Cluster Algebras: an organizing framework

1 Grassmannian background

The usual Grassmannian $\mathrm{Gr}_{k,n} = \mathrm{Gr}_{k,n}(F)$ is the set of all k-dimensional subspaces of an n-dimensional F-vector space. (F will almost always be $\mathbb{C}$.) The Plücker coordinates of such a subspace define the Plücker embedding $\mathrm{Gr}_{k,n}(F) \to \mathbb{P}^{\binom{n}{k}-1}$, and we will identify $\mathrm{Gr}_{k,n}$ with its image under this embedding to give it the structure of a projective variety. On an affine chart, points of $\mathrm{Gr}_{k,n}$ are $k \times n$ matrices like $[I_k|M_{k,n-k}]$ where I_k is the $k \times k$ identity matrix and $M_{k,n-k}$ is an arbitrary $k \times n-k$ matrix, so the dimension of $\mathrm{Gr}_{k,n}$ is $k(n-k)$. Its degree is more complicated to understand, and is given by the formula:

$$\deg \mathrm{Gr}_{k,n} = (k(n-k))! \prod_{i=1}^{k} \frac{(i-1)!}{(n-k+i-1)!}$$

In this work, we will focus on the subspaces $\mathrm{LGr}_n \subset \mathrm{Gr}_{n,2n}$ of *Lagrangian* subspaces of a $2n$-dimensional *symplectic* vector space. We call a vector space V symplectic if it is equipped with an alternating, non-degenerate bilinear form ω. A vector subspace $U \subset V$ is Lagrangian if $\omega|_U = \omega|_{U^\perp} = 0$. (Equivalently, if $U = U^\perp$. Note that Lagrangian subspaces are necessarily half-dimensional.) As before, the Plücker coordinates define an embedding $\mathrm{LGr}_n \to \mathbb{P}^{\binom{n}{k}-1}$, and we will identify LGr_n with its image under this embedding to give it the structure of a projective variety. On an affine chart, points of LGr_n are $n \times 2n$ matrices like $[I_n|\Sigma_{n,n}]$ where $\Sigma_{n,n}$ is an arbitrary symmetric $n \times n$ matrix, so the dimension of LGr_n is $\binom{n+1}{2}$. Again, the degree is more complicated, and is given by the formula:

$$\deg \mathrm{LGr}_n = \binom{n+1}{2}! \prod_{i=1}^{n} \frac{1}{(2i-1)^{n-i+1}}$$

There are many ways to study these objects, and we will focus on the use of cluster algebras to understand them.

2 Cluster background

A cluster algebra (of geometric type) is a commutative ring with additional, combinatorial structure. We begin with the definition of a quiver, which encodes the combinatorial data of a cluster algebra.

Definition 1. A *quiver* Q is a finite, loopless, directed graph without 2-cycles with a distinguished set of *mutable* vertices (the other vertices are *frozen.* A *quiver mutation* at a mutable vertex v produces a new quiver $\mu_v(Q)$ by the following sequence of steps:

1. For all paths $v_1 \to v \to v_2$, add an edge $v_1 \to v_2$ if v_1, v_2 are not both frozen.

2. Reverse all edges incident to v.

3. Delete a maximal set of 2-cycles.

Quiver mutation satisfies the following properties:

1. $\mu_v(\mu_v(Q)) = Q$

2. If v_1, v_2 are mutable vertices with no edges between them, then $\mu_{v_1} \circ \mu_{v_2} = \mu_{v_2} \circ \mu_{v_1}$.

Another way to represent a quiver is by a (signed) adjacency matrix, and these are generalized by extended skew-symmetrizable matrices.

Definition 2. We call an $m \times n$ ($m \geq n$) integer matrix $B = (b_{ij})$ *extended skew-symmetrizable* if there are $d_1, \ldots, d_n \in \mathbb{Z}_{>0}$ such that $d_i b_{ij} = -d_j b_{ji}$ for all $1 \leq i, j \leq n$. A *matrix mutation* at $1 \leq k \leq n$ produces a new extended skew-symmetrizable matrix $\mu_k(B)$ by the following:

$$(\mu_k(B))_{ij} = \begin{cases} -b_{ij} & i = k \text{ or} j = k \\ b_{ij} + b_{ik}b_{kj} & b_{ik} > 0 \text{ and } bkj > 0 \\ b_{ij} - b_{ik}b_{kj} & b_{ik} < 0 \text{ and } b_{kj} < 0 \\ b_{ij} & \text{else} \end{cases}$$

In the definition above, we think of the top $n \times n$ submatrix of B as the mutable vertices of a quiver Q, the bottom $(m-n) \times n$ submatrix of B as the frozen vertices, and matrix mutation as quiver mutation. Note that in general these matrices do not come from quivers. Matrix mutation satisfies the same properties as quiver mutation listed above. We now move to the algebraic data of a cluster algebra. Let $F = \mathbb{C}(x_1, \ldots, x_m)$.

Definition 3. A *labelled seed* (of geometric type) in F is a pair (x, B) where $x = (f_1, \ldots, f_m)$ is a set of algebraically independent generators for F (i.e. $F = \mathbb{C}(f_1, \ldots, f_m)$) and B is an $m \times n$ extended skew-symmetrizable matrix.

The *seed mutation in direction* k transforms a labelled seed (x, B) into a new labelled seed (x', B') as follows:

1. $x' = (x_1, \ldots, x'_k, \ldots, x_m)$, where x'_k is determined by

$$x_k x'_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik<0}} x_i^{-b_{ik}}$$

2. $B' = \mu_k(B)$

For a sequence $s \in [n]^i$, let $(x^s, B^s) := \mu_s(x, B) = \mu_{s_i} \circ \mu_{s_{i-1}} \circ \cdots \circ \mu_{s_1}(x, B)$ be the labelled seed obtained by mutation in directions $s_1, \ldots, s_i$.

We are now ready to define the cluster algebra associated to a labelled seed.

Definition 4. Let (x, B) be a labelled seed, and fix n, m as above. Let $R = \mathbb{C}[x_{n+1}, \ldots, x_m]$. The cluster algebra associated to the labelled seed (x, B) is $R[\mathcal{X}]$, where

$$\mathcal{X} = \bigcup_{i \in \mathbb{N}} \bigcup_{s \in [n]^i} \{x_1^s, \ldots, x_n^s\}$$

where $x^s = (x_1^s, \ldots, x_m^s)$ is as above. We call n the *rank* of the cluster algebra, and when $\mathcal{X}$ is finite, we say $R[\mathcal{X}]$ is of finite type.

