離散數學 期中考試 2023/11/102　　　　　　　　　　姓名:__________ 學號:____________

1. (6%) Suppose that the universal set is {1,2,3,4,5,6,7,8} and A={1,2,3,6,8}, B={6,7,8}, C={3,6}.
   (a) What is $\overline{(A \cup C)} \cup B$?
   (b) What is $C^2 \cap (A \cdot B)$?

Ans: (a) {4,5,6,7,8} (b) {(3,6), (6,6)}

2. (10%) Let A={∅,{∅}}. Determine whether each of the following statements is **true or false**.
   (圈選True(T)或False(F)即可)

(a) T F	∅ ∈ **P**(A)	(f) T F	{∅} ∈ A
(b) T F	∅ ⊆ **P**(A)	(g) T F	{{∅}} ⊆ **P**(A)
(c) T F	{∅} ⊆ **P**(A)	(h) T F	{{∅}} ⊆ A
(d) T F	{∅} ⊆ A	(i) T F	{{∅}} ∈ **P**(A)
(e) T F	{∅} ∈ **P**(A)	(j) T F	{{∅}} ∈ A

Ans:TTTTTTTTTF

3. (4%) What is the **cardinality** of the set {∅,{∅},{{∅}},(∅,∅), {∅,(∅,∅)}}? Ans:_____

Ans: 5

4. (8%)Suppose g: A → B and f: B → C where A = {1, 2, 3, 4}, B = {a, b, c}, C = {2, 8, 10}, and g and f are defined by g = {(1, b),(2, a),(3, b),(4, a)} and f = {(a, 8),(b, 10),(c, 2)}.
   (a) What is f^{-1}?
   (b) What is f ∘ g ?

Ans:(a){(8,a),(10, b),(2, c)}
(b) {(1,10),(2,8),(3,10),(4,8)}

5. (5%) Let x,y∈***R***. Determine whether each of the following statements is **true or false**.
   (圈選True(T)或False(F)即可)

   (a) T F　$\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$
   (b) T F　$\lfloor 2x \rfloor = 2\lfloor x \rfloor$
   (c) T F　$\lceil -x \rceil = -\lfloor x \rfloor$
   (d) T F　$\lfloor (x+1)/2 \rfloor = \lfloor x/2 \rfloor$
   (e) T F　$\lceil x \rceil \lceil y \rceil = \lceil xy \rceil$

Ans:TFTFF

6. (10%) Assume that the universe for x is all people and the universe for y is the set of all movies. Write the English statement using the following predicates and any needed quantifiers:
   S(x, y): x saw y; L(x, y): x liked y; A(y): y won an award; C(y): y is a comedy.
   (a) No comedy won an award.
   (b) Lois saw Casablanca, but didn't like it.
   (c) Some people have seen every comedy.
   (d) No one liked every movie he has seen.
   (e) Ben has never seen a movie that won an award.

Ans:(a) ∀y (C(y)→¬A(y)). (b) S(Lois, Casablanca) ∧ ¬L(Lois, Casablanca). (c) ∃x∀y [C(y)→S(x, y)]. (d) ¬∃x∀y [S(x, y)→L(x, y)]. (e) ¬∃y [A(y) ∧ S(Ben, y)] 另解:∀y (S(Ben, y)→¬A(y))

7. (6%)In questions (a)–(c), describe each sequence **recursively**. Include **initial conditions** and assume that the sequences begin with a_1.
   (a) a_n = 1+2+3+…+n
   (b) $a_n = 5^n$
   (c) 1, 101, 10101, 1010101, ...

Ans
   (a) $a_n = a_{n-1} + n$, $a_1 = 1$.
   (b) $a_n = 5a_{n-1}$, $a_1 = 5$.
   (c) $a_n = 100a_{n-1}+1$, $a_1 = 1$.

8. (4%) Let S be the set of bit strings defined recursively by
   (1) λ ∈ S ;and
   (2) if x ∈ S, 0x0 ∈ S and 1x1 ∈ S , where λ is the empty string.
   Find all strings in S of length **not exceeding 6.**

Ans: λ, 00, 11, 0000, 1001, 0110, 1111, 000000, 100001, 010010, 110011, 001100, 101101, 011110, 111111

9. (5%) Determine whether each of these sets is **Countable(可數) or Uncountable(不可數)**. (請圈選C或U)
   (a)C U　The real numbers between 0 and 2
   (b)C U　The real numbers with decimal representations of all 5's (e.g.,5.55, 55.555 . . .).
   (c)C U　The real numbers with decimal representations of all 1s or 2s. (e.g., 1.12, 1.1211, …).
   (d)C U　All bit strings not containing the bit 0.
   (e)C U　All positive integers divisible by 3 but not by 5

Ans:UCUCC

10. (6%) Let A= $\begin{bmatrix}1&0&1\\1&1&0\\0&0&1\end{bmatrix}$ and B= $\begin{bmatrix}0&1&1\\1&0&1\\1&0&1\end{bmatrix}$. Find the (a) **join**, (b) **meet**, and (c) **Boolean product** of these 2 **zero-one matrices**.
Ans:(計算略)

11. (10%) Show that the compound proposition $((q \wedge (p \to \neg q)) \to \neg p$ is a **tautology (恆真)**
(a) using a **truth table**.
(b) using **logical equivalences**;

Ans: (a)略 (b) $(q \wedge (p \to \neg q)) \to \neg p \Longleftrightarrow (q \wedge (\neg p \vee \neg q)) \to \neg p \Longleftrightarrow ((q \wedge \neg p) \vee (q \wedge \neg q)) \to \neg p \Longleftrightarrow (q \wedge \neg p) \to \neg p \Longleftrightarrow \neg(q \wedge \neg p) \vee \neg p \Longleftrightarrow (\neg q \vee p) \vee \neg p \Longleftrightarrow \neg q \vee (p \vee \neg p) \Longleftrightarrow \neg q \vee \mathbf{T} \Longleftrightarrow \mathbf{T}$.

