8.1 See Jupyter notebook.

8.2 See Jupyter notebook.

8.3

$$\begin{aligned}
\max_{x,y} \quad & 4x + 3y \\
\text{s.t.} \quad & 15x + 10y \leq 1800 \\
& 2x + 2y \leq 350 \\
& y \leq 200 \\
& x, y \geq 0
\end{aligned}$$

8.4

$$\begin{aligned}
\min_{x_{i,j}} \quad & 2x_{AB} + 5x_{AD} + 5x_{BC} + 9x_{BF} + 7x_{BE} + 2x_{BD} + 2x_{CF} + 3x_{EF} + 4x_{DE}, \quad i \neq j \\
\text{s.t.} \quad & x_{AB} + x_{AD} = 10 \\
& x_{BC} + x_{BD} + x_{BE} + x_{BF} - x_{AB} = 1 \\
& x_{CF} - x_{BC} = -2 \\
& x_{DE} - x_{AD} = -3 \\
& x_{EF} - x_{BE} - x_{DE} = 4 \\
& 0 - x_{CF} - x_{BF} - x_{EF} = -10 \\
& 0 \leq x_{i,j} \leq 6
\end{aligned}$$

8.5

(i)

$$\begin{aligned}
\max \quad & 3x_1 + x_2 \\
\text{s.t.} \quad & x_1 + 3x_2 + w_1 = 15 \\
& 2x_1 + 3x_2 + w_2 = 18 \\
& x_1 - x_2 + w_3 = 4 \\
& \mathbf{x}, \mathbf{w} \geq 0
\end{aligned}$$

$$
\begin{array}{ccccccc}
\zeta & = & & & 3x_1 & + & x_2 \\
\hline
w_1 & = & 15 & - & x_1 & - & 3x_2 \\
\hline
w_2 & = & 18 & - & 2x_1 & - & 3x_2 \\
\hline
w_3 & = & 4 & - & x_1 & + & x_2 \\
\hline
\hline
\zeta & = & 12 & + & 4x_2 & - & 3w_3 \\
\hline
w_1 & = & 11 & - & 4x_2 & + & w_3 \\
\hline
w_2 & = & 10 & - & 5x_2 & + & 2w_3 \\
\hline
x_1 & = & 4 & + & x_2 & - & w_3 \\
\hline
\hline
\zeta & = & 20 & - & \frac{4}{5}w_2 & - & \frac{7}{5}w_3 \\
\hline
w_1 & = & 3 & + & \frac{4}{5}w_2 & - & \frac{3}{5}w_3 \\
\hline
x_2 & = & 2 & - & \frac{1}{5}w_2 & + & \frac{2}{5}w_3 \\
\hline
x_1 & = & 6 & - & \frac{1}{5}w_2 & - & \frac{3}{5}w_3 \\
\hline
\end{array}
$$

$(x_1, x_2, w_1, w_2, w_3) = (6, 2, 0, 0, 0)$. The optimum point is $(6, 2)$ with an optimum value of 20.

(ii)

$$
\begin{aligned}
\max \quad & 4x + 6y \\
\text{s.t.} \quad & -x + y + w_1 = 11 \\
& x + y + w_2 = 27 \\
& 2x + 5y + w_3 = 90 \\
& x, y, \mathbf{w} \geq 0
\end{aligned}
$$

ζ	$=$			$4x$	$+$	$6y$
w_1	$=$	11	$+$	x	-	y
w_2	$=$	27	-	x	-	y
w_3	$=$	90	-	$2x$	-	$5y$
ζ	$=$	66	$+$	$10x$	-	$6w_1$
y	$=$	11	$+$	x	-	w_1
w_2	$=$	16	-	$2x$	$+$	w_1
w_3	$=$	35	-	$7x$	$+$	$5w_1$
ζ	$=$	116	$+$	$\frac{8}{7}w_1$	-	$\frac{10}{7}w_3$
y	$=$	16	-	$\frac{2}{7}w_1$	-	$\frac{1}{7}w_3$
w_2	$=$	6	-	$\frac{3}{7}w_1$	$+$	$\frac{2}{7}w_3$
x	$=$	5	$+$	$\frac{5}{7}w_1$	-	$\frac{1}{7}w_3$
ζ	$=$	132	-	$\frac{8}{3}w_2$	-	$\frac{2}{7}w_3$
y	$=$	12	$+$	$\frac{2}{3}w_2$	-	$\frac{1}{7}w_3$
w_1	$=$	14	-	$\frac{7}{3}w_2$	$+$	$\frac{2}{3}w_3$
x	$=$	15	-	$\frac{5}{3}w_2$	$+$	$\frac{1}{3}w_3$

