Graph Theory Fall 2021

Assignment 3

Due at 5:00 pm on Friday, September 17

Questions with a ($\star$) are each worth 1 bonus point for 453 students.

1. Let $n \geq 3$ be given and let C_n be the n-cycle whose vertices are $\{0,1,2,\ldots,n-1\}$. Recall that ab is an edge if and only if $a = b + 1$ mod n or $b = a + 1$ mod n.
   a. How many vertices does $C_n \times C_n$ have?

Given graphs G and H with vertex sets V_G and V_H, we have $V_{G\times H} = V_G \times V_H$. Therefore,

$$|V_{G\times H}| = |V_G| \cdot |V_H|.$$

For $G = H = C_n$, we have $|V_G| = |V_H| = n$ and so $|V_{G\times H}| = n^2$ in this case.

   b. Show that $C_n \times C_n$ is 4-regular. One way is to let (p,q) be any vertex of $C_n \times C_n$ and explicitly show that (p,q) is adjacent to exactly four vertices.

All arithmetic is modulo n.

If $(p,q) \in V_{C_n\times C_n}$, then since the neighbors of p are $p-1$ and $p+1$ and the neighbors of q are $q-1$ and $q+1$, the neighbors of (p,q) are

$$(p-1,q), \qquad (p+1,q), \qquad (p,q-1), \qquad (p,q+1)$$

and so (p,q) has degree 4. Since (p,q) is arbitrarily chosen, $C_n \times C_n$ is 4-regular.

Alternative solution. (This requires answering 2a correctly) Using question 2a, we know C_n is 2-regular and so $C_n \times C_n$ is $(2+2)$-regular.

   c. How many edges does $C_n \times C_n$ have?

From parts a and b, the total degree is $4n^2$. By the handshaking lemma, the number of edges is half this, or $2n^2$.

2. We discuss Cartesian products of regular graphs more generally.

   a. Let G be an a-regular graph and H be a b-regular graph. Show that $G \times H$ is an $(a+b)$-regular graph.

Let $(p,q) \in V_{G\times H}$ be arbitrarily chosen. Since $p \in G$ and G is a-regular, we may assume the neighbors of p are the a distinct vertices $v_1, v_2, \dots, v_a$. Similarly, we may assume the neighbors of q are the b distinct vertices $w_1, w_2, \dots, w_b$. Thus, the neighbors of (p,q) are precisely the distinct vertices

$$(v_i, q):\ 1 \le i \le a$$
$$(p, w_j):\ 1 \le j \le b$$

This is a collection of $a+b$ vertices. Since (p,q) was arbitrarily chosen, every vertex has degree $a+b$ and so $G \times H$ is $(a+b)$-regular.

   b. ($\star$) Let G be an a-regular graph. Show that G^k (this is the k-fold cartesian product $G \times G \times \cdots \times G$) is a ka-regular graph. If you use mathematical induction, you may assume that G^k is isomorphic to $G \times G^{k-1}$.

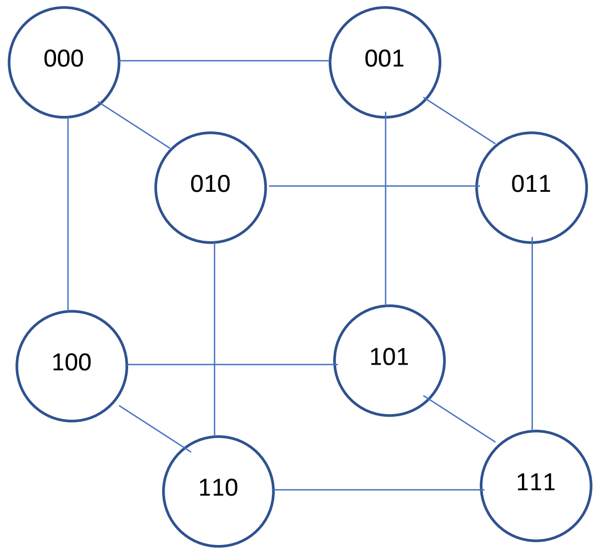
We proceed by mathematical induction on k, $k \ge 1$. For the base case, we note that $G^1 = G$, an a-regular graph. Since $a = 1a$, G^1 is $1a$-regular.

