Exercise 1

According to the Brock and Mirman model, households solve the following dynamic program:

$$V(K_t, z_t) = \max_{K_{t+1}} \ln(e^{z_t}K_t^\alpha - K_{t+1}) + \beta E_t\{V(K_{t+1}, z_{t+1})\}$$

with an associated Euler equation:

$$\frac{1}{e^{z_t}K_t^\alpha - K_{t+1}} = \beta E_t\left\{\frac{\alpha e^{z_{t+1}}K_{t+1}^{\alpha-1}}{e^{z_{t+1}}K_{t+1}^\alpha - K_{t+2}}\right\}$$

Observe that

$$\begin{aligned}
\frac{1}{e^{z_t}K_t^\alpha - Ae^{z_t}K_t^\alpha} &= \beta E_t\left\{\frac{\alpha e^{z_{t+1}}(Ae^{z_t}K_t^\alpha)^{\alpha-1}}{e^{z_{t+1}}(Ae^{z_t}K_t^\alpha)^\alpha - Ae^{z_{t+1}}(Ae^{z_t}K_t^\alpha)^\alpha}\right\}\\
\frac{1}{e_t^z K_t^\alpha(1-A)} &= \beta E_t\left\{\frac{\alpha e^{z_{t+1}}(Ae^{z_t}K_t^\alpha)^{\alpha-1}}{(e^{z_{t+1}}(Ae^{z_t}K_t^\alpha)^\alpha)(1-A)}\right\}\\
\frac{1}{e_t^z K_t^\alpha(1-A)} &= \beta E_t\left\{\frac{\alpha(Ae^{z_t}K_t^\alpha)^{-1}}{1-A}\right\}\\
\frac{1}{e_t^z K_t^\alpha} &= \beta\frac{\alpha}{(Ae^{z_t}K_t^\alpha)}\\
1 &= \beta\frac{\alpha}{A}\\
A &= \beta\alpha
\end{aligned}$$

where A is expressed as a function of the model's parameters. We know that this value is correct because...

Thus $K_{t+1} = Ae^{z_t}K_t^\alpha = \beta\alpha e^{z_t}K_t^\alpha$

Exercise 2

Using the model in section 3 of the notes, we consider the following functional forms:

$$
\begin{aligned}
u(c_t, \ell_t) &= \ln c_t + a \ln (1 - \ell_t) \\
F(K_t, L_t, z_t) &= e^{z_t} K_t^{\alpha} L_t^{1-\alpha}
\end{aligned}
$$

Observe that

$$
\begin{aligned}
u_c(c_t, \ell_t) &= \frac{1}{c_t} \\
u_\ell(c_t, \ell_t) &= -\frac{a}{1-\ell_t} \\
f_K(k_t, \ell_t, z_t) &= \alpha e^{z_t} \left(\frac{k_t}{\ell_t}\right)^{\alpha-1} \\
f_L(k_t, \ell_t, z_t) &= (1-\alpha) e^{z_t} \left(\frac{k_t}{\ell_t}\right)^{\alpha}
\end{aligned}
$$

With these in mind, the (seven) characterizing equations are as follows:

$$
\begin{gathered}
c_t = (1-\tau)[w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1} \\
\frac{1}{c_t} = \beta E_t \left\{ \frac{1}{c_{t+1}} [(r_{t+1} - \delta)(1-\tau)] + 1 \right\} \\
\frac{a}{1-\ell_t} = \frac{1}{c_t} w_t (1-\tau) \\
r_t = e^{z_t} \left(\frac{k_t}{\ell_t}\right)^{\alpha-1} \\
w_t = (1-\alpha) e^{z_t} \left(\frac{k_t}{\ell_t}\right)^{\alpha} \\
\tau [w_t \ell_t + (r_t - \delta) k_t] = T_t \\
z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \varepsilon_t^z; \quad \varepsilon_t^z \sim \text{i.i.d}(0, \sigma_z^2)
\end{gathered}
$$

with (seven) unknowns, $\{c_t, k_t, \ell_t, w_t, r_t, T_t, z_t\}$. We cannot use the same tricks as in Exercise 1 to solve for the policy function in this case because ...

