Graph Theory Fall 2021

Assignment 4

Due at 5:00 pm on Monday, September 27

Questions with a ($\star$) are each worth 1 bonus point for 453 students.

1. Recall that the adjacency matrix of a simple graph G with vertex set $\{v_1, v_2, \dots, v_n\}$ is the $n \times n$ matrix A with entries

$$A_{i,j} = \begin{cases} 1 & v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

A. Let $K_{3,4}$ be the complete bipartite graph with vertices $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and where vertices v_i and v_j are adjacent if and only if i and j have different parity (one of i or j is odd and the other is even.) What does the adjacency matrix A look like in this case?

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

B. Let $K_{3,4}$ be the complete bipartite graph with vertices $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and where vertices v_i and v_j are adjacent if and only if ($i \le 3$ and $j \ge 4$) or ($i \ge 4$ and $j \le 3$). What does the adjacency matrix A look like in this case?

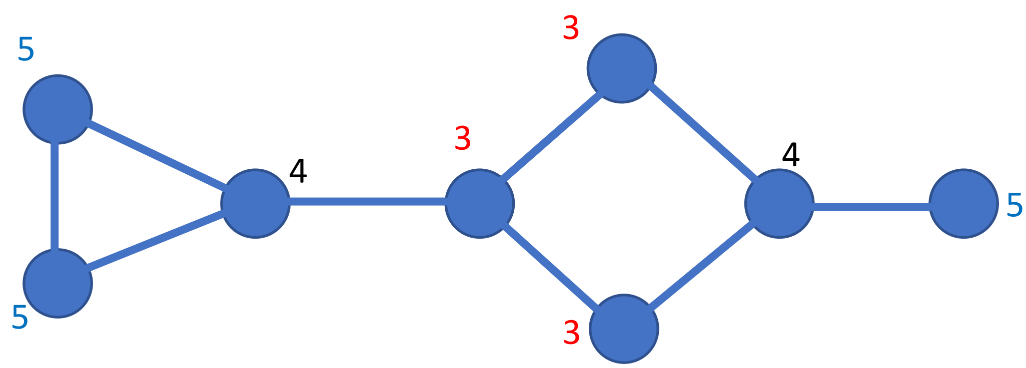
$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. We let G be a connected graph. For any vertex $v \in V$, define its **eccentricity** by the formula

$$\mathrm{ecc}(v) = \max\{D(u, v):\ u \in V\}.$$

This is the length of "longest among all shortest paths with v as an endpoint."

a. Let G be the graph drawn below. Label each vertex with its eccentricity.



b. The **diameter** of a graph is the maximum among the eccentricities of its vertices and the **radius** of a graph is the minimum among the eccentricities of its vertices. For the graph G drawn in part a, what is its diameter and radius?

$$\mathrm{diameter}(G) = 5;\ \ \mathrm{radius}(G) = 3$$

c. A **central vertex** is a vertex v such that $\mathrm{ecc}(v) = \mathrm{radius}(G)$. Which of the vertices in the graph G are central vertices?

The central vertices are those with eccentricity 3 (they're the ones with red labels).

d. A **peripheral vertex** is a vertex v such that $\mathrm{ecc}(v) = \mathrm{diameter}(G)$. Which of the vertices in graph G are peripheral vertices?

The peripheral vertices are those with eccentricity 5 (they're the ones with blue labels).

e. Explain why it is important for these definitions that G be a connected graph.

If G is not connected, then the eccentricity of every vertex is infinite, so very little information is conveyed via the radius or diameter.

f. Show that for any connected graph H,

$$\text{radius}(H) \leq \text{diameter}(H) \leq 2\ \text{radius}(H).$$

One inequality is quite easy and the second can be handled using a central vertex and the triangle inequality.

$$\begin{aligned} \text{radius}(H) &= \min\{\text{ecc}(v): v \in V\} \\ &\leq \max\{\text{ecc}(v): v \in V\} = \text{diameter}(H) \end{aligned}$$

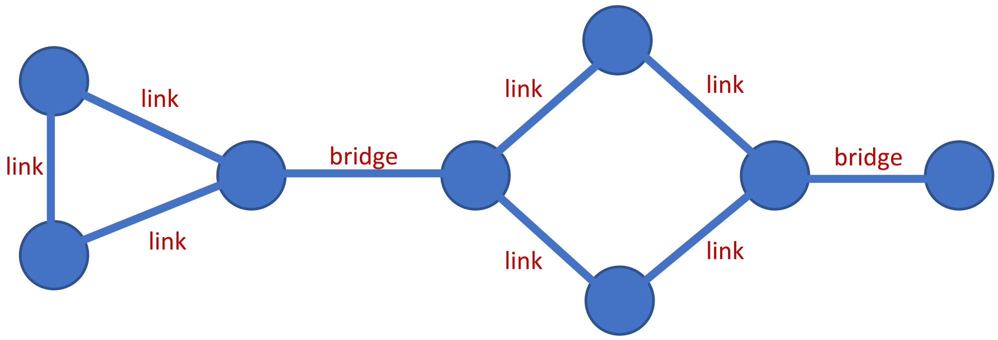
For $\text{diameter}(H) \leq 2\ \text{radius}(H)$, let u and w be vertices such that $D(u,w) = \text{diameter}(H)$ and let v be a central vertex; hence, $\text{ecc}(v) = \text{radius}(G)$. Then

$$\begin{aligned} \text{diameter}(H) &= D(u,w) \\ &\leq D(u,v) + D(v,w) \text{ by the triangle inequality} \\ &\leq \text{ecc}(v) + \text{ecc}(v) \\ &= 2\ \text{radius}(G) \end{aligned}$$

as desired.

3. Recall that a **bridge** is an edge whose deletion increases the number of components of a graph. Also, a **link** is another term for "non-bridge."

   a. In the graph G (same as in problem 2a) below, which edges are bridges and which edges are links?



   b. If you delete all of the bridges, how many components remain?

   Deleting the two bridges results in a graph with two additional components, for a total of three components.

   c. Suppose, instead, you deleted links one at a time until the remaining graph had no links. How many links could you delete in this process?

You can delete one link from the triangle subgraph and one link from the 4-cycle subgraph before all other edges become bridges.

4. Let G be a graph and x be a vertex of G. We say that $u \approx w$ if $D(u,x) = D(w,x)$. When we discuss trees, the equivalence classes under $\approx$ will be the levels of a tree.
   a. Show that the relation $\approx$ is reflexive.

Let u be any vertex. Then $D(u,x) = D(u,x)$ and so $u \approx u$.

   b. Show that the relation $\approx$ is symmetric.

Let u and v be vertices. Each of the following implies the next:

1) $u \approx v$
2) $D(u,x) = D(v,x)$
3) $D(v,x) = D(u,x)$
4) $v \approx u$

   c. Show that the relation $\approx$ is transitive.

Let u, v, and w be vertices. Each of the following implies the next:

1) $u \approx v$ and $v \approx w$
2) $D(u,x) = D(v,x)$ and $D(v,x) = D(w,x)$
3) $D(u,x) = D(w,x)$
4) $u \approx w$

   d. Suppose x has no loops and suppose ux is an edge. Briefly describe the equivalence class $[u]$ under $\approx$.

Since x has no loops and ux is an edge, we have $D(u,x) = 1$. Thus, if $v \in [u]$, we must have $D(v,x) = D(u,x) = 1$ as well, implying vx is an edge. Hence, $[u] = \{\text{all vertices adjacent to } x\}$.