1 Markov property

1. $P(X_{2n} = x_{2n}|X_{2n-2} = x_{2n-2}, \dots, X_0 = x_0) = \sum_{x_{2n-1}} P(X_{2n} = x_{2n}|X_{2n-1} = x_{2n-1}, X_{2n-2} = x_{2n-2}, \dots, X_0 = x_0)P(X_{2n-1} = x_{2n-1}|X_{2n-2} = x_{2n-2}, \dots, X_0 = x_0) = \sum_{x_{2n-1}} P(X_{2n} = x_{2n}|X_{2n-1} = x_{2n-1}, X_{2n-2} = x_{2n-2})P(X_{2n-1} = x_{2n-1}|X_{2n-2} = x_{2n-2}) = P(X_{2n} = x_{2n}|X_{2n-2} = x_{2n-2})$ so the even index rvs satisfy the Markov property. This makes sense intuitively since X_{2n} depends on X_{2n-1} and thus indirectly on X_{2n-2}. When X_{2n-1} is removed from the Markov chain, then the dependence on X_{2n-2} becomes direct by hiding the information from X_{2n-1}.

2. (a) Since X_0 follows the stationary distribution, then all X_i follow the stationary distribution. Thus $P(X_i = 3) = s_3$ so the distribution of how many are 3 is $Bin(10, s_3)$ which has expected value $10s_3$.

   (b) If $X_n = 1, 2$ then $Y_n = 0$ and if $X_n = 3$ then $Y_n = 2$. The transitions $3 \to 1, 2$ in X correspond to the transition $2 \to 0$ in Y. Analogously, $1, 2 \to 3$ corresponds to $0 \to 2$ and $1 \leftrightarrow 2$ is a self-edge from 0 to 0. (All self-transitions of X correspond to self transitions of Y as well.)

   Since Y identifies the transitions $1, 2 \to 3$, we can make Y Markov by making these probabilities equal and otherwise we can make Y fail to be Markov by making these probabilities different. If Y starts in state 0, we do not know the transition probility to state 2 unless we know whether the corresponding X starts in state 1 or 2.

3. (a) Consider the sequence 01. The 0 first indicates that the Y_0 starts at A and then the 1 indicates that Y_1 moves to B, so we know that $Y_2 = 1$ meaning a switch back to A. However, given only $Y_1 = 1$, we do not know the current state of X_1 so the probabilities for Y_2 also depend on Y_0.

   (b) Consider the sequence $01 \dots 1$ consisting of a zero followed by m ones. Then the same argument as last time applies, since knowing the initial zero fixes the transition probabilities while only knowing the next m 1's does not.

4. (a) Let X_i be the current block after jump i with state space $\{1, 2, 3\}$. The transition matrix is

$$\begin{array}{ccc} .5 & .5 & 0 \\ .5 & 0 & .5 \\ 0 & 0 & 1 \end{array}$$

   $E(J) = 1 + E(J|X_1 = 1)/2 + E(J|X_1 = 2)/2 = 1 + E(J)/2 + (E(J|X_1 = 2, X_2 = 3)/2 + E(J|X_1 = 2, X_2 = 1)/2 + 1)/2 = 1 + E(J)/2 + 1/2 + E(J)/4 \implies E(J) = 6.$

   (b) This problem is exactly the same as the expected number of flips needed to obtain two heads in a row when flipping a fair coin.

2 Stationary distribution

5. (a) $\begin{matrix} p & 1-p \\ 1-p & p \end{matrix}$.

   (b) Since both rows and columns sum to 1, this matrix has stationary distribution $(1/2, 1/2)$.

   (c) We could recognize this as the birth death chain / ehrenfest chain with two particles. However, explicitly, if $p = 1/2$, then $Q^n = Q$ for all n. Otherwise, rewrite the probabilities as $p = 1/2 + x$ and $1 - p = 1/2 - x$ with $-1/2 < x < 1/2$. Then $Q^2 = \begin{matrix} 1/2 + 2x^2 & 1/2 - 2x^2 \\ 1/2 - 2x^2 & 1/2 + 2x^2 \end{matrix}$. For these values of x, $2x^2 < |x|$ so Q^n converges to the all 1/2 matrix.

6. (a) $Q = \begin{matrix} .5 & .5 & 0 & 0 \\ .25 & .75 & 0 & 0 \\ 0 & 0 & .25 & .75 \\ 0 & 0 & .75 & .25 \end{matrix}$.

   (b) All states are recurrent and no states are transient.

