**Exercise 1.3**.

(1) $\mathcal{G}_1 = \{A : A \subset \mathbb{R}, \text{A open}\}$ is neither an algebra, nor a $\sigma$-algebra.

(2) $\mathcal{G}_2 = \{A : A$ is a finite union of intervals of the form $(a, b], (-\infty, b], \text{and } (a, \infty)\}$ is an algebra, but not a $\sigma$-algebra.

(3) $\mathcal{G}_3 = \{A : A$ is a coutable union of $(a, b], (-\infty, b],$ and $(a, \infty)\}$ is both an algebra and a $\sigma$-algebra

**Exercise 1.7**
$\mathcal{P}(X)$ is the largest possible $\sigma$-algebra because each element of it is a subset of $X$, including $X$ itself and $\emptyset$. Thus, it contains every possible combination of subsets of $X$. Now, $\{\emptyset, X\}$ is the smallest possible $\sigma$-algebra namely because by containing only a single element $\emptyset$, it also must contain its complement $X$. Hence $\{\emptyset, X\}$ is the smallest possible $\sigma$-algebra.

**Exercise 1.10**

*Proof.* Let $\{\mathcal{S}_\alpha\}$ be a family of $\sigma$-algebras on $X$.

i. Since $\emptyset \in \mathcal{S}_\alpha$, it follows that $\emptyset \in \cap_\alpha \mathcal{S}_\alpha$.

ii. Take an arbitrary set $A \in \cap_\alpha \mathcal{S}_\alpha$. Thus, $A \in \mathcal{S}_\alpha$ for some $\alpha$. Given that $\mathcal{S}_\alpha$ is an algebra, it must be that $A^c \in \cap_\alpha \mathcal{S}_\alpha$

iii. Now, take some arbitrary sets $A_1, A_2, A_3, ... \in \cap_\alpha \mathcal{S}_\alpha$. Hence, each of these sets constitute a $\sigma-$algebra. Therefore, $\cup_{i=1}^{\infty} A_i \in \cup_\alpha \mathcal{S}_\alpha$

Therefore, $\cap_\alpha \mathcal{S}_\alpha$ is also a $\sigma$-algebra □

**Exercise 1.17**

*Proof.* Let $(X, \mathcal{S}, \mu)$ be a measure space.

- Suppose $A, B \in \mathcal{S}$ and $A \subset B$. Observe that

  $$B = (B \cap A^c) \cup A$$

  is disjoint. So,

  $$\mu(B) = \mu(B \cap A^c) + \mu(A)$$

  where $\mu(B \cap A^c) \geq 0$. Thus,

  $$\mu(A) \leq \mu(B)$$

  and $\mu$ is monotonically increasing.

- Now suppose $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$. Suppose we have a disjoint sequence, namely

  $$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 \cap A_1^c \\ B_3 &= A_3 \cap (A_1^c \cup A_2^c) \\ &\vdots \\ B_n &= A_n \setminus (\cup_{i=1}^{j-1} A_i) \end{aligned}$$

  These sets cover the same area as $\cup_{i=1}^{\infty} A_i$, hence

  $$\begin{aligned} \mu(\cup_{i=1}^{\infty} A_i) &= \mu(\cup_{i=1}^{\infty} B_i) \\ &= \sum_{i=1}^{\infty} \mu(B_i) \end{aligned}$$

  by property $(ii)$. Now, by the monotonicity property, it follows that $\sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$. Therefore, $\mu(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$

$\square$

**Exercise 1.18.**

*Proof.* Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $B \in \mathcal{S}$.

i. Observe that

$$\begin{aligned}\lambda(\emptyset) &= \mu(A \cap \emptyset) \\ &= \mu(\emptyset) \\ &= 0\end{aligned}$$

ii. Now observe that

$$\begin{aligned}\lambda(\cup_{i=1}^{\infty} A_i) &= \mu((\cup_{i=1}^{\infty} A_i) \cap B) \\ &= \mu(\underbrace{\cup_{i=1}^{\infty} (A_i \cap B)}_{\text{disjoint union}}) \\ &= \sum_{i=1}^{\infty} \mu(A_i \cap B) \\ &= \sum_{i=1}^{\infty} \lambda(A_i)\end{aligned}$$

