1 Conditioning on evidence

1. Let S be spam and F be free money, then

$$P(S|F) = \frac{P(F|S)P(S)}{P(F|S)P(S) + P(F|\neg S)P(\neg S)} = \frac{.1 * .8}{.1 * .8 + .01 * .2} = \frac{8}{8.2} \approx .9756$$

2. Let I be identical, BB be twin boys, then

$$P(I|BB) = \frac{P(BB|I)P(I)}{P(BB|I)P(I) + P(BB|I^c)P(I^c)} = \frac{.5/3}{.5/3 + .5/3} = \frac{1}{2}.$$

3. Let S be smoker and L be lung cancer, then

$$P(S|L) = \frac{P(L|S)P(S)}{P(L)} = \frac{.216P(L|S)}{.216P(L|S) + (1 - .216)P(L|\neg S)} = \frac{.216}{.216 + .784/23} \approx .8637$$

4. (a)

$$P(K|R) = \frac{P(R|K)P(K)}{P(R|K)P(K) + P(R|\neg K)P(\neg K)} = \frac{p}{p + \frac{1-p}{n}}$$

   (b) $p + \frac{1-p}{n} \leq 1$ if $n \geq 1$ so the ratio is larger or equal to p. This makes sense because in order to get the question correct, Fred either had to know the answer or guess correctly. As n gets larger, the probability of guessing correctly goes to 0.

   The formula gives equality when $n = 1$, because then whether or not Fred gets the answer correct has no relationship with whether he knew the answer to begin with.

5. Let A_1, E_2, A_3 be ace spades first, eight clubs second, and any other ace third.

$$P(A_3|A_1, E_2) = \frac{P(A_1, E_2, A_3)}{P(A_1, E_2)} = (1/52) * (1/51) * (3/50)/(1/52) * (1/51) = 3/50.$$

   We could also have seen this directly since once the first two cards are known, there are 3/50 ways to get an ace next.

6. Let D be double headed and 7 be seven heads. Then

$$P(D|7) = \frac{P(7|D)P(D)}{P(7|D)P(D) + P(7|\neg D)P(\neg D)} = \frac{1}{1 + \frac{99}{2^7}} \approx .5639$$

7. (a) Let 7 be seven heads.

$$P(D|7) = \frac{P(7|D)P(D)}{P(7|D)P(D) + P(7|\neg D)P(\neg D)} = \frac{(.01 + .99/2^7)}{(.01 + .99/2^7) + 1/2^7} \approx .6942$$

   (b) Let C be chosen double heads.

$$P(C|7) = \frac{P(7|C)P(C)}{P(7|C)P(C) + P(7|\neg C)P(\neg C)} = \frac{1/200}{1/200 + 1/2^7 * 199/200} \approx .3914$$

8. Let D be defective, then

$$P(A|D) = \frac{P(D|A)P(A)}{P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)} = \frac{.01 * .5}{.01 * .5 + .02 * .3 + .03 * .2} \approx .2941$$

9. (a) A_i implies B means that $A_i \subset B$, so $P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = P(A_i)/P(B)$.

   (b) Since both A_1 and A_2 imply B, there is nothing to distinguish them just knowing that B occurred, so the equality of the prior probabilities must also result in equality of the conditional probabilities.

10. (a) $P(A_3|A_1) = P(A_3|A_1, A_2)P(A_2|A_1) + P(A_3|A_1, \neg A_2)P(\neg A_2|A_1) = .8*.8+.3*.2 = .7$

    (b) $P(A_3|\neg A_1) = P(A_3|\neg A_1, A_2)P(A_2|\neg A_1) + P(A_3|\neg A_1, \neg A_2)P(\neg A_2|\neg A_1) = .8 * .3 + .3 * .7 = .45$

    $P(A_3) = P(A_3|A_1)P(A_1) + P(A_3|\neg A_1)P(\neg A_1) = .75 * .7 + .45 * .25 = .6375$

11. $.6 = P(A|W) = \frac{P(W|A)P(A)}{P(W)} = \frac{.7p}{.7p+.3(1-p)} \implies p = \frac{9}{23} \approx .3913$

12. (a) $P(A_1|B_1) = \frac{P(B_1|A_1)P(A_1)}{P(B_1|A_1)P(A_1)+P(B_1|A_0)P(A_0)} = \frac{.9*.5}{.9*.5+.05*.5} \approx .9474$

    (b) $P(A_1|B_{110}) = \frac{P(B_{110}|A_1)P(A_1)}{P(B_{110}|A_1)P(A_1)+P(B_{110}|A_0)P(A_0)} = \frac{.9*.9*.1*.5}{.9*.9*.1*.5+.05*.05*.95*.5} \approx .9715$

13. (a) Let C be correct diagnosis. Then $P(C) = P(C|D)P(D) + P(C|\neg D)P(\neg D) = .95*.01+.95*.99 = .95$ for company A while $= 0*.01+1*.99 = .99$ for company B.

    (b) If the disease is fatal, it is much more important to identify when someone has the disease rather than fail to identify when someone does not have the disease. Company B's product does not help at all to determine whether a person has the disease.

    (c) $s(.01)+s(.99) = s$ so company A needs to have $s = .99$ sensitivity and specificity. If the sensitivity is 1, then $1 * .01 + s * .99 = .99 \implies s \geq \frac{98}{99}$. If the specificity is 1, then $s * .01 + 1 * .99 = .99 \implies s \geq 0$.

