1 PMFs and CDFs

1. To get $X = k+1$, we need no matches among k people and then a match with the $(k+1)st$ person, so $P(X = k+1) = \frac{365}{365}\frac{364}{365}\cdots\frac{365-k+1}{365}\frac{k}{365}$.

2. (a) To get $X = k+1$, we need k failures and one success, so $P(X = k+1) = \frac{1}{2^{k+1}}$.

   (b) The first trial is success or failure with equal probability. After the first trial, the situation is the same as the previous problem, so $P(X = k+2) = \frac{1}{2^{k+1}}$.

3. $P(Y \leq y) = P(\mu + \sigma X \leq y) = P(X \leq \frac{y-\mu}{\sigma}) = F(\frac{y-\mu}{\sigma})$.

4. The floor function is increasing and right-continuous, so F has the same properties. $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq n$, so F also has the correct limiting behavior. The corresponding PMF is $f(x) = \frac{1}{n}$ if $x \in \{1, \ldots, n\}$.

5. (a) This can be realized as the PMF of the random variable X defined as the number of heads before the first tail if we flip a coin until we see heads.

   (b) $F(X \leq k) = \sum_{i=0}^{k} \frac{1}{2^{i+1}} = \frac{1}{2}\frac{1-.5^{k+1}}{.5}$

6. Since $1+1/j$ is greater than 1, then $\log(1+1/j)$ is positive. Adding these all together gives $\log(2/1 * 3/2 * \cdots * 10/9) = \log(10) = 1$.

7. To reach level j as the highest, Bob must reach levels $1, \ldots, j$ and not $j+1$ (if it exists). $P(X = j) = p_1 \cdots p_{j-1}(1-p_j)$ for $1 \leq j \leq 6$ and $P(X = 7) = p_1 \cdots p_6$.

8. The most valuable prize is the max of the five chosen. Then $P(X = j) = \frac{\binom{j-1}{4}}{\binom{100}{5}}$.

9. (a) Right-continuity and increasing follow from the corresponding facts for F_1 and F_2. The limiting behavior follows because $F_1, F_2 \xrightarrow{x\to-\infty} 0, F_1, F_2 \xrightarrow{x\to\infty} 1$ and $p + (1-p) = 1$.

   (b) Let X be the random variable described in the problem. Then $P(X \leq x) = P(X \leq x|H)P(H) + P(X \leq x|T)P(T) = pF_1(x) + (1-p)F_2(x)$.

10. (a) No, since the sum $\sum_{i\geq 1} \frac{1}{i}$ diverges, there is no constant we could multiply by to rescale the total probability to be 1.

    (b) Yes, since the sum of the inverse squares is finite, $\sum_{i\geq 1} \frac{1}{n^2} \leq 1 + \int_1^\infty \frac{1}{x^2}dx = 1 - \frac{1}{x}|_1^\infty = 2$.

11. If x is not an integer, then $P(X < x) = P(X \leq x)$. If x is an integer, then $P(X < x) = P(X \leq x) - P(X = x)$.

12. (a) If $X < Y$ are constants (i.e. $X = x$ and $Y = y$ with probability 1), then the same inequality holds for their CDFs.

    (b) It is impossible due to the summation condition, since then the probabilities for X will not sum to 1.

13. $P(X = a) = \sum_z P(X = a|Z = z)P(Z = z) = \sum_z P(Y = a|Z = z)P(Z = z) = P(Y = a)$.

14. (a) $P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-\lambda}$. $P(X \geq 2) = 1 - P(X = 1) - P(X = 0) = 1 - e^{-\lambda}\lambda - e^{-\lambda}$.
    (b) $P(X = k|X \geq 1) = P(X = k)/P(X \geq 1) = (e^{-\lambda}\lambda^k/k!)/(1 - e^{-\lambda})$

2 Named distributions

15. The CDF is $F(x) = \frac{\lfloor x \rfloor}{n}$ for $0 \leq x \leq n$, 0 for $x \leq 0$ and 1 for $x \leq n$.

16. $P(X = x|X \in B) = 0$ if $x \notin B$ and $\frac{1}{|B|}$ if $x \in B$.

17. There must be at least 10 people who don't show up, so $\sum_{i=10}^{110} \binom{110}{i}(1-p)^i p^{110-i}$.

18. (a) A must win at least four games, so $\sum_{i=4}^{7} \binom{7}{i} p^i (1-p)^{7-i}$.
    (b) It does not matter, since if a team wins 4 games, then no matter what the remaining games result in, that team has already one. (This is why we can compute the probability as in the previous problem.)

