3.1

(i)

$$\begin{aligned}(||\mathbf{x}+\mathbf{y}||^2 - ||\mathbf{x}-\mathbf{y}||^2) &= \langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{x}\rangle + \langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{x}\rangle \\ &= 2\langle \mathbf{x},\mathbf{y}\rangle + 2\langle \mathbf{y},\mathbf{x}\rangle \\ &= 4\langle \mathbf{x},\mathbf{y}\rangle \\ \frac{1}{4}(||\mathbf{x}+\mathbf{y}||^2 - ||\mathbf{x}-\mathbf{y}||^2) &= \langle \mathbf{x},\mathbf{y}\rangle\end{aligned}$$

as desired.

(ii)

$$\begin{aligned}(||\mathbf{x}+\mathbf{y}||^2 + ||\mathbf{x}-\mathbf{y}||^2) &= ||\mathbf{x}||^2 + ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + ||\mathbf{y}||^2 \\ &= 2||\mathbf{x}||^2 + 2||\mathbf{y}||^2 \\ &= 2(||\mathbf{x}||^2 + ||\mathbf{y}||^2) \\ \frac{1}{2}(||\mathbf{x}+\mathbf{y}||^2 + ||\mathbf{x}-\mathbf{y}||^2) &= ||\mathbf{x}||^2 + ||\mathbf{y}||^2\end{aligned}$$

as desired.

3.2

$$\begin{aligned}&\frac{1}{4}(||\mathbf{x}+\mathbf{y}||^2 - ||\mathbf{x}-\mathbf{y}||^2 + i||\mathbf{x}-i\mathbf{y}||^2 - i||\mathbf{x}+i\mathbf{y}||^2) \\ &\qquad = \frac{1}{4}(4\langle \mathbf{x},\mathbf{y}\rangle) \\ &\qquad = \langle \mathbf{x},\mathbf{y}\rangle\end{aligned}$$

(Note that $i||\mathbf{x}-i\mathbf{y}||^2 - i||\mathbf{x}+i\mathbf{y}||^2 = i\langle \mathbf{x},\mathbf{x}\rangle - \langle \mathbf{y},\mathbf{x}\rangle + \langle \mathbf{x},\mathbf{y}\rangle + i\langle \mathbf{y},\mathbf{y}\rangle - i\langle \mathbf{x},\mathbf{x}\rangle + \langle \mathbf{y},\mathbf{x}\rangle - \langle \mathbf{x},\mathbf{y}\rangle - i\langle \mathbf{y},\mathbf{y}\rangle = 0$)

3.3

(i) $\cos\theta = \frac{\int_0^1 (x)(x^5)dx}{\sqrt{\int_0^1 x^2 dx}\sqrt{\int_0^1 x^{10} dx}} = \frac{1/7}{\sqrt{1/3}\sqrt{1/11}} = \frac{1/7}{\sqrt{1/33}} = \frac{\sqrt{33}}{7}$; $\boxed{\theta = \cos^{-1}\left(\frac{\sqrt{33}}{7}\right)}$

(ii) $\cos\theta = \frac{\int_0^1 x^6 dx}{\sqrt{\int_0^1 x^4 dx}\sqrt{\int_0^1 x^8 dx}} = \frac{1/7}{\sqrt{1/5}\sqrt{1/9}} = \frac{3\sqrt{5}}{7}$; $\boxed{\theta = \cos^{-1}\left(\frac{3\sqrt{5}}{7}\right)}$

3.8

(i) *Proof.* Let V be the inner product space $C([-\pi,\pi];\mathbb{R})$ with inner product

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt$$

Let $X = \operatorname{span}(S) \subset V$, where $S = \{\cos(t), \sin(t), \cos(2t), \sin(2t)\}$. Observe that

$$\begin{aligned}
\langle \cos(t), \sin(t)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\sin(t)dt = 0 \\
\langle \cos(t), \cos(2t)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\cos(2t)dt = 0 \\
\langle \cos(t), \sin(2t)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\sin(2t)dt = 0 \\
\langle \sin(t), \cos(2t)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(t)\cos(2t)dt = 0 \\
\langle \sin(t), \sin(2t)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(t)\sin(2t)dt = 0 \\
\langle \cos(2t), \sin(2t)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(2t)\sin(2t)dt = 0
\end{aligned}$$

Thus, each function is orthogonal to each other. Now, observe that

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\cos(t)dt &= 1 \\
\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(t)\sin(t)dt &= 1 \\
\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(2t)\cos(2t)dt &= 1 \\
\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(2t)\sin(2t)dt &= 1
\end{aligned}$$

Therefore, S is an orthonormal set. $\square$

(ii) $||t|| = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{1}{\pi}(\frac{2\pi^3}{3}) = \frac{2\pi^2}{3}$

(iii) Recall from part (i) that each component of S is orthogonal to one another. Using this fact, $\text{proj}_X(\cos(3t)) = 0$

(iv) Note that $\langle \cos(t), t\rangle = 0$ and $\langle \cos(2t), t\rangle = 0$.
Now, $\langle \sin(t), t\rangle \sin(t) = 2\sin(t)$ and $\langle \sin(2t), t\rangle \sin(2t) = -\sin(2t)$
Take the sum of each of these yields $\boxed{\text{proj}_X(t) = 2\sin(t) - \sin(2t)}$

3.9

Proof. $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ Now observe that

$$\begin{aligned} R_\theta^H R_\theta &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

By Theorem 3.2.15, the rotation is an orthonormal transformation. $\square$

3.10

(i) Let $Q \in M_n(\mathbb{F})$ be an orthonormal matrix with $x \in \mathbb{F}^n$. Observe that

$$\begin{aligned} \langle x, x\rangle &= \langle Qx, Qx\rangle \\ &= x^H Q^H Q x \\ &= \langle x, Q^H Q x\rangle \end{aligned}$$

and hence $x = Q^H Q x$. Thus $Q^H Q = I$, which implies that $Q^{-1} = Q^H$ since Q is square. Thus $QQ^H = I$.

