Von Newman stability analysis for 2D acoustic wave equation explicit

Von Newman stability analysis for acoustic wave equation explicit centered differences: 2nd order time and space (N 2)'th order:

$$U_{jk}^{n+1} = \left(\frac{\Delta t V_{jk}}{\Delta s}\right)^2 \left(\sum_{a=-N}^{N} w_a U_{j+ak}^n + \sum_{a=-N}^{N} w_a U_{jk+a}^n\right) + 2U_{jk}^n - U_{jk}^{n-1}$$
$$U_{jk}^{n+1} = \left(\frac{\Delta t V_{jk}}{\Delta s}\right)^2 \sum_{a=-N}^{N} w_a \left(U_{j+ak}^n + U_{jk+a}^n\right) + 2U_{jk}^n - U_{jk}^{n-1}$$

(1)

For forth order space, we have $N = 2$ and w is:

$$w = \frac{1}{12}[-1, 16, -30, 16, -1]$$

Can also be simplified to 1st order (N=1):

$$U_{jk}^{n+1} = \left(\frac{\Delta t V_{jk}}{\Delta s}\right)^2 \left(U_{j+1k}^n - 4U_{jk}^n + U_{jk+1}^n + U_{j-1k}^n + U_{jk-1}^n\right) + 2U_{jk}^n - U_{jk}^{n-1}$$

Using the discrete solution for 2D wave equation, where $i = \sqrt{-1}$, $n = n\Delta t$, $j = j\Delta x$ and $k = k\Delta z$. Last using $\Delta x = \Delta z = \Delta s$, follows that the discrete solution can be written as:

$$U_{jk}^n = e^{i(\omega t + px + qz)}$$
$$U_{jk}^n = \epsilon^n e^{i(pj\Delta s + qk\Delta s)}$$
$$U_{jk}^n = \epsilon^n e^{i\Delta s(pj+qk)}$$

(2)

Where ϵ is the growth factor, and should be $|\epsilon| \leq 1$ for stability.

Replacing (2) in (1), using the identities bellow and simplifying dividing both sides by U_{jk}^{n+1}

$$r = \frac{\Delta t V_{jk}}{\Delta s}$$

$$\phi_{j+l\,k+m} = e^{i\Delta s(pl+qm)}$$

$$\Omega = r^2 \sum_{a=-N}^{N} w_a \left(\phi_{j+ak} + \phi_{jk+a}\right)$$

(3)

we get:

$$1 = (\Omega + 2)\,\epsilon^{-1} - \epsilon^{-2}$$
$$\text{making } \epsilon^{-1} = \mu$$
$$\mu^2 - (\Omega + 2)\,\mu + 1 = 0$$
$$\mu = \frac{(\Omega + 2) \pm \sqrt{\Omega^2 + 4\Omega}}{2}$$

(4)

back to expand Ω defined in (3):

$$\Omega = r^2 \sum_{a=-N}^{N} w_a \left(\phi_{j+ak} + \phi_{jk+a}\right)$$
$$= r^2 \sum_{a=-N}^{N} w_a \left(e^{i\Delta s\, pa} + e^{i\Delta s\, qa}\right)$$

$$= r^2 \begin{pmatrix} \cdots & e^{-i\Delta s2p} + e^{-i\Delta s2q} & e^{-i\Delta sp} + e^{-i\Delta sq} & e^0 + e^0 & e^{i\Delta sp} + e^{i\Delta sq} & e^{i\Delta s2p} + e^{i\Delta s2q} & \cdots \end{pmatrix} \begin{pmatrix} \cdots \\ w_{-2} \\ w_{-1} \\ w_0 \\ w_1 \\ w_2 \\ \cdots \end{pmatrix}$$

Since w is even $w_a = w_{-a}$ and $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ we can rewrite as:

$$= r^2 \begin{pmatrix} \cdots & 2\cos(\Delta s2p) + 2\cos(\Delta s2q) & 2\cos(\Delta sp) + 2\cos(\Delta sq) & 2 \end{pmatrix} \begin{pmatrix} \cdots \\ w_2 \\ w_1 \\ w_0 \end{pmatrix}$$

For the simplest case 2nd order $N = 1$ we have $(w_1, w_0) = (1, -2)$

$$\begin{aligned} \Omega &= r^2 \left(2\cos(\Delta sp) + 2\cos(\Delta sq) - 4\right) \\ &= -4r^2 \left(\sin^2(\frac{\Delta sp}{2}) + \sin^2(\frac{\Delta sq}{2})\right) \end{aligned}$$

(5)

Note: $2\cos(\theta) - 2 = -4\sin^2(\theta)$.

