EVHS Algebra II Unit 3 Handouts

Polynomial relations (Part I)

Contents

Topic 1 Preamble: Review of The Ground Rules of Exponents and Polynomials

Objectives:

In this topic, you will learn how to use the properties of exponents, and some basic operations of the polynomials.

The Lesson

Objective 1:

Exponent Properties

The following table outlines the properties of exponents:

1. $a^0 = 1$	2. $a^{-1} = \frac{1}{a}$
3. $(ab)^x = a^x b^x$	4. $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
5. $a^x \cdot a^y = a^{x+y}$	6. $\frac{a^x}{a^y} = a^{x-y}$
7. $\left(a^x\right)^y = a^{xy}$	8. if $a^x = a^y$, then $x = y$

Ex1 Find the ratio of the volume of a sphere with radius of r to the surface area of the same sphere

Ex1.5 Use the following properties of exponents as given:

$$a^0 = 1$$

$$a^x \cdot a^y = a^{x+y}$$

Form a proof to show that $a^{-1} = \frac{1}{a}$

Ex2 Solve the following exponent equations (Find x and/or y)

a. $a^{x+1} = \left(a^{3x-1}\right)^{x}$	b. $4^{x} = 8^{2x-1}$
c. $36 \cdot 6^{3(x+y)} = 2^{4y} \cdot 3^{2x-6}$	d. $2 \cdot 10^{2x+y} = 5^{y+2} \cdot 8^{x-1}$

Objective 2:

Add/multiply polynomials (combine the like terms and distributive properties).

Ex3 $f(x)=2x+3;\ g(x)=3x^3-x+1; h(x)=x^4+\frac{1}{2}x^3-4x^2+2x+1; k(x)=-x+2x^2-1$

Use the given polynomials to find the followings

a. $f(x)g(x)-2h(x)$	b. $h(x)+\frac{1}{2}k(x)$
c. $f(x)k(x)-g(x)$	d. $f^2(x)$
e. $\left(f(x)+k(x)\right)^2$	f. $g(x)h(x)$

g. if $m(x)+g(x)=k^2(x)$, find $m(x)$	h. if $\begin{cases} m(x)+n(x)=f(x)+4x^2 \\ m(x)-n(x)=g(x)+k(x) \end{cases}$, Find $m(x)$ and $n(x)$

Exit ticket

if $\begin{cases} a(x)+2b(x)+c(x)=f(x) \\ b(x)-c(x)=g(x) \\ a(x)+2b(x)=h(x) \end{cases}$, and $a(x), b(x), c(x)$ are polynomials, find $a(x), b(x)$ and $c(x)$.

Topic 2 Factoring Polynomials (Completely)

Objective:

In this topic, you will learn a couple of techniques to factor a polynomial whenever it is factorable.

The Lesson

If a polynomial can be factored, (How would I know?),then it usually can be factored through the following 3 techniques:

- Factor by grouping: such as the following example,

  x^3-2x^2+x-2 $\qquad$ $6x^3-18x^2+2x-6$

- Factor by using patterns: Following table lists some of the most common patterns,

$a^2+2ab+b^2=(a+b)^2$	$a^2-2ab+b^2=(a-b)^2$
$a^3+3a^2b+3ab^2+b^3=(a+b)^3$	$a^3-3a^2b+3ab^2-b^3=(a-b)^3$
$a^3+b^3=(a+b)(a^2-ab+b^2)$	$a^3-b^3=(a-b)(a^2+ab+b^2)$
$a^2-b^2=(a+b)(a-b)$	$a^2+b^2+c^2+2ab+2bc+2ca=(a+b+c)^2$

  Here are some applications of these patterns

  x^6+1 $\qquad$ $16x^4-81$

- A mixed of the previous two strategies, such as

  x^4-2x^3+2x-1 $\qquad$ $-x^4+x^3-x+1$

Practice: Factor completely the following polynomials

1. $4-9x^2$	2. $(x+1)^2-1$
3. $4x^2+4x(x+1)+(x+1)^2$	4. $x^2-4+2(x+2)^2$
5. $1-x^2+(2+2x)^2$	6. $(x+1)^2-15(x+1)+56$

7. $x(y+2)-x-y-1$	8. $4x^2+4xy+y^2-4x-2y-3$
9. $x^2y^2-x^2-y^2-6xy+4$	10. $3ax^2-2x+3ax-2$
11. $a(b^2-c^2)-c(a^2-b^2)$	12. $xy^2-2xy-3x-y^2-2y-1$

Exit Ticket:

Factor completely: $x^5-3x^4-16x+48$

Topic 3 Long Division for polynomials

Objective:

In this topic, you will learn how to use the long division to divide two polynomials

The Lesson

When you divide $f(x)$ by a divisor $d(x)$, you will get a quotient $q(x)$ and a remainder $r(x)$, and the result of your division should be represented in the following format:

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}, \quad \text{where } \deg(r(x)) < \deg(d(x))$$

Example: Use long division to divide $f(x) = 2x^6 - 3x^5 + 7x^4 - 11x^3 + 5x^2 - 6x + 4$ by $d(x) = x^2 - 2x + 1$

Examples:

1. $(3x^4 - 5x^3 + 4x - 6) \div (x^2 - 3x + 5)$	2. $(2x^4 + 2x^2 + 1) \div (-2x + 1)$
3. $(6x^2 - 4x + x^3 - 2x^5) \div (x + x^2)$	4. $(x^3 + 5x^2 - 7x + 2) \div (x + 2)$