Now let $Q^H Q = I$ and let $x, y \in \mathbb{F}^n$. Observe that

$$\begin{aligned}\langle Qx, Qy\rangle &= x^H Q^H Q y \\ &= x^H y \\ &= \langle x, y\rangle\end{aligned}$$

Therefore Q is an orthonormal matrix.

(ii) *Proof.* Let $Q \in M_n(\mathbb{F})$ be an orthonormal matrix. Observe that

$$||x||^2 = \langle x, x\rangle = \langle Qx, Qx\rangle = ||Qx||^2$$

□

(iii) *Proof.* Let $Q \in M_n(\mathbb{F})$ be an orthonormal matrix. Observe that

$$\begin{aligned}QQ^H = Q^H Q &= I \\ &\Rightarrow Q^H = Q^{-1}\end{aligned}$$

Now observe that

$$\begin{aligned}(Q^H)^H &= Q \\ (Q^H)(Q^H)^H &= (Q)^H Q = I\end{aligned}$$

So $Q^H = Q^{-1}$ is an orthonormal matrix. □

(iv) Let $Q \in M_n(\mathbb{F})$ be an orthonormal matrix. We know from part(i) that

$Q^H Q = I$. Observe that

$$
\begin{aligned}
I &= \begin{bmatrix} q_1^H \\ q_2^H \\ \vdots \\ q_n^H \end{bmatrix} [q_1 q_2 ... q_n] \\
&= \begin{bmatrix} q_1^H q_1 & q_1^H q_2 & ... & q_1^H q_n \\ q_2^H q_1 & q_2^H q_2 & ... & q_2^H q_n \\ \vdots & \vdots & \ddots & \vdots \\ q_n^H q_1 & q_n^H q_2 & ... & q_n^H q_n \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 1 \end{bmatrix}
\end{aligned}
$$

Hence $q_i^H q_i = 1$ and $q_i^H q_j = 0$ for all $i \neq j$. Therefore, the columns of an orthonormal matrix $Q \in M_n(\mathbb{F})$ are orthonormal.

(v) Let $Q \in M_n(\mathbb{F})$ be an orthonormal matrix. We know from part (i) that $Q^H Q = I$. Furthermore, $\det(Q^H) = \det(Q)$, also $\det(Q^H Q) = \det(Q^H)\det(Q)$. Observe that

$$
\begin{aligned}
1 &= \det(I) \\
&= \det(Q^H Q) \\
&= \det(Q^H)\det(Q) \\
&= (\det(Q))
\end{aligned}
$$

That is $|\det(Q)| = 1$.
Now Let $Q = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ so $Q^T = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. Note that $\det(Q) = 1$. Observe that

$$
\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \neq I
$$

and Q is not orthonormal

(vi) Let $Q_1, Q_2 \in M_n(\mathbb{F})$ be orthonormal matrices. Hence, $Q_1^H Q_1 = I$ and $Q_2^H Q_2 = I$. Let $Q = Q_1 Q_2$. Observe that

$$\begin{aligned} Q^H Q &= (Q_1 Q_2)^H (Q_1 Q_2) \\ &= Q_2^H Q_1^H Q_1 Q_2 \\ &= Q_2^H Q_2 \\ &= I \end{aligned}$$

Therefore, $Q_1 Q_2$ is an orthonormal matrix.

3.11 When we apply the Gram-Schmidt orthonormalization process to a collection of linearly *dependent* vectors then the kth step will produce $\mathbf{0}$ since $\mathbf{x}_k$ is a linear combination of $\mathbf{x}_1, ..., \mathbf{x}_{k-1}$. Note also that by producing the zero vector or any multiple of it, the length of it will not be equal to one, which is a necessary condition to be orthonormal.

3.16

(i) *Proof.* Let $D_{n \times n}$ be a diagonal matrix with 1's along the diagonal, and where the entry of the last row and column is not 1. Note that $D = D^{-1}$, and let the QR decomposition of A be denoted by $A = QR$. It follows that $A = QR = QDD^{-1}R = \widetilde{Q}\widetilde{R}$. By exercise 3.10, it must be that $\widetilde{Q} = QD$ is orthonormal. Now, it is also the case that $\widetilde{R} = D^{-1}R$ is upper diagonal. Thus, we have that $A = \widetilde{Q}\widetilde{R}$ is a QR decomposition. Therefore, the QR decomposition is not unique. □

(ii) *Proof.* Let A be an invertible matrix. Now, suppose that there are two distinct QR decompositions for A, namely $A = QR = \hat{Q}\hat{R}$. Note that Q and $\hat{Q}$ are orthonormal, and that R and $\hat{R}$ have positive diagonal elements. By our assumption, it must be that $Q, \hat{Q}, R$, and $\hat{R}$ are all invertible. Thus $(\hat{Q})^{-1}Q = \hat{R}R^{-1}$. Now, on the LHS we have $(\hat{Q})^{-1}$ which is orthonormal by exercise 3.10 and $(\hat{Q})^{-1}Q$ is orthonormal. Given that $\hat{R}R^{-1}$ is upper triangular, then we have that $(\hat{Q})^{-1}Q = I = \hat{R}R^{-1}$. This, $\hat{Q} = Q$ and $\hat{R} = R$ and the QR decomposition is unique. □

