FOUTE
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TD5 SVM : Suite Cas non linéairement séparable.

$$y(x) = w^T\phi(x) + b$$

(PRIMAL) : $\text{Min } \frac{1}{2}\|w\|^2 + M\sum_{k=1}^{M} \varepsilon_k$

s-c $t^k(w^T\phi(x^k)+b) \geq 1-\xi_k \;,\; k=1,\dots M$ (

$w \in \mathbb{R}^n$ (2)

$b \in \mathbb{R}$ (3)

$\varepsilon_k \geq 0$ $\quad k=1,2,\dots M$ (4)

1) Fonction de Lagrange $L(w, b, \varepsilon, \lambda, \mu)$.

$$L(w,b,\varepsilon,\lambda,\mu) = \frac{1}{2}\|w\|^2 + M\sum_{k=1}^{M} \varepsilon_k - \sum_{k=1}^{M}\lambda_k\left[t^k(w^T\phi(x^k)+b) - 1 + \varepsilon_k\right]$$
$$- \sum_{k=1}^{M} \mu_k \varepsilon_k .$$

2) Expression de $\nabla_w L(w, b, \varepsilon, \lambda, \mu)$.

$$\nabla_w L = w_j + 0 - \sum_{k=1}^{M}\lambda_k t^k \phi(x^k) \quad \text{Car} \quad \frac{\partial w^T\phi(x^k)}{\partial w_k} = \phi(x^k).$$

$j = 1 \dots M.$

$$\nabla_w L = w - \sum_{k=1}^{M}\lambda_k t^k \phi(x^k).$$

L est minimisé si $\nabla_w L = 0$ D'après K-K-T.

$$\nabla_w L = 0 \implies w - \sum_{k=1}^{M}\lambda_k t^k \phi(x^k) = 0$$

$$\implies w = \sum_{k=1}^{M}\lambda_k t^k \phi(x^k).$$

3) Expression de $\nabla_b L(w, b, \varepsilon, \lambda, M)$

$$\nabla_b L = 0 + 0 - \sum_{k=1}^{N} \lambda_k t^k - 0$$

$$\nabla_b L = - \sum_{k=1}^{N} \lambda_k t^k.$$

$$\nabla_b L(w, b, \varepsilon, \lambda, \mu) = 0 \implies \sum_{k=1}^{N} \lambda_k t^k = 0.$$

4) Expressions de $\nabla_\varepsilon L(w, b, \varepsilon, \lambda, M)$

$$\nabla_\varepsilon L = 0 + M - 0 - \mu_k - \lambda_k \quad \text{Car} \quad \frac{\partial \sum \varepsilon_k}{\partial \varepsilon_k} = 1.$$

$$\nabla_\varepsilon L(w, b, \varepsilon, \lambda, \mu) = M - \mu_k - \lambda_k.$$

La solution optimale est obtenue à $\nabla_\varepsilon L = 0$

impliquant $M - \mu_k - \lambda_k = 0$

$$\implies \mu_k = M - \lambda_k.$$

5) Deduisons que le (DUAL) s'ecrit :

$$(\text{DUAL}): \ \text{Max} \quad e^T\lambda - \frac{1}{2}\lambda^T Q \lambda$$
$$\text{s.c} \quad \lambda^T t = 0$$
$$0 \leq \lambda \leq M.$$

où $Q = t^i t^j \phi^T(x^i)\phi(x^j)$., $\lambda \in \mathbb{R}^T$ $i, j \in \{1, \ldots, N\}$

on a : $w = \sum_{k=1}^{N} \lambda_k t^k \phi(x^k)$ et $\mu_k = M - \lambda_k$.

~~donc~~: on a : $g(\lambda) = \operatorname{Min} L(w, b, \varepsilon, \lambda, M)$.

Il suffit de trouver $g(\lambda)$ en remplaçant les solution optimale dans l'expression de la fonction de Lagrange 2).

on a : $\frac{1}{2}\|w\|^2 + M\sum_{k=1}^{N}\varepsilon_k - \sum_{k=1}^{N}\lambda_k\left[t^k(w^T\phi(x^k)+b)-1+\varepsilon_k\right] - \sum_{k=1}^{N}M_k\varepsilon_k$

$$\Longleftrightarrow \frac{1}{2}\left\|\sum_{k=1}^{N}\lambda_k t^k\phi(x^k)\right\|^2 - \sum\lambda_k\left[t^k\left(\left(\sum_{j=1}^{N}\lambda_j t_j\phi(x^j)\right)^T\phi(x^k)+b\right)-1+\varepsilon_k\right] + M\sum_{k=1}^{N}\varepsilon_k - \sum_{k=1}^{N}(M-\lambda_k)\varepsilon_k.$$

$$\Longleftrightarrow \frac{1}{2}\left[\left(\sum_{k=1}^{N}\lambda_k t^k\phi(x^k)\right)^T\left(\sum_{k=1}^{N}\lambda_k t^k\phi(x^k)\right)\right] - \sum_{k=1}^{N}\lambda_k\left[t^k\left[\left(\sum_{j=1}^{N}\lambda_j t_j\phi(x_j)\right)^T\phi(x^k)+b\right]\right] + \sum_{k=1}^{N} - \cancel{\sum_{k=1}^{N}\lambda_k\varepsilon_k} + \cancel{M\sum_{k=1}^{N}\varepsilon_k} - \cancel{M\sum_{k=1}^{N}\varepsilon_k} + \cancel{\sum_{k=1}^{N}\lambda_k\varepsilon_k}$$

$$\Longleftrightarrow \frac{1}{2}\left[\sum_{k=1}^{N}\left(\lambda_k t^k\phi(x^k)\right)^T\right]\left(\sum_{k=1}\lambda_k t^k\phi(x^k)\right) - \underbrace{\sum\lambda_k t^k}_{0} b - \sum_{k=1}^{N}\lambda_k t^k\left(\sum_{j=1}^{N}\lambda_j t^j\phi(x^j)^T Q(x^k)\right) + \sum_{k=1}^{N}\lambda_k$$

$$\Longleftrightarrow \sum\lambda_k - \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1}^{N}\lambda_k t^k\lambda_j t^j\phi(x^j)^T\phi(x^k).$$

Posons $k=i$. et $\sum_{k=1}^{N}\lambda_k = e^T\lambda$

Le problème Dual peut s'écrire :

$$(\text{DUAL}):\ \operatorname{Max}\ e^T\lambda - \frac{1}{2}\lambda^T Q\lambda$$
$$\text{s.c}\quad \lambda^T t = 0 \quad \text{et } 0 \le \lambda \le M.$$

car $\sum\lambda^i t^i = 0 \Rightarrow \lambda^T t = 0$.