1 Practice

Solution 1.1. Since Simpson's rule involves a degree 2 polynomial approximation, we first form the degree 2 Lagrange interpolating polynomial to $f(x)$ at the points

$$(a, f(a)), (\frac{a+b}{2}, f(\frac{a+b}{2})), (b, f(b)).$$

For convenience, we'll write $h = \frac{b-a}{2}$, so the points can be written as a, $\frac{a+b}{2} = a+h, b = a+2h$.

$$L_{2,0} = \frac{(x-(a+h))(x-(a+2h))}{(a-(a+h))(a-(a+2h))} \qquad L_{2,1} = \frac{(x-a)(x-(a+2h))}{(a+h-a)(a+h-(a+2h))}$$

$$L_{2,2} = \frac{(x-a)(x-(a+h))}{(a+2h-a)(a+2h-(a+h))}$$

Simplifying slightly:

$$L_{2,0} = \frac{(x-(a+h))(x-(a+2h))}{2h^2} \qquad L_{2,1} = \frac{(x-a)(x-(a+2h))}{-h^2}$$

$$L_{2,2} = \frac{(x-a)(x-(a+h))}{2h^2}$$

Then the interpolating polynomial is

$$P_2(x) = f(a)L_{2,0} + f(a+h)L_{2,1} + f(a+2h)L_{2,2}$$

so our integral approximation is

$$\int_a^b f(x)dx \approx \int_a^b P_2(x)dx = \sum_{i=0}^{2} f(a+ih) \int_a^b L_{2,i}dx.$$

Thus the integration weights come from integrating the Lagrange polynomials, which we now do:

$$\int_a^b L_{2,0}dx = \frac{1}{2h^2}\int_a^b (x-a-h)(x-a-2h)dx = \frac{1}{2h^2}\frac{2h^3}{3} = \frac{h}{3}.$$

$$\int_a^b L_{2,1}dx = \frac{1}{-h^2}\int_a^b (x-a)(x-a-2h)dx = \frac{1}{-h^2}\frac{-4h^3}{3} = \frac{4h}{3}.$$

$$\int_a^b L_{2,2}dx = \frac{1}{2h^2}\int_a^b (x-a)(x-a-h)dx = \frac{1}{2h^2}\frac{2h^3}{3} = \frac{h}{3}.$$

Combining all these weights, we obtain

$$\int_a^b f(x)dx \approx \frac{h}{3}f(a) + \frac{4h}{3}f(\frac{a+b}{2}) + \frac{h}{3}f(b)$$

$$= \frac{b-a}{6}\left(f(a) + 4f(\frac{a+b}{2}) + f(b)\right)$$

which is exactly Simpson's rule.

Solution 1.2. As we saw in the interpolation unit, the accuracy of an interpolating polynomial can become very poor especially near the endpoints of the interpolation interval if the points are equally spaced. Composite integration allows use to avoid this issue by splitting up the integration interval into many smaller subintervals and then interpolating over each of those subintervals individually.

Composite integration also allows us to avoid numerical roundoff errors which would occur when interpolating very many points (i.e. just think about the number of operations required to produce the Newton or Lagrange interpolating polynomials if the number of points is very large...).

2 Gaussian Quadrature

Solution 2.1. We start with $P_0(x) = 1$, $P_1(x) = x$. The next few are:

$$P_2(x) = xP_1(x) - \frac{1^2}{4 * 1^2 - 1} P_0(x) = x^2 - \frac{1}{3}.$$

$$P_3(x) = xP_2(x) - \frac{2^2}{4 * 2^2 - 1} P_1(x) = x^3 - \frac{x}{3} - \frac{4x}{15} = x^3 - \frac{3x}{5}$$

$$P_4(x) = xP_3(x) - \frac{3^2}{4 * 3^2 - 1} P_2(x) = x^4 - \frac{3x^2}{5} - \frac{9x^2}{35} + \frac{3}{35} = x^4 - \frac{6x^2}{7} + \frac{3}{35}$$

$$P_5(x) = xP_4(x) - \frac{4^2}{4 * 4^2 - 1} P_3(x) = x^5 - \frac{6x^3}{7} + \frac{3x}{35} - \frac{16x^3}{63} + \frac{48x}{315} = x^5 - \frac{10x^3}{9} + \frac{5x}{21}$$

