1. We know that $n = \sum_{j=1}^{\infty} n_j$ and $2(n-1) = total\ degree = \sum_{j=1}^{\infty} jn_j$

   This implies, $\sum_{j=1}^{n}(2-j)n_j = 2$

   $$n_1 + \sum_{j=2}^{n}(2-j)n_j = 2$$

   Since $k \geq 3$ and $n_1 < k$, $(2-j)$ is negative. We can get $n_j = 0$ $for\ all\ j > k$.

   Hence, for any $k \geq 3$, if a tree T has fewer than k leaves, then the maximum degree $\Delta(T)$ among the vertices of T must satisfy $\Delta(T) < k$.

2.

   a. We know that r is on the unique u, v-path, and r is the root of T. $D(u,v) = D(u,r) + D(r,v)$. Since the root has no parents, $D(u,r) = L(u), D(r,v) = L(v)$

      Hence, $D(u,v) = L(u) + L(v)$.

   b. We know that $L(v) = D(r,v)$ and $L(u) = D(r,u)$

      If $L(u) + L(v) = D(u,v)$, then $D(r,v) + D(r,u) = D(u,v)$

      Since the root r has no parents and u, v is a unique path, r must on the unique u, v-path.

   c. Since the height H of the rooted tree (T, r) is the maximum among all levels of its vertices, the longest level of any vertex in the rooted tree is H. Assume u and v are both leaves of rooted tree, and r is on the unique u, v-path. Then, $D(u,v)$ must be the longest path of any two vertices in rooted tree. $D(u,v) = D(r,u) + D(r,v) = L(u) + L(v) = H + H = 2H$. Hence, for any two vertices u and v, $D(u,v) \leq 2H$.

d. Assume u is a parent of some vertices and v is non-parents, and r is on the unique u, v path. According to the definition, $L(u) \leq H$ and $L(v) = H$. $D(u,v) = D(r,u) + D(r,v) = L(u) + L(v) \leq 2H.$ Hence, if $D(u,v) = 2H, then\ u\ and\ v\ must\ be\ non - parents.$

3.

a. Since (T, r) is a saturated rooted q-ary tree, every parent has q children. Assume T has b parents. The number of edges is equal to bq.

b. $n = bq + 1$

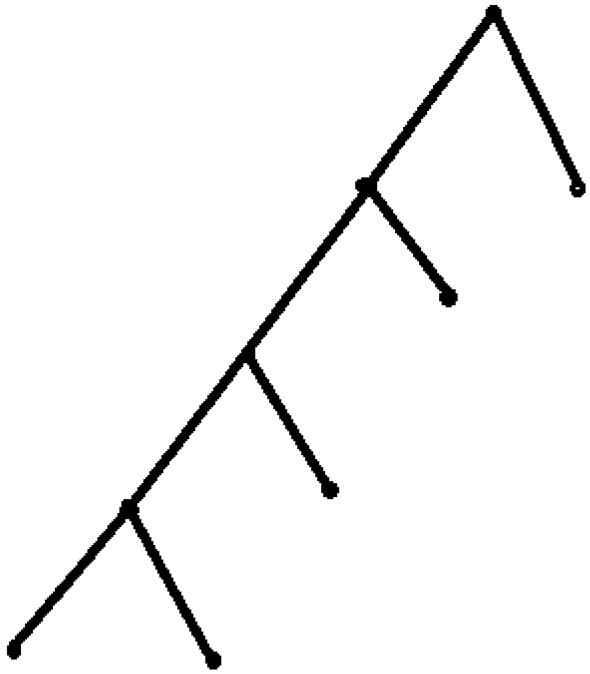
c. The number of non-parent vertices is $n - b = bq + 1 - b.$

4.

a. To approach upper bounds on H, we should find the minimum number of vertices on each level. For a rooted tree, the minimum number of vertices on each level is 1. Hence, the upper bound for H is 10^{12}.

The lower bound of H is 1.

b.



Let n be the number of vertices in T. $n = 10^{12} + 1$.

The lower bound of H is $\log_2(n+1) - 1 = \log_2(10^{12} + 2) - 1$

The upper bound of H is $\frac{10^{12}}{2}$.