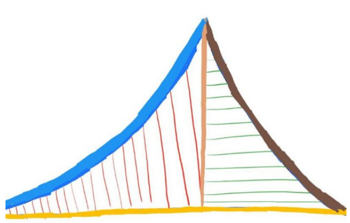
DIGITAL COMMUNICATION

Through Simulations

G. V. V. Sharma



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Contents

Introduction

This book introduces digital communication through probability.

Chapter 1

Maximum Likelihood Detection: BPSK

1.1. Maximum Likelihood

1.1.1 Generate equiprobable $X \in \{1, -1\}$.

1.1.2 Generate

$$Y = AX + N, \tag{1.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

1.1.3 Plot Y using a scatter plot.

1.1.4 Guess how to estimate X from Y.

1.1.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{1.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{1.3}$$

1.1.6 Find P_e assuming that X has equiprobable symbols.

1.1.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

1.1.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

1.1.9 Repeat the above exercise when

$$p_X(0) = p \tag{1.4}$$

1.1.10 Repeat the above exercise using the MAP criterion.

1.2. Examples

1.2.1 Consider a channel over which either symbol x_A or symbol x_B is transmitted. Let the output of the channel Y be the input to a maximum likelihood (ML) detector at the receiver. The conditional probability density for Y given x_A and x_B are:

$$f_{Y|x_A}(y) = e^{-(y+1)}u(y+1)$$
$$f_{Y|x_B}(y) = e^{(y-1)}(1 - u(y-1))$$

where $u(.)$ is the standard unit step function. the probability of symbol error for this system is (GATE EC 2022) **Solution:**

