離散數學 Quiz 4 姓名:______________ 學號:______________

1. If G(x) is the generating function for $a_0,a_1,a_2,a_3,\ldots$, describe **in terms of G(x)** the following **generating function** for each of the following sequences.
(a) (5%) $0,0,0,a_0,a_1,a_2,\ldots$.
(b) (5%) $a_0,0,a_1,0,a_2,0,a_3,0,a_4,\ldots$.
(c) (5%) $a_3,a_4,a_5,a_6,\ldots$.
Ans: (a)$x^3G(x)$ (b)$G(x^2)$ (c): $1/x^3\,(G(x)-a_0-a_1x-a_2x^2)$

2. Please write down the **sequence** generated by each of the following **generating functions**.
(a) (5%) $\dfrac{x^6-1}{x-1}$
(b) (5%) $\dfrac{1}{1-2x}$
(c) (5%) $\dfrac{-2}{(1-x)^2}$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$
$$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$$
$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$$

Ans: (a) 1,1,1,1,1 (b)1,2,4,8,16,...
(c) -2,-4,-6,...

3. (12%) What is the solution to the **recurrence relation** $a_n=-a_{n-1}+12a_{n-2}$ if $a_0=3$ and $a_1=2$?

$r^2+r-12=0$
$(r+4)(r-3)=0$
$r=-4$ or 3
$\therefore a_n=\alpha(-4)^n+\beta\cdot 3^n$

$a_0=3,\ \alpha+\beta=3$
$a_1=2,\ -4\alpha+3\beta=2$
$\begin{cases}\alpha+\beta=3\\-4\alpha+3\beta=2\end{cases}$ $\begin{cases}3\alpha+3\beta=9\\-4\alpha+3\beta=2\end{cases}$

$7\alpha=7,\ \alpha=1$
$\beta=2$
$\therefore a_n=(-4)^n+2\cdot 3^n$ #

4. (12%) Solve $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 1$, $a_1 = 5$.

$r^2-6r+9=0$
$(r-3)^2=0,\ r=3$
$\therefore a_n=\alpha 3^n+n\beta 3^n$

$a_0=1,\quad 1=\alpha\cdot 1+0\cdot\beta\cdot 1,\quad \alpha=1$
$a_1=5,\quad 5=3+3\beta,\quad 2=3\beta,\ \beta=\frac{2}{3}$
$\therefore a_n=3^n+n\cdot\frac{2}{3}\cdot 3^n$

5. Let X={A,B,C} and Y={1,2,3}. .
(a) (4%) How many functions f: $\mathbf{X^2} \to$ Y are **one-to-one**?
(b) (4%) How many functions f: $\mathbf{X^2} \to$ Y are **onto**?
(c) (4%) For a function f and a set S, we define f(S)={ f(i) | i ∈ S }. How many functions f: $\mathbf{X^2} \to$ Y are there such that $|f(X^2)|=2$? ⇒代表f(x)的值域剛好為Y的2個element
Ans:
(a) 0
(b) 3^9-C(3,1)2^9+C(3,2)1^9
(c) C(3,2) (2^9-C(2,1)1^9)

(c) C^3_2 ⇒ Y中選2個 element
2^9 ⇒ 所有可能的方法
$C^2_1\cdot 1^9$ ⇒ 只有對應到1個element的方法(需扣除)
$\therefore$ all: $C^3_2\times[2^9-(C^2_1\cdot 1^9)]$

6. Solve the given recurrence relation: $a_n = 3a_{n-1} - 2a_{n-2} - 5n + 3$ step by step.
(a) (2%) Write the **associated homogeneous recurrence relation**.
(b) (4%) Find the **general solution** (i.e., $a_n^{(h)}$) to the **associated** homogeneous recurrence relation.
(c) (4%)Find a **particular solution** (i.e., $a_n^{(p)}$) to the **given recurrence relation**.
(d) (2%)Now that the **general solution** to the given recurrence relation is $a_n=a_n^{(h)}+a_n^{(p)}$. Find the particular solution to the given recurrence relation **when a_0 = 2, a_1 = 8.**

(a) $a_n^{(h)}=3a_{n-1}-2a_{n-2}$

(b) $r^2-3r+2=0,\ (r-1)(r-2)=0,\ r=1$ or 2
$\therefore a_n^{(h)}=\alpha+\beta\cdot 2^n$ #

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(c) $F(n) = -5n+3$

∵ $S=1$, 和根相同, 重數為1.