Exercise 3

We now do the same as in Exercise 2, but instead

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a \ln(1 - \ell_t)$$

Note that firms have the same production function as in Exercise 2. Observe that

$$\begin{aligned}
u_c(c_t, \ell_t) &= \frac{1}{c_t^\gamma} \\
u_\ell(c_t, \ell_t) &= -\frac{a}{1-\ell_t} \\
f_K(k_t, \ell_t, z_t) &= \alpha e^{z_t}\left(\frac{k_t}{\ell_t}\right)^{\alpha-1} \\
f_L(k_t, \ell_t, z_t) &= (1-\alpha)e^{z_t}\left(\frac{k_t}{\ell_t}\right)^{\alpha}
\end{aligned}$$

With these in mind, the (seven) characterizing equations are as follows:

$$\begin{aligned}
c_t &= (1-\tau)[w_t\ell_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \\
\frac{1}{c_t^\gamma} &= \beta E_t\left\{\frac{1}{c_{t+1}^\gamma}[(r_{t+1} - \delta)(1-\tau)] + 1\right\} \\
\frac{a}{1-\ell_t} &= \frac{1}{c_t^\gamma} w_t(1-\tau) \\
r_t &= e^{z_t}\left(\frac{k_t}{\ell_t}\right)^{\alpha-1} \\
w_t &= (1-\alpha)e^{z_t}\left(\frac{k_t}{\ell_t}\right)^{\alpha} \\
\tau[w_t\ell_t + (r_t - \delta)k_t] &= T_t \\
z_t &= (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \varepsilon_t^z; \quad \varepsilon_t^z \sim \text{i.i.d}(0, \sigma_z^2)
\end{aligned}$$

with (seven) unknowns, $\{c_t, k_t, \ell_t, w_t, r_t, T_t, z_t\}$.

Exercise 4

Now, suppose the functional forms are as follows:

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a\,\frac{(1-\ell_t)^{1-\xi} - 1}{1-\xi}$$
$$F(K_t, L_t, z_t) = e^{z_t}[\alpha K_t^\eta + (1-\alpha)L_t^\eta]^{\frac{1}{\eta}}$$

Observe that

$$u_c(c_t, \ell_t) = \frac{1}{c_t^\gamma}$$
$$u_\ell(c_t, \ell_t) = -\frac{a}{(1-\ell_t)^\xi}$$
$$f_K(k_t, \ell_t, z_t) = \alpha k_t^{\eta-1} e^{z_t}[\alpha k_t^\eta + (1-\alpha)\ell_t^\eta]^{\frac{1-\eta}{\eta}}$$
$$f_L(k_t, \ell_t, z_t) = (1-\alpha)\ell_t^{\eta-1} e^{z_t}[\alpha k_t^\eta + (1-\alpha)\ell_t^\eta]^{\frac{1-\eta}{\eta}}$$

With these in mind, the (seven) characterizing equations are as follows:

$$c_t = (1-\tau)[w_t\ell_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1}$$
$$\frac{1}{c_t^\gamma} = \beta E_t\left\{\frac{1}{c_{t+1}^\gamma}[(r_{t+1} - \delta)(1-\tau)] + 1\right\}$$
$$\frac{a}{(1-\ell_t)^\xi} = \frac{1}{c_t^\gamma} w_t(1-\tau)$$
$$r_t = \alpha k_t^{\eta-1} e^{z_t}[\alpha k_t^\eta + (1-\alpha)\ell_t^\eta]^{\frac{1-\eta}{\eta}}$$
$$w_t = (1-\alpha)\ell_t^{\eta-1} e^{z_t}[\alpha k_t^\eta + (1-\alpha)\ell_t^\eta]^{\frac{1-\eta}{\eta}}$$
$$\tau[w_t\ell_t + (r_t - \delta)k_t] = T_t$$
$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \varepsilon_t^z;\;\; \varepsilon_t^z \sim \text{i.i.d}(0, \sigma_z^2)$$

with (seven) unknowns, $\{c_t, k_t, \ell_t, w_t, r_t, T_t, z_t\}$.