Decision in favour of x_A when

$$f_{Y|x_A}(y) > f_{Y|x_B}(y) \tag{1.5}$$

Decision in favour of x_B when

$$f_{Y|x_A}(y) < f_{Y|x_B}(y) \tag{1.6}$$

From the figure,

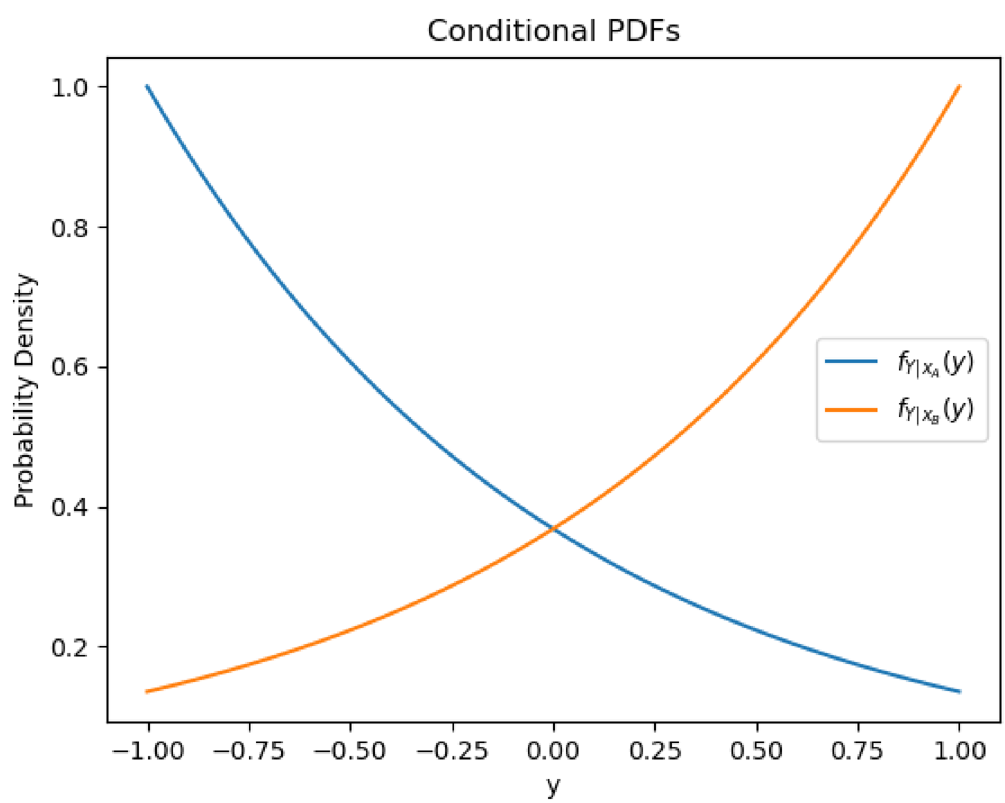

Figure 1.1: Conditional pdf

$$
\begin{cases}
f_{Y|x_A}(y) < f_{Y|x_B}(y) & , y < -1 \\
f_{Y|x_A}(y) > f_{Y|x_B}(y) & , -1 < y < 0 \\
f_{Y|x_A}(y) < f_{Y|x_B}(y) & , 0 < y < 1 \\
f_{Y|x_A}(y) > f_{Y|x_B}(y) & , y > 1
\end{cases} \tag{1.7}
$$

(a) When $0 < y < 1$.

In this interval, when x_A is transmitted, error occurs because the likelihood of observing Y given x_A is lower than the likelihood of observing Y given x_B, Therefore,

$$
\begin{aligned}
P_{e_{x_A}} &= \int_0^1 f_{Y|x_A}(y)\,dy && (1.8)\\
&= \int_0^1 e^{-(y+1)} u(y+1)\,dy && (1.9)\\
&= \int_0^1 e^{-(y+1)} dy && (1.10)\\
&= e^{-1} - e^{-2} && (1.11)\\
P_{e_{x_A}} &= 0.23 && (1.12)
\end{aligned}
$$

(b) When $-1 < y < 0$.

In this interval, when x_B is transmitted, error occurs because the likelihood of observing Y given x_A is higher than the likelihood of observing Y given x_B, Therefore,

$$P_{e_{x_B}} = \int_{-1}^{0} f_{Y|x_B}(y)\,dy \tag{1.13}$$
$$= \int_{-1}^{0} e^{(y-1)}\left(1 - u\left(y-1\right)\right) \tag{1.14}$$
$$= \int_{-1}^{0} e^{(y-1)} \tag{1.15}$$
$$= e^{-1} - e^{-2} \tag{1.16}$$
$$P_{e_{x_B}} = 0.23 \tag{1.17}$$

(c) When $y < -1$ and $y > 1$.

There are no errors in these intevals as the ML detectors can more reliably to make a decison. Therefore,

$$P_e = 0 \tag{1.18}$$

Hence, the total probability of error for this system can be given as,

$$P_e = \Pr\left(x_A\right) P_{e_{x_A}} + \Pr\left(x_B\right) P_{e_{x_B}} \tag{1.19}$$
$$= 0.23 \times \left(\Pr\left(x_A\right) + \Pr\left(x_B\right)\right) \tag{1.20}$$
$$= 0.23 \tag{1.21}$$

Chapter 2

Transformation of Random Variables

2.1. Gaussian to Other

2.1.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{2.1}$$

2.1.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{2.2}$$

find α.

2.1.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{2.3}$$

2.2. Conditional Probability

2.2.1 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{2.4}$$

for

$$Y = AX + N, \tag{2.5}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}\left(0, 1\right), X \in \left(-1, 1\right)$ for $0 \leq \gamma \leq 10$ dB.

2.2.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

2.2.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)p_X(x)\,dx \tag{2.6}$$

Find $P_e = E\left[P_e(N)\right]$.

2.2.4 Plot P_e in problems 2.2.1 and 2.2.3 on the same graph w.r.t γ. Comment.

Chapter 3

Bivariate Random Variables: FSK

3.1. Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{3.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{3.3}$$

3.1.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{3.4}$$

on the same graph using a scatter plot.

3.1.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

3.1.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{3.5}$$

with respect to the SNR from 0 to 10 dB.

3.1.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.

Chapter 4

Exercises

4.1. BPSK

1. The signal constellation diagram for BPSK is given by Fig. 4.1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received.