14. (a) $P(A|B)$ since if Peter's home is burglarized, then he is very likely to install an alarm.

    (b) $P(B|A)$ since one of the strongest catalysts for getting an alarm system might be getting burglarized.

    (c)

    $$P(A|B) - P(A|B^c) > 0 \iff P(AB)/P(B) - P(AB^c)/P(B^c) > 0$$
    $$\iff P(B^c)P(AB) - P(B)P(AB^c) > 0 \iff P(AB) - P(B)(P(AB) + P(AB^c)) > 0$$
    $$\iff P(AB) - P(A)P(B) > 0.$$

    Analogous manipulation shows that $P(B|A) > P(B|A^c) \iff P(AB) - P(AB) > 0$.

(d) Many people probably think that the probability of getting burglarized without an alarm is high without thinking about the fact that one of the causes for getting an alarm could be getting burglarized, which they already thought was high.

15. A, since $P(A) - P(AB) = P(A \setminus B)$ is smallest.

16.

$$P(A|B) \leq P(A) \iff P(AB) \leq P(A)P(B) \iff P(AB) \leq (P(AB)+P(AB^c))(1-P(B^c))$$

$$\iff 0 \leq P(AB^c)-P(A)P(B^c) \iff P(AB^c) \geq P(A)P(B^c) \iff P(A|B^c) \geq P(A).$$

17. (a) $P(B|A) = P(AB)/P(A) = 1 \iff P(AB) = P(A)$.

$$P(A^c|B^c) = P(A^cB^c)/P(B^c) = \frac{1 - P(A \cup B)}{1 - P(B)} = \frac{1 - P(A) - P(B) + P(AB)}{1 - P(B)}$$

$$= 1 - \frac{P(A) - P(AB)}{1 - P(B)} = 1$$

where in the last line we used $P(A) = P(AB)$.

(b) If A and B are independent, then we have from the previous computation

$$P(A^c|B^c) = 1 - \frac{P(A) - P(A)P(B)}{1 - P(B)} = 1 - P(A)$$

so if $P(A) \approx P(AB) \approx 1$, then $P(A^c|B^c) \approx 0$.

18. $P(A|B) = P(AB)/P(B) = (P(B) - P(A^cB))/P(B)$. Since $A^c \cap B \subset A^c$, then $P(A^cB) \leq P(A^c) = 1 - P(A) = 0 \implies P(A^cB) = 0$, so $P(A|B) = P(B)/P(B) = 1$.

19. If $A_1, \ldots, A_n$ are the only possibilities (so $\sum_i P(A_i) = 1$) and $P(A_i|E) = 0$ for all but one i_*, then $P(A_{i_*}|E) = 1$ no matter how small $P(A_{i_*})$ is.

20. (a) There are three cards left and we only want one of them, so $\frac{1}{3}$.

(b) There are 10 possibilities where one card is a queen and 2 where both are queens, so $\frac{1}{5}$.

(c) There are 6 possibilities and we only want two of them so $\frac{1}{3}$.

21. (a) There are four ways to get at least two heads, so $\frac{1}{4}$.

(b) There are only two options (since the slips of paper correspond to particular flips), so $\frac{1}{2}$.

22. There are three ways to draw a green marble, and in two of them the remaining marble is green, so the probability is $\frac{2}{3}$.

23. Yes, it is possible if the two pieces of evidence are incompatible with guilty. i.e. $E_1 \cap G$ and $E_2 \cap G$ are both nonempty while $E_1 \cap E_2 \cap G$ is empty.

24. Yes, it is possible. If $A_1 \cap A_2 \cap B$ is relatively large (i.e. almost equal to $A_1 \cap A_2$) while $A_1 \cap A_2 \cap C$ is relatively small (i.e. empty), then it is possible for the union conditioned on B to have smaller probability than the union conditioned on C.

25. (a) Let T_A be suspect A matching the blood type. Then

$$P(A|T_A) = \frac{P(T_A|A)P(A)}{P(T_A|A)P(A) + P(T_A|A^c)P(A^c)} = \frac{.5}{.5 + .1 * .5} = \frac{10}{11}.$$

(b) Let T_B be suspect B matching the blood type. Then

$$P(T_B|T_A) = P(T_B|T_A, B)P(B|T_A)+P(T_B|T_A, B^c)P(B^c|T_A) = P(B|T_A)+.1P(B^c|T_A)$$

$$= 1-.9P(B^c|T_A) = .1+.9P(B|T_A) = .1+.9\frac{P(T_A|B)P(B)}{P(T_A)} = .1+.9\frac{.1/2}{.1/2 + 1/2} = \frac{2}{11}$$

26. (a) $P(L|M_1) = \frac{P(M_1|L)P(L)}{P(M_1|L)P(L)+P(M_1|L^c)P(L^c)} = \frac{.9*.1}{.9*.1+.1*9} = \frac{1}{2}$

(b) $P(L|M_1, M_2) = \frac{P(M_1,M_2|L)P(L)}{P(M_1,M_2|L)P(L)+P(M_1,M_2|L^c)P(L^c)} = \frac{.9*.9*.1}{.9*.9*.1+.1*.1*.9} = .9$

(c) Yes, we know from the section that sequentially updating the information or updating all at once is equivalent via Bayes rule:

$$\tilde{P}(L|M_2) = \frac{\tilde{P}(L \cap M_2)}{\tilde{P}(M_2)} = \frac{P(L \cap M_2|M_1)}{P(M_2|M_1)} = \frac{P(L \cap M_2 \cap M_1)}{P(M_2 \cap M_1)}$$

27. Let G be guilty and E be evidence.

$$P(G|E) = \frac{P(G \cap E)}{P(E)} = \frac{p_2 p}{2p1p2} = \frac{p}{2p_1}$$

The probability decreases if $p_1 > \frac{1}{2}$.

