Question 1.

min = 0-59 : $6\times1+10=16$

$\frac{16}{60}=\frac{4}{15}$

hour = 1-12 $\Rightarrow \frac{4}{12}=\frac{1}{3}$

$\Rightarrow \mathbb{P}(\text{correct time}) = 1-\frac{4}{15}-\frac{1}{3} = 1-\frac{4+5}{15} = \frac{2}{5}$

Question 2

Denote $P(n)$ as probability of going bankrupt

here for simplicity, consider $P(\text{head})=\frac{1}{2}$, $P(\text{tail})=\frac{1}{2}$

if not, the solution is similar and can be solved using similar procedure.

$$\begin{cases} P(n)=\frac{1}{2}P(n-1)+\frac{1}{2}P(n+2) \\ P(0)=1 \end{cases}$$

$\Rightarrow P(n+2)=2P(n)-P(n-1)$;

$x^3=2x-1$ characteristic equation

$x_1=1,\ x_2=\frac{1}{2}(\sqrt{5}-1),\ x_3=\frac{1}{2}(-1-\sqrt{5})$; denote $1,\ \phi-1,\ -\phi$

=> General solution is $P(n) = a1^n + b(\phi-1)^n + c(-\phi)^n$ w.r.t. $a+b+c=1$ to match the condition $P(0)=1$,

c must be 0 (Otherwise $c>0$, the probability will be negative for large odd n and greater than 1 for large even n ; $c<0$, probability will be >1 for large odd n and negative for large even n)

$$\lim_{n\to\infty} P(n)=0$$

$$\therefore P(n)=(\phi-1)^n$$

$$P(50)=(\phi-1)^{50} = \left(\tfrac{1}{2}(\sqrt{5}-1)\right)^{50}$$

Now we demonstrate the case that $\mathbb{P}(head)=p$, $\mathbb{P}(tail)=1-p=q$

$$P(n)= p\,P(n+2)+q\,P(n-1),\ n\geq 1 \quad ; \ P(0)=1$$

$$p\cdot P(n+2)-P(n)+q\,P(n-1)=0$$

$$px^3-x+1-p=0 \quad \text{(also, characteristic equation)}$$

$$(x-1)[p(x^2+x+1)-1]=0$$

$$(x-1)[px^2+px+p-1]=0$$

$$x_1=1,\ x_{2,3}=\frac{1}{2p}\left(-p\pm\sqrt{4p-3p^2}\right) = -\frac{1}{2}\pm\frac{1}{2}\sqrt{\frac{4}{p}-3}$$

let $\phi=\frac{1}{2}+\frac{1}{2}\sqrt{\frac{4}{p}-3}$

$$P(n)=a\cdot 1^n+b(\phi-1)^n+c(-\phi)^n$$

$$P(0)=a+b+c=1$$

$$\lim_{n\to\infty}P(n)=0 \Rightarrow a=0$$

$c=0$, otherwise $P(n)>1$ or $P(n)<0$

$$\therefore b=1$$

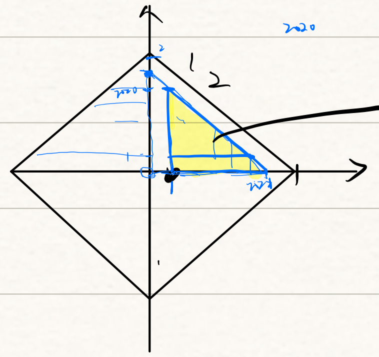
$$\therefore P(n)=(\phi-1)^n=\left(\frac{\sqrt{\frac{4}{p}-3}-1}{2}\right)^n$$

$$P(50)=\left(\tfrac{1}{2}\left(\sqrt{\tfrac{4}{p}-3}-1\right)\right)^{50}$$

QED

Q3.