3.17

Proof. Let $A \in M_{m\times n}$ have rank $n \leq m$, and let $A = \hat{Q}\hat{R}$ be a reduced QR decomposition. Note that $\hat{Q}$ is an $m \times n$ orthonormal matrix and $\hat{R}$ is an $n \times n$ upper-triangular matrix (see Remark 3.3.10). Observe that

$$\begin{aligned}
A^H A\mathbf{x} &= A^H\mathbf{b} \\
(\hat{Q}\hat{R})^H\hat{Q}\hat{R}\mathbf{x} &= (\hat{Q}\hat{R})^H\mathbf{b} \\
\hat{R}^H\hat{Q}^H\hat{Q}\hat{R}\mathbf{x} &= \hat{R}^H\hat{Q}^H\mathbf{b} \\
\hat{R}^H\hat{R}\mathbf{x} &= \hat{R}^H\hat{Q}^H\mathbf{b} \\
\hat{R}\mathbf{x} &= \hat{Q}^H\mathbf{b}
\end{aligned}$$

□

3.23 Let $(V, ||\cdot||)$ be a normed linear space. Observe that

$$\begin{aligned}
||x-y+y|| &\leq ||x-y|| + ||y|| \\
||x|| &\leq ||x-y|| + ||y|| \\
||x|| - ||y|| &\leq ||x-y||
\end{aligned}$$

Also observe that

$$\begin{aligned}
||y-x+x|| &\leq ||y-x|| + ||x|| \\
||y|| &\leq ||x-y|| + ||x|| \\
||y|| - ||x|| &\leq ||x-y||
\end{aligned}$$

Therefore, $\big|||x| - ||y|||\big| \leq ||x-y||$

3.24 Let $C([a,b];\mathbb{F})$ be the vector space of all continuous functions from $[a,b] \subset \mathbb{R}$ to $\mathbb{F}$.

(i) $||f||_{L^1} = \int_a^b |f(t)|dt$

- positivity: $|f(t)| \geq 0$ by definition. If $f(t) = 0$, then $|f(t) = 0$ and if $f(t) \neq 0$ then $|f(t) \neq 0$. Thus $\int_a^b |f(t)|dt$ is positive.
- scale preservation: Let $\alpha \in \mathbb{R}$ be a scalar. Note that $||\alpha f(t)||_{L^1} = \int_a^b |\alpha f(t)|dt = \int_a^b |\alpha||f(t)|dt = |\alpha| \int_a^b |f(t)|dt$

- triangle inequality:

$$\begin{aligned}
||f+g||_{L^1} &= \int_a^b |f(t)+g(t)|dt \\
&\leq \int_a^b |f(t)|+|g(t)|dt \\
&= \int_a^b |f(t)|dt + \int_a^b |g(t)|dt
\end{aligned}$$

Therefore, $||f||_{L^1}$ is a norm on $C([a,b];\mathbb{F})$.

(ii) $||f||_{L^2} = \left(\int_a^b |f(t)|^2 dt\right)^{\frac{1}{2}}$

- positivity: $|f(t)|$ is positive (see exercise 3.24 (i)). It follows that $|f(t)|^2$ is also positive. Hence $\int_a^b |f(t)|^2 dt$ is positive. Now, if $\int_a^b |f(t)|^2 dt = 0$ then $\left(\int_a^b |f(t)|^2 dt\right)^{\frac{1}{2}} = 0$. But if $\int_a^b |f(t)|^2 dt \neq 0$ then $\left(\int_a^b |f(t)|^2 dt\right)^{\frac{1}{2}} \neq 0$ and so $\left(\int_a^b |f(t)|^2 dt\right)^{\frac{1}{2}}$ is positive.
- scalar preservation: For some scalar $\alpha \in \mathbb{R}$ observe that

$$\begin{aligned}
||\alpha f||_{L^2} &= \left(\int_a^b |\alpha f(t)|^2 dt\right)^{\frac{1}{2}} \\
&= \left(\int_a^b |\alpha|^2 |f(t)|^2 dt\right)^{\frac{1}{2}} \\
&= \left(|\alpha|^2 \int_a^b |f(t)|^2 dt\right)^{\frac{1}{2}} \\
&= |\alpha| \left(\int_a^b |f(t)|^2 dt\right)^{\frac{1}{2}}
\end{aligned}$$

- triangle inequality: $||f+g||_{L^2} = \left(\int_a^b |f(t)+g(t)|^2\right)^{\frac{1}{2}}$. Observe that

$$\begin{aligned}(||f+g||_{L^2})^2 &= \int_a^b |f(t)+g(t)|^2 dt \\ &= \int_a^b |f(t)|^2 + 2|f(t)g(t)| + |g(t)|^2 dt \\ &= ||f||_{L^2} + 2|f(t)g(t)| + ||g||_{L^2} \\ &\leq ||f||_{L^2} + 2||f||_{L^2}||g||_{L^2} + ||g||_{L^2} \\ &= (||f||_{L^2} + ||g||_{L^2})^2\end{aligned}$$

  Hence $||f+g||_{L^2} \leq ||f||_{L^2} + ||g||_{L^2}$

Therefore $||f||_{L^2}$ is a norm on $C([a,b];\mathbb{F})$.

