1. (8%) P(x, y) is a predicate and the universe for the variables x and y is {1, 2, 3}.
   Suppose P(1, 3), P(2,1), P(2, 2), P(2, 3), P(3, 1), P(3,2) are true, and P(x, y) is false otherwise. Determine whether the following statements are TRUE(**T**) or FALSE(**F**). (請圈選T或F)
   (a) T F $\exists y \forall x P(x, y)$.
   (b) T F $\neg \exists x \exists y (P(x, y) \land \neg P(y, x))$.
   (c) T F $\forall y \exists x (P(x, y) \rightarrow P(y, x))$.
   (d) T F $\forall x \forall y (x \neq y \rightarrow (P(x, y) \lor P(y, x))$.

x \ y	1	2	3
1			T
2	T	T	T
3	T	T	

(c) $\forall y \exists x (P(x,y) \longrightarrow P(y,x)) \equiv \forall y \exists x (\neg P(x,y) \lor P(y,x))$

2. (8%) Determine whether each of the following statements is **true or false**.
   (圈選True(T)或False(F)即可)
   (a) T F f: **N**→**N**, $f(x)=2^x$ is one-to-one
   (b) T F f: **N**→**N**, $f(x)=2^x$ is onto
   (c) T F f: **N**→**Z⁺**, $f(x)=x+1$ is one-to-one
   (d) T F f: **N**→**Z⁺**, $f(x)=x+1$ is onto
   (e) T F f: **N**→**N**×**N**, $f(x)=(x, x+1)$ is one-to-one
   (f) T F f: **N**→**N**×**N**, $f(x)=(x, x+1)$ is onto (1,3), (2,2) 找不到
   (g) T F f: **R**→**R**×**R**, $f(x)=(-x^2, x^2)$ is one-to-one
   (h) T F f: **R**→**R**×**R**, $f(x)=(-x^2, x^2)$ is onto

3. (7%) Determine whether each of these sets is **Countable(可數) or Uncountable(不可數)**. (請圈選C或U)
   (a) C U (¼, ¾)
   (b) C U $\{(x,y) \in \mathbf{N} \times \mathbf{R} \mid xy=1\}$
   (c) C U The real numbers with decimal representations of all 1's (e.g.,1.11, 11.111 . . .).
   (d) C U The real numbers with decimal representations of all 5s or 6s. (e.g., 5.55, 5.5655, …).
   (e) C U All bit strings not containing the bit 1.
   (f) C U The set of irrational numbers.
   (g) C U The set of all functions from **N** to {0,1}.

4. (10%) Let A,B,C,D be sets and D={∅,{∅}}. Determine whether each of the following statements is **true or false**.
   (圈選True(T)或False(F)即可)
   (a) T F $(A \in B) \land (B \in C) \rightarrow (A \in C)$
   (b) T F $(A \in B) \land (B \in C) \rightarrow (A \subseteq C)$
   (c) T F $(A \in B) \land (B \subseteq C) \rightarrow (A \in C)$
   (d) T F $(A \in B) \land (B \subseteq C) \rightarrow (A \subseteq C)$

   (a)
   (b) A: {1} , B={{1},2} , C={{{1},2},3}

   (d) A:{1} , B={{1},2,3} C:{{1},2,3,4}

   (e) T F $\varnothing \subseteq D$
   (f) T F $\varnothing \in D$
   (g) T F $\{\varnothing\} \in D$
   (h) T F $\{\varnothing\} \subseteq \mathbf{P}(D)$
   (i) T F $\{\varnothing\} \in \mathbf{P}(D)$
   (j) T F $\{\{\{\varnothing\}\}\} \subseteq \mathbf{P}(D)$

   $P(D)= \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$

5. (6%) Suppose that the universal set is {1,2,3,4,5,6,7,8} and A={1,2,3,6,8}, B={6,7,8}, C={3,6}.
   (a) What is $\overline{A-C} \cup B$?
   (b) What is $B^2 \cap (A \cdot C)$?