Exercise 5

Now consider the following functional forms:

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$
$$F(K_t, L_t, z_t) = K_t^{\alpha}(L_t e^{z_t})^{1-\alpha}$$

and assume $\ell_t = 1$. Observe that

$$u_c(c_t) = \frac{1}{c_t^{\gamma}}$$
$$u_\ell(c_t) = 0$$
$$f_K(k_t, \ell_t, z_t) = \alpha\left(\frac{k_t}{e^{z_t}}\right)^{\alpha-1}$$
$$f_L(k_t, \ell_t, z_t) = (1-\alpha)\left(\frac{k_t}{e^{z_t}}\right)^{\alpha}$$

With these in mind, the (six) characterizing equations are as follows:

$$c_t = (1-\tau)[w_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1}$$
$$\frac{1}{c_t^{\gamma}} = \beta E_t\left\{\frac{1}{c_{t+1}^{\gamma}}[(r_{t+1} - \delta)(1-\tau)] + 1\right\}$$
$$r_t = \alpha\left(\frac{k_t}{e^{z_t}}\right)^{\alpha-1}$$
$$w_t = (1-\alpha)\left(\frac{k_t}{e^{z_t}}\right)^{\alpha}$$
$$\tau[w_t + (r_t - \delta)k_t] = T_t$$
$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \varepsilon_t^z; \quad \varepsilon_t^z \sim \text{i.i.d}(0, \sigma_z^2)$$

with (six) unknowns, $\{c_t, k_t, w_t, r_t, T_t, z_t\}$.
To write out the steady state version of these equations, just drop the t

subscript and put bars over as follows:

$$
\begin{aligned}
\bar{c} &= (1-\tau)[\bar{w} + (\bar{r}-\delta)\bar{k}] + \bar{T} \ \ \text{(note that } \bar{k}-\bar{k}=0)\\
\frac{1}{\bar{c}^{\gamma}} &= \beta E_t\left\{\frac{1}{\bar{c}^{\gamma}}[(\bar{r}-\delta)(1-\tau)]+1\right\}\\
\bar{r} &= \alpha\left(\frac{\bar{k}}{e^{\bar{z}}}\right)^{\alpha-1}\\
\bar{w} &= (1-\alpha)\left(\frac{\bar{k}}{e^{\bar{z}}}\right)^{\alpha}\\
\tau[\bar{w} + (\bar{r}-\delta)\bar{k}] &= \bar{T}\\
0 &= \varepsilon_t^z \ \ \text{(note that } \bar{z}-(1-\rho_z)\bar{z}-\rho_z\bar{z}=0)
\end{aligned}
$$

Now, to algebraically solve for the steady value of k as a function of the steady state value of z, we plug $\bar{r}$ into the characteristic equation $\frac{1}{\bar{c}^{\gamma}} = \beta E_t\left\{\frac{1}{\bar{c}^{\gamma}}[(\bar{r}-\delta)(1-\tau)]+1\right\}$. Observe that

$$
\begin{aligned}
\frac{1}{\bar{c}^{\gamma}} &= \beta\left[\frac{1}{\bar{c}^{\gamma}}\left[\left(\alpha\left(\frac{\bar{k}}{e^{\bar{z}}}\right)^{\alpha-1}-\delta\right)\left(1-\tau\right)\right]+1\right]\\
\frac{1-\beta}{\beta} &= \left(\alpha\left(\frac{\bar{k}}{e^{\bar{z}}}\right)^{\alpha-1}-\delta\right)\left(1-\tau\right)\\
\frac{1-\beta}{\alpha\beta(1-\tau)}+\frac{\delta}{\alpha} &= \left(\frac{\bar{k}}{e^{\bar{z}}}\right)^{\alpha-1}\\
&\boxed{\bar{k} = e^{\bar{z}}\left[\frac{1-\beta}{\alpha\beta(1-\tau)}+\frac{\delta}{\alpha}\right]^{\frac{1}{\alpha-1}}}
\end{aligned}
$$

For the numerical solution, see the corresponding notebook.