$|x|+|y|<2022$



for this part, we have $(1+2021)\times 2020\times\frac{1}{2}$

$= 2042220 = A$

origin

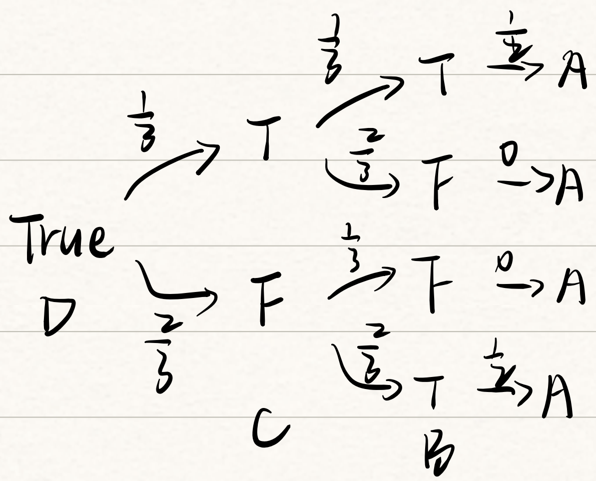
$A\times 4+1 = 8168881$

Q4. A, B, C, D four people

$P(\text{truth}) = \frac{1}{3}$

We start from D.

$P(C \text{ tells } D \text{ is actual true} \mid D \text{ says true}) = \frac{1}{3}$



$\frac{1}{3}\times\frac{1}{3}+\frac{2}{3}\times\frac{2}{3}=\frac{5}{9}$

$\therefore P(A \text{ is telling truth} \mid D \text{ says it is true}) = \frac{5}{9}$

QED

Q5. $f: \mathbb{Z} \to \mathbb{Z}$

$f(a-f(b)) = f(f(a)) - f(b) - 1$ for $\forall a, b \in \mathbb{Z}$

$f(a-f(b)) + 1 = f(f(a)) - f(b)$

set $a=0$, $f(-f(b)) = f(f(0)) - f(b) - 1$

$\because -f(b) \in \mathbb{Z}$

$\therefore f$ satisfies $f(x) = f(f(0)) + x - 1$ ①

set a & $f(b) = 0$.

$f(0) = f(f(0)) - 1$, plug back in ①

$\therefore f(x) = f(0) + 1 + x - 1 = f(0) + x$

$\Rightarrow f(b) = b + f(0)$

$\therefore f(a-b-f(0)) = f(f(a)) - b - f(0) - 1$

$a - b - f(0) + f(0) = a + 2f(0) - b - f(0) - 1$

$\therefore f(0) = 1$

Therefore, $f(x)$ should be $f(x) = x+1$

We can simply verify that it is true for $\forall a$ and b.

QED.

Q6.

9 identical coins → 1 light (denote coin 1, 2, ..., 9)

3 balance → 1 broken (denote balance A, B, C)

① seperate into 3, 3, 3 coins

$\underbrace{123,\ 456}_{A}$ $\underbrace{123,\ 456}_{B}$, $\underbrace{123,\ 456}_{C}$

According to the result we find the fake balance, (the one has different result

here for demonstration, without loss of generality, we say that

C is the broke balance, so we abandon it

Also, from previous result we know the weighted result

① if A and B's results are both equal

fake coin is in 789.

$\underbrace{7,\ 8}_{A}$ if 7=8, 9 is the fake one

if 7≠8, than the lighter one is fake

② if not equal, W.O.L.G assume A is lighter

fake coin is in 123 ⇒ similar procedure in ①

(B is lighter is also the same)

Q.E.D.

Q7.

(1) we consider this problem in three situation

① jockey is at corner.

W.O.L.G, it is assumed to be at a_1

$a_1 \to c_2 \to a_3 \to b_1$ □ $a_1 \to b_2$ is the similar procedure

② jocky it at edge but not frontier

W.O.L.G, it is assumed to be at a_2

$\begin{cases} a_2 \to c_1 \to b_3 \to a_1 \quad \square \quad a_2 \to c_3 \to b_1 \to a_3 \\ a_2 \to b_4 \to d_3 \to b_2 \end{cases}$

③ Jocky is not at edge nor at corner

W.O.L.G, it is assumed to be at b_2

$b_2 \to c_4 \to a_3 \to c_2$

Therefore, jocky could reach all its neighborhood in finite steps in any position of chess board

$\Rightarrow$ it could reach central area infinite steps

(2) Yes, as proved in (1)

jocky could reach all its neighborhood in finite steps in any position of chess board

$\therefore$ jocky starts from bottom left could moves to any other position in finite steps legally

(3) As showed in (1)(2), it is actually irreducible and positive recurrence markov chain

$\mathbb{P}$(return in finite steps) = 1

For better illustration. we put it into axes for better illustration.

Given vertex i of the graph

d_i = degree of vertex, # of edges connected to vertex

we can use balance equation

Q8. k positive divisor of n

1. Choose all prime numbers =

m^2-1