$(x, y, w_1, w_2, w_3) = (15, 12, 0, 0, 0)$. The optimum point is $(15, 12)$ with an optimum value of 132.

8.6

$$\begin{aligned}
\max_{x,y} \quad & 4x + 3y \\
\text{s.t.} \quad & 15x + 10y + w_1 = 1800 \\
& 2x + 2y + w_2 = 350 \\
& y + w_3 = 200 \\
& x, y, \mathbf{w} \geq 0
\end{aligned}$$

ζ	$=$			$4x$	$+$	$3y$
w_1	$=$	1800	-	$15x$	-	$10y$
w_2	$=$	300	-	$2x$	-	$2y$
w_3	$=$	200			-	y
ζ	$=$	450	$+$	x	-	$\frac{3}{2}w_2$
w_1	$=$	300	-	$5x$	$+$	$5w_2$
y	$=$	150	-	x	-	$\frac{1}{2}w_2$
w_3	$=$	50	$+$	x	$+$	$\frac{1}{2}w_2$
ζ	$=$	510	-	$\frac{1}{5}w_1$	-	$\frac{1}{2}w_2$
x	$=$	60	-	$\frac{1}{5}w_1$	$+$	w_2
y	$=$	90	$+$	$\frac{1}{5}w_1$	-	$\frac{3}{2}w_2$
w_3	$=$	110	-	$\frac{1}{5}w_1$	$+$	$\frac{3}{2}w_2$

The optimal choice of each toy to manufacture: $(x, y) = (60, 90)$ where $x :=$ GI Barb soldiers and $y :=$ Joey dolls. The maximal profit is \$510.

8.7

(i)

$$\begin{aligned} \max \quad & -x_0 \\ \text{s.t.} \quad & -4x_1 - 2x_2 + x_3 - x_0 = -8 \\ & -2x_1 + 3x_2 + x_4 - x_0 = 6 \\ & x_1 + x_5 - x_0 = 3 \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

ζ	$=$						-	x_0
x_3	$=$	-8	$+$	$4x_1$	$+$	$2x_2$	$+$	x_0
x_4	$=$	6	$+$	$2x_1$	-	$3x_2$	$+$	x_0
x_5	$=$	3	-	x_1			$+$	x_0
ζ	$=$	-8	$+$	$4x_1$	$+$	$2x_2$	-	x_3
x_0	$=$	8	-	$4x_1$	-	$2x_2$	$+$	x_3
x_4	$=$	14	-	$2x_1$	-	$5x_2$	$+$	x_3
x_5	$=$	11	-	$5x_1$	-	$2x_2$	$+$	x_3
ζ	$=$						-	x_0
x_1	$=$	2	-	$\frac{1}{2}x_2$	$+$	$\frac{1}{4}x_3$	-	$\frac{1}{4}x_0$
x_4	$=$	10	-	$4x_2$	$+$	$\frac{1}{2}x_3$	$+$	$\frac{1}{2}x_0$
x_5	$=$	1	$+$	$\frac{1}{2}x_2$	-	$\frac{1}{4}x_3$	$+$	$\frac{5}{4}x_0$
ζ	$=$	2	$+$	$\frac{3}{2}x_2$	$+$	$\frac{1}{4}x_3$		
x_1	$=$	2	-	$\frac{1}{2}x_2$	$+$	$\frac{1}{4}x_3$		
x_4	$=$	10	-	$4x_2$	$+$	$\frac{1}{2}x_3$		
x_5	$=$	1	$+$	$\frac{1}{2}x_2$	-	$\frac{1}{4}x_3$		
ζ	$=$	3	$+$	$2x_2$	-	x_5		
x_1	$=$	3			-	x_5		
x_4	$=$	12	-	$3x_2$	-	$2x_5$		
x_3	$=$	4	$+$	$2x_2$	-	$4x_5$		
ζ	$=$	11	-	$\frac{2}{3}x_4$	-	$\frac{7}{3}x_5$		
x_1	$=$	3			-	x_5		
x_2	$=$	4	-	$\frac{1}{3}x_4$	-	$\frac{2}{3}x_5$		
x_3	$=$	4	-	$\frac{2}{3}x_4$	-	$\frac{16}{3}x_5$		