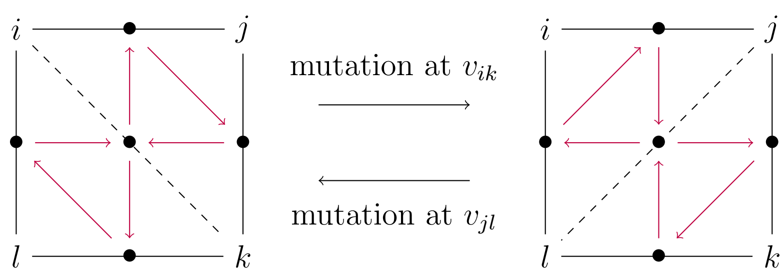
We say that a finitely-generated k-algebra A is a cluster algebra if it is a domain, and there exist $f_1, \ldots, f_m \in k(x_1, \ldots, x_m)$ and an extended skew-symmetrizable matrix B such that A is isomorphic to the cluster algebra associated to the labelled seed $((f_1, \ldots, f_m), B)$. Unfortunately, this cluster structure is not unique, and furthermore even though A is finitely generated, cluster structures on A are generally not of finite type.

3 Example: $\mathrm{Gr}_{2,n}$

To see why cluster structures on finitely generated algebras might be interesting, we will consider the Grassmannians $\mathrm{Gr}_{2,n}$ of 2-planes in n-space. Along the way, we hope to exemplify the slogan that cluster algebras provide an organizing framework for algebraic data.

Recall that the ideal of $\mathrm{Gr}_{2,n}$ is generated in degree 2 by the three-term Plücker relations $p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk}$ where $i < j < k < l$. We will now try to find a cluster structure on the homogeneous coordinate ring of $\mathrm{Gr}_{2,n}$.

Consider an $(n+3)$-gon whose vertices are labelled in clockwise order by $1, \ldots, n$. Let T be a triangulation of this $n+3$-gon, and label each edge in the triangulation by its vertices. Form a quiver Q_T from T by taking for each edge e of T a vertex v_e and for each triangle $t = (e_1, e_2, e_3)$ (labelled clockwise) of T a cycle $(v_{e_1}, v_{e_2}), (v_{e_2}, v_{e_3}), (v_{e_3}, v_{e_1})$ in Q_T. (Here, we neglect edges between frozen vertices.) Let $x_T = (p_{ij} \mid (i, j) \text{ is an edge of T})$. It can be shown that the homogeneous coordinate ring of $\mathrm{Gr}_{2,n}$ is isomorphic to the cluster algebra associated to $(x_T, B(Q_T))$, where $B(Q_T)$ is the extended skew-symmetrizable matrix associated to Q_T. We will first analyze what happens when we perform a mutation (quiver shown in purple).



This corresponds to the following change to the tuple x (note that $x'_{ik} = x_{jl}$):

$$x_{ik}x_{jl} = x_{il}x_{jk} + x_{ij}x_{kl}$$

which is exactly the three-term Plücker relation for $i < j < k < l$. Since the flip graph of a polygon is connected, and each edge appears in some triangulation, we get all Plücker coordinates, and we see all the Plücker relatiosn by performing these mutations, so the homogeneous coordinate ring of $\mathrm{Gr}_{2,n}$ is indeed a cluster algebra.

The flip graph of the triangulations of an $(n+3)$-gon is the 1-skeleton of the type A_n associahedron, which is the cluster complex associated to the cluster structure previously determined for $\mathrm{Gr}_{2,n}$. We will see this again later.

4 General Grassmannians

Unfortunately, the situation for general $\mathrm{Gr}_{k,n}$ is much more complicated. In a positive direction, Geiß, Leclerc, and Schroer showed that the homogeneous coordinate ring of $\mathrm{Gr}_{k,n}$ carries a cluster structure [GLS08]. (Actually, their result holds for flag varieties.) However, these cluster algebras are of finite type only in the cases $\mathrm{Gr}_{2,n}, \mathrm{Gr}_{3,6}, \mathrm{Gr}_{3,7}, \mathrm{Gr}_{3,8}$ [Sco06]. Interestingly, the finite type classifications for Grassmannians coincides with schön-ness [KT06], [Cor17]; does this hold for other Grassmannians? In particular, the Lagrangian Grassmannians LGr_3 and LGr_4 seem to be the only finitely generated Grassmannians in type C, and we would like to study schön-ness in these cases, and to see furthermore whether there is any connection in general to finite type cluster algebras.

5 Plabic Graphs

From now, the main object of focus will be *plabic graphs* (**pla**nar **bic**olored graphs).

5.1 Background

Definition 1. A *plabic graph* is a graph G with a planar embedding into a disk where each internal vertex (not on the boundary of the disk) is colored either white or black. For convenience, we require that G is bipartite (i.e. white vertices are adjacent only to black vertices, and vice versa), and we further assume that G has no parallel edges (in this case we say G is reduced).

Beginning at any boundary vertex, v, in a plabic graph, we form a path p_v by the following rule: when we reach a black vertex, turn maximally right (right with respect to the entering direction and the planar embedding) and when we reach a white vertex, turn maximally left. Stop when we reach a vertex on the boundary. Each such path p_v splits the graph into two components, and we label each face to the left of p_V by v.

We can also obtain a quiver from a plabic graph by taking its dual graph, setting vertices corresponding to faces adjacent to the boundary of the disk as frozen vertices, and the rest as mutable. We orient edges in the quiver so that going from the face corresponding to the tail to the face corresponding to the head, we see a white vertex to the left. We give an example of this for $\mathrm{Gr}_{3,6}$ below.