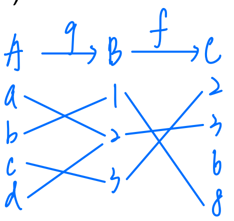
   $A-C = \{1,2,8\}$
   $\overline{A-C} = \{3,4,5,6,7\}$
   $\therefore \overline{A-C} \cup B = \{3,4,5,6,7,8\}$

   (a) {3,4,5,6,7,8} (b) {(6,6), (8,6)}

6. (8%) Determine the cardinality of each of the following sets:
   (a) A= {0, 2, 4, ..., 20}
   (b) ∅
   (c) B={{{}}}
   (d) {∅,{∅},{∅},(∅,∅), {∅,(∅,∅)}}

(a) = 11
(b) = 0
(c) = 1
(d) = 4

7. (8%) Suppose g: A → B and f: B → C where A = {a,b,c,d}, B = {1,2,3}, C = {2,3,6,8}, and g and f are defined by g = {(a, 2),(b, 1),(c, 3),(d, 2)} and f = {(1, 8),(2, 3),(3, 2)}.
   (a) Find f ◦ g.
   (b) Find f^{-1}.



(a)
{(a,3), (b,8), (c,2), (d,3)}

(b)
Not bijection. No f^{-1}

8. (5%) How many **satisfying assignments** are there for the following system specification: 4 #
(¬w∧x∧¬y∧¬z)∨(¬w∧¬x∧y)∨(¬w∧¬x∧z)

(1) $(\neg w \wedge x \wedge \neg y \wedge \neg z) \vee (\neg w \wedge \neg x \wedge y) \vee (\neg w \wedge \neg x \wedge z) \equiv \neg w \wedge [(x \wedge \neg y \wedge z) \vee (\neg x \wedge y) \vee (\neg x \wedge z)]$

x	y	z	¬y	¬x	x∧¬y∧z	¬x∧y	¬x∧z	(x∧¬y∧z)∨(¬x∧y)∨(¬x∧z)
T	T	T	F	F	F	F	F	F
T	T	F	F	F	F	F	F	F
T	F	T	T	F	T	F	F	T
T	F	F	T	F	F	F	F	F
F	T	T	F	T	F	T	T	T
F	T	F	F	T	F	T	F	T
F	F	T	T	T	F	F	T	T
F	F	F	T	T	F	F	F	F

(〇)
∵ ¬w 要為 T,
∴ w 為 F

(一)
x∧¬y∧z 為 T
∵ ¬y 為 T,
∴ x 為 T, y 為 F, z 為 T
① (T, F, T)

(二)
¬x∧y 為 T
∵ ¬x 為 T
∴ x 為 F, y 為 T, z 可以為 T or F
② (F, T, T)
③ (F, T, F)

(三)
¬x∧z 為 T
∵ ¬x 為 T
∴ x 為 F, z 為 T, y 可以為 T or F
(F, T, T)
④ (F, F, T)

9. (9%) Suppose the domain for x and y consists of all persons, and
   G(x): x is a girl;
   B(x): x is a boy;
   L(x,y): x likes y;
   P(x): x is good-looking.
   Write each of the following statements using these predicates and any needed quantifiers.
   (a) Every boy likes a girl.
   (b) Every girl likes all good-looking boys.
   (c) There are some girls who do not like any boys.

(a) $\forall x \exists y \, (B(x) \longrightarrow G(y) \wedge L(x,y))$
(b) $\forall x \forall y \, (G(x) \wedge B(y) \wedge P(y) \longrightarrow L(x,y))$
(c) $\exists x \forall y \, (G(x) \wedge B(y) \longrightarrow \neg L(x,y))$

10. (9%)In questions (a)–(c), describe each sequence **recursively**. Include **initial conditions** and assume that the sequences begin with a_1.
   (a) a_n = n!
   (b) 0.1, 0.11, 0.111, 0.1111, ...
   (c) $1^2, 2^2, 3^2, 4^2$,... (hint: 1+3+5+...+(2n-1)=n^2)

(a)
$a_n = a_{n-1} \times n$
$a_1 = 1$

(b)
$a_n = a_{n-1} + 10^{-n}$
$a_1 = 0.1$

(c)
$a_2 = a_1 + (2 \cdot 2 - 1)$
$a_3 = a_2 + (2 \cdot 3 - 1)$
$a_4 = a_3 + (2 \cdot 4 - 1)$