In this case, the optimal point is $(x_1, x_2) = (3, 4)$ with optimal value 11.

(ii)

$$\begin{aligned} \max \quad & -x_0 \\ \text{s.t.} \quad & 5x_1 + 3x_2 + x_3 - x_0 = 15 \\ & 3x_1 + 5x_2 + x_4 - x_0 = 15 \\ & 4x_1 - 3x_2 + x_5 - x_0 = -12 \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

ζ	$=$						-	x_0
x_3	$=$	15	-	$5x_1$	-	$3x_2$	$+$	x_0
x_4	$=$	15	-	$3x_1$	-	$5x_2$	$+$	x_0
x_5	$=$	-12	-	$4x_1$	$+$	$3x_2$	$+$	x_0
ζ	$=$	-12	-	$4x_1$	$+$	$3x_2$	-	x_5
x_3	$=$	27	-	x_1	-	$6x_2$	$+$	x_5
x_4	$=$	27	$+$	x_1	-	$8x_2$	$+$	x_5
x_0	$=$	12	$+$	$4x_1$	-	$3x_2$	$+$	x_5
ζ	$=$	$\frac{15}{8}$	-	$\frac{29}{8}x_1$	-	$\frac{3}{8}x_4$	-	$\frac{5}{8}x_5$
x_3	$=$	$\frac{27}{4}$	-	$\frac{7}{4}x_1$	$+$	$\frac{3}{4}x_4$	$+$	$\frac{1}{4}x_5$
x_2	$=$	$\frac{27}{8}$	$+$	$\frac{1}{8}x_1$	-	$\frac{1}{8}x_4$	$+$	$\frac{1}{8}x_5$
x_0	$=$	-$\frac{15}{8}$	$+$	$\frac{29}{8}x_1$	$+$	$\frac{3}{8}x_4$	$+$	$\frac{5}{8}x_5$

In this case, we have reached an optimum. But note that we cannot get $x_0 = 0$. Thus the solution to the problem is infeasible.

(iii)

ζ	$=$		-	$3x_1$	$+$	x_2
w_1	$=$	4			-	x_2
w_2	$=$	6	$+$	$2x_1$	-	$3x_2$
ζ	$=$	2	-	$\frac{7}{3}x_1$	-	$\frac{1}{3}w_2$
w_1	$=$	2	-	$2x_1$	$+$	w_2
x_2	$=$	2	$+$	$\frac{2}{3}x_1$	-	$\frac{1}{3}w_2$

In this case, the optimal point is $(x_1, x_2) = (0, 2)$ with optimal value 2.

8.8

$$\begin{aligned} \max_{x,y,z} \quad & -x-y-z \\ \text{s.t.} \quad & x,y,z \geq 0 \end{aligned}$$

8.9

$$\begin{aligned} \max_{x,y,z} \quad & x+y+z \\ \text{s.t.} \quad & x,y,z \geq 0 \end{aligned}$$

8.10

$$\begin{aligned} \max_{x,y,z} \quad & x+y+z \\ \text{s.t.} \quad & x+y+z \leq -1 \\ & = x,y,z \geq 0 \end{aligned}$$

8.11

$$\begin{aligned} \max_{x,y,z} \quad & x+y+z \\ \text{s.t.} \quad & x+y+z \geq 1 \\ & x+y+z \leq 4 \\ & x,y,z \geq 0 \end{aligned}$$