   Solution: The possible received symbols are

   $$y|s_0 = \sqrt{E_b} + n \tag{4.1}$$

   $$y|s_1 = -\sqrt{E_b} + n \tag{4.2}$$

   where the AWGN $n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

2. From Fig. 4.1 obtain a decision rule for BPSK

   Solution: The decision rule is

   $$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{4.3}$$

3. Repeat the previous exercise using the MAP criterion.

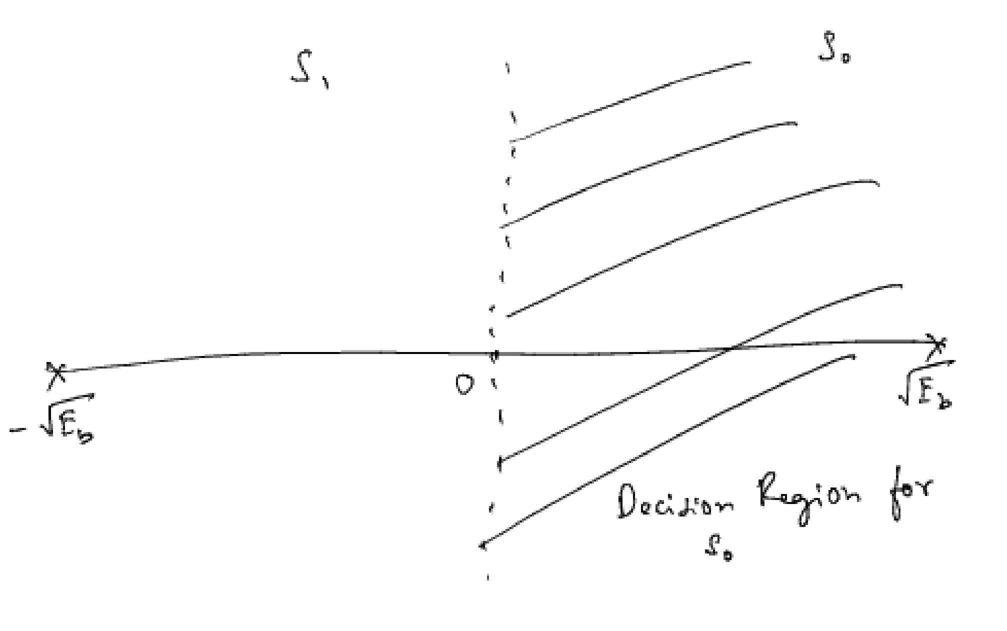

Figure 4.1:

4. Using the decision rule in Problem 2, obtain an expression for the probability of error for BPSK.

Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr\left(y < 0 | s_0\right) = \Pr\left(\sqrt{E_b} + n < 0\right) \tag{4.4}$$

$$= \Pr\left(-n > \sqrt{E_b}\right) = \Pr\left(n > \sqrt{E_b}\right) \tag{4.5}$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0,1)$. Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in

(4.5),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \tag{4.6}$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{4.7}$$

where $Q(x) \triangleq \Pr(w > x), x \geq 0$.

5. The PDF of $w \sim \mathcal{N}(0,1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{4.8}$$

and the complementary error function is defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt. \tag{4.9}$$

Show that

$$Q(x) = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{4.10}$$

6. Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.

