - \frac{1}{4}\right)\right) + \frac{1}{100}$$

$$(\text{Analytical field})\ \ \mathbf{E} = \left\langle -\frac{z \cdot y \cdot e^{-5(x^2+y^2+z^2)}}{\gamma(x)}\vec{i};\ \ \frac{-x \cdot z \cdot e^{-5(x^2+y^2+z^2)}\vec{j}}{\gamma(y)};\ \ \frac{x \cdot y \cdot e^{-5(x^2+y^2+z^2)}\vec{k}}{\gamma(z)} \right\rangle$$

On the interval [-1 , 1]. We don't know the exact solution for potentials A and ϕ but we can compute a pseudo-analytical solution:

$$\begin{bmatrix} \text{speye(nfaces)} & \text{GRAD} \\ \text{DIV} & 0 \end{bmatrix} \begin{bmatrix} A \\ \varphi \end{bmatrix} = \begin{bmatrix} E \\ 0 \end{bmatrix}$$

We can compute the source term:

$$J = (i\,\omega\,\mu)^{-1}\,\nabla x \nabla x\, E \;-\; \sigma E$$

Then solve for A and ϕ analytically…

n^3	∂A	∂φ	|DIV A|	$\|E_a - E_c\|$
8	2.53e-03	5.02e-04	2.78e-17	1.16e-03
16	1.59e-03	1.33e-04	5.55e-17	7.59e-04

Computing impedances

The inversion requires computation of data from the computed fields.

$$\mathbf{Q\,u = d}$$

Where u is a vector containing [A ; ϕ], d are measured impedances and Q a complicated operator…

From the fields we need to compute impedances:

$$\begin{bmatrix} E_x{}^x & E_x{}^y \\ E_y{}^x & E_y{}^y \end{bmatrix} = \begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix} \begin{bmatrix} H_x{}^x & H_x{}^y \\ H_y{}^x & H_y{}^y \end{bmatrix}$$

The things that Q must do:

1. **Compute H on edges**
2. **Average the fields to faces**
3. **Select fields at observation points**
4. **Solve for impedances**
5. **Repeat all for two polarization directions**

$$Z_{xy} = \frac{E_x{}^x H_y{}^y - E_x{}^y H_y{}^x}{H_y{}^y H_y{}^x - H_y{}^x H_x{}^y} \qquad Z_{xy} = \frac{E_x{}^y H_x{}^x - E_x{}^x H_x{}^y}{H_y{}^y H_y{}^x - H_y{}^x H_x{}^y}$$

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```
Qhx = SObsH * AVHx * CURL_f * SA / (1i*w*uo);
Qhy = SObsH * AVHy * CURL_f * SA / (1i*w*uo);

Qex = SObsEx * AVFvcz * AVC * SA;
Qey = SObsEy * AVFvcz * AVC * SA;

Zxx = @(ux, uy) ( (Qex*ux) .* (Qhy*uy) - (Qex*uy) .* (Qhx*ux) ) ./...
                ( (Qhy*uy) .* (Qhy*ux) - (Qhy*ux) .* (Qhx*uy) );

Zxy = @(ux, uy) ( (Qex*uy) .* (Qhx*ux) - (Qex*ux) .* (Qhx*uy) ) ./...
                ( (Qhy*uy) .* (Qhy*ux) - (Qhy*ux) .* (Qhx*uy) );

Zyx = @(ux, uy) ( (Qey*uy) .* (Qhy*ux) - (Qey*ux) .* (Qhy*uy) ) ./...
                ( (Qhx*uy) .* (Qhy*ux) - (Qhx*ux) .* (Qhy*uy) );