5. $(4x^5 - 5x^3 + 4) \div (x^2 - 3x - 2)$	6. $(4x^6 + 3x^4 + 4x^2 - 2) \div (2x - 1)$
7. $(x^4 + 4x^3 + 16x^2 - 36) \div (1 - x)$	8. $(4x^3 + 2x + 5) \div (8x - 2)$
9. $(3x^4 - 2x^3 + \frac{16}{3}x^2 - 6x + 12) \div (6x + 4)$	10. $(7x^4 - 4x^3 + 5x^2 - 6x - 2) \div (x^2 - 2x + 1)$

Topic 4 Synthetic division, factor theorem, and remainder theorem

Objectives:

In this topic, you will learn how to use synthetic division to divide a polynomial by a divisor $x-a$

Objective 1: Use synthetic division to divide a polynomial by x-a

$(4x^4+3x^3-5x+1)\div(x+2)$

$(2x^4-x+1)\div(x+3)$

Practice:

1. $(2x^3+x^2-8x+5)\div(x+3)$	2. $(\frac{2}{3}x^3-4x+x^2+1)\div(x+6)$
3. $(x^4+x^3+2)\div(x-\frac{1}{2})$	4. $\left(x^4+5x^3-2x^2+x-1\right)\div(x-5)$

Objective 2: Remainder theorem and Factor theorem

Remainder theorem:

If $f(x)$ is divided by $x-a$, then the remainder of the division is $R(x)=f(a)$ [which is a constant]

Factor theorem:

If $f(x)$ is divided by $x-a$, and the remainder of the division is 0, then $x-a$ is a factor of $f(x)$.

Example 2:

If $x+1$ is a factor of $f(x)=2x^2+ax+3$, Find variable a

What are the zeros of $f(x)=0$?

Example 3:

Given $f(x)=\frac{7}{3}x^5-\frac{3}{2}x^4+\frac{5}{6}x^3-\frac{4}{3}x^2+3x-\frac{1}{6}$

Evaluate

(1) $f(\frac{3}{2})$

(2) $f(6)$

Practice:

1. $f(x)=3x^3+4ax^2+2a^2x+1$ has a factor (x+1). Use factor theorem to find a.	2. From question 1, find all the zeros for $f(x)=0$
3. Divide $g(x)=x^3+4ax^2+ax-1$ by x+2, the remainder is -2, Find a.	4. Use g(x) from question 3, calculate $g(x)\div(x^2+x+1)$
5. Use $f(x)$ from question 1, find $f(-2)$	6. Use $g(x)$ from question 3, find $g(-\frac{1}{2})$

7. Let $h(x) = f(x) + g(x)$. Now find the remainder, if $h(x)$ is divided by $(x+2)$?	8. find $f(-\frac{1}{2})$
9. Combine the results from question 6 and 8, and use the experience you have with question 7, predict the remainder, if h(x) is divided by (x+2) without performing the division.	10. Form an opinion (Mathematically, this is called a "conjecture") about the experience you have from question 7 through question 9.

Prove your conjecture:

Objective 3:

With slight modification of the synthetic division, one can generalize the division process for divisors with the structures of $ax+b$.

Example: Use synthetic division to divide $f(x)=4x^4-2x^2+5x-6$ by $(2x-3)$

Step 1: divide all coefficients by a.

Step 2: use synthetic division to divide the modified $f(x)$ from step1 by $x-\frac{b}{a}$

Step 3: multiply the remainder by a.

Mr. Chen, please explain **<u>why</u>** this modification works? (Please.)

Practice: use synthetic division to find the quotient and the remainder of the operation

1. $(8x^2 + 34x - 1) \div (4x - 1)$	2. $(10x^3 - 81x^2 + 71x + 42) \div (2x - 3)$
3. $(3x^3 + 34x^2 + 72x - 64) \div (3x - 2)$	4. $(3x^3 - 2x^2 - 61x - 20) \div (3x + 2)$
5. $(2x^3 - 15x^2 + 34x - 21) \div (2x - 5)$	6. $(2x^3 + 17x^2 + 46x + 40) \div (2x + 5)$
7. $(30x^3 + 7x^2 - 39x + 14) \div (3x - 2)$	8. $(4x^3 - 2x^2 + x - 1) \div (2x - 1)$

Topic 5 Rational Zero Theorem

Objective:

In this topic, you will learn the rational zero theorem and use the theorem to find all possible rational zeros (or all the zeros)

The Lesson

Given that $(2x+3)$ is a factor of $f(x) = 56x^3 + 46x^2 - 97x - 60$.

Find all the zeros of $f(x)$.

Now observe the leading coefficient and the constant of $f(x)$, and the zeros you just found, to the table below:

leading coef =	all possible factors of the leading coefficient:
constant =	all possible factors for the constant

zero1 = zero2 = zero3 =

Conclusion of the observation:

Rational Zero Theorem:

If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ has integer coefficients, then every rational zero of f has the following form:

$$\frac{p}{q} = \frac{\text{factors of } a_0}{\text{factors of } a_n}$$

Ex 1 For the following polynomials, (a) List all possible rational zeros and (b) Find all zeros (including complex zeros)

a. $f(x) = x^3 + 2x^2 - 11x + 12$

b. $g(x) = 4x^4 - x^3 - 3x^2 + 9x - 10$

c. $h(x) = x^3 - 8x^2 + 11x + 20$

d. $k(x) = x^3 - 4x^2 - 15x + 18$

Practice:

1. $f(x) = 2x^3 + 7x^2 + 7x + 12$

2. $f(x) = 2x^5 + 9x^4 + 11x^3 - 21x^2 - 76x - 60$

3. $f(x) = 12x^4 - 52x^3 + 45x^2 + 13x - 12$

4. $f(x) = 30x^3 + 7x^2 - 39x + 14$

5. $f(x) = 2x^6 - x^5 + 11x^4 - 6x^3 + 4x^2 - 5x - 5$