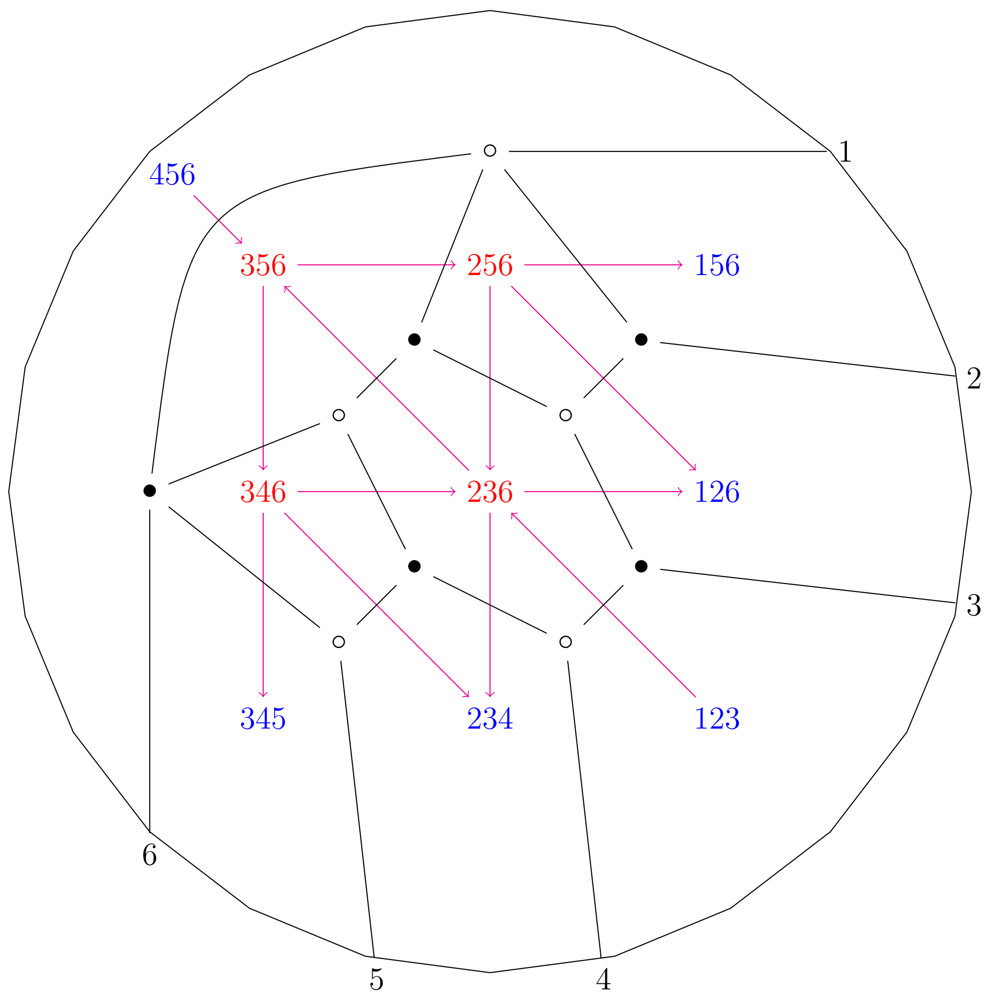
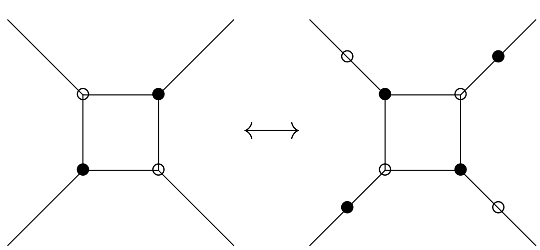
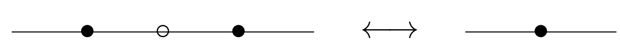


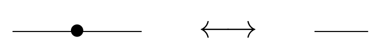
Figure 1: Plabic Graph (black) and quiver (blue vertices are frozen, red vertices are mutable, edges are magenta) for $\mathrm{Gr}_{3,6}$. As usual, we do not include edges between frozen (blue) vertices. Faces are labelled by Plücker coordinates. The trip permutation is $(4, 5, 6, 1, 2, 3)$.

Analogously to quiver mutations, we have the following local moves on plabic graphs:

1. Square move:



2. Vertex contraction/expansion 

3. Removal of degree two vertex 

We now collect some facts about plabic graph combinatorics.

Facts 2. Let G be a (reduced) plabic graph with boundary vertices labelled $1, \ldots, n$. Define the map $\pi_G : [n] \to [n]$ by setting $\pi_G(i)$ to be the endpoint of the path p_i (as above).

1. π_G is a bijection, which we call the *trip permutation* of G.

2. π_G is invariant under the three moves listed above.

3. Whenever two graphs G_1, G_2 have $\pi_{G_1} = \pi_{G_2}$, then there is a sequence of the three moves listed above transforming G_1 to G_2.

4. When G is a plabic graph corresponding to a Grassmannian $\mathrm{Gr}_{k,n}$, each interior face will be labelled by a unique k-element subset of $\{1, \ldots, n\}$. Furthermore, the set of Plücker coordinates corresponding to these labels together with the quiver coming from the dual graph form a labelled seed in the homogeneous coordinate ring of the Grassmannian.

In [RW17], Rietsch and Williams define two polytopes corresponding to a choice of plabic graph with trip permutation $\pi_{k,n} = (n-k+1, \ldots, n, 1 \ldots, n-k)$ (corresponding to the Grassmannian $\mathrm{Gr}_{k,n}$). One is the Newton-Okounkov body, which encodes algebraic data, and the other is the superpotential polytope, which encodes mirror symmetry data. They showed that these two polytopes coincide, and they give a concrete realization of the superpotential polytope via inequalities. This gives an algorithm for computing the Newton-Okounkov body in these cases, which is usually incredibly difficult.

In general (i.e. possibly after a sufficiently large dilation), the set of lattice points in the Newton-Okounkov body forms a *Khovanskii basis* for the homogeneous coordinate ring of the Grassmannian. When no dilation is necessary, the Plücker coordinates are in bijection with the lattice points of the Newton-Okounkov body, and thus the Plücker coordinates form a Khovanskii basis for the homogeneous coordinate ring of the Grassmannian with respect to the valuation of theorem 1.2 [RW17]. (In general, the valuation is more difficult to define when a dilation is required because we need the valuation of more than just Plücker coordinates.)

In the framework of Kaveh-Manon [KM16], this yields a *toric degeneration* of the Grassmannian from the associated graded ring to the valuation above, and that this ring is isomorphic to the homogeneous coordinate ring of an initial degeneration of the Grassmannian. We will be interested in questions about these initial degenerations and how they can be characterized using plabic graphs.

5.2 Plabic Graph Combinatorics

In this section, we describe how to obtain the valuations and initial degenerations from a plabic graph. We will restrict our attention to plabic graphs with trip permutation $\pi_{k,n}$.

Rietsch and Williams use plabic graphs to parametrize embeddings of tori into an open subvariety of $\mathrm{Gr}_{k,n}$, and they show that these embeddings give a transcendence basis for the rational function field of $\mathrm{Gr}_{k,n}$ by the coordinates corresponding to faces of the plabic graphs. They give a valuation in these coordinates using the *theta basis*, and they give a combinatorial description using *flows* with respect to a *perfect orientation*.

Definition 3. Let G be a plabic graph with boundary vertices labelled by $[n]$. A *perfect orientation* of G is an orientation of the edges of G such that:

1. Each internal black vertex has exactly one outgoing edge.
2. Each internal white vertex has exactly one incoming edge.

We call boundary vertices incident to an outgoing edge *sources* and those incident to an incoming edge *sinks*.

Facts 4. We collect some useful facts about perfect orientations:

1. Every reduced plabic graph has an acyclic perfect orientation.
2. Every reduced plabic graph with trip permutation $\pi_{k,n}$ has an ayclic perfect orientation with source set $(1, \dots, k)$.

Definition 5. Let G be a plabic graph with boundary vertices labelled by $[n]$, and O a perfect orientation of G. Fix $I = \{i_1, \dots, i_m\}, J = \{j_1, \dots, j_m\} \subset [n]$. A flow in G with respect to O from I to J is a set of vertex-disjoint paths $F = \{p_1, \dots, p_m\}$ such that each $i \in I$ is the beginning of some path and each $j \in J$ is the end of some path.