28. (a) $P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D)+P(T|D^c)P(D^c)}$

(b) When the disease is rare, then $P(D)$ is small and $P(D^c)$ is large, so the specificity $P(T|D^c)$ has a much larger impact on the final result than changes in the sensitivity (assuming the sensitivity is already reasonable).

29. Let GG be two girls and GC be girl with C. Then

$$P(GG|GC) = \frac{P(GC|GG)P(GG)}{P(GC|GG)P(GG) + 2P(GC|GB)P(GB)}$$

$$= \frac{(1-(1-p)^2)/4}{(1-(1-p)^2)/4 + 2p/4} = \frac{2-p}{4-p}$$

2 Independence and conditional independence

30. (a) These are not independent, since if A is older than B, then A is more likely to be old.

    (b) Given that A is older than C, there are three possible orderings: BAC, ABC, ACB. In two of them, A is older than B so the probability is $\frac{2}{3}$.

31. This event must have probability $P(A) = P(A)^2 \implies P(A) = 0, 1$.

32. (a) $P(A > B)$ is $\frac{2}{3}$. $P(B > C)$ is $\frac{2}{3}$. $P(C > D)$ is $\frac{1}{3} + \frac{2}{3}\frac{1}{2} = \frac{2}{3}$. $P(D > A)$ is $\frac{1}{2} + \frac{1}{2}\frac{1}{3} = \frac{2}{3}$.

    (b) Yes, since $P(A > B, B > C) = \frac{2}{3}\frac{2}{3} = P(A > B)P(B > C)$. No, since if $B > C$, then $C = 2$ and $P(C > D|B > C) = \frac{1}{2} \neq \frac{2}{3}$.

33. (a) $P(A = D) = \frac{1}{2^{100}}$.

    (b) A is a subset of B if everybody who is Alice's friend is also Bob's friend, so the probability is $\frac{1}{2^{|A|}}$.

    (c) $A \cup B = C$ if nobody is friends with neither Bob nor Alice, so the probability is $(\frac{3}{4})^{|C|}$.

34. (a) A and B are not independent because whether or not the man gets into an accident gives information about whether or not he is a good driver.

    (b) $P(G|A^c) = \frac{P(A^c|G)P(G)}{P(A^c|G)P(G)+P(A^c|G^c)P(G^c)} = \frac{g(1-p_1)}{g(1-p_1)+(1-g)(1-p_2)}$.

    (c) $P(B|A^c) = P(B|A^c, G)P(G|A^c) + P(B|A^c, G^c)P(G^c|A^c) = p_1 P(G|A^c) + p_2(1 - P(G|A^c))$.

35. (a) Let W be win first game. Then $P(W) = P(W|B)P(B)+P(W|I)P(I)+P(W|M)P(M) = .9/3 + .5/3 + .3/3 = 17/30 = .5\overline{6}$

    (b) Let $W2$ be win second game. Then $P(W2|W) = P(W2|B, W)P(B|W)+P(W2|I, W)P(I|W)+ P(W2|M, W)P(M, W)$. For $X = B, I, M$ we have $P(X|W) = \frac{P(W|X)P(X)}{P(W)} = \frac{10}{17}P(W|X)$, so $P(B|W) = \frac{9}{17}, P(I|W) = \frac{5}{17}$, $P(M|W) = \frac{3}{17}$. Thus

    $$P(W2|W) = .9\frac{9}{17} + .5\frac{5}{17} + .3\frac{3}{17} \approx .6765$$

    (c) Assuming independence implies that the outcomes of the games don't tell you anything about the skill of your opponent, which seem sunreasonable. On the other hand, assuming conditional independence implies that once you know the skill of your opponent, winning or losing a game doesn't tell you anything more about the skill of your opponent. This is a more reasonable assumption, although it could be further strengthened.

36. (a) It's hard to be good at both math and baseball, so most students will focus on one at the detriment of the other.

(b) $P(AB|A \cup B) = \frac{P(AB)}{P(A \cup B)} = \frac{P(A)P(B)}{P(A \cup B)} \neq \frac{P(A)}{P(A \cup B)} \frac{P(B)}{P(A \cup B)}$ as long as $P(A \cap B) > 0$ and $P(A \cup B) < 1$.

Furthermore, given the assumptions, we have

$$P(A \cup B) < 1 \iff P(A \cup B)P(A \cap B) < P(A \cap B) \iff P(A \cap B)/P(B) < P(A)/P(A \cap B)$$

$$\iff P(A \cap B \cap (A \cup B))/P(B \cap (A \cup B)) < P(A \cap (A \cup B))/P(A \cap (A \cup B))$$

$$\iff P(A|B, C) < P(A|C).$$

37. (a) $P(W) = P(W|D_1, D_2)P(D_1, D_2) + P(W|D_1, D_2^c)P(D_1, D_2^c) + P(W|D_1^c, D_2)P(D_1^c, D_2) + P(W|D_1^c, D_2^c)P(D_1^c, D_2^c) = p_1p_2 + p_1(1 - p_2) + (1 - p_1)p_2 + w_0(1 - p_1)(1 - p_2)$.

