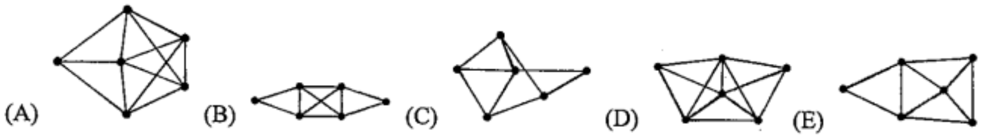
離散數學 期末考試 2023/01/12　　　　　　　　　　　　姓名:____________ 學號:______________

一、單選題(每題5%)

1. **If 50 objects are placed into N boxes, then there is at least one box containing at least K objects. Which of the following (N,K) pairs make the above statement true?**
(A) (3,18) (B) (70,3) (C) (15,5) (D) (7,8) (E) (51,2)
Ans:(D)
2. **Which one of the following graphs has an Euler Path but has not an Euler Cycle?**



Ans: (A)或(E)都對
3. **Which solves $a_n = a_{n-1} + 6a_{n-2}$ for a_n if a_0 = A and a_1 = B?**
(A) $\frac{1}{5}[(-3)^n(2A-B)+2^n(3A+B)]$ (B) $\frac{1}{5}[(-3)^n(2A-B)+2^n(3A-B)]$
(C) $\frac{1}{5}[(-2)^n(3A-B)+3^n(2A+B)]$ (D) $\frac{1}{5}[(-2)^n(3A+B)+3^n(2A+B)]$
(E) $\frac{1}{5}[(-2)^n(3A-B)+3^n(2A-B)]$
Ans:(C)
4. **The generating function in partial fraction decomposition for the recurrence equation $a_n = -a_{n-1} + 6a_{n-2}$ for a_n in terms of a_0 = A and a_1 = B is**
(A) $\frac{1}{5}\left[\frac{2A+B}{1-3x}+\frac{3A-B}{1+2x}\right]$ (B) $\frac{1}{5}\left[\frac{2A+B}{1-3x}+\frac{3A+B}{1+2x}\right]$ (C) $\frac{1}{5}\left[\frac{2A-B}{1-3x}+\frac{3A-B}{1+2x}\right]$ (D) $\frac{1}{5}\left[\frac{3A-B}{1-2x}+\frac{2A-B}{1+3x}\right]$ (E) $\frac{1}{5}\left[\frac{3A+B}{1-2x}+\frac{2A-B}{1+3x}\right]$
Ans:(E)
5. **The number of positive integer solutions of $x_1+x_2+...+x_n=r$ equals**
(A) $\binom{r-1}{n-1}$ **(B)** $\binom{n+r-1}{n-1}$ **(C)** $\binom{r}{n}$ **(D)** r^n **(E)** $\frac{1}{r+1}\binom{2n}{n}$
Ans:(A)
6. **If |A|=m, How many anti-symmetric relations on A are there?**
(A) $2^{(m^2-m)/2}$ **(B)** $2^{(m^2+m)/2}$ **(C)** $2^{m^2}-2^{(m^2+m)/2}$ **(D)** $2^m(3^{(m^2-m)/2})$ **(E)** $2^{(m^2-m)}$
Ans:(D)
7. **For integers a and b define a R b if 2a+3b=5n for some integer n. Which of the following claims about relation R is true?**
(A) R is not reflexive
(B) R is symmetric
(C) R is anti-symmetric
(D) R is not transitive
(E) R is a total order
Ans: (B))
8. **S = {1, 2, 3, 4, 5, 6, 7, 8, 9}. x, y∈S. Let x R y if x| y. We can conclude that**
(A) (S, R) is a total order;
(B) (S, R) does not have a maximal element;
(C) (S, R) is a lattice;
(D) (S, R) has a least element;
(E) {4,6} has a least upper bound.
Ans: (D)
9. **How many ways to put 5 distinguishable(可區別的) objects into 4 indistinguishable(不可區別的) boxes if empty boxes are allowed (可以有空盒子)? (hint: stirling number)**
(A) 10 (B) 41 (C) 51 (D) 52 (E) 60

Ans:(C)
S(n,j)=j*S(n-1,j)+S(n-1,j-1)

j / n	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	3	1				

4	1	7	6	1			
5	1	15	25	10	1		

S(5,1)+S(5,2)+S(5,3)+S(5,4)=51

10. What is the next larger 4-combination of the set {1,2,3,4,5,6} after {1,2,5,6}?
(A) {1,2,5,7} (B) {1,3,5,6} (C) {1,3,4,5} (D) {2,3,4,5} (E) {1,4,5,6}

ANS: (C)

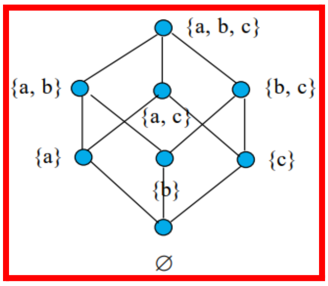
二、計算證明題

11. (8%) {{1,2}, {3}}is a partition of A={1, 2, 3}. Find the equivalence relation R on A such that R's different equivalence classes form the same partition of A.