(iii) $||f||_{L^\infty} = \sup_{x\in[a,b]} |f(x)|$

- positivity: $|f(x)| \geq 0$ by definition. If $f(x) = 0$ then $|f(x)| = 0$, and if $f(x) \neq 0$ then $|f(x) \neq 0$. Thus $\sup_{x\in[a,b]} |f(x)|$ is positive.
- scalar preservation: For some scalar $\alpha \in \mathbb{R}$ observe that

$$\begin{aligned}||\alpha f||_{L^\infty} &= \sup_{x\in[a,b]} |\alpha f(x)| \\ &= \sup_{x\in[a,b]} |\alpha||f(x)| \\ &= |\alpha| \sup_{x\in[a,b]} |f(x)|\end{aligned}$$

- triangle inequality: Observe that

$$\begin{aligned}||f+g||_{L^\infty} &= \sup_{x\in[a,b]} |f(x)+g(x)| \\ &\leq \sup_{x\in[a,b]} (|f(x)| + |g(x)|) \\ &= \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |g(x)|\end{aligned}$$

Therefore, $||f||_{L^\infty}$ is a norm on $C([a,b]);\mathbb{F}$.

3.26 Reflexive: Note that $||x||_a = ||x||_a$. Given $0 \leq m \leq M$ we have $m||x||_a \leq ||x||_a \leq M||x||_a$ where $||x||_a \leq ||x||_a \leq ||x||_a$ for $m, M = 1$. Thus, $||x||_a \sim ||x||_a$.
Symmetric: Suppose $||x||_a \sim ||x||_b$. There exists $m, M \in R$ with $0 < m \leq M$ such that $m||x||_a \leq ||x||_b \leq M||x||_a$. Hence $\frac{1}{M}||x||_b \leq ||x||_a \leq \frac{1}{m}||x||_b$. Thus $||x||_b \sim ||x||_a$
Transitive: Suppose $||x||_a \sim ||x||_b$ and $||x||_b \sim ||x||_c$. There exists $m_1, M_1, m_2, M_2 \in \mathbb{R}$ with $0 < m_1 \leq M_1$ and $0 < m_2 \leq M_2$ such that $m_1||x||_a \leq ||x||_b \leq M_1||x||_a$ and $m_2||x||_b \leq ||x||_c \leq M_2||x||_b$. Now $m_2||x||_b \leq ||x||_c$ so $m_1m_2||x||_a \leq ||x||_c$ and $||x||_c \leq M_2||x||_b$. So $||x||_c \leq M_1M_2||x||_a$. Putting it all together we have $m_1m_2||x||_a \leq ||x||_c \leq M_1M_2||x||_a$. Thus $||x||_a \sim ||x||_c$.
Therefore, topological equivalence is an equivalence relation.

(i)

$$|x_1|^2 + |x_2|^2 + .. + |x_n|^2 \leq (|x_1| + |x_2| + ... + |x_n|)^2$$
$$(|x_1|^2 + |x_2|^2 + .. + |x_n|^2)^{\frac{1}{2}} \leq |x_1| + |x_2| + ... + |x_n|$$
$$||x||_2 \leq ||x||_1$$

Now, $|\langle x, \mathbf{1}\rangle| \leq ||x||_2||\mathbf{1}||_2$ (Cauchy-Schwartz) where
$||\mathbf{1}||_2 = (1^2 + 1^2 + ... + 1^2)^{\frac{1}{2}} = \sqrt{n}$. So, $||x||_1 \leq \sqrt{n}||x||_2$.
Therefore, $||x||_2 \leq ||x||_1 \leq \sqrt{n}||x||_2$

(ii) Let $\in \mathbb{F}^n$. Now WLOG assume that for $1 \leq k \leq n$ we have that $|x_k| = ||x||_\infty$. Now, $||x||_\infty = |x_k| = (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}} = ||x||_2$. It follows that $||x||_\infty \leq ||x||_2$.
Now, $(||x||_2)^2 \sum_{i=1}^n \leq n|x_k|^2$, and hence we have that $||x||_2 \leq \sqrt{n}|x_k| = \sqrt{n}||x||_\infty$. Putting what we have altogether, we have that $||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty$

3.28 Let A be an $n \times n$ matrix.

(i) *Proof.* By exercise 3.26 observe that

$$\frac{1}{\sqrt{n}||x||_2} \leq \frac{1}{||x||_1} \leq \frac{1}{||x||_2}$$

Furthermore, we have that

$$||Ax||_2 \leq ||Ax||_1 \leq \sqrt{n}||Ax||_2$$

Hence, it follows that

$$||A||_1 \geq \frac{||Ax||_1}{||x||_1} \geq \frac{||Ax||_2}{\sqrt{n}||x||_2}$$

by definition. It follows that

$$||A||_1 \geq \frac{1}{\sqrt{n}}||A||_2$$

Now, observe that

$$\sqrt{n}||A||_2 \geq \frac{\sqrt{n}||Ax||_2}{||x||_2} \geq \frac{||Ax||_1}{||x||_1}$$

by definition. It follows that

$$\sqrt{n}||A||_2 \geq ||A||_1$$

Now, by combining what we have derived, we have that

$$\frac{1}{\sqrt{n}}||A||_2 \leq ||A||_1 \leq \sqrt{n}||A||_2$$

□

(ii) *Proof.* By exercise 3.26 observe that

$$\frac{1}{\sqrt{n}||x||_\infty} \leq \frac{1}{||x||_2} \leq \frac{1}{||x||_\infty}$$

