離散數學 Quiz 4 姓名:______________ 學號:______________

1. (15%) At a party, 7 gentlemen check their hats. In how many ways can their hats be returned so that
   (a)**none** of the gentlemen receives his own hat?
   (b)**at least one** of the gentlemen receives his own hat?
   (c)**exactly two** gentlemen receive their own hats?
   Ans:

2. (15%) Find the number of **onto** functions from set P = {a, b, c, d} to set Q = {u, v, w}
   Ans:

3. (20%) Find a formula for a **generating function** for
   (a) −1, −1, −1, ...
   (b) −1, −2, −3, ...
   (c) 0, 0, 0, 0, −1, −2, −3, ...
   (d) 3, 2, 1, 0, −1, −2, −3, ...

   Ans:

4. (15%) What is the solution to the **recurrence relation** $a_n=8a_{n-1}+9a_{n-2}$ if $a_0=3$ and $a_1=7$?
   Ans:

5. (15%) Solve $a_n = 10a_{n-1} - 25a_{n-2}$, $a_0 = 3$, $a_1 = 4$.
   Ans:

6. (15%) Solve $a_n = 3a_{n-1} + 2^n$, with initial condition $a_0 = 2$.
   Ans:

7. (15%) Solve $a_n = 3a_{n-1} - 2$, $a_0 = 4$, using generating functions(生成函數).

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   Ans:
   (a) 錯位排列D_7 = 1854.
   (b) 7! − D_7 = 7! − 1854 = 3186.
   (c) 觀念: 令 A_i 表集合 {1, 2, . . . , n} 所有排列中恰有 i 個數字待在原位者. 則 $|A_i|$ = C(n, i) × D_{n-i}。 $|A_2|$ = C(7, 2) D_5 =21*44=924
2. (15%) Find the number of **onto** functions from set P = {a, b, c, d} to set Q = {u, v, w}
   Ans:The number of onto functions = 3^4– C(3,1)$(3-1)^4$ + C(3,2)$(3-2)^4$= 81 – 48 + 3=36

3. (20%) Find a formula for a **generating function** for
   (a) −1, −1, −1, ...
   (b) −1, −2, −3, ...
   (c) 0, 0, 0, 0, −1, −2, −3, ...
   (d) 3, 2, 1, 0, −1, −2, −3, ...
   Ans: (a) $\frac{-1}{1-x}$ (b)$\frac{-1}{(1-x)^2}$(c)$\frac{-x^4}{(1-x)^2}$(d)$3+2x+x^2+\frac{-x^4}{(1-x)^2}$
4. (15%) What is the solution to the **recurrence relation** $a_n=8a_{n-1}+9a_{n-2}$ if $a_0=3$ and $a_1=7$?
   Ans:
   此線性齊次遞迴關係之特徵方程式為$r^2-8r-9=0$
   => (r-9)(r+1)=0 => r=-1或r=9 (兩相異特徵根)
   =>其解的型式必為$a_n=c(-1)^n+d(9)^n$, c,d為常數
   又$a_0=3=c+d$, $a_1=7=-c+9d$, 解聯立方程式得c=2, d=1
   因此可得解為$a_n=2(-1)^n+9^n$,

5. (15%) Solve $a_n = 10a_{n-1} - 25a_{n-2}$, $a_0 = 3$, $a_1 = 4$.
   Ans:
   此線性齊次遞迴關係之特徵方程式為$r^2-10r+25=0$
   => $(r-5)^2=0$ => r=5 (2重根)
   =>其解的型式必為$a_n=c(5)^n+dn(5)^n$, c,d為常數
   又$a_0=3=c$, $a_1=4=5c+5d$, 解聯立方程式得c=3, d=-11/5
   因此可得解為$a_n=3(5)^n-(11/5)n(5)^n$,

6. (15%) Solve $a_n = 3a_{n-1} + 2^n$, with initial condition $a_0 = 2$.
   Ans:
   (1)與此遞迴關係相關之線性齊次遞迴關係為$a_n^{(h)} = 3a_{n-1}^{(h)}$
   =>此線性齊次遞迴關係之特徵方程式為r-3=0 => r=3
   =>其解的型式必為$a_n^{(h)}=c(3)^n$, c為常數

   (2)滿足$a_n^{(p)} = 3a_{n-1}^{(p)} + 2^n$的一特定解型式為$a_n^{(p)}=d(2^n)$, d為常數
   =>$d(2^n)=3(d(2^{n-1}))+2^n$=>2d=3d+2=>d=-2
   亦即$a_n^{(p)}=-2(2^n)=-2^{n+1}$

   因此原題所述遞迴關係之一般解的型式必為$a_n=a_n^{(h)}+a_n^{(p)}=c(3)^n-2^{n+1}$
   又$a_0=2=c-2$=>c=4
   可得解為$a_n=4(3)^n-2^{n+1}$

7. (15%) Solve $a_n = 3a_{n-1} - 2$, $a_0 = 4$, using generating functions(生成函數).

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Ans: (用生成函數解才給分)

令 $G(x)=\sum_{n=0}^{\infty} a_n x^n.$

$$
\begin{aligned}
a_n &= 3a_{n-1}-2 && \text{(recurrence relation to be solved)}\\
a_n x^n &= 3a_{n-1}x^n - 2x^n && \text{(multiply by } x^n\text{)}\\
\sum_{n=1}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} 3a_{n-1}x^n - \sum_{n=1}^{\infty} 2x^n && \text{(sum from } n=1 \text{ to } \infty\text{)}\\
G(x)-a_0 &= 3x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} - 2x\sum_{n=1}^{\infty} x^{n-1} && \text{(substitute for } G(x)\text{; rewrite sums)}\\
G(x)-4 &= 3x\sum_{n=0}^{\infty} a_n x^n - 2x\sum_{n=0}^{\infty} x^n && \text{(change indices of summation)}\\
G(x)-4 &= 3xG(x)-\frac{2x}{1-x} && \text{(evaluate sums)}\\
G(x)(1-3x) &= 4-\frac{2x}{1-x} && \text{(combine like terms)}\\
G(x) &= \frac{4-6x}{(1-x)(1-3x)} && \text{(solve for } G(x)\text{)}\\
G(x) &= \frac{1}{1-x}+\frac{3}{1-3x} && \text{(rewrite as sum of two fractions)}\\
G(x) &= \sum_{n=0}^{\infty} x^n + 3\sum_{n=0}^{\infty} 3^n x^n && \text{(rewrite as infinite sums)}\\
G(x) &= \sum_{n=0}^{\infty} (1+3^{n+1})x^n. && \text{(combine sums)}
\end{aligned}
$$

因此其解為a_n=1+3^{n+1}