Solution: The following code

```
codes/bpsk_ber.py
```

yields Fig. 4.2

7. Show that

$$Q(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta \tag{4.11}$$

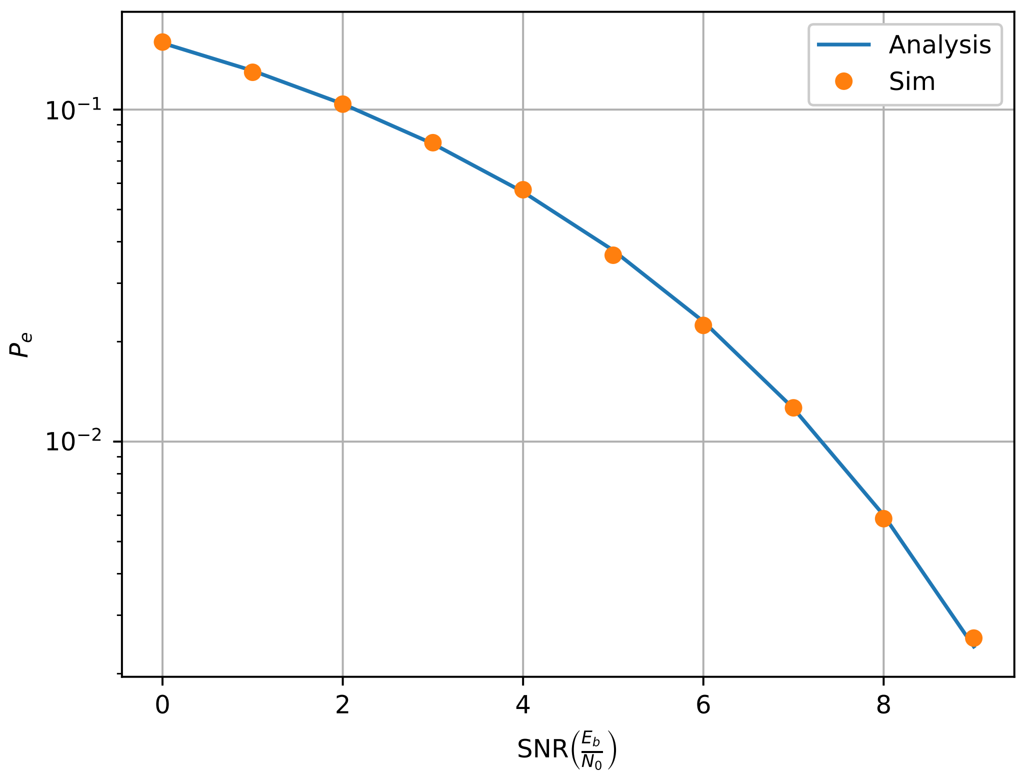

Figure 4.2:

4.2. Coherent BFSK

1. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 4.3. Obtain the equations for the received symbols.

   Solution: The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{4.12}$$

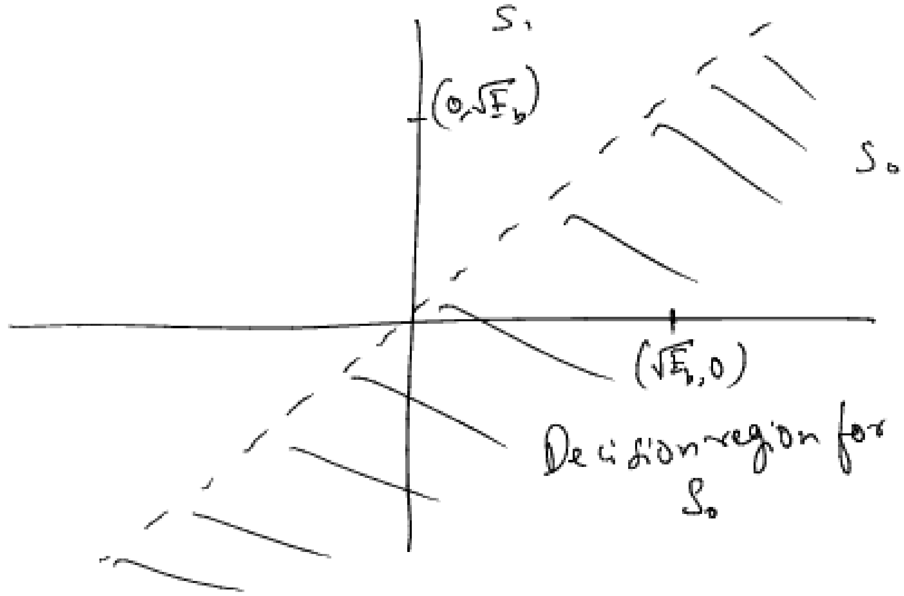

Figure 4.3:

and

$$\mathbf{y}|s_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{4.13}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

2. Obtain a decision rule for BFSK from Fig. 4.3.

   Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{4.14}$$

3. Repeat the previous exercise using the MAP criterion.

4. Derive and plot the probability of error. Verify through simulation.

4.3. QPSK

1. Let

$$\mathbf{r} = \mathbf{s} + \mathbf{n} \tag{4.15}$$

where $\mathbf{s} \in \{s_0, s_1, s_2, s_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix}, \tag{4.16}$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_b} \end{pmatrix}, \tag{4.17}$$

$$E\left[\mathbf{n}\right] = \mathbf{0}, E\left[\mathbf{n}\mathbf{n}^T\right] = \sigma^2\mathbf{I} \tag{4.18}$$

(a) Show that the MAP decision for detecting $\mathbf{s}_0$ results in

$$|r|_2 < r_1 \tag{4.19}$$

(b) Express $\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right)$ in terms of r_1, r_2. Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as

$$\rho = \frac{E\left[\left(X - \mu_x\right)\left(Y - \mu_y\right)\right]}{\sigma_x \sigma_y} \tag{4.20}$$

X and Y are said to be uncorrelated if $\rho = 0$

(c) Show that if X and Y are uncorrelated Verify this numerically.