Furthermore, we have that

$$||Ax||_\infty \leq ||Ax||_2 \leq \sqrt{n}||Ax||_\infty$$

Hence, it follows that

$$||A||_2 \geq \frac{||Ax||_2}{||x||_2} \geq \frac{||Ax||_\infty}{\sqrt{n}||x||_\infty}$$

by definition. It follows that

$$||A||_2 \geq \frac{1}{\sqrt{n}}||A||_\infty$$

Now, observe that

$$\sqrt{n}||A||_\infty \geq \frac{\sqrt{n}||Ax||_\infty}{||x||_\infty} \geq \frac{||Ax||_2}{||x||_2}$$

by definition. It follows that

$$\sqrt{n}||A||_\infty \geq ||A||_2$$

Now, by combining what we have derived, we have that

$$\frac{1}{\sqrt{n}}||A||_\infty \leq ||A||_2 \leq \sqrt{n}||A||_\infty$$

$\square$

3.29

Proof. Let $Q \in M_n(\mathbb{F})$ be an orthonormal matrix. Observe that

$$\begin{aligned} ||Q|| &= \sup_{x \neq 0} \frac{||Qx||_2}{||x||_2} \\ &= \sup_{x \neq 0} \frac{||x||_2}{||x||_2} \\ &= 1 \end{aligned}$$

as desired.
Now, Let $x \in \mathbb{F}^n$. Observe that $||R_x|| = \sup_{A:||A|| \neq 0} \left(\frac{||Ax||_2}{||A||} \right)$. But $||A|| =$

$\sup_{y\neq 0}\left(\frac{||Ay||_2}{||y||_2}\right) \geq \frac{||Ax||_2}{||x||_2}$. So, it follows that $||R_x|| \leq \sup_{A:||A||\neq 0}\left(\frac{||Ax||_2||x||_2}{||Ax||_2}\right) = ||x||_2$. Given A orthonormal, then this actually holds with equality. Hence $||R_x|| = ||x_2||$ since $||Ax||_2 = ||x||_2$ and $||A|| = 1$ □

3.30

Proof.

- Positivity: Let $A \in M_n(\mathbb{F})$. Since $||\cdot||$ is a matrix norm, it follows that $||A||_S = ||SAS^{-1}|| \geq 0$. Now, since $||\cdot||$ is a matrix norm, it also follows that $||A||_S = ||SAS^{-1}|| = 0$ iff $SAS^{-1} = 0$. But given that S is invertible implies that $A = 0$.

- Scale Preservation: Let $\alpha \in \mathbb{R}$. Given that $||\cdot||$ is a matrix norm, observe that

$$\begin{aligned}||\alpha A||_S &= ||\alpha SAS^{-1}|| \\ &= \alpha||SAS^{-1}|| \\ &= \alpha||A||_S\end{aligned}$$

- Let $B \in M_n(\mathbb{F})$. Given that $||\cdot||$ is a matrix norm, observe that

$$\begin{aligned}||A+B||_s &= ||S(A+B)S^{-1}|| \\ &= ||SAS^{-1} + SBS^{-1}|| \\ &\leq ||SAS^{-1}|| + ||SBS^{-1}|| \\ &= ||A||_S + ||B||_S\end{aligned}$$

By definition 3.5.15, any norm $||\cdot||$ on the finite-dimensional vector space $M_n(\mathbb{F})$ that satisfies the submultiplicative property is called a *matrix norm*. Observe that

$$\begin{aligned}||AB||_S &= ||SABS^{-1}|| \\ &= ||SAS^{-1}SBS^{-1}|| \\ &\leq ||SAS^{-1}||||SBS^{-1}|| \\ &= ||A||_S||B||_S\end{aligned}$$

Thus, $||AB||_S \leq ||A||_S||B||_S$, and the submultiplicative property is satisfied. Therefore by definition 3.5.15, $||\cdot||_S$ is a matrix norm on M_n $\square$

3.37 Let $p \in V$ such that $p = ax^2 + bx + c$. We can express this as a vector in $\mathbb{R}^3$, namely (a, b, c). Now, we need to find a unique $q \in V$ such that $L[p] = p'(1) = p'q = 2a + b$. Thus, $q = (2, 1, 0)$

3.38 Let $p \in V$ such that $p = ax^2 + bx + c$. Given that $p = (a, b, c)^T$, and $p' = D(p) = (0, 2a, b)^T$, it follows that $D = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, of which the Hermitian is represented by $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

3.39

Proof. let V and W be finite-dimensional inner-product spaces.

(i) Let $S, T \in \mathscr{L}(V; W)$. Observe that

$$\langle (S+T)(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, (S+T)^*(\mathbf{w}) \rangle$$

which follows from the definition of adjoint. Now,

$$\begin{aligned} \langle (S+T)(\mathbf{v}), \mathbf{w} \rangle &= \langle S(\mathbf{v}) + T(\mathbf{v}), \mathbf{w} \rangle \\ &= \langle S(\mathbf{v}), \mathbf{w} \rangle + \langle T(\mathbf{v}), \mathbf{w} \rangle \\ &= \langle \mathbf{v}, S^*(\mathbf{w}) \rangle + \langle \mathbf{v}, T^*(\mathbf{w}) \rangle \\ &= \langle \mathbf{v}, (S^* + T^*)(\mathbf{w}) \rangle \end{aligned}$$

Hence, $(S+T)^* = S^* + T^*$ Now, observe that for $\alpha \in \mathbb{F}$

$$\langle (\alpha T)(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, (\alpha T)^*(\mathbf{w}) \rangle$$