Auxiliary problem (for which **0** is feasible):

$$\max \quad -w \tag{0.1}$$

$$\text{s.t.} \quad -x-y-z-w \leq -1 \tag{0.2}$$

$$x+y+z-w \leq 4 \tag{0.3}$$

$$x,y,z,w \geq 0 \tag{0.4}$$

8.12

$$\begin{aligned} \max\ & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{s.t.}\ & 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 + x_5 = 0 \\ & 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 + x_6 = 0 \\ & x_1 + x_7 = 1 \\ & \mathbf{x} \geq 0 \end{aligned}$$

ζ	=		-	$10x_1$	-	$57x_2$	-	$9x_3$	-	$24x_4$
x_5	=		-	$0.5x_1$	+	$1.5x_2$	+	$0.5x_3$	-	x_4
x_6	=		-	$0.5x_1$	+	$5.5x_2$	+	$2.5x_3$	-	$9x_4$
x_7	=	1	-	x_1						
ζ	=		-	$27x_2$	+	x_3	-	$44x_4$	-	$20x_5$
x_1	=			$3x_2$	+	x_3	-	$2x_4$	-	$2x_5$
x_6	=			$4x_2$	+	$2x_3$	-	$8x_4$	+	x_5
x_7	=	1	-	$3x_2$	-	x_3	+	$2x_4$	+ $2x_5$	
ζ	=	1	-	$30x_2$	-	$42x_4$	-	$18x_5$	-	x_7
x_1	=	1							-	x_7
x_6	=	2	-	$2x_2$	-	$4x_4$	+	$5x_5$	-	$2x_7$
x_3	=	1	-	$3x_2$	+	$2x_4$	+	$2x_5$	- x_7	

In this case, the optimal point is $(x_1, x_2, x_3, x_4) = (1, 0, 1, 0)$ with optimal value 1.

8.15

Proof. Let $\mathbf{x}$ be a primal feasible point, and let $\mathbf{y}$ be any feasible point of the dual. Observe that

$$\begin{aligned} \mathbf{c}^T\mathbf{x} &= \mathbf{x}^T\mathbf{c} \\ &\leq \mathbf{x}^T A^T \mathbf{y} \\ &= \mathbf{y}^T A\mathbf{x} \\ &\leq \mathbf{y}^T\mathbf{b} \\ &= \mathbf{b}^T\mathbf{y} \end{aligned}$$

by the dual and primary problems. $\square$

8.17

Proof. Observe that by definition 8.6.1, the primal problem can be expressed as

$$\begin{aligned} \max \ & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

with dual problem

$$\begin{aligned} \min \ & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} \ & A^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{aligned}$$

Now observe that the dual of the dual is given by

$$\begin{aligned} \min \ & -\mathbf{c}^T\mathbf{x} \\ \text{s.t.} \ & -A\mathbf{x} \leq -\mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

where $(-A^T)^T = -A$.
If we take the max of the dual of the dual instead of the min, we take the negative of it which yields the primal problem. $\square$

8.18 The linear problem is given by

$$\begin{aligned} \max \ & x_1 + x_2 \\ \text{s.t.} \ & 2x_1 + x_2 \leq 3 \\ & x_1 + 3x_2 \leq 5 \\ & 2x_1 + 3x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

With dual problem

$$
\begin{aligned}
\min\ & 3y_1 + 5y_2 + 4y_3 \\
\text{s.t.}\ & 2y_1 + y_2 + 2y_3 \geq 1 \\
& y_1 + 3y_2 + 3y_3 \geq 1 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$

Solving these by the Simplex Algorithm yields the optimizers (1.25, 0.5) and (0.25, 0, 0.25) respectively. Plugging these into the respected objective functions yields

$$
\begin{aligned}
1.25 + 0.5 &= 1.75 \\
3(0.25) + 5(0) + 4(0.25) &= 1.75
\end{aligned}
$$

Clearly, $1.75 = 1.75$ and the optimal values are equal.