    (b) $P(D_i|W) = P(W|D_i)P(D_i)/P(W) = p_i/P(W)$.
    $P(D_1, D_2|W) = P(W|D_1, D_2)P(D_1, D_2)/P(W) = p_1p_2/P(W)$.

    (c) They are not conditionally independent given W unless $P(W) = 1$.

    (d) They are not conditionally independent, because having weird symptoms and not disease 1 would imply having disease 2. In other words, $P(D_1|D_2^c, W) = 1$ whereas $P(D_1|D_2^c) = p_1$ since disease 1 and disease 2 are independent.

38. Let $w = p_{23}p_{64}p_{65} \prod_{i \neq 23,64,65}(1 - p_i)$ and $w' = r_{23}r_{64}r_{65} \prod_{i \neq 23,64,65}(1 - r_i)$. Then $P(spam|W_{23}W_{64}W_{65}W_i^c) = wp/(wp + w'(1 - p))$.

3 Monty Hall

39. (a) Given that you switch, then to win you have to not pick the car and then pick the car, so $\frac{6}{7} * \frac{1}{3} = \frac{2}{7}$ chance of winning, which is better than the initial $\frac{1}{7}$.

    (b) Given that you switch, then to win you have to not pick the car and then pick the car, so $\frac{n-1}{n} * \frac{1}{n-m-1}$, which is better than the initial $\frac{1}{n}$.

40. (a) The unconditional probabilility is still $\frac{2}{3}$ since we still have a $\frac{2}{3}$ chance of not choosing the car on the first go.

    (b) Let C_i represent the car behind door i. $P(W|M_2) = P(W|C_1, M_2)P(C_1|M_2) + P(W|C_2, M_2)P(C_2|M_2) + P(W|C_3, M_2)P(C_3|M_2)$. Monty cannot open door 2 if the car is there so $P(C_2|M_2) = 0$. Switching cannot win if the car is behind door 1 so $P(W|C_1, M_2) = 0$. Finally, $P(W|C_3, M_2) = 1$ since we always win if the car is not behind door 1. Thus $P(W|M_2) = P(C_3|M_2) = \frac{P(M_2|C_3)P(C_3)}{P(M_2)} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{p}{3}} = \frac{1}{1+p}$.

    (c) Using the analogous reasoning as the previous part, we arrive at

$$P(W|M_3) = P(C_2|M_3) = \frac{P(M_3|C_2)P(C_2)}{P(M_3|C_2)P(C_2) + P(M_3|C_1)P(C_1)} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1-p}{3}} = \frac{1}{2-p}.$$

41. The contestant wins if and only if there is a car behind door 3, so we need to compute $P(C_3|M_2) = P(M_2|C_3)P(C_3)/P(M_2|C_3)P(C_3) + P(M_2|C_1)P(C_1) + P(M_2|C_2)P(C_2)$. After some calculations ($P(M_2|C_1) = \frac{p}{2} + \frac{(1-p)}{2}$, $P(M_2|C_3) = p + \frac{1-p}{2}$, and $P(M_2|C_2) = 0$), I get $\frac{1+p}{2+p}$.

42. (a) $P(W|C_1) = 0$ since the contestant will be given a choice to switch and will switch. Thus $P(W) = P(W|C_2)/3 + P(W|C_3)/3$. The latter two cases are symmetric, so consider C_2. $P(W|C_2) = (1-p)*0 + p*1 = p$, so $P(W) = 2p/3$.

    If $p = 0$, then the contestant can never win since they will not get another chance if they choose a goat and they will switch if they pick a car. If $p = 1$, then the contestant will have $2/3$ probability of winning because this is the original Monty Hall problem.

    (b) The contestant gets the car if and only if the car is behind door 3, so we compute $P(C_3|M_2) = P(M_2|C_3)P(C_3)/P(M_2)$. If car behind door 3, then Monty has to choose to open door with probability p, so $P(M_2|C_3) = p$. Thus we compute

$$P(C_3|M_2) = \frac{P(M_2|C_3)P(C_3)}{P(M_2|C_3)P(C_3) + P(M_2|C_1)P(C_1) + P(M_2|C_2)P(C_2)} = \frac{p/3}{p/3 + 1/6}.$$

43. (a) If Monty reveals the goat, then I either picked the car or computer to begin with. I have no additional information, so it is probably .5.

    (b) The probability that the car is behind door 1 is:

$$P(C_1|comp) = \frac{P(comp|C_1)P(C_1)}{P(comp|C_1)P(C_1) + P(comp|G_1)P(G_1) + P(comp|comp1)P(comp1)}$$

$$\frac{q/3}{q/3 + p/3} = \frac{q}{p+q} = q$$

    so if q is larger than .5, then you should switch.

44. (a) $P(C3|M2) = \frac{P(M2|C3)P(C3)}{P(M2|C3)P(C3)+P(M2|C1)P(C1)+P(M2|C2)P(C2)} = \frac{p3/2}{p3/2+p1} = \frac{p_3}{p_3+2p_1}$. The contestant should switch if $2p_1 > p_3$.

    (b) $P(C3|M2) = \frac{P(M2|C3)P(C3)}{P(M2|C3)P(C3)+P(M2|C1)P(C1)+P(M2|C2)P(C2)} = \frac{p_3/2}{p_3/2+p_1/2} = \frac{p_3}{p_3+p_1}$. The contestant should switch if $p_1 > p_3$.

    (c) Nothing distinguishes doors 1 and 3 a priori, so we can just switch p_1 and p_3 above.