Zyy = @(ux, uy) ( (Qey*ux) .* (Qhx*uy) - (Qex*uy) .* (Qhx*ux) ) ./...
                ( (Qhx*uy) .* (Qhy*ux) - (Qhx*ux) .* (Qhy*uy) );
```

…tested on half-space

For a simple 1D problem we can compute the apparent conductivity from the Zxy term (background ρ = 1) :

$$\boldsymbol{\sigma_{app}} = \frac{|\boldsymbol{Zxy}|^{-1}}{\boldsymbol{\omega\mu}}$$

Z_{xx} & Z_{yy}		Z_{xy} & Z_{yx}		
1.0e-017 *		Impxy =		app_con =
(1,1)	-0.5110 + 0.0864i	(1,1)	0.0073 + 0.0086i	0.9876
(2,1)	-0.5188 + 0.1271i	(2,1)	0.0073 + 0.0086i	0.9876
(3,1)	-0.4660 + 0.0702i	(3,1)	0.0073 + 0.0086i	0.9876
(4,1)	-0.4016 + 0.0161i	(4,1)	0.0073 + 0.0086i	0.9876
(5,1)	-0.3492 + 0.0005i	(5,1)	0.0073 + 0.0086i	0.9876
(6,1)	-0.3161 + 0.0098i	(6,1)	0.0073 + 0.0086i	0.9876
(7,1)	-0.3149 - 0.0201i	(7,1)	0.0073 + 0.0086i	0.9876
(8,1)	-0.2729 - 0.0648i	(8,1)	0.0073 + 0.0086i	0.9876
(9,1)	-0.2758 - 0.1779i	(9,1)	0.0073 + 0.0086i	0.9876
(10,1)	-0.3228 - 0.2082i	(10,1)	0.0073 + 0.0086i	0.9876

Very small

Off- diagonals Equal

$\boldsymbol{\sigma \approx \sigma_{app}}$

Code vs. Physics

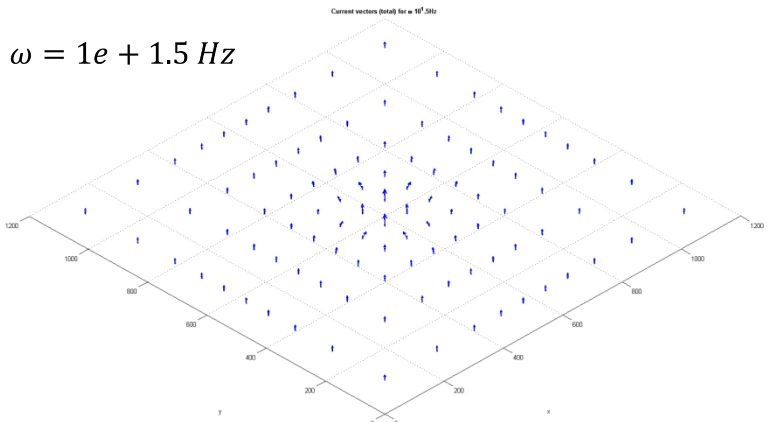
We have been asked:
" For a given conductor, how would the current change as a function of frequencies…"

From experience, geophysicists know that good conductors become insulators at high frequencies.

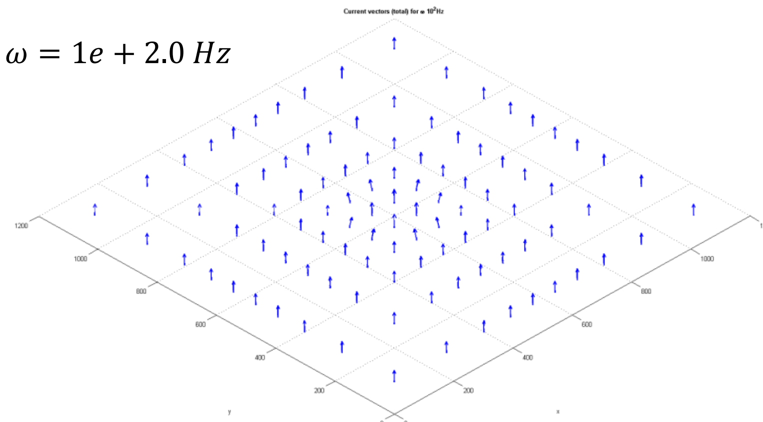
Simple problem:

- **Single conductivity anomaly near the surface (avoid skin depth attenuation)**
- **Range of frequencies**
- **Plot current density vectors**

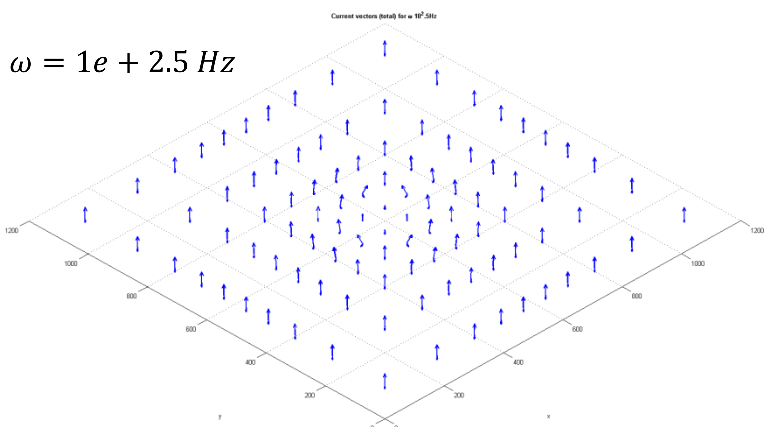
$\omega = 1e + 1.5\ Hz$



$\omega = 1e + 2.0\ Hz$



$\omega = 1e + 2.5\ Hz$



$\omega = 1e + 3.0\ Hz$