(d) Show that X and Y are independent, i.e. $p_{XY}(x,y) = p_X(x)p_Y(y)$.

(e) Show that $X, Y \sim \mathcal{N}\left(0, 2\sigma^2\right)$.

(f) Show that $\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right) = \Pr\left(X < A, Y < A\right)$.

(g) Find $\Pr\left(X < A, Y < A\right)$.

(h) Verify the above through simulation.

4.4. M-PSK

1. Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \cos\left(\frac{2\pi i}{M}\right) \end{pmatrix}, i = 0, 1, \dots M-1$. Let

$$\mathbf{r}|s_0 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \tag{4.21}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

(a) Substituting

$$r_1 = R\cos\theta \tag{4.22}$$

$$r_2 = R\sin\theta \tag{4.23}$$

show that the joint pdf of R, θ is

$$p\left(R, \theta\right) = \frac{R}{\pi N_0}\exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{4.24}$$

(b) Show that

$$\lim_{\alpha\to\infty}\int_0^\infty (V-\alpha)\,e^{-(V-\alpha)^2}\,dV = 0 \tag{4.25}$$

$$\lim_{\alpha\to\infty}\int_0^\infty e^{-(V-\alpha)^2}\,dV = \sqrt{\pi} \tag{4.26}$$

(c) Using the above, evaluate

$$\int_0^\infty V\exp\left\{-\left(V^2-2V\sqrt{\gamma}\cos\theta+\gamma\right)\right\}\,dV \tag{4.27}$$

for large values of γ.

(d) Find a compact expression for

$$I = 1-\sqrt{\frac{\gamma}{\pi}}\int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta}\cos\theta\,d\theta \tag{4.28}$$

(e) Find $P_{e|\mathbf{s}_0}$.

4.5. Noncoherent BFSK

4.5.1 Show that

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\,d\theta \tag{4.29}$$

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos(\theta-\phi)}\,d\theta \tag{4.30}$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{m_1\cos\theta+m_2\sin\theta}\,d\theta = I_0\left(\sqrt{m_1^2+m_2^2}\right) \tag{4.31}$$

where the modified Bessel function of order n (integer) is defined as

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}\cos n\theta\, d\theta \tag{4.32}$$

4.5.2 Let

$$\mathbf{r}|0 = \sqrt{E_b}\begin{pmatrix}\cos\phi_0\\ \sin\phi_0\\ 0\\ 0\end{pmatrix} + \mathbf{n}_0, \mathbf{r}|1 = \sqrt{E_b}\begin{pmatrix}0\\ 0\\ \cos\phi_1\\ \sin\phi_1\end{pmatrix} + \mathbf{n}_1 \tag{4.33}$$

where $\mathbf{n}_0, \mathbf{n}_1 \sim \mathcal{N}\left(\mathbf{0}, \frac{N_0}{2}\mathbf{I}\right)$.

(a) Taking $\mathbf{r} = (r_1, r_2, r_3, r_4)^T$,, find the pdf $p(\mathbf{r}|0, \phi_0)$ in terms of $r_1, r_2, r_3, r_4, \phi, E_b$ and N_0. Assume that all noise variables are independent.

(b) If ϕ_0 is uniformly distributed between 0 and 2π, find $p(\mathbf{r}|0)$. Note that this expression will no longer contain ϕ_0.

(c) Show that the ML detection criterion for this scheme is

$$I_0\left(k\sqrt{r_1^2 + r_2^2}\right) \underset{1}{\overset{0}{\gtrless}} I_0\left(k\sqrt{r_3^2 + r_4^2}\right) \tag{4.34}$$

where k is a constant.

(d) The above criterion reduces to something simpler. Can you guess what it is? Justify your answer.