Exercise 6

Now consider the following functional forms:

$$
\begin{aligned}
u(c_t, \ell_t) &= \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a \, \frac{(1-\ell_t)^{1-\xi} - 1}{1-\xi} \\
F(K_t, L_t, z_t) &= K_t^{\alpha} (L_t e^{z_t})^{1-\alpha}
\end{aligned}
$$

Observe that

$$
\begin{aligned}
u_c(c_t, \ell_t) &= \frac{1}{c_t^{\gamma}} \\
u_\ell(c_t, \ell_t) &= -\frac{a}{(1-\ell_t)^{\xi}} \\
f_K(k_t, \ell_t, z_t) &= \alpha \left(\frac{k_t}{\ell_t e^{z_t}} \right)^{\alpha-1} \\
f_L(k_t, \ell_t, z_t) &= (1-\alpha) \left(\frac{k_t}{\ell_t e^{z_t}} \right)^{\alpha}
\end{aligned}
$$

With these in mind, the (seven) characterizing equations are as follows:

$$
\begin{gathered}
c_t = (1-\tau)[w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1} \\
\frac{1}{c_t^{\gamma}} = \beta E_t \left\{ \frac{1}{c_{t+1}^{\gamma}} [(r_{t+1} - \delta)(1-\tau)] + 1 \right\} \\
\frac{a}{(1-\ell_t)^{\xi}} = \frac{1}{c_t^{\gamma}} w_t (1-\tau) \\
r_t = \alpha \left(\frac{k_t}{\ell_t e^{z_t}} \right)^{\alpha-1} \\
w_t = (1-\alpha) \left(\frac{k_t}{\ell_t e^{z_t}} \right)^{\alpha} \\
\tau [w_t \ell_t + (r_t - \delta) k_t] = T_t \\
z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \varepsilon_t^z; \;\; \varepsilon_t^z \sim \text{i.i.d}(0, \sigma_z^2)
\end{gathered}
$$

with (seven) unknowns, $\{c_t, k_t, \ell_t, w_t, r_t, T_t, z_t\}$.

The steady state versions of these equations are as follows:

$$
\begin{aligned}
\bar{c} &= (1-\tau)[\bar{w}\bar{\ell} + (\bar{r}-\delta)\bar{k}] + \bar{T} \quad \text{(note that } \bar{k}-\bar{k}=0\text{)} \\
\frac{1}{\bar{c}^{\gamma}} &= \beta E_t \left\{ \frac{1}{\bar{c}^{\gamma}} [(\bar{r}-\delta)(1-\tau)] + 1 \right\} \\
\frac{a}{(1-\bar{\ell})^{\xi}} &= \frac{1}{\bar{c}^{\gamma}} \bar{w}(1-\tau) \\
\bar{r} &= \alpha \left(\frac{\bar{k}}{\bar{\ell} e^{\bar{z}}} \right)^{\alpha-1} \\
\bar{w} &= (1-\alpha) \left(\frac{\bar{k}}{\bar{\ell} e^{\bar{z}}} \right)^{\alpha} \\
\tau[\bar{w}\bar{\ell} + (\bar{r}-\delta)\bar{k}] &= \bar{T} \\
0 &= \varepsilon_t^z \quad \text{(note that } \bar{z} - (1-\rho_z)\bar{z} - \rho_z \bar{z} = 0\text{)}
\end{aligned}
$$

For the numerical solution, see the corresponding notebook.