19. Let X be the number of drawn games. Then X is distributed as $\text{Bin}(n, .6)$ so the PMF is $P(X = k) = \binom{n}{k}.6^k.4^{n-k}$.

20. (a) The number of winning tickets is distributed as $\text{Bin}(3, p)$.
    (b) The probability of at least one ticket winning is probability of each ticket winning, minus probability of two tickets winning, plus probability of three tickets winning $3p - 3p^2 + p^3$.

    The probability of at least one winning ticket is 1 minus the probability of no ticket winning, so $1 - (1-p)^3 = 1 - (1 - 3p + 3p^2 - p^3) = 3p - 3p^2 + p^3$.
    (c) If p is small, then $p^2, p^3 \approx 0$, so the probability of having at least one winning ticket is $\approx 3p$.

21. $X - Y$ can have negative values, so it is not binomial.

22. (a) $P(X = k) = P(X = k|C_1)P(C_1) + P(X = k|C_2)P(C_2) = \frac{1}{2}\binom{n}{k}(p_1^k(1-p_1)^{n-k} + p_2^k(1-p_2)^{n-k})$.
    (b) If $p_1 = p_2$, then the above reduces to $\binom{n}{k}p^k(1-p)^{n-k}$.
    (c) When n is very large, there will be two distinct "local maxima" of the PMF of X, whereas a binomial distribution would only have one maximum probability at $\frac{n}{2}$.

23. Let I_j be the indicator of the j^{th} person voting for Kodos, V_j be the indicator of the j^{th} person voting, R_j be the indicator of the j^{th} person registering, and $X = \sum_j I_j$. Then $P(I_j = 1) = P(I_j = 1|V_j = 1)P(V_j = 1) = p_3 P(V_j = 1|R_j = 1)P(R_j = 1) = p_3 p_2 p_1$. We have simplifications in each case since if a person does not vote, then they

cannot vote for Kodos and if a person does not register, then they cannot vote. Thus $X \sim \text{Bin}(n, p_1p_2p_3)$.

24. (a) Let Y be the number of heads in 8 fair coin tosses. Then $P(X = k|HH) = P(Y = k-2)$.

    (b) Let H_2 be at least two heads. $P(X = k) = P(X = k|H_2)P(H_2) + P(X = k|H_2^c)P(H_2^c)$. If $k \geq 2$, then $P(X = k|H_2^c) = 0$ so $p(X = k|H_2) = P(X = k)/P(H_2)$. $P(H_2) = 1 - \frac{11}{2^{10}}$. Else, if $k \leq 1$, then $P(X = k|H_2) = 0$.

25. (a) X and $n - X$ are identically distributed and Y and $n+1-Y$ are identically distributed. Thus the two probabilities are equal.

    (b) $P(X < Y) = P(X + 1 > Y) = P(X \geq Y)$. Furthermore, $P(X < Y) + P(X \geq Y) = 1$ since $X < Y$ and $X \geq Y$ are disjoint an exhaustive. Thus $P(X < Y) = \frac{1}{2}$.

26. From the story proof, $n - X$ is the number of black balls, so $n - X$ is distributed as $\text{HGeom}(b, w, n)$.

27. If I_j is the indicator variable of the j^{th} card being in the correct position, then it is Bernoulli with probability $\frac{1}{n}$. However, the I_j are not independent, so X is not Binomial. X is also not hypergeometric, for example it is impossible to have $X = n-1$ while it is possible to have $X + n - 2$ or $X = n$.

28. Let H be the number of hatches. Then $H \sim \text{Bin}(n, p)$. Let S be number of survives and I_j the indicator of whether the j^{th} chick survives (so $S = \sum_j I_j$). Then $P(I_j = 1) = P(I_j = 1|H_j = 1)P(H_j) = pr$. Thus $S \sim \text{Bin}(n, pr)$.

29. Given a fixed number of successes k, the probability of any outcome with k successes is $p^k q^{n-k}$. Then $P(outcome|ksuccesses) = P(ksuccesses, outcome)/P(ksuccesses) = P(outcome)/P(ksuccesses)$.

30. (a) There are m men and n women, and we select t random employees and see how many women are among them. This is $\text{HGeom}(n, m, t)$.

    (b) The number of women promoted is distributed as $\text{Bin}(n, p)$, the number of women who are not promoted is distributed as $\text{Bin}(n, 1-p)$, and the distribution of the number of employees who are promoted is $\text{Bin}(m + n, p)$.

    (c) Let X be the number of women promoted and Y be the number of men promoted, both of which are binomially distributed by the previous part. Then the conditional distribution of X given $X + Y = t$ is $\text{HGeom}(n, m, t)$.