12. (6%) Use **proof by contradiction(矛盾)** to show that $\forall n \in \boldsymbol{Z}$, if $4|n^3$, then n is even. ([註] x|y : x可整除y)

Ans: 假設n為奇數，則n=2k+1 for some $k \in \boldsymbol{Z}$ => $n^3=(2k+1)^3=8k^3+12k^2+6k+1=4(2k^3+3k^2+k)+2k+1$ => n^3為奇數無法被4整除→← 因此n為偶數

13. (8%) $\forall n \in \boldsymbol{Z}^+$, use **Mathematic induction (數學歸納法)** to show that $9|n^3+(n+1)^3+(n+2)^3$.

Ans:令命題函數P(n): $9|n^3+(n+1)^3+(n+2)^3$，本題欲證$\forall n \in \boldsymbol{Z}^+$, P(n) 都成立

Basis step: P(1): $9|1^3+2^3+3^3=36$ 成立

Induction step: 在此步驟中我們必須證明P(k)→P(k+1)

假設P(k)成立，i.e., $9|k^3+(k+1)^3+(k+2)^3$ (Induction Hypothesis),

則$(k+1)^3+(k+2)^3+(k+3)^3=(k+1)^3+(k+2)^3+k^3+9k^2+27k+27=\underline{k^3+(k+1)^3+(k+2)^3}+\underline{9(k^2+3k+3)}$

⇨ $9|(k+1)^3+(k+2)^3+(k+3)^3$

⇨ P(k+1)成立 (到此已完成歸納步驟)

由數學歸納法得知$\forall n \in \boldsymbol{Z}^+$, P(n)成立，i.e., $9|n^3+(n+1)^3+(n+2)^3$

14. (8%) Use **strong induction (強歸納法)** to show that postage of 24 cents or more can be achieved by using only 5-cent and 7-cent stamps.

Ans:令命題函數P(n): n cents郵資可只用5-cent和7-cent郵票湊成，則本題欲證$\forall n \in Z^+$, $n \ge 24$，P(n)成立

Basis step:

P(24)成立，因為24=5+5+7+7
P(25)成立，因為25=5+5+5+5+5
P(26)成立，因為26=5+7+7+7
P(27)成立，因為27=5+5+5+5+7
P(28)成立，因為28=7+7+7+7

Induction step: 在此步驟中我們必須證明$(P(24) \wedge P(25) \wedge \ldots \wedge P(k)) \to P(k+1)$, for $k \ge 28$

假設P(j)成立for $24 \le j \le k$, i.e., j cents郵資可只用5-cent和7-cent郵票湊成 (Induction Hypothesis),
則因K+1=(k-4)+5，根據歸納假設，k-4 cents郵資可只用5-cent和7-cent郵票湊成,
再加上一張5 cents郵票即可湊成k+1cents郵資，故P(k+1)亦成立! (到此已完成歸納步驟)

由數學歸納法得知$\forall n \in \boldsymbol{Z}^+$,, $n \ge 24$，P(n)成立，i.e., 24 cents(含)以上郵資可只用5-cent和7-cent郵票湊成

15. (8%) Show that $\sqrt{2}$ is **irrational**(無理數)

Ans: (略)詳如課程投影片

16. (8%)The set of **full binary trees** (FBT)can be defined recursively by these steps.
BASIS STEP: There is a full binary tree consisting of only a single vertex r.
RECURSIVE STEP: If T_1 and T_2 are disjoint full binary trees, there is a full binary tree, denoted by $T_1 \cdot T_2$, consisting of a root r together with edges connecting the root to each of the roots of the left subtree T_1 and the right subtree T_2.
Prove through **structure induction** that every full binary tree has an odd number of nodes.

Ans: <u>Basis Step</u>: 此步驟將證明遞迴定義中basis step所定義的元素有奇數個節點
單一節點的Full Binary Tree (FBT) 其節點數為奇數，成立

<u>Inductive Step</u>:

根據遞迴定義中recursive step: 若T_1和$T_2 \in$ FBT, 則$T_1 \cdot T_2 \in$ FBT
此步驟將證明在T_1和T_2節點數為奇數的前提下$T_1 \cdot T_2$節點數一定是奇數

<u>Inductive Hypothesis:</u>

T_1節點數=2p+1, for some integer p
T_2節點數=2q+1, for some integer q

則:
$T_1 \cdot T_2$ 節點數=(2p+1)+(2q+1)+1=2(p+q+1)+1=2k+1, for k=p+q+1
即$T_1 \cdot T_2$ 節點數必為奇數，此完成歸納步驟之證明

由結構歸納法得知 every full binary tree has an odd number of nodes