For the inductive step, assume G^{k-1} is $(k-1)a$-regular. From 2a, we note that $G \times G^k$ is $(a+(k-1)a)$-regular. Hence, $G \times G^{k-1}$ is ka-regular. Since G^k is isomorphic to $G \times G^{k-1}$, it follows that G^k is ka-regular.

3. Let Q_3 be the 3-cube with vertex set

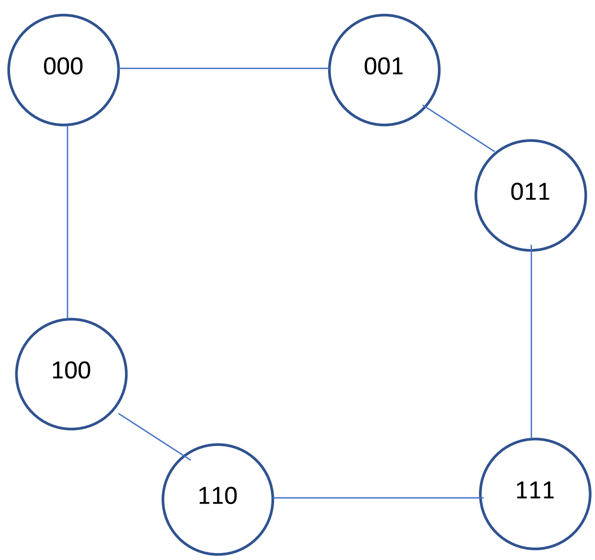
$$V = \{000,001,010,011,100,101,110,111\}.$$

A drawing helps here:



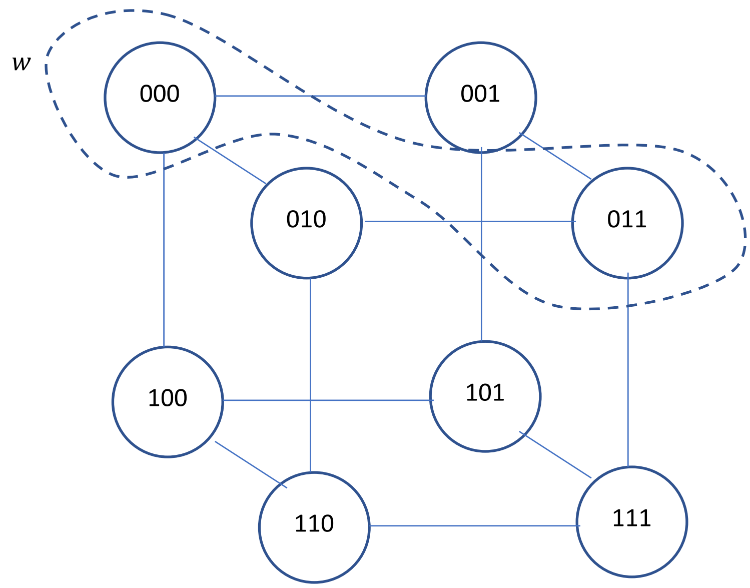
a. Give an example of two vertices such that if we delete them both, the graph G that remains is a 6-cycle.

It turns out that any pair of vertices that disagree on all three coordinates would work. Here, if we delete 010 and 101, we obtain the following graph:



Deleting two adjacent vertices in Q_3 results in a 6-cycle with an extra edge.

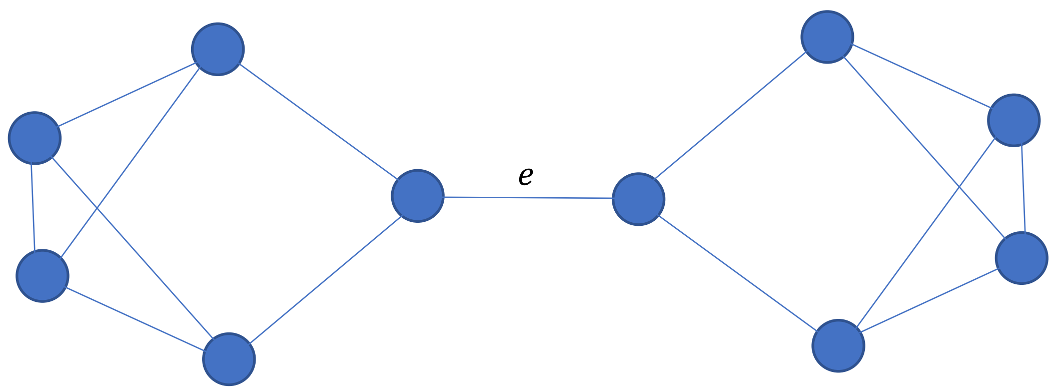
b. Suppose we start with Q_3 and produce the graph H by identifying vertices 000 and 011; we call the resulting vertex w. What is the degree of w in H?