Therefore, $\lambda(A) = \mu(A \cap B)$ is also a measure $(X, \mathcal{S})$. □

**Exercise 1.20**

*Proof.* Let $\mu$ be a measure on $(X, \mathcal{S})$. Suppose that $\{A_n\}$ is a decreasing sequence of measurable sets. That is

$$A_1 \supset A_2 \supset A_3 ..., A_i \in \mathcal{S}$$

where $\mu(A_1) < \infty$. Now, we examine the case for when $A = \cap_{i=1}^{\infty} A_i = \emptyset$ (note that if we replace $A_n$ with $A_n \setminus A$ then the general cases reduces). Observe that

$$\begin{aligned}A_1 &= (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup ... \\ A_2 &= (A_2 \setminus A_3) \cup (A_3 \setminus A_4) \cup ... \\ &\vdots \\ A_n &= (A_n \setminus A_{n+1}) \cup (A_{n+1} \setminus A_{n+2}) \cup ...\end{aligned}$$

By the subadditivity property, we have that

$$\mu(A_1) = \sum_{i=1}^{\infty} \mu(A_i \setminus A_{i+1})$$
$$\mu(A_2) = \sum_{i=2}^{\infty} \mu(A_i \setminus A_{i+1})$$
$$\vdots$$
$$\mu(A_n) = \sum_{i=n}^{\infty} \mu(A_i \setminus A_{i+1})$$

Note that the series $\mu(A_1) = \sum_{i=1}^{\infty} \mu(A_i \setminus A_{i+1})$ converges; hence, the remainder of $\mu(A_n) = \sum_{i=n}^{\infty} \mu(A_i \setminus A_{i+1})$ as $n \to \infty$ approaches 0. Thus,

$$\begin{aligned}\lim_{n\to\infty} \mu(A_n) &= 0 \\ &= \mu(\emptyset)\end{aligned}$$

$\square$

**Exercise 2.10.**
Since $\mu^*$ is an outer measure, we know it is countably subadditive. By the countable subadditivity property, we can express
$\mu^* \geq \mu^*(B \cap E) + \mu^*(B \cap E^c)$ as
$\mu^* \leq \mu^*(B \cap E) + \mu^*(B \cap E^c)$. Since the weak inequality can be expressed in both directions, we can replace it by $\mu^* = \mu^*(B \cap E) + \mu^*(B \cap E^c)$.

**Exercise 2.14.**
By **Definition 1.11**, we know that $\sigma(\mathcal{O}) :=$ the smallest $\sigma-$algebra containing all the open sets of $X$. We refer to this as the Borel $\sigma-$algebra of $X$ denoted by $\mathcal{B}(X)$. By **Theorem 2.8** (Carathéodory Construction), we know that $\mathcal{M}$ is a $\sigma-$algebra. Moreover, $\mathcal{M}$ is the collection of all Lebesgue measurable sets, which contains open sets. Thus, since $\mathcal{B}(X)$ is the intersection of $\sigma-$algebras containing open sets, it follows that $\mathcal{B}(X) \subset \mathcal{M}$.

**Exercise 3.1.**

*Proof.* Let $X \subset \mathbb{R}$ where $\{x_i\}_{i=1}^{\infty}$ represents each element of $X$. Let $\varepsilon > 0$. Define $A_n = (x_n - \frac{\varepsilon}{2^n}, x_n + \frac{\varepsilon}{2^n})$. So, $X \subset \cup_{i=1}^{\infty} A_i$. Observe that $\mu(\cup_{i=1}^{\infty}) \leq \sum_{i=1}^{\infty} 2^{1-i} = 2\varepsilon$. Therefore, $X$ has measure zero. $\square$

**Exercise 3.4.**
The set must be preserved under complements. So we can replace $\{x \in X : f(x) < a\}$ with $\{x \in X : f(x) \geq a\}$ since these are complements to each other. Now, we need to show that we can replace $<$ with $\leq$ or $>$ (which are complements of each other).

Claim: $f^{-1}((-\infty, a)) \in \mathcal{M} \Leftrightarrow f^{-1}((-\infty, a]) \in \mathcal{M}$.

