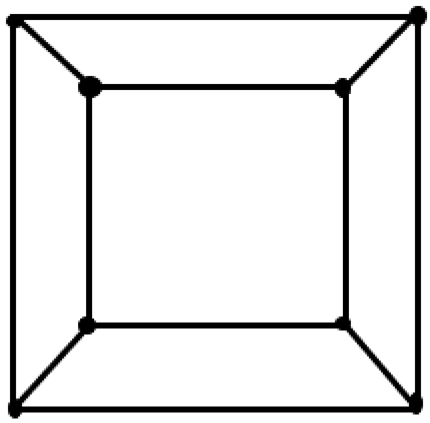
1.

A. Q_3 $have\ 8\ vertices, 12 edges\ and\ 6\ regions.$

According to Euler's formula for spherical graphs,

$$n - m + r = 8 - 12 + 6 = 2$$

Thus, Q_3 can be drawn on the plane.



B. Suppose Q_4 is planar and let H be a spherical drawing of Q_4. The graph H has n=16 vertices and m=32 edges. From Euler,

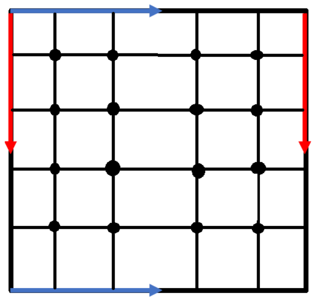
$$n - m + r = 2$$

$$16 - 9 + r = 2$$

$$r = 18$$

Since G is simple and bipartite, each region will contribute at least 4 to the total edge count. So the total edge count must be at least 4*18=72. But the total edge count is 2m=64 which is not at least 72, and so Q_4 is not planar.

C.



D. For Q_5 we have n=32 vertices and m=80 edges. The total edge count is 2m=160. Since Q_5 is simple and bipartite, the total edge count must be at least 4r. This means $160 \geq 4r, r \leq 40$. Feed this information into Euler's formula we can get:

$$n - m + r = 2 - 2g(Q_5)$$

$$32 - 80 + r = 2 - 2g(Q_5)$$

$$2g(Q_5) = 50 - r$$

$$g(Q_5) = \frac{50-r}{2}$$

$$Since\ r \leq 40,\ \ g(Q_5) \geq 5$$

E. For Q_k we have $n = 2^k$ vertices and $m = k2^{k-1}$ edges. The total edge count is $2m = 2k2^{k-1} = k2^k$. Since Q_k is simple and bipartite, the total edge count must be at least 4r. This means $k2^k \geq 4r, r \leq \frac{k2^k}{2^2}$, $r \leq k2^{k-2}$. Feed this information into Euler's formula we can get:

$$n - m + r = 2 - 2g(Q_k)$$

$$2^k - k2^{k-1} + r = 2 - 2g(Q_k)$$

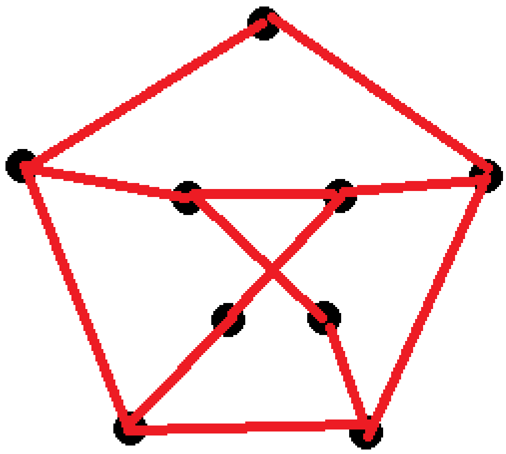
$$2g(Q_k) \geq 2 - 2^k + k2^{2k-1} - k2^{k-2}$$

$$g(Q_k) \geq 1 - 2^{k-1} + k2^{2K-2} - k2^{k-3}$$

$$Since\ k \geq 0,\ \ g(Q_k) \geq 1 - \tfrac{1}{2} = 1/2$$

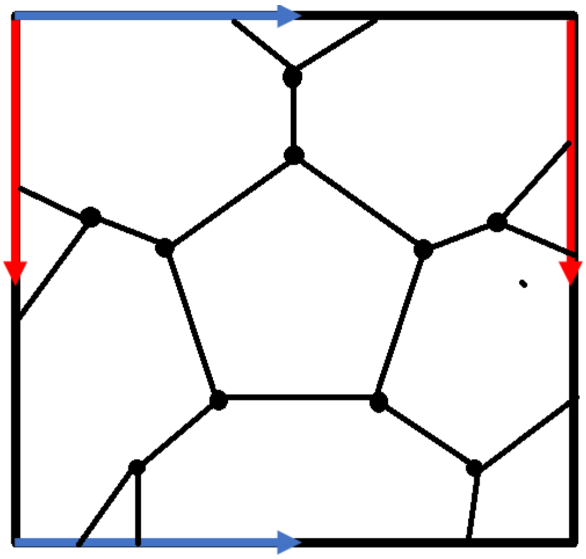
2.

A.



The above graph H is a subgraph of P. Also, H is homeomorphic to $K_{3,3}$. According to Kuratowski's theorem, we know that a graph is non-planar if and only if one of its sub-graph is homeomorphic to $K_{3,3}\ or\ K_5$. Hence graph P is non-planar.

B.



3.