1 Counting

1. There are 11 letters: 1M, 4I, 4S, 2P so we can permute them in 11! ways but we don't care if we permuate the Ms, Is, Ss, or Ps, so we divide 11!/(1!4!4!2!).
2. (a) There are 8 choices for first number and 10 choices for the remaining 6, so $8 \cdot 10^6$.
   (b) We need to subtract the number of phone numbers beginning with 911: $8 \cdot 10^6 - 10^4$
3. (a) There are 10 choices on Monday, 9 on Tuesday, etc.: $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$.
   (b) There are 10 choicse on Monday, 9 on Tuesday, 9 on Wednesday, etc.: $10 \cdot 9^4$.
4. (a) There will be $\binom{n}{2}$ games, so there are $2^{\binom{n}{2}}$ possible outcomes if no ties are allowed.
   (b) $\binom{n}{2}$ games are played in total.
5. (a) There are n rounds, since each round eliminates half the players.
   (b) There are $2^{n-1} + 2^{n-2} + \cdots + 2^0 = 2^n - 1$ games.
   (c) There are $2^n - 1$ games since all but one player needs to be eliminated.
6. There are 20! ways to line everyone up and pair them off 2 at a time. Since we only care about which pairs are formed and not the order of the pairs themselves (though we do care about the order within a pair to distinguish white and black), we only need to divide by 10!: $\frac{20!}{10!}$.
7. (a) We can arrange the 7 games in 7! ways, and since we don't only care about the numbers of each outcome, divide the overcounting by 3!2!2!: $\frac{7!}{3!2!2!}$.
   (b) B can end up with 3 points by: winning 3 times, winning 2 times and drawing 2 times, winning 1 time an drawing 4 times, or winning 0 times and drawing 6 times. This gives $\binom{7}{3} + \binom{7}{2}\binom{5}{2} + \binom{7}{1}\binom{6}{4} + \binom{7}{6}$.
   (c) A can only win by 4 to 3 if there have been seven games, so we only need to count the number of ways for A to win by 4 to 3. Let N denote the result from part (b). Then we need to subtract from this the number of ways for A to get 4 points before the seventh game. To do this, A can either win 4 of the first 6, win 3 and draw 2, or win 2 and draw 4. There are $\binom{6}{4} + \binom{6}{3}\binom{3}{2} + \binom{6}{2}\binom{4}{4}$ ways for this to happen, so we need to subtract this from N.
8. (a) We can permute all 12 people, splitting them by first 2, next 5, then final 5. However, we then need to divide by $2!5!5! \cdot 2!$ for the number of ways to permute people in each team as well as permuting the two five member teams: $\frac{12!}{2!5!5!2!}$
   (b) Using the same idea, $\frac{12!}{4!4!4!3!}$.
9. (a) The path has length $110 + 111 = 221$ so we just need to specify the up steps: $\binom{221}{110}$.
   (b) We first go through $(110, 111)$ in $\binom{221}{110}$ ways. Then we go from $(110, 111)$ to $(210, 211)$ in $\binom{200}{100}$ ways.

10. (a) We subtract the number of ways to avoid choosing a statistics course $\binom{20}{7} - \binom{15}{7}$.
    (b) This overcounts situations where we take more than one statistics courses.

11. (a) For each of the n inputs, we have m choices: m^n
    (b) For each of n inputs, we can choose any previously not chosen element of B: $m(m-1)\dots(m-n+1)$.

12. (a) We need to say which 13 cards go to player A: $\binom{52}{13}$.
    (b) We choose 13 cards for each player successively, so $\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$.
    (c) Each player has to have different cards.

13. We need to say how many copies of each of the 52 cards we get, so we are looking for the number of ways to distribute 10 objects (cards) among 52 bins (card types): $\binom{10+52-1}{52-1}$.

14. The number of ways to order a single pizza is $N = 4 \cdot 2^8$. Then there are $\binom{N}{2} + N$ ways to order two pizzas.

2 Story

15. The number of subsets of an n-element set can be counted as the number of 0 element subsets, 1 element subsets, etc.

16.

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!}{k!(n-k+1)!}(n-k+1+k)$$

$$= \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}$$

We can count k-subsets of an $(n+1)$-element set by whether they include a distinguished element or not.

17. The number of n element subsets of $\{1,\dots,2n\}$ can be counted as the number of subsets with 0 elements from $\{1,\dots,n\}$, 1 element from $\{1,\dots,n\}$, etc.

18. The number of n element subsets of $\{1,\dots,2n\}$ with a distinguished element in $\{1,\dots n\}$ can be counted as the number of subsets with k elements from $\{1,\dots,n\}$ and a distinguished element from the k element subset for k from 0 to n.