   (c) Since the chain is reducible, any suitably weighted linear combination of the stationary distributions for the two parts is going to be another stationary distribution. For the left chain, a stationary distribution is $(1/3, 2/3)$ while for the right chain, a stationary distribution is $(1/2, 1/2)$. Thus there are infinitely many stationary distributions of the form $(a/3, 2a/3, b/2, b/2)$ where $a + b = 1$.

7. (a) Since X_0 is generated according to the stationary distribution, then $X_i \sim s$ for all i so the expected number of times Drogon is home is $25s_0$.

   (b) Since the translation between W_n and (X_n, Y_n, Z_n) is bijective, it suffices to show that the triple is a Markov chain (in the sense that $P(X_n = x_n, Y_n = y_n, Z_n = z_n | X_i, Y_i, Z_i) = P(X_n, Y_n, Z_n | X_{n-1}, Y_{n-1}, Z_{n-1})$), which is true because each X, Y, Z is itself a Markov chain and the dragons independently explore the world.

   (c) The probability that all three dragons are at home is s_0^3 so the expected time is $1/s_0^3$.

8. Each state transitions with uniform probability to one of $\binom{52}{2}$ other states. In particular, every state is symmetric with every other so the stationary distribution must be $1/52!$ with each state equally likely.

3 Reversibility

9. (a) $Y_n = |X_n|$ is a Markov chain, since the transitions of Y are well-defined. The possible states are $0, 1, 2, 3$ with $0 \to 1, 3 \to 2$ with probability 1 and $1 \to 0, 2$ and $2 \to 1, 3$ with probability $1/2$.

(b) No, this is not a Markov chain, since if we only know $sgn(X_n) = 1$, the state could be in any of $1, 2, 3$ and whether a transition to $sgn(X_{n+1}) = 0$ is possible could depend on previous information.

(c) This can also technically be modeled as a birth-death chain. However, we can also easily get it by looking at degrees: the proportions of "particles" in $-2, -1, 0, 1, 2$ must be the same and the proportions in $-3, 3$ must be the same and half of $-2, -1, 0, 1, 2$. Thus the stationary distrbution is $(1/12, 1/6, 1/6, 1/6, 1/6, 1/6, 1/12)$.

(d) We can add a transition back and forth $3 \to -3$ and $-3 \to 3$ both with probability $1/2$ and modifying the transition probabilities to be $1/2$ for $-3 \to -2, 3 \to 2$. This will make the entire Markov chain a cycle.

10. (a) For $i \neq j$, $q_{ij} = 1/d_i \min(d_i/d_j, 1) = \min(1/d_j, 1/d_i)$ since we first uniformly choose an edge adjacent to v_i and then select whether to make the transition. Otherwise set $q_{ii} = 1 - \sum_{j \neq i} q_{ij}$.

    (b) The transitions are symmetric, since $q_{ij} = \min(1/d_j, 1/d_i) = \min(1/d_i, 1/d_j) = q_{ji}$. Thus the transition matrix is symmetric so the uniform distrbution $(1/M, \dots 1/M)$ is stationary.

11. (a) All transitions are symmetric, so the uniform distribution $(1/7, \dots, 1/7)$ is stationary.

    (b) This is the same as the previous chain with $-3, -2, -1, 0, 1, 2, 3$. The stationary distribution is proportional to the degree of each vertex: $(1/12, 1/6, 1/6, 1/6, 1/6, 1/6, 1/12)$.

12. (a) The Y's are a Markov chain since the transition matrix is Q^{365}. (Equivalently, given Y_n, there is no other relevant information for computing Y_{n+1} in previous Y's.)

    (b) The degree argument works here, so the ends have relative probability 2, one from the end has relative probability 3, and all the interior ones have relative probability 4. The total sum is 110 so the stationary distribution is $1/110(2, 3, 4, \dots, 4, 3, 2)$.

13. The degree argument works here. The four corners have degree 3, the remaining 24 edge squares have degree 5, and the interior squares have degree 8. Thus the stationary distribution has probabilities proportional to $3, 5, 8$ with normalizing constant 420.

14. (a) The Markov chain is irreducible for every piece except the bishop, since the bishop if started on a light square must remain on a light square and vice versa.

    The Markov chain is aperiodic except for the knight, which has period 2 since the knight must visit an opposite color square each move.

    (b) The uniform distribution is stationary for the rook, since from any (i, j) the rook can move to any of 14 other squares with equal probability.

    (c) By the previous problem (13), the corners have probability $3/420 = 1/140$ in the station ary distribution, so the expected number of turns to return is 140.