    (d) Nothing distinguishes doors 1 and 3 a priori, so we can just switch p_1 and p_3 above.

45. (a) The contestant will lose in configurations GGG, CGG which have probability q^2, so the probability of winning is $1 - q^2$.

    (b) The given situation can occur in configurations GGG,GGC,CGG,CGC, and the contestant only wins in the configurations GGC and CGC. The probability of winning is thus $(q^2p + qp^2)/(\frac{q^3}{2} + \frac{3}{2}q^2p + qp^2) = \frac{qp(q+p)}{\frac{q^2}{2}(q+p)+qp(q+p)} = \frac{qp}{q^2/2+qp}$

46. (a) Other than the situation where the car is behind door 1 (in which case the contestant will never get the car), all three other cases are symmetric, with probability .5 of getting the car. Thus there is $\frac{3}{8}$ chance of winning.

(b) If door 1 contains the apple, then Monty will never reveal the apple. If door 1 contains the goat, Monty will reveal the apple with probability p. If door 1 contains the book, Monty will reveal the apple with probability $1-p$. If door 1 contains the car, Monty will reveal the apple with probability $1-p$. Thus Monty will reveal the apple with probability $\frac{2-p}{4}$.

(c) $P(W|A) = P(W|A, C_1)P(C_1|A) + P(W|A, G_1)P(G_1|A) + P(W|A, B_1)P(B_1|A)$. Since contestant always switches, $P(W|A, C_1) = 0$. In either of the two remaining cases, the contestant will win with probability $\frac{1}{2}$, so we need to compute $P(G_1|A)$ and $P(B_1|A)$.

$$P(G_1|A) = \frac{P(A|G_1)P(G_1)}{P(A)} = \frac{p}{2-p} \qquad P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A)} = \frac{1-p}{2-p}$$

so the final answer is $\frac{1}{4-2p}$.

47. (a) To win, have to select car with first guess, so $\frac{1}{4}$.

(b) To win, have to not select car first time, so $\frac{3}{4}$.

(c) To win, have to not select car first time, and then choose correctly from two choices, so $\frac{3}{8}$.

(d) To win, have to either select car first time or not select car first time and not select the car after first switch, so $\frac{1}{4} + \frac{3}{4} * \frac{1}{2} = \frac{5}{8}$.

(e) The stay switch strategy seems to be the best.

4 first step analysis and gambler's ruin

48. (a) $p_0 = 1, p_k = 0$ for $k < 0$, $p_n = \frac{1}{6}(p_{n-1} + \cdots + p_{n-6})$.

(b) $p_0 = 1, p_1 = \frac{1}{6}, p_2 = \frac{1}{6} + \frac{1}{36}, p_3 = \frac{1}{6} + \frac{2}{36} + \frac{1}{216}, p_4 = \frac{1}{6} + \frac{3}{36} + \frac{3}{216} + \frac{1}{6^4}, p_5 = \frac{1}{6} + \frac{4}{36} + \frac{6}{216} + \frac{4}{6^4} + \frac{1}{6^5}, p_6 = \frac{1}{6} + \frac{5}{36} + \frac{10}{216} + \frac{10}{6^4} + \frac{5}{6^5} + \frac{1}{6^6}, p_7 = \frac{6}{36} + \frac{15}{216} + \frac{20}{6^4} + \frac{15}{6^5} + \frac{6}{6^6} + \frac{1}{6^7}$.

(c) First, the probability eventually limits to a constant sequence due to the averaging, since each entry is strictly between the maximum and minimum of the previous six. (Note this does not rigorously prove the limiting behavior, but provides the intuition.)

Since we started with all the probability weight initially at zero, then since the average roll is $\frac{1}{6}(1+2+3+4+5+6) = 3.5$, we can expect that for large enough n, the probability of seeing a particular running sum is $\frac{1}{3.5}$.

49. (a) $P(A_2) = q_1 q_2 + (1-q_1)(1-q_2) = 2q_1 q_2 + 1 - q_1 - q_2 = 2(q_1 - \frac{1}{2})(q_2 - \frac{1}{2}) + \frac{1}{2}$

(b) $P(A_n) = P(A_{n-1})q_n + (1-P(A_{n-1}))p_n = (\frac{1}{2} + 2^{n-2}b_1 \cdots b_{n-1})q_n + (\frac{1}{2} - 2^{n-2}b_1 \cdots b_{n-1})p_n = \frac{1}{2} + 2^{n-2}(b_1 \cdots b_{n-1})(q_n - p_n)$. Since $q_n - p_n = 2\frac{2q_n - 1}{2} = 2b_n$, we obtain $P(A_n) = \frac{1}{2} + 2^{n-1}b_1 \cdots b_n$.

(c) If p_i is $\frac{1}{2}$, then the above formula reduces to $P(A_n) = \frac{1}{2}$. This makes sense: excluding the i^{th} trial, there is either an odd or even number of successes. If there is an odd number of successes, then we need to get a success in trial i with probability .5 and if there is an even number of successes, then we need to get a failure in trial i with probability .5.

If $p_i = 0$ for all i, then $q_i = 1$ and $b_i = .5$, so $P(A_n) = 1$, since there are always 0 successes and hence always an even number of successes. If $p_i = 1$ for all i, then $q_i = 0$ and $b_i = -.5$, so $P(A_n) = \frac{1}{2} + \frac{(-1)^n}{2}$, since there are always n successes and hence even if n is even and odd if n is odd.