19. The number of 5 element subsets of $\{1,\dots,n+3\}$ can be counted by considering the middle element, which can be any of $3,4,\dots,n+1$. If the middle number is $k+1$, then we need to select 2 numbers from $\{1,\dots,k\}$ and 2 numbers from $\{k+2,\dots,n+3\}$.

20. (a) The number of $(k+1)$-element subset of $\{1,\ldots,n+1\}$ can be counted by first declaring what the largest element is and then for each choice, choosing the remaining k elements from the set of elements smaller than the largest one.

    (b)
$$\sum_{k=30}^{50}\binom{k+5-1}{5-1} = \sum_{k=4}^{54}\binom{k}{4} - \sum_{k=4}^{33}\binom{k}{4} = \binom{55}{5} - \binom{34}{5}$$

21. (a) The number of unordered set partitions of $\{1,\ldots,n+1\}$ into k nonempty disjoint subsets is equal to the number of unordered set partitions of $\{1,\ldots,n\}$ into $k-1$ nonempty disjoint subsets with one subset consisting of only $n+1$ plus the number of unordered set partitions of $\{1,\ldots,n\}$ into k nonempty subsets and we choose one of the k subsets to place $n+1$ in.

    (b) The number of unordered set partitions of $\{1,\ldots,n+1\}$ into $k+1$ nonempty disjoint subsets is equal to the sum over j of the number of ways to choose $n-j$ elements to be in the same subset as $n+1$ and then partitioning the remaining j elements into k nonempty disjoint subsets.

22. (a) The number of pairs that can be formed from a $n+1$ element set is equal to the sum of the number of pairs where the largest element is $2, 3, \ldots$

    (b) Could do by hockeystick using: The number of quadruples of numbers between 1 and n where the last number is the (weakly) largest number can be instead counted by cases considering the number of distinct elements chosen. For each k, we choose $6\binom{k}{3}$ for 3 distinct numbers with ordering, $6\binom{k}{2}$ for 2 distinct elements with ordering and choosing which of the elements to double, and finally $\binom{k}{1}$ for 1 distinct element. Then hockey stick identity summing over k from 0 to n gives the desired result.

    However, hint is more direct. The number of quadruples of numbers between 0 and n where the last number is the largest is equal to $6\binom{n+1}{4}$ (choose 4 numbers from 0 to n, the largest is the final entry and we can order the remaining 3 numbers arbitrarily) plus $6\binom{n+1}{4}$ (choose 3 numbers from 0 to n, the largest is the final entry and we need to decide among the remaining two elements which one to double and what order to use) plus $\binom{n+1}{2}$ (choose 2 numbers, the largest is the final entry and the remaining number is repeated 3 times).

3 naive definition of probability

23. There are 7 possiblities for three consecutive floors and all are symmetric. Thus we consider only the probability of 2,3,4 and multiply by 7. There are 6 ways for the buttons 234 to be pressed, since the order doesn't matter, so there are 42 ways for consecutive numbers in total out of 9^3 total possible outcomes (not all distinct).

24. This problem has two interpretations: if the children are distinct then we are considering birth orders out of 6!=720 ways whereas if only the gender is important, then there are only $\binom{6}{3} = 20$ birth orders.

    Using the second interpretation, we get $\frac{1}{20}$. Using the first interpretation, we get $\frac{36}{720} = \frac{1}{20}$, so I guess it didn't matter in the end.

25. If the robberies are considered distinct, there are 6^6 possiblities. Otherwise, there are $\binom{11}{6}$ possiblities. The probability that some district had more than one robbery is 1 minus the probability that no district had more than one robbery, i.e. one robbery per district.

    In the first interpretation, there are 6! possibilities while in the second interpretation there is 1 possibility. I believe it should be $\frac{6!}{6^6}$.

26. (a) This problem is similar to the birthday problem if we consider the 1 million residents as the possible birthdays and the 1000 sampled individuals as the people whose birthdays we are tracking.

    (b) This is 1 minus the probability that nobody is chosen more than once: $1 - 1000!\binom{1000000}{1000}\frac{1}{1000000^{1000}}$

27. This is 1 minus the probability that no location has more than one phone number stored there, i.e. $1 - k!\binom{n}{k}\frac{1}{n^k}$.

28. This is 1 minus the probability that no timeslot has more than one statistics class: $1 - \frac{10 \cdot 9 \cdot 8}{10^3}$.

29. (a) >: 22 = 6+6+6+4 = 6+6+5+5 while 21 = 6+6+6+3 = 6+5+5+5 = 6+6+5+4 so there are more ways to achieve 21.