$\therefore a_n = a_{n-1} + (2n-1)$
$a_1 = 1$

11. (8%) Show that the compound proposition $(\neg p \wedge (p \vee q)) \rightarrow q$ is a **tautology (恆真)**
(a) using a **truth table**.
(b) using **logical equivalences**;

(a)

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$\neg p \wedge (p \vee q) \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

(b)
$(\neg p \wedge (p \vee q)) \longrightarrow q$
$\equiv ((\underbrace{\neg p \wedge p}_{F}) \vee (\neg p \wedge q)) \longrightarrow q$
$\equiv (\neg p \wedge q) \longrightarrow q$
$\equiv \neg(\neg p \wedge q) \vee q \equiv p \vee \underbrace{\neg q \vee q}_{T} \equiv p \vee T \equiv T$ #

12. (5%) A staff member of a small hospital said, "The hospital staff consists of 16 doctors and nurses, including me. The following facts apply to the staff members; whether you include me or not does not make any difference."

The staff consists of:
- more nurses than doctors; $FN + MN > FD + MD \Rightarrow MN + FN \geq 9$, $FD + MD \leq 7$
- more male doctors than female nurses; $MD > FN \Rightarrow MD \geq 6$
- more female nurses than male nurses; $FN > MN$, $FN \geq 5$
- at least one female doctor. $FD \geq 1$

Give the sex and job of the speaker.

	N	D
M	6	4
F	1	5

13. (6%) Give a **recursive definition** of the set S if S is
(a) the set of **positive even integers**.
(b) the set of nonempty **bit strings** that are **palindromes**(回文; 順/倒讀都一樣).

(a) Basis step : $2 \in S$
Recursive step ; if $x \in S$, then $x+2 \in S$

(b) Basis step : $0 \in S$ and $1 \in S$ and $00 \in S$ and $11 \in S$
Recursive step : if $x \in S$, then $0x0 \in S$ and $1x1 \in S$

14. (8%) use **Mathematical induction (數學歸納法)** to prove that for all $n \in \mathbf{N}$, 5 divides $n^5 - n$.
Ans:令命題函數P(n): $5 \mid n^5-5$, 本題欲證 $\forall n \in \boldsymbol{N}$, P(n) 都成立
Basis step: P(0): $5 \mid 0^5 - 0$, P(0) 成立
Induction step: 在此步驟中我們必須證明P(k)→P(k+1)
假設P(k)成立, i.e., $5 \mid k^5-k$ (Induction Hypothesis),
則$(k+1)^5-(k+1)=k^5+5k^4+10k^2+5k+1-k-1=k^5-k+5(k^4+2k^2+k+)=5m+5(k^4+2k^2+k)= 5(m+k^4+2k^2+k)$
⇨ $5 \mid (k+1)^5-k$
⇨ P(k+1)成立 (到此已完成歸納步驟)
由數學歸納法得知 $\forall n \in \boldsymbol{N}$, P(n)成立, i.e., $5 \mid n^5-n$

15. (8%) Use **strong induction (強歸納)** to show that a class with n≥12 students can be divided into groups of 4 or 5.
Ans:令命題函數P(n): n人 可依4/5人一組完成分組, 則本題欲證明 $\forall n \in Z^{+,\ n\geq 12,\ P(n)成立}$
Basis step:
P(12)成立, 因為12=4+4+4
P(13)成立, 因為13=4+4+5
P(14)成立, 因為14=4+5+5
P(15)成立, 因為15=5+5+5
Induction step: 在此步驟中我們必須證明(P(12)∧P(13)∧…∧P(k))→P(k+1), for k≥12
假設P(j)成立for12≤j≤k, i.e., j 人可依`4/5人一組完成分組 (Induction Hypothesis),
則因K+1=(k-4)+5, 根據歸納假設, k-4 人可依`4/5人一組完成分組,
再加上5人一組則共k+1人, 故P(k+1)亦成立! (到此已完成歸納步驟)
由數學歸納法得知 $\forall n \in \boldsymbol{Z}^{+}$, , n≥12 , P(n) 成立 , i.e., 12 人(含)以上可依 4/5 人一組完成分組