50. (a) Let C be Calvin winning. Then

$$P(C) = P(C|WW)P(WW) + P(C|LL)P(LL) + P(C|WL)P(WL) + P(C|LW)P(LW)$$

$$= p^2 + 2pqP(C) \implies P(C) = \frac{p^2}{1-2pq}$$

(b) One can interpret this as a Gambler's ruin problem where the starting position is $i = 2$, the game ends if we reach 4 (Calvin wins) or 0 (Hobbes wins) and we move right with probability p and left with probability $1-p$. Applying the Gambler's ruin solution and setting $x = \frac{q}{p}$, we obtain

$$\frac{1+x}{1+x+x^2+x^3} = \frac{p^2}{1-2pq}$$

51. This is another Gambler's ruin problem, with $n = 1000002$ and starting position $i = 1000000$. Since $p = \frac{1}{3}$ and $q = \frac{2}{3}$, we have $\frac{q}{p} = 2$, so we obtain

$$\frac{2^{1000000}-1}{2^{1000002}-1} < \frac{1}{4}$$

52. Each dollar corresponds to k increments, so we have $n = kN$ with starting position ik. The probability of moving to the right is p while the probability of moving to the left is $q = 1-p$. Thus the probability of A winning is

$$\frac{x^{ki}-1}{x^{kN}-1} \approx \frac{1}{x^{k(N-i)}} \to 0$$

53. There are three possible outcomes: the wolf reaches the opposite point last, the wolf reaches the opposite point from the right before reaching from the left, or the wolf reaches the opposite point from the left before reaching from the right. We can compute the probability of the final two events using two equivalent Gambler's ruin problems with $N = 99$ and $i = 49$. Since $p = \frac{1}{2}$, we have $\frac{49}{99}$ chance of either situation happening, so the probability that the wolf reaches the opposite point last is $\frac{1}{99}$. (In fact, it seems by symmetry that this answer does not depend on i...)

54. (a) At the beginning, p_k is the probability that we ever make it k steps to the right. After one step, we are either $k-1$ steps or $k+1$ steps from k, so $p_k = pp_{k-1}+qp_{k+1}$ with $p_0 = 1$.

    (b) In gambler's ruin, the probability that the game never ends is zero, so we can consider the event that the drunk reaches k as the union of the events $A_1, A_2, \ldots$ where A_i is the probability of reaching k before $-i$ starting at 0. In order to reach $-i$, the drunk must reach each $-1, -2, \ldots, -(i-1)$ first so $A_i \subset A_{i+1}$ for all i. By Gambler's ruin,

$$\frac{N-k}{N} \xrightarrow{N\to\infty} 1, \frac{x^{N-k}-1}{x^N-1} \xrightarrow{N\to\infty, x<1} 1, \frac{x^{N-k}-1}{x^N-1} \xrightarrow{N\to\infty, x>1} \frac{1}{x^k} = \frac{p^k}{q^k}.$$

5 Simpson's paradox

55. (a) $P(A) = P(A|C)P(C)+P(A|C^c)P(C^c) < P(B|C)P(C)+P(B|C^c)P(C^c) = P(B)$.

    (b) In the case of Simpson's rule the weights are different while here they are the same in both expansions.

56. (a) I ivory dealer, B, H, C boots, hat, check ivory. L lots of ivory. S hurt Stampy.

    (b) Lisa: $P(I|B, H, C)$ is high and $P(S|I)$ is high. Homer: $P(S|L) < P(S|L^c)$.

    (c) Even if the ivory dealer has a large amount of ivory and personally doesn't desire to acquire more, they are selling to people who don't have a large amount of ivory.

57. (a) C_1 has 20 green/70 red gummy and M_1 has 10 red gummy. C_2 has 10 green and M_2 has 80 green/10 red.

    (b) C_1 has 2/7 green and M_1 has 0/10 green. C_2 has 10/10 green and M_2 has 8/9 green. However, $C_1 + C_2$ has 3/10 green while $M_1 + M_2$ has 80/100 green. The point is that while C_2 is higher green than M_2, the overall amounts for M_2 far outweight the overall amounts for C_2.

58. (a) $P(A|B) = P(AB)/P(B) = P(A)$ and $P(A|B^c) = P(AB^c)/P(B^c) = P(A)$ if A and B are independent.

    (b) May be true, but doesn't follow in obvious way like other cases. Would definitely be true if A and C were independent given B, but just knowing independence of A and C does not seem to be sufficient.

    (c) $P(A|B) = P(A|B, C)P(C|B)+P(A|B, C^c)P(C^c|B) = P(A|B, C)P(C)+P(A|B, C^c)P(C^c)$ $< P(A|B^c, C)P(C|B^c) + P(A|B^c, C^c)P(C^c|B^c) = P(A|B^c)$ where we used independence of B and C.

59. (a) reuse same ideas as constructing all the simpson's paradoxes so far...

    (b) It is possible since the probability of living in a blue state (i.e. the weighting probabilities in LOTP) has changed.