    (b) =: the three letter word doesn't care about the middle letter, so the probability of both is just the probability that one randomly selected number is equal to another.

30. the first half of the letters (including the middle if n is odd) can be chosen freely while the remainder are then determined: $\frac{26^{\lceil \frac{n}{2} \rceil}}{26^n}$

31. Of the n previously tagged elks, we must select k and out of the $N - n$ remaining elks, we must select the remaining $m - k$: $\frac{\binom{n}{k}\binom{N-n}{m-k}}{\binom{N}{m}}$.

32. There are $\binom{4}{2}$ possibilities. It is impossible to get an odd number correct. For 0 and 4 correct, the probability is $\frac{1}{6}$ and for 2 the probability is $\frac{4}{6}$.

33. (a) There are $\binom{g+r}{r}$ total sequences of balls, and the number of sequences where the first draw is green is equal to $\binom{g-1+r}{r}$ which is the same as the number of sequences where the second draw is green.

    (b)

$$\frac{\binom{g+r-1}{r}}{\binom{g+r}{r}} = \frac{g}{g+r}$$

(c) Ignoring denominators, the equality of the probabilities can be written as

$$\binom{14}{r} + \binom{14}{g} = \binom{14}{r-1} + \binom{14}{g-1} \implies \frac{g(g-1)+r(r-1)}{g!r!} = \frac{2rg}{r!g!}$$

$$\iff (g-r)^2 = g+r = 16 \iff g-r = \pm 4 \implies g=10, r=6; r=10, g=6.$$

34. (a) There are $\binom{52}{5}$ hands total. The number of flushes in an individual suit is $\binom{13}{5}-1$, so $4(\binom{13}{5}-1)/\binom{52}{5}$.

    (b) First choose the 3 ranks $\binom{13}{3}$ then choose 2 out of those three ranks to be pairs $\binom{3}{2}$. For each pair there are $\binom{4}{2}$ options, so $\binom{13}{3}\binom{3}{2}\binom{4}{2}^2 \cdot 4/\binom{52}{5}$.

35. Since there are 13 cards, this means that one suit will have four cards while the other three suits contribute three cards each. $4\binom{13}{4}\binom{13}{3}^3$.

36. There are 6^{30} total results, $\frac{30!}{5!5!5!5!5!5!}/6^{30}$.

37. (a) We only need to consider the orderings of all the specified cards: $\frac{15!}{3!4!4!4!}/\frac{16!}{4!4!4!4!}$. Even simpler: there are four types of cards with equal probabilities, so there is $\frac{1}{4}$ chance the first one is an ace.

    (b) We only need to consider the orderings of all the specified cards: $3!\frac{12!}{3!3!3!3!}/\frac{16!}{4!4!4!4!}$ and simplifying yields $4^2 \cdot 3!/(15 \cdot 14 \cdot 13)$.

38. (a) Choose an arbitrary position for Tyrion in 12 ways, then a position for Cersei in 2 ways, then the remaining people in 10! ways to get $\frac{2}{11}$.

    (b) There are $\binom{12}{2}$ ways to put T+C in (not caring about the order in which they sit), and they are next to each other in 12 of them.

39. There are $\binom{2n}{k}$ committees. To have j married couples, we first choose j married couples in $\binom{n}{j}$ ways and the remaining $k-2j$ can be chosen one per couple in $2^{k-2j}\binom{n-j}{k-2j}$ ways.

40. (a) There are n^k possible sequences and $\binom{n}{k}$ increasing sequences.

    (b) There are n^k possible sequences and $\binom{n+k-1}{n-1}$ weakly increasing sequences.

41. There are n^n total outcomes and $n!(n-1)$ outcomes where exactly one box is empty. (This undercounts the number of ways since it assume that we fill up the $n-1$ boxes first, but we could instead double up before filling all the boxes.)

    Instead: first choose empty box in n ways, then choose pair of balls to go in same box in $\binom{n}{2}$ ways. Then place $(n-1)$ "objects" ($n-2$ balls and one pair of balls) into $n-1$ bins in $(n-1)!$ ways

$$\frac{n\binom{n}{2}(n-1)!}{n^n}$$

42. There are $\frac{26!}{(26-k)!}$ length k norepeatwords, so the probability is

$$\frac{1}{\frac{1}{26!}+\frac{1}{25!}+\cdots+\frac{1}{0!}} \approx \frac{1}{e},$$

which comes from the series expansion of $e = \sum_{n \geq 0} \frac{1}{n!} \approx \frac{1}{26!} + \cdots + \frac{1}{0!}$.