(d) This follows from problem 10, since the construction here is equivalent. Under this construction transition matrix is symmetric, so the modified chain now has a uniform stationary distribution.

15. This is a birth-death chain, so we could use the given result. Instead, we will argue by balancing the incoming and outgoing proportions at each node (i.e. using the reversibility condition). Let x be the stationary probability of state 1. Then the stationary probability of 2 is $2x$. Then the stationary probability of state 3 is $4x$. Then the stationary probability of state 4 is $8x$ and finally the stationary probability of state 4 is $16x$. This is due to the fact that at each left-right transition, there is twice as much probability to the right as there is to the left. Thus $1 = x + 2x + 4x + 8x + 16x = 31x$ so the normalizing constant is $1/31$ and the stationary distribution is $(1/31, 2/31, 4/31, 8/31, 16/31)$.

16. (a) The number of black balls can change by at most one each time, so we only need to consider transitions of the form $q_{i,i-1}, q_{i,i}, q_{i,i+1}$. The total number of black balls is N, so there are $N - X_n$ black balls (and thus X_n white balls) in the second urn. In order to decrease the number of black balls in the first urn, we need to select a black ball in the first urn and a white ball in the second, which has probability $(i/N)^2$. In order to leave the number unchanged, we need to select the same color ball in each urn, which has probability $2(i/N)(N-i)/N$. In order to increase the number, we need to select a white ball in the first urn and a black ball in the second, which has probability $((N-i)/N)^2$.

    (b) The reversibility condition is $s_i q_{ij} = s_j q_{ji}$. Computing $s_i q_{i,i-1} = \binom{N}{i}\binom{N}{N-i}/\binom{2N}{N}(i/N)^2 = \binom{N}{i-1}\binom{N}{N-i+1}/\binom{2N}{N}((N-i+1)/N)^2$. By symmetry this accounts for all cases, so this distribution is stationary.

17. By symmetry, $s_i = s_j$ for all $i, j \neq 0$. The reversibility condition implies $s_j q_{j0} = s_j r = s_0 q_{0j} = s_0 p \implies s_0 = s_j r/p$. Together with $1 = 110 s_j + s_0 = 110 s_j + s_j r/p = s_j(110 + r/p) \implies s_j = 1/(110 + r/p)$ and $s_0 = r/p/(110 + r/p)$.

18. This Markov chain is reversible, since the ratio of all 2-cycles is constant 2 in the "up" direction. Thus if the stationary probability of 1 is x, then we can take $2x$ for $2, 3$ and $4x$ for $4, 5, 6, 7$.

19. (a) It seems easier to make a two state chain wiht states A, B. The transition probabilities are $A \to A$ with probability a, $B \to B$ with probability b, $A \to B$ with probability $1-a$ and $B \to A$ with probability $1-b$.

    (b) This chain is reversible.

    (c) The stationary distribution is $s_A = (1-b)/(2-a-b)$ and $s_B = (1-a)/(2-a-b)$.

    (d) The probability of success when independently randomly choosing for each visitor is $1/2(a+b)$. The probability of success using the stationary distribution is $b(1-a)/(2-a-b) + a(1-b)/(2-a-b)$. Comparing these two boils down to using the AM-GM inequality on $(a^2+b^2)/2 \geq \sqrt{a^2b^2} = ab$. (Details: $(b - ab + a - ab)/(2-a-b) > (a+b)/2 \iff a + b - 2ab > (a+b) - (a+b)^2/2 \iff$

$-4ab > -a^2 - 2ab - b^2 \iff -2ab > -a^2 - b^2 \iff a^+b^2 > 2ab$ which is true by AMGM.)

20. (a) $q_{ij} = w_{ij}/\sum_k w_{ik}, q_{ji} = w_{ji}/\sum_k w_{jk}$. Thus, defining $v_i = \sum_j w_{ij}$ satisfies $v_i q_{ij} = v_j q_{ji}$ so the Markov chain is reversible with stationary distributions proportional to v_i.

    (b) Consider a reversible Markov chain with M states and stationary distribution s and transition matrix Q, thus satisfying $s_i q_{ij} = s_j q_{ji}$. Then in analogy with the previous part, define a graph on M nodes with an edge $i \leftrightarrow j$ of weight $s_i q_{ij}$. The transition probabilities of the random walk are $q'_{ij} = s_i q_{ij}/\sum_k s_i q_{ik} = q_{ij}$ so this agrees with the original Markov chain.