We can also write (5) using $\beta = \left(\sin^2(\frac{\Delta sp}{2}) + \sin^2(\frac{\Delta sq}{2})\right)$ as :

$$\Omega = -4r^2\beta$$

Replacing back to (4) :

$$\begin{aligned} \mu &= \frac{(\Omega + 2) \pm \sqrt{\Omega^2 + 4\Omega}}{2} \\ \mu &= -2r^2\beta + 1 \pm 2\sqrt{r^2\beta(r^2\beta - 1)} \end{aligned}$$

I am a little lost how to find if $|\mu| >= 1$ or what limitations I have in r for this requirement, that is the same as needing $|\epsilon| <= 1$.

Is there any easier alternative to Von Newman that also could be applied to the general explicit form in (1) ?

pde finite-difference stability fourier-analysis discretization

edited Jul 22 '13 at 18:23

asked May 11 '13 at 4:21
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1 Answer

After more than 2 months and no answer. I post my own answer this is as far as I could get (not final answer though).

I found the general formula for stability criteria (in a paper[1]). That is given by:

$$r \leq \frac{2}{\sqrt{\sum_{a=-N}^{N}(|w_a^1| + |w_a^2|)}}$$

With $r = \frac{V\Delta t}{\Delta s}$ and w_a is the centered finite differences weights and the indexes 1 e 2 refer to the x and y dimensions.

But I couldn't get to this general formula, I could just get to the criteria to the 2nd order that was the post N=1.

Not certain if this a proof by contradiction. (Also forgive my bad math I am really eager to learn)

Suppose $\Delta > 0$ condition holds for $|\epsilon| \leq 1$ that using $\epsilon^{-1} = \mu$ means $|\mu| \geq 1$. Thus this requires for $r^2\beta - 1 > 0$ to be $r > \frac{1}{\sqrt{\beta}}$ that can be satisfied by using $r = \frac{1}{\sqrt{\beta}} + \psi$ with $\psi > 0$ positive, real.

Going back to for the first root μ', we have:

$$\begin{aligned}\mu^{'} &= -2r^2\beta+1+2\sqrt{r^2\beta(r^2\beta-1)}\\ &= -2\left(1+\frac{2\psi\beta}{\sqrt{\beta}}+\psi^2\beta\right)+1+2\sqrt{\left(1+\frac{2\psi\beta}{\sqrt{\beta}}+\psi^2\beta\right)\left[\left(1+\frac{2\psi\beta}{\sqrt{\beta}}+\psi^2\beta\right)-1\right]}\\ &= -2\left(1+2\psi\sqrt{\beta}+\psi^2\beta\right)+1+2\sqrt{\left(1+2\psi\sqrt{\beta}+\psi^2\beta\right)\left[\left(1+2\psi\sqrt{\beta}+\psi^2\beta\right)-1\right]}\\ &= -2A+1+2\sqrt{A^2-A}\end{aligned}$$

With:

$$A=\left(1+2\psi\sqrt{\beta}+\psi^2\beta\right)$$

Note that since $\psi>1$ then $A>1$ always. Using the requirement for stability:

$$\begin{aligned}|\mu^{'}| &\geq 1\\ \left|-2A+1+2\sqrt{A^2-A}\right| &\geq 1\end{aligned}$$

To satisfy the inequality, two possibilities

$$\begin{aligned}-2A+1+2\sqrt{A^2-A} &\leq -1\\ -2A+1+2\sqrt{A^2-A} &\geq 1\\ -2A+2\sqrt{A^2-A} &\leq -2\ (2)\\ -2A+2\sqrt{A^2-A} &\geq 0\ (3)\end{aligned}$$

At (2) for $A>1$ left hand side cannot hold, always >-2. At (3) for $A>1$ also cannot hold, $-1>lefthandside>-2$ We don't even need to look at the second root.

