So far, we have learned how to solve initial value problems by applying various numerical schemes. Another interesting class of differential equations is the class of *boundary value problems*. The simplest kind of boundary value problem is the 2-point BVP, which is specified by the following data:

$$y'' = f(x, y, y') \qquad x \in [a, b] \qquad y(a) = \alpha \qquad y(b) = \beta.$$

Note that in the context of an initial value problem, we would have been given complete information at a: $y(a) = y_0$ and $y'(a) = y_0'$ whereas in this problem we are instead given partial information at both a and b.

One of the most conceptually simple methods is called the *shooting method*. The idea is the following:

1. Choose an arbitrary initial guess $y'(a) = t_0$ and solve the initial value problem

$$y'' = f(x, y, y') \qquad x \in [a, b] \qquad y(a) = \alpha \qquad y'(a) = t_0$$

   using any method that we've discussed in class.

2. If the resulting approximation at b is sufficiently close to β, then return this solution and say that we are satisfied.

3. Otherwise, if the resulting approximation at b is too far from β, then choose another $y'(a) = t_i$ and try again.

Question 0.1. How can we select t_i at each round to try so that we can guarantee that our approximations at b will eventually converge to β?

A very reasonable idea here is to try one of the rootfinding strategies. Let $y(b, t)$ represent the value at $x = b$ of the exact solution $y(x, t)$ to the initial value problem

$$y'' = f(x, y, y') \qquad x \in [a, b] \qquad y(a) = \alpha \qquad y'(a) = t.$$

Then we are looking for a zero of the function $g(t) = y(b, t) - \beta$. We could use bisection, but each evaluation of $y(b, t)$ requires solving an initial value problem so is very expensive. As a result, we would really like to minimize the number of steps required to obtain convergence, so we should use a more sophisticated method such as Newton's method. To apply Newton's method, we should write the root-finding problem in fixed point form

$$t = t - \frac{g(t)}{g'(t)} = h(t).$$

In order to compute $g'(t)$, we have

$$g'(t) = \frac{\partial y}{\partial t}(b, t).$$

However, it is very unclear how to evaluate $\frac{\partial y}{\partial t}(b, t)$ to perform iterations so we need to devise a stragey for approximating this quantity.

To do this, we can take a time derivative of the equation $y'' = f(x, y, y')$:

$$\frac{\partial y''}{\partial t} = \frac{\partial f}{\partial x}(x, y, y')\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}(x, y, y')\frac{\partial y}{\partial t} + \frac{\partial f}{\partial y'}(x, y, y')\frac{\partial y'}{\partial t}.$$

However, note that x does not depend on t, so that $\frac{\partial x}{\partial t} = 0$. Thus

$$\frac{\partial y''}{\partial t} = \frac{\partial f}{\partial y}(x, y, y')\frac{\partial y}{\partial t} + \frac{\partial f}{\partial y'}(x, y, y')\frac{\partial y'}{\partial t}.$$

Commuting the derivatives in t with the derivatives in x,

$$\left(\frac{\partial y}{\partial t}\right)'' = \frac{\partial f}{\partial y}(x, y, y')\frac{\partial y}{\partial t} + \frac{\partial f}{\partial y'}(x, y, y')\left(\frac{\partial y}{\partial t}\right)'$$

and renaming $z(x, t) = \frac{\partial y}{\partial t}$ we obtain the new initial value problem

$$z'' = \frac{\partial f}{\partial y}(x, y, y')z + \frac{\partial f}{\partial y'}(x, y, y')z' \qquad x \in [a, b] \qquad z(a, t) = 0 \qquad z'(a, t) = 1.$$

This initial value problem may seem insoluble at first because we don't get approximations for y and y' along the way. However, the point is that we already solved an IVP for y and y', so we can reuse those approximations here. Thus, in order to apply Newton iterations for the shooting method, we need to use the following procedure.

1. Guess an initial t_0 and solve the following IVP using some numerical scheme

$$y'' = f(x, y, y') \qquad x \in [a, b] \qquad y(a) = \alpha \qquad y'(a) = t_0.$$

2. Using the approximations found in the previous step, solve the following IVP using some numerical scheme

$$z'' = \frac{\partial f}{\partial y}(x, y, y')z + \frac{\partial f}{\partial y'}(x, y, y')z' \qquad x \in [a, b] \qquad z(a, t) = 0 \qquad z'(a, t) = 1$$

3. Compute the Newton update

$$t_k = t_{k-1} - \frac{y(b, t_{k-1})}{z(b, t_{k-1})}$$

4. Repeat the previous steps for as long as desired.

Question 0.2. Newton's method complicates things pretty heavily here. Compare the benefits vs. drawbacks of using Newton's method over bisection.

Question 0.3. How would you choose an initial value t_0?