The resulting vertex w is adjacent to 001, 010, 100, and 111 and so $\deg(w) = 4$. Remember that we do not allow for parallel edges or loops to form as the result of identifying vertices.

c. ($\star$) What is the smallest number of edges we could delete from Q_3 so that the resulting graph is disconnected? Justify your answer.

This is the hardest question on the assignment. It was popular to answer that since the graph is 3-regular, that it would require deleting 3 edges to disconnect the graph. The graph below is 3-regular, but one can disconnect the graph by deleting the single edge e:



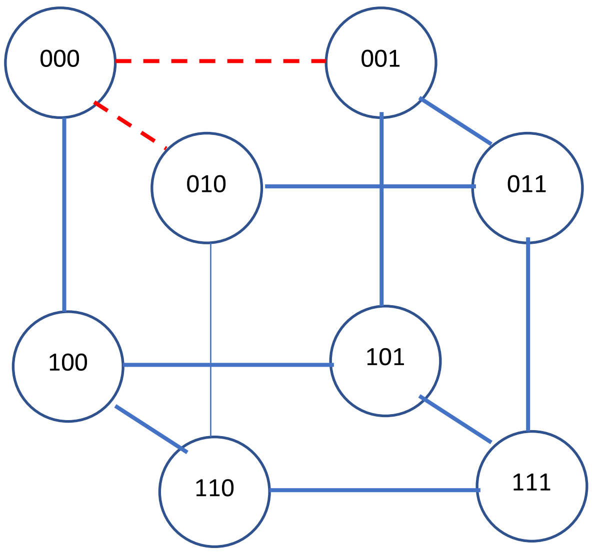
One clever approach is to show that if it takes deleting k edges to disconnect Q_k, then it takes deleting at least $k+1$ edges to disconnect $Q_k \times P_2 = Q_{k+1}$.

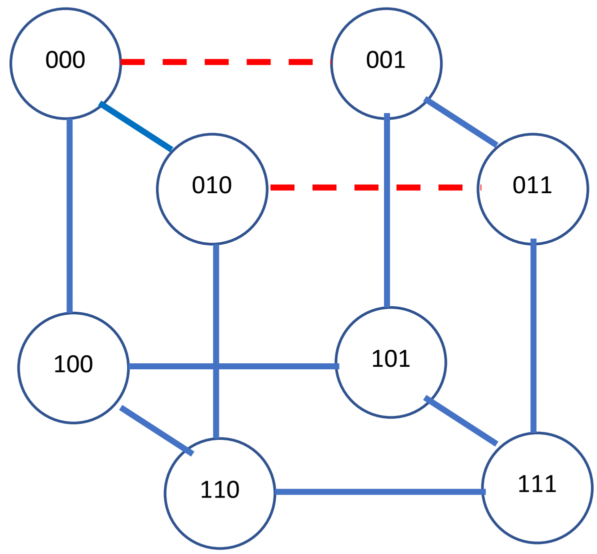
Theorem. For $k \geq 1$, it takes deleting k edges to disconnect Q_k.

Proof. We proceed by mathematical induction on k. For the base case, when $k = 1$, we have $Q_1 = K_2$, a single edge joining two vertices. We must delete the 1 edge to disconnect them. Thus, the result holds for $k = 1$.

