Linear Algebra and Numerical Optimization

Project pitch

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Outline

What the project is about?

- Choose a project topic (or suggest your own) related to linear algebra or numerical optimization that is of interest to you
- search for literature covering this topic and find out
  - what problem the method solves
  - where the method(s) is (are) used
  - what are possible algorithms, their pros and cons
  - choose up to two algorithms, describe what they do; give pseudocode
  - discuss complexity of the algorithms and their limitations
  - implement the algorithm(s) numerically in Python, Matlab, R, ...
  - compare the efficiency and accuracy against available benchmarks
  - give a summary: what you achieved, what caused main difficulties etc
- your report must cover the above (plus literature list) and be approx. 8–10 pages long (without appendices)
- give the link to the page with your code and tested examples

Grading system

- The project will contribute up to 20% of the total grade item More exactly, $20 = 2 + 4 + 3 + 4 + 3 + 4$, where
  - 2% for the interim report
  - $4\% + 3\%$ is our grade for your report, implementation, and presentation
  - $4\% + 3\%$ will be the peer review evaluation of your project and presentation by your classmates
  - 4% for the quality of your peer review of the other project
- What is evaluated?
  - description of the method (tasks, approaches, challenges, efficiency)
  - literature review
  - theoretical justification of the algorithm
  - code and its performance
  - overall presentation
  - etc

Timing (tentative)

- Choose a topic and form a team of up to 3: 15 Dec 2022
- Interim report 16 Jan 2023
- Final report and presentation ** Feb 2023
- Peer reviews ** Feb 2023

In the interim report, you should submit a 2-page account of your current progress with the final project. This must include:

- a clear statement of the topic, aim, and the task
- description of the problem you will discuss
- sketch of the possible approaches
- short explanations of the pros and cons of your chosen method
- a short discussion of the planned research steps and/or numerical realization

Inverting matrices

- What methods exist for finding A^{-1}?
- Most use factorizations: LU, QR, SVD, Cholesky etc
  - $A = LU$ with lower/upper triangular L and U
  - $A = QR$ with orthogonal Q and upper-triangular R
  - $A = U\Sigma V^\top$ the SVD decomposition
  - $A = LL^\top$ for nonnegative definite or $LDL^\top$
- Iterative methods
- Applicability
- Pros and cons
- Efficient methods

Generalized inverses

- Solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} := A^{-1}\mathbf{b}$ if A^{-1} exists
- If A is singular or non-square $m \times n$, can use pseudo-inverse
- There is no B such that $AB = I_m$ and $BA = I_n$ if $m \neq n$
- The ranks of AB and BA are at most $r = \operatorname{rank} A$
- A^+ is pseudo-inverse of A if
$$AA^+ \approx I_r \oplus 0 \text{ and } A^+A \approx I_r \oplus 0$$
- Questions:
    - how to find A^+?
    - Is it unique?
    - Are there other pseudo-inverses?
    - Where are they used?
    - Efficient methods ...

Factorization methods for solving $A\mathbf{x} = \mathbf{b}$

- There are lots of them: LU, QR, SVD, Cholesky etc
- What methods are best in which cases?
- What are their pros and cons?
- etc

Iterative methods for solving $A\mathbf{x} = \mathbf{b}$

- We'll discuss the idea of the approach:

$$\text{rewrite } A\mathbf{x} = \mathbf{b} \text{ as } \mathbf{x} = B\mathbf{x} + \mathbf{c}$$
$$\text{use the iteration scheme } \mathbf{x}_{n+1} = B\mathbf{x}_n + \mathbf{c}$$

- What methods are available?
- What are their pros and contras?
- What method is the most efficient and when can it be applied
- etc

Iterative methods for eigenvalues/eigenvectors

- Mentioned the power method
- May not be the best
- There are modifications
  - inverse power method
  - adaptive power method
  - ...

A $25,000,000,000 eigenvector

- This project addresses the PageRank method of ranking websites
- mathematically, Internet surfing is described by a Markov process
- the leading eigenvector of the corresponding Markov matrix contains site ranks
- the eigenvector equation is

$$r_i = \frac{1-q}{N} + q \sum p_{ij} r_j$$

- what methods are there for PageRank calculation?

Eigenvalue computation in the 21$^{\mathrm{st}}$ century

A paper with this title (with 20$^{\mathrm{th}}$ in place of 21$^{\mathrm{st}}$) gives a review of the methods developed for eigenvalue/eigenvector computation

Matrix factorization

There are lots of matrix factorizations that are widely used in practice:

- *LU*
- *QR*
- *SVD*
- Cholesky
- Gram–Schmidt
- Householder reflection
- Givens rotation
- non-negative matrix factorization etc

Choose any of them and discuss where and how it is used

Singular value decomposition and image compression

One of the most striking applications of the SVD is for low-rank matrix approximations. This is especially important for image compression, because they often allow significant compression without quality loss.

How this works, what is the best rank etc is the topic of this project

Principal component analysis

Mathematically, principal component analysis is reduction of the quadratic form to its principal axes, i.e., diagonalization of a symmetric matrix by an orthogonal one.

PCA is widely used to reduce dimensionality of data, optimization problems etc.

PCA is behind the so called “eigenface” approach to face detection

Linear least squares

Linear least squares problems arise in many different contexts—not only as the method of solving $A\mathbf{x} = \mathbf{b}$ but also for e.g. classification. There are many different techniques and approaches; e.g., among the most efficient is the one based on the Householder reflections

- *QR*
- *SVD*
- Cholesky
- Gram–Schmidt
- Householder reflection
- Givens rotation etc

Non-linear least squares

Non-linear least squares problems have enormously many applications; they are generalization of the linear least square problem: like the latter, they minimize the sum of squared residuals.

Within this project, you can explore typical applications and approaches

Numerical optimization

Optimization problems arise almost everywhere in applications. There are many different approaches developed for their solutions

Within this project, you can explore typical applications of optimization problems and approaches of finding optimal solution

Linear algebra in machine learning

Many ideas and methods of machine learning are just the ones of linear algebra in disguise, e.g.,

- non-linear support vector machines
- linear/nonlinear regression
- Kestroke dynamics: how Coursera servers (used to) identify us?
- ...

This topic is about spotting and describing LA in ML

Linear algebra in statistics

Linear algebra is used in statistics while e.g. studying covariance matrices.
There are, however, many other applications, and this topic invites you to a “hunting” tour through statistical methods to spot linear algebra there.

Have fun with LA!