6 Mixed practice

60. (a) $P(D|T) = P(T|D)P(D)/P(T|D)P(D) + P(T|D^c)P(D^c)$
    $P(T|D) = P(T|D,A)P(A|D) + P(T|D,B)P(B|D)$ and analogously for D^c.

$$= \frac{(a_1+b_1)p/2}{(a_1+b_1)p/2 + (a_2+b_2)(1-p)/2}$$

    (b) $P(A|T) = P(T|A)P(A)/P(T) = P(T|A)/(P(T|A) + P(T|B))$.
    $P(T|-) = P(T|-,D)P(D|-) + P(T|-,D^c)P(D^c|-)$.

$$P(A|T) = \frac{a_1p + (1-a_1)(1-p)}{a_1p + (1-a_1)(1-p) + b_1p + (1-b_1)(1-p)}$$

61. (a) $P(D|T) = P(T|D)P(D)/P(T) = \frac{a^np}{a^np+b^n(1-p)}$.
    (b) $P(D|T) = P(T|D)P(D)/P(T)$. $P(T|D) = P(T|D,G)P(G|D)+P(T|D,G^c)P(G^c|D)$.
    $P(T|D^c) = P(T|D^c,G)P(G|D^c) + P(T|D^c,G^c)P(G^c|D^c)$

$$P(D|T) = \frac{(1+a_0^n)p/2}{(1+a_0^n)p/2 + (1+b_0^n)(1-p)/2}$$

62. (a) Let C_1, C_2 be child i has disease and M be mother has disease. Then

$$P(C_1^c, C_2^c) = P(C_1^c, C_2^c|M)P(M) + P(C_1^c, C_2^c|M^c)P(M^c) = \frac{9}{12}$$

    (b) No, because knowing whether the elder child has the disease gives information about whether the mother has it.
    (c)

$$P(M|C_1^c, C_2^c) = \frac{P(C_1^c, C_2^c|M)P(M)}{P(C_1^c, C_2^c|M)P(M) + P(C_1^c, C_2^c|M^c)P(M^c)} = \frac{1/12}{1/12 + 2/3} = \frac{1}{9}$$

63. The probability discussed here is a conditional probability $P(HHH/TTT|HH/TT)$. Otherwise, we cannot apply the naive definition of probability since the outcomes are not equally weighted. There are only two ways to get HHH or TTT but six ways to get HHT or TTH.

64. (a) Since it is with replacement, we can just ignore the red balls, so we are asking for the probability that the first ball we see if green. This has probability $\frac{g}{g+b}$.
    (b) Again, we can just ignore the red balls and count the ways to arrange the green and blue balls. If there were n green and blue balls total, then they could be arranged in $\binom{n}{gn/(g+b)}$ ways and there are $\binom{n-1}{gn/(g+b)-1}$ ways where a green ball comes first. Let $k = \frac{gn}{g+b}$,

$$\binom{n-1}{k-1}\Big/\binom{n}{k} = \frac{k}{n} = \frac{g}{g+b}.$$

(c) We can group together all types not equal to i or j into one megatype $[n] \setminus \{i,j\}$, and let i correspond to green, j correspond to blue, and $[n] \setminus \{i,j\}$ correspond to red. Then the answer to part a gives $\frac{p_i}{p_i+p_j}$.

65. (a) There are 200 different sequences of draws (distinguished by the position of the you lose) and all are equally likely.

    (b) The probability of drawing you lose if going first is $\frac{v}{v+199w}$. The probability of drawing you lose second is $\frac{199w}{v+199w}\frac{v}{v+198w}$. To see who has a higher chance of losing, we compare

$$\frac{v}{v+199w} > \frac{v}{v+199w}\frac{199w}{v+198w} \iff v > w$$

    so it's better to go first if $v \leq w$ and else better to go second.

66. If we assume the numbers 94-99 are roughly equally likely, then there are the most ways (6) to roll exactly 100. This seems like a reasonable assumption, since 94 is large enough that by that point the probability of getting any individual number is close to $\frac{2}{7}$ as we computed before.

67. (a) This is the same as the fraction of sequences of c,g,j where the final symbol is g. This is the same as the previous problem, so the answer is $\frac{c}{c+g+j}$.

    (b) The last donut must be chocolate (with probability $\frac{c}{c+g+j}$) and then before that the last of jelly and glazed must be jelly (with probability $\frac{j}{g+j}$). $(P(J,C) = P(J|C)P(C)$.) so $\frac{cj}{(c+g+j)(g+j)}$.

68. (a) $(P(D|C)/P(D^c|C))/(P(D|C^c)P(D^c|C^c)) = (P(D|C)/P(D|C^c))(P(D^c|C^c)/P(D^c|C))$. If the disease is rare, then $P(D^c|C), P(D^c|C^c) \approx 1$ so we get the approximation $(P(D|C)/P(D|C^c))(P(D^c|C^c)/P(D^c|C)) \approx (P(D|C)/P(D|C^c)) = RR$

    (b)

$$\frac{P(D|C)P(D^c|C^c)}{P(D|C^c)P(D^c|C)} = \frac{P(DC)P(D^cC^c)}{P(DC^c)P(D^cC)} = \frac{p_{11}p_{00}}{p_{10}p_{01}}$$

    (c) In terms of the previous problem, swapping C and D corresponds to swapping the order of the two indices of each p.

69. (a) $P(Y) = P(Y|D)P(D) + P(Y|D^c)P(D^c) = pd + (1-p)(1-d)$

    (b) The worst choice would be $p = .5$, since $.5d + .5(1-d) = .5$ giving no information about d.

    (c) Let Y be say yes, D be drug question. Then $P(Y) = P(Y|D)P(D) + P(Y|D^c)P(D^c) = pd + \frac{1-p}{4} \implies d = \frac{1}{p}(y + \frac{p-1}{4})$.

70. (a) They are both correct. However, Fred's friend is not accounting for the fact that they are probably more interested in the number of heads rather than the specific sequence. So 92 heads is the least likely number of heads (tied with 0 heads) and e.g. getting 46 heads would be much more likely.