This implies that $\Delta>0$ doesn't satisfy the stability criteria.

Now suppose $\Delta=0$ condition holds} for $|\epsilon|\leq 1$ that using $\epsilon^{-1}=\mu$ means $|\mu|\geq 1$. Thus this requires that $r=\frac{1}{\sqrt{\beta}}$ \

For booth roots μ, we have:

$$\begin{aligned}\mu &= -2r^2\beta+1+2\sqrt{r^2\beta(r^2\beta-1)}\\ &= -2+1\\ &= -1\end{aligned}$$

That clearly holds.

Finally suppose $\Delta<0$ condition holds for $|\epsilon|\leq 1$ Thus this requires $r^2\beta-1<0$ that can be satisfied by using $r=\frac{1}{\sqrt{\beta}}-\psi$ with $\psi>0$ positive, real.

Again going back for the first root μ', we have:

$$\begin{aligned}\mu^{'} &= -2r^2\beta+1+2\sqrt{r^2\beta(r^2\beta-1)}\\ &= -2\left(1-2\psi\sqrt{\beta}+\psi^2\beta\right)+1+2\sqrt{\left(1-2\psi\sqrt{\beta}+\psi^2\beta\right)\left[\left(1-2\psi\sqrt{\beta}+\psi^2\beta\right)-1\right]}\end{aligned}$$

Rearranging due the imaginary part

$$\begin{aligned}&= -2\left(1-2\psi\sqrt{\beta}+\psi^2\beta\right)+1+2i\sqrt{\left[1-\left(1-2\psi\sqrt{\beta}+\psi^2\beta\right)\right]\left(1-2\psi\sqrt{\beta}+\psi^2\beta\right)}\\ &= -2A+1+2i\sqrt{A^2-A}\end{aligned}$$

With $i=\sqrt{-1}$ imaginary unit and:

$$A=\left(1-2\psi\sqrt{\beta}+\psi^2\beta\right)$$

Note that since $\psi>1$ then $A<1$ always. Then using the requirement for stability and complex number modulus (the other root is just conjugate of this so same modulus):

$$\begin{aligned}|\mu| &\geq 1\\ \left|-2A+1+2i\sqrt{A^2-A}\right| &\geq 1\\ \sqrt{(-2A+1)^2+4(A^2-A)} &\geq 1\\ \sqrt{1} &\geq 1\end{aligned}$$

So this condition also holds.

Thus the solution is $r\leq\frac{1}{\sqrt{\beta}}$ that is maximum given $\beta=2$ and then $r\leq\frac{1}{\sqrt{2}}$ That agrees with general formula presented first for N=1.

[1] A stability formula for Lax-Wendroff methods with fourth-order in time and general-order in space for the scalar wave equation - pg T38 - Geophysics Vol. 76 No. 2 2011

answered Jul 22 '13 at 18:22

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1 Not an explicit answer per se, but if you're unaware you'll find it extremely easy to simply compute a contour plot of your growth factor ϵ for various values of r to determine where in the complex plane a given method is stable. As you obviously know, finding a closed-form relation for the stability limits for anything but the most trivial methods quickly becomes an algebraic nightmare, and I don't believe it's a very informative exercise in the end anyway. – Aurelius Aug 25 '13 at 19:47

Thanks @aurelius that's indeed a good idea I wasn't aware. That's indeed a algebraic nightmare, in the near future for more complex stuff maybe I will try the contour plots. thanks – eusoubrasileiro Aug 27 '13 at 0:28

1 No problem, if you have Matlab handy here's a sample for a couple simple time integration methods: spitfire.princeton.edu/stability.m – Aurelius Aug 28 '13 at 14:13