To work with plabic graphs G for $\mathrm{Gr}_{k,n}$, we make the following assumptions:

1. G is reduced.
2. The boundary vertices of G are labelled $1, \dots, n$ in clockwise order with respect to the planar embedding in a disk.
3. The trip permtuation π_G is $\pi_{k,n} = (n-k+1, \dots, n, 1, \dots, n-k)$.
4. Perfect orientations of G are acyclic and have source set $\{1, \dots, k\}$.

We are now ready to define the valuations of the Plücker coordinates. Let G be a plabic graph with perfect orientation O as above. Let $J \subset n$ be any face label of G (so $|J| = k$).

Consider a Plücker coordinate p_I. For any flow F_I from $\{1, \dots, k\}$ to I, we define $v_{F_I,J}$ to be the number of paths in F_I the face labelled by J appears to the left of. Then we define a valuation $\mathrm{val}_G : \mathbb{C}(\mathrm{Gr}_{k,n}) \to \mathbb{Z}^{n(n-k)+1}$ on the Plücker coordinates p_I by $\mathrm{val}_G(p_I)_J = \min_{F_I}\{v_{F_I,J}\}$. Although this valuation seems to depend upon the choice of perfect orientation O, it is in fact independent of the chosen perfect orientation. Furthermore, there is a purely combinatorial formula for $\mathrm{val}_G(p_I)_J$ in terms of the young diagrams corresponding to I and J (see theorem 1.2 in [RW17]).

When the Plücker coordinates form a Khovanskii basis for this valuation, the matrix with columns given by the valuations of the Plücker coordinates corresponds to the valuation matrix M in [KM16] and val_G coincides with the valuation ν_M defined in [KM16]. Now we have a toric degeneration with homogeneous coordinate ring $\mathrm{gr}_{\mathrm{val}_G}(\mathbb{C}[\mathrm{Gr}_{k,n}])$ coming from the framework of Kaveh-Manon. In the next section, we explore the connection between this toric degeneration and initial degenerations of $\mathrm{Gr}_{k,n}$ coming from Gröbner theory.

We now give an example of the above by computing this information for $\mathrm{Gr}_{3,6}$ using the same plabic graph as above.

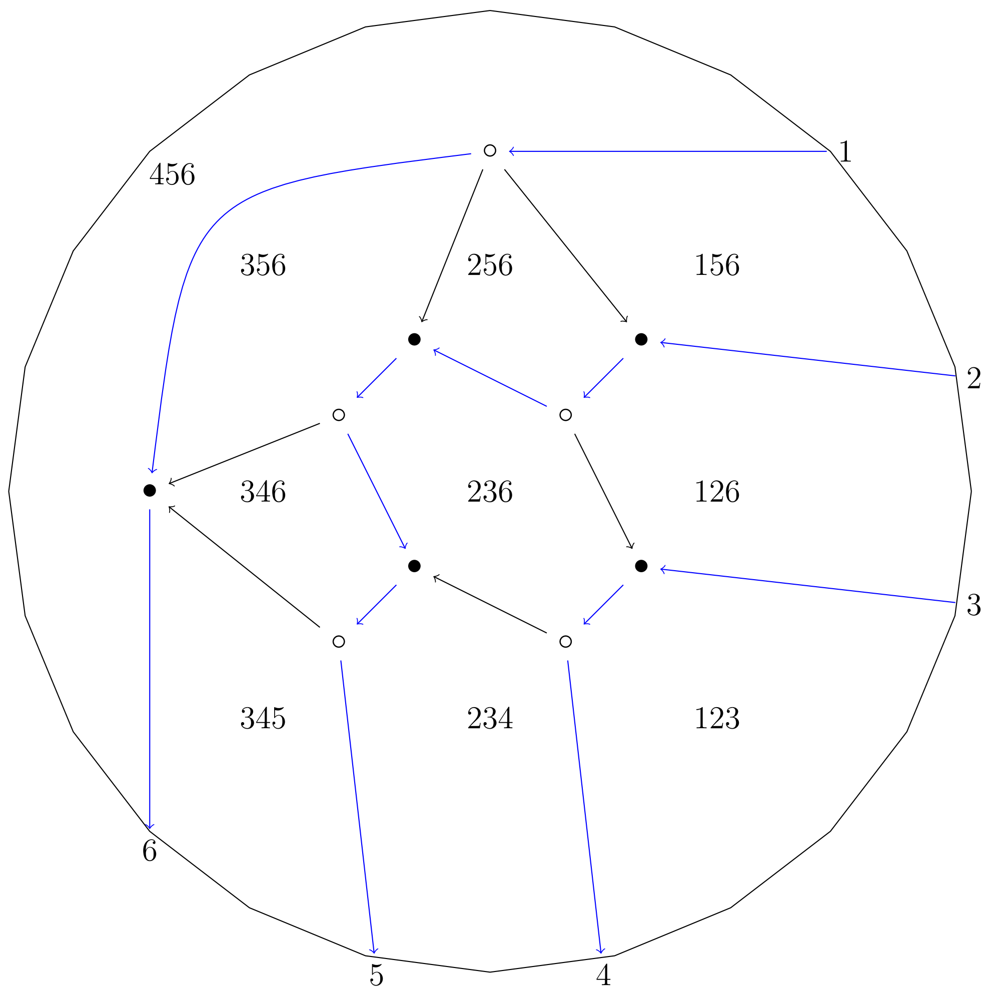


Figure 2: Plabic Graph with perfect orientation and flow from $\{1,2,3\}$ to $\{4,5,6\}$. There is a unique flow in this case, and this gives $\mathrm{val}_G(p_{456}) = (3,2,1,2,2,1,1,1,1,0)$ (where the coordinates of the plabic graph are lexicographically ordered). Note that our conventions for plabic graphs always force the last coordinate to be 0.

We think of this valuation as the "fine data." We can also define a "coarser valuation" (the degree map of [BFF⁺16]) which assigns to each Plücker coordinate an integer rather than a vector. For each flow F_I above, we define v_{F_I} to be the sum over the internal faces (not incident to the boundary disk) of the number of paths that internal face is to the left of, and then we set $\deg(p_I) = \min_{F_I}\{v_{F_I}\}$.

5.3 Symmetric Plabic Graphs

Since the Lagrangian Grassmannian is a (relatively simple) subspace of the Grassmannian, we might expect not much modification to the previous setup will be necessary. So far, we have indexed Plücker coordinates by $I \subset \binom{[n]}{k}$. We may also index Plücker coordinates by Young diagrams by the following procedure. A lattice path in $\mathbb{R}^2$ going only down and left starting from $(n-k, 0)$ and ending at $(0, -k)$ has exactly n steps ($n-k$ horizontal, and k vertical steps), and we label them in order from $1, \dots, n$. To $I \in \binom{[n]}{k}$, we associate the Young diagram whose boundaries are the lattice path whose vertical steps are in I, the x-axis, and the y-axis.

