

1-10題每題3分

In questions 1–5 suppose $P(x, y)$ is a predicate and the universe for the variables x and y is $\{1, 2, 3\}$. Suppose $P(1,3)$, $P(2, 2)$, $P(2, 3)$, $P(3, 1)$, $P(3, 2)$ are true, and $P(x, y)$ is false otherwise. Determine whether the following statements are True or False. (請直接勾選答案)

1. True False $\forall x \exists y P(x, y)$.
2. True False $\exists x \forall y P(x, y)$.
3. True False $\neg \exists x \exists y (P(x, y) \wedge \neg P(y, x))$.
4. True False $\forall y \exists x (P(x, y) \rightarrow P(y, x))$
5. True False $\forall x \forall y (x \neq y \rightarrow (P(x, y) \vee P(y, x)))$.

In questions 6–10 determine whether the proposition is TRUE(T) or FALSE(F). (請直接勾選答案)

6. True False If $1 < 0$, then $3 = 4$.
7. True False If it is raining, then it is raining.
8. True False $1 + 1 = 2$ if and only if $2 + 2 = 3$.
9. True False If $2 + 1 = 3$, then $1 = 3 - 2$.
10. True False If $2 + 2 = 3$ or $2 + 2 = 4$, then $1 + 1 = 1$ and $1 + 1 = 2$.
11. (8%) How many satisfying assignments are there for the following system specification?
 $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$
12. (8%) $x \in \mathbb{Z}^+$, if $1285^x - 1116^x = 637^x$. Find x .
13. (8%) Find a proposition using only p, q, \neg , and the connective \vee that has the following truth table.

p	q	?
T	T	T
T	F	F
F	T	T
F	F	T

14. (8%) Find a compound proposition involving the propositional variables p, q , and r that is true when exactly two of p, q , and r are true and is false otherwise.
15. (8%) Prove or disprove that the following compound proposition is a tautology.

$$((p \vee q) \wedge \neg p) \rightarrow q$$

16. (12%) Suppose the variables x and y represent real numbers, and
 $L(x, y) : x < y$;
 $Q(x, y) : x = y$;
 $E(x) : x$ is even;
 $I(x) : x$ is an integer.

Write each of the following statements using these predicates and any needed quantifiers.

- (a) Every integer even is an integer.
- (b) If $x = y$, then x is greater than or equal to y .
- (c) There is no smallest real number.
17. (8%) What are the converse and the contrapositive of the statement “If it rains, then I get wet”?
18. (8%) Prove or disprove that $(p \rightarrow r) \wedge (q \rightarrow r)$ and $(p \vee q) \rightarrow r$ are equivalent.
19. (8%) Given the lemma: if $(p, q) = 1$, then $(p, q^2) = 1$. Prove that all the solutions to the equation $x^2 = x + 1$ are irrational (無理數). [註: (p, q) 表 p, q 的最大公因數]
20. (8%) A staff member of a small hospital said, “The hospital staff consists of 13 doctors and nurses, including me. The following facts apply to the staff members; whether you include me or not does not make any difference.”
The staff consists of:
 - more nurses than doctors;
 - more male doctors than male nurses;
 - more male nurses than female nurses;
 - at least one female doctor.
Give the sex and job of the speaker.

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p	q	? $\neg p \vee q$
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14. (8%) Find a compound proposition involving the propositional variables p, q , and r that is true when exactly two of p, q , and r are true and is false otherwise.
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Give the sex and job of the speaker.

1. 2

P	q	r	(p-q)	(q-r)	(p-q) ∧ (q-r)	(p-q) ∧ (q-r) ∧ (r-q)
T	T	T	F	F	F	F
T	T	F	F	T	F	F
T	F	T	T	F	F	F
T	F	F	T	T	T	F
F	T	T	F	F	F	F
F	T	F	F	T	F	F
F	F	T	T	F	F	F
F	F	F	T	T	T	F

12. $x=2$
根據韋達定理，當整數 $n > 2$ 時， $x^2 - 1116x + 1285 = 0$ 沒有正整數解。
 \therefore 帶入最式成立， \therefore 故 $x=2$ ，所以 $x=2$ 。

13. $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$

P	q	r	$p \vee \neg q$	$q \vee \neg r$	$r \vee \neg p$	$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$
T	T	T	F	F	F	F
T	T	F	F	T	F	F
T	F	T	T	F	F	F
T	F	F	T	T	T	T
F	T	T	T	F	F	F
F	T	F	T	T	T	T
F	F	T	T	F	F	F
F	F	F	T	T	T	F

15. 證偽。
 $((p \vee q) \wedge \neg p) \rightarrow q$
 $\equiv ((p \vee q) \wedge \neg p) \rightarrow q$
 $\equiv ((p \vee q) \wedge \neg p) \rightarrow q$
 $\equiv \neg((p \vee q) \wedge \neg p) \vee q$
 $\equiv \neg(p \vee q) \vee q$
 $\equiv (\neg p \wedge \neg q) \vee q$
 $\equiv \neg p \vee (\neg q \vee q)$
 $\equiv \neg p \vee \text{True}$
 $\equiv \text{True}$
 \therefore proposition is tautology.

16.
(a) $\forall x (E(x) \rightarrow I(x))$
(b) $\forall x \forall y (Q(x, y) \rightarrow \neg L(x, y))$
(c) $\forall y \exists x L(x, y)$

17. 證偽。
converse: 如果 I get wet, then it rains.
contrapositive: 如果 I don't get wet, then it doesn't rain.

18. 參考答案
 $(p \vee q) \rightarrow r$
 $\equiv \neg(p \vee q) \vee r$
 $\equiv (\neg p \wedge \neg q) \vee r$
 $\equiv (\neg p \vee r) \wedge (\neg q \vee r)$
 $\equiv (\neg p \vee r) \wedge (\neg q \vee r)$

19. 證偽。
19. lemma: $(p, q)=1 \Rightarrow (p, q^2)=1$
假設 x 是有理數，則 $x = \frac{a}{b}$ ，其中 a, b 互質。則 $x^2 = x + 1$
 $\frac{a^2}{b^2} = \frac{a}{b} + 1$
 $\frac{a^2}{b^2} - \frac{a}{b} = 1$
 $\frac{a^2 - ab}{b^2} = 1$
 $a^2 - ab = b^2$
 $a(a - b) = b^2$
 $\therefore a$ 是 b^2 的倍數，故 $a = kb$ ，其中 k 是正整數。
 $(kb)^2 - kb^2 = b^2$
 $k^2 b^2 - kb^2 = b^2$
 $b^2(k^2 - k) = b^2$
 $k^2 - k = 1$
 $k^2 - k - 1 = 0$
 $k = \frac{1 \pm \sqrt{5}}{2}$
 $\therefore k$ 不是整數，故 x 不是有理數，故 x 是無理數。