*Proof.* ($\Rightarrow$) By **Exercise 3.1**, we know that countable sets have measure zero. Consider a set consisting of a single set $\{a\}$, which is countable. Hence $\{a\} \in \mathcal{M}$ and $f^{-1}(a) \in \mathcal{M}$. Thus, $f^{-1}((-\infty, a]) = f^{-1}((-\infty, a)) \cup f^{-1}(a) \in \mathcal{M}$.
($\Leftarrow$) Now, let $f^{-1}((-\infty, a]) \in \mathcal{M}$. Since $\mathcal{M}$ is closed under finite intersections and complements, we have that $f^{-1}((-\infty, a)) = f^{-1}((-\infty, a]) \cap f^{-1}(a^c) \in \mathcal{M}$.
Therefore $f^{-1}((-\infty, a)) \in \mathcal{M} \Leftrightarrow f^{-1}((-\infty, a]) \in \mathcal{M}$, and we can replace the set with any of the aforementioned sets. $\square$

**Exercise 3.7.**

*Proof.* Let $f, g, \{f_n\}_{n\in\mathbb{N}}, \{g_n\}_{n\in\mathbb{N}}$ be measurable functions on $X, \mathcal{M}$, and let $F : (Im(f), Im(g)) \to \mathbb{R}$ be continuous. Assume $F(f(x), g(x))$ is measurable. That is, the composition of two measurable functions is measurable.

Suppose $F(f(x) + g(x)) = f(x) + g(x)$. Hence, $F$ is continuous and measurable by **Theorem 3.6**. Thus $f + g$ is measurable.

Now, suppose $F(f(x) + g(x)) = f(x)g(x)$. Hence, $F$ is continuous and measurable by **Theorem 3.6**. This $f \cdot g$ is measurable.

Since $f$ and $g$ are measurable functions on $X, \mathcal{M}$, then $\{x \in X : f(x) < a\} \in \mathcal{M}$ and $\{x \in X : g(x) < a\} \in \mathcal{M}$ for all $a \in \mathbb{R}$. Observe that $\{x \in X : \max(f(x), g(x)) < a\} = \{x \in X : f(x) < a\} \cap \{x \in X : g(x) < a\}$. Now, we know that $\mathcal{M}$ is closed under countable intersections, and so it follows that $\{x \in X : \max(f(x), g(x)) < a\} \in \mathcal{M}$. Thus $\max(f(x), g(x))$ is measurable.

To prove that $\min(f, g)$ is measurable uses a similar to that $max(f, g)$ (see above). But, $\{x \in X : \min(f(x), g(x)) > a\} = \{x \in X : f(x) > a\} \cup \{x \in X : g(x) > a\}$. Given that $\mathcal{M}$ is closed under countable intersections, we have that $\{x \in X : \min(f(x), g(x)) > a\} \in \mathcal{M}$. Thus, $\min(f(x), g(x))$ is measurable.

Now, $\{x \in X : |f(x)| > a\} = \{x \in X : f(x) < -a\} \cup \{x \in X : f(x) > a\}$, which are both in $\mathcal{M}$, which is closed with respect to countable unions. Thus, $\{x \in X : |f(x)| > a \in \mathcal{M}\}$, and $|f(x)|$ is measurable. $\square$

**Exercise 3.14.**

*Proof.* Let $f$ be a function where $f : X \to \mathbb{R}$. Let $\varepsilon > 0$ be given. Since we are given that $f$ is bounded, we know that $|f(x)| \leq M$ for all $x \in X$. So, $x \in E_i^M$ for some $i$ and for all $x \in X$. Hence there is some $N \in \mathbb{N}$ with $N \geq M$ such that $\frac{1}{2^N} < \varepsilon$. Thus, for all $x \in X$ and $n \geq N$ we have that $||s_n(x) - f(x)|| < \varepsilon$. Therefore, the convergence is uniform. $\square$

**Exercise 4.13.**

*Proof.* Let $f$ be measurable, $||f|| < M$ on $E \in \mathcal{M}$, and $\mu(E) < \infty$. From **Remark 4.10**, we know that $||f|| = f^+ + f^-$, where $f^+ > 0$ and $f^- > 0$. So, $f^+ < M$ and $f^- < M$. Hence, $\int_E f^+ d\mu < \infty$ and $\int_E f^- d\mu < \infty$ which implies that $f \in \mathscr{L}^1(\mu, E)$ $\square$

**Exercise 4.14.**

*Proof.* By the contrapositive statement, suppose there exists some $\hat{E}$ where $f$ is infinite on $\hat{E} \subset E$ and $\mu(\hat{E}) > 0$. Thus $\int_{\hat{E}} f d\mu = \infty$ which implies that

