Graph Theory Fall 2021

Assignment 2

Due at 5:00 pm on Wednesday, September 8

1. Among all simple graphs (no loops, no parallel edges) with $n = 100$ vertices, determine (with justification) the minimum possible and the maximum possible values for m, the number of edges such a graph could have.

Since empty graphs have no loops or parallel edges, they are simple. Thus, the minimum value for m is 0. For the maximum value, this is achieved by joining every pair of distinct vertices. Since there are $\binom{100}{2} = \frac{100(99)}{2} = 4950$ such pairs of vertices, the maximum value for m is 4950.

2. Let G be the simple graph with vertex set

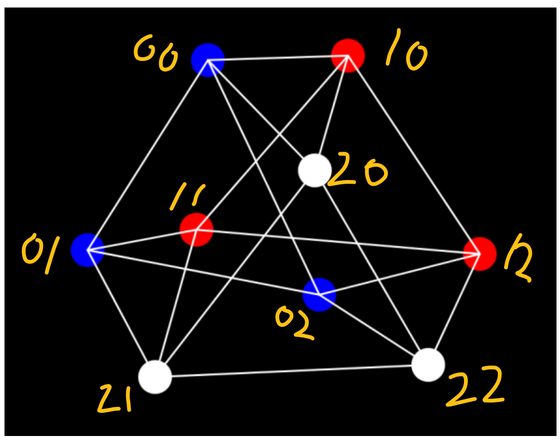
$$V = \{00, 01, 02, 10, 11, 12, 20, 21, 22\}; \ \ |V| = 9$$

and where vertices ab and cd are joined by an edge when <u>exactly one</u> of the following conditions holds (so there are no loops):

$$a = c \ \text{or} \ b = d.$$

A. Sketch G; you are allowed to do this by hand and then copy your sketch electronically into your PDF submission.

One possibility:



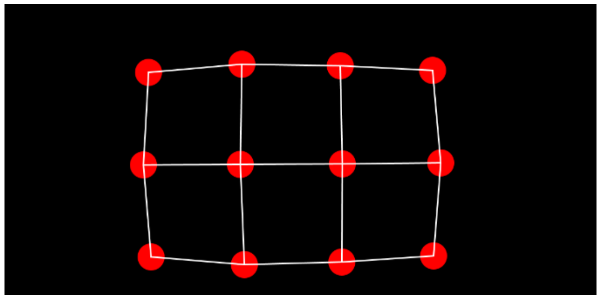
B. Determine m, the number of edges of G.

Every vertex (a, b) is joined to $(a, b+1), (a, b+2), (a+1, b)$, and $(a+2, b)$ where the addition is modulo 3. Hence, every vertex has degree 4. There are nine vertices, so the graph G has total degree

$$9 \cdot 4 = 36$$

The number of edges is half this, and so $m = 18$.

3. The “grid graph” $P_{r,s}$ is the cartesian product of the two paths P_r and P_s. For instance, $P_{3,4}$ is drawn below:



A. In terms of r and s, find a formula for the number of vertices of the grid graph $P_{r,s}$.

   There are rs vertices in $P_{r,s} = P_r \times P_s$.

B. In terms of r and s, find a formula for the number of edges of the grid graph $P_{r,s}$.

   Each of the s copies of P_r has $r-1$ edges while each of the r copies of P_s has $s-1$ edges, for a total of $s(r-1)+r(s-1)$ edges.

4. Recall that a graph G is “cubic” if and only if it is 3-regular.
   A. Show that there exists no cubic graph with an odd number of vertices.

      If a cubic graph G existed with $2k + 1$ vertices, then its total degree would be $3(2k + 1) = 6k + 3 = 2(3k + 1) + 1$, an odd number. The total degree of any graph is twice the number of edges, an even number, and so the total degree of G would be simultaneously odd and even. No such integer exists, and so G does not exist.

   B. For every integer $n \geq 3$, show that there exists a simple cubic graph (no loops, no parallel edges) with $2n$ vertices. One way to do this is to produce a construction, i.e., give a set of $2n$ vertices and a recipe for when vertices are joined by edges for constructing such graphs.

      The quickest example to give is $C_n \times P_2$. Every vertex (a, b) of this graph is adjacent to $(a - 1, b)$, $(a + 1, b)$, and $(a, b + 1)$ where addition in the first coordinate is modulo n and in the second coordinate is modulo 2.