21. (a) The cat chain has stationary probabilities $(1/2, 1/2)$ since its distribution is symmetry. Since the mouse moves to room 1 with twice the probability that it moves to room 2, the stationary distribution is $(2/3, 1/3)$.

    (b) Since we are assuming that cat and mouse move independently, then $Z_0, Z_1, \dots, Z_n$ is a Markov chain. (It obeys the Markov property: knowing the next move only depends on the current state.)

    (c) Let X_{12} be the expected time until the cat eats the mouse when C1M2 and conversely X_{21}. Then $X_{12} = 1 + .32(0) + .12(0) + .08X_{12} + .48X_{21} \implies .92X_{12} - .42X_{21} = 1$ and $X_{21} = 1 + .56(0) + .06(0) + .14X_{21} + .24X_{12} \implies -.24X_{12} + .86X_{21} = 1$.

22. (a) This is the same as the undirected network, so we just need to consider the degrees to figure out the stationary distribution. Since A_n, B_n are independent, then $P(A_n = i, B_n = j) = P(A_n = i)P(B_n = j)$. The long term distribution limits to the stationary distribution, so $\lim_{n\to\infty} P(A_n = i)P(B_n = j) = d_i d_j/(\sum_k d_k)^2$.

    (b) This is the same as the cat and mouse problem just with many more rooms. $t_{i,k} = 1 + p_{ik}p_{kk}(0) + p_{ii}p_{ki}(0) + \sum_{j,l} p_{ij}p_{kl}t_{j,l}$ for any i, k.

23. There is probability .2 of moving from state 0 to state 1 and then .2 probability of moving from state 1 to state 2. There are no backwards moves, so T is the sum of two iid $FS(.2)$ random variables. Hence $E(T) = 10$ and $Var(T) = 2(.8/.04) = 40$.

24. (a) There is no way to transition from $3, 4, 5, 6$ to $1, 2$, so the chain will never return to state 1 after leaving state 2 to state 3. The probability that this happens (without first returning to state 1) is .25 (first go to state 2 then go to state 3), so the distribution of the number of times the chain returns to state 1 is $Geom(.25)$. (More explicitly, we the three possibilities frmo state 1 are self transition, go to state 2 then back to state 1, go to state 2 then to state 3, the last of which has probability .25.)

    (b) All transition probabilities (except 2 to 3) are symmetric, and there is no way to reach states 1,2 after reaching states 3,4,5,6, so the long term fraction of time in state 3 is 1/4.

25. (a) All states are transient except M, which is recurrent. Since the probability of transitioning to state M (in some number of steps) is nonzero from all states, then the chain will always eventually reach M at which point no other states are reachable.

(b) The limit is the matrix with ones all along the rightmost column since for large enough n, all paths eventually converge to M.

(c) $(Q^n)_{i,j}$.

(d) Let I_k be the indicator of whether the chain is in state j at time k. Then the desired number is $\sum_{k=0}^{n} I_k$ and the expected value is $\sum_{k=0}^{n} E(I_k) = \sum_{k=0}^{n}(Q^k)_{i,j}$.

(e) By Taylor expansion, $(I-R)^{-1} = I + R + R^2 + \cdots$. Since the last row and column of Q only affect the last row and column of Q in the powers Q^k, then the result of deleting the last row and column of Q^k is R^k, so by the previous problem the (i,j) entry of R^k is the probability of being in state j at time k given that the chain started in state i at time 0 and thus the (i,j) entry of $\sum_{k\geq 0} R^k$ is the expected number of times that the chain is in state j before absorption given that it started in state i.

26. (a) It is impossible to "end up" in a state (grid cell) corresponding to the start of a ladder or chute. Thus, only the states $1, 3, 4, 7, 9$ are possible so we only need to consider these 5 states.

(b) It is impossible to avoid visiting square 7 on the way to square 9. Thus, we can consider the first time we reach state 7. From there, we will either end the game with probability .5 or start over (until we reach state 7 again) with probability .5. Thus the distribution is $1 + Geom(.5) = FS(.5)$, so the mean is 2 and the variance is 2.

(c) The game ends when we reach state 9 from state 1. $P(L \leq k)$ is equal to the sum of the probabilities that we reach state 9 in $1 \leq i \leq k$ moves. Thus since state 9 is absorbing, the median duration is the smallest k such that the upper right entry of Q^k is at least .5. This occurs for $k = 7$.

(d) The total duration is also equal to the total amount of time not spent in the absorbing state, so the expected duration is the expected time in any transient state, which is the sum of the expected time in each transient state. By the previous problem, this is the sum of the entries of $(I-R)^{-1}$ which is 31.2.