Additionally,

$$\begin{aligned}\langle(\alpha T)(\mathbf{v}),\mathbf{w}\rangle &= \langle \alpha T(\mathbf{v}),\mathbf{w}\rangle\\ &= \alpha\langle \mathbf{v}, T^*(\mathbf{w})\rangle\\ &= \langle \mathbf{v}, \bar{\alpha}T^*(\mathbf{w})\rangle\end{aligned}$$

Hence, $(\alpha T^*) = \bar{\alpha}T^*$

(ii) Let $S \in \mathscr{L}(V;W)$. Observe that

$$\langle (S^*)^*(\mathbf{v}),\mathbf{w}\rangle = \langle \mathbf{v}, (S^*)^*(\mathbf{w})\rangle$$

Additionally,

$$\begin{aligned}\langle S^*(\mathbf{w}),\mathbf{v}\rangle &= \overline{\langle \mathbf{v}, S^*(\mathbf{w})\rangle}\\ &= \overline{\langle S(\mathbf{v}),\mathbf{w}\rangle}\\ &= \langle \mathbf{w}, S(\mathbf{v})\rangle\end{aligned}$$

Hence, $(T^*)^* = T$

(iii) Let $S, T \in \mathscr{L}(V)$. Observe that

$$\langle (ST)(\mathbf{v}),\mathbf{w}\rangle = \langle \mathbf{v}, (ST)^*(\mathbf{w})\rangle$$

Additionally,

$$\begin{aligned}\langle (ST)(\mathbf{v}),\mathbf{w}\rangle &= \langle S(T(\mathbf{v})),\mathbf{w}\rangle\\ &= \langle T(\mathbf{v}), S^*(\mathbf{w})\rangle\\ &= \langle \mathbf{v}, T^*(S^*(\mathbf{w})\rangle\\ &= \langle \mathbf{v}, (T^*S^*)(\mathbf{w})\rangle\end{aligned}$$

Hence, $(ST)^* = T^*S^*$

(iv) Let $T \in \mathscr{L}(V)$ and suppose T is invertible. Observe that

$$\begin{aligned}(T^*)^{-1}T^* &= I \\ ((T^*)^{-1}T^*)^* &= I^*\end{aligned}$$

But $I^* = I$. From property (iii) we have that $T^{**}((T^*)^{-1})^* = I$. Note that $T^{**} = T$, so we have that

$$\begin{aligned}T((T^*)^{-1})^* &= I \\ ((T^*)^{-1})^* &= T^{-1} \\ (T^*)^{-1} &= (T^{-1})^*\end{aligned}$$

□

3.40 Let $M_n(\mathbb{F})$ be endowed with the Frobenius inner product. We are given that any $A \in M_n(\mathbb{F})$ defines a linear operator on $M_n(\mathbb{F})$ by left multiplication.

(i) *Proof.* Let $B, C \in M_n(\mathbb{F})$. Observe that

$$\begin{aligned}\langle B, AC\rangle &= \langle \mathrm{tr}(B^H AC)\rangle \\ &= \mathrm{tr}((A^H B)^H C) \\ &= \langle A^H B, C\rangle\end{aligned}$$

Thus, $A^* = A^H$ □

(ii) *Proof.* Let $A_1, A_2, A_3 \in M_n(\mathbb{F})$. Observe that

$$\begin{aligned}\langle A_2, A_3A_1\rangle &= \mathrm{tr}(A_2^H A_3 A_1) \\ &= \mathrm{tr}(A_1 A_2^H) \\ &= \mathrm{tr}((A_2 A_1^H)^H A_3) \\ &= \langle A_2 A_1^H, A_3\rangle\end{aligned}$$

From part(i), $A_1^H = A_1^*$. Thus, $\langle A_2, A_3A_1\rangle = \langle A_2A_1^*, A_3\rangle$ □

(iii) *Proof.* Let $A, B, C \in M_n(\mathbb{F})$. Given the linear operator definition provided in the exercise, we have that

$$\langle B, AC - CA \rangle = \langle B, AC \rangle - \langle B, CA \rangle$$

By part (ii) we have that

$$\langle B, CA \rangle = \langle BA^*, c \rangle$$

Note also that

$$\begin{aligned}
\langle B, AC \rangle &= \operatorname{tr}(B^H AC) \\
&= \operatorname{tr}((A^H B)^H C) \\
&= \langle A^H B, C \rangle \\
&= \langle A^* B, C \rangle
\end{aligned}$$

Therefore, all of this together means that we have $(T_A)^* = T_{A^*}$ □

3.44

Proof. Let $A \in M_{m\times n}(\mathbb{F})$ and $\mathbf{b} \in \mathbb{F}^m$. First, observe that the *Fredholm alternative* is equivalent to $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} \in \mathbb{F}$ iff for all $y \in \mathscr{N}(A^H)$ we have that $\langle \mathbf{y}, \mathbf{b} \rangle = 0$. Now, assume $A\mathbf{x} = \mathbf{b}$ has a solution $x \in \mathbb{F}$, and let $y \in \mathscr{N}(A^H)$. Observe that

$$\begin{aligned}
\langle \mathbf{y}, \mathbf{b} \rangle &= \langle \mathbf{y}, A\mathbf{x} \rangle \\
&= \langle A^H \mathbf{y}, \mathbf{x} \rangle \\
&= \langle 0, \mathbf{x} \rangle \\
&= 0
\end{aligned}$$

