離散數學 第8章小考　　　　　　　　　　　　姓名: ________________ 學號:______________

1. (10%) Suppose |A| = 8 and |B| = 4. Find the number of functions f:A→B that are onto.

   Ans:

2. (10%) How many positive integers not exceeding 1000 are not divisible by either 4 or 6?

   Ans:

3. (10%) List the **derangements(錯位)** of the set {1, 2, 3, 4}.

   Ans:

4. Acomputer system considers a string of decimal digits a **valid** codeword if it contains an **even number of 0** digits. For instance, 1230407869 is a 10-digit valid codeword, whereas 1209045608 is not valid. Let a_n be the number of valid n-digit codewords.

   (a) (5%) Find a recurrence relation for a_n.

   (b) (2%) What is (are) the initial condition(s)?

   Ans:

5. (20%) Find a **closed form** for the generating function for each of the following sequences.

   [**hint**: A **close form** for the generation function for sequence 1, 1, 1, 1, … is 1/(1-x).]

   (a) 4, 8, 16, 32, 64, ...

   (b) 1, 0, 1, 0, 1, 0, 1, 0, ...

   (c) 2, 0, 0, 2, 0, 0, 2, 0, 0, 2, ...

   (d) 2, 4, 6, 8, 10, 12, ...

   Ans:

6. (15%) Find the solution of the linear homogeneous recurrence relation $a_n = 7a_{n-1} - 6a_{n-2}$ with $a_0 = -1$ and $a_1 = 4$.

   Ans:

7. (25%) Consider the recurrence relation $a_n = -a_{n-1} + n$.

   (a) Write the **associated homogeneous recurrence relation**.

   (b) Find the **general solution** to the **associated homogeneous recurrence relation**. (即$a_n^{(h)}$)

   (c) Find a **particular solution** to the given recurrence relation. (即$a_n^{(p)}$)

   (d) Write the **general solution** to the given recurrence relation.

   (e) Find the **solution** to the given recurrence relation when $a_0 = 1$.

   Ans:

8. (15%) Use **generating function** to **solve** $a_n = 3a_{n-1} + 2^n$, $a_0 = 5$.

   Ans:

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1. (10%) Suppose |A| = 8 and |B| = 4. Find the number of functions f:A→B that are onto.

   Ans: $4^8 - \binom{4}{1}3^8 + \binom{4}{2}2^8 - \binom{4}{3}1^8$.

2. (10%) How many positive integers not exceeding 1000 are not divisible by either 4 or 6?

   Ans: $1000-\lfloor 1000/4\rfloor-\lfloor 1000/6\rfloor+\lfloor 1000/12\rfloor=1000-250-166+83=667$.

3. (10%) List the **derangements(錯位)** of the set {1, 2, 3, 4}. $4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)=9$

   Ans: 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321

4. Acomputer system considers a string of decimal digits a **valid** codeword if it contains an **even number of 0** digits. For instance, 1230407869 is a 10-digit valid codeword, whereas 1209045608 is not valid. Let a_n be the number of valid n-digit codewords.

   (a) (5%) Find a recurrence relation for a_n.

   (b) (2%) What is (are) the initial condition(s)?

   Ans: (a) $a_n=9a_{n-1}+(10^{n-1}-a_{n-1})=8a_{n-1}+10^{n-1}$ (b) $a_1=9$

5. (20%) Find a **closed form** for the generating function for each of the following sequences.
   [**hint**: A **close form** for the generation function for sequence 1, 1, 1, 1, … is 1/(1-x).]

   (a) 4, 8, 16, 32, 64, ...

   (b) 1, 0, 1, 0, 1, 0, 1, 0, ...

   (c) 2, 0, 0, 2, 0, 0, 2, 0, 0, 2, ...

   (d) 2, 4, 6, 8, 10, 12, ...

   Ans:

   $\frac{4}{1-2x}$. $4+8x+16x^2+\cdots=4(1+2x+4x^2+\cdots)$

   $\frac{1}{1-x^2}$. $1+x^2+x^4+\cdots$

   $\frac{2}{1-x^3}$. $2+2x^3+2x^6+\cdots=2(1+x^3+x^6+\cdots)$

   $\frac{2}{(1-x)^2}$. $2+4x+6x^2+\cdots=2(1+2x+3x^2+\cdots)$

6. (15%) Find the solution of the linear homogeneous recurrence relation $a_n = 7a_{n-1} - 6a_{n-2}$ with $a_0 = -1$ and $a_1 = 4$.

   Ans:

   The characteristic equation is $r^2-7r+6=(r-1)(r-6)=0$. The characteristic roots are $r=1$ and $r=6$. The solutions are of the form $a_n=c_1\cdot 1^n+c_2\cdot 6^n=c_1+c_2\cdot 6^n$. Since $a_0=-1$ and $a_1=4$ we have $c_1+c_2=-1$ and $c_1+6c_2=4$. Subtracting the first equation from the second gives $5c_2=5$, so $c_2=1$. This implies that $c_1+1=-1$, so $c_1=-2$. Hence the solution is $a_n=-2+6^n$.

7. (25%) Consider the recurrence relation $a_n = -a_{n-1} + n$.

   (a) Write the **associated homogeneous recurrence relation**.

   (b) Find the **general solution** to the **associated homogeneous recurrence relation**. (即$a_n^{(h)}$)

   (c) Find a **particular solution** to the given recurrence relation. (即$a_n^{(p)}$)

   (d) Write the **general solution** to the given recurrence relation.

   (e) Find the **solution** to the given recurrence relation when $a_0 = 1$.

   Ans: $r^n=-r^{n-1}$, $r=-1$ $\qquad n=(1\cdot n+0)\cdot 1^n$

   (a) $a_n=-a_{n-1}$. (b) $a_n=c(-1)^n$. (c) $a_n=\frac{n}{2}+\frac{1}{4}$. (d) $a_n=\frac{n}{2}+\frac{1}{4}+c(-1)^n$. (e) $a_n=\frac{n}{2}+\frac{1}{4}+\frac{3}{4}(-1)^n$.

   $an+b=-(a(n-1)+b)+n$, $a=\frac{1}{2}$, $b=\frac{1}{4}$

8. (15%) Use **generating function** to **solve** $a_n = 3a_{n-1} + 2^n$, $a_0 = 5$.

   Ans: $a_n = 7\cdot 3^n - 2\cdot 2^n$. (必須用生成函數的方法解才給分)