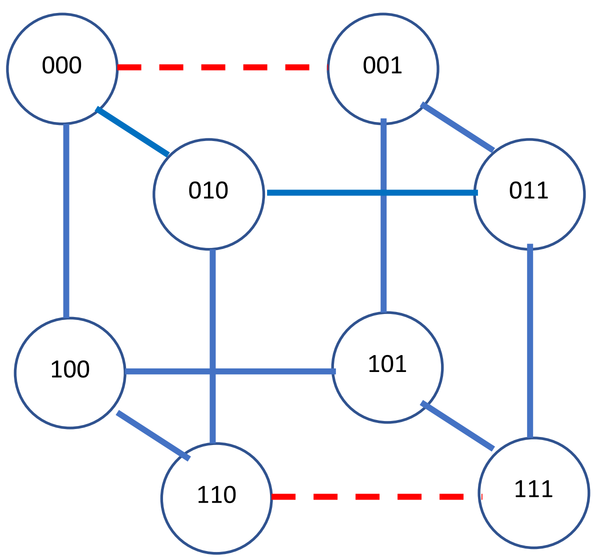
For the inductive step, suppose we delete only k edges from $Q_{k+1} \cong Q_k \times P_2$, resulting in the graph G. Notice that there are $2^k > k$ edges that are copies of the P_2 graph, so at least one of these will remain. If both copies of Q_k remain connected, then a copy of P_2 that remains can be used to connect them, rendering the graph G connected. If one of the copies of Q_k is disconnected, then all k of the deleted edges must have been used to disconnect it, and so the other copy of Q_k is left intact as are all copies of P_2. If u and v are in different components of the disconnected copy of Q_k, then there is a u, v-path u, e_u, P, e_v, v in G where P is a u', v'-path in the intact copy of Q_k. Here, we're assuming e_u is the copy of P_2 joining u to u' and e_v is the copy of P_2 joining v to v'. All other possibilities of pairs of vertices are easy to handle and so G is connected.

A more mundane approach (symmetry is your friend here!) is to note that deleting 2 edges on the same face leaves the opposite face and all perpendicular edges intact, so the resulting graph is connected. This handles the case where the two deleted edges share an endpoint and the case where the two deleted edges are in the same direction on the same face.

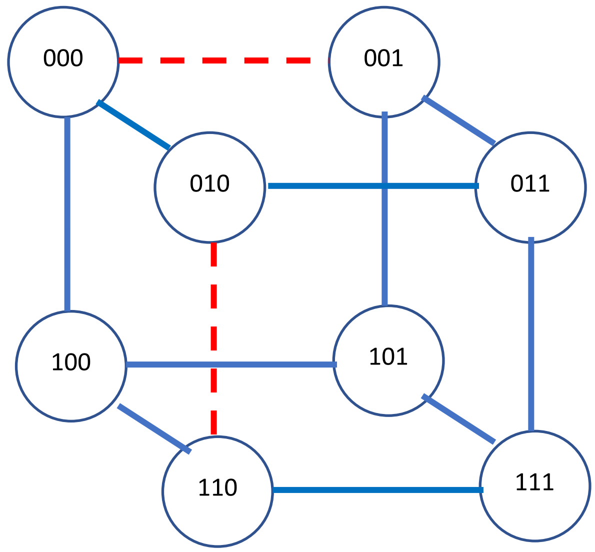




Deleting two edges that join two opposite faces leaves the two opposite faces intact and connected to each other (twice over).



Finally, deleting two “skew” edges as depicted below leaves the remaining graph connected:



4. Let G be a finite graph with at least one edge.
   a. Show that for any $k \geq 1$, if W is a walk in G of length k, then there exists a walk W' in G of length $k+1$. This shows that G cannot have a longest walk.

If G has a walk W of length $k \geq 1$ (this means e_1 exists):

$$W\colon\ v_0, e_1, v_1, \dots, e_k, v_k$$

then G has a walk W' of length $k+1$:

$$W\colon\ v_1, e_1, v_0, e_1, v_1, \dots, e_k, v_k$$

   b. Show that there is some upper bound b on the length of a path in G. This means that if W is a walk of length $k > b$, then a vertex must be repeated in W. This shows that G must have a longest path.

We claim $b = |V| - 1$ is an upper bound.

If W is a walk of length $k > |V| - 1$, then W includes $k + 1 > |V|$ vertices in the sequence. Since there are only $|V|$ distinct vertices, W must repeat a vertex.