(b) $P(F|92H) = P(92H|F)P(F)/P(92H) = \frac{2^{-92}p}{2^{-92}p+(1-p)}$

(c) $(2^{-92}+1)p > \frac{1}{2^{-92}+1}$.
$p < \frac{1}{1+5\cdot 2^{-90}-2^{-92}}$

71. (a) $p_{ij} = \frac{j}{n}p_{i-1,j} + \frac{n-j+1}{n}p_{i-1,j-1}$

(b) First define $p_{i,0} = 0$ for all i, $p_{i,j} = 0$ for all $j > i$, and $p_{1,1} = 1$. Then iteratively compute $p_{2,1}, p_{2,2}, p_{3,1}, p_{3,2}, p_{3,3}, \ldots, p_{i,1}, \ldots, p_{i,i}$, etc.

72. (a) $p_n = aa_n + b(1-a_n) = (a-b)a_n + b$.
$a_{n+1} = aa_n + (1-b)(1-a_n) = (a+b-1)a_n + 1 - b$

(b) $p_{n+1} = (a-b)((a+b-1)a_n + 1 - b) + b = (a+b-1)(a-b)a_n + (a-b)(1-b) + b$
$= (a+b-1)(p_n - b) + (a-b)(1-b) + b = (a+b-1)p_n - 2ab + b + a$

(c) The probability of success in the first trial is $\frac{a+b}{2} = p_0$ and plugging in repeatedly,

$$p_0 = \frac{a+b}{2}$$

$$p_1 = (a+b-1)p_0 + (a+b-2ab)$$

$$p_2 = (a+b-1)^2 p_0 + (a+b-1)(a+b-2ab) + (a+b-2ab)$$

$$p_n = (a+b-1)^n p_0 + (a+b-2ab)\sum_{i=0}^{n-1}(a+b-1)$$

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty}(a+b-1)^n p_0 + (a+b-2ab)\sum_{i=0}^{n-1}(a+b-1)$$

Note that $a+b-1 \leq 1$ with equality iff $a = b = 1$. In that case, the limit is equal to 1. Furthermore, $a+b-1 \geq -1$ with equality iff $a = b = 0$. In that case, the limit is equal to 0. Thus, suppose $-1 < a+b-1 < 1$. Then $(a+b-1)^n \xrightarrow{n\to\infty} 0$ and $\sum_{i=0}^{n-1}(a+b-1) = \frac{1}{1-(a+b-1)} = \frac{1}{2-a-b}$, so the limit converges to

$$\frac{a+b-2ab}{2-a-b}.$$

73. (a) $P(AA) = P(AA|AA,AA)P(AA,AA) + 2P(AA|AA,Aa)P(AA,Aa) + P(AA|Aa,Aa)P(Aa,Aa)$
$= p^4 + 2p^3(1-p) + p^2(1-p)^2 = p^2(p^2 + 2p(1-p) + (1-p)^2) = p^2$
By symmetry (swap p and $1-p$), $P(aa) = (1-p)^4 + 2p^3(1-p) + p^2(1-p)^2 = (1-p)^2(p^2 + 2p(1-p) + (1-p)^2) = (1-p)^2$.
Since these genotypes are exhaustive, what is left must be $P(Aa) = 2p(1-p)$. Thus Hardy-Weinberg is stable in the sense of preserving the relative poportion of genotypes.

(b) Let h be homozygous and H be heterozygous. Then

$$P(h|hh) = \frac{1}{P(hh)}(2P(h|AA,aa)P(AA,aa)+P(h|AA,AA)P(AA,AA)+P(h|aa,aa)P(aa,aa))$$

$$= \frac{p^4 + (1-p)^4}{(p^2 + (1-p)^2)^2}$$

There is only one way to be heterozygous, and each parent must contribute a different gene, so $P(H|HH) = \frac{1}{2}$.

(c) Let H be child heterozygous, ha be child aa, and ma, fa be mother and father aa. Then we want $P(H|ha^c, ma^c, fa^c) = \frac{P(H,ha^c,ma^c,fa^c)}{P(ha^c,ma^c,fa^c)}$. The top has (child, mother, father) are $(Aa, AA, AA), (Aa, AA, Aa), (Aa, Aa, AA), (Aa, Aa, Aa)$ while the bottom has these four and $(AA, AA, AA), (AA, AA, Aa), (AA, Aa, AA), (AA, Aa, Aa)$. The probability in the numerator is $0, p^3(1-p), p^3(1-p), 2p^2(1-p)^2$ and the denominator also includes $p^4, p^3(1-p), p^3(1-p), p^2(1-p)^2$. Thus we have

$$\frac{2p^2(1-p)}{2p^2(1-p)+p^2}$$

74. (a) There are $\binom{52}{4}$ ways to choose the positions of four aces, while there are $\binom{51}{3}$ ways to choose the positions of the four aces if the first ace must be immediately followed by the second. The ratio between these is 1/13. This makes sense since pairing the first two aces elimintes one of the 52 choices for the other cards while eliminating only one of the four choices for the aces.

(b) $P(B|C_j) = P(BC_j)/P(C_j) = \binom{52-j-1}{2}/\binom{52-j}{3} = \frac{3}{52-j}$.

(c) $P(B) = \sum_{j=1}^{49} P(B|C_j)P(C_j)$

(d) See part (a), $\frac{1}{13}$.