31. (a) If she has no ability, then all $\binom{6}{3}$ choices are equally likely. In order to get at least 2 correct, there are $\binom{3}{2}\binom{3}{1} + \binom{3}{3}$ ways, so the probability is $10/20 = \frac{1}{2}$. (I guess this makes sense, at least 2/3 correct means at most 1/3 incorrect and at most 1 correct means at least 2/3 incorrect.)

    (b) Let A be milk first and B be lady says milk first. Then $\frac{P(A|B)}{P(A^c|B)} = \frac{P(B|A)P(A)}{P(B|A^c)P(A^c)} = p_1/(1-p_2)$.

32. (a) The key terms have two taggings: first by whether they appear on the exam and second by whether Evan studies them. Then $X \sim \text{HGeom}(10, 90, s) = \text{HGeom}(s, 100 - s, 10)$.

    (b) $P(X \geq 7) = P(X = 7) + \cdots + P(X = 10) = \frac{1}{\binom{100}{75}}(\binom{10}{7}\binom{90}{68} + \binom{10}{8}\binom{90}{67} + \binom{10}{9}\binom{90}{66} + \binom{10}{10}\binom{90}{65}) = 0.7853844(R) = 0.7853843668965791(python)$

33. (a) Let I_j be the indicator of at least one proofreader catching the j^{th} typo. Then $P(I_j = 1) = 1 - P(I_j = 0) = 1 - (1 - p_1)(1 - p_2) = 1 - (1 - p_1 - p_2 + p_1 p_2) = p_1 + p_2 - p_1 p_2$. Then $X \sim \text{Bin}(n, p_1 + p_2 - p_1 p_2)$.

    (b) In this situation, we can consider tagging the typos both with who is reading as well as whether they are caught by Prue. Thus $X_1 \sim \text{HGeom}(n, n, t)$.

34. (a) Let Y be the number of statistics majors in the sample. Since X is binomial, each student has independent probability p of being a statistics major, so $Y \sim \text{Bin}(m, p)$. (If I_j is the indicator of student j being a statistics major, then $X = \sum_j I_j$ and $Y = \sum_{j \in S} I_j$ where S is the sample of size m.)

    (b) See previous part.

35. (a) Let A be the number of questions A gets right. Then $A \sim \text{Bin}(m, p_1)$.

    (b) Let B be the number of questions B gets right. Then $B \sim \text{Bin}(n, p_2)$. The total number is $A + B$ which has PMF $P(A + B = k) = \sum_{i=0}^{k} P(A = i)P(B = k - i)$. This is a binomial distribution if and only if $p_1 = p_2$ or $m = 0$ or $n = 0$.

    (c) Let A be A wins. If A gets the first question right, then A wins. If A gets the first question wrong, then A loses if B gets the second question right and returns to the original scenario if A gets the second question wrong. Thus $P(A) = p_1 + (1 - p_1)(1 - p_2)P(A) \implies P(A) = \frac{p_1}{p_1 + p_2 - p_1 p_2}$.

36. (a) $X \sim \text{Bin}(n, .5)$, so the probability of a tie is $\binom{n}{n/2}.5^n$.

    (b) $n!/(n/2)!^2 * .5^n \approx \sqrt{2\pi n}(n/e)^n / \sqrt{\pi n}^2 (n/2e)^n * 2^n = 1/\sqrt{(\pi/2)n}$

37. (a) This method only detects an odd number of errors, so there are undetected errors if and only if there are 2 errors or 4 errors: $\binom{5}{2}.1^2.9^3 + \binom{5}{4}.1^4.9^1$.

    (b) $\sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n}{2i} p^{2i}(1 - p)^{n-2i}$.

    (c) Let b be as suggested. Then $(a + (1 - p)^n) + b = (p + (1 - p))^n = 1$ and $(a + (1 - p)^n) - b = (1 - p - p)^n = (1 - 2p)^n$. Thus $b = 1 - (a + (1 - p)^n)$ so $(a + (1 - p)^n) - (1 - (a + (1 - p)^n)) = 2a + 2(1 - p)^n - 1 = (1 - 2p)^n \implies a = \frac{1 + (1 - 2p)^n}{2} - (1 - p)^n$.

3 Independence of r.v.s

38. (a) X is any RV and $Y = X + 1$.

(b) X is negative number of heads and Y is number of heads plus one.

39. X be discrete uniform, and Y be an arbitrary permutation of X. If X and Y are independent, then it is impossible since $P(X = Y) = \sum_{i=1}^{10} P(X = i)P(Y = i) > 0$.

40. (a) If $P(X = Y) = 1$, then $P(X = i) = P(Y = i)$ for all i.