4 axioms of probability

43. Let $C = A \cap B$. Then $P(A) + P(B) = P(A \setminus C) + P(B \setminus C) + 2P(C)$. Since $P(A \setminus C) + P(C) + P(B \setminus C) \leq 1$, then $P(A) + P(B) - 1 \leq P(C)$. These are equal if $A \cup B$ is the entire sample space.

    $P(C) \leq P(A \cup B)$ since $C \subset A \cup B$. These are equal if $A \cap B = A \cup B$, i.e. if $A = B$.

    $P(A \cup B) \leq P(A) + P(B)$ by inclusion-exclusion with equality if A and B are disjoint.

44. $B = A \cup (B - A)$ (since $A \subset B$, which is not true in general) and both are disjoint, so $P(A) + P(B - A) = P(B)$.

45. $A \Delta B = A \setminus B \cup B \setminus A$ which are disjoint, so $P(A \Delta B) = P(A \setminus B) + P(B \setminus A) = P(A) - P(A \cap B) + P(B) - P(A \cap B)$.

46. $P(B_k) = P(C_k) - P(C_{k+1})$.

47. (a) let S be results of two coin flips, A be event first coin is heads, B be event second coin is heads.

    (b) A and B are independent if the area of $A \cap B$ is equal to the area of A times the area of B. Not independent: disjoint rectangles. Independent: rectangle with boundaries $[0, 1] \times [0, \frac{1}{2}]$ and vice versa.

    (c) $A \cup B$ and $A^c \cap B^c$ are disjoint and their union is the entire sample space. It remains to show that A^c and B^c are independent.

    $$P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B)$$

    $$= 1 - P(A) - P(B) + P(A)P(B) = (1 - P(A))(1 - P(B)) = P(A^c)P(B^c)$$

48. If $P_A(A) + P_A(B) > P_A(A \cup B)$, then sell an A and B certificate and buy an $A \cup B$ certificate. If A or B occurs, then we will cash in the $A \cup B$ certificate and have the A or B certificate cashed against us, leading to 0 net gain. However, since $P_A(A) + P_A(B) > P_A(A \cup B)$, we were able to buy the $A \cup B$ certificate for less than we sold both the A and B certificates.

    Reverse the trade if $P_A(A) + P_A(B) < P_A(A \cup B)$.

5 Inclusion-Exclusion

49. Let A_i be the event that i does not appear. The probability of A_I is $\frac{(6-|I|)^n}{6^n}$ for I a subset of $\{1, \ldots, 6\}$. Then the probability that at least one of the values never appears is $P(A_1 \cup \cdots A_6) = \sum_{i=1}^{6} (-1)^{i+1} \binom{6}{i} \frac{(6-i)^n}{6^n}$.

50. Let A_s be the event that the hand is void in suit s. Then $P(A_I) = \binom{52-13|I|}{13}$ and $P(A_h \cup A_s \cup A_c \cup A_d) = \sum_{i=1}^{4} (-1)^{i+1} \binom{4}{i} \binom{52-13i}{13}$.

51. Let A_s be the event that season s does not appear. Then the probabilty that all four seasons occur at least once is 1 minus the probabilty that at least one season does not appear, which is $\sum_{i=1}^{4}(-1)^{i+1}\binom{4}{i}\frac{(4-i)^7}{4^7}$.

52. Number the students $1,\ldots,20$ and let A_i be the event that student i sits in the same seat. Then the probability that no one sits in the same seat is 1 minus the probability that at least one student sits in the same seat, which is $\sum_{i=1}^{20}(-1)^{i+1}\binom{20}{i}\frac{(20-i)!}{20!}$

53. (a) Each position can be one of 62 characters, so there are 62^8 total passwords minus 36^8 passwords which do not use a lowercase letter.

    (b) There are 62^8 total passwords minus $2\cdot 36^8$ passwords which have no lowercase letters or no uppercase letters plus 10^8 passwords which only have letters because these were double subtracted.

    (c) $62^8 - 2\cdot 36^8 - 52^8 + 10^8 + 2\cdot 26^8 - 0^8$.

54. The probability of i days having no classes is $\frac{\binom{30-6i}{7}}{\binom{30}{7}}$ so the probabilty of having class every day is 1 minus the probability of having at least one day with no classes, which is $\sum_{i=1}^{5}(-1)^{i+1}\binom{5}{i}\frac{\binom{30-6i}{7}}{\binom{30}{7}}$.

55. (a) There are 37 people, so there are $\binom{37}{5}$ possible committees. If there are exactly 3 sophomores there are $\binom{15}{3}$ options and then we need to select 2 people from the remaining 22 $\binom{22}{2}$.