which makes sense since $\mathbf{y} \in \mathscr{N}(A^H) \Rightarrow A^H\mathbf{y} = 0$.
Now, assume that $\langle \mathbf{y}, \mathbf{b} \rangle = 0$ for all $\mathbf{y} \in \mathscr{N}(A^H)$. Furthermore, assume by way of contradiction that there is no solution for $A\mathbf{x} = \mathbf{b}$. Hence $\mathbf{b} \notin \mathscr{R}(A) \Rightarrow \mathbf{b} \in \mathscr{R}(A^H)$. So $\langle \mathbf{b}, \mathbf{b} \rangle = 0$, which happens when $\mathbf{b} = 0$. Now, for $A\mathbf{x} = \mathbf{b}$ and $\mathbf{b} = 0$, then $\mathbf{x} = 0$. Then by contradiction, $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} \in \mathbb{F}$. □

3.45

Proof. Consider the vector space $M_n(\mathbb{R})$ with the Frobenius inner product. Let $A \in \text{Skew}_n(\mathbb{R})$, and let $B \in \text{Sym}_n(\mathbb{R})$. Observe that

$$\begin{aligned}
\langle A, B \rangle &= \langle -A, B \rangle \\
&= -\langle A, B \rangle \\
&= \text{tr}(AB)
\end{aligned}$$

Furthermore, observe that

$$\begin{aligned}
\langle A, B \rangle &= \langle B, A \rangle \\
&= \text{tr}(B^T A) \\
&= \text{tr}(BA) \\
&= \text{tr}(AB)
\end{aligned}$$

with $B^T = B$ since B is symmetric. Note that this follows from the fact that inner products are symmetric with respect to the reals. Now, we have that

$$-\text{tr}(AB) = \text{tr}(AB)$$

and hence it must be that $\text{tr}(AB) = 0$. It follows that $\langle A, B \rangle = 0$ hence $A \in \text{Sym}_n(\mathbb{R})^\perp$.
Now let $A \in \text{Sym}_n(\mathbb{R})^\perp$ and also let $B \in \text{Sym}_n(\mathbb{R})$. So $\text{tr}(A^T B) = 0$. Suppose we have $A + A^T \in \text{Sym}_n(\mathbb{R})$ which follows from $(A + A^T)^T = (A + A^T)$.

Observe that

$$\begin{aligned}\langle A + A^T, B\rangle &= \langle A, B\rangle \\ &= \langle A^T, B\rangle \\ &= \mathrm{tr}(A^T B) + \mathrm{tr}(AB) \\ &= \mathrm{tr}(AB) \\ &= \mathrm{tr}(AB^T) \\ &= \mathrm{tr}(B^T A) \\ &= \mathrm{tr}((A^T B)^T) \\ &= \mathrm{tr}(A^T B) \\ &= 0\end{aligned}$$

But if we have that $B = A + A^T$ it must be that $\langle A + A^T, , A + A^T\rangle \geq 0$ from the positivity definition of inner products. We showed that this holds as a strict equality being equal to 0. That is, $A + A^T = 0$ and so $A^T = -A$. Hence, $A \in \mathrm{Skew}_n(\mathbb{R})$. Therefore, $\mathrm{Skew}_n(\mathbb{R})^\perp = \mathrm{Skew}_n(\mathbb{R})$ □

3.46

(i) Let $\mathbf{x} \in \mathscr{N}(A^H A)$. Observe that

$$\begin{aligned}A^H A\mathbf{x} &= A^H(A\mathbf{x}) \\ &= 0\end{aligned}$$

and $A\mathbf{x} \in \mathscr{N}(A^H)$. Now $\mathscr{R}(A)$ is the set of all possible linear combinations of the columns of the matrix A, and so $A\mathbf{x} \in \mathscr{R}(A)$.

(ii) Let $\mathbf{x} \in \mathscr{N}(A)$, hence $A\mathbf{x} = 0$. It follows that $A^H A\mathbf{x} = 0$ and $\mathbf{x} \in \mathscr{N}(A^H A)$.
Now, let $\mathbf{x} \in \mathscr{N}(A^H A)$, hence $A^H A\mathbf{x} = 0$. Suppose we have $\langle A\mathbf{x}, A\mathbf{x}\rangle = \mathbf{x}^H A^H A\mathbf{x} = 0$. Thus $||A\mathbf{x}|| = 0$ and so $A\mathbf{x} = 0$.

(iii) Suppose A has dimension j and rank n. It follows that $\mathscr{N}(A)$ has dimension $j - n$. From part (ii), we know that $\mathscr{N}(A^H A) = \mathscr{N}(A)$ and so it must be that $\mathscr{N}(A^HA)$ has dimension $j-n$. Thus A^HA has rank n as well. Therefore A and A^HA have that same rank.

(iv) Let A have linear independent columns. This implies that A has full rank. We know that A^HA is square, and by part (iii) A and A^HA have the same rank. Therefore, A^HA is nonsingular.