To find the roots, we first factor out an x from $P_5(x)$ (corresponding to the root $x = 0$)

$$P_5(x) = x\left(x^4 - \frac{10x^2}{9} + \frac{5}{21}\right).$$

Next, we note that the degree 4 factor is a polynomial in x^2, so we apply the quadratic formula

$$(x^2)^2 - \frac{10}{9}(x^2) + \frac{5}{21} = 0 \implies x^2 = \frac{\frac{10}{9} \pm \sqrt{\frac{100}{81} - \frac{20}{21}}}{2} \implies x = \pm\sqrt{\frac{\frac{10}{9} \pm \sqrt{\frac{100}{81} - \frac{20}{21}}}{2}}.$$

Solution 2.2. Let x_0, x_1, x_2, x_3, x_4 denote the five roots (in increasing order) found in the previous problem. According to Gaussian integration, we use these as our integration points, and we compute the weights by integrating the squares of the corresponding Lagrange polynomials. Computing the integrals (exactly because they are polynomials) yields

$$[w_0, w_1, w_2, w_3, w_4] =$$

$$\frac{21}{50}\frac{5\sqrt{10}+\sqrt{7}}{7\sqrt{10}+4\sqrt{7}}, \frac{21}{50}\frac{5\sqrt{10}-\sqrt{7}}{7\sqrt{10}-4\sqrt{7}}, 128/225, \frac{21}{50}\frac{5\sqrt{10}-\sqrt{7}}{7\sqrt{10}-4\sqrt{7}}, \frac{21}{50}\frac{5\sqrt{10}+\sqrt{7}}{7\sqrt{10}+4\sqrt{7}}$$

Solution 2.3. Given an arbitrary integral

$$\int_a^b f(x)dx$$

we first want to transform the integral into one over $[-1, 1]$ instead so that we can apply our previous results. In order to do this, we make the substitution

$$x = \frac{b-a}{2}y + \frac{a+b}{2} \implies \int_a^b f(x)dx = \int_{-1}^1 f\left(\frac{b-a}{2}y + \frac{a+b}{2}\right)\frac{b-a}{2}dy.$$

Now using the previous methods, we obtain

$$\int_{-1}^1 f\left(\frac{b-a}{2}y + \frac{a+b}{2}\right)\frac{b-a}{2}dy = \frac{b-a}{2}\sum_{k=0}^{4} w_k f\left(\frac{b-a}{2}x_k + \frac{a+b}{2}\right).$$

Thus, we should take the new weights $u_k = \frac{b-a}{2}w_k$ and the new points $y_k = \frac{b-a}{2}x_k + \frac{a+b}{2}$.

Solution 2.4. See attached code. This method correctly integrates the monomials $1, x, \ldots, x^9$ as expected.

3 Adaptive integration

Solution 3.1. See attached code. This method correctly integrates the monomials $1, x, \ldots, x^9$ as expected.

Solution 3.2.

tol	result	function evals
10^{-1}	1.2913519470201165	15
10^{-2}	1.2913519470201165	15
10^{-3}	1.2913519470201165	15
10^{-4}	1.2913030086885464	35
10^{-5}	1.2912860663635768	115
10^{-6}	1.2912859981503026	175
10^{-7}	1.2912859970672157	255
10^{-8}	1.2912859970630315	315
10^{-9}	1.291285997062966	375
10^{-10}	1.2912859970627468	475
10^{-11}	1.2912859970626647	615
10^{-12}	1.2912859970626642	735
10^{-13}	1.2912859970626636	955
10^{-14}	1.2912859970626636	1195

Note that the number of function evaluations required for each successive digit increases quite rapidly.

Solution 3.3.

accuracy	result	function evals
10^{-4}	1.2912784672311213	101
10^{-5}	1.291284096676604	201
10^{-6}	1.2912851493591513	301
10^{-6}	1.291285519285477	401
10^{-6}	1.2912856909014652	501
10^{-6}	1.2912857842652148	601
10^{-6}	1.2912858406209289	701
10^{-6}	1.2912858772274383	801
10^{-7}	1.2912859023405678	901
10^{-7}	1.2912859203129659	1001
10^{-7}	1.2912859336160816	1101
10^{-7}	1.2912859437377502	1201
10^{-7}	1.2912859516171549	1301
10^{-7}	1.2912859578708384	1401
10^{-7}	1.2912859629171316	1501
10^{-7}	1.2912859670479813	1601
10^{-7}	1.2912859704721311	1701

Note that the number of function evaluations is twice the number of subintervals. (Technically, it should be $2n+1$, but for simplicity I just dropped the extra evaluation.) While it's reasonable to obtain a relatively low accuracy with composite Simpson's rule, we see from the table above that it is very difficult to increase the accuracy further. This highlights the strong advantage of adaptive methods over even composite methods.