For a point $p \in \mathrm{Gr}_{n,2n}$ (now with coordinates indexed by young diagrams), Rachel Karpman showed in [Kar15] that $p_T \in \mathrm{LGr}_n$ if and only if for any coordinate p_μ, we have $p_\mu = p_{\mu^*}$, where μ^* is the transpose Young diagram. On the plabic graph level, this manifests as the following symmetry condition:

Definition 6. A *symmetric plabic graph* is a plabic graph with a distinguished diamater d of the disk such that:

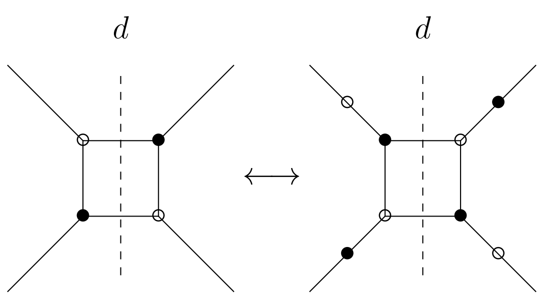
1. The endpoints of the diameter lie between vertices $2n, 1$ and between vertices $n, n+1$.

2. No vertex of G lies on d, though edges may cross d.

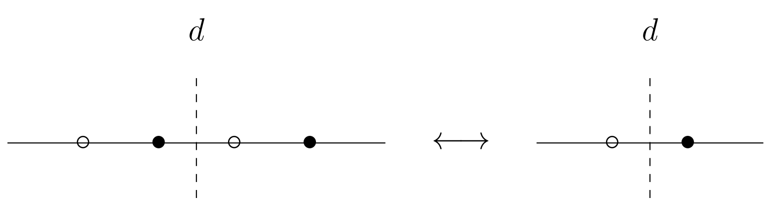
3. Reflection through d gives the same graph G but with vertex colors reversed.

In order to preserve symmetry, we must change the moves allowed slightly so that they preserve the symmetry with respect to d. If G is a symmetric plabic graph, and we can perform an ordinary move (one of the three defined above) on G, then we consider two cases. If the affected region lies entirely on one side of d, then we must perform the move on both sides of d. Otherwise, we have the following two new moves:

1. Symmetric square move:



2. Symmetric vertex contraction/expansion



Note that each of the moves above can be obtained by performing one or more sequential moves on the undirected plabic graph G. As a consequence, we can think of studying plabic graphs for LGr_n simply as studying those plabic graphs for $\mathrm{Gr}_{n,2n}$ that are symmetric. Note that the Plücker coordinate valuation or the degree as defined above will also exhibit this symmetry by having certain Plücker coordinates equal. The other definitions above persist.

5.4 Positivity

In [Pos06], Postnikov defined the *boundary measurement map* associated to a plabic graph G, which he showed (Theorem 12.7) defines a map $\mathrm{Meas}_G : \mathbb{R}_{\geq 0}^{F(G)-1} \to \mathbb{R}_{\geq 0}^{\binom{n}{k}}$ which is surjective onto a positroid cell (corresponding to G) in the Grassmannian. Here $F(G)$ denotes the number of faces of G. We now recall this definition.

Let G be a (reduced) plabic graph with trip permutation $\pi_{k,n}$ and all the assumptions of section 5.2, with face labels L and corresponding variables x_l. Let $\mathrm{left}(F, l)$ be the number of paths in F which contain l to the left. For a flow F from $\{1, \ldots, k\}$ to $I \in \binom{[n]}{k}$, we define the *weight* of F to be the product $\prod_{l \in L} x_l^{\mathrm{left}(F,l)}$. Finally, we define the flow polynomial f_I by the sum $f_I = \sum_{F \text{ flow from } [k] \text{ to } I} \mathrm{weight}(F)$. (By construction, this polynomial always has positive coefficients.) Now, Meas_G will be given coordinate-wise by these polynomials, where the $I \in \binom{[n]}{k}$ coordinate of Meas_G is the polynomial f_I. Even though there are $F(G)$ face labels, our convention forces one label never to appear in any flow polynomial.

Karpman showed the analogous result for the Lagrangian Grassmannian [Kar15].

6 Tropical Geometry

Although tropical varieties are interesting in their own right, we will mainly be interested in using them to organize information about initial ideals, and studying the parts we can see from plabic graphs.

6.1 Valuations and Degrees

In this section, we fix a term order $\prec$ on $R = \mathbb{C}[p_I \mid I \in \binom{[n]}{k}]$. For an $m \times \binom{n}{k}$ matrix M, define $\alpha \prec_M \beta$ when $M\alpha \prec M\beta$ for $\alpha, \beta \in \mathbb{Z}^{\binom{n}{k}}$. The initial form $\mathrm{in}_M(f)$ of $f \in R$ is the sum of $\prec_M$-minimal terms of f. For an ideal $I \subset R$, we define $\mathrm{in}_M(I)$ to be the ideal generated by initial forms $\mathrm{in}_M(f)$ for $f \in I$.

In general, for a rational matrix M and homogeneous $I \subset R$, there is a single rational vector w with $\in_M (I) = \in_w (I)$, and furthermore if the rows of M all lie in the same cone of the Gröbner fan of I, we may take u to be the sum of the rows of M (see e.g. [KM16]).

In our case, more seems to be true. We state our conjectures for the finite type Grassmannians ($\mathrm{Gr}_{2,n}$, $\mathrm{Gr}_{3,n}$ for $n = 6, 7, 8$, $\mathrm{LGr}_3, \mathrm{LGr}_4$), when there exists a unique coarsest fan structure, but expect some of them to hold in greater generality.

Conjecture 1. Fix a plabic graph G with trip permutation $\pi_{k,n}$. If the Plücker coordinates form a Khovanskii basis with respect to val_G, then we set M to be the matrix whose columns are the valuations of the Plücker coordinates. In this case, the rows of M corresponding to frozen (boundary) faces lie in the lineality space of the tropical variety, and the rows of M corresponding to the mutable (interior) faces are rays of the same cone in the tropical variety.

Moreover, in this case, the vector of degrees of Plücker coordinates gives an interior point of the cone corresponding to the rows of the matrix M.

Given this, we may ask what types of cones correspond to a plabic graph, and how local moves on plabic graphs manifest in the tropical variety.

Conjecture 2. Fix a plabic graph G with trip permutation $\pi_{k,n}$. The vector of degrees of Plücker coordinates lies in a maximal cone of the tropical (Lagrangian) Grassmannian $\mathrm{Gr}_{k,n}$ with its coarsest fan structure. Note that in the Gröbner fan structure, the point may not lie on a full dimensional cone (see e.g. [BFF$^+$16] where a plabic graph for $\mathrm{Gr}_{3,6}$ gives a point that lies on a face of type GG).

It is in general difficult to understand the coarsest fan structure on a tropical variety, and we woule like to see if there is any further connection between cluster algebras and the fan structure. For example, in LGr_3, there are non-simplicial cones which are triangulated in the Gröbner fan structure, and the vectors obtained form plabic graphs in these cones land on the facets in the Gröbner fan structure, but in the interior in the coarsest fan structure.