3.47 Assume A is an $m \times n$ matrix of rank n. Let $P = A(A^HA)^{-1}A^H$

(i)

$$\begin{aligned} P^2 &= A(A^HA)^{-1}A^HA(A^HA)^{-1}A^H && (0.1) \\ &= A(A^HA)^{-1}A^H && (0.2) \\ & && (0.3) \end{aligned}$$

(ii)

$$\begin{aligned} P^H &= (A(A^HA)^{-1}A^H)^H \\ &= A((A^HA)^{-1})^HA^H \\ &= A((A^HA)^H)^{-1}A^H \\ &= A(A^HA)^{-1}A^H \\ &= P \end{aligned}$$

(iii)

$$\text{rank}(P) = \text{rank}(A(A^HA)^{-1}A^H)$$

Now because P is idempotent by (i) we have that

$$\begin{aligned} \text{tr}(A(A^HA)^{-1}A^H) &= \text{tr}(A^HA(A^HA)^{-1}) \\ &= \text{tr}(I) \end{aligned}$$

where I is $n \times n$, so it has rank n. Therefore, rank(P)=n.

3.48 Let $P(A) = \frac{A+A^T}{2}$ be the map $P : M_n(\mathbb{R}) \to M_n(\mathbb{R})$

(i) Let $A, b \in M_n(\mathbb{R})$ with scalars $x, y \in \mathbb{R}$. Observe that

$$\begin{aligned} P(xA+yB) &= \frac{(xA+yB)+(xA+yB)^T}{2} \\ &= x\frac{A+A^T}{2} + y\frac{B+B^T}{2} \\ &= xP(A)+yP(B) \end{aligned}$$

and P is linear as desired.

(ii)

$$\begin{aligned} P^2(A) &= \frac{P(A)+P(A)^T}{2} \\ &= \frac{\frac{A+A^T}{2}+\frac{A+A^T}{2}}{2} \\ &= \frac{A+A^T}{2} \\ &= P(A) \end{aligned}$$

and $P^2 = P$ as desired.

(iii) We know that P^* must satisfy $\langle A, P(B)\rangle = \langle P^*(A), B\rangle$ for all $A, B \in M_n(\mathbb{R})$. Now suppose $A = B$. Observe that

$$\begin{aligned} \langle A, P(A)\rangle &= \langle P^*A, A\rangle \\ &= \operatorname{tr}((P^*(A))^T A) \\ &= \operatorname{tr}((A^T P^*(A))^T) \\ &= \operatorname{tr}(A^T P^*(A)) \\ &= \langle A, P^*(A)\rangle \end{aligned}$$

and $P = P^*$ as desired.

(iv) Let $A \in \mathscr{N}(P)$. Hence $P(A)A = 0$. Observe that

$$\begin{aligned} 0 &= \frac{A + A^T}{2} A \\ &= \frac{AA + A^T A}{2} \end{aligned}$$

Hence $AA = -A^T A$. Thus $-A = A^T$ and so $A \in \text{Skew}_n(\mathbb{R})$.
Now suppose $A \in \text{skew}_n(\mathbb{R})$. Hence $A^T = -A$. Observe that

$$\begin{aligned} P(A)A &= \frac{A + A^T}{2} A \\ &= \frac{AA + A^T A}{2} \\ &= \frac{AA - AA}{2} \\ &= 0 \end{aligned}$$

and $A \in \mathscr{N}(P)$ and so $\mathscr{N}(P) = \text{Skew}_n(\mathbb{R})$.

(v) Let $A \in \text{Sym}_n(\mathbb{R})$, hence $A^T = A$. Observe that

$$\begin{aligned} P(A) &= \frac{A + A^T}{2} \\ &= \frac{2A}{A} \\ &= A \end{aligned}$$

and $A \in \mathscr{R}(P)$.
Now let $A \in \mathscr{R}(P)$. So there is some $B \in M_n(\mathbb{R})$ where $P(B) = A$. That is $\frac{B+B^T}{2} = A$. Observe that

$$\begin{aligned} A^T &= \left(\frac{B + B^T}{2}\right)^T \\ &= \frac{B + B^T}{2} \\ &= A \end{aligned}$$

and $A^T = A$. Thus $A \in \mathrm{Sym}_n(\mathbb{R})$. Therefore, $\mathscr{R}(P) = \mathrm{Sym}_n(\mathbb{R})$.

(vi) Observe that

$$\begin{aligned}
||A - P(A)||_F^2 &= \langle A - P(A), A - P(A) \rangle \\
&= \langle A - \frac{A + A^T}{2}, A - \frac{A + A^T}{2} \rangle \\
&= \langle \frac{A - A^T}{2}, \frac{A - A^T}{2} \rangle \\
&= \mathrm{tr}\left(\left(\frac{A - A^T}{2} \right)^T \left(\frac{A - A^T}{2} \right) \right) \\
&= \mathrm{tr}\left(\frac{A^T - A}{2} \frac{A - A^T}{2} \right) \\
&= \mathrm{tr}\left(\frac{A^T A - A^2 - (A^T)^2 + AA^T}{4} \right) \\
&= \mathrm{tr}\left(\frac{A^T A - A^2 - A^2 + A^T A}{4} \right) \\
&= \mathrm{tr}\left(\frac{A^T A - A^2}{2} \right) \\
&= \frac{\mathrm{tr}(A^T A) - \mathrm{tr}(A^2)}{2}
\end{aligned}$$

Taking the square root of both sides yields the desired result.

3.50 $rx^2 + sy^2 = 1 \Leftrightarrow y^2 = \frac{1}{s} - \frac{r}{s}x^2$. Hence

$$A = \begin{bmatrix} 1 & x_1^2 \\ 1 & x_2^2 \\ \vdots & \vdots \\ 1 & x_n^2 \end{bmatrix}$$

$$x = \begin{bmatrix} \frac{1}{s} \\ \frac{-r}{s} \end{bmatrix}$$

$$b = \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_n^2 \end{bmatrix}$$

and solve $A^H A x = A^H b$ for r and s.