$\int_{\hat{E}} f^+ d\mu = \infty$. Now since $\hat{E} \subset E$ we have that $\infty = \int_{\hat{E}} f^+ d\mu < \int_E f^+ d\mu = \infty$, and $f \notin \mathscr{L}^1(\mu, E)$. $\square$

**Exercise 4.15.**

*Proof.* Let $f, g \in \mathscr{L}^1(\mu, E)$. Let $B(f) = \{s : 0 \leq s \leq f, \text{s simple , measurable}\}$, and suppose that $f \leq g$. Hence, we have that $f^+ \leq g^+$ and also $f^- \geq g^-$. Hence $B(f^+) \subset B(g^+)$ as well as $B(g^-) \subset B(f^-)$ which imply that

$$\begin{aligned} \int_E f^+ d\mu &\leq \int_E g^+ d\mu \\ \text{and} \int_E f^- d\mu &\geq \int_E g^- d\mu \end{aligned}$$

Observe that

$$\begin{aligned} \int_E f d\mu &= \int_E f^+ d\mu - \int_E f^- d\mu \\ &\leq \int_E g^+ d\mu - \int_E g^- d\mu \\ &= \int_E g d\mu \end{aligned}$$

Therefore, $\int_E f d\mu \leq \int_E g d\mu$ as desired. $\square$

**Exercise 4.16.**

*Proof.* Let $s(x)$ be a simple function where $s(x) = \sum_{i=1}^N c_i \chi_{E_i}$ with $E_i \in \mathcal{M}$. Now suppose $A \subset E \subset \mathcal{M}$. By the monotonicity property, it follows that $\mu(A \cap E_i) \leq \mu(E \cap E_i)$ for every $i$. Hence (See **Definition 4.1**),

$$\begin{aligned} \int_A s d\mu &= \sum_{i=1}^N c_i \mu(A \cap E_i) \\ &\leq \sum_{i=1}^N c_i \mu(E \cap E_i) \\ &= \int_E s d\mu \end{aligned}$$

Now observe that

$$\int_A f d\mu = \sup\{\int_A s d\mu : 0 \leq s \leq f, \text{s simple, measurable}\}$$
$$\int_E f d\mu = \sup\{\int_E s d\mu : 0 \leq s \leq f \text{s simple, measurable}\}$$

which follows from **Definition 4.2**. Note though that $\int_A f d\mu \leq \int_E f d\mu$ since $s$ was arbitrary. Now, since $f \in \mathscr{L}^1(\mu, E)$, then $\int_E ||f|| d\mu < \infty$. Hence, it follows that $\int_E f d\mu < \infty$, and $\int_A f d\mu < \infty$. Thus, $\int_A f^+ d\mu < \infty$ as well as $\int_A f^- d\mu < \infty$. Therefore, $f \in \mathscr{L}^1(\mu, A)$ □

**Exercise 4.21.**

*Proof.* Let $A, B \in \mathcal{M}$, $B \subset A$, and $\mu(A - B) = 0$. Suppose that $f \in \mathscr{L}^1$. So we have $\int_{A-B} f d\mu = 0$ from **Proposition 4.6**. Furthermore, $\mu_1(A) = \int_A f^+ d\mu$ and also $\mu_2(A) = \int_A f^- d\mu$ are measures on $\mathcal{M}$ (note that $f^+$ and $f^-$ are both nonnegative $\mathcal{M} measurable$ since $f \in \mathscr{L}^1$). Thus, we have that

$$\begin{aligned}\int_A f d\mu &= \int_A f^+ d\mu - \int_A f^- d\mu \\ &= \mu_1(A) - \mu_2(A)\end{aligned}$$

Assume $A = (A - B) \cup B$. Observe that $\mu_i(A) = \mu_i(A - B) + \mu_i(B)$ with $i = 1, 2, 3, ....$ Since $\mu(A - B) = 0$ it follows that $\mu_i(A) = \mu_i(B)$ for $i = 1, 2, 3, ....$ Hence,

$$\begin{aligned}\int_A f d\mu &= \mu_1(B) - \mu_2(B) \\ &= \int_B f d\mu\end{aligned}$$

Therefore, by implication we have that $\int_A f d\mu \leq \int_B f d\mu$ □