Conjecture 3. Let C_G be a cone in the tropical (Lagrangian) Grassmannian corresponding to a plabic graph G via the vector of degrees of Plücker coordinates. If G' is obtained from G by a square move, then $C_{G'}$ and C_G are connected in codimension 1.

If this is true, then we can obtain a substructure of the tropical variety corresponding to plabic graphs. What kind of substructure is this?

We now ask a few questions.

Question 4. Rietsch and Williams note that when the Plücker coordinates do not form a Khovanskii basis, the Newton-Okounkov body has a non-integer vertex. On the algebraic side, this seems connected to the primality of the corresponding cone in the tropical variety. (A prime cone is one such that the initial ideal corresponding to that cone is prime.)

We would like to understand this correspondence more exactly: can this non-integrality or non-primeness be detected in the plabic graph? If non-integrality and non-primeness do not coincide, can this be detected in the plabic graph?

In particular, when the cone is prime, we get a toric degeneration of the (Lagrangian) Grassmannian. Otherwise, we get a variety with a toric component. Can we detect isomorphism classes of these toric varieties from the plabic graph?

Question 5. In the quiver corresponding to a plabic graph, there are more moves than there are moves on the plabic graph. Some of these moves do not give a plabic graph, but are still interesting in the cluster structure of the Grassmannian. What role do these play in the tropical variety? Is it still the case that this mutation finds an adjacent cone in the tropical variety?

6.2 Positivity

We return to the question of positivity. We restrict ourselves to the real Grassmannian, so that positivity makes sense. The totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{tnn}$ consists of those points of the Grassmannian where all Plücker coordinates have the same sign (nonnegative or nonpositive). Postnikov ([Pos06]) showed that this positive part admits a stratification by *positroid cells*, each of which is the (closure of the) image of the boundary measurement map for some plabic graph G.

On the tropical side, we define $\mathrm{Trop}(\mathrm{Gr}_{k,n})_+$ to be the set of vectors $w \in \mathbb{R}^{\binom{n}{k}}$ such that $\mathrm{in}_w(I_{k,n})$ contains no polynomial with only positive coefficients. We now ask what is the relation between $\mathrm{Gr}_{k,n}^{tnn}$ and $\mathrm{Trop}(\mathrm{Gr}_{k,n})_+$.

In order to do this, we recall the setup of section 3 in [PS04]. Let $f : \mathbb{R}^d \to \mathbb{R}^n$ be a (coordinatewise) polynomial map. We say that f is positive if the coefficients of each entry of f are positive, and we say that f is surjectively positive if the restriction to the positive orthant $f_{\geq 0} : \mathbb{R}^d_{\geq 0} \to \mathbb{R}^n_{\geq 0} \cap \operatorname{im}(f)$ is surjective. The tropicalization of f is $g : \mathbb{R}^d \to \mathbb{R}^n$ defined by replacing the usual arithmetic operations in f with tropical operations. The Newton polytope of f is defined to be the Minkowski sum of the Newton polytopes of its coordinates. In this setup, [PS04] give the following theorem:

Theorem 6. *If f is surjectively positive, then the image of g is the same as the positive part of the tropical variety of the variety corresponding to the image of f.*

For us, since the boundary measurement map is surjective onto its corresponding positroid cell, and these cells stratify the positive part of the (Lagrangian) Grassmannian, we can study $\mathrm{Trop}(\mathrm{Gr}_{k,n})_+$ by understanding these boundary measurement maps. Furthermore, there is one top dimensional positroid cell, so it is enough to understand this the boundary measurement map corresponding to this cell. We carry this out in the case LGr_4.

6.3 $\mathrm{Trop}(\mathrm{LGr}_4)_+$

First, we need to write down all the flow polynomials for the top positroid cell in LGr_4. We draw the corresponding plabic graph, and find all the $\binom{2*4=8}{4} = 70$ flow polynomials (not written here) and their corresponding Newton polytopes. These flow polynomials are the coordinates of a map $f : \mathbb{R}^{10} \to \mathbb{R}^{70}$, where the domain corresponds to the 10 distinct face labels (excluding the face that never appears in any flow polynomial) of the plabic graph for LGr_4, and the range corresponds to the 70 Plücker coordinates (some of which will be the identified in the Lagrangian Grassmannian).

We then take the common refinement, F, of the 70 normal fans corresponding to these Newton polytopes. This is a pure 10-dimensional fan with 4-dimensional lineality space, 132 rays, 1890 maximal cones, and f-vector $(132, 1302, 4698, 7870, 6232, 1890)$.

Finally, we tropicalize f (replace $+$ with min and $\times$ with usual $+$), and consider the image of F under the tropicalized map $\mathrm{trop}(f)$. The first coordinate of any point in the image of F is always 1 (the flow $\{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ is always the empty flow, so the first entry corresponding to Plücker coordinate p_{1234} of f is always the constant 1 function). Thus we consider the map $g : \mathbb{R}^{10} \to \mathbb{R}^{69}$ obtained by forgetting the first entry of $\mathrm{trop}(f)$. Then the image of F under $\mathrm{trop}(f)$ is the cone over the image of F under $\mathrm{trop}(g)$ embedded in $\mathbb{R}^{70}$ by setting the first coordinate to be 1.

In this case, the image of F under $\mathrm{trop}(g)$ is a fan isomorphic to the normal fan above, and the image of f is a cone over this fan. The image of $\mathrm{trop}(f)$ gives a 7 dimensional fan in $\mathbb{R}^{70}$ with lineality space of dimension 4. The image of $\mathrm{trop}(f)$ is not pure, and has f-vector $(132, 1586, 6541, 12267, 11054, 4283, 409)$. In this case, we confirm the conjecture of [SW03].

7 Data

We collect here the data produced. We have computed all plabic graphs for the Grassmannians $\mathrm{Gr}_{3,n}$ for $n = 6, 7, 8$ and $\mathrm{Gr}_{4,8}$ as well as for Lagrangian Grassmannians $\mathrm{LGr}_3, \mathrm{LGr}_4$.