(e) Show that

$$P_{e|0} = \Pr\left(r_1^2 + r_2^2 < r_3^2 + r_4^2 | 0\right) \tag{4.35}$$

(f) Show that the pdf of $Y = r_3^2 + r_4^2$ id

$$p_Y(y) = \frac{1}{N_0} e^{-\frac{y}{N_0}}, y > 0 \tag{4.36}$$

(g) Find

$$g(r_1, r_2) = \Pr\left(r_1^2 + r_2^2 < Y | 0, r_1, r_2\right). \tag{4.37}$$

(h) Show that $E\left[e^{-\frac{X^2}{2\sigma^2}}\right] = \frac{1}{\sqrt{2}} e^{-\frac{\mu^2}{4\sigma^2}}$ for $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

(i) Now show that

$$E\left[g(r_1, r_2)\right] = \frac{1}{2} e^{-\frac{E_b}{2N_0}}. \tag{4.38}$$

4.5.3 Let $U, V \sim \mathcal{N}\left(0, \frac{k}{2}\right)$ be i.i.d. Assuming that

$$U = \sqrt{R} \cos \Theta \tag{4.39}$$

$$V = \sqrt{R} \sin \Theta \tag{4.40}$$

(a) Compute the jacobian for U, V with respect to X and Θ defined by

$$J = \det \begin{pmatrix} \frac{\partial U}{\partial R} & \frac{\partial U}{\partial \Theta} \\ \frac{\partial V}{\partial R} & \frac{\partial V}{\partial \Theta} \end{pmatrix} \tag{4.41}$$

(b) The joint pdf for R, Θ is given by,

$$p_{R,\Theta}(r, \theta) = p_{U,V}(u, v) \left. J \right|_{u=\sqrt{r}\cos\theta, v=\sqrt{r}\sin\theta} \tag{4.42}$$

Show that

$$p_R(r) = \begin{cases} \frac{1}{k}e^{-\frac{r}{k}} & r > 0, \\ 0 & r < 0, \end{cases} \tag{4.43}$$

assuming that Θ is uniformly distributed between 0 to 2π.

(c) Show that the pdf of $Y = R_1 - R_2$, where R_1 and R_2 are i.i.d. and have the same distribution as R is

$$p_Y(y) = \frac{1}{2k}e^{-\frac{|y|}{k}} \tag{4.44}$$

(d) Find the pdf of

$$Z = p + \sqrt{p}\left[U\cos\phi + V\sin\phi\right] \tag{4.45}$$

where ϕ is a constant.

(e) Find $\Pr(Y > Z)$.

(f) If $U \sim \mathcal{N}\left(m_1, \frac{k}{2}\right), V \sim \mathcal{N}\left(m_2, \frac{k}{2}\right)$, where m_1, m_2, k are constants, show that the pdf of

$$R = \sqrt{U^2 + V^2} \tag{4.46}$$

is

$$p_R(r) = \frac{e^{-\frac{r+m}{k}}}{k} I_0\left(\frac{2\sqrt{mr}}{k}\right), \quad m = \sqrt{m_1^2 + m_2^2} \tag{4.47}$$

(g) Show that

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{4^n (n!)^2} \tag{4.48}$$

(h) If

$$p_Z(z) = \begin{cases} \frac{1}{k} e^{-\frac{z}{k}} & z \geq 0 \\ 0 & z < 0 \end{cases} \tag{4.49}$$

find $\Pr(R < Z)$.

4.6. Craig's Formula and MGF

4.6.1 The Moment Generating Function (MGF) of X is defined as

$$M_X(s) = E\left[e^{sX}\right] \tag{4.50}$$

where X is a random variable and $E[\cdot]$ is the expectation.

(a) Let $Y \sim \mathcal{N}(0, 1)$. Define

$$Q(x) = \Pr(Y > x), x > 0 \tag{4.51}$$

Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2 \sin^2 \theta}} \, d\theta \tag{4.52}$$

(b) Let $h \sim \mathcal{CN}\left(0, \frac{\Omega}{2}\right), n \sim \mathcal{CN}\left(0, \frac{N_0}{2}\right)$. Find the distribution of $|h|^2$.

(c) Let

$$P_e = \Pr\left(\Re\left\{h^* y\right\} < 0\right), \text{ where } y = \left(\sqrt{E_s} h + n\right), \tag{4.53}$$

Show that

$$P_e = \int_0^\infty Q\left(\sqrt{2x}\right) p_A(x)\, dx \tag{4.54}$$

where $A = \frac{E_s |h|^2}{N_0}$.

(d) Show that

$$P_e = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} M_A\left(-\frac{1}{\sin^2\theta}\right) d\theta \tag{4.55}$$

(e) compute $M_A(s)$.

(f) Find P_e.

(g) If $\gamma = \frac{\Omega E_s}{N_0}$, show that $P_e < \frac{1}{2\gamma}$.