    (b)
$$1-\frac{1}{\binom{37}{5}}\left(\binom{22}{5}+\binom{25}{5}+\binom{27}{5}-\binom{10}{5}-\binom{12}{5}-\binom{15}{5}+\binom{0}{5}\right)$$

6 Mixed Practice

56. (a) $>$, $5!^2 < 4!6!$

    (b) $>$ same as before

    (c) $<$ there are 6 possible birthday arrangements for distinct birthdays

    (d) $<$ Martin loses if the first roll is tails and may not win if the first roll is heads.

57. The probability that I share no molecules with Caesar is $\left(\frac{10^{44}-10^{22}}{10^{44}}\right)^{10^{22}} = (1-\frac{1}{10^{22}})^{10^{22}} \approx \frac{1}{e}$.

58. (a) The inspector will test 9 if and only if they have found 0, 1, or 2 defectives or the 9th one is the 3rd defective. The total number of inspection outcomes is $\binom{12}{3}$. The four cases we found above are $\binom{9}{9}+\binom{9}{1}\binom{3}{2}+\binom{9}{2}\binom{3}{1}+\binom{8}{2}$.

(b) The inspector will test 10 if and only if they have found 1 or 2 defectives or the 10th one is the 3rd defective. The total number of inspection outcomes is $\binom{12}{3}$. The two cases we found above are $\binom{10}{1} + \binom{10}{2}\binom{2}{1} + \binom{9}{2}$.

59. (a) We just care how many chocolate bars each child gets: $\binom{24}{9}$.

    (b) The remaining 5 can be distributed freely $\binom{14}{9}$.

    (c) Each chocolate bar has 10 possibilities: 10^{15}.

    (d) Let A_i be the event that child i receives no chocolate bars. Then A_I has probability $(10-i)^{15}/10^{15}$, so we get $10^{15} - \sum_{i=1}^{10}(-1)^{i+1}\binom{10}{i}(10-i)^{15}$.

60. (a) There are n^n bootstrap samples.

    (b) There are $\binom{2n-1}{n-1}$ bootstrap samples where order doesn't matter.

    (c) There is only one way to get all a_i (which is the least likely), but there are $n!$ ways to get $a_1, \ldots, a_n$ (which is the most likely). The ratio is $n!$.

    There is 1 unordered bootstrap sample where each element is sampled once, while there are n unordered bootstrap samples where one element is sampled n times. The ratio of these probabilities is $(n-1)!$.

61. $\frac{1}{2}$ at each "random" point, that person can either take their own seat with probability $\frac{1}{n}$ (i.e. reset order), choose the last person's seat with probability $\frac{1}{n}$, or cause chaos with probability $\frac{2}{n}$. However, if they cause chaos, then the next person with a choice will face the same decision, leading to the same probability choices for them.

    Even cleaner: the last person can only sit in the first person's seat or their own, each with equal probabilities. (This is actually the same argument as the previous one I think...)

62. (a) We need to compute 1 minus the probability of no birthday match. So everyone needs to be born on distinct days, and the probability of any k subset of days is $k!p_{i_1}\cdots p_{i_k}$ so we need $k!e_k(p_1, \ldots, p_{365})$.

    (b) If all the probability is concentrated on one day, then the probability of birthday matches is 1. As long as some day is more likely than others, more people are likely to be born on that day than other days increasing the chance of a match.

    (c) In the three terms

    $$x_1x_2e_{k-2}(x_3, \ldots, x_n) + (x_1+x_2)e_{k-1}(x_3, \ldots, x_n) + e_k(x_3, \ldots, x_n)$$

    the first one gives all degree k monomials with both x_1 and x_2, the middle term gives all degree k monomials with one of x_1 and x_2, and the final term gives all monomials with no x_1 or x_2. This covers all cases, establishing the equality.

$$\begin{aligned} e_k(r_1, \ldots, r_{365}) &= r_1r_2e_{k-2}(r_3, \ldots, r_n) + (r_1+r_2)e_{k-1}(r_3, \ldots, r_n) + e_k(r_3, \ldots, r_n) \\ &= \frac{(p_1+p_2)^2}{4}e_{k-2}(p_3, \ldots, p_n) + (p_1+p_2)e_{k-1}(p_3, \ldots, p_n) + e_k(p_3, \ldots, p_n) \end{aligned}$$

From here I will consider only the coefficient of the first term:

$$(p_1^2 + 2p_1p_2 + p_2^2) = (p_1^2 + p_2^2) + 2p_1p_2 \geq 4p_1p_2 \implies r_1r_2 \geq p_1p_2$$

which implies the desired equality since these polynomials have a negative sign in front of them.