1 The 1D heat equation

1.1 Finite difference methods

Solution 1.1. The Taylor expansion is

$$\begin{aligned}
u(x_i, t_{j+1}) &= u(x_i, t_j) + \frac{\partial u}{\partial t}(x_i, t_j)(t_{j+1} - t_j) + \frac{\partial^2 u}{\partial t^2}(x_i, \mu_j)\frac{(t_{j+1} - t_j)^2}{2} \\
&= u(x_i, t_j) + \frac{\partial u}{\partial t}(x_i, t_j)k + \frac{\partial^2 u}{\partial t^2}(x_i, \mu_j)\frac{k^2}{2}.
\end{aligned}$$

Rearranging gives

$$\frac{\partial u}{\partial t}(x_i, t_j) = \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{k} - \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \mu_j).$$

Solution 1.2. The Taylor expansion is

$$\begin{aligned}
u(x_i, t_{j-1}) &= u(x_i, t_j) + \frac{\partial u}{\partial t}(x_i, t_j)(t_{j-1} - t_j) + \frac{\partial^2 u}{\partial t^2}(x_i, \mu_j)\frac{(t_{j-1} - t_j)^2}{2} \\
&= u(x_i, t_j) - \frac{\partial u}{\partial t}(x_i, t_j)k + \frac{\partial^2 u}{\partial t^2}(x_i, \mu_j)\frac{k^2}{2}.
\end{aligned}$$

Rearranging gives

$$\frac{\partial u}{\partial t}(x_i, t_j) = \frac{u(x_i, t_j) - u(x_i, t_{j-1})}{k} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \mu_j).$$

Solution 1.3. The Taylor expansion is

$$\begin{aligned}
u(x, t_j) = u(x_i, t_j) &+ \frac{\partial u}{\partial x}(x_i, t_j)(x - x_i) + \frac{\partial^2 u}{\partial x^2}(x_i, t_j)\frac{(x - x_i)^2}{2} \\
&+ \frac{\partial^3 u}{\partial x^3}(x_i, t_j)\frac{(x - x_i)^3}{6} + \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j)\frac{(x - x_i)^4}{24}
\end{aligned}$$

Plugging in $x = x_{i+1}$ yields

$$u(x_{i+1}, t_j) = u(x_i, t_j) + \frac{\partial u}{\partial x}(x_i, t_j)h + \frac{\partial^2 u}{\partial x^2}(x_i, t_j)\frac{h^2}{2} + \frac{\partial^3 u}{\partial x^3}(x_i, t_j)\frac{h^3}{6} + \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j)\frac{h^4}{24}$$

Plugging in $x = x_{i-1}$ yields

$$u(x_{i-1}, t_j) = u(x_i, t_j) - \frac{\partial u}{\partial x}(x_i, t_j)h + \frac{\partial^2 u}{\partial x^2}(x_i, t_j)\frac{h^2}{2} - \frac{\partial^3 u}{\partial x^3}(x_i, t_j)\frac{h^3}{6} + \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j)\frac{h^4}{24}$$

Adding these two equations gives

$$u(x_{i+1}, t_j) + u(x_{i-1}, t_j) = 2u(x_i, t_j) + h^2\frac{\partial^2 u}{\partial x^2}(\xi_i, t_j) + \frac{h^4}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, t_j)$$

and rearranging gives

$$\frac{\partial^2 u}{\partial x^2}(\xi_i, t_j) = \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, t_j)$$

1.2 FTCS

Solution 1.4. Rearranging the difference equation gives

$$w_{i,j+1} = (1 - \frac{2\alpha^2 k}{h^2})w_{i,j} + \frac{\alpha^2 k}{h^2}(w_{i+1,j} + w_{i-1,j})$$

Solution 1.5. Writing out the system of equations we found before explicitly, we obtain

$$\begin{aligned}
w_{1,j+1} &= (1 - \frac{2\alpha^2 k}{h^2})w_{1,j} + \frac{\alpha^2 k}{h^2}w_{2,j} \\
w_{2,j+1} &= (1 - \frac{2\alpha^2 k}{h^2})w_{2,j} + \frac{\alpha^2 k}{h^2}(w_{3,j} + w_{1,j}) \\
&\vdots \\
w_{m-1,j+1} &= (1 - \frac{2\alpha^2 k}{h^2})w_{m-1,j} + \frac{\alpha^2 k}{h^2}(w_{m-2,j})
\end{aligned}$$

Let $\lambda = \frac{\alpha^2 k}{h^2}$. Then writing this in matrix form, we get

$$\begin{pmatrix} w_{1,j+1} \\ w_{2,j+1} \\ \vdots \\ w_{m-1,j+1} \end{pmatrix} = \begin{pmatrix} 1-2\lambda & \lambda & 0 & \dots & \dots & 0 \\ \lambda & 1-2\lambda & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & & \ddots & \vdots \\ \vdots & \ddots & & & & 0 \\ \vdots & & \ddots & & & \lambda \\ 0 & \dots & \dots & 0 & \lambda & 1-2\lambda \end{pmatrix} \begin{pmatrix} w_{1,j} \\ w_{2,j} \\ \vdots \\ w_{m-1,j} \end{pmatrix}$$

Solution 1.6. Since the inequality needs to hold for all i, we can consider the case $i = 1$ for which the cosine term is maximized.

$$\begin{aligned}
\left|1 - 4\frac{\alpha^2 k}{h^2}\cos\left(\frac{\pi}{2m}\right)^2\right| \leq 1 &\iff -1 \leq 1 - 4\frac{\alpha^2 k}{h^2}\cos\left(\frac{\pi}{2m}\right)^2 \leq 1 \\
&\iff -2 \leq -4\frac{\alpha^2 k}{h^2}\cos\left(\frac{\pi}{2m}\right)^2 \leq 0 \\
&\iff 0 \leq \frac{\alpha^2 k}{h^2}\cos\left(\frac{\pi}{2m}\right)^2 \leq \frac{1}{2}
\end{aligned}$$

Since $0 < \frac{i\pi}{2m} < \frac{\pi}{2}$ as i goes from 1 to $m-1$, cosine is positive for all values of i that we will consider. We should also ask that the inequality holds for all $m \to \infty$, meaning that the cosine can become arbitrarily close to 1. Thus

$$\iff 0 \leq \frac{\alpha^2 k}{h^2} \leq \frac{1}{2}.$$

Thus, the condition for stability is that h and k must satisfy

$$\alpha^2 k \leq \frac{h^2}{2}.$$