    (b) It is not possible unless X and Y are trivial, since if $j \neq k$ but j in support of X and k in support of Y, then $0 = P(X = j, Y = k) \neq P(X = j)P(Y = k)$.

41. No, for example if $Z = X$.

42. X and Y have the same distribution, but $X < Y$ with probability $\frac{6}{7}$.

43. (a) Yes, to have $P(X < Y) \geq \frac{b-1}{b}$, we can set $n = b$, or let n be infinite.

    (b) Yes, the answers change. Since for independent random variables with the same distribution, $P(X < Y) = P(Y < X) \leq \frac{1}{2}$.

44. (a) $X \oplus Y = 0$ if $X = Y$ (which has probability $\frac{p+q}{2} = \frac{1}{2}$) and $X \oplus Y = 1$ if $X \neq Y$ (which has probability $\frac{p+q}{2} = \frac{1}{2}$), so $X \oplus Y \sim \text{Bern}(.5)$.

    (b) $P(X \oplus Y = i|X = j) = P(j \oplus Y = i) = \frac{1}{2} = P(X \oplus Y = i)$ so $X \oplus Y$ and X are independent no matter what p is. $P(X \oplus Y = i|Y = j) = P(X \oplus j = i) \neq \frac{1}{2}$ so $X \oplus Y$ and Y are not independent unless $p = \frac{1}{2}$.

    (c) Iinduction on (a) shows that $Y_J \sim \text{Bern}(1/2)$. For any $Y_J, Y_{J'}$, we can write $Y_J = (Y_J \oplus Y_{J'}) \oplus Y_{J'}$ and $Y_J \oplus Y_{J'}$ is of the form $Y_{J''}$ for $J'' = J \Delta J'$ denoting the symmetric difference. Thus by (b) Y_J and $Y_{J'}$ are independent. However, not all the Y_J are independent since knowing $Y_{\{i\}}$ for all i determines all the Y_J.

4 Mixed practice

45. (a) Let B be new treatment better. Then

$$P(B|15) = \frac{P(15|B)P(B)}{P(15)} = \frac{2\binom{20}{15}.6^{15}.4^5}{2\binom{20}{15}.6^{15}.4^5 + \binom{20}{15}.5^{20}}$$

    (b) We update to use p and $1 - p$ instead of $2/3$. Let I_j be the indicator of whether the treatment is more effective on patient j. Then $P(I_j = 1|15) = P(I_j = 1|B, 15)P(B|15) + P(I_j = 1|B^c, 15)P(B^c|15) = .6p + .5(1 - p) = .5 + .1p$.

    Thus the desired random variable is a mixture of $Y = pX_1 + (1 - p)X_2$ where $X_1 \sim \text{Bin}(20, .6)$ and $X_2 \sim \text{Bin}(20, .5)$.

46. (a) p_k is the probability of A having k more losses than twice the number of wins. Each gamble can be thought of as an experiment, and we stop if we ever reach 0 but otherwise keep going forever. Thus the union of all the events $\cup_{i \geq 1} A_i$ is the probability that we have more than twice as many losses as wins, and the union is contained within A_1.

(b) $p_0 = 1$. $2p_k = p_{k-1} + p_{k+2}$. This difference equation has the characteristic equation $x^3 - 2x + 1 = 0$, which factors as $(x-1)(x^2+x-1)$, thus whose roots are $x = 1, \frac{-1\pm\sqrt{5}}{2}$. Thus the general solution is $p_k = a(1)^n + b(\frac{-1+\sqrt{5}}{2})^n + c(\frac{-1-\sqrt{5}}{2})^n$. When $n = 0$, we have $a + b + c = 1$.

As $n \to \infty$, we have $(\frac{-1+\sqrt{5}}{2})^n \to 0$ and $|(\frac{-1-\sqrt{5}}{2})^n| \to \infty$. In particular, since $a(1)^n + b(\frac{-1+\sqrt{5}}{2})^n$ is finite and probabilities must lie between 0 and 1, then $c = 0$. Furthermore, since $p_n \to 0$, then $a = 0$ since otherwise p_n will converge to a nonzero constant. Thus $b = 1$.

(c) $p_1 = \varphi - 1$.

47. (a) The probability is 0 for $m < \frac{n}{2}$ and 1 for $m > n$. If we consider a success as taking from tray 1 and a failure as taking from tray 2, then we want the probability that $X \sim \text{Bin}(n, .5)$ lies between $a = \max(n-m, 0)$ and $b = \min(n, m)$ (inclusive), so $pbinom(b, n, .5) - pbinom(a-1, n, .5)$.

(b) 8, 60, 531, 5098 for $n = 10, 100, 1000, 10000$.