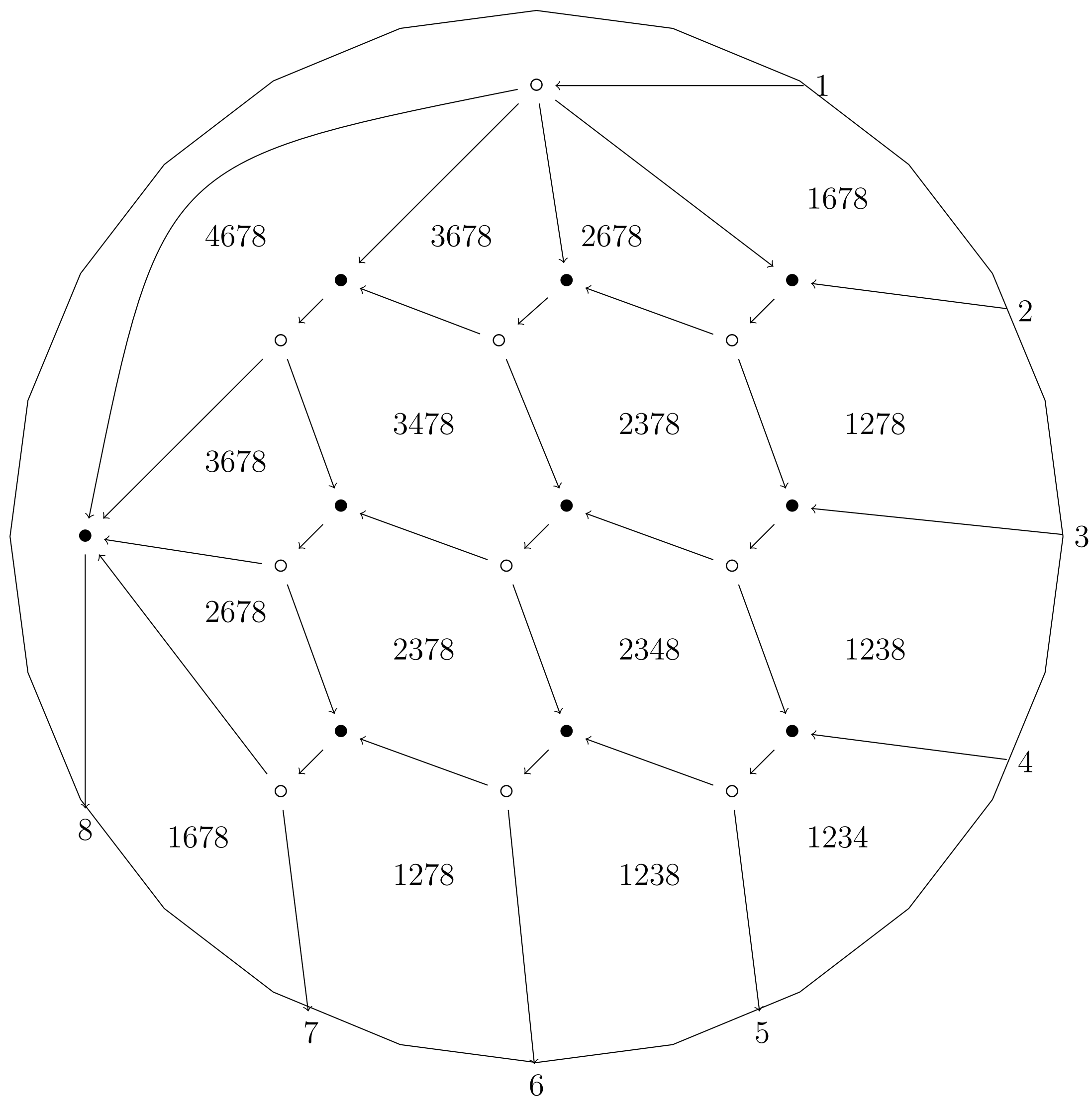


Figure 3: Symmetric plabic graph for LGr_4 corresponding to top positroid cell with perfect orientation. Last region has label 5678, but never appears to the left of any path.

The respective numbers of plabic graphs are $34, 256, 2136, 5470$ for the Grassmannians and $12, 194$ for the Lagrangian Grassmannians. We also have the corresponding weight vectors.

For $\mathrm{Gr}_{3,6}$, we confirmed the computations in [BFF$^+$16] that there are exactly 2 non-binomial initial ideals coming from the 34 weight vectors, and furthermore that these coincide with the plabic graphs giving non-integral Newton-Okounkov bodies listed in [RW17]. Furthermore, we began computations for $\mathrm{Gr}_{3,7}$ and found 42 non-binomial initial ideals. This differs slightly from the 43 non-integral Newton-Okounkov bodies found in [RW17]. It would be interesting to study the remaining ideal, which we guess will be binomial, but non-prime.

In the case of LGr_3, we have further data for the 12 plabic graphs.

7.1 LGr_3

We computed the tropical variety $\mathrm{Trop}(\mathrm{LGr}_3)$. In this case, we have a unique coarsest fan structure with 35 rays, 153 facets, and an f-vector $(35, 151, 153)$. Since the Lagrangian Grassmannian is a subspace of the Grassmannian given by linear relations of the form $p_\mu - p_\lambda$, we can choose to eliminate some variables because tropicalizing these relations just moves to a linear subspace of the ambient space of the tropicali variety. If we perform this elimination, then this fan is 7-dimensional with a 4-dimensional lineality space in a space of ambient dimension 14. Otherwise, the lineality space will be 10-dimensional and the ambient space will have dimension 20. We label its rays by the numbers $0, \ldots, 34$, and note that the weight vectors given below live in this larger space. We label cones in this fan by their rays.

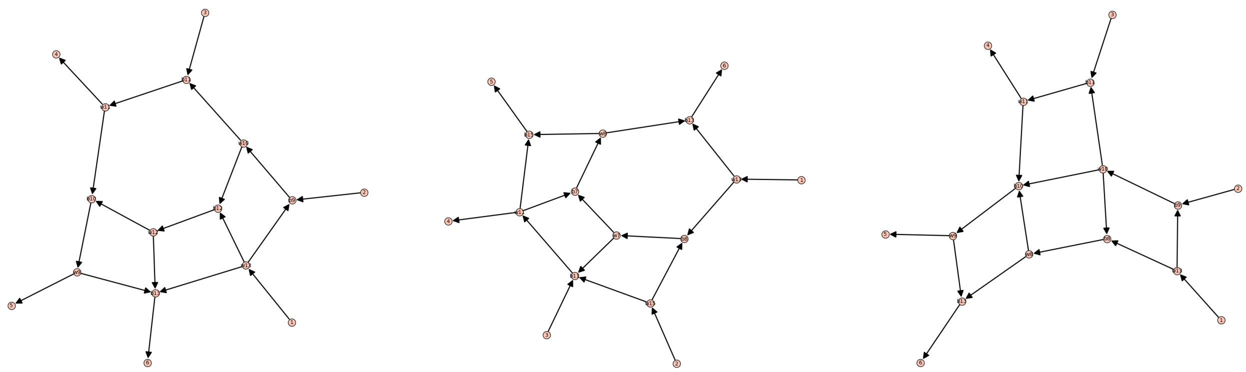


Figure 4: (0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 0, 0, 0, 1, 1, 2, 2, 2, 3, 5) (left) in cone (1,5,6)
(0, 0, 0, 1, 0, 0, 1, 3, 3, 3, 1, 1, 2, 3, 3, 3, 3, 3, 3, 7) (middle) in cone (1,6,8)
(0, 0, 0, 0, 0, 0, 0, 1, 1, 3, 0, 0, 0, 1, 1, 3, 3, 3, 4, 5) (right) in cone (0,1,5)