Ans: R = {(1,1), (2,2), (3,3), (1,2), (2,1)}

12. (8%) Let A={a,b,c}. Construct the Hasse diagram of the poset (*P*(A), ⊆).

Ans:



13. (8%) **Let G=(V, E) be an undirected graph. If the incidence matrix (關聯矩陣)of G is as follows:**

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}. \end{array}$$

Find the adjacency matrix(鄰接矩陣) of G.

Ans:

$$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array}\begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

14. (8%) The coefficient(係數) of $x^2y^{\cancel{10}\,n-2}$ in the expansion of $(3x - y)^n$ is 594. What is n?

Ans: $C(n,2)x3^2x(-1)^{n-2}=594 \Rightarrow n=12$

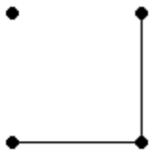
原題目n-2誤植為10，同學直接10+2=12後帶入 $C(12,2)x3^2x(-1)^{10}=594$無誤即可!

15. (8%) For each of the following sequences determine whether there is a simple graph whose vertices have these degrees. If the answer is yes, draw such a graph. If the answer is no, explain why no such graph exists. (序列各項代表各點的degree)

(a) 0, 1, 1, 2

(b) 1, 2, 3, 3, 4

Ans:



(a) Yes,

(b) No, the sum of degrees is odd, which conflicts with the handshaking theorem !

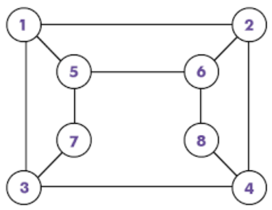
16. (8%)

(a) (5%) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take 1, 2, or 3 steps at a time.

(b) (3%) What are the initial conditions?

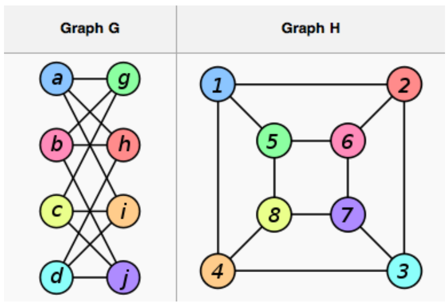
Ans: (a) $a_n=a_{n-1}+a_{n-2}+a_{n-3}$ (b)$a_0=1$, $a_1=1$, $a_2=2$,

17. (8%) Is the following graph bipartite? Justify your answer.



Ans: The graph is bipartite. The vertex set can be partitioned into {1,4,6,7} and {2, 3, 5,8}. There are no edges connecting a vertex in one set and a vertex in the other set.

18. (8%) Decide whether the graphs G and H are isomorphic(同構). Prove that your answer is correct!



Ans: The graphs are isomorphic. An isomorphism between G and H: f(a)=1, f(b)=6, f(c)=8, f(d)=3, f(g)=5, f(h)=2, f(i)=4, f(j)=7.

19. (8%) A={1,2,3,4,5,6,7}

(a) How many derangements of A are there?

(b) How many derangements of A begin with the integers 1, 2, and 3, in some order?

(c) How many ways can the digits 1,2,3,4,5,6,7 be arranged so that no odd digit is in its original position?

Ans:

(a) D_7 =7! (1-1/1!+1/2!-1/3!+1/4!-1/5!+1/6!-1/7!)=1854

(另解: $D_n=(n-1)(D_{n-1}+D_{n-2})$, $D_1=0$,$D_2=1$, 疊代可得D_7 =1854

(b) $D_3 x D_4$=2x9=18

(c) 7!-C(4,1)x6!+C(4,2)x5!-C(4,3)x4!+C(4,4)*3!=2790

20. (8%) Show that if we take n + 1 numbers from the set {1, 2, . . . , 2n}, then some pair of numbers will have no factors in common(沒有公因數). [Hint: The Pigeonhole Principle]

Ans: Note that consecutive numbers (such as 3 and 4) don't have any factors in common.
Let n pigeonholes be the following sets: {1, 2}, {3, 4}, . . . , {2n − 1, 2n}
pigeons are the n + 1 numbers we're choosing from the set {1, 2, . . . , 2n}.
By the pigeonhole principle, two of our n + 1 numbers will be in the same pigeonhole
Since the above sets were chosen to contain pairs of consecutive numbers, this means that we'll have a pair of consecutive numbers. This means we'll have a pair of numbers with no factors in common.