∴特殊解的一般形式為 $n^1\cdot(cn+d)\cdot 1^n = n(cn+d)$

帶入 $a_n^{(p)} = 3a_{n-1}^{(p)} - 2a_{n-2}^{(p)} - 5n+3$

$$cn^2+dn = 3(c(n-1)^2+d(n-1)) - 2(c(n-2)^2+d(n-2)) - 5n+3$$
$$= 3(cn^2-2cn+c+dn-d) - 2(cn^2-4cn+4c+dn-2d) - 5n+3$$
$$= 3cn^2-6cn+3c+3dn-3d-2cn^2+8cn-8c-2dn+4d-5n+3$$

$$\cancel{cn^2}+\cancel{dn} = \cancel{cn^2}+2cn-5c+\cancel{dn}+d-5n+3$$
$$0 = 2cn-5c+d-5n+3$$
$$= (2cn-5n)+(d-5c+3)$$
$$= n\underbrace{(2c-5)}_{0}+\underbrace{(d-5c+3)}_{0},\ c=\frac{5}{2},\ d=\frac{19}{2}$$

∴ $a_n^{(p)} = n\left(\frac{5}{2}n+\frac{19}{2}\right)$ #

(d) $a_n = a_n^{(h)} + a_n^{(p)}$
$= \alpha+\beta 2^n + n\left(\frac{5}{2}n+\frac{19}{2}\right)$

$a_0=2$, $\alpha+\beta=2$

$a_1=8$, $\alpha+2\beta+12=8$

$\begin{cases}\alpha+\beta=2\\ \alpha+2\beta=-4\end{cases}$ ∴ $\beta=-6$, $\alpha=8$

∴ $a_n = 8-6\times 2^n + n\left(\frac{5}{2}n+\frac{19}{2}\right)$ #

7. (12%) An office manager has 4 employees and 9 projects to be completed. In how many ways can the projects be assigned to the employees so that **each employee works on at least one project**.

Ans:

$$\boxed{4^9-\binom{4}{1}3^9+\binom{4}{2}2^9-\binom{4}{3}1^9}$$

8. How many ways can the digits 1,2,3,4,5,6 be arranged so that
(a) (6%) **none** of the digits is in its original position?
(b) (6%) no **even digit** is in its original position?

Ans:
(a) 錯位排列D_6 = 265
(b) 6!-C(3,1)5!+C(3,2)4!-C(3,3)3!=426

(b) $C^3_1\cdot 5!$ ⇒ 1個偶數在原位,其餘的去排列
$C^3_2\cdot 4!$, $C^3_3\cdot 3!$ ⇒ 依此類推

9. (12%) Solve $a_n = 3a_{n-1} + 2^n + 5$, $a_0 = 1$, using **generating functions**(限用生成函數).

$$a_n = 3a_{n-1} + 2^n + 5$$
$$\sum_{k=0}^{\infty} a_n x^k = \sum_{k=0}^{\infty} 3a_{n-1}x^k + \sum_{k=0}^{\infty} 2^n x^k + \sum_{k=0}^{\infty} 5x^k$$
$$G(x) - a_0 = 3x\sum_{k=1}^{\infty} a_{n-1}x^{k-1} + 2x\sum_{k=1}^{\infty} 2^{n-1}x^{k-1} + 5x\sum_{k=1}^{\infty} x^{k-1}$$
$$G(x) - 1 = 3x\cdot G(x) + \frac{2x}{1-2x} + \frac{5x}{1-x}$$
$$G(x) - 3xG(x) = \frac{2x}{1-2x} + \frac{5x}{1-x} + 1$$
$$G(x)(1-3x) = \frac{2x}{1-2x} + \frac{5x}{1-x} + 1$$
$$G(x) = \boxed{\frac{2x}{(1-2x)(1-3x)}} + \boxed{\frac{5x}{(1-x)(1-3x)}} + \frac{1}{1-3x}$$

紅色: $\alpha(1-3x)+\beta(1-2x)=2x$
$\alpha-3\alpha x+\beta-2\beta x=2x$
$\begin{cases}\alpha+\beta=0\\ -3\alpha-2\beta=2\end{cases} \Rightarrow \begin{cases}3\alpha+3\beta=0\\ -3\alpha-2\beta=2\end{cases}$, $\beta=2$, $\alpha=-2$

綠色: $\alpha(1-3x)+\beta(1-x)=5x$
$\alpha-3\alpha x+\beta-\beta x=5x$
$\begin{cases}\alpha+\beta=0\\ -3\alpha-\beta=5\end{cases}$ $-2\alpha=5$, $\alpha=-\frac{5}{2}$, $\beta=\frac{5}{2}$

$$∴ G(x) = \frac{-2}{1-2x} + \frac{2}{1-3x} + \frac{-\frac{5}{2}}{1-x} + \frac{\frac{5}{2}}{1-3x} + \frac{1}{1-3x}$$
$$= \frac{-2}{1-2x} + \frac{\frac{11}{2}}{1-3x} + \frac{-\frac{5}{2}}{1-x}$$
$$= \sum_{n=0}^{\infty} -2\times 2^n x^n + \sum_{n=0}^{\infty}\frac{11}{2}\cdot 3^n\cdot x^n + \sum_{n=0}^{\infty} -\frac{5}{2}\cdot x^n$$
$$= -2\cdot 2^n + \frac{11}{2}\cdot 3^n - \frac{5}{2} = -2^{n+1} + \frac{11}{2}\cdot 3^n - \frac{5}{2}$$ #