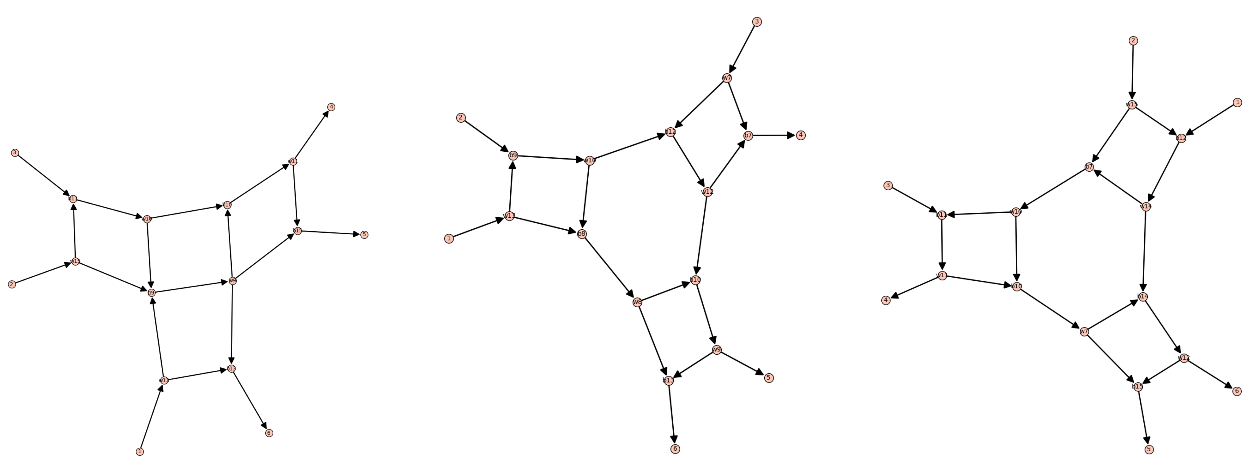


Figure 5: (0, 0, 0, 2, 0, 0, 2, 3, 3, 3, 2, 2, 3, 3, 3, 3, 3, 3, 3, 7) (left) in cone (6,8,10)
(0, 0, 1, 1, 1, 1, 1, 1, 1, 4, 1, 1, 1, 1, 1, 4, 4, 4, 5, 5) (middle) in cone (0,4,5,17), this initial ideal is not binomial
(0, 0, 0, 1, 0, 0, 1, 1, 2, 4, 1, 1, 2, 2, 3, 4, 4, 4, 4, 5) (right) in cone (0,1,12)

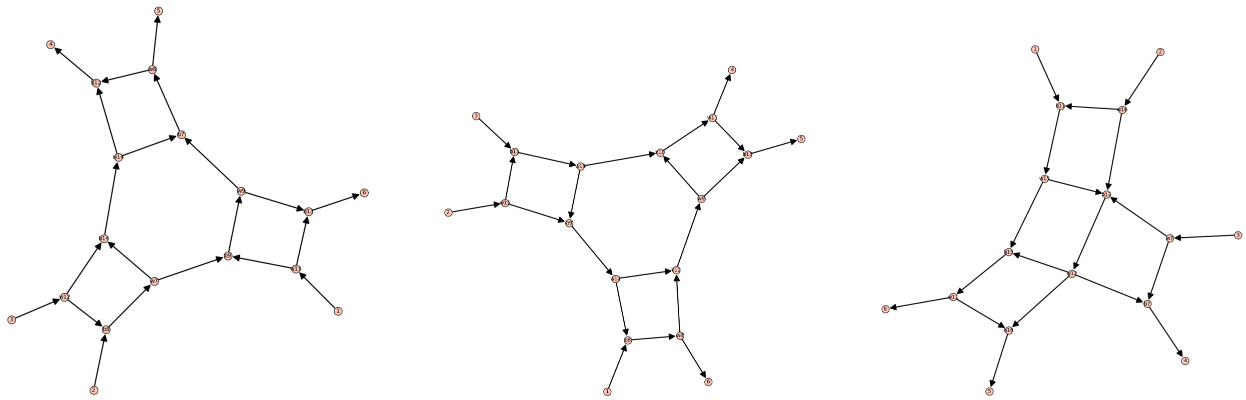


Figure 6: (0, 0, 1, 3, 1, 2, 3, 3, 3, 4, 3, 3, 3, 3, 3, 4, 4, 4, 5, 7) (left) in cone (6,10,28)
(0, 0, 0, 3, 0, 0, 3, 3, 4, 4, 3, 3, 4, 4, 4, 4, 4, 4, 4, 7) (middle) in cone (3,8,10,19), this initial ideal is not binomial
(0, 0, 1, 2, 1, 1, 2, 1, 2, 5, 2, 2, 3, 2, 3, 5, 5, 5, 5, 5) (right) in cone (0,12,17)

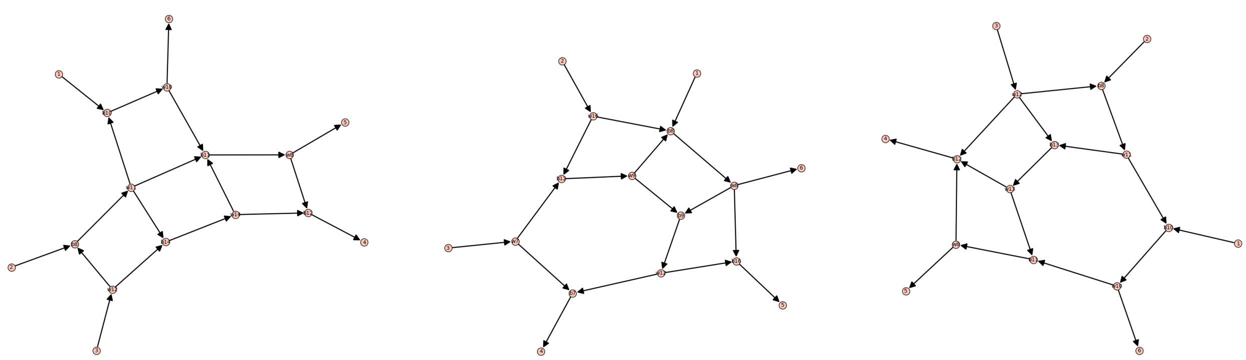


Figure 7: (0, 0, 1, 4, 1, 2, 4, 3, 4, 5, 4, 4, 4, 4, 4, 5, 5, 5, 6, 7) (left) in cone (10,19,28)
(0, 0, 1, 3, 1, 1, 3, 1, 3, 5, 3, 3, 4, 3, 4, 5, 5, 5, 5, 5) (middle) in cone (12,17,19)
(0, 0, 2, 4, 2, 3, 4, 3, 4, 6, 4, 4, 4, 4, 4, 6, 6, 6, 7, 7) (right) in cone (17,19,28)

We would like to carry out similar analysis for other Grassmannians for which we have the tropical variety.

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