

1. (10%) Professor Lai gives a **quiz** that has **8 True-or-False questions(是非題)**. What is the **minimum number of students** that must be in Lai's class in order to **guarantee(保證)** that **at least 2 answer sheets are identical (至少有2張答案卷相同)**? (Assume that no answers are left blank.)

2. (16%) Suppose $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d, e, f, g\}$
 (a) Find the number of functions $f: A \rightarrow B$.
 (b) Find the number of **one-to-one** functions $g: A \rightarrow B$.

Ans:

(a)

(b)

3. (12%)
 (a) Find the **next 4 (緊接著4個)** permutation in lexicographic order after 142536.
 (b) Find the **next 4** 4-combinations of the set $\{1, 2, 3, 4, 5, 6, 7\}$ after $\{1, 2, 4, 7\}$.

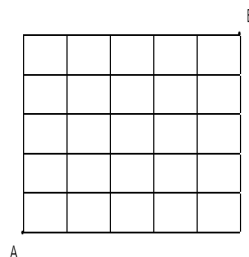
Ans:

(a)

(b)

4. (10%) The figure at the right shows a 5-block by 5-block grid of streets. Find the number of ways in which you can **go from point A to point B**, where at each stage you can **only go right or up**. (You are not allowed to go left or down.) For example, one allowable route from A to B is: **Right, Right, Up, Right, Up, Up, Right, Right, Up, Up**.

Ans:



5. (10%) Find the coefficient of x^7y^5 in the expansion of $(2x - 3y)^{12}$.

Ans:

6. (20%)

- (a) (10%) Find the number of solutions to $x + y + z = 32$, where x, y , and z are nonnegative integers.
 (b) (5%) Answer part (a), but assume that $y \geq 10$.
 (c) (5%) Answer part (a), but assume that $x \leq 20$

Ans: (a) (b) (c)

7. (10%) Find the number of permutations of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 **either start with '1' or end with '9'**.

Ans:

8. (10%) In how many ways can we distribute 8 red balls and 10 blue balls among 5 children so that **each child receives at least one blue ball**?





































Ans:

9. (10%) Let $X \subseteq \{1, 2, \dots, 99\}$ and $|X| = 10$. Show that it is possible to select two **disjoint nonempty proper subsets** Y, Z of X such that $\sum(y | y \in Y) = \sum(z | z \in Z)$. [hint: The Pigeonhole Principle]

Ans:

10. (10%)How many ways are there to distribute 7 **distinguishable**(可區別的) **objects** into 4 **indistinguishable**(不可區別的) **boxes** so that each of the boxes contains at least one object?

Ans: _____

1. (10%) Professor Lai gives a **quiz** that has **8 True-or-False questions(是非題)**. What is the **minimum number of students** that must be in Lai's class in order to **guarantee(保證)** that **at least 2 answer sheets are identical (至少有2張答案卷相同)**? (Assume that no answers are left blank.)

Ans: 鴿籠原理

There are 2^8 possible answer sheets. Therefore $2^8 + 1$ is the minimum number that will guarantee three identical answer sheets.

2. (16%) Suppose $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d, e, f, g\}$
 (a) Find the number of functions $f: A \rightarrow B$.
 (b) Find the number of **one-to-one** functions $g: A \rightarrow B$.

Ans:

(a) 7^5

(b) $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$

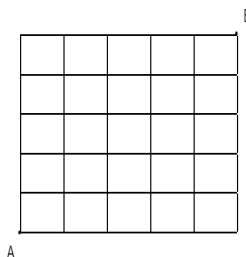
3. (12%)
 (a) Find the **next 4** (緊接著4個) permutation in lexicographic order after 142536.
 (b) Find the **next 4** 4-combinations of the set $\{1, 2, 3, 4, 5, 6, 7\}$ after $\{1, 2, 4, 7\}$.

Ans:

(a) 142563, 142635, 142653, 143256

(b) $\{1, 2, 5, 6\}$, $\{1, 2, 5, 7\}$, $\{1, 2, 6, 7\}$, $\{1, 3, 4, 5\}$

4. (10%) The figure at the right shows a 5-block by 5-block grid of streets. Find the number of ways in which you can **go from point A to point B**, where at each stage you can **only go right or up**. (You are not allowed to go left or down.) For example, one allowable route from A to B is: **Right, Right, Up, Right, Up, Up, Right, Right, Up, Up**.



Ans: $C(10, 5)$

5. (10%) Find the coefficient of x^7y^5 in the expansion of $(2x - 3y)^{12}$.

Ans: $C(12, 7) \cdot 2^7 \cdot (-3)^5$

6. (20%)

- (a) (10%) Find the number of solutions to $x + y + z = 32$, where x, y , and z are nonnegative integers.
 (b) (5%) Answer part (a), but assume that $y \geq 10$.
 (c) (5%) Answer part (a), but assume that $x \leq 20$

Ans: (a) $C(34, 2)$ (b) $C(24, 2)$ (c) $C(34, 2) - C(13, 2)$

7. (10%) Find the number of permutations of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 **either start with '1' or end with '9'**.

Ans: $8! + 8! - 7!$

8. (10%) In how many ways can we distribute 8 red balls and 10 blue balls among 5 children so that **each child receives at least one blue ball**?

Ans: $C(12, 4) \cdot C(9, 4)$

9. (10%) Let $X \subseteq \{1, 2, \dots, 99\}$ and $|X| = 10$. Show that it is possible to select two **disjoint nonempty proper subsets** Y, Z of X such that $\sum(y | y \in Y) = \sum(z | z \in Z)$. [hint: The Pigeonhole Principle]

Ans: $|P(X) - \emptyset - X| = 2^{10} - 2 = 1022$, 令 $P(X) - \emptyset = \{S_1, S_2, \dots, S_{1022}\}$

$$\forall i, 1 \leq \sum(s | s \in S_i) \leq 91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99 = 855$$

根據鴿籠原理, 必可找到 $\lceil 1023/855 \rceil = 2$ 個 nonempty proper subset Y, Z , $Y \neq Z$, $\sum(y | y \in Y) = \sum(z | z \in Z)$

若 $Y \cap Z = \emptyset$, 則 Y, Z 即所求

若 $Y \cap Z \neq \emptyset$, 則 $Y' = Y - (Y \cap Z)$, $Z' = Z - (Y \cap Z)$ 為所求

10. (10%) How many ways are there to distribute 7 **distinguishable**(可區別的) **objects** into 4 **indistinguishable**(不可區別的) **boxes** so that each of the boxes contains at least one object?

Ans: (stirling number)

$$S(n,j)=j \cdot S(n-1,j)+S(n-1,j-1)$$

j \ n	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	3	1				
4	1	7	6	1			
5	1	15	25	10	1		
6	1	31	90	65	15	1	
7	1	63	301	350	140	21	1

$$S(7,4)=4 \times S(6,4)+S(6,3)=4 \times 65+90=350$$